MANIN PRODUCTS, KOSZUL DUALITY, LODAY ALGEBRAS AND DELIGNE CONJECTURE

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Dedicated to Jean-Louis Loday, on the occasion of his sixtieth birthday[1]

Abstract. In this article we give a conceptual definition of Manin products in any category endowed with two coherent monoidal products. This construction can be applied to associative algebras, non-symmetric operads, operads, colored operads, and properads presented by generators and relations. These two products, called black and white, are dual to each other under Koszul duality functor. We study their properties and compute several examples of black and white products for operads. These products allow us to define natural operations on the chain complex defining cohomology theories. With these operations, we are able to prove that Deligne's conjecture holds for a general class of operads and is not specific to the case of associative algebras. Finally, we prove generalized versions of a few conjectures raised by M. Aguiar and J.-L. Loday related to the Koszul property of operads defined by black products. These operads provide infinitely many examples for this generalized Deligne's conjecture.

Introduction

In his works on quantum groups and non-commutative geometry, Yu. I. Manin defined two products in the category of quadratic algebras. An associative algebra A is called *quadratic* if it is isomorphic to a quotient algebra of the form $A = T(V)/(R)$, where $T(V)$ is the free algebra on V and where (R) is the ideal generated by $R \subset V^{\otimes 2}$. Let $A = T(V)/(R)$ and $B = T(W)/(S)$ be two quadratic algebras. Any quadratic algebra generated by the tensor product $V \otimes W$ is determined by a subspace of $(V \otimes W)^{\otimes 2}$. Since $R \subset V^{\otimes 2}$ and $S \subset W^{\otimes 2}$, one has to introduce the isomorphism $(23) : V \otimes V \otimes W \otimes W \to V \otimes W \otimes V \otimes W$ defined by the permutation of the second and third terms. The black and white products were defined by Manin as follows

$$\begin{aligned} A \bullet B &:= T(V \otimes W)/\big((23)(R \otimes S)\big), \\ A \circ B &:= T(V \otimes W)/\big((23)(R \otimes W^{\otimes 2} + V^{\otimes 2} \otimes S)\big). \end{aligned}$$

Since (23) is an isomorphism, many properties of the algebras A and B remain true for their black and white products. For instance, the white product of two quadratic algebras is equal to their degreewise tensor product $A \otimes B := \oplus_{n \geq 0} A_n \otimes B_n$. Therefore, one can apply the method of J. Backelin [Bac] to prove that the white product of two Koszul algebras is again a Koszul algebra.

Koszul duality theory is a homological algebra theory developed by S. Priddy [Pri70] in 1970 for quadratic algebras. To a quadratic algebra $A = T(V)/(R)$ generated by a finite dimensional vector space V, one can associate the *Koszul dual algebra* $A^! := T(V^*)/(R^\perp)$. Under this finite dimensional hypothesis, we have $(A \circ B)^! = A^! \bullet B^!$, that is black and white constructions are dual to each other under Koszul duality functor. The main result of Manin is the following adjunction in the category of finitely generated quadratic algebras

$$\mathrm{Hom}_{Q.\,Alg}(A \bullet B^!,\, C) \cong \mathrm{Hom}_{Q.\,Alg}(A,\, B \circ C).$$

[1]The title of this paper can be read "How to use Manin's products to prove Deligne's conjecture for Loday algebras with Koszul property"

2000 *Mathematics Subject Classification.* 18D50, 18D10, 20C30, 55P48.
Keywords and phrases. Monoidal categories, 2-monoidal categories, Algebras, Operads, Properads, Manin's black and white products, Koszul duality, Cohomology operations, Deligne's conjecture

Using the general properties of internal cohomomorphisms, Manin proved that $A \bullet A^!$ is a Hopf algebra and was able to realize well known quantum groups as black products of an algebra with its Koszul dual. For more properties of Manin's products for quadratic algebras, we refer the reader to the book of A. Polishchuk and L. Positselski [PP05].

Koszul duality theory was later generalized to binary quadratic operads by V. Ginzburg and M. Kapranov [GK95] in 1994. This generalization comes from the fact that an operad, like an associative algebra, is a monoid in a monoidal category. A quadratic operad $\mathcal{P} = \mathcal{F}(V)/(R)$ is a quotient of a free operad by an ideal generated by a sub-$\mathbb{S}$-module R of $\mathcal{F}_{(2)}(V)$, the part of weight 2 of $\mathcal{F}(V)$. Let $\mathcal{P} = \mathcal{F}(V)/(R)$ and $\mathcal{Q} = \mathcal{F}(W)/(S)$ be two quadratic operads. A quadratic operad generated by the tensor product $V \otimes W$ is determined by a subspace of $\mathcal{F}_{(2)}(V \otimes W)$. Since $R \subset \mathcal{F}_{(2)}(V)$ and $S \subset \mathcal{F}_{(2)}(W)$, we need a map from $\mathcal{F}_{(2)}(V) \otimes \mathcal{F}_{(2)}(W)$ to $\mathcal{F}_{(2)}(V \otimes W)$. In the binary case, Ginzburg and Kapranov mentioned in [GK95'] two maps $\Psi : \mathcal{F}_{(2)}(V) \otimes \mathcal{F}_{(2)}(W) \to \mathcal{F}_{(2)}(V \otimes W)$ and $\Phi : \mathcal{F}_{(2)}(V \otimes W) \to \mathcal{F}_{(2)}(V) \otimes \mathcal{F}_{(2)}(W)$ and defined the black and white products for binary quadratic operads as follows.

$$\begin{aligned} \mathcal{P} \bullet \mathcal{Q} &:= \mathcal{F}(V \otimes W)/\big(\Psi(R \otimes S)\big), \\ \mathcal{P} \circ \mathcal{Q} &:= \mathcal{F}(V \otimes W)/\big(\Phi^{-1}(R \otimes W^{\otimes 2} + V^{\otimes 2} \otimes S)\big). \end{aligned}$$

When the operad $\mathcal{P} = \mathcal{F}(V)/(R)$ is a binary quadratic operad generated by a finite dimensional $\mathbb{S}$-module V, they defined a *Koszul dual operad* by the formula $\mathcal{P}^! := \mathcal{F}(V^\vee)/(R^\perp)$, where $V^\vee(2) := V^*(2) \otimes \mathrm{sgn}_{\mathbb{S}_2}$ is the dual representation twisted by the signature representation. As in the case of algebras, they proved that $(\mathcal{P} \circ \mathcal{Q})^! = \mathcal{P}^! \bullet \mathcal{Q}^!$ and they showed the adjunction

$$\mathrm{Hom}_{Bin.Q.\,Op.}(\mathcal{P} \bullet \mathcal{Q}^!,\, \mathcal{R}) \cong \mathrm{Hom}_{Bin.Q.\,Op.}(\mathcal{P},\, \mathcal{Q} \circ \mathcal{R}),$$

in the category of finitely generated binary quadratic operads.

From the properties of black and white products for associative algebras and binary quadratic operads, a few natural questions arise. Where do the functors Ψ and Φ conceptually come from ? Is the black or white product of two binary Koszul operads still a Koszul operad ? Can one do non-commutative geometry with an operad of the form $\mathcal{P} \bullet \mathcal{P}^!$? One can also add : is it possible to recover classical operads as black or white products of more simple operads? Can black and white products help to describe the natural operations acting on cohomological spaces ? The aim of this paper is to answer these questions.

Let us recall that Koszul duality theory of associative algebras and binary quadratic operads was extended to various other monoidal categories in the last few years. The following diagram shows these monoidal categories were Koszul duality holds. They are represented by the name of their monoids.

$$\begin{tikzcd}
\text{Associative Algebras} \arrow[r, tail] \arrow[dr] & \text{Non-symmetric Operads} \arrow[d, "\Sigma"] & \\
& \text{Operads} \arrow[r, tail] \arrow[d, tail] & \text{Colored operads} \\
& \text{Dioperads} \arrow[d, tail] & \\
& \text{Properads} &
\end{tikzcd}$$

Koszul duality for dioperads was proved by W.L. Gan in [Gan03], it was proved by P. Van der Laan in [VdL] for colored operads and by the author for properads in [Val1]. A *properad* is an object slightly smaller than a prop which encodes faithfully a large variety of algebraic structures like bialgebras or Lie bialgebras, for instance (see Appendix A for more details). We would like to emphasize that the Koszul dual that appears naturally, without finite dimensional assumptions, is a comonoid (coalgebra, cooperad, coproperad, etc ...). See Section 2 for more details.

To answer the first question about the conceptual definition of the functors Ψ and Φ, we introduce a new notion of category endowed with 2 coherent monoidal products. It is called *2-monoidal category* in Section 1. This definition generalizes previous notions given by A. Joyal and R. Street in [JS93] in the framework of braided tensor categories and by C. Balteanu, Z. Fiedorowicz, R. Schwänzl and R. Vogt in [BFSV03] in the framework of iterated monoidal category and iterated loop spaces. All the examples given above are monoids in a 2-monoidal category. In a 2-monoidal category, we define the functors Ψ and Φ by universal properties. This allows us to define white products for monoids presented by generators and relations in Section 3. Since the Koszul dual is a comonoid, we define a black product for comonoids presented by generators and relations. (This notion is introduced and detailed in Appendix B).

The white product defined here coincides with the one of Y. Manin for quadratic algebras $A \circ B$, with the one of Ginzburg-Kapranov for binary quadratic operads $\mathcal{P} \circ \mathcal{Q}$ and with the one of R. Berger, M. Dubois-Violette and M. Wambst [BDVW03] for N-homogenous algebras. Note that the white product is defined without homogenous assumption. Therefore, one could apply them to non-homogenous cases. In this sense, it would be interesting to study the properties of the white products of Artin-Schelter algebras [AS87, LPWZ04].

Under finite dimensional assumptions, the twisted linear dual of the Koszul dual cooperad gives the Koszul dual operad defined by [GK95]. Using this relation, we define a black product for operads in Section 4. We do several computations of black and white products and show that some classical operads can be realized as products of simpler operads. All these examples are products of Koszul operads and the result is again a Koszul operad. This fact is not true in general and we provide a counterexample in Section 4.5. Whereas this property holds for associative algebras, it is not true here because the functors Φ and Ψ are not isomorphisms.

We extend the adjunction of Manin and Ginzburg-Kapranov and prove that $\mathcal{P} \bullet \mathcal{P}^!$ is a Hopf operad. Since operads are non-linear generalizations of associative algebras, the notion of Hopf operad can be seen as a non-linear generalization of bialgebras. Hopf operads of the form $\mathcal{P} \bullet \mathcal{P}^!$ can provide new examples of "quantum groups", in the philosophy of [Man88]. This adjunction also allows us to understand the algebraic structures on tensor products or spaces of morphisms of algebras. For instance, it gives a description of the structure of cohomology spaces.

Non-symmetric operads are operads without the action of the symmetric group. One can symmetrize a non-symmetric operad to get an operad. (It corresponds to the functor Σ in the diagram above). The image of a non-symmetric operad under functor is called a *regular* operad. We define black $\blacksquare$ and white $\square$ square products for regular operads as the image of black and white products of non-symmetric operads in Section 5. In the case of binary quadratic regular operads, the black square product given here corresponds to the one introduced by K. Ebrahimi-Fard and L. Guo in [EFG05] (see also J.-L. Loday [Lod04]). We prove the same kind of results for regular operads and square products than the ones for operads and Manin's products.

The adjunction for black and white square products allows us to construct natural operations on the chain complex defining the cohomology of an algebra over a non-symmetric (regular) operad. The example of associative algebras is very classical. Since the introduction of this (co)chain complex by Hochschild in 1945, it has been extensively studied. M. Gerstenhaber proved in the sixties that the cohomology of any associative algebra is endowed with two coherent products : the commutative cup product and a Lie bracket. This structure is now called a *Gerstenhaber algebra*. (Gerstenhaber also used this Lie bracket to study deformations of associative algebras.

This led to the work of Kontsevich on deformation-quantization of Poisson manifold thirty years latter). In homotopy theory, there is a topological operad, formed by configurations of disks in the plane and called the *little disks operad*, whose action allows to recognize two-fold loop spaces. In 1976, F. Cohen proved that the operad defined by the homology of the little disks operad is equal to the operad coding Gerstenhaber algebras. Therefore the Hochschild cohomology space is an algebra over the homology of the little disks operad. This surprising link between algebra, topology and geometry led Deligne to formulate the conjecture that this relation can be lifted on (co)chain complexes, that is the singular chain complex of the little disks operad acts on the Hochschild (co)chain complex of an associative algebra. This conjecture was proved by several researchers using different methods. In the present paper, we take a transversal approach. We prove that Deligne's conjecture holds for a general class of operads and is not specific to the case of associative algebras. Using Manin's products, we construct operations on the chain complex of any algebra over an operad of this class. (To be more precise, finitely generated binary non-symmetric Koszul operads form this class). Since these operations verify the same relations than the ones on the Hochschild (co)chain complex, Deligne's conjecture is then proved with the same methods.

Since the white square product is the Koszul dual of the black square product, we can compute the Koszul duals of operads defined by black square product. The first example is the operad $\mathcal{Q}uad = \mathcal{D}end \blacksquare \mathcal{D}end$ defined by M. Aguiar and J.-L. Loday in [AL04]. Using the explicit description of its Koszul dual and the method of partition posets of [Val2], we prove that it is Koszul over $\mathbb{Z}$, which answers a conjecture of Aguiar-Loday. Actually, with the same methods we show that the families $\mathcal{D}end^{\blacksquare n}$, $\mathcal{D}ias^{\square n}$ and $\mathcal{T}ri\mathcal{D}end^{\blacksquare n}$, $\mathcal{T}rias^{\square n}$ are Koszul over $\mathbb{Z}$. These families provide infinitely many examples for which Deligne's conjecture hold over $\mathbb{Z}$.

Appendix A is a survey on the notions of operads and properads. Appendix B yields a categorical approach of algebra with monoids and comonoids ((co)ideal, (co)modules).

Unless stated otherwise, we work over a field k of characteristic 0.

Contents

1. 2-monoidal categories

In this section, we define the general framework of *2-monoidal category* verified by the examples studied throughout the text. The notion of 2-monoidal category given here is a lax and more general version of the one given by A. Joyal and R. Street in [JS93] and the one given by C. Balteanu, Z. Fiedorowicz, R. Schwänzl and R. Vogt in [BFSV03].

1.1. Monoidal category.

We recall briefly the definitions of *monoidal category*, *lax monoidal functor* and *monoid* in order to settle the notations for the next section. We refer to the book of S. MacLane [MacL98] Chapter VII and to the article of J. Bénabou [Be63] for full references on the subject.

Definition (Monoidal category)**.** A monoidal category $(\mathcal{A}, \boxtimes, I, \alpha, r, l)$ is a category $\mathcal{A}$ equipped with a bifunctor $\boxtimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and a family of isomorphisms

$$\alpha_{A,B,C} \ : \ (A \boxtimes B) \boxtimes C \xrightarrow{\sim} A \boxtimes (B \boxtimes C),$$

for every A, B and C in $\mathcal{A}$. These isomorphisms are supposed to verify the pentagon axiom. For every object A in $\mathcal{A}$, there exists two isomorphisms $l_a \ : \ I \boxtimes A \to A$ and $r_a \ : \ A \boxtimes I \to A$ compatible with α.

Example. Let $(\mathcal{A}, \boxtimes, I, \alpha, r, l)$ be a monoidal category. The cartesian product $\mathcal{A} \times \mathcal{A}$ is a monoidal category where the monoidal product $\boxtimes^2$ is defined by $(A, B) \boxtimes^2 (A', B') := (A \boxtimes B, A' \boxtimes B')$. The unit is (I, I). The associative isomorphisms are given by

$$\begin{array}{c} \big((A, B) \boxtimes^2 (A', B')\big) \boxtimes^2 (A'', B'') = \big((A \boxtimes A') \boxtimes A'', (B \boxtimes B') \boxtimes B''\big) \\ \Big\downarrow {\scriptstyle (\alpha_{A,A',A''}, \alpha_{B,B',B''})} \\ (A, B) \boxtimes^2 \big((A', B') \boxtimes^2 (A'', B'')\big) = \big(A \boxtimes (A' \boxtimes A''), B \boxtimes (B' \boxtimes B'')\big). \end{array}$$

The other isomorphisms are $l_{(A,\, B)} := (l_A, l_B)$ and $r_{(A,\, B)} := (r_A, r_B)$.

Definition (Monoid)**.** Let $(\mathcal{A},\,\boxtimes,\,I)$ be a monoidal category. A *monoid* $(M,\,\mu,\,u)$ is an object M of $\mathcal{A}$ endowed with two morphisms : an associative *product* $\mu\ :\ M\boxtimes M\to M$ and a *unit* $u\ :\ I\to M$.

Definition (Lax monoidal functor)**.** A *lax monoidal functor* is a functor F between two monoidal categories $(\mathcal{A},\,\boxtimes_{\mathcal{A}},\,I_{\mathcal{A}})\to(\mathcal{B},\,\boxtimes_{\mathcal{B}},\,I_{\mathcal{B}})$ such that there exists a map $\iota\ :\ I_{\mathcal{B}}\to F(I_{\mathcal{A}})$ and a natural transformation

$$\varphi_{A,\,A'}\ :\ F(A)\boxtimes_{\mathcal{B}}F(A')\to F(A\boxtimes_{\mathcal{A}}A'),$$

for every A, A' in $\mathcal{A}$. This natural transformation is supposed to be compatible with the associativity and the units of the monoidal categories :

- Associativity condition : For every A, A' and A'' in $\mathcal{A}$, the following diagram is commutative

$$\begin{tikzcd}
\big(F(A)\boxtimes_{\mathcal{B}}F(A')\big)\boxtimes_{\mathcal{B}}F(A'') \arrow[rr, "\alpha^{\mathcal{B}}_{F(A),F(A'),F(A'')}"] \arrow[d, "\varphi_{A,A'}\boxtimes_{\mathcal{B}}id"] & & F(A)\boxtimes_{\mathcal{B}}\big(F(A')\boxtimes_{\mathcal{B}}F(A'')\big) \arrow[d, "id\boxtimes_{\mathcal{B}}\varphi_{A',A''}"] \\
F(A\boxtimes_{\mathcal{A}}A')\boxtimes_{\mathcal{B}}F(A'') \arrow[d, "\varphi_{A\boxtimes_{\mathcal{A}}A',A''}"] & & F(A)\boxtimes_{\mathcal{B}}F(A'\boxtimes_{\mathcal{A}}A'') \arrow[d, "\varphi_{A,A'\boxtimes_{\mathcal{A}}A''}"] \\
F\big((A\boxtimes_{\mathcal{A}}A')\boxtimes_{\mathcal{A}}A''\big) \arrow[rr, "F\left(\alpha^{\mathcal{A}}_{A,A',A''}\right)"] & & F\big(A\boxtimes_{\mathcal{A}}(A'\boxtimes_{\mathcal{A}}A'')\big).
\end{tikzcd}$$

- Unit condition : For every A in $\mathcal{A}$, the following diagram is commutative

$$\begin{tikzcd}
I_{\mathcal{B}}\boxtimes_{\mathcal{B}}F(A) \arrow[r, "\iota\boxtimes_{\mathcal{B}}F(A)"] \arrow[drr, "l^{\mathcal{B}}_{F(A)}"'] & F(I_{\mathcal{A}})\boxtimes_{\mathcal{B}}F(A) \arrow[r, "\varphi_{I_{\mathcal{A}},A}"] & F(I_{\mathcal{A}}\boxtimes_{\mathcal{A}}A) \arrow[d, "F(l^{\mathcal{A}}_{A})"] \\
 & & F(A).
\end{tikzcd}$$

The same statement holds on the right hand side.

The purpose of the definition of lax monoidal functors is to preserve monoids.

Proposition 1 ([Be63])**.** *Let $F\ :\ (\mathcal{A},\,\boxtimes_{\mathcal{A}},\,I_{\mathcal{A}})\to(\mathcal{B},\,\boxtimes_{\mathcal{B}},\,I_{\mathcal{B}})$ be a lax monoidal functor and let $(M,\,\mu,\,u)$ be a monoid in $\mathcal{A}$. The image of M under F is a monoid in $\mathcal{B}$. The product $\widetilde{\mu}$ is defined by*

$$\widetilde{\mu}\ :\ F(M)\boxtimes_{\mathcal{B}}F(M)\xrightarrow{\varphi_{M,M}}F(M\boxtimes_{\mathcal{A}}M)\xrightarrow{F(\mu)}F(M).$$

And the unit $\widetilde{u}$ is defined by

$$\widetilde{u}\ :\ I_{\mathcal{B}}\xrightarrow{\iota}F(I_{\mathcal{A}})\xrightarrow{F(u)}F(M).$$

1.2. **Definition of lax 2-monoidal category.** Motivated by the examples treated in the sequel, we define a general notion of category with two compatible monoidal products.

Definition (Lax 2-monoidal category)**.** A *lax 2-monoidal category* is a category $(\mathcal{A},\,\boxtimes,\,I,\,\otimes,\,K)$, such that both $(\mathcal{A},\,\boxtimes,\,I)$ and $(\mathcal{A},\,\otimes,\,K)$ are monoidal categories and such that the bifunctor $\otimes\ :\ \mathcal{A}\times\mathcal{A}\to\mathcal{A}$ is a lax monoidal functor with respect to the monoidal products $\boxtimes^2$ and $\boxtimes$.

The last assumption of the definition describes the compatibility between the two monoidal structures. The next proposition makes it more explicit.

Proposition 2. *A* lax 2-monoidal category *is a category $(\mathcal{A},\,\boxtimes,\,I,\,\otimes,\,K)$, such that both $(\mathcal{A},\,\boxtimes,\,I)$ and $(\mathcal{A},\,\otimes,\,K)$ are monoidal categories. These two monoidal structures are related by a natural transformation called the* interchange law

$$(A\otimes A')\boxtimes(B\otimes B')\xrightarrow{\varphi_{A,A',B,B'}}(A\boxtimes B)\otimes(A'\boxtimes B'),$$

where A, A', B and B' are in $\mathcal{A}$. This interchange law is supposed to be compatible with the associativity of the first monoidal product $\boxtimes$, that is

$$\begin{tikzcd}
\big((A\otimes A')\boxtimes(B\otimes B')\big)\boxtimes(C\otimes C') \arrow[r, "\alpha^{\boxtimes}_{A\otimes A',B\otimes B',C\otimes C'}"] \arrow[d, "\varphi_{A,A',B,B'}\boxtimes id"] & (A\otimes A')\boxtimes\big((B\otimes B')\boxtimes(C\otimes C')\big) \arrow[d, "id\boxtimes\varphi_{B,B',C,C'}"] \\
\big((A\boxtimes B)\otimes(A'\boxtimes B')\big)\boxtimes(C\otimes C') \arrow[d, "\varphi_{A\boxtimes B,A'\boxtimes B',C,C'}"] & (A\otimes A')\boxtimes\big((B\boxtimes C)\otimes(B'\boxtimes C')\big) \arrow[d, "\varphi_{A,A',B\boxtimes C,B'\boxtimes C'}"] \\
\big((A\boxtimes B)\boxtimes C\big)\otimes\big((A'\boxtimes B')\boxtimes C'\big) \arrow[r, "\alpha^{\boxtimes}_{A,B,C}\otimes\alpha^{\boxtimes}_{A',B',C'}"] & \big(A\boxtimes(B\boxtimes C)\big)\otimes\big(A'\boxtimes(B'\boxtimes C')\big),
\end{tikzcd}$$

where $\alpha^{\boxtimes}_{A,B,C}$ is the associativity morphism for the monoidal product $\boxtimes : (A\boxtimes B)\boxtimes C \xrightarrow{\sim} A\boxtimes(B\boxtimes C)$.

There exists a map $\iota : I \to I\otimes I$ such that for every A and A' in $\mathcal{A}$, the following diagram is commutative

$$\begin{tikzcd}
I\boxtimes(A\otimes A') \arrow[r, "\iota\boxtimes id"] \arrow[drr, "l^{\boxtimes}_{(A\otimes A')}"'] & (I\otimes I)\boxtimes(A\otimes A') \arrow[r, "\varphi_{I,I,A,A'}"] & (I\otimes A)\boxtimes(I\otimes A') \arrow[d, "l^{\otimes}_{A}\boxtimes l^{\otimes}_{A'}"] \\
 & & F(A).
\end{tikzcd}$$

The same statement holds on the right hand side.

PROOF. The proof is a straightforward application of the definition. □

Proposition 3. *Let $(\mathcal{A}, \boxtimes, I, \otimes, K)$ be a lax 2-monoidal category. Consider two $\boxtimes$-monoids M and N in $\mathcal{A}$. Their $\otimes$-product $M\otimes N$ is a $\boxtimes$-monoid.*

PROOF. It is a direct corollary of Definition 2 and Proposition 1. □

Motivated by the example of braided monoidal categories, A. Joyal and R. Street gave the first notion of a category endowed with two compatible monoidal products in [JS93]. In their definition, the monoidal categories are non-necessarily strict but the interchange law is supposed to be a natural isomorphism. This last condition forces the two monoidal products to be isomorphic.
In order to model n-fold loop spaces, C. Balteanu, Z. Fiedorowicz, R. Schwänzl and R. Vogt introduced in [BFSV03] the notions of *n-fold monoidal category.* Their notion of 2-fold monoidal category is, in some sense, a lax version of the one given by Joyal and Street since they do not assume the interchange law to be an isomorphism. But they require the monoidal structures to be strict and the two units are equal.
In the definition of a lax 2-monoidal category, we do not ask the monoidal structures to be strict. The two units need not be isomorphic. And the interchange law is not an isomorphism. Therefore, the notion given here is a lax version of the one of Joyal-Street and the one of Balteanu-Fiedorowicz-Schwänzl-Vogt. The definition of lax 2-monoidal category was suggested by our natural examples, that we make explicit in Section 1.4.

1.3. **Definition of 2-monoidal category.** Working in the opposite category, we get the dual notion of *colax 2-monoidal category.* Finally, we call a *2-monoidal category* a category which is both lax and colax 2-monoidal.

Definition (Comonoid)**.** A *comonoid* C is a monoid in the opposite category. It is endowed with two morphisms : a coassociative *coproduct* $C \to C\boxtimes C$ and a *counit* $C \to I$.

Definition (Colax monoidal functor)**.** A *colax monoidal functor* is a functor F between two monoidal categories $(\mathcal{A}, \boxtimes_{\mathcal{A}}, I_{\mathcal{A}}) \to (\mathcal{B}, \boxtimes_{\mathcal{B}}, I_{\mathcal{B}})$ such that there exists a map $I_{\mathcal{B}} \leftarrow F(I_{\mathcal{A}})$ and a natural transformation

$$\psi_{A,\,A'}\ :\ F(A)\boxtimes_{\mathcal{B}} F(A') \leftarrow F(A\boxtimes_{\mathcal{A}} A').$$

This natural transformation is supposed to be compatible with the associativity and the units of the monoidal categories. Explicitly, these compatibilities are given by the reversed diagrams defining a lax monoidal functor.

The purpose of the definition of colax monoidal functors is to preserve comonoids.

Proposition 4 (Be)**.** *Let $F\,:\,(\mathcal{A},\,\boxtimes_{\mathcal{A}},\,I_{\mathcal{A}}) \to (\mathcal{B},\,\boxtimes_{\mathcal{B}},\,I_{\mathcal{B}})$ be a colax monoidal functor and let C be a comonoid in $\mathcal{A}$. The image of C under F is a comonoid in $\mathcal{B}$.*

Definition (Colax 2-monoidal category)**.** A *colax 2-monoidal category* is a category $(\mathcal{A},\,\boxtimes,\,I,\,\otimes,\,K)$, such that both $(\mathcal{A},\,\boxtimes,\,I)$ and $(\mathcal{A},\,\otimes,\,K)$ are monoidal categories and such that the bifunctor $\otimes\,:\,\mathcal{A}\times\mathcal{A}\to\mathcal{A}$ is a colax monoidal functor.

A category $(\mathcal{A},\,\boxtimes,\,I,\,\otimes,\,K)$ is a colax 2-monoidal category if it is endowed with natural transformations, called *the interchange laws*,

$$(A\otimes A')\boxtimes(B\otimes B') \xleftarrow{\psi_{A,A',B,B'}} (A\boxtimes B)\otimes(A'\boxtimes B'),$$

verifying the same commutative diagram than the one defining a lax 2-monoidal category, with the maps φ replaced by the maps ψ.

Proposition 5. *Let $(\mathcal{A},\,\boxtimes,\,I,\,\otimes,\,K)$ be a colax 2-monoidal category. Consider two $\boxtimes$-comonoids M and N in $\mathcal{A}$. Their $\otimes$-product $M\otimes N$ is a $\boxtimes$-comonoid.*

Proof. It is a direct corollary of Definition 1.3 and Proposition 4. □

Definition (2-monoidal category)**.** A *2-monoidal category* is a category $(\mathcal{A},\,\boxtimes,\,I,\,\otimes,\,K)$, such that both $(\mathcal{A},\,\boxtimes,\,I)$ and $(\mathcal{A},\,\otimes,\,K)$ are two monoidal categories and such that the bifunctor $\otimes\,:\,\mathcal{A}\times\mathcal{A}\to\mathcal{A}$ is a lax and colax monoidal functor.

Definition (Strong 2-monoidal category)**.** A *strong 2-monoidal category* is a 2-monoidal category where the bifunctor $\otimes\,:\,\mathcal{A}\times\mathcal{A}\to\mathcal{A}$ is a strong monoidal functor, that is the interchange laws are isomorphisms.

