Trigonometric dynamical r-matrices over Poisson Lie base*

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Dedicated to the memory of Joseph Donin

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Abstract

Let $\mathfrak{g}$ be a finite dimensional complex Lie algebra and $\mathfrak{l} \subset \mathfrak{g}$ a Lie subalgebra equipped with the structure of a factorizable quasitriangular Lie bialgebra. Consider the Lie group $\mathrm{Exp}\,\mathfrak{l}$ with the Semenov-Tjan-Shansky Poisson bracket as a Poisson Lie manifold for the double Lie bialgebra $\mathfrak{D}\mathfrak{l}$. Let $\mathcal{N}_{\mathfrak{l}}(0) \subset \mathfrak{l}$ be an open domain parameterizing a neighborhood of the identity in $\mathrm{Exp}\,\mathfrak{l}$ by the exponential map. We present dynamical r-matrices with values in $\mathfrak{g} \wedge \mathfrak{g}$ over the Poisson Lie base manifold $\mathcal{N}_{\mathfrak{l}}(0)$.

Key words: Dynamical Yang-Baxter equation, Poisson Lie groups, Poisson Lie manifolds.

1 Introduction

The classical dynamical Yang-Baxter equation (CDYBE) is a differential equation which generalizes the algebraic or ordinary classical Yang-Baxter equation (CYBE) associated with an arbitrary Lie algebra $\mathfrak{g}$. Contrary to the CYBE, a solution of CDYBE is a function of the so called dynamical parameter. Up to recently, the space of parameters was taken to be the dual $\mathfrak{l}^*$ of a Lie subalgebra $\mathfrak{l}$ in $\mathfrak{g}$, [BDF, F, ES, EV]. In [DM1, DM2], the DYBE was

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formulated for any quadruple $(\mathfrak{g}, \mathfrak{l}, \mathfrak{l}^*, L)$, where $\mathfrak{l} \subset \mathfrak{g}$ is a pair of Lie algebras, $(\mathfrak{l}, \mathfrak{l}^*)$ is a Lie bialgebra, and L is a special Poisson Lie (PL) manifold, the space of parameters.

A particular case of the PL version of the CDYBE for a quasitriangular Lie bialgebra $\mathfrak{l} = \mathfrak{g}$ and L being the group space $\mathrm{Exp}\,\mathfrak{g}$ appeared in [BFP, FM] in connection with a factorization problem in the chiral WZW model. In the present paper we give examples of dynamical r-matrices for the quadruple $(\mathfrak{g}, \mathfrak{l}, \mathfrak{l}^*, L)$, where $\mathfrak{g}$ is a complex Lie algebra and $(\mathfrak{l}, \mathfrak{l}^*)$ a factorizable quasitriangular Lie bialgebra. As a base manifold L we take a domain $\mathcal{N}_{\mathfrak{l}}(0)$ in $\mathfrak{l}$ parameterizing a neighborhood of the identity in the group $\mathrm{Exp}\,\mathfrak{l}$ equipped with the Semenov-Tjan-Shansky (STS) Poisson bracket, [S].

We apply the Etingof-Varchenko approach of base reduction, [EV], to the PL CDYBE. Then we employ, within the PL setting, the idea of Etingof-Schiffmann which is used for constructing generalized Alekseev-Meinrenken dynamical r-matrices in [ES]. In this way we obtain PL dynamical r-matrices that are in between the r-matrices of [BFP] and trigonometric r-matrices of [EV, S].

2 Dynamical Yang-Baxter equation on PL base manifold

Let $(\mathfrak{l}, \mathfrak{l}^*)$ be a finite dimensional Lie bialgebra and let δ denote the cobracket $\mathfrak{l} \to \mathfrak{l} \wedge \mathfrak{l}$, [Dr1]. Let $\mathfrak{D}\mathfrak{l} = \mathfrak{l} \bowtie \mathfrak{l}^*_{op}$ be the double Lie algebra with the canonical invariant symmetric tensor $\theta := \sum_i (\eta^i \otimes \xi_i + \xi_i \otimes \eta^i)$, where $\{\xi_i\} \subset \mathfrak{l}$ and $\{\eta^i\} \subset \mathfrak{l}^*$ are the dual bases. Recall from [DM1] that a $\mathfrak{D}\mathfrak{l}$-manifold is called an $\mathfrak{l}$-base manifold if the Casimir element θ generates the zero bidifferential operator via the action of $\mathfrak{D}\mathfrak{l}$. It follows that a base manifold is equipped with a $\mathfrak{D}\mathfrak{l}$-PL structure induced by the r-matrix $\sum_i \eta^i \otimes \xi_i \in (\mathfrak{D}\mathfrak{l})^{\otimes 2}$ via the action of $\mathfrak{D}\mathfrak{l}$.

In the present paper, by a function on L we understand an analytical or meromorphic function. Let $\mathfrak{g}$ be a Lie algebra containing $\mathfrak{l}$ as a subalgebra. Let us call a function $r\colon L \to \wedge^2\mathfrak{g}$ quasi-invariant if

$$\xi \triangleright r(\lambda) + [\xi \otimes 1 + 1 \otimes \xi, r(\lambda)] \;=\; -\delta(\xi), \quad \xi \in \mathfrak{l}, \tag{1}$$

where δ is the cobracket on $\mathfrak{l}$ and $\triangleright$ denotes the $\mathfrak{l}$-action on functions on L by vector fields. If the Lie bialgebra $\mathfrak{l}$ is coboundary, i.e. $\delta(\xi) := [\xi \otimes 1 + 1 \otimes \xi, r_{\mathfrak{l}}]$ for some $r_{\mathfrak{l}} \in \wedge^2\mathfrak{l}$, then a function $r\colon L \to \wedge^2\mathfrak{g}$ is quasi-invariant if and only if the function $r' := r + r_{\mathfrak{l}}$ is invariant, i.e.