1.4. **Examples of 2-monoidal categories.** In the this section, we study the relation between the composition product $\boxtimes$ and the Hadamard product $\otimes_H$ in the category of $\mathbb{S}$-bimodules and in the subcategories of $\mathbb{S}$-modules and k-modules. These notions are recalled in Appendix A.

Proposition 6. *The categories (k-Mod, $\otimes_k$, k), ($\mathbb{S}$-Mod$,\circ,\,I$) and ($\mathbb{S}$-biMod$,\boxtimes,\,I$) endowed with the Hadamard tensor products are 2-monoidal categories. The first one is a strong 2-monoidal category and it is a full sub-2-monoidal category of the second one, which is a full sub-2-monoidal category of the last one.*

Proof.

- In the first category, the two monoidal products are equal , that is $\boxtimes=\otimes=\otimes_k$. The interchange laws are given by the twisting isomorphism $(23)\,:\,V_1\otimes V_2\otimes V_3\otimes V_4 \xrightarrow{\sim} V_1\otimes V_3\otimes V_2\otimes V_4$.

- In the category of $\mathbb{S}$-modules with $\boxtimes = \circ$, $\otimes = \otimes_H$ the first interchange law $\varphi_{V,V',W,W'}$ map comes from the well defined natural map

$$(V \otimes V') \circ (W \otimes W')(n) :=$$
$$\left(\bigoplus_{i_1+\cdots+i_l=n} (V \otimes V')(l) \otimes_k \left((W \otimes W')(i_1) \otimes_k \cdots \otimes_k (W \otimes W')(i_l)\right) \otimes_{\mathbb{S}_{i_1}\times\cdots\times\mathbb{S}_{i_l}} k[\mathbb{S}_n] \right)_{\mathbb{S}_l}$$
$$\downarrow$$
$$\left(\bigoplus_{i_1+\cdots+i_l=n} V(l) \otimes_k \left(W(i_1) \otimes_k \cdots \otimes_k W(i_l)\right) \otimes_{\mathbb{S}_{i_1}\times\cdots\times\mathbb{S}_{i_l}} k[\mathbb{S}_n] \right)_{\mathbb{S}_l} \bigotimes$$
$$\left(\bigoplus_{i_1+\cdots+i_l=n} V'(l) \otimes_k \left(W'(i_1) \otimes_k \cdots \otimes_k W'(i_l)\right) \otimes_{\mathbb{S}_{i_1}\times\cdots\times\mathbb{S}_{i_l}} k[\mathbb{S}_n] \right)_{\mathbb{S}_l} =$$
$$(V \circ W) \otimes (V' \circ W')(n).$$

  The other map corresponds to the transpose of this one. It is well defined on invariants instead of coinvariants. Since we work over a field k of characteristic 0, we use the classical isomorphism between invariants and coinvariants to fix this.
- In the last case, which includes the two first, the interchange law map is the direct generalization of the one written above. Its explicit description is

$$\big(V \otimes V'\big) \boxtimes \big(W \otimes W'\big)(m,\, n) :=$$
$$\bigoplus_{N\in\mathbb{N}^*} \Big(\bigoplus_{\bar{l},\, \bar{k},\, \bar{\jmath},\, \bar{\imath}} k[\mathbb{S}_m] \otimes_{\mathbb{S}_{\bar{l}}} \big(V \otimes V'\big)(\bar{l},\, \bar{k}) \otimes_{\mathbb{S}_{\bar{k}}} k[\mathbb{S}^c_{\bar{k},\, \bar{\jmath}}] \otimes_{\mathbb{S}_{\bar{\jmath}}} \big(W \otimes W'\big)(\bar{\jmath},\, \bar{\imath}) \otimes_{\mathbb{S}_{\bar{\imath}}} k[\mathbb{S}_n] \Big)_{\mathbb{S}_b^{\mathrm{op}} \times \mathbb{S}_a}$$
$$\downarrow$$
$$\bigoplus_{N\in\mathbb{N}^*} \Big(\bigoplus_{\bar{l},\, \bar{k},\, \bar{\jmath},\, \bar{\imath}} \big(k[\mathbb{S}_m] \otimes_{\mathbb{S}_{\bar{l}}} V(\bar{l},\, \bar{k}) \otimes_{\mathbb{S}_{\bar{k}}} k[\mathbb{S}^c_{\bar{k},\, \bar{\jmath}}] \otimes_{\mathbb{S}_{\bar{\jmath}}} W(\bar{\jmath},\, \bar{\imath}) \otimes_{\mathbb{S}_{\bar{\imath}}} k[\mathbb{S}_n]\big) \otimes$$
$$\big(k[\mathbb{S}_m] \otimes_{\mathbb{S}_{\bar{l}}} V'(\bar{l},\, \bar{k}) \otimes_{\mathbb{S}_{\bar{k}}} k[\mathbb{S}^c_{\bar{k},\, \bar{\jmath}}] \otimes_{\mathbb{S}_{\bar{\jmath}}} W'(\bar{\jmath},\, \bar{\imath}) \otimes_{\mathbb{S}_{\bar{\imath}}} k[\mathbb{S}_n]\big) \Big)_{\mathbb{S}_b^{\mathrm{op}} \times \mathbb{S}_a} =$$
$$\downarrow$$
$$\bigoplus_{N\in\mathbb{N}^*} \Big(\bigoplus_{\bar{l},\, \bar{k},\, \bar{\jmath},\, \bar{\imath}} k[\mathbb{S}_m] \otimes_{\mathbb{S}_{\bar{l}}} V(\bar{l},\, \bar{k}) \otimes_{\mathbb{S}_{\bar{k}}} k[\mathbb{S}^c_{\bar{k},\, \bar{\jmath}}] \otimes_{\mathbb{S}_{\bar{\jmath}}} W(\bar{\jmath},\, \bar{\imath}) \otimes_{\mathbb{S}_{\bar{\imath}}} k[\mathbb{S}_n] \Big)_{\mathbb{S}_b^{\mathrm{op}} \times \mathbb{S}_a} \otimes$$
$$\bigoplus_{N\in\mathbb{N}^*} \Big(\bigoplus_{\bar{l},\, \bar{k},\, \bar{\jmath},\, \bar{\imath}} k[\mathbb{S}_m] \otimes_{\mathbb{S}_{\bar{l}}} V'(\bar{l},\, \bar{k}) \otimes_{\mathbb{S}_{\bar{k}}} k[\mathbb{S}^c_{\bar{k},\, \bar{\jmath}}] \otimes_{\mathbb{S}_{\bar{\jmath}}} W'(\bar{\jmath},\, \bar{\imath}) \otimes_{\mathbb{S}_{\bar{\imath}}} k[\mathbb{S}_n] \Big)_{\mathbb{S}_b^{\mathrm{op}} \times \mathbb{S}_a} =$$
$$\big(V \boxtimes W\big) \otimes \big(V' \boxtimes W'\big)(m,\, n).$$

Note that the first map preserves the shape of the underlying graph of the composition, whereas the second one does not. Therefore, this interchange law map is injective but not an isomorphism. The reverse natural transformation $\big(V \boxtimes W\big) \otimes \big(V' \boxtimes W'\big)(m,\, n) \twoheadrightarrow \big(V \otimes V'\big) \boxtimes \big(W \otimes W'\big)(m,\, n)$ is given by the projection on pairs of composition of $\big(V \boxtimes W\big) \otimes \big(V' \boxtimes W'\big)$ based on the same 2-levelled graph (see A.3). To such pairs, it is straightforward to associate an element of $\big(V \otimes V'\big) \boxtimes \big(W \otimes W'\big)$. This map is the transpose of the first one. It is the composite of an epimorphism with an isomorphism, therefore it is an epimorphism. □

REMARK. In the same way, we can also show that the underlying category of non-symmetric operads, $\frac{1}{2}$-props [MV03], dioperads [Gan03], colored operads are 2-monoidal categories. We refer to [Val3] Section 5 and to [Mar06] Section 9 for surveys of these notions.

1.5. **Bimonoids.** In this section, we define the notion of *bimonoid* that generalizes the notion of *bialgebra* in any lax 2-monoidal category.

Let $(\mathcal{A}, \boxtimes, I, \otimes, K)$ be a lax 2-monoidal category. Proposition 1 shows that the category of $\boxtimes$-monoids, denoted by $\mathcal{M}\mathrm{on}^{\boxtimes}_{\mathcal{A}}$, is a monoidal category for the monoidal product $\otimes$.

Definition (Bimonoid)**.** A *bimonoid* is a comonoid in the monoidal category $(\mathcal{M}\mathrm{on}^{\boxtimes}_{\mathcal{A}}, \otimes, K)$.

Examples. The examples of the categories (k-Mod, $\otimes_k$, k), ($\mathbb{S}$-Mod, $\circ$, I) and ($\mathbb{S}$-biMod, $\boxtimes$, I) endowed with the Hadamard tensor products, give the following notions.

- In the case of k-modules, we find the classical notion of *bialgebras*.
- In the case of $\mathbb{S}$-modules, we find the notion of *Hopf operads*. We refer the reader to the recent preprint of M. Aguiar and S. Mahajan [AM06] for a study of *Hopf monoids* in the category of species which is a very close notion.
- In the case of $\mathbb{S}$-bimodules, this generalizes the notion of Hopf operads to properads. We call them *Hopf properads*.

When $\mathcal{P}$ is a Hopf operad, the category of $\mathcal{P}$-algebras is stable under the tensor product(see A).

Proposition 7. *Let $\mathcal{P}$ be a Hopf properad. The tensor product $A \otimes B$ of two $\mathcal{P}$-gebra is again a $\mathcal{P}$-gebra.*

Proof. The proof is straightforward. □

2. Koszul duality pattern

We work in the abelian monoidal category $\big(\text{dg-}\mathbb{S}\text{-biMod}, \boxtimes, I\big)$ of dg-$\mathbb{S}$-bimodules (see Appendix A). A monoid in this category is called a (dg-)*properad*. Since the abelian monoidal categories of differential graded vector spaces and dg-$\mathbb{S}$-modules are abelian monoidal subcategories of dg-$\mathbb{S}$-bimodules, the sequel includes the cases of (dg-)associative algebras and (dg-)operads. In the following of the text, we will implicitly work in the differential graded context without writing "dg", for the sake of simplicity. We use a very general language since most of what follows can be generalized to another examples (colored operads, non-symmetric operads, for instance). Denote by $\mathcal{F}(V)$ the free properad (monoid) on V and by $\mathcal{F}^c(V)$ the cofree connected coproperad (comonoid) on V (see Appendix A for more details).

The Koszul dual coproperad is usually defined by the top homology groups of the bar construction. The purpose of this section is to prove that the construction of the Koszul dual can be described with a pure categorical or algebraic point of view. This section is a generalization of Section 2.4 of E. Getzler and J.D.S. Jones preprint [GJ94].

2.1. **Quadratic (co)properads.** Let $(V,\, R)$ be a *quadratic data*, that is $R \subset \mathcal{F}_{(2)}(V)$. Since the underlying $\mathbb{S}$-bimodule of the free properad $\mathcal{F}(V)$ and the cofree connected coproperad $\mathcal{F}^c(V)$ are isomorphic [2], we consider the following sequence in $\mathbb{S}$-biMod

$$R \rightarrowtail \mathcal{F}_{(2)}(V) \rightarrowtail \mathcal{F}(V) \cong \mathcal{F}^c(V) \twoheadrightarrow \mathcal{F}^c_{(2)}(V) \twoheadrightarrow \mathcal{F}^c_{(2)}(V)/R =: \overline{R}.$$

A quadratic data will be written (V, R) or equivalently $(V, \overline{R})$. To such a sequence, we can naturally define a quotient properad of $\mathcal{F}(V)$ and a subcoproperad of $\mathcal{F}^c(V)$ (see Appendix B.4).

Definition (Quadratic properad generated by V and R)**.** The *quadratic properad generated by V and R* is the quotient properad of $\mathcal{F}(V)$ by the ideal generated by $R \rightarrowtail \mathcal{F}(V)$. We denote it by $\mathcal{P}(V,\, R) = \mathcal{F}(V)/(R)$.

Definition (Quadratic coproperad generated by V and $\overline{R}$)**.** The *quadratic coproperad generated by V and $\overline{R}$* is the subcoproperad of $\mathcal{F}^c(V)$ generated by $\mathcal{F}^c(V) \twoheadrightarrow \overline{R}$. We denote it by $\mathcal{C}(V,\, \overline{R})$.

For example, the quadratic coalgebra generated by $(V, \overline{R})$ is equal to

$$\mathcal{C}(V, \overline{R}) = k \oplus V \oplus \bigoplus_{n\geq 2} \bigcap_{i=0}^{n-2} V^{\otimes i} \otimes R \otimes V^{\otimes n-2-i}.$$

Remark. We proved in [Val1] Corollary 7.5 that when a properad is Koszul, it is necessarily quadratic. Therefore, there is no restriction to study only the quadratic case.

[2] This should also come from the fact that the colored operad coding properads is Koszul-autodual.

2.2. **Definition of the Koszul dual revisited.** Koszul duality theory comes from homological algebra, when one tries to find small resolutions (minimal models) of algebraic structures (associative algebras, operads, properads, colored operads, for instance).

The Koszul dual cooperad of an operad $\mathcal{P}$ is defined by the top homology of the bar construction $\mathcal{B}(\mathcal{P})$ (see [Fre04] Section 5 and [GJ94] Section 2.4). In [Val1] Section 7, we used the same idea to define the Koszul dual coproperad of a properad. The purpose of this section is to prove that the Koszul dual coproperad is a quadratic coproperad and to prove the dual statement.

Let $\mathcal{P} = \mathcal{P}(V,\, R)$ be a quadratic properad. Recall from [Val1] Section 4 that the *bar construction* $\mathcal{B}(\mathcal{P})$ of $\mathcal{P}$ is the chain complex defined on $\mathcal{F}^c(s\overline{\mathcal{P}})$ by the unique coderivation δ which extends the partial composition of $\mathcal{P}$. Dually, the *cobar construction* of a coproperad is the chain complex $\mathcal{F}(s^{-1}\bar{\mathcal{C}})$, where the differential d is the unique derivation which extends the partial composition coproduct of $\mathcal{C}$.

When $\mathcal{P} = \mathcal{P}(V,\, R)$ is a quadratic properad, it is weight graded. Denote this grading by (ω). In this case, the bar construction of $\mathcal{P}$ decomposes with respect to this grading. The part of weight (ω) of $\mathcal{B}_\bullet(\mathcal{P})$ begins with

$$\mathcal{B}_\bullet(\mathcal{P})_{(\omega)} \ :\ 0 \to \mathcal{F}^c_{(\omega)}(s\mathcal{P}_{(1)}) \xrightarrow{\delta} \cdots .$$

Let $\mathcal{P}^{¡}_{(\omega)}$ be its top homology group $H_\omega\left(\mathcal{B}_\bullet(\mathcal{P})_{(\omega)}\right)$ and $\mathcal{P}^{¡} := \bigoplus_{(\omega)} \mathcal{P}^{¡}_{(\omega)}$. Using $H_\omega\left(\mathcal{B}_\bullet(\mathcal{P})_{(\omega)}\right) = \ker\delta$, we proved in [Val1] Proposition 7.2, that $\mathcal{P}^{¡}$ is a subcoproperad of $\mathcal{F}^c(s\mathcal{P}_{(1)}) \cong \mathcal{F}^c(V)$.

Dually, let $\mathcal{C} = \mathcal{C}(V,\, \overline{R})$ be a quadratic coproperad. It is a connected coproperad, that is weight graded and such that $\mathcal{C}_{(0)} = k$. Once again, its cobar construction is the direct sum of subcomplexes indexed by the weight

$$\Omega_\bullet(\mathcal{C})_{(\omega)} \ :\ \cdots \xrightarrow{d} \mathcal{F}_{(\omega)}(s^{-1}\mathcal{C}_{(1)}) \to 0.$$

Define $\mathcal{C}^{¡}$ to be the top homology groups of the cobar construction of $\mathcal{C}$, that is $\mathcal{C}^{¡} := \bigoplus_{(\omega)} H_{-\omega}\left(\Omega_\bullet(\mathcal{C})_{(\omega)}\right)$. Since $H_{-\omega}\left(\Omega_\bullet(\mathcal{C})_{(\omega)}\right) = \operatorname{coker} d$, $\mathcal{C}^{¡}$ is a quotient properad of $\mathcal{F}(s^{-1}\mathcal{C}_{(1)}) \cong \mathcal{F}(V)$.

Theorem 8. *Let $(V,\, R)$ be a quadratic data. Denote by s^2R the image of R in $F_{(2)}(sV)$ and by $s^{-2}\overline{R}$ the quotient of $\mathcal{F}^c_{(2)}(s^{-1}V)$ by $s^{-2}R$.*
The Koszul dual coproperad of $\mathcal{P}(V,R)$ is equal to $\mathcal{P}(V,R)^{¡} = \mathcal{C}(sV, s^2\overline{R})$. Dually, the Koszul dual properad of $\mathcal{C}(V,\overline{R})$ is equal to $\mathcal{C}^{¡} := \mathcal{P}(s^{-1}V,\, s^{-2}R)$. Therefore, we have $\mathcal{P}^{¡¡} = \mathcal{P}$ and $\mathcal{C}^{¡¡} = \mathcal{C}$.

Proof. The cobar construction of $\mathcal{C}$ has the following form

$$\Omega_\bullet(\mathcal{C})_{(\omega)} \ :\ \cdots \to \mathcal{F}_{(\omega)}(s^{-1}\mathcal{C}_{(1)} + \underbrace{s^{-1}\mathcal{C}_{(2)}}_{1}) \xrightarrow{d} \mathcal{F}_{(\omega)}(s^{-1}\mathcal{C}_{(1)}) \to 0,$$

where $\mathcal{F}_{(\omega)}(s^{-1}\mathcal{C}_{(1)} + \underbrace{s^{-1}\mathcal{C}_{(2)}}_{1})$ stands for the sub-$\mathbb{S}$-bimodule of $\mathcal{F}_{(\omega)}(s^{-1}\mathcal{C}_{(1)} + s^{-1}\mathcal{C}_{(2)})$ composed by graphs with $\omega - 1$ vertices indexed by elements of $s^{-1}\mathcal{C}_{(1)}$ and just one vertex indexed by an element of $s^{-1}\mathcal{C}_{(2)}$. The image of d is the kernel of the cokernel $\mathcal{F}_{(\omega)}(s^{-1}V) \twoheadrightarrow \mathcal{C}^{¡}_{(\omega)}$ of d. Since $\mathcal{C}^{¡}$ is a quotient properad of $\mathcal{F}(s^{-1}V)$, $\operatorname{Im} d \rightarrowtail \mathcal{F}(s^{-1}V)$ is an ideal monomorphism. From the shape of $\Omega_\bullet(\mathcal{C})$, we see that the image of d is made of graphs indexed by $s^{-1}V$ with at least one subgraph graph in $s^{-2}R$. Therefore, the image of d is equal to the image of μ^2 : $\mathcal{F}(s^{-1}V) \boxtimes (\mathcal{F}(s^{-1}V) + \underline{s^{-2}R}) \boxtimes \mathcal{F}(s^{-1}V)$, that is the ideal generated by $s^{-2}R$ by Proposition 57 of Appendix A.
We dualize the arguments (in the opposite category) to get the dual statement. The last assertion is easily verified. $\square$

A properad is called a *Koszul properad* when the homology of its bar construction is concentrated in top dimension, that is when $H_\bullet\big(\mathcal{B}(\mathcal{P})\big) = \mathcal{P}^{¡}$.

2.3. **Relation with the Koszul dual properad.** To an $\mathbb{S}$-bimodule M, we associate its linear dual $M^* := \{M(m,n)^*\}_{m,n}$. The linear dual * of a coproperad $(\mathcal{C}, \Delta)$ is always a properad : define the composition product by the formula $\mathcal{C}^* \boxtimes \mathcal{C}^* \to (\mathcal{C} \boxtimes \mathcal{C})^* \xrightarrow{\Delta^t} \mathcal{C}^*$. But we need a finite dimensional assumption on the underlying $\mathbb{S}$-bimodule to have the dual result. The main explanation for such a phenomenon is that there exists a map $V^* \otimes W^* \to (V \otimes W)^*$, which is an isomorphism when V and W are finite dimensional vector spaces.

Definition (Locally finite $\mathbb{S}$-bimodule)**.** An $\mathbb{S}$-bimodule M is *locally finite* if for every m and n in $\mathbb{N}$, the dimension of the module $M(m,n)$ is finite over k.

Proposition 9. *When V is a locally finite $\mathbb{S}$-bimodule, the linear dual of the quadratic coproperad $\mathcal{C}(V, \overline{R})$ generated by V and $\overline{R}$ is the quadratic properad $\mathcal{F}(V^*)/(R^\perp)$, where $R^\perp \subset \mathcal{F}_{(2)}(V)^* \cong \mathcal{F}_{(2)}(V^*)$.*

Proof. The image under * of the terminal object (see Appendix B.4)

$$\begin{tikzcd}
\mathcal{F}^c_{(2)}(V)/R & & \\
\mathcal{F}^c(V)/\mathcal{C}(V,\, R) \arrow[u, two heads] & \mathcal{F}^c(V) \arrow[l, two heads] \arrow[ul, two heads] & \mathcal{C}(V,\, \overline{R}) \arrow[l, tail] \arrow[ull, bend right, two heads, "0"']
\end{tikzcd}$$

gives the initial object of

$$\begin{tikzcd}
\left(\mathcal{F}^c_{(2)}(V)/R\right)^* \arrow[d, tail] \arrow[dr] \arrow[drr, bend left, "0"] & & \\
(\mathcal{F}^c(V)/\mathcal{C}(V,\, R))^* \arrow[r, tail] & \mathcal{F}^c(V)^* \arrow[r, two heads] & \mathcal{C}(V,\, R)^*.
\end{tikzcd}$$

Since V is locally finite, we can identify $(\mathcal{F}^c(V))^*$ with the free properad on $V^* : \mathcal{F}(V^*)$. (The (co)free (co)properad on V is given by a direct sum of particular tensor powers of V). Therefore, $\left(\mathcal{F}^c_{(2)}(V)/R\right)^*$ is isomorphic to the orthogonal of R, that is $R^\perp := \{f \in \mathcal{F}_{(2)}(V)^* \cong \mathcal{F}_{(2)}(V^*) \mid f_R = 0\}$. We conclude by the uniqueness property of the initial object. $\square$

When $\mathcal{P}(V,R)$ is a quadratic properad generated by a locally finite $\mathbb{S}$-bimodule, we consider the linear dual of the Koszul dual coproperad $\mathcal{P}^{\text{¡}*}$. By Proposition 9, we have $\mathcal{P}(V,R)^{\text{!`}*} = \mathcal{P}(s^{-1}V^*, s^{-2}R^\perp)$. In the case of finitely generated associative algebra, it is the definition given by S. Priddy [Pri70]. In the case of binary quadratic operads, V. Ginzburg and M. Kapranov ([GK95] Section 2) defined a twisted Koszul dual operad by the formula $\mathcal{P}^! := \mathcal{P}(V^\vee, R^\perp)$, where $M^\vee(n) := M^*(n) \otimes \mathrm{sgn}_{\mathbb{S}_n}$. The reason for this lies in Quillen functors which are the bar and cobar constructions between $\mathcal{P}$-algebras and $\mathcal{P}^{\text{¡}}$-coalgebras (see [Qui69] and [GJ94] Section 2). The bar construction of a $\mathcal{P}$-algebra A is the cofree $\mathcal{P}^{\text{¡}}$-coalgebra on the suspension of A, that is $\mathcal{P}^{\text{¡}}(sA)$. When A is concentrated in degree 0, we have

$$\mathcal{P}^{\text{¡}}(sA) = \bigoplus_{n \geq 1} \mathcal{P}^{\text{¡}}(n) \otimes_{\mathbb{S}_n} (sA)^{\otimes n} = \bigoplus_{n \geq 1} s^n \mathcal{P}^{\text{¡}}(n) \otimes \mathrm{sgn}_{\mathbb{S}_n} \otimes_{\mathbb{S}_n} A^{\otimes n}.$$

We define the *suspension* operad by $S(n) := s^{n-1}.k \otimes \mathrm{sgn}_{\mathbb{S}_n}$, with the signature action of the symmetric group. Actually, S is equal to the operad of endomorphisms of $s^{-1}k$, that is $S = \mathrm{End}(s^{-1}k)$. We have $\mathcal{P}^{\text{¡}}(sA) = s(S \otimes \mathcal{P}^{\text{¡}})(A)$. Up to suspensions, $\mathcal{P}^{\text{¡}}(sA)$ is the cofree "$\mathcal{P}^!$-coalgebra" on A. The operad $\mathcal{P}$ is Koszul if and only if $\mathcal{P}^{\text{¡}}$ is Koszul, which is also equivalent to $\mathcal{P}^!$ is Koszul.

3. Manin products

The aim of this section is to provide a general and intrinsic framework for the definitions of Manin's black and white products. We first give the conceptual definition of Manin's white product of monoids in any lax 2-monoidal category. Then, we dualize the arguments to get the notion of black product of comonoids in any colax 2-monoidal category.

We make explicit all the constructions in the category of $\mathbb{S}$-bimodules. But they remain valid in general 2-monoidal categories with mild assumptions (existence of the free monoid, cofree comonoid, for instance). These constructions also hold for non-symmetric operads (see Section 5) and colored operads, for instance. We denote the vertical connected composition product of $\mathbb{S}$-bimodules and the Hadamard horizontal tensor $\otimes_H$ by $\otimes$, to lighten the notations.

3.1. **A canonical map between free monoids.** V. Ginzburg and M.M. Kapranov mentioned in [GK95'] a morphism of operads $\Phi : \mathcal{F}(V \otimes W) \to \mathcal{F}(V) \otimes \mathcal{F}(W)$ "which reflects the fact that the tensor product of an $\mathcal{F}(V)$-algebra and an $\mathcal{F}(W)$-algebra is an $\mathcal{F}(V \otimes W)$-algebra". We describe and extend this map Φ to a more general setting.

Proposition 10. *Let $(\mathcal{A}, \boxtimes, I, \otimes, K)$ be a lax 2-monoidal category such that $(\mathcal{A}, \boxtimes, I)$ admits free monoids. There exists a natural morphism of monoids $\Phi : \mathcal{F}(V \otimes W) \to \mathcal{F}(V) \otimes \mathcal{F}(W)$.*

Proof. Let V and W be two objects in $\mathcal{A}$. There is a natural map $u_{\mathcal{F}(V)} \otimes u_{\mathcal{F}(W)} : V \otimes W \to \mathcal{F}(V) \otimes \mathcal{F}(W)$. Using Proposition 3, we know that $\mathcal{F}(V) \otimes \mathcal{F}(W)$ is a monoid for $\boxtimes$. By the universal property of the free monoid on $V \otimes W$, there exists a unique morphism of monoids Φ which factors the previous map

$$\begin{tikzcd} V \otimes W \arrow[r, "u_{V\otimes W}"] \arrow[dr, "u_v \otimes u_W"'] & \mathcal{F}(V \otimes W) \arrow[d, dashed, "\exists ! \Phi"] \\ & \mathcal{F}(V) \otimes \mathcal{F}(W). \end{tikzcd}$$

$\square$

Examples.

- When $\mathcal{A}$ is the category of k-modules, the map Φ is the direct sum of the isomorphisms $(V \otimes W)^{\otimes n} \cong V^{\otimes n} \otimes W^{\otimes n}$ induced by the twisting map.
- In the category of $\mathbb{S}$-modules, the map Φ corresponds to the injective morphism of operads $\mathcal{F}(V \otimes W) \rightarrowtail \mathcal{F}(V) \otimes \mathcal{F}(W)$ mentioned in [GK95'].
- For $\mathbb{S}$-bimodules, the previous construction gives a morphism of properads between the free properad $\mathcal{F}(V \otimes W)$and the Hadamard product $\mathcal{F}(V) \otimes \mathcal{F}(W)$. Once again, this map is always injective but not an isomorphism in general.

One remark before to conclude this section. The purpose of this paragraph was to show that the definition of the map Φ is canonical and does not depend on the bases of the modules involved here. Now, if we choose a basis for the free operad, for instance, we can make the map Φ more explicit. In this case, the image of a tree T with vertices indexed by elements of $V \otimes W$ under Φ is the tensor product of the same tree T with vertices indexed by the corresponding elements of V with the tree T whose vertices are indexed by the corresponding elements of W.

3.2. **Definition of the white product.** In this section we define the white product for every pair of properads defined by generators and relations. When the two properads are quadratic, the resulting white product is again quadratic. Since an associative algebra is an operad and an operad is a properad, this construction summarizes what can be found in the literature. In the case of quadratic associative algebras, it corresponds to the original notions introduced by Y. Manin [Man88] and in the case of binary quadratic operads, it corresponds to the definitions of V. Ginzburg and M. Kapranov [GK95, GK95'].

The properties of the morphism Φ lead directly to the definition of the white product. Let $\mathcal{P}$ and $\mathcal{Q}$ be two properads defined by generators and relations, $\mathcal{P} = \mathcal{F}(V)/(R)$ and $\mathcal{Q} = \mathcal{F}(W)/(S)$. And denote the projections $\pi_{\mathcal{P}} : \mathcal{F}(V) \twoheadrightarrow \mathcal{P}$ and $\pi_{\mathcal{Q}} : \mathcal{F}(W) \twoheadrightarrow \mathcal{Q}$.
Consider the following composite of morphisms of properads

$$\pi_{\mathcal{P}} \otimes \pi_{\mathcal{Q}} \circ \Phi : \mathcal{F}(V \otimes W) \rightarrowtail \mathcal{F}(V) \otimes \mathcal{F}(W) \twoheadrightarrow \mathcal{P} \otimes \mathcal{Q}.$$

Since it is a morphism of properads, its kernel is an ideal of $\mathcal{F}(V \otimes W)$. It is the ideal generated by $\Phi^{-1}(R \otimes \mathcal{F}(W) + \mathcal{F}(V) \otimes S)$ in $\mathcal{F}(V \otimes W)$.

Definition (White product)**.** Let $\mathcal{P} = \mathcal{F}(V)/(R)$ and $\mathcal{Q} = \mathcal{F}(W)/(S)$ be two properads defined by generators and relations. The quotient properad

$$\mathcal{P} \circ \mathcal{Q} := \mathcal{F}(V \otimes W)/\big(\Phi^{-1}(R \otimes \mathcal{F}(W) + \mathcal{F}(V) \otimes S)\big)$$

is called the *white product of* $\mathcal{P}$ and $\mathcal{Q}$.

The definition of the white product of two properads shows that the morphism Φ factors through a natural morphism of properads $\bar{\Phi} : \mathcal{P} \circ \mathcal{Q} \to \mathcal{P} \otimes \mathcal{Q}$. In the abelian category $\mathbb{S}$-bimodules, $\bar{\Phi}$ is the image of $\pi_{\mathcal{P}} \otimes \pi_{\mathcal{Q}} \circ \Phi$. Hence, it is a monomorphism.

$$\begin{tikzcd}
\mathcal{F}(V \otimes W) \arrow[r, tail, "\Phi"] \arrow[drr, two heads] & \mathcal{F}(V) \otimes \mathcal{F}(W) \arrow[r, two heads, "\pi_{\mathcal{P}} \otimes \pi_{\mathcal{Q}}"] & \mathcal{P} \otimes \mathcal{Q} \\
 & & \mathcal{P} \circ \mathcal{Q} \arrow[u, tail, "\bar{\Phi}"']
\end{tikzcd}$$

Let A be a $\mathcal{P}$-gebra and B a $\mathcal{Q}$-gebra, since the tensor $A \otimes B$ is a $\mathcal{P} \otimes \mathcal{Q}$-gebra, we get the following result.

Proposition 11. *The tensor product $A \otimes B$ is a gebra over the white product $\mathcal{P} \circ \mathcal{Q}$.*

Example. Let $\mathcal{P}$ and $\mathcal{Q}$ be two operads. The tensor product of a $\mathcal{P}$-algebra with a $\mathcal{Q}$-algebra is a $\mathcal{P} \otimes \mathcal{Q}$-algebra. We can partially dualize this statement. Let C be a $\mathcal{Q}$-coalgebra and A be a $\mathcal{P}$-algebra, the space of morphism $\mathrm{Hom}_k(C, A)$ is $\mathcal{P} \otimes \mathcal{Q}$-algebra (see [BM03] Proposition 1.1). It is also a $\mathcal{P} \circ \mathcal{Q}$-algebra by Proposition 11. As explained by G. Barnich, R. Fulp, T. Lada, and J. Stasheff in [BFLS00], when C is a cocommutative coalgebra, $\mathrm{Hom}_k(C, A)$ is always a $\mathcal{P}$-algebra. This comes from the fact that $\mathcal{C}om$ is the unit object for $\otimes$ and $\circ$. Motivated by structures appearing in Lagrangian field theories in physics, these authors studied the algebraic structures of $\mathrm{Hom}_k(C, A)$ when C is a coassociative coalgebra and A a Lie algebra or a Poisson algebra. Since $\mathrm{Hom}_k(C, A)$ is a $\mathcal{P} \circ \mathcal{Q}$-algebra, Manin's white product for operads gives a way to describe such structures.

The white product is a construction that preserves the grading of the properads.

Proposition 12. *If $S \subset \bigoplus_{\omega=0}^{N} \mathcal{F}_{(\omega)}(V)$ and $R \subset \bigoplus_{\omega=0}^{M} \mathcal{F}_{(\omega)}(W)$, the white product of $\mathcal{P}$ and $\mathcal{Q}$ is a properad generated by $V \otimes W$ with relations in $\bigoplus_{\omega=0}^{\max(N,\,M)} \mathcal{F}_{(\omega)}(V \otimes W)$.*
If S and R are homogenous of weight N, that is $S, R \subset \mathcal{F}_{(N)}(V)$, then $\mathcal{P} \circ \mathcal{Q}$ is once again a properad defined by homogenous relations of weight N.

Proof. It comes from the definition of the morphism Φ which preserves the grading. $\square$

Examples.

- Let A and B be two quadratic associative algebras. The white product $A \circ B$ is equal to $T(V \otimes W)/\big((23)(R \otimes W^{\otimes 2} + V^{\otimes 2} \otimes S)\big)$, which is the definition given by Manin in [Man88, Man87]. It is isomorphic via $\bar{\Phi}$ to the Hadamard (or Segre) product $A \otimes_H B := \bigoplus_n A_{(n)} \otimes_k B_{(n)}$. This crucial property allowed J. Backelin to prove in his thesis [Bac] that the white product of two Koszul algebras is a Koszul algebra.

- An associative algebra $A = T(V)/(R)$ is *N-homogenous* if $R \subset V^{\otimes N}$. R. Berger, M. Dubois-Violette and M. Wambst generalized Manin's black and white products to N-homogenous algebras in [BDVW03]. For two N-homogenous algebras, the definition given above coincide with their definition. Note that the definition given here can be applied to non-homogenous algebras. The class of Artin-Schelter algebras [AS87] provide interesting examples of non-homogenous algebras. It would be interesting to study the properties of the white product of such algebras, for instance the ones of global dimension 4 of [LPWZ04].
- When $\mathcal{P}$ and $\mathcal{Q}$ are binary quadratic operads, the modules $\mathcal{F}_{(2)}(V)$ and $\mathcal{F}_{(2)}(W)$ are equal to $\mathcal{F}(V)(3)$ and $\mathcal{F}(W)(3)$. In that case, we get $R \otimes \mathcal{F}(W) = R \otimes \mathcal{F}(W)(3)$ and $\mathcal{F}(V) \otimes S = \mathcal{F}(V)(3) \otimes S$. This construction is the original one described by Ginzburg and Kapranov in [GK95, GK95']. Note that in this case, the white product is not, in general, equal to the Hadamard product. (The morphism $\bar{\Phi}$ is not an isomorphism in general). A direct consequence of this fact is that the white product of two Koszul operads is not necessarily a Koszul operad again. See section 4.5 for a counterexample.

3.3. **The black product.** We dualize the arguments and work in the opposite category. This gives the definition *black product* of coproperads.

Proposition 13. *Let $(\mathcal{A}, \boxtimes, I, \otimes, K)$ be a colax 2-monoidal category such that $(\mathcal{A}, \boxtimes, I)$ admits cofree comonoids. There exists a natural morphism of comonoids $\Psi : \mathcal{F}^c(V \otimes W) \leftarrow \mathcal{F}^c(V) \otimes \mathcal{F}^c(W)$.*

Definition (Black product)**.** The black product of two coproperads $\mathcal{C}(V, \overline{R})$ and $\mathcal{C}(W, \overline{S})$ is the image of the morphism of comonoids $\Psi \circ (\iota \otimes \iota)$

$$\begin{tikzcd}
\mathcal{F}^c(V \otimes W) & \mathcal{F}^c(V) \otimes \mathcal{F}^c(W) \arrow[l, two heads, "\Psi"'] & \mathcal{C}(V, \overline{R}) \otimes \mathcal{C}(W, \overline{S}) \arrow[l, tail, "\iota\otimes\iota"'] \arrow[d, "\bar{\Psi}"] \\
 & & \mathcal{C}(V, \overline{R}) \bullet \mathcal{C}(W, \overline{S}). \arrow[ull, tail]
\end{tikzcd}$$

It is equal to $\mathcal{C}(V, \overline{R}) \bullet \mathcal{C}(W, \overline{S}) = \mathcal{C}(V \otimes W, \overline{\Psi(R \otimes S)})$.

Black and white constructions are dual to each other under linear duality.

Theorem 14. *Let (V, R) and (W, S) be two quadratic data, with V and W locally finite. We have the following isomorphism of properads $(\mathcal{C}(V, R) \bullet \mathcal{C}(W, S))^* \cong \mathcal{C}(V, R)^* \circ \mathcal{C}(W, S)^*$.*

Proof. Since ψ is the transpose of φ, we have $\Psi = {}^t\Phi_{V^*, W^*}$, up to isomorphism like $\big(\mathcal{F}(V^*) \otimes \mathcal{F}(W^*)\big)^* \cong \mathcal{F}(V) \otimes \mathcal{F}(W)$. Therefore, we get $\Psi(R \otimes S)^\perp = \Phi^{-1}_{V^*, W^*}(R^\perp \otimes \mathcal{F}(W^*) + \mathcal{F}(V^*) \otimes S^\perp)$. By Proposition 9, we have

$$\begin{aligned}
(\mathcal{C}(V, R) \bullet \mathcal{C}(W, S))^* &\cong \mathcal{P}(V^* \otimes W^*, \Psi(R \otimes S)^\perp) \\
&\cong \mathcal{P}(V^* \otimes W^*, \Phi^{-1}_{V^*, W^*}(R^\perp \otimes \mathcal{F}(W^*) + \mathcal{F}(V^*) \otimes S^\perp)) \\
&\cong \mathcal{P}(V^*, R^\perp) \circ \mathcal{P}(W^*, S^\perp) \\
&\cong \mathcal{C}(V, R)^* \circ \mathcal{C}(W, S)^*.
\end{aligned}$$

□

One of the main interest of the classical notions of black and white products is that one gives the other via the Koszul dual functor. In the next sections, we define a black product for monoids (operad and non-symmetric operads). The translation of Theorem 14 in this framework will give the relation with Koszul dual functor.

4. Manin products for operads

In this section, we study Manin products for (symmetric) operads. We first give a sufficient condition for the white product to be equal to the Hadamard product. Then, we recall the bases used to describe binary quadratic operads and their Koszul dual operad. We refer the reader to [GK95], [Lod96] and [MSS02] for complete references. We make our constructions explicit for binary quadratic operads in order to do computations. The linear dual version of the black product for cooperads defines a product for operads which corresponds to the definition of Ginzburg and Kapranov, where we make the signs precise. We give an example of a pair of Koszul operads such that their products is not Koszul. This shows that black and white products for operads do not behave like black and white products for associative algebras. Following Y. Manin, we prove that $\mathcal{P}\bullet\mathcal{P}^!$ is always a Hopf operad. Finally, we describe the relation between unary operators and black products.

4.1. **Relation between the Hadamard product and the white product.** We saw in the previous section that the composite $(\pi_\mathcal{P}\otimes\pi_\mathcal{Q})\circ\Phi$ factors through its image $\bar{\Phi}\,:\,\mathcal{P}\circ\mathcal{Q}\rightarrowtail\mathcal{P}\otimes\mathcal{Q}$. Therefore, $\bar{\Phi}$ is an isomorphism if and only if the composite $(\pi_\mathcal{P}\otimes\pi_\mathcal{Q})\circ\Phi$ is an epimorphism. We shall give a sufficient condition for this.

Consider the case of binary quadratic operads, that is quadratic operads generated by binary operations ($V(n)=0$ for $n\neq 2$). In this case, the free operad on V is given by (non-planar) binary trees with vertices labelled by operations of V. Denote by $\mathbb{T}$ such a tree with $n-1$ vertices and the induced *label morphism* by $\mathcal{L}^V_\mathbb{T}\,:\,V^{\otimes(n-1)}\to\mathcal{F}(V)(n)$.

Proposition 15. *Let $\mathcal{P}$ be a binary quadratic operad such that for every $n\geq 3$ and every binary tree $\mathbb{T}$ with $n-1$ vertices, the composite $\pi_\mathcal{P}\circ\mathcal{L}^V_\mathbb{T}\,:\,V^{\otimes(n-1)}\to\mathcal{F}(V)(n)\twoheadrightarrow\mathcal{P}(n)$ is surjective.*
For every binary quadratic operad $\mathcal{Q}$, the white product $\mathcal{P}\circ\mathcal{Q}$ is equal to the Hadamard product $\mathcal{P}\otimes\mathcal{Q}$.

Proof. It is enough to prove that $(\pi_\mathcal{P}\otimes\pi_\mathcal{Q})\circ\Phi$ is an epimorphism. Let $p\otimes q$ be an elementary tensor of $\mathcal{P}(n)\otimes\mathcal{Q}(n)$, where $\mathcal{Q}=\mathcal{F}(W)/(S)$. The element q of $\mathcal{Q}(n)$ can be written $q=\sum_{i=1}^k\pi_\mathcal{Q}\circ\mathcal{L}^W_{\mathbb{T}_i}(w^i_1,\,\ldots,\,w^i_{n-1})$, with $\{\mathbb{T}_i\}$ a finite set of trees and $\{w^i_j\}$ elements of $W(2)$. By the assumption, there exists $v^i_1,\,\ldots,\,v^i_{n-1}$ in $V(2)$ such that $p=\pi_\mathcal{P}\circ\mathcal{L}^V_{\mathbb{T}_i}(v^i_1,\,\ldots,\,v^i_{n-1})$, for every $\mathbb{T}_i$. Therefore, we have $p=\dfrac{1}{k}\displaystyle\sum_{i=1}^k\pi_\mathcal{P}\circ\mathcal{L}^V_{\mathbb{T}_i}(v^i_1,\,\ldots,\,v^i_{n-1})$. Finally, it shows that

$$p\otimes q=(\pi_\mathcal{P}\otimes\pi_\mathcal{Q})\circ\Phi\left(\frac{1}{k}\sum_{i=1}^k\mathcal{L}^{V\otimes W}_{\mathbb{T}_i}(v^i_1\otimes w^i_1,\,\ldots,\,v^i_{n-1}\otimes w^i_{n-1})\right).$$

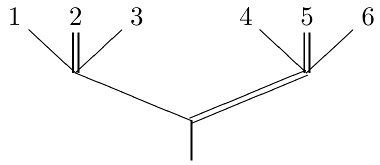
$\square$

The condition of this proposition means that every operation of $\mathcal{P}$ can be written by any type of composition of generating operations. In the next corollary, we show that the operads $\mathcal{C}om$, $\mathcal{P}erm$ and $\mathcal{C}om\mathcal{T}rias$ are examples of such operads. Recall briefly that $\mathcal{C}om$ is the operad for commutative algebras. The operad $\mathcal{P}erm$ was introduced by F. Chapoton in [Cha01] and $\mathcal{C}om\mathcal{T}rias$ was defined in [Val2] Appendix A.

Corollary 16. *For every binary quadratic operad $\mathcal{Q}$, we have*

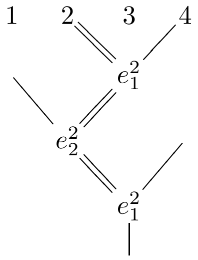
- $\mathcal{C}om\circ\mathcal{Q}=\mathcal{C}om\otimes\mathcal{Q}=\mathcal{Q}$. *The operad $\mathcal{C}om$ is neutral for the white product in the category of binary quadratic operads.*
- $\mathcal{P}erm\circ\mathcal{Q}=\mathcal{P}erm\otimes\mathcal{Q}$ *and* $\mathcal{C}om\mathcal{T}rias\circ\mathcal{Q}=\mathcal{C}om\mathcal{T}rias\otimes\mathcal{Q}$

Proof. The operad $\mathcal{C}om$ is generated by $V(2)=k$ with trivial action of $\mathbb{S}_2$ and the associativity relation. Hence, we have only one commutative operation with arity n, that is $\mathcal{C}om(n)=k$. Therefore, for every tree $\mathbb{T}$, the morphism $\mathcal{L}_\mathbb{T}$ is a surjection on k and $\mathcal{C}om\circ\mathcal{Q}=\mathcal{C}om\otimes\mathcal{Q}=\mathcal{Q}$. The operad $\mathcal{P}erm$ corresponds to commutative operations with one input emphasized (see [Val2] 4.2 and [CV06] 1.3.2). In arity n, we have n operations $\mathcal{P}erm(n)=k.e^n_1\oplus\cdots k.e^n_n$ where e^n_i corresponds to the corolla with n inputs such that the ith input (or branch) is emphasized. The composition of corollas gives a corolla where the leaf emphasized is the one with a path to the root via emphasized branches.



$\mapsto\ e_5^6$

Let $\mathbb{T}$ be a binary tree with $n-1$ vertices. To get e_i^n, it is enough the look at the unique path from the ith leaf to the root and index the vertices on this path with the relevant operations.



$\mapsto\ e_2^4$

The operations of $\mathcal{C}om\mathcal{T}rias(n)$ are corollas with at least one leaf emphasized and the proof is the same. $\square$

This corollary shows that the Hadamard product of one operad $\mathcal{C}om$, $\mathcal{P}erm$ or $\mathcal{C}om\mathcal{T}rias$ with any other binary quadratic operad is again a binary quadratic operad. For $\mathcal{C}om$, the result is obvious. In the particular case of $\mathcal{P}erm$ this result was proved directly by F. Chapoton in [Cha01]. For every binary quadratic operad $\mathcal{Q}$, he constructed by hand a quadratic operad isomorphic to $\mathcal{P}erm \otimes \mathcal{Q}$. This construction is actually the white product $\mathcal{P}erm \circ \mathcal{Q}$.

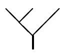
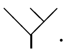
Proposition 17. *We have $\mathcal{P}erm \circ \mathcal{A}s = \mathcal{D}ias$.*

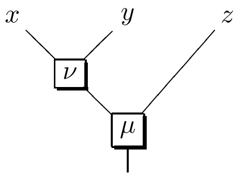
Proof. Using the complete description of $\mathcal{P}erm$, $\mathcal{A}s$ and $\mathcal{D}ias$, Chapoton proved in [Cha01] that $\mathcal{P}erm \otimes \mathcal{A}s = \mathcal{D}ias$. Apply the previous corollary to conclude. $\square$

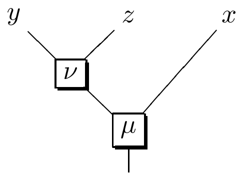
4.2. **Binary quadratic operad and Koszul dual operad.** The preceding section gives a method for computing the white product for a particular class of operads. When we cannot apply this method, we need the explicit form of the products to compute them. In this section, we describe a basis for binary quadratic operads and their Koszul dual operads.

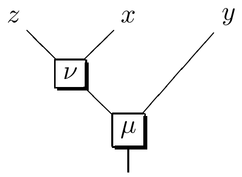
Recall that the free operad $\mathcal{F}(V)$ on V is given by trees with the vertices indexed by elements of V, with respect to the action of the symmetric groups. When V is an $\mathbb{S}_2$-module, that is a module over the symmetric group $\mathbb{S}_2$, we have $\mathcal{F}_{(2)}(V) = \mathcal{F}(V)(3)$, the part with 3 inputs of the free operad on V which is isomorphic to

$$\mathcal{F}(V)(3) = \big(V \otimes_{\mathbb{S}_2} (V \otimes k \oplus k \otimes V)\big) \otimes_{\mathbb{S}_2} k[\mathbb{S}_3],$$

where the summand $V \otimes (V \otimes k)$ corresponds to the compositions on the left  and the summand $V \otimes (k \otimes V)$ corresponds to the compositions on the right . Since the action of $\mathbb{S}_2$ maps one to the other, we choose the one on the left and $\mathcal{F}(V)(3)$ is isomorphic to the induced representation $\mathrm{Ind}_{\mathbb{S}_2 \times \mathbb{S}_1}^{\mathbb{S}_3} (V \otimes (V \otimes k))$. Therefore, $\mathcal{F}(V)(3)$ can be identify with 3 copies of $V \otimes V$ represented by the following types of tree :

I : 

II : 

III : 

.

Denote them by $\mu \circ_{\mathrm{I}} \nu$, $\mu \circ_{\mathrm{II}} \nu$ and $\mu \circ_{\mathrm{III}} \nu$.
The action of the permutation (12) is given by $(\mu \circ_{\mathrm{I}} \nu)^{(12)} = \mu \circ_{\mathrm{I}} \nu^{(12)}$, $(\mu \circ_{\mathrm{II}} \nu)^{(12)} = \mu \circ_{\mathrm{III}} \nu^{(12)}$, $(\mu \circ_{\mathrm{III}} \nu)^{(12)} = \mu \circ_{\mathrm{II}} \nu^{(12)}$ and the action of (132) is given by $(\mu \circ_{\alpha} \nu)^{(123)} = \mu \circ_{(\alpha+\mathrm{I})} \nu$.

Remark. This basis is different from the one in [GK95] p. 228. The one given here has nice symmetric properties with respect to the action of $\mathbb{S}_3$ that we will use in 4.4 to simplify the computations.

The dual representation V^* of an $\mathbb{S}_n$-module V is the vector space $V^* = \mathrm{Hom}(V, k)$ endowed with the following right action of the symmetric group. For $f : V \to k$ and $\sigma \in \mathbb{S}_n$, we have $(f^\sigma)(x) := f(x^{\sigma^{-1}})$. We will need to twist the dual representation by the signature, that is $V^\vee := V^* \otimes \mathrm{sgn}_{\mathbb{S}_n}$.

Let V be an $\mathbb{S}$-module concentrated in arity 2, that is an $\mathbb{S}_2$-module. When V is a finite dimensional k-vector space, denote by μ, ν, η, $\zeta, \ldots$ one of its basis, stable by the action of $\mathbb{S}_2$, and by μ^*, ν^*, η^*, $\zeta^*, \ldots$ the dual basis. Therefore $\mu^\vee = \mu^*$, $\nu^\vee = \nu^*, \eta^\vee = \eta^*$, $\zeta^\vee = \zeta^*, \ldots$ forms a basis of $V^\vee$ such that $(\mu^\vee)^{(12)} = -(\mu^{(12)})^\vee$.We define the following non-degenerate bilinear form

$$\mathcal{F}(V)(3) \otimes \mathcal{F}(V^\vee)(3) \xrightarrow{<\,,\,>} k$$
$$< \mu \circ_\alpha \nu,\, \eta^\vee \circ_\beta \zeta^\vee > := \begin{cases} 1 & \text{if } \alpha = \beta,\ \mu = \eta \text{ and } \nu = \zeta, \\ 0 & \text{otherwise.} \end{cases}$$

For a sub-$\mathbb{S}_3$-module R of $\mathcal{F}(V)(3)$, we consider its orthogonal $R^\perp := \{\Omega \in \mathcal{F}(V^\vee)(3) \mid\ < \omega,\, \Omega >= 0,\ \forall\, \omega \in R\}$ for this bilinear form.
Since the action of $\mathbb{S}_3$ on the bilinear form $<\,,\,>$ is given by the signature $< \omega^\sigma,\, \Omega^\sigma >= \mathrm{sgn}(\sigma).< \omega,\, \Omega >$ we have that $R^\perp$ is a sub-$\mathbb{S}_3$-module of $\mathcal{F}(V^\vee)(3)$. Note that the non-degenerate bilinear form $<\,,\,>$ defines an isomorphism of $\mathbb{S}_3$-modules from $\mathcal{F}(V)(3)^\vee$ to $\mathcal{F}(V^\vee)(3)$.

Recall from 2.3 that under finite dimensional assumptions, the Koszul dual operad of $\mathcal{F}(V)/(R)$ is $\mathcal{P}^! = \mathcal{F}(V^\vee)/(R^\perp)$. This bilinear form provides a method for computing it. The canonical isomorphism $(V^\vee)^\vee \cong V$ induces $(R^\perp)^\perp \cong R$ and $\left(\mathcal{P}^!\right)^! = \mathcal{P}$.

Examples. The operad $\mathcal{C}om$ for commutative (associative) algebras $(A, *)$ is generated by the one dimensional $\mathbb{S}_2$-module $V := k.*$ with trivial action. Denote by $t_1 = * \circ_{\mathrm{I}} *$, $t_2 = * \circ_{\mathrm{II}} *$ and $t_3 = * \circ_{\mathrm{III}} *$ the elements of the basis of $\mathcal{F}(V)(3)$. The associativity relation is the quadratic relation $t_1 = t_2 = t_3$. Therefore, the operad $\mathcal{C}om$ has the following presentation $\mathcal{C}om = \mathcal{F}(k.*)/(t_1 - t_2,\, t_2 - t_3)$.

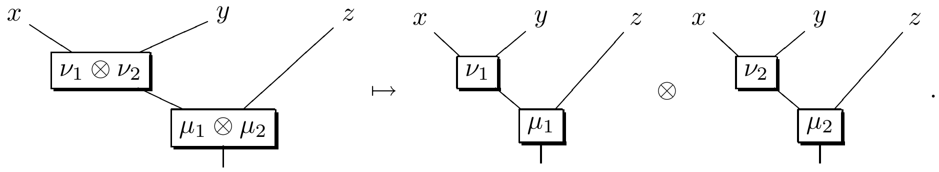
The operad $\mathcal{L}ie$ for Lie algebras $(L, [\,,\,])$ is generated by the one dimensional $\mathbb{S}_2$-module $V' := k.[\,,\,]$ where the action is given by the signature. If we denote by t'_1, t'_2 and t'_3 the elements of the basis of $\mathcal{F}(V')(3)$, the Jacobi relation corresponds to $t'_1 + t'_2 + t'_3 = 0$ and the operad $\mathcal{L}ie$ is given by $\mathcal{L}ie = \mathcal{F}(k.[\,,\,])/(t'_1 + t'_2 + t'_3)$.

Under the identification $V' \cong V^\vee$, we have $((t_1 - t_2).k \oplus (t_2 - t_3).k)^\perp = (t'_1 + t'_2 + t'_3).k$. Therefore we get $\mathcal{C}om^! = \mathcal{L}ie$ (and $\mathcal{L}ie^! = \mathcal{C}om$).

4.3. **Definition of the black product for operads.** Using the notions of the previous section, we define a black product for binary quadratic operads.

The definition of the white product is based on the morphism Φ (see 3.1). For binary quadratic operads, this morphism $\Phi : \mathcal{F}(V \otimes W)(3) \to \mathcal{F}(V)(3) \otimes \mathcal{F}(W)(3)$ is the componentwise projection.

For instance, for compositions of type I, we have



We describe a general method that will be applied later in other cases.

When V is finite dimensional, the Koszul dual of binary quadratic operad $\mathcal{F}(V)/(R)$ can be defined by means of a particular non-degenerate bilinear form on $\mathcal{F}(V)(3) \otimes \mathcal{F}(V^\vee)(3)$ (see 4.2) denoted by $<,>_V$. For the moment, we do not need its explicit description. Since this bilinear form is non-degenerate, it induces an isomorphism $\theta_V : \mathcal{F}(V)(3) \xrightarrow{\simeq} \mathcal{F}(V^\vee)(3)^\vee$. Let V and W be two finite dimensional k-modules. Define the morphism Ψ by the following commutative diagram

$$\begin{tikzcd}
\mathcal{F}(V)(3) \otimes \mathcal{F}(W)(3) \otimes k.\mathrm{sgn}_{\mathbb{S}_3} \arrow[r, "\Psi"] \arrow[d, "\theta_V \otimes \theta_W \otimes \mathrm{sgn}"] & \mathcal{F}(V \otimes W \otimes k.\mathrm{sgn}_{\mathbb{S}_2})(3) \\
\mathcal{F}(V^\vee)(3)^\vee \otimes \mathcal{F}(W^\vee)(3)^\vee \otimes k.\mathrm{sgn}_{\mathbb{S}_3} \arrow[d, "\simeq"] & \mathcal{F}((V \otimes W \otimes k.\mathrm{sgn}_{\mathbb{S}_2})^\vee)(3)^\vee \arrow[u, "\theta^{-1}_{V \otimes W \otimes \mathrm{sgn}}"'] \\
\big(\mathcal{F}(V^\vee)(3) \otimes \mathcal{F}(W^\vee)(3)\big)^\vee \arrow[r, "{}^t\Phi_{V^\vee,\, W^\vee}"] & \mathcal{F}(V^\vee \otimes W^\vee)(3)^\vee, \arrow[u, "\simeq"']
\end{tikzcd}$$

where $\simeq$ stands for the natural isomorphism for the linear dual of a tensor product, since the modules are finite dimensional. The morphism Ψ defined here is a twisted version of the one defined in 3.3.
Recall that $\Phi_{V^\vee,\, W^\vee}$ is the morphism $\mathcal{F}(V^\vee \otimes W^\vee) \to \mathcal{F}(V^\vee) \otimes \mathcal{F}(W^\vee)$.