$$\xi \triangleright r'(\lambda) + [\xi \otimes 1 + 1 \otimes \xi, r'(\lambda)] \;=\; 0, \quad \xi \in \mathfrak{l}. \tag{2}$$

Definition 2.1. A quasi-invariant function $r\colon L \to \wedge^2\mathfrak{g}$ is called a dynamical r-matrix over the base L if

$$\sum_i \mathrm{Alt}\big(\xi_i \otimes \nabla_{\eta^i} r(\lambda)\big) + \mathrm{CYB}\big(r(\lambda)\big) \;=\; Z, \qquad (3)$$

where ∇_η is the vector field on L generated by $\eta \in \mathfrak{l}^*_{op}$, $Z \subset (\wedge^3\mathfrak{g})^{\mathfrak{g}}$ is an invariant element, and

$$\begin{aligned} \mathrm{CYB}(A) &:= [A_{12}, A_{13}] + [A_{12}, A_{23}] + [A_{13}, A_{23}], \quad A \in \mathfrak{g} \otimes \mathfrak{g}, \\ \mathrm{Alt}(B) &:= B_{123} - B_{213} + B_{312}, \quad B \in \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}. \end{aligned}$$

Note that Definition 2.1 differs from the definition given in [DM1, DM2] by the change of sign $r \to -r$. Sometimes we will use the term PL dynamical r-matrix in order to distinguish it from the particular case of abelian $\mathfrak{l}^*$.

Remark 2.2. Suppose $(\mathfrak{l}, \mathfrak{l}^*)$ is the trivial bialgebra, with zero δ (abelian $\mathfrak{l}^*$). Then replacement Z by $\epsilon^2 Z$ results in the transformation $r(\lambda) \to \epsilon r(\epsilon\lambda)$ of the dynamical r-matrix, the solution to (3).

Remark 2.3. A particular case of equation (3) for $\mathfrak{l} = \mathfrak{g}$ and $L = \mathrm{Exp}\,\mathfrak{l}$ appeared in an equivalent form in the paper [BFP] devoted to the chiral WZW model. Motivated by equivariant deformation quantization on G-spaces, equation (3) was deduced in [DM1]. There, $\mathfrak{g}$ was assumed to be a Lie bialgebra containing $\mathfrak{l}$ as a sub-bialgebra. This assumption is relaxed in Definition 2.1, where $\mathfrak{g}$ is just a Lie algebra. The present definition of dynamical r-matrix was formulated in [DM2]. It was shown that such dynamical r-matrices define coboundary Lie bialgebroids of a special type.

We have the following Poisson characterization of dynamical r-matrices. Let $G = \mathrm{Exp}\,\mathfrak{g}$ be the connected simply connected Lie group corresponding to $\mathfrak{g}$. For $\zeta \in \mathfrak{g}$ let ζ^l and ζ^r denote the left and right invariant vector fields

$$\zeta^l f(g) = \frac{d}{dt} f(ge^{t\zeta})|_0, \quad \zeta^r f(g) = \frac{d}{dt} f(e^{t\zeta}g)|_0, \quad f \in C^\infty(G).$$

Consider the following two sided action of the Lie algebra $\mathfrak{l}$ on the Cartesian product $L\times G\times L$. The left action affects the first copy of L and the group space G (by the left invariant vector fields). The right action of $\mathfrak{l}$ is defined on the right copy of L via the principal anti-automorphism (antipode) and on G by the left invariant vector fields.

Let $r\colon L \to \mathfrak{g} \wedge \mathfrak{g}$ be an equivariant function. We define a bivector field π on $L \times G \times L$ as follows. When restricted to the left copy of L, π coincides with the Poisson structure on

L. When restricted to the right copy of L, π coincides with the opposite Poisson structure on L. Further, we set

$$\pi(f,a) \;:=\; \sum_i (\eta^i \triangleright f)\big(\xi_i^l + \xi_i^r\big)(a), \tag{4}$$

$$\pi(a,b) \;:=\; \big(-r^{l,l}(\lambda') + r^{r,r}(\lambda'')\big)(a,b) \tag{5}$$

for $f \in C^\infty(L\times L)$ and $a,b \in C^\infty(G)$; here $(\lambda',\lambda'') \in L\times L$. The superscripts l and r denote the left- and right-invariant vector fields.

We consider the direct sum $\mathfrak{l}\oplus\mathfrak{l}_{op}$ as a Lie bialgebra with the dual Lie algebra $\mathfrak{l}^*\oplus\mathfrak{l}^*$.

Proposition 2.4. *The bivector field π is an $\mathfrak{l}\oplus\mathfrak{l}_{op}$-Poisson Lie structure on $L\times G\times L$ if and only if r is a dynamical r-matrix.*

This assertion was stated in [DM1], Proposition 3.28, in a more restrictive setting, cf. Remark 2.3. A proof can be adopted from the proof of Theorem 9.6 from [DM1]. Note that we changed the sign of r here, as compared to [DM1, DM2]. Proposition 2.4 is analogous to Proposition 1.8 from [EEM]. Note that the definition of dynamical r-matrix from [DM1] (cf. Remark 2.3) goes over to the definition of [EEM] for the case $L = \mathrm{Exp}\,\mathfrak{l}$ if one subtracts from r its value at the identity of $\mathrm{Exp}\,\mathfrak{l}$.