Lemma 18. *Let $\mathcal{P} = \mathcal{F}(V)/(R)$ and $\mathcal{Q} = \mathcal{F}(W)/(S)$ be two binary quadratic operads such that V and W are finite dimensional. The orthogonal of $\Psi(R \otimes S)$ for $<\,,\,>_{V \otimes W \otimes sgn}$ is $\Phi^{-1}_{V^\vee,\, W^\vee}(R^\perp \otimes \mathcal{F}(W^\vee) + \mathcal{F}(V^\vee) \otimes S^\perp)$.*

Proof. By definition of the transpose of $\Phi_{V^\vee,\, W^\vee}$, we have

$$\begin{aligned}
<\Psi(r \otimes s),\, X>_{V \otimes W \otimes \mathrm{sgn}} &= \; < r \otimes s, \Phi_{V^\vee,\, W^\vee}(X) >_{(\mathcal{F}(V) \otimes \mathcal{F}(V^\vee)) \times (\mathcal{F}(W) \otimes \mathcal{F}(W^\vee))} \\
&= \; (< r, - >_V \,.\, < s, - >_W) \circ \Phi_{V^\vee,\, W^\vee}(X),
\end{aligned}$$

for every $(r, s) \in R \times S$ and every $X \in \mathcal{F}\big((V \otimes W \otimes k.\mathrm{sgn}_{\mathbb{S}_2})^\vee\big)$
Therefore, we have $\Psi(R \otimes S)^\perp =$

$$\begin{aligned}
&= \Big\{ X \in \mathcal{F}((V \otimes W \otimes k.\mathrm{sgn}_{\mathbb{S}_2})^\vee)(3) \mid \forall (r,\, s) \in R \times S \quad <\Psi(r \otimes s),\, X>_{V \otimes W \otimes k.\mathrm{sgn}_{\mathbb{S}_2}} = 0 \Big\} \\
&= \{ X \in \mathcal{F}(V^\vee \otimes W^\vee)(3) \mid \forall (r,\, s) \in R \times S \quad (< r, - >_{V'} \,.\, < s, - >_{W'}) \circ \Phi_{V^\vee, W^\vee}(X) = 0 \} \\
&= \big\{ X \in \mathcal{F}(V^\vee \otimes W^\vee)(3) \mid \Phi_{V^\vee, W^\vee}(X) \in R^\perp \otimes \mathcal{F}(W^\vee) + \mathcal{F}(V^\vee) \otimes S^\perp \big\} \\
&= \Phi^{-1}_{V^\vee, W^\vee}(R^\perp \otimes \mathcal{F}(W^\vee) + \mathcal{F}(V^\vee) \otimes S^\perp).
\end{aligned}$$

□

Definition (Black product for operads)**.** Let $\mathcal{P} = \mathcal{F}(V)/(R)$ and $\mathcal{Q} = \mathcal{F}(W)/(S)$ be two binary quadratic operads with finite dimensional generating spaces. Define their black product by the formula

$$\mathcal{P} \bullet \mathcal{Q} \quad = \quad \mathcal{F}(V \otimes W \otimes k.\mathrm{sgn}_{\mathbb{S}_2})/\big(\Psi(R \otimes S)\big).$$

Proposition 19. *For binary quadratic operads generated by finite dimensional $\mathbb{S}_2$-modules, this definition of black product verifies $\big(\mathcal{P} \bullet \mathcal{Q}\big)^! = \mathcal{P}^! \circ \mathcal{Q}^!$ and corresponds to the one of Ginzburg and Kapranov* [GK95'].

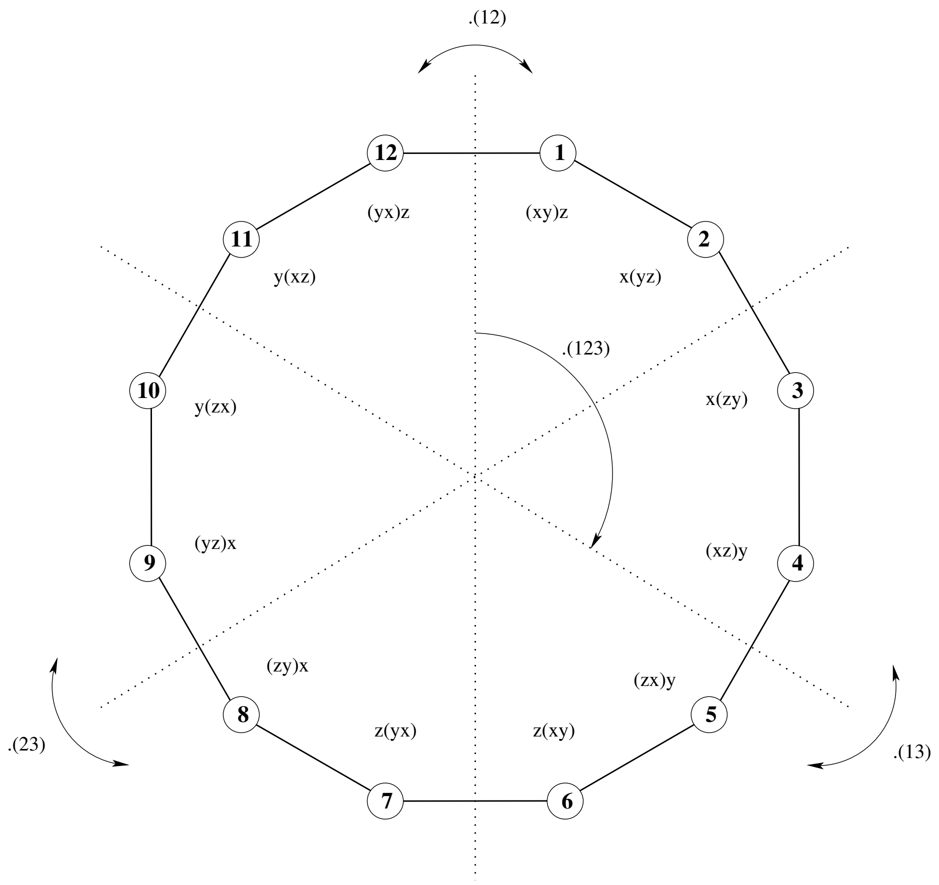


FIGURE 1. The permutoassociahedron

PROOF. It is a direct corollary of Lemma 18. □

Since $\mathcal{C}om$ is the neutral element for $\circ$, we have that $\mathcal{L}ie$ is the neutral element for $\bullet$.

4.4. **Examples.** We make explicit some computations of black and white products. For the definitions of the various operads encountered in this section, we refer the reader to [Lod01].

In order to compute black and white products for operads where the space of generators V is equal to $k[\mathbb{S}_2] = \mu.k \oplus \mu'.k$, with $\mu.(12) = \mu'$, we will adopt the following convention. Denote by $v_1, \ldots, v_{12}$ the 12 elements of $\mathcal{F}(V)(3)$.

1	$\mu \circ_{\mathrm{I}} \mu \;\leftrightarrow\; (xy)z$	**5**	$\mu \circ_{\mathrm{III}} \mu \;\leftrightarrow\; (zx)y$	**9**	$\mu \circ_{\mathrm{II}} \mu \;\leftrightarrow\; (yz)x$
2	$\mu' \circ_{\mathrm{II}} \mu \;\leftrightarrow\; x(yz)$	**6**	$\mu' \circ_{\mathrm{I}} \mu \;\leftrightarrow\; z(xy)$	**10**	$\mu' \circ_{\mathrm{III}} \mu \;\leftrightarrow\; y(zx)$
3	$\mu' \circ_{\mathrm{II}} \mu' \;\leftrightarrow\; x(zy)$	**7**	$\mu' \circ_{\mathrm{I}} \mu' \;\leftrightarrow\; z(yx)$	**11**	$\mu' \circ_{\mathrm{III}} \mu' \;\leftrightarrow\; y(xz)$
4	$\mu \circ_{\mathrm{III}} \mu' \;\leftrightarrow\; (xz)y$	**8**	$\mu \circ_{\mathrm{II}} \mu' \;\leftrightarrow\; (zy)x$	**12**	$\mu \circ_{\mathrm{I}} \mu' \;\leftrightarrow\; (yx)z$

This labelling corresponds to the labelling of the permutoassociahedron [Kap93]. Figure 1 represents it with the action of the symmetric group $\mathbb{S}_3$.

An associative algebra is a vector space with a binary associative operation, that is $\mu(\mu(a,\, b),\, c) = \mu(a,\, \mu(b,\, c))$. With these notations, the relations of associativity of the operad $\mathcal{A}s$ become $v_i - v_{i+1}$, for $i = 1, 3, 5, 7, 9, 11$. A *(right) preLie algebra* is a vector space with a binary operation such that its associator is right symmetric, that is $\mu(\mu(a,\, b),\, c) - \mu(a,\, \mu(b,\, c)) = \mu(\mu(a,\, c),\, b) - \mu(a,\, \mu(c,\, b))$. This relation corresponds to $v_i - v_{i+1} + v_{i+2} - v_{i+3}$ for $i = 1, 5, 9$ with our conventions. The operation of a $\mathcal{P}erm$*-algebra* verifies $\mu(\mu(a,\, b),\, c) = \mu(a,\, \mu(b,\, c)) = \mu(a,\, \mu(c,\, b))$ which gives here

$v_i = v_{i+1} = v_{i+2} = v_{i+3}$ for $i = 1, 5, 9$. Note that $\mathcal{P}re\mathcal{L}ie$ is the Koszul dual of $\mathcal{P}erm$ and vice versa (*cf.* [CL01]).

We now give an example of computation.

Theorem 20. *We have $\mathcal{P}re\mathcal{L}ie \bullet \mathcal{C}om = \mathcal{Z}inb$, $\mathcal{P}re\mathcal{L}ie \bullet \mathcal{A}s = \mathcal{D}end$ and $\mathcal{P}erm \circ \mathcal{L}ie = \mathcal{L}eib$.*

PROOF. Denote by ν the commutative generating operation of $\mathcal{C}om$ and by w_1, w_5, w_9 the related elements of $\mathcal{F}(\nu.k)(3)$. We write the associativity relation of ν : $w_1 - w_5 = 0$ and $w_5 - w_9 = 0$. We have

$$\begin{array}{llcl}
(1) & \Psi\left((v_1 - v_2 + v_3 - v_4) \otimes (w_1 - w_5)\right) & = & \Psi\left(v_1 \otimes w_1 + v_4 \otimes w_5\right) \\
(2) & \Psi\left((v_1 - v_2 + v_3 - v_4) \otimes (w_5 - w_9)\right) & = & \Psi\left((v_2 - v_3) \otimes w_9 - v_4 \otimes w_5\right) \\
(3) & \Psi\left((v_5 - v_6 + v_7 - v_8) \otimes (w_1 - w_5)\right) & = & \Psi\left((v_7 - v_6) \otimes w_1 - v_5 \otimes w_5\right) \\
(4) & \Psi\left((v_5 - v_6 + v_7 - v_8) \otimes (w_5 - w_9)\right) & = & \Psi\left(v_5 \otimes w_5 + v_8 \otimes w_9\right) \\
(5) & \Psi\left((v_9 - v_{10} + v_{11} - v_{12}) \otimes (w_1 - w_5)\right) & = & \Psi\left(-v_{12} \otimes w_1 + (v_{10} - v_{11}) \otimes w_5\right) \\
(6) & \Psi\left((v_9 - v_{10} + v_{11} - v_{12}) \otimes (w_5 - w_9)\right) & = & \Psi\left((v_{11} - v_{10}) \otimes w_5 - v_9 \otimes w_9\right)
\end{array}$$

The action of (132) sends (1) to (4), (3) to (6) and (5) to (2). The image of (1) under (13) is (3). Therefore, we only need to make (1) and (2) explicit. If we identify $(\mu.k \oplus \mu'.k) \otimes \nu.k \otimes k.\mathrm{sgn}_{\mathbb{S}_2}$ with $\gamma.k \oplus \gamma'.k$ via the isomorphism of $\mathbb{S}_2$-modules

$$\begin{array}{rcl}
\mu \otimes \nu \otimes 1 & \mapsto & \gamma \\
\mu' \otimes \nu \otimes 1 & \mapsto & -\gamma',
\end{array}$$

the morphism Ψ becomes

$$\begin{array}{rcl}
\Psi((\mu \circ_{\mathrm{I}} \mu) \otimes (\nu \circ_{\mathrm{I}} \nu)) = \Psi(v_1 \otimes w_1) & = & \gamma \circ_{\mathrm{I}} \gamma = z_1 \quad \text{and} \\
\Psi((\mu' \circ_{\mathrm{II}} \mu) \otimes (\nu \circ_{\mathrm{II}} \nu)) = \Psi(v_2 \otimes w_1) & = & -\gamma' \circ_{\mathrm{I}} \gamma = -z_2.
\end{array}$$

The image of the other elements are obtained from these two by the action of $\mathbb{S}_3$. For instance, we have $\Psi(v_3 \otimes w_1) = -z_3$, $\Psi(v_4 \otimes w_1) = z_4$ and $\Psi(v_5 \otimes w_5) = z_5$.
We get

$$\begin{array}{rcl}
\Psi\left(v_1 \otimes w_1 + v_4 \otimes w_5\right) & = & \gamma \circ_{\mathrm{I}} \gamma - \gamma \circ_{\mathrm{III}} \gamma' \\
\Psi\left((v_2 - v_3) \otimes w_9 - v_4 \otimes w_5\right) & = & -\gamma' \circ_{\mathrm{II}} \gamma - \gamma' \circ_{\mathrm{II}} \gamma' + \gamma \circ_{\mathrm{III}} \gamma'.
\end{array}$$

Finally, if we represent the operation $\gamma(x, y)$ by $x \star y$, we have

$$\begin{array}{rcl}
(x \star y) \star z & = & (x \star z) \star y \\
(x \star z) \star y & = & x \star (z \star y) + x \star (y \star z),
\end{array}$$

where we recognize the axioms of a Zinbiel algebra (*cf.* [Lod01]).

The two other identities are obtained by Koszul duality using Proposition 19. From Proposition 17 $\mathcal{P}erm \circ \mathcal{A}s = \mathcal{D}ias$, we get $\mathcal{P}re\mathcal{L}ie \bullet \mathcal{A}s = \left(\mathcal{P}erm \circ \mathcal{A}s\right)^! = \left(\mathcal{D}ias\right)^! = \mathcal{D}end$. The last equality $\mathcal{P}erm \circ \mathcal{L}ie = \mathcal{L}eib$ is the Koszul dual of the first one $\mathcal{P}re\mathcal{L}ie \bullet \mathcal{C}om = \mathcal{Z}inb$. $\square$

Jean-Louis Loday defined the operad $\mathcal{D}end$ by two operations such that their sum is an associative product (see [Lod01]). In the same way, he defined the operad $\mathcal{Z}inb$ with one product such that its symmetrized version is a associative (and commutative) product. This process is often called a *splitting of associativity*. Proposition 20 shows that we can interpret the operation $\mathcal{P}re\mathcal{L}ie \bullet -$ as a natural way of splitting the associativity.

A commutative algebra is an associative algebra. Therefore, we have a morphism of operads $\mathcal{A}s \to \mathcal{C}om$. Since a commutative algebra is a $\mathcal{P}erm$-algebra and a $\mathcal{P}erm$-algebra is an associative algebra, the previous morphism factors through $\mathcal{A}s \to \mathcal{P}erm \to \mathcal{C}om$. Similarly, a Zinbiel algebra is a dendriform algebra $\mathcal{D}end \to \mathcal{Z}inb$. We can factor this morphism by a new operad $\mathcal{P}reLie \bullet \mathcal{P}erm$ using the functor $\mathcal{P}reLie \bullet -$

$$\begin{tikzcd} \mathcal{A}s \arrow[r] & \mathcal{P}erm \arrow[r] & \mathcal{C}om & {} \arrow[d, bend left, "\mathcal{P}reLie\,\bullet -"] \\ \mathcal{D}end \arrow[r] & \mathcal{P}reLie \bullet \mathcal{P}erm \arrow[r] & \mathcal{Z}inb. & {} \end{tikzcd}$$

We describe this new type of algebra.

Theorem 21. *An algebra over the operad $\mathcal{P}reLie \bullet \mathcal{P}erm$ is a dendriform algebra such that the two operations $\prec$ and $\succ$ verify the two extra relations*

$$\begin{aligned} & x \prec (y \prec z) + x \prec (y \succ z) = x \prec (z \prec y) + x \prec (z \succ y) \\ & x \succ (y \prec z) = x \succ (z \succ y). \end{aligned}$$

Using the notation $x * y := x \prec y + x \succ y$, we sum up the 5 relations of a $\mathcal{P}reLie \bullet \mathcal{P}erm$-algebra by

$$\begin{cases} (x \prec y) \prec z = x \prec (y * z) \\ (x \succ y) \prec z = x \succ (y \prec z) \\ (x * y) \succ z = x \succ (y \succ z) \\ x \prec (y * z) = x \prec (z * y) \\ x \succ (y \prec z) = x \succ (z \succ y). \end{cases}$$

A $\mathcal{P}erm$-algebra is an associative algrebra which is symmetric on the right. A $\mathcal{P}reLie \bullet \mathcal{P}erm$-algebra is a dendriform algebra with right-symmetric relations.

Proof. Denote by ω the generating operation of the operad $\mathcal{P}erm$ and by $w_1, \ldots, w_{12}$ the related elements of $\mathcal{F}(\omega.k \oplus \omega'.k)(3)$. The space of relations $\Psi(R \otimes S)$ is generated by the elements $\Psi\left((v_i - v_{i+1} + v_{i+2} - v_{i+3}) \otimes (w_j - w_{j+1})\right)$, for $i \in \{1, 5, 9\}$ and $j \in \{1, 2, 3, 5, 6, 7, 9, 10, 11\}$. Reduce the computations using the action of $\mathbb{S}_3$ (the symmetries can be seen on the permutoassociahedron), it remains 5 relations among which 3 correspond to the following ones

$$\begin{aligned} &(7) & \Psi\left((v_1 - v_2 + v_3 - v_4) \otimes (w_1 - w_2)\right) &= \Psi\left(v_1 \otimes w_1 + (v_2 - v_3) \otimes w_2\right) \\ &(8) & \Psi\left((v_1 - v_2 + v_3 - v_4) \otimes (w_5 - w_6)\right) &= \Psi\left(-v_4 \otimes w_5 - v_1 \otimes w_6\right) \\ &(9) & \Psi\left((v_1 - v_2 + v_3 - v_4) \otimes (w_7 - w_8)\right) &= \Psi\left(v_1 \otimes w_7 + (v_2 - v_3) \otimes w_8\right). \end{aligned}$$

Identify the representation $(\mu.k \oplus \mu'.k) \otimes (\omega.k \oplus \omega'.k) \otimes k.\mathrm{sgn}_{\mathbb{S}_2}$ with the two copies of $k[\mathbb{S}_2]$: $\alpha.k[\mathbb{S}_2] \oplus \beta.k[\mathbb{S}_2] = \alpha.k \oplus \alpha'.k \oplus \beta.k \oplus \beta'.k$ via the isomorphism of $\mathbb{S}_2$-modules

$$\begin{array}{rcl} \mu \otimes \omega \otimes 1 \mapsto \alpha & \text{and} & \mu' \otimes \omega \otimes 1 \mapsto -\beta, \\ \mu' \otimes \omega' \otimes 1 \mapsto -\alpha' & \text{and} & \mu \otimes \omega' \otimes 1 \mapsto \beta'. \end{array}$$

The morphism Ψ becomes

$$\begin{aligned} \Psi((\mu \circ_{\mathrm{I}} \mu) \otimes (\omega \circ_{\mathrm{I}} \omega)) = \Psi(v_1 \otimes w_1) &= \alpha \circ_{\mathrm{I}} \alpha, \\ \Psi((\mu' \circ_{\mathrm{II}} \mu) \otimes (\omega' \circ_{\mathrm{II}} \omega)) = \Psi(v_2 \otimes w_2) &= -\alpha' \circ_{\mathrm{II}} \alpha \quad \text{and} \\ \Psi((\mu' \circ_{\mathrm{II}} \mu') \otimes (\omega' \circ_{\mathrm{II}} \omega)) = \Psi(v_3 \otimes w_2) &= -\alpha' \circ_{\mathrm{II}} \beta, \end{aligned}$$

for instance. Hence, the relations (7), (8) and (9) are

$$\begin{aligned} & \alpha \circ_{\mathrm{I}} \alpha - \alpha' \circ_{\mathrm{II}} \alpha - \alpha' \circ_{\mathrm{II}} \beta \\ & \alpha \circ_{\mathrm{III}} \beta - \beta' \circ_{\mathrm{I}} \alpha \\ & \beta' \circ_{\mathrm{I}} \beta' - \beta \circ_{\mathrm{II}} \beta' - \beta \circ_{\mathrm{II}} \alpha' \end{aligned}$$

If we represent the operation $\alpha(x, y)$ by $x \prec y$ and $\beta(x, y)$ by $x \succ y$, these 3 relations become

$$\begin{aligned} & (x \prec y) \prec z = x \prec (y \prec z) + x \prec (y \succ z) \\ & (z \succ x) \prec y = z \succ (x \prec y) \\ & z \succ (y \succ x) = (z \succ y) \succ x + (z \prec y) \succ x, \end{aligned}$$

which are the axioms defining dendriform algebras [Lod01].

The two other relations are

$$\Psi\left((v_1 - v_2 + v_3 - v_4) \otimes (w_2 - w_3)\right) = -\alpha' \circ_{\mathrm{II}} \alpha + \alpha' \circ_{\mathrm{II}} \beta' - \alpha' \circ_{\mathrm{II}} \beta + \alpha' \circ_{\mathrm{II}} \alpha', \tag{10}$$

$$\Psi\left((v_1 - v_2 + v_3 - v_4) \otimes (w_6 - w_7)\right) = \beta' \circ_{\mathrm{I}} \alpha - \beta' \circ_{\mathrm{I}} \beta'. \tag{11}$$

And they give after identification

$$\begin{aligned} x \prec (y \prec z) + x \prec (y \succ z) &= x \prec (z \prec y) + x \prec (z \succ y) \\ x \succ (y \prec z) &= x \succ (z \succ y). \end{aligned}$$

□

A $\mathcal{P}re\mathcal{L}ie \bullet \mathcal{P}erm$-algebra is a $\mathcal{P}erm$-algebra with splitting of the associativity relation.

Proposition 22. *Let $(A, \prec, \succ)$ be a $\mathcal{P}re\mathcal{L}ie \bullet \mathcal{P}erm$-algebra. With the operation $* := \prec + \succ$, the vector space $(A, *)$ becomes a $\mathcal{P}erm$-algebra.*

Proof. Consider the sum of the relations. □

Since a $\mathcal{P}re\mathcal{L}ie$-algebra gives a $\mathcal{L}ie$-algebra by anti-symmetrization of the product, we have a morphism of operads $\mathcal{L}ie \xrightarrow{\lambda} \mathcal{P}re\mathcal{L}ie$. Taking the black product of this morphism with an operad $\mathcal{P}$, we get a morphism of the form $\mathcal{P} = \mathcal{L}ie \bullet \mathcal{P} \xrightarrow{\lambda \bullet \mathcal{P}} \mathcal{P}re\mathcal{L}ie \bullet \mathcal{P}$. A $\mathcal{P}re\mathcal{L}ie \bullet \mathcal{P}$-algebra has twice more generating operations then $\mathcal{P}$ and this morphism corresponds to take the sum of them. Denote it by $\lambda \bullet \mathcal{P} = +$. In the previous cases, we had

$$\begin{tikzcd} \mathcal{A}s \arrow[r] \arrow[d, "+"] & \mathcal{P}erm \arrow[r] \arrow[d, "+"] & \mathcal{C}om \arrow[d, "+"] \\ \mathcal{D}end \arrow[r] & \mathcal{P}re\mathcal{L}ie \bullet \mathcal{P}erm \arrow[r] & \mathcal{Z}inb. \end{tikzcd}$$

Therefore, the black product with $\mathcal{P}re\mathcal{L}ie$ is a general splitting of the relations.

One interesting property of the black and white products is to recover classical operads and morphisms between them by means of products from simpler operads. We have the dual diagram of operads

$$\begin{tikzcd} \mathcal{A}s & \mathcal{P}re\mathcal{L}ie \arrow[l] & \mathcal{L}ie \arrow[l] \\ \mathcal{D}ias \arrow[u] & \mathcal{P}re\mathcal{L}ie \circ \mathcal{P}erm \arrow[l] \arrow[u] & \mathcal{L}eib. \arrow[l] \arrow[u] \end{tikzcd}$$

The operad $\mathcal{P}re\mathcal{L}ie$ allows to factor the map $\mathcal{A}s \leftarrow \mathcal{L}ie$. The notion of $\mathcal{P}re\mathcal{L}ie$-algebra is important and has application in deformation theory and differential geometry for instance (see [CL01]). The second row $\mathcal{D}ias \leftarrow \mathcal{L}eib$ was introduced by J.-L. Loday with a view forward applications in algebraic K-theory (see the introduction of [Lod01]). The operad $\mathcal{D}ias$ appears naturally when one tries to build a bicomplex in algebraic K-theory with the same form then the one in cyclic homology (the additive counterpart of algebraic K-theory). Since the operad $\mathcal{P}re\mathcal{L}ie \circ \mathcal{P}erm$ factors $\mathcal{D}ias \rightarrow \mathcal{L}eib$, we expect the operad $\mathcal{P}re\mathcal{L}ie \circ \mathcal{P}erm$ to appear in these fields in the future.

Recall from [CL01], that a basis for $\mathcal{P}re\mathcal{L}ie(n)$ is given by the set of rooted trees with n vertices labelled by $\{1, \ldots, n\}$. From Corollary 16, we have $\mathcal{P}re\mathcal{L}ie \circ \mathcal{P}erm = \mathcal{P}re\mathcal{L}ie \otimes \mathcal{P}erm$. Therefore, a basis for $\mathcal{P}re\mathcal{L}ie \circ \mathcal{P}erm(n)$ is provided by the set of rooted trees with n vertices labelled by $\{1, \ldots, n\}$ with one vertex emphasized. We leave to the reader to describe the composition map of this operad. (Use the composition of $\mathcal{P}re\mathcal{L}ie$ based on rooted trees given in [CL01] with the fact that only the insertion of a tree in an emphasized vertex keeps a vertex emphasized).

4.5. **A counterexample.** In this section, we show that the category of Koszul operads is not stable by white and black products. We exhibit a pair of Koszul operads whose black product is not Koszul.

Consider the *nilpotent* operad $\mathcal{N}$ defined by a generating skew-symmetric binary operation such that every composition of it vanishes.

Lemma 23. *The operad $\mathcal{P}reLie \bullet \mathcal{N}$ is equal to the quadratic operad generated by a binary operation $\diamond$ with the following relations : $(x \diamond y) \diamond z = 0$ and $x \diamond (y \diamond z) = x \diamond (z \diamond y)$, for every $x,\ y,\ z$.*

Proof. We use the same notations v_i for the space R of relations of the operad $\mathcal{P}reLie$. The space S of relations of the nilpotent operad is generated by w_1, w_5 and w_9. By symmetry of the relations, we only have to compute the three terms

$$\Psi\left((v_1 - v_2 + v_3 - v_4) \otimes w_1\right) = \Psi(v_1 \otimes w_1) = \diamond \circ_{\mathrm{I}} \diamond, \tag{12}$$

$$\Psi\left((v_1 - v_2 + v_3 - v_4) \otimes w_5\right) = \Psi((-v_2 + v_3) \otimes w_5) = -\diamond' \circ_{\mathrm{II}} \diamond + \diamond' \circ_{\mathrm{II}} \diamond', \tag{13}$$

$$\Psi\left((v_1 - v_2 + v_3 - v_4) \otimes w_9\right) = \Psi(-v_4 \otimes w_9) = \diamond \circ_{\mathrm{III}} \diamond. \tag{14}$$

They correspond to

$$(x \diamond y) \diamond z = 0, \quad x \diamond (y \diamond z) = x \diamond (z \diamond y) \quad \text{and} \quad (x \diamond z) \diamond y = 0.$$

$\square$

Theorem 24. *The operad $\mathcal{P}reLie \bullet \mathcal{N}$ is not Koszul.*

Proof. Because of its relations, the operad $\mathcal{P}reLie \bullet \mathcal{N}$ has no operations in arity n for n greater than 4, that is $(\mathcal{P}reLie \bullet \mathcal{N})\,(n) = 0$ for $n \geq 4$. Recall that the Poincaré series of an operad $\mathcal{P}$ is defined by $f_{\mathcal{P}}(x) := \sum_{n \geq 1} \dfrac{\dim(\mathcal{P}(n))}{n!} x^n$ (see [GK95] Section 3 or [Lod01] Appendix B.5.c.). When an operad $\mathcal{P}$ is Koszul, its Poincaré series and the Poincaré series of its dual verify the equation $f_{\mathcal{P}^!}(-f_{\mathcal{P}}(-x)) = x$ ([GK95] Formula (3.3.2)). The Poincaré series $-f_{\mathcal{P}reLie \bullet \mathcal{N}}(-x)$ is $x - x^2 + \frac{1}{2}x^3$. Its inverse for the composition is

$$x + x^2 + \frac{3}{2}x^3 + \frac{5}{2}x^4 + \frac{17}{4}x^5 + 7x^6 + \frac{21}{2}x^7 + \frac{99}{8}x^8 + \frac{55}{16}x^9 \underline{- \frac{715}{16}x^{10}} + \cdots .$$

Since the 10^{th} coefficient is negative, this series does not correspond to the Poincaré series of an operad. Therefore the operad $\mathcal{P}reLie \bullet \mathcal{N}$ is not Koszul. $\square$

The operad $\mathcal{P}reLie$ is Koszul (see [CL01] for a proof in characteristic 0 and [CV06] for a more general one). Any nilpotent operad is Koszul (the Koszul dual is a free operad, which is Koszul). So the operad $\mathcal{P}reLie \bullet \mathcal{N}$ is the black product of two Koszul operads which is not a Koszul operad. This result comes from the fact the morphism Ψ (and the morphism Φ) is not an isomorphism in general. The morphism Ψ is a projection and kills part of the relations. Therefore, the coherence between the relations, expressed by the Koszul property, does not hold anymore.

4.6. **Adjunction.** In this section, we generalize the main result of [Man87] about the adjunction between the black and the white products to k-ary quadratic operads.

Let k be an integer greater than 2. Consider the category of *k-ary quadratic operads*, that is quadratic operads generated by a finite dimensional $\mathbb{S}$-module concentrated in arity k. A morphism between two k-ary quadratic operads $\mathcal{F}(V)/(R)$ and $\mathcal{F}(W)/(S)$ is a morphism induced by a map of $\mathbb{S}_k$-modules $V(k) \to W(k)$. Denote this category by *k.q-Op*.
One can generalize the basis and the non-degenerate bilinear form of 4.2 for the binary case to the k-ary case. Then Lemma 18 and Proposition 19 also hold in k.q-Op, which defines black products in this category. Recall from V. Gnedbaye [Gne97] the notion of *k-Lie algebra*, that is a module endowed with a k-ary antisymmetric bracket satisfying a generalized Jacobi relation. We denote the associated operad by $\mathcal{L}ie^{<k>}$. Gnedbaye proved that $\mathcal{L}ie^{<k>}$ is the Koszul dual operad of $\mathcal{C}om^{<k>}$ (denoted $stAs^{<k>}$ in [Gne97]), where a $\mathcal{C}om^{<k>}$-algebra is module equipped with a k-ary commutative and totally associative operation.

Proposition 25. *The black and white products endow the category of k-ary quadratic operads with a structure of symmetric monoidal category, where the operad $\mathcal{L}ie^{<k>}$ is the unit object for $\bullet$ and the operad $\mathcal{C}om^{<k>}$ is the unit object for $\circ$.*

Proof. The same arguments as in 4.1 show that for a k-ary quadratic operad $\mathcal{P}$, we have $\mathcal{C}om^{<k>} \circ \mathcal{P} = \mathcal{C}om^{<k>} \otimes \mathcal{P} = \mathcal{P}$. If $n \in (k-1).\mathbb{N}+1$, $\mathcal{C}om^{<k>}(n) = k$, otherwise $\mathcal{C}om^{<k>}(n) = \mathcal{P}(n) = 0$. The rest of the proof is straightforward. $\square$

Theorem 26. *There is a natural isomorphism* $\mathrm{Hom}_{k.q.Op}(\mathcal{P} \bullet \mathcal{Q},\, \mathcal{R}) \cong \mathrm{Hom}_{k.q.Op}(\mathcal{P},\, \mathcal{Q}^! \circ \mathcal{R})$. *Hence, the tensor category of k-ary quadratic operads with the black product $\bullet$ is endowed with an internal* Hom *object denoted* $\mathrm{hom}_\bullet(\mathcal{Q},\, \mathcal{R}) := \mathcal{Q}^! \circ \mathcal{R}$. *Dually,* $cohom(\mathcal{P},\, \mathcal{Q}) := \mathcal{P} \bullet \mathcal{Q}^!$ *defines an internal* coHom *object in* $(k.q.Op, \circ, \mathcal{C}_k)$.

Proof. Let $\mathcal{P} = \mathcal{F}(V)/(R)$, $\mathcal{Q} = \mathcal{F}(W)/(S)$ and $\mathcal{R} = \mathcal{F}(X)/(T)$ be three k-ary quadratic operads. There is a one-to-one bijection between maps $f : V \otimes W \otimes \mathrm{sgn} \to X$ and maps $g : V \to W^\vee \otimes X$. It remains to show that $\mathcal{F}(f)\big(\Psi(R \otimes S)\big) \subset T$ is equivalent to $\mathcal{F}(g)(R) \subset \Phi^{-1}(S^\perp \otimes \mathcal{F}(X) + \mathcal{F}(W^\vee) \otimes T)$. By Lemma 18, we have

$$\begin{aligned} < \mathcal{F}(g)(R), \Phi^{-1}(S^\perp \otimes \mathcal{F}(X) + \mathcal{F}(W^\vee) \otimes T)^\perp >_{W^\vee \otimes X} &= < \mathcal{F}(g)(R), \Psi(S \otimes T^\perp) >_{W^\vee \otimes X} \\ &= < \mathcal{F}(f)\left(\Psi(R \otimes S)\right), T^\perp >_X, \end{aligned}$$

which concludes the proof. $\square$

For another point of view on this type of adjunction and coHom objects in another operadic setting, we refer the reader to D. Borisov and Y.I. Manin [BM06].

Corollary 27. *Let $\mathcal{P}$ be a k-ary quadratic operad. The operad* $\mathrm{end}(\mathcal{P}) := \mathcal{P} \bullet \mathcal{P}^!$ *is a comonoid in* $(k.q.Op, \circ, \mathcal{C}_k)$.

Proof. The proof comes from general methods of coHom objects. $\square$

Composing Δ with $\bar{\Phi} : \mathrm{end}(\mathcal{P}) \circ \mathrm{end}(\mathcal{P}) \to \mathrm{end}(\mathcal{P}) \otimes \mathrm{end}(\mathcal{P})$, we get that $\mathcal{P}^! \bullet \mathcal{P}$ is a comonoid for the tensor product, that is a Hopf operad (see 1.5).

Theorem 28. *For every k-ary quadratic operad $\mathcal{P}$, the operad* $\mathrm{end}(\mathcal{P}) = \mathcal{P}^! \bullet \mathcal{P}$ *is a Hopf operad.*

The first example is $\mathcal{C}om = \mathcal{C}om \bullet \mathcal{L}ie$. Other examples are $\mathcal{P}re\mathcal{L}ie \bullet \mathcal{P}erm$, $\mathcal{Z}inb \bullet \mathcal{L}eib$.

In [Man88], Yu. I. Manin proved the equivalent theorem for quadratic algebras. This allowed him to realize quantum groups as black products of an algebra with its Koszul dual algebra. In this spirit, the previous theorem gives a method to get new "quantum groups", that is Hopf operads.

The tensor product of a Lie algebra with a commutative algebra is again a Lie algebra (Courant algebras for instance). This result can be widely generalized. Let $\mathcal{P}$ be a k-ary quadratic operad. For any $\mathcal{P}^!$-algebra A and any $\mathcal{P}$-algebra B, their tensor product $A \otimes B$ is a $\mathcal{L}ie^{<k>}$ algebra (see [Lod01] Appendix B.5.a. for a proof in the binary case and see [GW00] Theorem 2.3 for a proof in the ternary case). In the language of operads, it means that there exists a morphism of operads $\mathcal{L}ie^{<k>} \xrightarrow{l} \mathcal{P}^! \otimes \mathcal{P}$. In the particular case of $\mathcal{P} = \mathcal{L}eib$ and $\mathcal{P}^! = \mathcal{Z}inb$, J.-L. Loday and I. Dokas refined this result and proved in [DL05] that the previous map factors through $\mathcal{P}re\mathcal{L}ie$. We now give a conceptual proof of the existence of the map from $\mathcal{L}ie^{<k>}$ to $\mathcal{P}^! \otimes \mathcal{P}$ and show that is always comes from a composite with the white product.

Proposition 29. *For every k-ary quadratic operad $\mathcal{P}$, there is a canonical morphism of operads $\mathcal{L}ie^{<k>} \xrightarrow{i} \mathcal{P}^! \circ \mathcal{P}$, defined by the commutative diagram*

$$\begin{tikzcd} \mathcal{L}ie^{<k>} \arrow[rr, "l"] \arrow[dr, "i"'] & & \mathcal{P}^! \otimes \mathcal{P} \\ & \mathcal{P}^! \circ \mathcal{P}. \arrow[ur, "\bar{\Phi}"'] & \end{tikzcd}$$

Proof. Apply Theorem 40 to the triple of operads $\mathcal{L}ie^{<k>}$, $\mathcal{P}$ and $\mathcal{P}$. We get a natural isomorphism $\mathrm{Hom}_{k.\mathrm{q.Op}}(\mathcal{L}ie^{<k>} \bullet \mathcal{P},\, \mathcal{P}) \cong \mathrm{Hom}_{k.\mathrm{q.Op}}(\mathcal{L}ie^{<k>},\, \mathcal{P}^! \circ \mathcal{P})$. Since $\mathcal{L}ie^{<k>}$ is the unit object for $\bullet$, we have $\mathrm{Hom}_{k.\mathrm{q.Op}}(\mathcal{P},\, \mathcal{P}) \cong \mathrm{Hom}_{k.\mathrm{q.Op}}(\mathcal{L}ie^{<k>},\, \mathcal{P}^! \circ \mathcal{P})$. Define $\mathcal{L}ie^{<k>} \xrightarrow{i} \mathcal{P}^! \circ \mathcal{P}$ to be the image of the identity of $\mathcal{P}$ under this isomorphism. $\square$

4.7. **Cohomology operations.** In this section, we recall the definition of the intrinsic Lie bracket on the chain complex defining the cohomology theories for algebras over a Koszul operad. We use the previous section to define another Lie bracket on the same space. Because of the symmetries, this operation vanishes on cohomology.

Let $(\mathcal{P}, \mu^{\mathcal{P}})$ and $(\mathcal{Q},\, \mu^{\mathcal{Q}})$ be two augmented dg-operads and let $\rho\ :\ \mathcal{P} \to \mathcal{Q}$ be a morphism of augmented dg-operads of degree 0. This morphism makes $\mathcal{Q}$ a module over $\mathcal{P}$. Denote by $\mu^{\mathcal{P}}_{(1,1)}$ the partial composition of $\mathcal{P}$, that is the composition of two non-trivial operations of $\mathcal{P}$.

Definition (Derivation)**.** A homogenous morphism $\partial : \mathcal{P} \to \mathcal{Q}$ is a *homogenous derivation of* ρ if
$$\partial \circ \mu^{\mathcal{P}}_{(1,\,1)} = \mu^{\mathcal{Q}}_{(1,\,1)} \circ (\partial \otimes \rho) + \mu^{\mathcal{Q}}_{(1,\,1)} \circ (\rho \otimes \partial).$$
This formula, applied to elements $p_1 \otimes p_2$ of $\mathcal{P} \otimes \mathcal{P}$, where p_1 and p_2 are homogenous elements of $\mathcal{P}$, gives
$$\partial \circ \mu^{\mathcal{P}}(p_1 \otimes p_2) \quad = \quad \mu^{\mathcal{Q}}\big(\partial(p_1) \otimes \rho(p_2)\big) + (-1)^{|\partial||p_1|}\mu^{\mathcal{Q}}\big(\rho(p_1) \otimes \partial(p_2)\big).$$
A *derivation* is a sum of homogenous derivations. The set of homogenous derivations with respect to ρ of degree n is denoted $\mathrm{Der}^n_\rho(\mathcal{P},\, \mathcal{Q})$ and the set of derivations is denoted $\mathrm{Der}^\bullet_\rho(\mathcal{P},\, \mathcal{Q})$ or simply $\mathrm{Der}(\mathcal{P},\, \mathcal{Q})$ when the morphism ρ is obvious.

We recall the definition of the cohomology of $\mathcal{P}$-algebras, when $\mathcal{P}$ is a Koszul operad. Let A be a $\mathcal{P}$-algebra, that is there is a morphism of operads $\mathcal{P} \xrightarrow{\phi} \mathrm{End}(A)$. Denote by $\Omega(\mathcal{P}^{¡}) = (\mathcal{F}(s^{-1}\overline{\mathcal{P}}^{¡}), \bar{\partial})$ the cobar construction of $\mathcal{P}^{¡}$, where the differential $\bar{\partial}$ is the unique derivation which extends the partial coproduct of the Koszul dual cooperad $\mathcal{P}^{¡}$. Since $\mathcal{P}$ is a Koszul operad, $\Omega(\mathcal{P}^{¡})$ is a quasi-free resolution of $\mathcal{P}$.

$$\begin{tikzcd} \Omega(\mathcal{P}^{¡}) \arrow[r, "\sim"] \arrow[dr, "\rho"'] & \mathcal{P} \arrow[d, "\phi"] \\ & \mathrm{End}(A). \end{tikzcd}$$

Lemma 30. *Let $(\mathcal{R},\, \bar{\partial}) \xrightarrow{\varepsilon} \mathcal{P}$ be a resolution of $\mathcal{P}$ and let f be an homogenous derivation of degree n in $\mathrm{Der}^n_\rho\, (\mathcal{R},\, \mathrm{End}(A))$. One has $f \circ \bar{\partial} \in \mathrm{Der}^{n-1}_\rho\, (\mathcal{R},\, \mathrm{End}(A))$.*

Proof. The degree of $f \circ \bar{\partial}$ is $n-1$. It remains to show that $f \circ \bar{\partial}$ is a derivation. Since $(\mathcal{R}, \bar{\partial})$ is a dg-operad, we have
$$\begin{aligned} f \circ \bar{\partial} \circ \mu^{\mathcal{R}} \quad &= \quad f \circ \mu^{\mathcal{R}} \circ (\bar{\partial} \otimes \mathrm{Id} + \mathrm{Id} \otimes \bar{\partial}) \\ &= \quad \mu^{\mathrm{End}(A)} \circ (f \otimes \rho + \rho \otimes f) \circ (\bar{\partial} \otimes \mathrm{Id} + \mathrm{Id} \otimes \bar{\partial}) \\ &= \quad \mu^{\mathrm{End}(A)} \circ \big((f \circ \bar{\partial}) \otimes \rho + \rho \otimes (f \circ \bar{\partial})\big)\big). \end{aligned}$$
Since $\mathcal{R}$ is concentrated in non-negative degree and A is concentrated in degree 0, the composite $\rho \circ \bar{\partial} = \phi \circ \varepsilon \circ \bar{\partial}$ is null. □

The deformation theory of the map $\mathcal{P} \xrightarrow{\phi} \mathrm{End}(A)$ is studied via the following cochain complex defined by M. Markl in [Mar96a]. The cohomology of a $\mathcal{P}$-algebra A is defined on the space of derivations of ρ (see also [Qui70], [Mar96b] and [KS00]).

Definition. The *cohomology of a $\mathcal{P}$-algebra A* is defined by the (deformation) chain complex
$$C^\bullet_{\mathcal{P}}(A) := \big(\mathrm{Der}^\bullet_\rho\, (\Omega(\mathcal{P}^{¡}),\, \mathrm{End}(A))\,, \partial\big),$$
where the differential ∂ is the pullback by $\bar{\partial}$, that is $\partial(f) := f \circ \bar{\partial}$.

Since $\Omega(\mathcal{P}^{¡})$ is a free operad, we have

$\mathrm{Der}^\bullet_\rho\, (\Omega(\mathcal{P}^{¡}),\, \mathrm{End}(A)) \cong \mathrm{Hom}^\bullet_{\mathbb{S}}(\mathcal{F}(s^{-1}\overline{\mathcal{P}}^{¡}), \mathrm{End}(A)) \cong \mathrm{Hom}^{\bullet-1}_{\mathbb{S}}(\overline{\mathcal{P}}^{¡}, \mathrm{End}(A)) \cong \mathrm{Hom}^{\bullet-1}_{k}(\overline{\mathcal{P}}^{¡}(A), A),$

where $\mathrm{Hom}_{\mathbb{S}}(M,N)$ denotes the set of $\mathbb{S}$-equivariant maps between the $\mathbb{S}$-modules M and N.

As in the paper of M. Kontsevich and Y. Soibelman [KS00], we can consider the augmented chain complex $\mathrm{Hom}^\bullet_{\mathbb{S}}(\mathcal{P}^{\text{¡}}, \mathrm{End}(A)) \cong \mathrm{Hom}^\bullet_k(\mathcal{P}^{\text{¡}}(A), A)$. Up to a shift of degree, the last space corresponds to the Hochschild (co)chain complex for associative algebras, Harrison cohomology of commutative algebras and Chevalley-Eilenberg for Lie algebras. Notice that in the literature, this cohomology is called the cohomology of A with coefficient in A. Since this chain complex is defined to control the deformation of the morphism Φ, that is the structure of $\mathcal{P}$-algebra on A, we call it the cohomology $\mathcal{P}$ with coefficient in A or simply the cohomology of A, once the operad is chosen.

In these three cases, the chain complex is a dg-Lie algebra whose bracket is often called the *intrinsic bracket* (see J. Stasheff [Sta93]). The space $\mathrm{Hom}_k(\mathcal{P}^{\text{¡}}, \mathrm{End}(A))$ of morphisms from a dg-cooperad to a dg-operad is an $\mathbb{S}$-module with the action by conjugation, that is $(f.\sigma)(p) := \big(f(p.\sigma^{-1})\big).\sigma$. Moreover, it is a dg-operad, called the *convolution operad* in [BM03] Section 1. On the direct sum of the $\mathbb{S}_n$-modules of an operad, one can define a preLie product $\star$ whose anti-symmetrization gives a Lie bracket. When the operad is the convolution operad $\mathrm{Hom}^\bullet_k(\mathcal{P}^{\text{¡}}, \mathrm{End}(A))$, the preLie product is a degree 0 operation given by

$$f \star g \ := \ \mathcal{P}^{\text{¡}} \xrightarrow{\Delta'} \mathcal{P}^{\text{¡}} \otimes \mathcal{P}^{\text{¡}} \xrightarrow{f\otimes g} \mathrm{End}(A) \otimes \mathrm{End}(A) \xrightarrow{\mu_A} \mathrm{End}(A)$$

where Δ' is the partial coproduct of the cooperad $\mathcal{P}^{\text{¡}}$. The intrinsic Lie bracket is defined by $[f,g] := f\star g - (-1)^{|f||g|} g\star f$. The space of $\mathbb{S}$-equivariant morphisms $\mathrm{Hom}_{\mathbb{S}}(\mathcal{P}^{\text{¡}}, \mathrm{End}(A))$ is equal to the space of invariants $\mathrm{Hom}_k(\mathcal{P}^{\text{¡}}, \mathrm{End}(A))^{\mathbb{S}}$ with respect to the action by conjugation. It is a subspace of the convolution operad $\mathrm{Hom}_k(\mathcal{P}^{\text{¡}}, \mathrm{End}(A))$ stable under the preLie product $\star$. (See for instance [VdL] for a proof of this in the coinvariant context. Since we work over a field k of characteristic 0, the isomorphism between invariants and coinvariants allows us to conclude.) The induced Lie bracket on $C^\bullet_{\mathcal{P}}(A) = \mathrm{Hom}_{\mathbb{S}}(\mathcal{P}^{\text{¡}}, \mathrm{End}(A))$ defines an intrinsic Lie bracket on cohomology.

When $\mathcal{P} = \mathcal{A}s$, it is exactly the structure defined by M. Gerstenhaber in [Ger63] and when $\mathcal{P} = \mathcal{L}ie$ it is the Lie bracket of Nijenhuis and Richardson, which controls the formal deformations of $\mathcal{P}$-algebra structure (see D. Balavoine [Bal97] Section 4).

Let A be a $\mathcal{P}$-algebra and C be a $\mathcal{P}^{\text{¡}}$-coalgebra, we have by Proposition 29 that $\mathrm{Hom}_k(C, A)$ is naturally endowed with a structure of $\mathcal{P}^! \circ \mathcal{P}$-algebra and $\mathcal{L}ie^{<k>}$-algebra (see also Section 3.2). Applied to $C = \mathcal{P}^{\text{¡}}(A)$, this result gives that the chain complex $C^\bullet_{\mathcal{P}}(A)$ is a $\mathcal{P}\circ\mathcal{P}^!$-algebra and a $\mathcal{L}ie^{<k>}$-algebra. In the binary case, it means that $\mathrm{Hom}^\bullet_k(\mathcal{P}^{\text{¡}}(A), A)$ is equipped with another Lie bracket $\{\,,\}$ of degree -1. Let α be a morphism of degree -1 defined as follows

$$\alpha \ : \ \mathcal{P}^{\text{¡}} \twoheadrightarrow \mathcal{P}^{\text{¡}}_{(1)} = \mathcal{P}^{\text{¡}}(2) = s\mathcal{P}(2) \to \mathcal{P}(2) \rightarrowtail \mathcal{P} \xrightarrow{\Phi} \mathrm{End}(A).$$

It is a *twisting cochain*, that is α is solution to the Maurer-Cartan equation $\alpha\star\alpha = 0$, when $\mathcal{P}$ and A are concentrated in degree 0 (see Section 2.3 of Getzler-Jones [GJ94]). The Lie bracket $\{f,g\}$ is equal to

$$\mathcal{P}^{\text{¡}} \xrightarrow{\Delta} \mathcal{P}^{\text{¡}} \circ \mathcal{P}^{\text{¡}} \twoheadrightarrow \mathcal{P}^{\text{¡}}(2) \otimes_{\mathbb{S}_2} \mathcal{P}^{\text{¡}\otimes 2} \xrightarrow{\alpha\otimes(f\otimes g + (-1)^{|f||g|} g\otimes f)} \mathrm{End}(A)\otimes\mathrm{End}(A) \xrightarrow{\mu_A} \mathrm{End}(A).$$

Note that in the binary case, the latter Lie bracket $\{\,,\}$ is not equal to the intrinsic Lie bracket $[\,,]$. For instance, there is a shift of degree between the two.

Lemma 31. *For every f and g in $C^\bullet_{\mathcal{P}}(A)$, we have*

$$\partial(f) = [f,\alpha] \quad \text{and} \quad \{f,g\} = \partial f\star g + (-1)^{|f|} f\star\partial g - \partial(f\star g).$$

PROOF. The proof is straightforward and left to the reader. □

Equipped with the intrinsic Lie bracket $[\,,]$, $C^\bullet_{\mathcal{P}}(A)$ becomes a dg-Lie algebra. The second formula shows that the Lie bracket $\{\,,\}$ vanishes on cohomology. This result and formula can be explained as follows. The preLie product comes from the partial composition of the operad. The general composition product of an operad defines symmetric braces. Since the partial composition of the operad generates the global one, the preLie product generates the symmetric braces. (See also J.-M. Oudom and D. Guin [OG04] and [LM05] for a proof of this result). Therefore, we cannot expect to have other products than the intrinsic Lie bracket in general. In particular examples, it would be interesting to see if the structure of $\mathcal{P} \circ \mathcal{P}^!$-algebra induces a non-trivial structure on cohomology. We will see in 5.3 how to refine this study when the operad is not symmetric (regular).

5. Black and white square-products for regular operads

K. Ebrahimi-Fard and L. Guo in [EFG05] and J.-L. Loday in [Lod04] defined and used an analog of Manin's black product for regular operads that they called the *black square product*. In this section, we give the conceptual definitions of Manin's black and white square products for regular operads. They are not equal to the black and white "circle"-products in the category of operads. Actually, they come from the black and white products in the category of non-symmetric operads.

5.1. **Definitions of non-symmetric and regular operads.** Recall that a *non-symmetric operad* is an operad without the actions of the symmetric groups. From a non-symmetric operad $\{\mathcal{P}'_n\}_{n\in\mathbb{N}^*}$, we can associate an $\mathbb{S}$-module by the collection of the free $\mathbb{S}_n$-modules on $\mathcal{P}(n) := \mathcal{P}'_n \otimes_k k[\mathbb{S}_n]$. The composition product for the operad $\mathcal{P}$ is defined from the non-symmetric one. Such an operad is called a *regular operad*. Denote Σ this functor from non-symmetric operads to operads. Therefore, the category of regular operads is the image of Σ and is equivalent to the category of non-symmetric operads. Denote by U the inverse functor :

$$\text{Non-symmetric Operads} \underset{\Sigma}{\overset{U}{\rightleftarrows}} \text{Regular Operads.}$$

Let $\mathcal{P} = \mathcal{F}(V)/(R)$ be a binary quadratic regular operad. In that case, we have that V and R are regular modules, that is $V = V' \otimes_k k[\mathbb{S}_2]$ and $R = R' \otimes_k k[\mathbb{S}_3]$. The non-symmetric operad $\mathcal{P}' = U(\mathcal{P})$ is once again binary and quadratic. It is given by $\mathcal{P}' = F(V')/(R')$.

5.2. **Definitions of black and white square-products.** A non-symmetric operad is a monoid in the category of non-negative graded modules with a non-symmetric version of $\circ$ (see Appendix A). Under the Hadamard product, this category forms a 2-monoidal category. Hence, we can apply arguments of section 3 and consider the morphism Φ and the induced white product for non-symmetric operads. From two binary quadratic regular operads $\mathcal{P} = \mathcal{F}(V)/(R)$ and $\mathcal{Q} = \mathcal{F}(W)/(S)$, we study the associated white product

$$U(\mathcal{P}) \circ U(\mathcal{Q}) := \mathcal{F}(V' \otimes W')/\big(\Phi^{-1}(R' \otimes \mathcal{F}(W') + \mathcal{F}(V') \otimes S')\big).$$

The idea is now to come back to the category of regular operads using the functor Σ.

Definition (White square-product)**.** The *white square-product* of two binary quadratic regular operads $\mathcal{P}$ and $\mathcal{Q}$ is defined by the following formula

$$\mathcal{P} \,\square\, \mathcal{Q} := \Sigma(U(\mathcal{P}) \circ U(\mathcal{Q})).$$

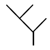
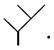
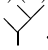
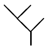
More explicitly, the white square-product of $\mathcal{P}$ and $\mathcal{Q}$ is equal to $\mathcal{P} \,\square\, \mathcal{Q} = \mathcal{F}(V' \otimes W' \otimes k[\mathbb{S}_2])/\big((\Phi^{-1}(R' \otimes \mathcal{F}(W') + \mathcal{F}(V') \otimes S')) \otimes k[\mathbb{S}_3]\big)$.
Note that the definition given above does not correspond to Definition 3.1 of K. Ebrahimi-Fard and L. Guo in [EFG05] (See Remark below.)

Proposition 32. *Let A be a $\mathcal{P}$-algebra and B a $\mathcal{Q}$-algebra, their tensor product $A \otimes B$ is an algebra over the white square-product $\mathcal{P} \,\square\, \mathcal{Q}$.*

Proof. The proof is the same than Proposition 11. ☐

Let V be an $\mathbb{S}_2$-module. The part $\mathcal{F}(V)(3)$ with 3 inputs of the free operad on V is isomorphic to

$$\mathcal{F}(V)(3) = \big(V \otimes_{\mathbb{S}_2} (V \otimes k \oplus k \otimes V)\big) \otimes_{\mathbb{S}_2} k[\mathbb{S}_3],$$

where the summand $V \otimes (V \otimes k)$ corresponds to the compositions on the left  and the summand $V \otimes (k \otimes V)$ corresponds to the compositions on the right . When V is a sum of regular representations $V = V' \otimes k[\mathbb{S}_2]$, we have $\mathcal{F}(V)(3) = \big(V' \otimes (V' \otimes k) \oplus V' \otimes (k \otimes V')\big) \otimes k[\mathbb{S}_3]$. Therefore, $\mathcal{F}(V)(3)$ can be identify with 2 copies of $V' \otimes V'$ represented by the following types of tree  and . These two copies correspond to the part of arity 3 of the free non-symmetric operad on V'. We denote the first composition based on the pattern  by $\mu \circ_1 \nu$ and the second one based on  by $\mu \circ_2 \nu$, where μ is below ν.

In the appendix B of [Lod01], J.-L. Loday described the non-degenerate bilinear form $<\,,\,>$ for regular operads. It comes from the following one for non-symmetric operads.

$$\begin{gathered}\mathcal{F}(V')(3) \otimes \mathcal{F}(V'^*)(3) \xrightarrow{<\,,\,>} k \\ < \mu \circ_1 \nu,\, \zeta \circ_1 \xi >:= +\zeta(\mu).\xi(\nu) \\ < \mu \circ_2 \nu,\, \zeta \circ_2 \xi >:= -\zeta(\mu).\xi(\nu),\end{gathered}$$

the other products being null.

We define the black product of binary non-symmetric operad like in 4.3 (the non-degenerate bilinear form is given below). Applying the same ideas, we have the analog of Lemma 18 and Proposition 19.

Lemma 33. *Let $\mathcal{P} = \mathcal{F}(V' \otimes k[\mathbb{S}_2])/(R' \otimes k[\mathbb{S}_3])$ and $\mathcal{Q} = \mathcal{F}(W' \otimes k[\mathbb{S}_2])/(S' \otimes k[\mathbb{S}_3])$ be two regular operads such that the V' and W' are finite dimensional. The orthogonal of $\Psi(R' \otimes S')$ for $<\,,\,>$ is $\big(\Phi_*^{-1}(R'^{\perp} \otimes \mathcal{F}(W'^*) + \mathcal{F}(V'^*) \otimes S'^{\perp})\big)$.*

Definition (Black square product)**.** Let $\mathcal{P}' = \mathcal{F}(V')/(R')$ and $\mathcal{Q}' = \mathcal{F}(W')/(S')$ be two binary quadratic non-symmetric operads with finite dimensional generating spaces. Define their black product by the formula

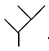
$$\mathcal{P}' \bullet \mathcal{Q}' \quad = \quad \mathcal{F}(V' \otimes W')/\big(\Psi(R' \otimes S')\big).$$

The *black square product* of two binary quadratic regular operads is defined by

$$\mathcal{P} \blacksquare \mathcal{Q} := \Sigma(U(\mathcal{P}) \bullet U(\mathcal{Q})).$$

Proposition 34. *For binary quadratic regular operads generated by finite dimensional modules, this definition of black product verifies $\big(\mathcal{P} \blacksquare \mathcal{Q}\big)^! = \mathcal{P}^! \,\square\, \mathcal{Q}^!$.*

Finally, we can use the particular form of the bilinear product $<\,,\,>$ to make explicit the morphism Ψ and show that the black square-product defined here corresponds to the one of [Lod04] and [EFG05].