3 Poisson Lie CDYBE on group manifold

Suppose that the Lie bialgebra $\mathfrak{l}$ is quasitriangular, i.e. coboundary with an r-matrix $r_\mathfrak{l} \in \wedge^2\mathfrak{l}$ and an element $\Omega_\mathfrak{l} \in (S^2\mathfrak{l})^\mathfrak{l}$ such that $\mathrm{CYB}(r_\mathfrak{l} + \frac{1}{2}\Omega_\mathfrak{l}) = 0$. Consider the connected simply connected Lie group $\mathrm{Exp}\,\mathfrak{l}$ corresponding to the Lie bialgebra $\mathfrak{l}$ and put $L = \mathrm{Exp}\,\mathfrak{l}$. According to [S], there is a PL structure on L considered as an $\mathrm{Exp}\,\mathfrak{l}$ -manifold with respect to conjugation. In fact, L is an $\mathfrak{l}$-base manifold, [DM1]. The action of double Lie algebra $\mathfrak{D}\mathfrak{l}$ on L is defined as follows. For $\xi\in\mathfrak{l}$ denote by ξ^l and ξ^r the vector fields acting by

$$\xi^l f(g) = \frac{d}{dt} f(ge^{t\xi})|_0, \quad \xi^r f(g) = \frac{d}{dt} f(e^{t\xi}g)|_0, \quad f\in C^\infty(L).$$

Then $\mathfrak{D}\mathfrak{l}$ acts on L by the vector fields

$$\xi^l - \xi^r \quad \text{for} \quad \xi\in\mathfrak{l} \quad \text{and} \quad r_\mathfrak{l}^r(\eta) - r_\mathfrak{l}^l(\eta) + \frac{1}{2}\big(\Omega_\mathfrak{l}^l(\eta) + \Omega_\mathfrak{l}^r(\eta)\big) \quad \text{for} \quad \eta\in\mathfrak{l}^*_{op}. \tag{6}$$

Here and further on the elements from $\mathfrak{l}\otimes\mathfrak{l}$ are considered as linear operators from $\mathfrak{l}^*$ to $\mathfrak{l}$ via pairing with the first tensor component.

From now on we assume the Lie bialgebra $(\mathfrak{l},\mathfrak{l}^*)$ to be factorizable. This means that the Casimir element $\Omega_\mathfrak{l}$ defines an isomorphism $\mathfrak{l}^*\to\mathfrak{l}$ of $\mathfrak{l}$-modules. Let $\{\xi_i\}\subset\mathfrak{l}$ be an orthonormal base with respect to the form $\Omega_\mathfrak{l}^{-1}$. Then the following proposition holds.

Proposition 3.1. *A quasi-invariant function $r\colon L \to \wedge^2\mathfrak{g}$ is a dynamical r-matrix if and only if the invariant function $r' := r + r_{\mathfrak{l}}$ satisfies the equation*

$$\sum_i \mathrm{Alt}\big(\xi_i \otimes \nabla'_{\xi_i} r'(\lambda)\big) + \mathrm{CYB}\big(r'(\lambda)\big) \;=\; Z - Z_l, \tag{7}$$

where $Z_{\mathfrak{l}} := \mathrm{CYB}(r_{\mathfrak{l}})$ and the vector field ∇'_ξ is defined to be $\nabla'_\xi := \frac{1}{2}(\xi^l + \xi^r)$ for $\xi \in \mathfrak{l}$.

Proof. The elements r' and $r_{\mathfrak{l}}$ satisfy the equality

$$\begin{aligned}\mathrm{CYB}(r' - r_{\mathfrak{l}}) &= \mathrm{CYB}(r') + \mathrm{CYB}(r_{\mathfrak{l}}) - \mathrm{Alt}\big([r_{\mathfrak{l}}^{12} + r_{\mathfrak{l}}^{13}, r'_{23}]\big)\\ &= \mathrm{CYB}(r') + \mathrm{CYB}(r_{\mathfrak{l}}) + \mathrm{Alt}\big(\xi_i \otimes r_{\mathfrak{l}}(\eta^i) \triangleright r'\big).\end{aligned} \tag{8}$$

Here we used invariance of r' with respect to the adjoint action of $\mathfrak{l}$. Now recall that the vector field $r_{\mathfrak{l}}(\eta^i)\triangleright$ on L equals $r_{\mathfrak{l}}(\eta^i)^l - r_{\mathfrak{l}}(\eta^i)^r$. It remains to substitute (8) into (3) and take into account the specific form of the $\mathfrak{l}^*_{op}$-vector fields on L given by (6). □

Remark 3.2. Equation (7) is more general than (3) since it makes sense for any self-dual Lie algebra $\mathfrak{l}$, not necessarily factorizable quasitriangular. It was derived in [BFP, FM] for the case $\mathfrak{l} = \mathfrak{g}$. A solution to this equation for $\mathfrak{l} = \mathfrak{g}$ (in fact, unique, up to a gauge equivalence) was found in [BFP].

4 Trigonometric dynamical r-matrix over abelian base

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{h}$ a Cartan subalgebra in $\mathfrak{g}$. Denote by $\Delta(\mathfrak{g})$ the root system of $\mathfrak{g}$ and by the $\Delta_+(\mathfrak{g})$ the subset of positive roots. Let $\{e_\alpha\}$, $\alpha \in \Delta(\mathfrak{g})$ be a base of root vectors from the root subspaces $\mathfrak{g}_\alpha$ normalized to $(e_\alpha, e_{-\alpha}) = 1$ for all $\alpha \in \Delta_+(\mathfrak{g})$ with respect to a non-degenerate invariant inner product on $\mathfrak{g}$. Let $\Omega_{\mathfrak{g}}$ denote the split-Casimir of this form,

$$\Omega_{\mathfrak{g}} := \sum_i x_i \otimes x_i + \sum_{\alpha\in\Delta(\mathfrak{g})} e_\alpha \otimes e_{-\alpha},$$

where $\{x_i\}$ is an orthonormal base of $\mathfrak{h}$. Recall from [EV], Theorem 3.1, that the meromorphic function $\mathfrak{h} \ni \lambda \mapsto \rho(\mathfrak{g}, \epsilon, \lambda) \in \wedge^2\mathfrak{g}$ defined by

$$\rho(\mathfrak{g}, \epsilon, \lambda) := \sum_{\alpha\in\Delta(\mathfrak{g})} \frac{\epsilon}{2}\coth\Big(\frac{\epsilon}{2}(\alpha, \lambda)\Big) e_\alpha \otimes e_{-\alpha} \tag{9}$$

is a solution to the DYBE with $Z = Z^\epsilon_{\Omega_{\mathfrak{g}}} := \frac{\epsilon^2}{4}[\Omega^{12}_{\mathfrak{g}}, \Omega^{23}_{\mathfrak{g}}]$. Here and further on we identify $\mathfrak{h}$ and $\mathfrak{h}^*$ via the restriction of the Killing form.

We may assume $\mathfrak{g}$ to be reductive and extend this solution to a meromorphic function on $\mathfrak{h}$ using projection along the center of $\mathfrak{g}$.

5 Base reduction in trigonometric case

It is shown in [EV] that the dynamical Yang-Baxter r-matrix admits a reduction of base when $\mathfrak{l}$ is a reductive Lie algebra with the trivial Lie bialgebra structure (abelian $\mathfrak{l}^*$). We will show that an analogous statement also holds if $\mathfrak{l}$ is a factorizable Lie bialgebra.

Let $\mathfrak{l}$ be a reductive subalgebra of a complex Lie algebra $\mathfrak{g}$. Suppose that $\mathfrak{l}$ is equipped with a factorizable quasitriangular Lie bialgebra structure. Denote by $r_\mathfrak{l} \subset \wedge^2\mathfrak{l}$ its classical r-matrix and by $\Omega_\mathfrak{l} \in (S^2\mathfrak{l})^\mathfrak{l}$ the corresponding Casimir element. Let $\mathcal{N}_\mathfrak{l}(0) \subset \mathfrak{l}$ be an open domain parameterizing a neighborhood of the identity in $\mathrm{Exp}\mathfrak{l}$ by the exponential map (we require the operator $1-\mathrm{ad}\xi$ to be non-degenerate for $\xi \in \mathcal{N}_\mathfrak{l}(0)$). We put the $\mathfrak{l}$-base manifold L to be $\mathcal{N}_\mathfrak{l}(0) \subset \mathfrak{l}$ equipped with the STS Poisson bracket. Let $\mathfrak{h}_\mathfrak{l}$ denote the Cartan subalgebra of $\mathfrak{l}$. The open domain $\mathcal{N}_{\mathfrak{h}_\mathfrak{l}}(0) := \mathcal{N}_\mathfrak{l}(0) \cap \mathfrak{h}_\mathfrak{l} \subset \mathfrak{h}_\mathfrak{l}$ is a base manifold for $\mathfrak{h}_\mathfrak{l}$.

Theorem 5.1. *A quasi-invariant function $r\colon \mathcal{N}_\mathfrak{l}(0) \to \wedge^2\mathfrak{g}$ is a dynamical r-matrix over $\mathcal{N}_\mathfrak{l}(0)$ if and only if the function $\tilde r(\lambda) := r|_{\mathfrak{h}_\mathfrak{l}}(\lambda)+\rho(\lambda)+r_\mathfrak{l}$, where $\rho(\lambda)=\rho(\mathfrak{g},1,\lambda)$, $\lambda \in \mathcal{N}_{\mathfrak{h}_\mathfrak{l}}(0)$, is a dynamical r-matrix over the abelian base $\mathfrak{h}_\mathfrak{l}$.*

Proof. Equation (3) on the quasi-invariant function r is equivalent to equation (7) on the invariant function $r' := r+r_\mathfrak{l}$. We assume in (7) that the orthonormal base $\{\xi_k\}$ is compatible with the root decomposition. Equation (7) gives rise to the following equation for the restriction of r' to $\mathfrak{h}_\mathfrak{l}$:

$$\sum_{i=1}^{\mathrm{rk}\,\mathfrak{l}} \mathrm{Alt}\Bigl(x_i \otimes \nabla'_{x_i} r'(\lambda)\Bigr) + \sum_{\alpha\in\Delta(\mathfrak{l})} \mathrm{Alt}\Bigl(e_\alpha \otimes \nabla'_{e_{-\alpha}} r'(\lambda)\Bigr) + \mathrm{CYB}\bigl(r'(\lambda)\bigr) = Z_\mathfrak{g} - \mathrm{CYB}(r_\mathfrak{l}) \quad (10)$$

where $\lambda \in \mathfrak{h}_\mathfrak{l}$. For all $\xi \in \mathfrak{l}$ the vector fields ∇'_ζ, on $\mathcal{N}_\mathfrak{l}(0)$ read

$$\nabla'_\xi f(\lambda) = \partial f(\lambda)(\frac{1}{2}\mathrm{ad}\lambda)\coth\bigl(\frac{1}{2}\mathrm{ad}\lambda\bigr)\xi,$$

where ∂f denotes the differential of a function f. On the other hand, $\xi \triangleright f(\lambda) = \partial f(\lambda)(\mathrm{ad}\lambda)\xi$. This implies, by $\mathfrak{l}$-invariance of r',

$$e_\alpha \otimes \nabla'_{e_{-\alpha}} r'(\lambda) = \frac{1}{2}\coth\bigl(\frac{1}{2}(\alpha,\lambda)\bigr)e_\alpha \otimes [e_{-\alpha}\otimes 1 + 1 \otimes e_{-\alpha}, r'(\lambda)], \quad \alpha \in \Delta(\mathfrak{l}), \quad \lambda \in \mathfrak{h}_\mathfrak{l}.$$