Proposition 35. *Under the same hypotheses, let $r \otimes s$ be an elementary tensor of $R' \otimes S'$. Denote $r = r_1 + r_2$, where r_1 is the part of r corresponding to the compositions of the form*  *and r_2 is the part of r corresponding to the compositions of the form* *. In the same way, write $s = s_1 + s_2$. The image of $r \otimes s$ under Ψ is $\Psi(r \otimes s) = \Phi^{-1}(r_1 \otimes s_1) - \Phi^{-1}(r_2 \otimes s_2)$.*

Proof. Note that $r_1 \otimes s_1$ and $r_2 \otimes s_2$ belong to $\operatorname{Im}\Phi$. For $X \in \mathcal{F}(V'^* \otimes W'^*)(3)$, denote the image of X under Φ_* by $\Phi_*(X) = \sum \Phi_{V'^*}(X) \otimes \Phi_{W'^*}(X)$. More precisely, we decompose the image of X under Φ_* with the two types of compositions $\Phi_*(X) = \Phi_*(X_1 + X_2) = \sum_1 \Phi_{V'^*}(X_1) \otimes$

$\Phi_{W'^*}(X_1) + \sum_2 \Phi_{V'^*}(X_2) \otimes \Phi_{W'^*}(X_2)$. We have

$$\begin{aligned}
& < \Phi^{-1}(r_1 \otimes s_1) - \Phi^{-1}(r_2 \otimes s_2),\, \mathcal{F}(\Xi)(X) >_{V' \otimes W'} \\
= & < \Phi^{-1}(r_1 \otimes s_1),\, \mathcal{F}(\Xi)(X)_1 >_{V' \otimes W'} - < \Phi^{-1}(r_2 \otimes s_2),\, \mathcal{F}(\Xi)(X)_2 >_{V' \otimes W'} \\
= & \sum_1 < r_1,\, \Phi_{V'^*}(X_1) >_{V'} . < s_1,\, \Phi_{W'^*}(X_1) >_{W'} + \sum_2 < r_2,\, \Phi_{V'^*}(X_2) >_{V'} . < s_2,\, \Phi_{W'^*}(X_2) >_{W'} = \\
= & \sum_1 < r,\, \Phi_{V'^*}(X_1) >_{V'} . < s,\, \Phi_{W'^*}(X_1) >_{W'} + \sum_2 < r,\, \Phi_{V'^*}(X_2) >_{V'} . < s,\, \Phi_{W'^*}(X_2) >_{W'} = \\
= & \sum < r,\, \Phi_{V'^*}(X) >_{V'} . < s,\, \Phi_{W'^*}(X) >_{W'} \\
= & \Big(\sum < r,\, - >_{V'} . < s,\, - >_{W'} \Big) \circ \Phi_*(X).
\end{aligned}$$

$\square$

Corollary 36. *The black-square product defined here is equal to the one defined in* [EFG05] *and in* [Lod04].

Remark. The white square-product is equal to $\mathcal{P} \square \mathcal{Q} := \mathcal{F}(V' \otimes W' \otimes k[\mathbb{S}_2]) / \big((\Phi^{-1}(R' \otimes \mathcal{F}(W') + \mathcal{F}(V') \otimes S')) \otimes k[\mathbb{S}_3] \big)$ and the black square-product to $\mathcal{P} \blacksquare \mathcal{Q} := \mathcal{F}(V' \otimes W' \otimes k[\mathbb{S}_2]) / \big((\Psi(R' \otimes S')) \otimes k[\mathbb{S}_3] \big)$. The definition of $\square$ proposed by K. Ebrahimi-Fard and L. Guo in [EFG05] corresponds to $\Psi(R' \otimes \mathcal{F}(W') + \mathcal{F}(V') \otimes S')$ instead of $\Phi^{-1}(R' \otimes \mathcal{F}(W') + \mathcal{F}(V') \otimes S')$. We have $\Phi^{-1}(R' \otimes \mathcal{F}(W') + \mathcal{F}(V') \otimes S') \subset \Psi(R' \otimes \mathcal{F}(W') + \mathcal{F}(V') \otimes S')$. But the second module can be slightly bigger than the first one (see the example of $\mathcal{D}ias \square \mathcal{D}ias$ on page 309 of [EFG05]). This explains why the white square-product defined in [EFG05] is not the Koszul dual of the black square-product.

With the explicit form of the black square-product, we get the following property which is Proposition 2.4 of [Lod04].

Proposition 37 ([Lod04])**.** *For two binary quadratic regular operads $\mathcal{P}$ and $\mathcal{Q}$, there exists a canonical epimorphism $\mathcal{P} \blacksquare \mathcal{Q} \twoheadrightarrow \mathcal{P} \square \mathcal{Q}$.*

Proof. We have to show that $\Phi \circ \Psi\,(R' \otimes S') \subset R' \otimes \mathcal{F}(W')(3) + \mathcal{F}(V')(3) \otimes S'$. Let $r \otimes s$ be an elemental tensor of $R' \otimes S'$. Denote $r \otimes s = (r_1 + r_2) \otimes (s_1 + s_2)$. From Proposition 35, we get $\Phi \circ \Psi(r \otimes s) = r_1 \otimes s_1 - r_2 \otimes s_2 = (r_1 + r_2) \otimes s_1 - r_2 \otimes (s_1 + s_2) \in R' \otimes \mathcal{F}(W')(3) + \mathcal{F}(V')(3) \otimes S'$. $\square$

The proposition means that any $\mathcal{P} \square \mathcal{Q}$-algebra is a $\mathcal{P} \blacksquare \mathcal{Q}$-algebra. This result together with Proposition 32, gives the following corollary (Proposition 3.3 of [EFG05]).

Corollary 38 ([EFG05])**.** *For any $\mathcal{P}$-algebra A and $\mathcal{Q}$-algebra B, their tensor $A \otimes B$ is a $\mathcal{P} \blacksquare \mathcal{Q}$-algebra.*

Remark. The operads χ^+ and χ^- discovered by J.-L. Loday in [Lod96] factors this projection

$$\begin{tikzcd}
\mathcal{D}end \blacksquare \mathcal{D}ias \arrow[r, two heads] \arrow[rr, leftrightarrow, bend right, "!"'] & \chi^{\pm} \arrow[loop above, "!"] \arrow[r, two heads] & \mathcal{D}end \square \mathcal{D}ias.
\end{tikzcd}$$

Going from the left to the right, there is one more relation each time. (The dimensions of the spaces of relations is 15, 16 and 17 respectively).

5.3. **Adjunction.** We can apply the same methods as in Section 4.6 to prove the same kind of adjunction for black and white square products for regular operads. Consider the category of *k-ary quadratic regular operads* denoted by *$k.q$-Reg.* One can extend black and white square products in this category. Recall from [Gne97] that a *totally associative k-ary algebra* is a module equipped with a regular k-ary operation such that all the quadratic compositions are equal. Denote the corresponding operad by $T\mathcal{A}s^{<k>}$. Dually, a *partially associative k-ary algebra* is a module equipped with a regular k-ary operation such that the sum of all quadratic compositions is zero.

Denote the corresponding operad by $P\mathcal{A}s^{<k>}$. Gnedbaye proved that these two operads are Koszul dual to each other.

Proposition 39. *The black and white square products endow the category of k-ary quadratic regular operads with a structure of symmetric monoidal category, where the operad $P\mathcal{A}s^{<k>}$ is the unit object for $\blacksquare$ and the operad $T\mathcal{A}s^{<k>}$ is the unit object for $\square$.*

Theorem 40. *There is a natural isomorphism* $\mathrm{Hom}_{k.q.Reg}(\mathcal{P} \blacksquare \mathcal{Q},\, \mathcal{R}) \cong \mathrm{Hom}_{k.q.Reg}(\mathcal{P},\, \mathcal{Q}^! \square \mathcal{R})$.

Proposition 41. *For every k-ary quadratic regular operad $\mathcal{P}$, there is a canonical morphism of operads $P\mathcal{A}s^{<k>} \xrightarrow{i} \mathcal{P}^! \square \mathcal{P}$, defined by the commutative diagram*

$$\begin{tikzcd} P\mathcal{A}s^{<k>} \arrow[rr, "l"] \arrow[dr, "i"'] & & \mathcal{P}^! \otimes \mathcal{P} \\ & \mathcal{P}^! \square \mathcal{P}. \arrow[ur, "\bar{\Phi}"'] & \end{tikzcd}$$

This proposition can be seen as a refinement of Proposition 29. When $\mathcal{P}$ is a k-ary regular quadratic operad, the map $\mathcal{L}ie^{<k>} \to \mathcal{P}^! \otimes \mathcal{P}$ factors through $P\mathcal{A}s^{<k>}$, where the morphism $\mathcal{L}ie^{<k>} \to P\mathcal{A}s^{<k>}$ is induced by the anti-symmetrization of the k-ary partially associative product as in the binary case.

5.4. **Non-symmetric cohomology operations.** In this section, we refine the arguments of Section 4.7 for non-symmetric (regular) operads. This gives non-vanishing natural operations on the deformation chain complex of any algebras over such operads. More precisely, we prove that, under some assumptions, the (co)chain complex defining the cohomology of algebras is a multiplicative operad.

Recall from [GV95] that an *operad with multiplication* is a non-symmetric operad $\mathcal{P}$ endowed with a morphism $\mathcal{A}s \to \mathcal{P}$. Let $\mathcal{P}$ be a finitely generated binary non-symmetric Koszul operad. Following Section 4.7, the chain complex defining the cohomology of a $\mathcal{P}$-algebra A is equal to $C^\bullet_{\mathcal{P}}(A) = \mathrm{Hom}^\bullet_k(\mathcal{P}^{¡}, \mathrm{End}(A))$ which is a non-symmetric (convolution) operad. By Proposition 41, there is a morphism of operads $\mathcal{A}s \to \mathcal{P}^! \otimes \mathcal{P}$. Since $\mathcal{P}^{¡} = \mathcal{P}^{!*}$, we have

$$\mathcal{A}s \to \mathcal{P}^! \otimes \mathcal{P} \cong \mathrm{Hom}_k(\mathcal{P}^{¡}, \mathcal{P}) \xrightarrow{\Phi_*} \mathrm{Hom}_k(\mathcal{P}^{¡}, \mathrm{End}(A)).$$

These results form the following proposition.

Proposition 42. *For every finitely generated binary non-symmetric Koszul operad $\mathcal{P}$ and every $\mathcal{P}$-algebra A, the chain complex defining its cohomology $C^\bullet_{\mathcal{P}}(A)$ is an operad with multiplication.*

The multiplication $\mathcal{A}s \to \mathcal{P}$ of an operad $\mathcal{P}$ allows us to define a canonical cosimplicial structure on it (see [MS02] Section 3) and then a differential map d by alternate summation (see [GV95] Formula (5)). Denote by m the image of the associative operation. The face maps $d^i : \mathcal{P}(n) \to \mathcal{P}(n+1)$ are defined by

$$d^i(p) := \begin{cases} m \circ_2 p & \text{if} \quad i = 0 \\ p \circ_i m & \text{if} \quad 0 < i < n+1 \\ m \circ_1 p & \text{if} \quad i = n+1. \end{cases}$$

The differential d is equal to $d(f) := m \star f - (-1)^{|f|} f \star m = [m, f]$.

Lemma 43. *With the same assumptions, the differential $\partial(f)$ on $C^\bullet_{\mathcal{P}}(A)$ is equal to $(-1)^{|f|} d(f)$. Hence, the chain complex $C^\bullet_{\mathcal{P}}(A)$ is always cosimplicial.*

Proof. The image of the associative operation in $\mathrm{Hom}_k(\mathcal{P}^{¡}, \mathrm{End}(A))$ is the map $\alpha : \mathcal{P}^{¡}(2) \to \mathrm{Hom}(A^{\otimes 2}, A)$ defined in Section 4.7. We prove in Lemma 31 that the differential on $C^\bullet_{\mathcal{P}}(A)$ is equal to $\partial(f) = [f, \alpha] = (-1)^{|f|} d(f)$. $\square$

Therefore, the chain complex $C^\bullet_{\mathcal{P}}(A)$ is endowed with two types of operations : *braces* operations induced by the non-symmetric operadic structure and an associative operation called the *cup product* coming from the properties of Manin's products. In [GV95], M. Gerstenhaber and A.A. Voronov defined the notion of *homotopy G-algebra* which gives the compatibility between these types of operations. Their purpose was to describe the operations acting of the chain complex of Hochschild cohomology of an associative algebra. Actually, the structure of homotopy G-algebra on the deformation chain complex and the structure of Gerstenhaber algebra on cohomology is universal among finitely generated binary non-symmetric Koszul operads.

Corollary 44. *For every finitely generated binary non-symmetric Koszul operad $\mathcal{P}$ and every $\mathcal{P}$-algebra A, the chain complex $C^\bullet_{\mathcal{P}}(A)$ is a homotopy G-algebra and the cohomology space $H^\bullet_{\mathcal{P}}(A)$ is a Gerstenhaber algebra.*

Proof. Apply Theorem 3 of [GV95] which asserts that any multiplicative operad induces a homotopy G-algebra on the direct sum of its components. To prove the second part, apply the computations of the proof of Corollary 5 of [GV95]. □

5.5. **Generalized Deligne's conjecture.** Finally, we extend and prove Deligne's conjecture to any algebra over a finitely generated binary non-symmetric Koszul operad, which includes the original case of associative algebras.

Here is a brief review on cohomology operations and Deligne's conjecture. In 1945, G. Hochschild defined in [Hoc45] the chain complex $C^\bullet_{\mathcal{P}}(A)$ for associative algebras and proved the existence of the cup product. In 1963, M. Gerstenhaber introduced in [Ger63] the preLie product $\circ$ and the induced Lie bracket on the chain complex. He gave some formulas between them and the cup product that allowed him to prove that this induces a graded Poisson structure, now called Gerstenhaber algebra, on cohomology. Note that the definition of the preLie product is based upon a preLie system $\{\circ_i\}_i$ whose axioms correspond the the definition of the partial product of an operad. In 1964, Gerstenhaber used in [Ger64] this Lie bracket to study the deformations of associative algebras. (This will directly lead in 1997 to the theorem of deformation-quantization of Poisson manifold by M. Kontsevich [Kon97]). In 1985, T. Kadeishvili introduced multi-linear operations called *braces* on $C^\bullet_{\mathcal{P}}(A)$ in [Kad88] (see also E. Getzler [Get93]) which correspond to the whole operadic structure. In 1995, Gerstenhaber and Voronov described the relations between the braces operations and the cup product in [GV95].

The little disk operad $\mathcal{D}_2$ is a topological operad defined by configurations of disks on the plane. In 1976, F. Cohen showed that the homology operad $H_\bullet(\mathcal{D}_2)$ is equal to the operad coding Gerstenhaber algebras [Coh76]. This led P. Deligne to make the following wish "I would like the complex computing Hochschild cohomology to be an algebra over [the singular chain operad of the little disks] or a suitable version of it" in [Del93]. By suitable version of it, he meant another operad homotopically equivalent to $\mathcal{D}_2$. This conjecture can be seen as a lifting on the level of chain complexes of the result of F. Cohen. In 1999, J.E. McClure and J.H. Smith gave a prove of this conjecture in the following way. First, they construct a topological operad $\mathcal{C}$ whose chain version acts on any multiplicative operad. Then, they show that this operad is equivalent to the little disks operad. This proof with Proposition 42 shows that Deligne conjecture can be generalized to any finitely generated binary non-symmetric Koszul operads and is not specific to the case of associative algebras.

Theorem 45. *For every finitely generated binary non-symmetric Koszul operad $\mathcal{P}$ and every $\mathcal{P}$-algebra A, the chain complex $C^\bullet_{\mathcal{P}}(A)$ is an algebra over an operad equivalent to the singular chains of the little disks operad.*

Proof. Since $C^\bullet_{\mathcal{P}}(A)$ is an operad with multiplication, the operad $\mathcal{C}$ of [MS02] acts on it. And this operad is weakly equivalent to the little disks operad by Theorem 3.3 of [MS02]. □

Notice that the non-symmetric case is very different from the symmetric one. The Lie bracket $\{\,,\}$ described in Section 4.7 vanishes on cohomology. When the algebra is modelled by a non-symmetric operad, this Lie bracket is the symmetrization of an associative operation, the cup-product, which is not necessarily trivial on cohomology.

In [Mar05], M. Markl defined the notion of *natural operations* on cohomology and asked a few questions and conjectures about the operad $\mathcal{B}_{\mathcal{P}}$ generated by these operations. Here we have proved that, for binary non-symmetric Koszul operads, the Gerstenhaber operad imbeds into $\mathcal{B}_{\mathcal{P}}$. For more precise statements depending on the operad $\mathcal{P}$, one has to work with $\mathcal{P}^{!} \circ \mathcal{P}$. In the symmetric case, operations of $\mathcal{P}^{!} \circ \mathcal{P}$ could give non-trivial operations in cohomology.

5.6. **The operad $\mathcal{Q}uad$ and its Koszul dual.** In this section, we study the example of black square-product $\mathcal{Q}uad = \mathcal{D}end \blacksquare \mathcal{D}end$ introduced by M. Aguiar and J.-L. Loday in [AL04]. We prove that the Koszul dual of $\mathcal{Q}uad$ is the operad $\mathcal{Q}uad^{!} = \mathcal{P}erm \otimes \mathcal{D}ias = \mathcal{P}erm \circ \mathcal{D}ias$.

The operad $\mathcal{D}end$ is a split of one associative product $\star$ into two products $\prec$ and $\succ$, $\star = \prec + \succ$. The operad $\mathcal{Q}uad$ was defined by M. Aguiar and J.-L. Loday in [AL04] as a split of an associative product $\star$ into four products $\nearrow$, $\searrow$, $\swarrow$ and $\nwarrow$, that is $\star = \nearrow + \searrow + \swarrow + \nwarrow$. It was proved in [EFG05] that this operad $\mathcal{Q}uad$ is equal to the black square-product $\mathcal{D}end \blacksquare \mathcal{D}end$. Therefore one can interpret the splitting of associativity with the black square-product with $\mathcal{D}end$. At the end of their paper, M. Aguiar and J.-L. Loday raised one question "what is the Koszul dual of the operad $\mathcal{Q}uad$?" and two conjectures. The first conjecture deals with the dimensions of the $\mathbb{S}_n$-modules $\mathcal{Q}uad(n)$ and the second one is that the operad $\mathcal{Q}uad$ is Koszul. In the rest of this section, we answer these questions.

The previous section give a direct answer to the first question.

Proposition 46. *The Koszul dual of $\mathcal{Q}uad$ is equal to $\mathcal{Q}uad^{!} = \mathcal{D}ias \,\square\, \mathcal{D}ias$.*

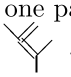
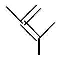
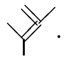
Proof. Since the Koszul dual of $\mathcal{D}end$ is the operad $\mathcal{D}end^{!} = \mathcal{D}ias$ ([Lod01] Proposition 8.3), we have

$$\mathcal{Q}uad^{!} = (\mathcal{D}end \blacksquare \mathcal{D}end)^{!} = \mathcal{D}end^{!} \,\square\, \mathcal{D}end^{!} = \mathcal{D}ias \,\square\, \mathcal{D}ias,$$

from Proposition 34. $\square$

It remains to use the explicit form of the white square-product to describe $\mathcal{D}ias \,\square\, \mathcal{D}ias$.

Theorem 47. *The operad $\mathcal{Q}uad^{!} = \mathcal{D}ias \,\square\, \mathcal{D}ias$ is isomorphic to $\mathcal{P}erm \otimes \mathcal{D}ias = \mathcal{P}erm \otimes \mathcal{P}erm \otimes \mathcal{A}s$.*

Proof. Denote the basis of $\mathcal{D}end'(2)$ by $\prec .k \oplus \succ .k$ and its dual basis, the basis of $\mathcal{D}ias'(2)$, by $\dashv .k \oplus \vdash .k$. The induced basis of $\mathcal{Q}uad'(2) = (\mathcal{D}end \blacksquare \mathcal{D}end)'(2)$ is $\{\prec \otimes \prec, \prec \otimes \succ, \succ \otimes \prec, \succ \otimes \succ\}$ and the induced basis of $\mathcal{Q}uad^{!\prime}(2) = (\mathcal{D}ias \,\square\, \mathcal{D}ias)'(2)$ is $\{\dashv \otimes \dashv, \dashv \otimes \vdash, \vdash \otimes \dashv, \vdash \otimes \vdash\}$. The relations of $\mathcal{D}ias$ are easy to remember. Represent the operation $\dashv$ by the tree  and the operation $\vdash$ by the tree . Any element of $\mathcal{F}(\dashv .k \oplus \vdash .k)(3)$ can be seen as a tree with exactly one path from one leaf to the root. For example, the composition $\dashv \circ_1 \vdash$ corresponds to the tree  . To get the relations of $\mathcal{D}ias$, identify the trees with paths from the same leaf. For instance, we have $\dashv \circ_1 \vdash = \vdash \circ_2 \dashv$, which corresponds to  = . The relations of $\mathcal{D}ias$ are

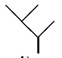
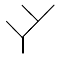
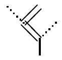
$$\left\{\begin{array}{ll} \dashv \circ_1 \dashv = \dashv \circ_2 \dashv = \dashv \circ_2 \vdash & (L) \\ \dashv \circ_1 \vdash = \vdash \circ_2 \dashv & (M) \\ \vdash \circ_1 \dashv = \vdash \circ_1 \vdash = \vdash \circ_2 \vdash & (R), \end{array}\right.$$

where the first line corresponds to the Left leaf, the second one to the Middle leaf and the last one to the Right leaf. For simplicity, denote these compositions and relations by

$$\left\{\begin{array}{l} L_1 = L_2' = L_2'' \\ M_1 = M_2 \\ R_1' = R_1'' = R_2. \end{array}\right.$$

The operad $\mathcal{D}ias$ is equal to $\mathcal{F}(V' \otimes k[\mathbb{S}_2])/(R' \otimes k[\mathbb{S}_3])$. One can see that the following relations are elements of $(R \otimes \mathcal{F}(V') + \mathcal{F}(V') \otimes R) \cap \operatorname{Im} \Phi$

$$\begin{cases} (L,L) & L_1 \otimes L_1 = L_2' \otimes L_2' = L_2'' \otimes L_2' = L_2' \otimes L_2'' = L_2'' \otimes L_2'' \\ (L,M) & L_1 \otimes M_1 = L_2' \otimes M_2 = L_2'' \otimes M_2 \\ (L,R) & L_1 \otimes R_1' = L_1 \otimes R_1'' = L_2' \otimes R_2 = L_2'' \otimes R_2 \\ (M,M) & M_1 \otimes M_1 = M_2 \otimes M_2 \end{cases}$$

The other ones are obtained by the symmetries  $\leftrightarrow$  and $a \otimes b \leftrightarrow b \otimes a$. We get 23 linearly independent elements in $(R\otimes\mathcal{F}(V')+\mathcal{F}(V')\otimes R)\cap \operatorname{Im}\Phi$. Since the dimension of $\mathcal{Q}uad'(3)$ is $23 = 32 - 9$, we know that these elements form a basis of $(R \otimes \mathcal{F}(V') + \mathcal{F}(V') \otimes R) \cap \operatorname{Im}\Phi$. Hence, they give the relations defining $\mathcal{Q}uad^!$.
Interpret these relations in the same way as the ones of $\mathcal{D}ias$. An element of $\mathcal{F}\big((\dashv .k \oplus \vdash .k) \otimes (\dashv .k \oplus \vdash .k)\big)(3)$ can be seen as a tree with two kind of paths, one given by the left side of $\otimes$ and the second one by the right side of $\otimes$. For instance, the tree  represents $(\dashv \otimes \vdash) \circ_1 (\vdash \otimes \dashv)$, where the left side corresponds to $=$ and the right side to $\cdots$. This produces two indexes for the leaves. With this identification, the relations of $\mathcal{Q}uad^!$ mean that any elements written with trees such that the same leaves are indexed by the same "colors" are equal. Therefore, a basis for $\mathcal{Q}uad^!(n)$ is given by planar corollas with n leaves indexed by two colors. The composition of such trees is a corollas and to know which leaf is indexed by which color, follow the path of the same color. As a consequence, we have $\mathcal{D}ias \,\square\, \mathcal{D}ias = \mathcal{P}erm \otimes \mathcal{D}ias$. (A basis for the operad $\mathcal{D}ias$ is given by corollas with one leaf emphasized. Tensoring with $\mathcal{P}erm$ induces another independent index of the leaves. And the compositions are the same.) $\square$

Remark. More generally, we get the duals of the operad coding octo-algebras of P. Leroux [Le03] and its follow-up. Since $\mathcal{O}cto = \mathcal{D}end^{\blacksquare 3}$, we get $\mathcal{O}cto^! = \mathcal{P}erm^{\otimes 3} \otimes \mathcal{A}s$ and, for any $n \in \mathbb{N}$, $\big(\mathcal{D}end^{\blacksquare n}\big)^! = \mathcal{D}ias^{\square n} = \mathcal{P}erm^{\otimes n} \otimes \mathcal{A}s$.

Corollary 48. *The dimensions of the components of the Koszul dual of $\mathcal{Q}uad$ are equal to*

$$dim\,(\mathcal{Q}uad^!(n)) = n^2.n!.$$

Proof. We have $\dim(\mathcal{Q}uad^!(n)) = \dim(\mathcal{P}erm(n) \otimes_k \mathcal{D}ias(n)) = n^2.n!$. $\square$

Proposition 17 gives that $\mathcal{D}ias \,\square\, \mathcal{D}ias = \mathcal{P}erm \otimes \mathcal{D}ias = \mathcal{P}erm \circ \mathcal{D}ias = \mathcal{P}erm \otimes \mathcal{P}erm \otimes \mathcal{A}s = \mathcal{P}erm \circ \mathcal{P}erm \circ \mathcal{A}s$. We have $\mathcal{D}ias = \Sigma(\mathcal{P}erm)$ and $\mathcal{D}ias \,\square\, \mathcal{D}ias = \Sigma\big(\mathcal{P}erm \circ \mathcal{P}erm\big) = \Sigma\big(\mathcal{P}erm^{\otimes 2}\big)$. By duality, we get another form for $\mathcal{Q}uad$.

Corollary 49. *We have $\mathcal{Q}uad = \mathcal{P}erm \bullet \mathcal{P}erm \bullet \mathcal{A}s$*

5.7. **Koszulity of $\mathcal{Q}uad$ and other operads defined by square products.** Aguiar and Loday made in [AL04] the conjecture that the operad $\mathcal{Q}uad$ is Koszul. We show this statement using poset's method of [Val2]. More generally, we prove that the operads of the form $\mathcal{D}end^{\blacksquare n}$ and $\mathcal{D}ias^{\square n}$ are Koszul. P. Leroux introduced in [Ler04] the operad $\mathcal{E}nnea = \mathcal{T}ri\mathcal{D}end \blacksquare \mathcal{T}ri\mathcal{D}end$. We prove the same results for the family $\mathcal{T}ri\mathcal{D}end^{\blacksquare n}$ and $\mathcal{T}rias^{\square n}$. All these families provide infinitely many examples of the generalized Deligne's conjecture proved in Theorem 45.

In order to study the homological properties of the algebras over an operad, it is crucial to prove that the operad is Koszul. We refer the reader to the paper of B. Fresse [Fre04] or to the book of M. Markl, S. Shnider and J. Stasheff [MSS02] for a full treatment of the subject. Since an operad is Koszul if and only if its Koszul dual is Koszul, we work with the simplest one to prove that the pair is a pair of Koszul operads. In the case of the operads $\mathcal{Q}uad$ and $\mathcal{Q}uad^! = \mathcal{P}erm \otimes \mathcal{D}ias$, we will prove that the Koszul property holds for the last one, $\mathcal{P}erm \otimes \mathcal{D}ias$.

Let $\mathcal{P}$ be an algebraic operad coming from an operad in the category of Sets. For instance, it is the case when the relations defining the operads only involve equalities between two terms and no linear combination. The operads $\mathcal{A}s$, $\mathcal{C}om$, $\mathcal{P}erm$ and $\mathcal{D}ias$ are of this type. In [Val2], we defined a family of partition type posets associated to such an operad $\mathcal{P}$ and proved that the operad is Koszul over $\mathbb{Z}$ and over any field k if and only each maximal intervals of the posets are Cohen-Macaulay.

We saw in the proof of Theorem 47 that $\mathcal{Q}uad^! = \mathcal{P}erm \otimes \mathcal{D}ias = \mathcal{P}erm \otimes \mathcal{P}erm \otimes \mathcal{A}s$. Therefore, $\mathcal{Q}uad^!$ is a set operad with basis $\{(i,\, j,\, \sigma) \mid 1 \le i,\, j \le n,\, \sigma \in \mathbb{S}_n\}$. The partitions associated to $\mathcal{Q}uad^!$ are of the form $\big(\sigma(1), \ldots, \overline{\sigma(i)}, \ldots, \underline{\sigma(j)}, \ldots, \sigma(n)\big)$, where $1 \le i,\, j \le n$ and $\sigma \in \mathbb{S}_n$. The order between the $\mathcal{Q}uad^!$-partitions is given by the refinement of partitions with respect to the two indexes. For instance, we have $\{(\overline{3},\, \underline{1},\, 5),\, (\overline{\underline{2}},\, 4)\} \le \{(3,\, \underline{1},\, 5,\, \overline{2},\, 4)\}$.