Observe that $\nabla'_{x_i} = \partial_i := \frac{\partial}{\partial x_i}$ at $\lambda \in \mathfrak{h}_\mathfrak{l}$. So equation (10) can be rewritten as

$$\sum_{i=1}^{\mathrm{rk}\,\mathfrak{l}} \mathrm{Alt}\Bigl(x_i \otimes \partial_i r'(\lambda)\Bigr) + \mathrm{Alt}\bigl([\rho_{12}+\rho_{13}, r'_{23}]\bigr) + \mathrm{CYB}\bigl(r'(\lambda)\bigr) = Z_\mathfrak{g} - \mathrm{CYB}(r_\mathfrak{l})$$

or, upon the substitution $r' = \tilde r - \rho$, as

$$\sum_{i=1}^{\mathrm{rk}\,\mathfrak{l}} \mathrm{Alt}\Bigl(x_i \otimes \partial_i \tilde r(\lambda)\Bigr) - \sum_{i=1}^{\mathrm{rk}\,\mathfrak{l}} \mathrm{Alt}\Bigl(x_i \otimes \partial_i \rho(\lambda)\Bigr) + {} + \mathrm{Alt}\bigl([\rho_{12}+\rho_{13}, \tilde r_{23}-\rho_{23}]\bigr) + \mathrm{CYB}(\tilde r - \rho) = Z_\mathfrak{g} - \mathrm{CYB}(r_\mathfrak{l}). \quad (11)$$

Since ρ is skew, we find $\mathrm{CYB}(\tilde{r}-\rho)$ to be equal to

$$\mathrm{CYB}(\tilde{r})+\mathrm{CYB}(\rho)-([\tilde{r}_{12},\rho_{13}]+[\tilde{r}_{12},\rho_{23}]+[\tilde{r}_{13},\rho_{23}]+[\rho_{12},\tilde{r}_{13}]+[\rho_{12},\tilde{r}_{23}]+[\rho_{13},\tilde{r}_{23}])$$
$$=\mathrm{CYB}(\tilde{r})+\mathrm{CYB}(\rho)-\mathrm{Alt}\big([\rho_{12}+\rho_{13},\tilde{r}_{23}]\big).$$

Also, it easy to see that $\mathrm{Alt}\big([\rho_{12}+\rho_{13},\rho_{23}]\big)=2\mathrm{CYB}(\rho)$. Taking this into account, we rewrite (11) as

$$\sum_{i=1}^{\mathrm{rk}\,\mathfrak{l}}\mathrm{Alt}\Big(x_i\otimes\partial_i\tilde{r}(\lambda)\Big)+\mathrm{CYB}(\tilde{r})-\sum_{i=1}^{\mathrm{rk}\,\mathfrak{l}}\mathrm{Alt}\Big(x_i\otimes\partial_i\rho(\lambda)\Big)-\mathrm{CYB}\big(\rho(\lambda)\big)=Z_{\mathfrak{g}}-\mathrm{CYB}(r_{\mathfrak{l}}).$$

We have $\mathrm{CYB}(r_{\mathfrak{l}})=-\mathrm{CYB}(\Omega_{\mathfrak{l}})=\frac{1}{4}[\Omega_{\mathfrak{l}}^{12},\Omega_{\mathfrak{l}}^{23}]=Z_{\mathfrak{l}}^{\epsilon=1}$. Since ρ solves the CDYBE for $\epsilon=1$, this equation reduces to the CDYBE for the function $\tilde{r}(\lambda)=r|_{\mathfrak{h}_{\mathfrak{l}}}(\lambda)+\rho(\lambda)+r_{\mathfrak{l}}$, were λ varies in $\mathcal{N}_{\mathfrak{h}_{\mathfrak{l}}}(0)$.

Conversely, suppose the function $\tilde{r}(\lambda)$ satisfies the CDYBE over base $\mathcal{N}_{\mathfrak{h}_{\mathfrak{l}}}(0)$. Then the function $r':=\tilde{r}|_{\mathcal{N}_{\mathfrak{h}_{\mathfrak{l}}}(0)}-\rho$ satisfies equation (7). Observe that equation (7) is $\mathfrak{l}$-invariant and, by assumption, r' is the restriction of an invariant function defined on $\mathcal{N}_{\mathfrak{l}}(0)$. Therefore the function r' solves (7) when restricted to semisimple elements from $\mathcal{N}_{\mathfrak{l}}(0)$ and hence everywhere in $\mathcal{N}_{\mathfrak{l}}(0)$ since semisimple elements are dense in $\mathfrak{l}$. □

Theorem 5.1 allows to construct a trigonometric dynamical r-matrix over a Levi subalgebra in a complex simple Lie algebra. Recall once again that we use identification $\mathfrak{g}\otimes\mathfrak{g}\simeq\mathfrak{g}^*\otimes\mathfrak{g}\simeq\mathrm{End}(\mathfrak{g})$ via the invariant pairing on $\mathfrak{g}$.

Corollary 5.2. *Let $\mathfrak{l}$ be a Levi subalgebra in a complex simple Lie algebra $\mathfrak{g}$ and let $\mathfrak{l}^{\perp}$ be the orthogonal complement of $\mathfrak{l}$ in $\mathfrak{g}$ with respect to the Killing form. The meromorphic function $r\colon\mathcal{N}_{\mathfrak{l}}(0)\ni\lambda\mapsto f_{\mathfrak{l}}(\mathrm{ad}\lambda)\oplus f_{\mathfrak{l}^{\perp}}(\mathrm{ad}\lambda)\in\mathrm{End}(\mathfrak{l})\oplus\mathrm{End}(\mathfrak{l}^{\perp})\subset\mathrm{End}(\mathfrak{g})$ with*

$$\begin{aligned} f_{\mathfrak{l}}(\lambda) &= \frac{1}{2}\coth\big(\frac{1}{2}\mathrm{ad}\lambda\big)-\frac{\epsilon}{2}\coth\big(\frac{\epsilon}{2}\mathrm{ad}\lambda\big)-r_{\mathfrak{l}},\\ f_{\mathfrak{l}^{\perp}}(\lambda) &= -\frac{\epsilon}{2}\coth\big(\frac{\epsilon}{2}\mathrm{ad}\lambda\big). \end{aligned}$$

is a PL dynamical r-matrix for $Z_{\mathfrak{g}}^{\epsilon}=\frac{\epsilon^2}{4}[\Omega_{\mathfrak{g}}^{12},\Omega_{\mathfrak{g}}^{23}]$.

Proof. Since $\mathfrak{g}$ is simple, the restriction of the matrix $\mathrm{ad}\lambda\in\mathrm{End}(\mathfrak{g})$ to the invariant subspace $\mathfrak{l}^{\perp}$ is invertible for λ belonging to a dense open subset in $\mathcal{N}_{\mathfrak{l}}(0)$. Hence $\lambda\mapsto r(\lambda)$ is a correctly defined meromorphic function on $\mathcal{N}_{\mathfrak{l}}(0)$. The function $r+r_{\mathfrak{l}}$ is $\mathfrak{l}$-invariant and $r|_{\mathfrak{h}_{\mathfrak{l}}}+r_{\mathfrak{l}}+\rho(\mathfrak{l},1,\lambda)=\rho(\mathfrak{g},\epsilon,\lambda)$ for $\lambda\in\mathfrak{h}_{\mathfrak{l}}$. □

6 Generalized Balog-Fehér-Palla dynamical r-matrices

Let $\mathfrak{g}$ be a finite dimensional complex Lie algebra and $B\colon \mathfrak{g} \to \mathfrak{g}$ an automorphism of order n. Then $\mathfrak{g} = \oplus_{j\in\mathbb{Z}/n\mathbb{Z}}\, \mathfrak{g}_j$, where $\mathfrak{g}_j := \ker(B - e^{2i\pi j/n})$. The Lie algebra $\mathfrak{g}_0$ acts on $\mathfrak{g}_j$ for all j.

Let $\mathfrak{g}$ be equipped with an ad- and B-invariant form; denote by $\Omega_\mathfrak{g} \in (S^2\mathfrak{g})^\mathfrak{g}$ the corresponding Casimir element. According to [ES], there exists a dynamical r-matrix over the base $\mathfrak{l}^*$ for the trivial Lie bialgebra $\mathfrak{l}$ (abelian $\mathfrak{l}^*$). Under the identification $\mathfrak{g}\otimes\mathfrak{g} \simeq \mathfrak{g}^*\otimes\mathfrak{g} \simeq \mathrm{End}(\mathfrak{g})$, it is given by the invariant function $\rho\colon \mathcal{N}_\mathfrak{l}(0) \to \mathrm{End}(\mathfrak{g})$, $\rho(A)|_{\mathfrak{g}_j} = f_j(\mathrm{ad}A)$, with

$$f_0(s) = \frac{1}{s} - \frac{1}{2}\coth\bigl(\frac{1}{2}s\bigr), \tag{12}$$
$$f_j(s) = -\frac{1}{2}\coth\bigl(\frac{1}{2}s + \frac{i\pi j}{n}\bigr), \quad j \neq 0. \tag{13}$$

This solution corresponds to $Z_\mathfrak{g} = \frac{1}{4}[\Omega_\mathfrak{g}^{12}, \Omega_\mathfrak{g}^{23}]$. For $B = \mathrm{id}$, formulas (12-13) give the rational-trigonometric dynamical r-matrix of Alekseev-Meinrenken, [AM].

The trigonometric analog of this construction is as follows. Consider the analytical function $r'_{trig}\colon \mathcal{N}_\mathfrak{l}(0) \to \mathrm{End}(\mathfrak{g})$, $r'_{trig}(A)|_{\mathfrak{g}_j} = f_j^{trig}(\mathrm{ad}A)$, with

$$f_0^{trig}(s) = \frac{1}{2}\coth\bigl(\frac{1}{2}s\bigr) - \frac{\epsilon}{2}\coth\bigl(\frac{\epsilon}{2}s\bigr), \tag{14}$$
$$f_j^{trig}(s) = -\frac{\epsilon}{2}\coth\bigl(\frac{\epsilon}{2}s + \frac{i\pi j}{n}\bigr), \quad j \neq 0. \tag{15}$$

Let $\Omega_\mathfrak{l}$ denote the Casimir element of the restriction of the invariant form to $\mathfrak{l}$.

Theorem 6.1. *The equivariant function r'_{trig} satisfies equation (7) for $Z^\epsilon_\mathfrak{g} = \frac{\epsilon^2}{4}[\Omega_\mathfrak{g}^{12}, \Omega_\mathfrak{g}^{23}]$ and $Z_\mathfrak{l} = \frac{1}{4}[\Omega_\mathfrak{l}^{12}, \Omega_\mathfrak{l}^{23}]$.*

Proof. The proof is an appropriate modification of the proof of Theorem A.1 of [ES]. At the first step one considers $\mathfrak{g} = \mathfrak{l}\oplus\ldots\oplus\mathfrak{l}$, the direct sum of n-copies of a reductive Lie algebra $\mathfrak{l}$ and B the cyclic permutation of these copies. One proves that the function $r'_{trig}|_{\mathfrak{h}_l} + \rho$, where $\rho(\lambda) := \rho(\mathfrak{l}, 1, \lambda)$ from (9), is a dynamical r-matrix over the $\mathfrak{h}_l$ for $Z = Z^\epsilon_\mathfrak{g}$. At $\epsilon = 1$, this function coincides with the corresponding function from [ES], see the proof of Proposition A.1; for $\epsilon \neq 1$ cf. Remark 2.2. By Theorem 5.1, r' solves equation (7) for these specific $\mathfrak{g}$ and B. The case of general $\mathfrak{g}$ is derived from this one similarly to [ES]. Namely, define a map $W\colon \mathcal{N}_\mathfrak{l}(0) \to \wedge^3\mathfrak{g}$ setting

$$W(A) = \mathrm{Alt}\bigl(\nabla' r'_{trig}(A)\bigr) + \mathrm{CYB}\bigl(r'_{trig}(A)\bigr) - Z^\epsilon_\mathfrak{g} + \frac{1}{4}[\Omega_\mathfrak{l}^{12}, \Omega_\mathfrak{l}^{13}].$$