Lemma 50. *For each $n \in \mathbb{N}$, the maximal intervals of the poset $\Pi_{\mathcal{P}erm \otimes \mathcal{D}ias}(n)$ associated to the operad $\mathcal{P}erm \otimes \mathcal{D}ias$ are totally semi-modular.*

PROOF. The proof is the same than Lemma 1.10, 1.15 and 2.6 of [CV06]. ☐

Theorem 51. *The operad $\mathcal{Q}uad$ is Koszul over $\mathbb{Z}$.*

PROOF. The maximal intervals of the posets $\Pi_{\mathcal{P}erm \otimes \mathcal{D}ias}(n)$ are totally semi-modular. Therefore they are Cohen-Macaulay over $\mathbb{Z}$ by [Bac76, Far79] . One can see that the operad $\mathcal{P}erm \otimes \mathcal{D}ias$ is a basic set operad (see [Val2] Page 6). Then we can apply Theorem 9 of [Val2]. ☐

Corollary 52. *The dimensions of the homogenous components of $\mathcal{Q}uad$ are equal to*

$$dim(\mathcal{Q}uad(n)) = (n-1)! \sum_{j=n}^{2n-1} \binom{3n}{n+1+j} . \binom{j-1}{j-n}.$$

PROOF. When an operad $\mathcal{P}$ is Koszul, there are relations between the dimensions of $\mathcal{P}(n)$ and the dimensions of $\mathcal{P}^!(n)$ (see [GK95] Theorem (3.3.2) or [Lod01] Appendix B.5.c.). Use these relations with Corollary 48 to conclude. ☐

More generally, we have seen that, for every $n \in \mathbb{N}$, $\big(\mathcal{D}end^{\blacksquare\, n}\big)^! = \mathcal{P}erm^{\otimes n} \otimes \mathcal{A}s$, which is a basic set operad. The related partitions have the same form than the ones for $\mathcal{Q}uad^!$ but with n types of indices instead of 2.

Theorem 53. *For every n, the operad $\mathcal{D}end^{\blacksquare\, n}$ is Koszul over $\mathbb{Z}$.*

PROOF. Apply the same arguments. ☐

J.-L. Loday and M. Ronco introduced in [LR04] the pair of Koszul dual operads $\mathcal{T}rias$ and $\mathcal{T}ri\mathcal{D}end$. A $\mathcal{T}rias$-algebra is a $\mathcal{D}ias$-algebra with an extra operation. In [Val2], we defined a commutative analogue of $\mathcal{T}rias$ which we denoted by $\mathcal{C}om\mathcal{T}rias$. The $\mathcal{C}om\mathcal{T}rias$-partitions are partitions with at least one element of each block emphasized. The $\mathcal{T}rias$-partitions are ordered partitions with at least one element in the block emphasized. Using the same ideas than before, we have to following results. The operad $\mathcal{T}rias^{\square\, n} \cong \mathcal{C}om\mathcal{T}rias^{\otimes n} \otimes \mathcal{A}s$. The maximal intervals of $\Pi_{\mathcal{T}rias^{\square\, n}}$ are totally semi-modular.

Theorem 54. *For every n, the operads $\mathcal{T}rias^{\square\, n}$ and $\mathcal{T}ri\mathcal{D}end^{\blacksquare\, n}$ are Koszul over $\mathbb{Z}$.*

Recall from [EFG05] Proposition 3.5 that $\mathcal{T}ri\mathcal{D}end \blacksquare \mathcal{T}ri\mathcal{D}end$ is isomorphic to the $\mathcal{E}nnea$ operad defined by P. Leroux in [Ler04]. The previous theorem gives that the $\mathcal{E}nnea$ operad is Koszul over $\mathbb{Z}$.

From this result, we get four infinite families of operads for which Deligne's conjecture holds.

Corollary 55. *Let $\mathcal{P}$ be an operad of the form $\mathcal{D}end^{\blacksquare\, n}$, $\mathcal{D}ias^{\square\, n}$, $\mathcal{T}ri\mathcal{D}end^{\blacksquare\, n}$ or $\mathcal{T}rias^{\square\, n}$. Then for any $\mathcal{P}$-algebra A, the chain complex $C^\bullet_{\mathcal{P}}(A)$ is an algebra over an operad equivalent to the singular chains of the little disks operad.*

PROOF. By the previous theorems, these operads are finitely generated binary non-symmetric and Koszul. Then apply Theorem 45. ☐

Notice that the poset's method of [Val2] allowed us to prove that these operads are Koszul over $\mathbb{Z}$. Since the proof of Deligne's conjecture [MS02] also works over the ring of integers, this last corollary holds over $\mathbb{Z}$.

Appendix A. Associative algebras, operads and properads

This appendix is a short survey on the notions of associative algebras, operads and properads which are the main examples of 2-monoidal categories treated in this text. For a complete treatment of the subject, we refer the reader to [Val1].

A.1. **Associative algebras.** Associative algebras, operads and properads are monoids in some monoidal categories.

Let k be the ground field and let $(k\text{-Mod}, \otimes_k,\ k)$ be the monoidal category of k-modules equipped with the tensor product over k.

Definition (Associative algebra)**.** A monoid $(A,\ \mu,\ \eta)$ in $(k\text{-Mod}, \otimes_k,\ k)$ is an *associative algebra.* The product $\mu\,:\,A \otimes_k A \xrightarrow{\mu} A$ is associative and $k \xrightarrow{\eta} A$ is the unit of A.

The product of elements $a_1, \ldots,\ a_l$ of A can be represented by an indexed branch, see Figure 2

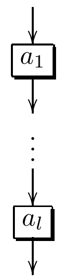


Figure 2. Product of $a_1,\ \ldots,\ a_l$.

Example. Let M be a k-module. Denote by $\mathrm{End}(M) := \mathrm{Hom}_k(M,\ M)$ the space of endomorphisms of M. With the composition of endomorphisms, $\mathrm{End}(M)$ is an associative algebra.

An *associative coalgebra* is a comonoid in $(k\text{-Mod}, \otimes_k,\ k)$, that is a monoid in the opposite category.

A.2. **Operads.** An $\mathbb{S}$-*module* is a collection $\{\mathcal{P}(n)\}_{n\in\mathbb{N}^*}$ of right modules over the symmetric group $\mathbb{S}_n$. In the category of $\mathbb{S}$-modules, one defines a monoidal product by the following formula

$$\mathcal{P} \circ \mathcal{Q}(n) := \bigoplus_{1\leqslant l \leqslant n} \left(\bigoplus_{i_1+\cdots+i_l=n} \mathcal{P}(l) \otimes_k \big(\mathcal{Q}(i_1) \otimes_k \cdots \otimes_k \mathcal{Q}(i_l)\big) \otimes_{\mathbb{S}_{i_1}\times\cdots\times\mathbb{S}_{i_l}} k[\mathbb{S}_n] \right)_{\mathbb{S}_l},$$

where the coinvariants are taken with respect to the action of the symmetric group $\mathbb{S}_l$ given by $(p \otimes_k q_1 \ldots q_l \otimes_k \sigma)^\nu := p^\nu \otimes_k q_{\nu(1)} \ldots q_{\nu(l)} \otimes_k \bar{\nu}^{-1}.\sigma$ for $p \in \mathcal{P}(l)$, $q_j \in \mathcal{Q}(i_j)$, $\sigma \in \mathbb{S}_n$ and $\nu \in \mathbb{S}_l$, such that $\bar{\nu}$ is the induced block permutation.

The notion of $\mathbb{S}$-module is used to model the multi-linear operations acting on some algebras. The monoidal product $\circ$ reflects the compositions of operations and can be represented by 2-levelled trees whose vertices are indexed by the elements of $\mathcal{P}$ and $\mathcal{Q}$, see Figure 3. The unit of this monoidal category is given by the $\mathbb{S}$-module $I = (k,\ 0,\ 0,\ \ldots)$, which corresponds to the identity operation represented by $|$.

Definition (Operad)**.** A monoid $(\mathcal{P}, \mu, \eta)$ in $(\mathbb{S}\text{-Mod}, \circ,\ I)$ is called an *operad.* The associative product $\mu\,:\,\mathcal{P}\circ\mathcal{P} \to \mathcal{P}$ is called the *composition product* and $\eta\,:\,I \to \mathcal{P}$ is the *unit.*

Example. Let M be a k-module and consider $\mathrm{End}(M) := \bigoplus_{n\in\mathbb{N}^*} \mathrm{Hom}_k(M^{\otimes n},\ M)$. The permutation of the inputs of a morphism in $\mathrm{Hom}_k(M^{\otimes n},\ M)$ makes $\mathrm{End}(M)$ into an $\mathbb{S}$-module. With the natural composition of morphisms, $\mathrm{End}(M)$ is an operad called the *endomorphism operad.*

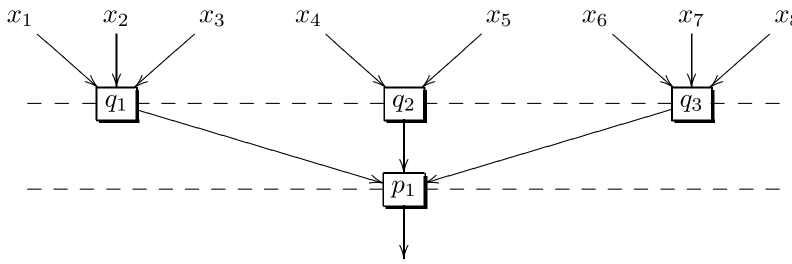


FIGURE 3. The monoidal product $\mathcal{P} \circ \mathcal{Q}$.

A *cooperad* is a comonoid in $(\mathbb{S}\text{-Mod}, \circ, I)$.

To every k-module V, one can associate an $\mathbb{S}$-module $\widetilde{V} := (V, 0, 0, \ldots)$ concentrated in arity 1. This defines an embedding of k-Mod into $\mathbb{S}$-Mod. One can check that this embedding is compatible with the monoidal products, that is $\widetilde{V \otimes W} = \widetilde{V} \circ \widetilde{W}$. Therefore, $(k\text{-Mod}, \otimes_k, k)$ is a full monoidal subcategory of $(\mathbb{S}\text{-Mod}, \circ, I)$. Thus every associative algebra is an operad.

We can forget the action of the symmetric groups and work in the category of $\mathbb{N}^*$-graded vector spaces. This category is endowed with a monoidal product, a non-symmetric analog of the previous one. We still denote it by $\circ$:

$$\mathcal{P} \circ \mathcal{Q}(n) := \bigoplus_{1 \leqslant l \leqslant n} \left(\bigoplus_{i_1 + \cdots + i_l = n} \mathcal{P}(l) \otimes_k \big(\mathcal{Q}(i_1) \otimes_k \cdots \otimes_k \mathcal{Q}(i_l)\big) \right).$$

Definition (Non-symmetric operad)**.** A monoid in this monoidal category is called a *non-symmetric operad*.

We can also define a notion of operads with colors indexing the inputs and the outputs. The composition of such operations have to fit with the colors. Such operads are called *colored operads* (cf. [VdL, BM05]).

A.3. **Properads.** We are going to pursue this generalization. Elements of an associative algebra can be seen as operations with one input and one output (see Figure 2). Elements of an operad represent operations with multiple inputs but one output. To model operations with multiple inputs and multiple outputs, one uses the notion of $\mathbb{S}$-*bimodule*. An $\mathbb{S}$-*bimodule* is a collection $\{\mathcal{P}(m,n)\}_{m,\, n \in \mathbb{N}^*}$ of modules over the symmetric groups $\mathbb{S}_n$ on the right and $\mathbb{S}_m$ on the left. In this category, we define a monoidal product based on the composition of operations indexing the vertices of a 2-levelled directed connected graph, see Figure 4.

Let a and b be the number of vertices on the first and on the second level respectively. Let N be the number of edges between the first and the second level. To an a-tuple of integers $\bar{\imath} := (i_1, \ldots, i_a)$, we associate $|\bar{\imath}| := i_1 + \cdots + i_a$. Given two a-tuples $\bar{\imath}$ and $\bar{\jmath}$, we denote by $\mathcal{Q}(\bar{\jmath}, \bar{\imath})$ the tensor product $\mathcal{Q}(j_1, i_1) \otimes \cdots \otimes \mathcal{Q}(j_a, i_a)$ and we denote by $\mathbb{S}_{\bar{\imath}}$ the image of the direct product of the groups $\mathbb{S}_{i_1} \times \cdots \times \mathbb{S}_{i_n}$ in $\mathbb{S}_{|\bar{\imath}|}$.

Definition (Connected permutations)**.** Let N be an integer. Let $\bar{k} = (k_1, \ldots, k_b)$ be a b-tuple and $\bar{\jmath} = (j_1, \ldots, j_a)$ be a a-tuple such that $|\bar{k}| = k_1 + \cdots + k_b = |\bar{\jmath}| = j_1 + \cdots + j_a = N$.
A $(\bar{k}, \bar{\jmath})$-*connected permutation* σ is a permutation of $\mathbb{S}_N$ such that the graph of a geometric representation of σ is connected if one gathers the inputs labelled by $j_1 + \cdots + j_i + 1, \ldots, j_1 + \cdots + j_i + j_{i+1}$, for $0 \leq i \leq a-1$, and the outputs labelled by $k_1 + \cdots + k_i + 1, \ldots, k_1 + \cdots + k_i + k_{i+1}$, for $0 \leq i \leq b-1$. The set of $(\bar{k}, \bar{\jmath})$-connected permutations is denoted by $\mathbb{S}^c_{\bar{k}, \bar{\jmath}}$.

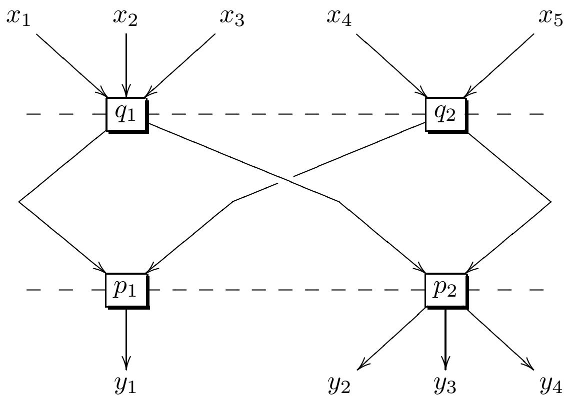
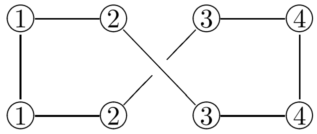


FIGURE 4. Composition of operations with multiple inputs and multiple outputs.

Example. Consider the permutation (1324) in $\mathbb{S}_4$ and take $\bar{k} = (2, 2)$ and $\bar{\jmath} = (2, 2)$. If one links the inputs 1, 2 and 3, 4 and the outputs 1, 2 and 3, 4, it gives the following connected graph



Therefore, the permutation (1324) is $((2, 2),\, (2, 2))$-connected.

Let $\mathcal{P}$ and $\mathcal{Q}$ be two $\mathbb{S}$-bimodules, their monoidal product is given by the formula

$$\mathcal{P} \boxtimes_c \mathcal{Q}(m,\, n) := \bigoplus_{N \in \mathbb{N}^*} \left(\bigoplus_{\bar{l},\, \bar{k},\, \bar{\jmath},\, \bar{\imath}} k[\mathbb{S}_m] \otimes_{\mathbb{S}_{\bar{l}}} \mathcal{P}(\bar{l},\, \bar{k}) \otimes_{\mathbb{S}_{\bar{k}}} k[\mathbb{S}^c_{\bar{k},\, \bar{\jmath}}] \otimes_{\mathbb{S}_{\bar{\jmath}}} \mathcal{Q}(\bar{\jmath},\, \bar{\imath}) \otimes_{\mathbb{S}_{\bar{\imath}}} k[\mathbb{S}_n] \right)_{\mathbb{S}_b^{\mathrm{op}} \times \mathbb{S}_a},$$

where the second direct sum runs over the b-tuples $\bar{l}$, $\bar{k}$ and the a-tuples $\bar{\jmath}$, $\bar{\imath}$ such that $|\bar{l}| = m$, $|\bar{k}| = |\bar{\jmath}| = N$, $|\bar{\imath}| = n$ and where the coinvariants correspond to the following action of $\mathbb{S}_b^{\mathrm{op}} \times \mathbb{S}_a$:

$$\begin{aligned} &\theta \otimes p_1 \otimes \cdots \otimes p_b \otimes \sigma \otimes q_1 \otimes \cdots \otimes q_a \otimes \omega \sim \\ &\theta\, \tau_{\bar{l}}^{-1} \otimes p_{\tau^{-1}(1)} \otimes \cdots \otimes p_{\tau^{-1}(b)} \otimes \tau_{\bar{k}}\, \sigma\, \nu_{\bar{\jmath}} \otimes q_{\nu(1)} \otimes \cdots \otimes q_{\nu(a)} \otimes \nu_{\bar{\imath}}^{-1}\, \omega, \end{aligned}$$

for $\theta \in \mathbb{S}_m$, $\omega \in \mathbb{S}_n$, $\sigma \in \mathbb{S}^c_{\bar{k},\, \bar{\jmath}}$ and for $\tau \in \mathbb{S}_b$ with $\tau_{\bar{k}}$ the corresponding block permutation, $\nu \in \mathbb{S}_a$ and $\nu_{\bar{\jmath}}$ the corresponding block permutation. The unit I for this monoidal product is given by

$$\begin{cases} I(1,\, 1) := k, & \text{and} \\ I(m,\, n) := 0 & \text{otherwise.} \end{cases}$$

We denote by $(\mathbb{S}\text{-biMod}, \boxtimes,\, I)$ this monoidal category.

REMARK. We need to restrict compositions to connected graphs and connected permutations in order to get a monoidal category (see Proposition 1.6 of [Val1]).

Definition (Properad)**.** A *properad* is a monoid in the monoidal category $(\mathbb{S}\text{-biMod}, \boxtimes,\, I)$.

Example. Let M be a k-module and consider $\mathrm{End}(M) := \bigoplus_{m,\, n \in \mathbb{N}^*} \mathrm{Hom}_k(M^{\otimes n},\, M^{\otimes m})$. The permutation of the inputs and the outputs of an element of $\mathrm{Hom}_k(M^{\otimes n},\, M^{\otimes m})$ makes $\mathrm{End}(M)$ into an $\mathbb{S}$-bimodule. Once again, $\mathrm{End}(M)$, endowed with the natural (connected) composition of morphisms, is a properad.

A comonoid in $(\mathbb{S}\text{-biMod}, \boxtimes,\, I)$ is called a *coproperad.*

To an $\mathbb{S}$-module V, we associate an $\mathbb{S}$-bimodule $\widetilde{V}$ defined by

$$\begin{cases} \widetilde{V}(1, n) := V(n) & \text{and} \\ \widetilde{V}(m, n) := 0 & \text{for } m > 1. \end{cases}$$

This defines an embedding of monoidal categories, that is $\widetilde{V \circ W} = \widetilde{V} \boxtimes \widetilde{W}$. The category $(\mathbb{S}\text{-Mod}, \circ, I)$ is a full monoidal subcategory of $(\mathbb{S}\text{-biMod}, \boxtimes, I)$. Hence, an operad is a properad. Since the notion of properad includes the one of associative algebras and operads, we work in this general framework throughout the text. We resume these notions in the following table

Monoidal category :	$(k\text{-Mod}, \otimes_k)$	$\rightarrowtail$	$(\mathbb{S}\text{-Mod}, \circ)$	$\rightarrowtail$	$(\mathbb{S}\text{-biMod}, \boxtimes)$
Monoid :	Associative algebra	$\rightarrowtail$	Operad	$\rightarrowtail$	Properad.

Remark that the first monoidal product $\otimes_k$ is bilinear and symmetric, the second one $\circ$ is only linear on the left and no symmetry. The third one $\boxtimes$ has no linear nor symmetric properties in general.

A.4. **$\mathcal{P}$-gebras.** In this section, we precise the previous analogy with multi-linear operations and recall the notion of an (al)gebra over a properad.

Let (A, μ, u) be an associative algebra and let M be a k-module. Recall that a structure of *module* over A on M is given by a morphism of associative algebras $\phi : A \to \mathrm{End}(M)$. More generally, we have the following definition.

Definition ($\mathcal{P}$-gebras)**.** Let $\mathcal{P}$ be a properad and let M be a k-module. A structure of *$\mathcal{P}$-gebra* on M is a morphism of properads $\phi : \mathcal{P} \to \mathrm{End}(M)$.

When $\mathcal{P}$ is an operad, this corresponds to the notion of algebra over $\mathcal{P}$ or *$\mathcal{P}$-algebra* (see V. Ginzburg and M. Kapranov [GK95]). There is an operad $\mathcal{A}s$ such that the category of $\mathcal{A}s$-algebras is equal to the category of non-unital associative algebras, an operad $\mathcal{C}om$ such that the category of $\mathcal{C}om$-algebras is equal to the category of non-unital commutative associative algebras and an operad $\mathcal{L}ie$ such that the category of $\mathcal{L}ie$-algebras is equal to the category of Lie algebras.

Categories of "algebras" defined by products and coproducts (multiple outputs), cannot be modelled by operads, one has to use properads. Recall from [Val1], that there is a properad $\mathcal{B}i$ such that the category of $\mathcal{B}i$-gebras is equal to the category of non-unital non-counital bialgebras and there exists a properad $\mathcal{B}i\mathcal{L}ie$ such that category of $\mathcal{B}i\mathcal{L}ie$-gebras is equal to the category of Lie bialgebras, for instance.

Remark. Following the article of J-.P. Serre [Ser93], we choose to call a *gebra* any algebraic structure like modules over an associative algebra, associative algebras, Lie algebras, commutative algebras or bialgebras, Lie bialgebras, etc ...

This point of view on algebraic structures allows us to understand and describe general properties between different types of gebras. Constructions on the levels of operads or properads induce general relations between the related types of gebras.

A.5. **Free and quadratic properad.** The forgetful functor U from the category of properads to the category of $\mathbb{S}$-bimodules has a left adjoint $\mathcal{F}$.

$$U : \text{Properads} \rightleftarrows \mathbb{S}\text{-biMod} : \mathcal{F}.$$

We gave an explicit construction of it in [Val3] by means of a particular colimit. For every $\mathbb{S}$-bimodule V, it provides the free properad $\mathcal{F}(V)$ on V. It is given by the direct sum of connected directed graphs with the vertices indexed by elements of V. The composition product μ is simply defined by the grafting of graphs. Therefore, the number of vertices is preserved by μ and it induces a natural graduation denoted $\mathcal{F}_{(\omega)}(V)$ and called the *weight*.

Remark that, when V is a k-module, we find the tensor algebra $T(V)$ on V, which is the free associative algebra on V. When V is an $\mathbb{S}$-module, we get the free operad in terms of indexed trees like in [GK95] Section 2.1.

We can generalize the notion of ideal for an associative algebra to ideals for operads and properads (see Appendix B). Let V be an $\mathbb{S}$-bimodule and R be a sub-$\mathbb{S}$-bimodule of $\mathcal{F}(V)$, we consider the ideal generated by R in $\mathcal{F}(V)$ and we denote it by (R). As usual, the quotient $\mathcal{F}(V)/(R)$ has a natural structure of properads. When $R \subset \mathcal{F}_{(2)}(V)$, the quotient properad is called a *quadratic properad*. When V is a k-module, this definition corresponds to the notion of *quadratic algebra* (see Y. Manin [Man88]) and when V is an $\mathbb{S}$-module it corresponds to the notion of *quadratic operad* of [GK95].

Examples. The symmetric and the exterior algebras are natural examples of quadratic algebras. The operads $\mathcal{A}s$, $\mathcal{C}om$ and $\mathcal{L}ie$ are the most common quadratic operads (see [GK95]). The properads $\mathcal{B}i\mathcal{L}ie$ of Lie bialgebras, $\varepsilon\mathcal{B}i$ of infinitesimal bialgebras and $\mathcal{F}rob$ of Frobenius bialgebras are quadratic properads (see [Val1] Section 2.9).

Since R is homogenous of weight 2, the quotient properad $\mathcal{F}(V)/(R)$ is graded by the weight.

Dually, there is a *connected cofree coproperad* denoted $\mathcal{F}^c(V)$ (see [Val1] Section 2.8).

A.6. **Hadamard tensor product.** We define another monoidal product in the category of $\mathbb{S}$-bimodules.

Definition (Hadamard product of $\mathbb{S}$-bimodules)**.** Let V and W be two $\mathbb{S}$-bimodules. Their Hadamard product is defined by $(V \otimes_H W)(m,\, n) := V(m,\, n) \otimes_k W(m,\, n)$.

When V and W are $\mathbb{S}$-modules, the Hadamard product is equal to $(V \otimes_H W)(n) := V(n) \otimes_k W(n)$. When it is obvious that in the context we are dealing with the Hadamard product, we simply denote it by $\otimes$.

This monoidal product is bilinear and symmetric. The unit of the Hadamard product is the $\mathbb{S}$-bimodule K defined by $K(m,\, n) := k$, with trivial action of $\mathbb{S}_n$ and $\mathbb{S}_m$, for all n, m (and $K(n) = k$ for $\mathbb{S}$-modules). (The properad K models commutative and cocommutative Frobenius algebras).

Appendix B. Categorical algebra

The aim of this section is to define the notion of "ideal of a monoid" in a modern, categorical point of view. Working in the opposite category, we get the dual notion for comonoids. The other purpose of this categorical treatment is to characterize the ideal "generated by" and its dual notion.

B.1. **Definition of the "ideal" notions.** In this section we define the notions of *ideal monomorphism* and *ideal subobject* of monoids. Dually, we define the notions of *coideal epimorphism* and *coideal quotient*.

Let us work in an abelian monoidal category, that is an abelian category $\mathcal{A}$ endowed with a monoidal product $\boxtimes$. Consider the subcategory $\mathcal{M}\mathrm{on}_{\mathcal{A}}$ whose objects ar monoids in $\mathcal{A}$. One natural question now is to ask whether $\mathcal{M}\mathrm{on}_{\mathcal{A}}$ is still an abelian category. The answer is no because the class of monomorphisms is too wide, for instance. Recall that in an abelian category the class of monomorphisms is equal to the class of kernels. Every morphism $A \xrightarrow{f} B$ in $\mathcal{M}\mathrm{on}_{\mathcal{A}}$

admits a kernel $i : K \rightarrowtail A$ is $\mathcal{A}$. The following diagram is commutative

$$\begin{tikzcd} K \boxtimes K \arrow[r, "i\boxtimes i"] \arrow[d, dotted, "\mu_K"] & A \boxtimes A \arrow[r, "f\boxtimes f"] \arrow[d, "\mu_A"] & B \boxtimes B \arrow[d, "\mu_B"] \\ K \arrow[r, tail, "i"] & A \arrow[r, "f"] & B, \end{tikzcd}$$

where μ_A and μ_B stand for the product of the monoid A and B respectively. Since $(f\boxtimes f)\circ(i\boxtimes i) = (f\circ i)\boxtimes(f\circ i) = 0$, the composite $f\circ\mu_A\circ(i\boxtimes i)$ is equal to 0. By the universal property of the kernel i, there exists an associative map $\mu_K : K\boxtimes K \to K$ making K into a monoid and $i : K \rightarrowtail A$ a morphism in $\mathcal{M}\mathrm{on}_{\mathcal{A}}$. Hence kernels exist in $\mathcal{M}\mathrm{on}_{\mathcal{A}}$ and every kernel is a monomorphism. Actually, K has more properties than just being a submonoid of A (see B.2), which explains why not all monomorphisms are kernels. On the other hand, let $I \rightarrowtail A$ be a monomorphism of monoids, its cokernel A/I in $\mathcal{A}$ is not necessarily a monoid. Following Kummer's language, we restrict our attention to *ideal monomorphisms*, that is monomorphisms that are kernels in $\mathcal{M}\mathrm{on}_{\mathcal{A}}$.

Definition (Ideal monomorphism)**.** Let $I \rightarrowtail A \twoheadrightarrow Q$ be an exact sequence in $\mathcal{A}$, where A is a monoid. In other words, $I \rightarrowtail A$ is the kernel of $A \twoheadrightarrow Q$ and $A \twoheadrightarrow Q$ is the cokernel of $I \rightarrowtail A$.
The monomorphism $I \rightarrowtail A$ in $\mathcal{M}\mathrm{on}_{\mathcal{A}}$ is an *ideal monomorphism* if $A \twoheadrightarrow Q$ is a morphism in $\mathcal{M}\mathrm{on}_{\mathcal{A}}$.

In this case, we say that I is an *ideal* (subobject) of A and Q is naturally a *quotient* monoid, also denoted by A/I.

Dually, recall that a comonoid in $\mathcal{A}$ is a monoid in the opposite category $\mathcal{A}^{\mathrm{op}}$. If we dualize the previous arguments in the opposite category, we can see that the category $\mathcal{C}\mathrm{omon}_{\mathcal{A}}$ of comonoids in $\mathcal{A}$ is not an abelian category because the class of epimorphisms is too big. The cokernel in $\mathcal{A}$ of a morphism in $\mathcal{C}\mathrm{omon}_{\mathcal{A}}$ is a morphism in $\mathcal{C}\mathrm{omon}_{\mathcal{A}}$ (and even more), but the kernel in $\mathcal{A}$ of a morphism in $\mathcal{C}\mathrm{omon}_{\mathcal{A}}$ is not necessarily a morphism in $\mathcal{C}\mathrm{omon}_{\mathcal{A}}$. Therefore, every epimorphism of comonoids is not a cokernel. Once again, we call *coideal epimorphisms*, the epimorphisms that are cokernels.