This is an analog of the function W from Appendix of [ES]. Then one defines $K_{ij}(A, X, Y) := \bigl(\mathrm{id}\otimes X\otimes Y, W(A)\bigr) \in \mathfrak{g}_{i+j}$ for all $i, j \in \mathbb{Z}/n\mathbb{Z}$, $A \in \mathfrak{l}$, $X \in \mathfrak{g}_i$ and $Y \in \mathfrak{g}_j$, which are expressed by universal Lie series in A, X, and Y for each pair i, j. For $K_{ij}(A, X, Y)$ one can prove the same assertions as Proposition A.2 and Lemma A.2 of [ES] and find that they vanish for all $A \in \mathfrak{l}$, $X \in \mathfrak{g}_i$, and $Y_j \in \mathfrak{g}_j$. Thus $W(A) = 0$ and this proves the theorem. $\square$

Corollary 6.2. *Suppose $\mathfrak{l} = \ker B$ has a quasitriangular structure $(r_{\mathfrak{l}}, \Omega_{\mathfrak{l}})$. Then $r_{trig} = r'_{trig} - r_{\mathfrak{l}}$ is a PL dynamical r-matrix.*

For $B = \mathrm{id}$, formulas (14-15) give the trigonometric PL dynamical r-matrix of [BFP].

7 Concluding remarks

In conclusion, we make some comments on the dynamical r-matrices over general base.

7.1. Let us call an $\mathfrak{l}$-base manifold homogeneous, if it is $\mathfrak{D}\mathfrak{l}$-homogeneous. The following proposition gives a local description of homogeneous base manifolds.

Proposition 7.1. *Local homogeneous $\mathfrak{l}$-base manifolds are in one-to-one correspondence with Lie subalgebras $\mathfrak{k} \subset \mathfrak{D}\mathfrak{l}$ containing their orthogonal complement $\mathfrak{k}^{\perp}$ with respect to the canonical inner product in $\mathfrak{D}\mathfrak{l}$. Over the complex field, such Lie subalgebras are exactly those containing a lagrangian (maximal isotropic) subspace.*

This proposition implies that base manifolds are quite numerous. The simplest examples are $\mathfrak{k} = \mathfrak{l}$ and $\mathfrak{k} = \mathfrak{l}^*$, which are themselves lagrangian.

All the examples of dynamical r-matrices known (including those constructed in the present paper) correspond to $\mathfrak{k} \simeq \mathfrak{l}$. It is interesting to find dynamical r-matrices over other bases.

7.2. The homogeneous base manifolds corresponding to $\mathfrak{k} \supset \mathfrak{l}$ are alike, to a certain extent. Each point of such a manifold has a neighborhood with a normal system of coordinates $\{\lambda_i\}$, where $\mathfrak{l}$ acts by linear (coadjoint) transformations. The local ring of formal functions on a normal neighborhood of a point admits a decreasing $\mathfrak{l}$-invariant filtration by powers of the maximal ideal. The dual Lie algebra acts on the normal neighborhood by vector fields $\xi^i \mapsto \partial_{\lambda_i} +$ terms of higher degrees. All this makes such base manifolds close to $\mathfrak{l}^*$ for the trivial Lie bialgebra $\mathfrak{l}$ (abelian $\mathfrak{l}^*$). It is possible to develop a classification theory of dynamical r-matrices over such base manifolds along the line of [EV, ES].

7.3. In [Dr2], Drinfeld established a correspondence between local homogeneous Poisson Lie $\mathfrak{g}$-manifolds with lagrangian Lie algebras $\mathfrak{f}$ in $\mathfrak{D}\mathfrak{g}$. The same objects describe Lie quasi-bialgebras, the infinitesimal deformations of $\mathcal{U}(\mathfrak{f})$ in the class of quasi-Hopf algebras, [Dr3]. Drinfeld set up a problem of finding relations between quasi-Hopf algebras and quantized

Poisson Lie manifolds. A partial answer to this problem is achieved in the framework of dynamical Yang-Baxter equation. The covered cases correspond to $\mathfrak{f} \simeq \mathfrak{g}$.

Let $\mathfrak{g}$ be a Lie bialgebra and $\mathfrak{l}$ its Lie sub-bialgebra. Let $\mathcal{U}_\hbar(\mathfrak{g})$ and $\mathcal{U}_\hbar(\mathfrak{l})$ be quantized universal enveloping algebras with $\mathcal{U}_\hbar(\mathfrak{l}) \hookrightarrow \mathcal{U}_\hbar(\mathfrak{g})$. One may admit $\mathfrak{g}$ or $\mathfrak{l}$ to be trivial Lie bialgebras; then $\mathcal{U}_\hbar(\mathfrak{g})$ or $\mathcal{U}_\hbar(\mathfrak{l})$ would be the ordinary universal enveloping algebras extended by $\mathbb{C}[[\hbar]]$. Let $\mathcal{L}$ be a base algebra over $\mathcal{U}_\hbar(\mathfrak{l})$, [DM1]. It plays a role of a function algebra on a quantized base manifold. Given a dynamical twist $\mathcal{F} = 1 \otimes 1 \otimes 1 + O(\hbar)$ with values in $\mathcal{U}_\hbar(\mathfrak{g}) \otimes \mathcal{U}_\hbar(\mathfrak{g}) \otimes \mathcal{L}$ and a $\mathcal{U}_\hbar(\mathfrak{l})$-invariant character λ of $\mathcal{L}$ one can reduce $\mathcal{F}$ to $\mathcal{F}_\lambda \in \mathcal{U}_\hbar(\mathfrak{g}) \otimes \mathcal{U}_\hbar(\mathfrak{g})$ and interpret it as a quasi-Hopf algebra twist of $\mathcal{U}_\hbar(\mathfrak{g})$.