Definition (Coideal epimorphism)**.** Let $I \rightarrowtail C \twoheadrightarrow Q$ be an exact sequence in $\mathcal{A}$, where C is a comonoid. The epimorphism $C \twoheadrightarrow Q$ in $\mathcal{C}\mathrm{omon}_{\mathcal{A}}$ is a *coideal epimorphism* if $I \rightarrowtail C$ is a morphism in $\mathcal{C}\mathrm{omon}_{\mathcal{A}}$.

In this case, the subobject $I \rightarrowtail C$ is naturally a *subcomonoid* of C and the quotient Q is called a *coideal quotient*.

Remark. The term *coideal* is already used in the literature, but stands for a (coideal) subobject $J \rightarrowtail C$ (or a monomorphism) in $\mathcal{A}$ of a comonoid C such that its cokernel in $\mathcal{A}$ is a morphism in $\mathcal{C}\mathrm{omon}_{\mathcal{A}}$. It is equivalent to ask that the quotient C/J is a comonoid.
This notion does not correspond to the dual of the notion of ideal, where "dual" means "in the opposite category".

B.2. **Relation with the classical definition.** We now relate this definition of ideal with the classical notion. Let $I \overset{\iota}{\rightarrowtail} A \overset{\pi}{\twoheadrightarrow} Q$ be an sequence in $\mathcal{M}\mathrm{on}_{\mathcal{A}}$, exact in $\mathcal{A}$. Denote by μ_A and μ_Q the products of A and Q respectively. The morphism π is a morphism in $\mathcal{M}\mathrm{on}_{\mathcal{A}}$, means that the following diagram commutes

$$\begin{tikzcd} A \boxtimes A \arrow[r, "\pi\boxtimes\pi"] \arrow[d, "\mu_A"] & Q \boxtimes Q \arrow[d, "\mu_Q"] \\ A \arrow[r, "\pi"] & Q. \end{tikzcd}$$

Let $\kappa : K_I \rightarrowtail A\boxtimes A$ be the kernel of $\pi\boxtimes\pi$ in $\mathcal{A}$.

Proposition 56. *Let $\mathcal{A}$ be a monoidal category such that the monoidal product $\boxtimes$ preserves epimorphisms. A monomorphism $I \rightarrowtail A$ is an ideal monomorphism if and only if the composite $\pi\circ\mu_A\circ\kappa$ is equal to* 0*.*

Proof.
(⇒) It comes from $\pi \circ \mu_A \circ \kappa = \mu_Q \circ (\pi \boxtimes \pi) \circ \kappa = 0$
(⇐) From the hypothesis, we have that $\pi \boxtimes \pi$ is an epimorphism. Therefore, it is the cokernel of κ and by the universal property of the cokernel, there exists a morphism $\mu_Q : Q \boxtimes Q \to Q$ such that $\pi \circ \mu_A$ factors through $\pi \boxtimes \pi$. It is then straightforward to check that μ_Q defines an associative product on Q. ☐

The extra assumption, that the monoidal product has to preserve epimorphisms, is verified in every cases studied in this paper. For a proof for this fact, we refer to Proposition 1, Proposition 10 and Section 5 of [Val3].
The problem is now to make explicit the kernel $\kappa : K_I \rightarrowtail A \boxtimes A$ of $\pi \boxtimes \pi$.

Definition (Multilinear part)**.** The *multilinear part* in X of $A \boxtimes (X \oplus Y) \boxtimes B$ is defined either

- by the cokernel of $A \boxtimes Y \boxtimes B \xrightarrow{A \boxtimes i_y \boxtimes B} A \boxtimes (X \oplus Y) \boxtimes B$
- or by the kernel of $A \boxtimes (X \oplus Y) \boxtimes B \xrightarrow{A \boxtimes \pi_y \boxtimes B} A \boxtimes Y \boxtimes B$,

since i_Y is a section of π_Y, that is $\pi_y \circ i_Y = \mathrm{Id}_Y$, these two objects are naturally isomorphic. We denote it $A \boxtimes (\underline{X} \oplus Y) \boxtimes B$.

It corresponds to elements of $A \boxtimes (X \oplus Y) \boxtimes B$ with at least one element of X in between.
Suppose that we are working in an abelian category $\mathcal{A}$ such that every short exact sequence splits, that is $\begin{tikzcd} I \arrow[r, tail, "\iota"] & A \arrow[r, two heads, shift right, "\pi"'] & Q \arrow[l, dotted, shift right, "\exists"'] \end{tikzcd}$ or equivalently $A \cong I \oplus Q$. Once again, this condition is verified in every category studied here since they are categories of representations of finite groups over a field of characteristic 0.

Proposition 57. *In a monoidal abelian category such that the monoidal product preserves epimorphisms and where every short exact sequence splits, we have*

$$K_I = A \boxtimes (A + \underline{I}) + (A + \underline{I}) \boxtimes A,$$

where $A \boxtimes (A + \underline{I}) := Im \left(A \boxtimes (A \oplus \underline{I}) \rightarrowtail A \boxtimes (A \oplus I) \xrightarrow{A \boxtimes (A + \iota)} A \boxtimes A \right)$.

Proof. It is enough to prove that $A \boxtimes (Q + \underline{I}) + (Q + \underline{I}) \boxtimes A$ is the kernel of $\pi \boxtimes \pi$. We have the following commutative diagram

$$\begin{tikzcd}
Q \boxtimes (Q \oplus \underline{I}) \arrow[r, tail] & Q \boxtimes A \arrow[r, two heads, "Q \boxtimes \pi"] & Q \boxtimes Q \\
A \boxtimes (Q \oplus \underline{I}) \arrow[u, dotted, two heads, "\exists!\, \theta_2"] \arrow[r, tail, "i_1"] \arrow[d, tail] & A \boxtimes A \arrow[u, "\pi \boxtimes A"'] \arrow[ur, "\pi \boxtimes \pi"] \arrow[r, two heads, "A \boxtimes \pi"] & A \boxtimes Q \arrow[u, "\pi \boxtimes Q"'] \\
A \boxtimes (Q \oplus \underline{I}) \oplus (Q \oplus \underline{I}) \boxtimes A \arrow[ur, "i_1 \oplus i_2"] & (Q \oplus \underline{I}) \boxtimes A \arrow[l, tail] \arrow[u, tail, "i_2"'] \arrow[r, dotted, two heads, "\exists!\, \theta_2"] & (Q \oplus \underline{I}) \boxtimes Q, \arrow[u, tail]
\end{tikzcd}$$

where the two dotted arrows exist by the property of kernels applied to the first line and last column. Since $A \boxtimes (Q + \underline{I}) + (Q + \underline{I}) \boxtimes A$ is by definition the image of the morphism $i_1 \oplus i_2$, it remains to show that $\pi \boxtimes \pi$ is the cokernel of $i_1 \oplus i_2$. The assumption that every short exact sequence splits implies that the maps θ_1 and θ_2 are epimorphisms. It is then straightforward to check that $\pi \boxtimes \pi$ is the cokernel of $i_1 \oplus i_2$. ☐

Since $\iota : I \rightarrowtail A$ is the kernel of $\pi : A \twoheadrightarrow Q$, there exists a morphism $\bar{\mu}$ making the following diagram to commute

$$\begin{tikzcd} A \boxtimes (A + \underline{I}) + (A + \underline{I}) \boxtimes A \arrow[r, tail] \arrow[d, "\bar{\mu}"] & A \boxtimes A \arrow[d, "\mu_A"] & \\ I \arrow[r, tail, "\iota"] & A \arrow[r, two heads, "\pi"] & Q. \end{tikzcd}$$

Hence, we get $\pi \circ \iota \circ \bar{\mu} = 0$. When the monoidal product is additive on the left and on the right, we have $A \otimes (A + \underline{I}) = A \otimes I$ and $(A + \underline{I}) \otimes A = I \otimes A$. In this case, the notion of ideal corresponds to the classical one.

Dually, let $I \overset{\iota}{\rightarrowtail} C \overset{\pi}{\twoheadrightarrow} Q$ be a sequence in $\mathcal{C}\mathrm{omon}_{\mathcal{A}}$, exact in $\mathcal{A}$. Denote by $\gamma : C \boxtimes C \twoheadrightarrow CoK_Q$ be the cokernel of $\iota \boxtimes \iota$. Note that when every short exact sequence splits, we have $\mathrm{CoK}_Q \cong K_I \cong C \boxtimes (C + \underline{I}) + (C + \underline{I}) \boxtimes C$.

Proposition 58. *Let $\mathcal{A}$ be a monoidal category such that the monoidal product $\boxtimes$ preserves monomorphisms. An epimorphism $C \twoheadrightarrow Q$ is an coideal epimorphism if and only if the composite $\gamma \circ \Delta_C \circ \iota$ is equal to 0.*

Proof. We work in the opposite category and we apply Proposition 56. ☐

In the case of a coassociative coalgebra $C = I \oplus Q$, it means that the composite $(\pi \otimes C \oplus C \otimes \pi) \circ \Delta_C \circ \iota : I \to Q \otimes C \oplus C \otimes Q$ is null. In other words, we have $\Delta_C(c) \in I \otimes I$, for $c \in I$.

B.3. **Various notions of modules.** We recall briefly the various notions of modules and relate one of them to the notion of ideal.

Definition (Module)**.** An object M of $\mathcal{A}$ is a *left module* over a monoid A if there exists a map $A \boxtimes M \to M$ compatible with the product of A. Dually, there is a notion of *right module*. And A compatible left and right action defines a *bimodule*.

At first sight, the biadditive case could lead to the following definition : I is an ideal of A if it is a bimodule over A : $A \boxtimes I \to I$ and $I \boxtimes A \to I$. The main problem with such a notion is that A/I is not a monoid when $\boxtimes$ is not biadditive. Instead of that, one has to consider a linearized version of module.

Definition. An object M of $\mathcal{A}$ is called a *multilinear left module over A* if it is endowed with a map $A \boxtimes (A \oplus \underline{M}) \to M$ compatible with the product and the unit of A.

We have a similar notion on the right hand sight and a notion of *multilinear bimodule*. If we use this language, I is an ideal of A if and only if I a multilinear bimodule with the action induced by μ_A.

Remark. The same notion arises from the work of D. Quillen on (co)homology theories [Qui70]. The coefficient for these theories are *abelian group objects*. When one wants to make explicit Quillen (co)homology of monoids, these coefficients are exactly linear version of modules. We refer the reader to the paper of H.J. Baues, M. Jibladze and A. Tonks [BJT97] for a complete description in the case of operads, or more generally when the monoidal product is additive only on one side.

Dualize these definitions to get the notions of *comodules* and *(multi)linear* comodules over a comonoid C.

B.4. **"generated by".** Following this categorical point of view, we define and make explicit the notions of *ideal generated by* and *subcomonoid generated by*.

Let $\xi : R \rightarrowtail A$ be a subobject of A in $\mathcal{A}$, where A is a monoid. We are going to consider the "cokernel" $A \twoheadrightarrow Q$ of ξ in $\mathcal{M}\mathrm{on}_{\mathcal{A}}$, that is the universal epimorphism of monoids such that the composite $R \rightarrowtail A \twoheadrightarrow Q$ vanishes. The resulting quotient monoid Q is the largest quotient of A with relations in R.

Since our leitmotiv is to treat together ideals and quotient monoids, we would rather use the following presentation. Consider the category $\mathcal{S}_\xi$ of sequences $(\mathbf{S}) : I \rightarrowtail A \twoheadrightarrow Q$ in $\mathcal{M}\mathrm{on}_{\mathcal{A}}$, exact in $\mathcal{A}$ such that the composite $R \rightarrowtail A \twoheadrightarrow Q$ is equal to 0. Since $I \rightarrowtail A$ is the kernel of $A \twoheadrightarrow Q$, this last condition is equivalent to the existence of a morphism $\iota : R \rightarrowtail I$ in $\mathcal{A}$ such that the following diagram commutes

$$\begin{tikzcd}
R \arrow[d, tail, dotted, "\exists\,\iota"'] \arrow[dr, tail, "\xi"] \arrow[drr, bend left, "0"] & & \\
I \arrow[r, tail] & A \arrow[r, two heads] & Q.
\end{tikzcd}$$

Let $(\mathbf{S}') : J \rightarrowtail A \twoheadrightarrow O$ be another object of $\mathcal{S}_\xi$, the morphisms between $(\mathbf{S})$ and $(\mathbf{S}')$ correspond to the pair of morphisms $(i,\, p)$ in $\mathcal{M}\mathrm{on}_{\mathcal{A}}$ such that the following diagram commutes

$$\begin{tikzcd}
R \arrow[d, tail] \arrow[dr, tail, "\xi"] \arrow[dd, tail, bend right] & & O \\
I \arrow[r, tail] \arrow[d, tail, dotted, "i"'] & A \arrow[r, two heads] \arrow[ur, two heads] & Q \arrow[u, two heads, dotted, "p"'] \\
J \arrow[ur, tail] & &
\end{tikzcd}$$

Definition (Ideal generated by R)**.** Let $\mathcal{A}$ be a category such that for every monoid A and every subobject $\xi : R \rightarrowtail A$, the category $\mathcal{S}_\xi$ admits an initial object $\bar{\mathbf{S}} : (R) \rightarrowtail A \twoheadrightarrow A/(R)$.
In this case, (R) is called the *ideal generated by* R and $A/(R)$ is the induced quotient monoid.

Remark. The terminal object of this category always exists and is given by the sequence $A \rightarrowtail A \twoheadrightarrow 0$.

If we dualize the previous arguments in the opposite category, we get the same kind of diagram but with C comonoid instead of A monoid.

Let $\xi : S \twoheadleftarrow C$ be a quotient of C in $\mathcal{A}$, where C is a comonoid. We aim to consider the largest subcomonoid of C vanishing on S. This notion is given by "kernel" $S \twoheadleftarrow C$ of ξ in $\mathcal{C}\mathrm{omon}_{\mathcal{A}}$, that is the universal monomorphism of comonoids such that the composite $S \twoheadleftarrow C \leftarrowtail Q$ is equal to 0.

Consider the category $\mathcal{S}_\xi$ of sequences $(\mathbf{S}) : I \twoheadleftarrow C \leftarrowtail Q$ in $\mathcal{C}\mathrm{omon}_{\mathcal{A}}$, exact in $\mathcal{A}$ such that the composite $S \twoheadleftarrow C \leftarrowtail Q$ is equal to 0. There exists a morphism $\pi : I \twoheadrightarrow S$ such that the diagram commutes

$$\begin{tikzcd}
S & & \\
I \arrow[u, two heads, dotted, "\exists\,\pi"] & C \arrow[l, two heads] \arrow[ul, two heads, "\xi"] & Q. \arrow[l, tail] \arrow[ull, bend right, "0"']
\end{tikzcd}$$

Let $(\mathbf{S}') : J \twoheadleftarrow C \leftarrowtail O$ be another object of $\mathcal{S}_\xi$, the morphisms between $(\mathbf{S})$ and $(\mathbf{S}')$ correspond to the pair of morphisms $(i,\, p)$ in $\mathcal{C}\mathrm{omon}_{\mathcal{A}}$ such that the following diagram commutes

$$
\begin{tikzcd}
S & & O \arrow[dl, tail] \arrow[d, dotted, tail, "i"] \\
I \arrow[u, two heads] & C \arrow[ul, two heads, "\xi"'] \arrow[l, two heads] \arrow[dl] & Q \arrow[l, tail] \\
J \arrow[u, dotted, two heads, "p"] \arrow[uu, bend left] & &
\end{tikzcd}
$$

Definition (Subcomonoid generated by S)**.** Let $\mathcal{A}$ be a category such that for every comonoid C and every quotient $\xi : S \twoheadleftarrow C$, the category $\mathcal{S}_\xi$ admits a terminal object $\bar{\mathbf{S}} : (S) \twoheadleftarrow C \hookleftarrow C(S)$. In this case, $C(S)$ is called the *subcomonoid of C generated by S* and (S) is the induced coideal quotient.

REMARK. The initial object is the sequence $C \twoheadleftarrow C \hookleftarrow 0$.

B.5. **Ideal generated = free multilinear bimodule.** Since the notion of ideal is equivalent to the notion of multilinear bimodule, the ideal of A generated by R is the free A-multilinear bimodule on R.

Proposition 59. *The ideal generated by R in A is given by the image*

$$Im\left(A \boxtimes (A + \underline{R}) \boxtimes A \xrightarrow{\mu^2} A\right).$$

PROOF. Using Proposition 57, we have that it is an ideal of A. It is easy to see that any ideal containing R also contains $\mathrm{Im}\left(A \boxtimes (A + \underline{R}) \boxtimes A \xrightarrow{\mu^2} A\right)$. □

If we dualize the arguments, we have the explicit form of the subcomonoid of C generated by R.

Proposition 60. *Let $S \twoheadleftarrow C$ be an epimorphism in $\mathcal{A}$. The subcomonoid of C generated by S is given by the kernel of*

$$Ker\left(C \xrightarrow{\Delta^2} C^{\boxtimes 3} \xrightarrow{proj} C \boxtimes (C + \underline{S}) \boxtimes C\right).$$

PROOF. Dualize. □

ACKNOWLEDGEMENTS

First of all, I would like to thank Marcelo Aguiar to have introduced me to Manin's products. I am grateful to Jean-Louis Loday, Damien Calaque and Martin Markl for useful discussions. Many thanks to Anton Korochkine for stimulating discussions about the counterexample in Section 4.5.

I am grateful to the Max Planck Institute for Mathematics in Bonn for the excellent working conditions I found there during my stay in May and June 2006 where I finished this paper.

REFERENCES

[AL04] M. Aguiar, J.-L. Loday, *Quadri-algebras*, J. Pure Appl. Algebra **191** (2004), no. 3, 205-221.

[AM06] M. Aguiar, S. Mahajan, *Hopf monoids in species and associated Hopf algebras*, private communication.

[AS87] M. Artin and W.F. Schelter, *Graded algebras of global dimension* 3, Adv. in Math. **66** (1987), no. 2, 171-216.

[Bac] J. Backelin, *A distributive property of augmented algebras, and some related homological results*, Ph. D. thesis, University of Stockholm.

[Bac76] K. Baclawski, *Homology and combinatorics of ordered sets*, Ph.D. thesis, Harvard Univ., Cambridge, Mass., 1976.

[Bal97] D. Balavoine, *Deformations of algebras over a quadratic operad*, Operads: Proceedings of Renaissance Conferences (Hartford, CT/Luminy, 1995) (Providence, RI), Contemp. Math., vol. 202, Amer. Math. Soc., 1997, pp. 207–234.

[BFSV03] C. Balteanu, Z. Fiedorowicz, R. Schwänzl, R. Vogt, *Iterated monoidal categories*, Advances in Math. **176** (2003), 277-349.

[BFLS00] G. Barnich, R. Fulp, T. Lada, and J. Stasheff, *Algebra structures on* $\mathrm{Hom}(C, L)$, Comm. Algebra **28** (2000), no. 11, 5481–5501.

[BJT97] H. J. Baues, M. Jibladze, A.Tonks, *Cohomology of monoïds in monoïdal categories*, in "Operads: proceedings of renaissance conferences (Hartford/Luminy, 1995)", 137-165, Contemp. Math. **202**, Amer. Math. Soc., Providence, RI, (1997).

[Be63] J. Bénabou, *Catégories avec multiplication*, C. R. Acad. Sci. Paris **256** (1963), 1887-1890.

[BM03] C. Berger and I. Moerdijk, *Axiomatic homotopy theory for operads*, Comment. Math. Helv. **78** (2003), no. 4, 805–831.

[BM05] C. Berger, I. Moerdijk, *Resolution of coloured operads and rectification of homotopy algebras*, preprint, `math.AT/0512576`.

[BDVW03] R. Berger, M. Dubois-Violette, M. Wambst, *Homogeneous algebras*, J. Algebra **261** (2003), no. 1, 172-185.

[BM06] D. Borisov, Y.I. Manin, *Intercohomomorphisms for operads*, preprint.

[Cha01] F. Chapoton, *Un endofoncteur de la catégorie des opérades*, in "Dialgebras and Related Operads", Lecture Notes in Mathematics **1763**, Springer-Verlag, 2001.

[CL01] F. Chapoton, M. Livernet, *Pre-Lie algebras and the rooted trees operad*, Internat. Res. Notices, **8**, (2001), 395-408.

[CV06] F. Chapoton, B. Vallette, *Pointed and multi-pointed partitions of type A and B*, J. Algebraic Combin., **23**, (2006), no. 4, 295-316.

[Coh76] F.R. Cohen, *The homology of* $\mathcal{C}_{n+1}$*-spaces,* $n \geq 0$, The homology of iterated loop spaces, Lecture Notes in Math., vol. 533, Springer-Verlag, 1976, pp. 207-351.

[Del93] P. Deligne, *Letter to Stasheff, Gerstenhaber, May, Schechtman, Drinfeld.*

[DL05] I. Dokas, J.-L. Loday, *On restricted Leibniz algebras*, preprint, `math.RA/0506473`

[EFG05] K. Ebrahimi-Fard, L. Guo, *On products and duality of binary quadratic regular operads*, J. Pure Appl. Algebra **200** (2005), no. 3, 293-317.

[Far79] F. D. Farmer, *Cellular homology for posets*, Math. Japon. **23** (1978/79), no. 6, 607-613.

[Fre04] B. Fresse, *Koszul duality of operads and homology of partition posets*, in "Homotopy theory : Relations with algebraic geometry, group cohomology and algebraic K-theory", Contemp. Math. **346**, AMS (2004).

[Gan03] W. L. Gan, *Koszul duality for dioperads*, Math. Res. Lett. **10** (2003), no. 1, 109-124.

[Ger63] M. Gerstenhaber, *The cohomology structure of an associative ring*, Ann. of Math. (2) **78** (1963), 267–288.

[Ger64] M. Gerstenhaber, *On the deformation of rings and algebras*, Ann. of Math. **79** (1964), 59-103.

[GV95] M. Gerstenhaber and A. A. Voronov, *Homotopy G-algebras and moduli space operad*, Internat. Math. Res. Notices (1995), no. 3, 141–153 (electronic).

[Get93] E. Getzler,*Cartan homotopy formulas and the Gauss-Manin connection in cyclic homology*, Israel Math. Conf. Proc. **7** (1993), 65-78.

[GJ94] E. Getzler, J.D.S. Jones, *Operads, homotopy algebra and iterated integrals for double loop spaces*, preprint, `arXiv:hep-th/9403055` (1994).

[GK95] V. Ginzburg, M.M. Kapranov, *Koszul duality for operads*, Duke Math. J. **76** (1995), 203-272.

[GK95'] V. Ginzburg, M.M. Kapranov, *Erratum to "Koszul duality for operads"*, Duke Math. J. **80** (1995), 293.

[Gne97] A. V. Gnedbaye, *Opérades des algèbres* $(k+1)$*-aires*, Operads: Proceedings of Renaissance Conferences (Hartford, CT/Luminy, 1995) (Providence, RI), Contemp. Math., vol. 202, Amer. Math. Soc., 1997, 83-113.

[GW00] A. V. Gnedbaye, M. Wambst, *Jordan triples and operads*, J. Algebra **231** (2000), no. 2, 744-757.

[Hoc45] G. Hochschild, *On the cohomology groups of an associative algebra*, Ann. of Math. (2) **46** (1963) no. 1, 58-67.

[JS93] A. Joyal, R. Street, *Braided tensor categories*, Advances in Math. **102** (1993), 20-78.

[Kad88] T. Kadeishvili, *the structure of* $A(\infty)$*-algebra and the Hochschild and Harrison cohomologies*, Trudy Tbiliss. Mat. Inst. Razmadze Akad. Nauk Gruzin. SSR **91** (1988), 19-27.

[Kap93] M.M. Kapranov, *The permutoassociahedron, Mac Lane's coherence theorem and asymptotic zones for the KZ equation*, J. Pure Appl. Algebra **85** (1993), no. 2, 119-142.

[Kon97] M. Kontsevich, *Deformation quantization of Poisson manifolds*, Preprint IHES, (1997).

[KS00] M. Kontsevich and Y. Soibelman, *Deformations of algebras over operads and the Deligne conjecture*, Conférence Moshé Flato 1999, Vol. I (Dijon), Math. Phys. Stud., vol. 21, Kluwer Acad. Publ., Dordrecht, 2000, pp. 255–307.

[LM05] T. Lada and M. Markl, *Symmetric brace algebras*, Appl. Categ. Structures **13** (2005), no. 4, 351–370.

[Le03] P. Leroux, *On some remarkable operads constructed from Baxter operators*, preprint, `math.QA/0311214`.

[Ler04] P. Leroux, *Ennea-algebras*, J. Algebra **281** (2004), no. 1, 287–302.

[Lod96] J.-L. Loday, *La renaissance des opérades*, Séminaire Bourbaki (Exp. No. 792), Astérisque **237** (1996), 47-74.

[Lod04] J.-L. Loday, *Completing the operadic butterfly*, preprint, `math.RA/0409183`.

[Lod01] J.-L. Loday, *Dialgebras*, in "Dialgebras and Related Operads", Lecture Notes in Mathematics **1763**, Springer-Verlag, 2001.

[LR04] J.-L. Loday and M. Ronco, *Trialgebras and families of polytopes*, Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory, Contemp. Math., vol. 346, Amer. Math. Soc., Providence, RI, 2004, pp. 369–398.

[LPWZ04] D.-M. Lu, J. H. Palmieri, Q.-S. Wu, J. J. Zhang, *Regular algebras of dimension 4 and their A-infinity Ext-algebras*, preprint, `math.RA/0411497`.

[MS02] J. E. McClure and J. H. Smith, *A solution of Deligne's Hochschild cohomology conjecture*, Recent progress in homotopy theory (Baltimore, MD, 2000), Contemp. Math., vol. 293, Amer. Math. Soc., Providence, RI, 2002, pp. 153–193.

[MacL98] S. Mac Lane, *Categories for the working mathematician (second edition)*, Graduate Texts in Mathematics **5**, Springer Verlag, 1998

[MM04] Anita Majumdar and Goutam Mukherjee, *Dialgebra cohomology as a G-algebra*, Trans. Amer. Math. Soc. **356** (2004), no. 6, 2443–2457 (electronic).

[Man87] Yu. I. Manin, *Some remarks on Koszul algebras and quantum groups*, Ann. Inst. Fourier **37** (1987), no. 4, 191-205.

[Man88] Yu. I. Manin, *Quantum groups and noncommutative geometry*, Université de Montréal, Centre de Recherches Mathématiques, Montréal, PQ, 1988.

[Mar96a] M. Markl, *Model for operads*, Comm. Algebra **24** (1996), no. 4, 1471-1500.

[Mar96b] M. Markl, *Cotangent cohomology of a category and defomrations*, J. Pure Appl. Algebra **113** (1996), 195-218.

[Mar05] M. Markl, *Cohomology operations and the deligne conjecture*, preprint, `math.AT/0506170`.

[Mar06] M. Markl, *Operads and Props*, preprint, `math.AT/0601129`, to appear in Handbook of Algebra.

[MSS02] M. Markl, S. Shnider, J. Stasheff, *Operads in algebra, topology and physics*, Mathematical Surveys and Monographs **96**, American Mathematical Society, 2002.

[MV03] M. Markl, A. A. Voronov, *PROPped up graph cohomology*, preprint `arXiv:math.QA/0307081` (2003).

[NR67] A. Nijenhuis and R. W. R. Jr., *Deformations of Lie algebra structures*, J. Math. Mech. **17** (1967), 89–105.

[OG04] J.-M. Oudom and D. Guin, *On the lie envelopping algebra of a pre-Lie algebra*, preprint, `arXiv:math.QA/0404457` (2004).

[PP05] A. Polishchuk, L. Positselski, *Quadratic algebras*, University Lecture Series 37, Providence, RI American Mathematical Society (AMS) xii, 159 p.

[Pri70] S. Priddy, *Koszul resolutions*, Trans. Amer. Math. Soc. **152** (1970), 39-60.

[Qui69] D. Quillen, *Rational homotopy theory*, Ann. of Math. (2) **90** (1969), 205-295.

[Qui70] D. Quillen, *On the (co-) homology of commutative rings*, Applications of Categorical Algebra (Proc. Sympos. Pure Math., Vol. XVII, New York, 1968), Amer. Math. Soc., Providence, R.I., 1970, pp. 65–87.

[Ser93] J.-P. Serre, *Gèbres*, Enseign. Math. (2) **39** (1993), no. 1-2, 33-85.

[Sta93] Jim Stasheff, *The intrinsic bracket on the deformation complex of an associative algebra*, J. Pure Appl. Algebra **89** (1993), no. 1-2, 231–235.

[Val1] B. Vallette, *A Koszul duality for props*, preprint, `math.AT/0411542`, to appear in Trans. Amer. Math. Soc..

[Val2] B. Vallette, *Homology of generalized partition posets*, preprint, `math.AT/0405312` to appear in J. of Pure Appl. Algebra.

[Val3] B. Vallette, *Free monoid in monoidal abelian categories*, preprint, `math.CT/0411543`.

[VdL] P. Van der Laan, *Koszul duality for colored operads*, Ph. D. Thesis.

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