The dual algebra $\mathcal{U}^*_\hbar(\mathfrak{g})$ is a quantum analog of functions on the group. It is a two sided module over $\mathcal{U}_\hbar(\mathfrak{g})$. Apply the quasi-Hopf algebra twist to $\mathcal{U}^*_\hbar(\mathfrak{g})$ from the left. The result will be a quasi-associative algebra. The deviation from associativity is controlled by an associator canonically constructed out of $\mathcal{F}_\lambda$. This associator reduces to identity on the trivial $\mathcal{U}_\hbar(\mathfrak{l})$-submodule in $\mathcal{U}^*_\hbar(\mathfrak{g})$. So the subspace of $\mathcal{U}_\hbar(\mathfrak{l})$-invariants in $\mathcal{U}^*_\hbar(\mathfrak{g})$ becomes an associative algebra. This algebra preserves the right $\mathcal{U}_\hbar(\mathfrak{g})$-action intact. In this way it is an equivariant quantization of the Poisson Lie homogeneous $\mathfrak{g}$-manifold with the stabilizer $\mathfrak{l}$.

7.4. The dynamical twists have been constructed for all pairs $(\mathfrak{g}, \mathfrak{l})$ where $\mathfrak{g}$ is simple Lie algebra and $\mathfrak{l}$ is a Levi subalgebra in $\mathfrak{g}$, [DM1, EE, EEM]. The choice of Levi subalgebras among other reductive Lie subalgebras is conditioned by the following. For Levi subalgebras, there exists a powerful computational tool, the generalized Verma modules. Categorically, the construction works as follows. Parabolic induction establishes an additive isomorphism between two module categories over the monoidal category of $\mathcal{U}_\hbar(\mathfrak{g})$-modules. In [DM1] this isomorphism is called dynamical adjoint functor (to the restriction functor from $\mathcal{U}_\hbar(\mathfrak{g})$-modules to $\mathcal{U}_\hbar(\mathfrak{l})$-modules). A dynamical adjoint functor induces a new multiplication on morphisms in the dynamical extension of the category of $\mathcal{U}_\hbar(\mathfrak{g})$-modules over $\mathcal{U}_\hbar(\mathfrak{l})$-modules. This new multiplication is equivalent to a dynamical twist.

We believe that reductive Lie subalgebras other than Levi can be treated using more sophisticated modules than the generalized Verma modules.

7.5. We conclude that for a good deal of important cases the Drinfeld problem has been solved within the theory of dynamical Yang-Baxter equation. The natural question is about limitations of this approach. This concerns both quantum and classical setting. First of all, what is the exact correspondence between dynamical r-matrices over Poisson Lie bases and Poisson Lie manifolds in Drinfeld's description?

References

[AM] A. Alekseev, E. Meinrenken: *The non-commutative Weil algebra*, Invent. Math. **135** (2000) 135–172.

[BDF] J. Balog, L. Dabrowski, and L. Fehér: *Classical r-matrix and exchange algebra in WZNW and Toda field theories*, Phys. Lett. **B 244** (1990) 227–234.

[BFP] J. Balog, L. Fehér, and L. Palla: *The chiral WZNW phase space and its Poisson-Lie groupoid*, Phys. Lett. **B 463** (1999) 83–92.

[Dr1] V. Drinfeld: *Quantum Groups*, in Proc. Int. Congress of Mathematicians, Berkeley, 1986, ed. A. V. Gleason, AMS, Providence (1987) 798–820.

[Dr2] V. Drinfeld: *On Poisson homogeneous spaces of Poisson-Lie groups*, Theor. Math. Phys. **95** (1993) #2, 524–525.

[Dr3] V. Drinfeld: *Quasi-Hopf algebras*, Leningrad Math. J. **1** (1990) # 6, 1419–1457.

[DM1] J. Donin, A. Mudrov: *Dynamical Yang-Baxter equation and quantum vector bundles*, accepted to Commun. Math. Phys., arXiv: math.QA/0306028.

[DM2] J. Donin, A. Mudrov: *Quantum groupoids associated with dynamical categories*, math.QA/0311116.

[EE] B. Enriquez, P. Etingof: Quantization of classical dynamical r-matrices withnonabelian base, arXiv:math.QA/0311224.

[EEM] B. Enriquez, P. Etingof, I. Marshall: *Quantization of some Poisson-Lie dynamical r-matrices and Poisson homogeneous spaces*, arXiv: math.QA/0403283.

[ES] P. Etingof, O. Schiffmann: *On the moduli space of classical dynamical r-matrices*, Math. Res. Lett. **8** (2001) 157–170.

[EV] P. Etingof, A. Varchenko: *Geometry and classification of solutions to the classical dynamical Yang-Baxter equation*, Commun. Math. Phys. **192** (1998) 77–120.

[F] G. Felder: *Conformal field theories and integrable models associated to elliptic curves*, Proc. ICM Zurich, (1994) 1247–1255.

[FM] L. Fehér and I. Marshall: *On a Poisson-Lie analogue of the classical dynamical Yang-Baxter equation for self dual Lie algebras*, Lett. Math. Phys. **62** (2002) 51–62.

[S] M. Semenov-Tian-Shansky: *Poisson-Lie Groups, Quantum Duality Principle, and the Quantum Double*, Contemp. Math. **175** (1994) 219–248.