COMPLETE T^n-INVARIANT HYPERKÄHLER $4n$-MANIFOLDS

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Abstract. We prove the following classification results: **1.** A connected simply connected complete hyperkähler $4n$-manifold of finite topological type equipped with an effective tri-Hamiltonian action of T^n is isometric (in fact isomorphic as a tri-Hamiltonian T^n-manifold) to a hyperkähler quotient of a flat $\mathbb{H}^{d+m}$ by $T^{d-n}\times\mathbb{R}^m$. **2.** A compact connected and simply connected 3-Sasakian manifold of dimension $4n-1$ with an effective action of T^n by 3-Sasakian isometries is isometric to a 3-Sasakian quotient of a sphere by a torus. **3.** A compact connected quaternion-Kähler $4n$-manifold with positive scalar curvature and an effective isometric action of T^{n+1} is isometric to $\mathbb{H}P^n$ or $Gr_2(\mathbb{C}^{n+2})$.

Hyperkähler metrics in dimension $4n$ with n commuting tri-holomorphic Killing vector fields form a large class of explicitly known Ricci-flat metrics. The flat $\mathbb{H}^n$, the Taub-NUT, and the Eguchi-Hanson metric are of this form. So are the multi-Eguchi-Hanson metrics (a.k.a. ALE-spaces of type A_k) and their Taub-NUT-like deformations, both due to Gibbons and Hawking. Higher dimensional examples include the Calabi metrics on $T^*\mathbb{C}P^n$ and the asymptotic metrics on moduli spaces of magnetic monopoles.

A powerful method of constructing such metrics was given by Lindström and Roček [19] (see also [15, 20]). It associates a hyperkähler metric having n commuting tri-Hamiltonian (meaning Hamiltonian for all three Kähler forms) Killing vector fields to every real-valued function on an open subset of $\mathbb{R}^3\otimes\mathbb{R}^n$ which is *polyharmonic*, i.e. harmonic on any affine 3-dimensional subspace of the form $a+\mathbb{R}^3\otimes\mathbb{R}v$, $a\in\mathbb{R}^3\otimes\mathbb{R}^n$, $v\in\mathbb{R}^n$. This construction gives locally *all* hyperkähler metric with n commuting tri-Hamiltonian Killing vector fields. On the other hand, the global properties of the resulting metrics - in particular completeness - have not been investigated.

There is another construction - that of hyperkähler quotient - which, while in this case more restrictive, has the advantage that the global properties of the resulting manifold are accessible. In particular, a hyperkähler quotient of a complete manifold by a compact Lie group is complete. A basic example is the hyperkähler quotient of $\mathbb{H}\times(S^1\times\mathbb{R}^3)$ by the diagonal circle action which yields the Taub-NUT metric. More generally, we obtain complete hyperkähler $4n$-manifolds with a tri-Hamiltonian (hence isometric) T^n-action by taking hyperkähler quotients of quaternionic vector spaces by tori. This class of manifolds has been investigated in detail in [7]. They are closely related to projective toric varieties. In particular, they are diffeomorphic to a union of cotangent bundles of finitely many such varieties glued together.

There is an interesting relation to symplectic geometry and Delzant's theorem [11] which states that a compact symplectic $2n$-manifold with a Hamiltonian T^n-action is completely determined, as a Hamiltonian T^n-manifold, by the image of the moment map. Hyperkähler $4n$-manifolds with a tri-Hamiltonian T^n-action can be thought of as the quaternionic analogue of that situation. Since now we cannot have compactness, we impose the next best thing: completeness of the metric.

By the above-mentioned construction of Lindström and Roček there are plenty of hyperkähler manifolds with a tri-Hamiltonian T^n-action. We shall see that the situation is very different if we require the manifold to be complete (and Riemannian). We first make a definition.

Definition 1. Let M, M' be two hyperkähler manifolds with tri-Hamiltonian actions of T^n and let μ, μ' be the chosen moment maps. We say that M and M' are isomorphic as tri-Hamiltonian hyperkähler T^n-manifolds, if there is a triholomorphic T^n-equivariant isometry $f : M \to M'$ such that $\mu = \mu' \circ f$.

The main result of this paper is the following:

Theorem 1. *Let M^{4n} be a connected complete hyperkähler manifold of finite topological type with an effective tri-Hamiltonian T^n-action. Then:*

(i) *If M is simply connected, then M is isomorphic, as a tri-Hamiltonian hyperkähler T^n-manifold, to a hyperkähler quotient of some flat $\mathbb{H}^d \times \mathbb{H}^m$, $m \leq n$, by $T^{d-n} \times \mathbb{R}^m$.*

(ii) *If M is not simply connected, then M is the product of a flat $\left(S^1 \times \mathbb{R}^3\right)^l$, $1 \leq l \leq n$, and a $4(n-l)$-dimensional manifold described in part (i).*

Remarks . 1. By a finite topological type we mean that the total Betti number is finite. There are examples [2, 13] of complete T^n-invariant hyperkähler manifolds not satisfying this condition.
2. In part (i), T^{d-n} acts on $\mathbb{H}^d$ as a subtorus of the diagonal maximal torus T^d in $Sp(d)$, while $\mathbb{R}^m$ acts on $\mathbb{H}^m \simeq \left(\mathbb{R}^m\right)^4$ by translations on the first factor and on $\mathbb{H}^d$ via an injective linear map $\rho : \mathbb{R}^m \to \mathrm{Lie}\,(T^d) = \mathbb{R}^d$.
3. Any *tri-symplectic* T^n-action on a simply-connected M is tri-Hamiltonian.
4. In part (i), we can assume that the level set of the moment map used to obtain M from $\mathbb{H}^{d+m}$ is smooth (this condition is always assumed in [7] and by no means follows from the smoothness of M - see [9] for the 3-Sasakian case).

Corollary 2. *Let M be a simply connected 4-dimensional complete hyperkähler manifold with an effective tri-symplectic action of a circle. If $b_2(M) = k$, then M is isomorphic either to an ALE-space of type A_k (i.e. a multi-Eguchi-Hanson space) or to its Taub-NUT-like deformation (i.e. to the hyperkähler quotient by $\mathbb{R}$ of the product of such a space with $\mathbb{H}$).*

We emphasize that, for $n > 1$, the n-dimensional toral hyperkähler quotients of $\mathbb{H}^d$ are also quite well understood [7]. We know their integer cohomology, their homotopy type, generic complex structure and we have explicit formulas for the Kähler potential and the metric.

The previous corollary suggests the following definition.

Definition 2. Let M^{4n} be as in Theorem 1. A Taub-NUT deformation (of order m) of M is the hyperkähler quotient of $M \times \mathbb{H}^m$ by $\mathbb{R}^m$ where $\mathbb{R}^m$ acts on M via an injective linear map $\rho : \mathbb{R}^m \to \mathrm{Lie}\,(T^n) = \mathbb{R}^n$.

Such a deformation M' is canonically T^n-equivariantly diffeomorphic to M by a diffeomorphism f which respects the hyperkähler moment maps μ, μ', i.e. $\mu = \mu' \circ f$.

We have the following hyperkähler analogue of Delzant's theorem.

Theorem 3. *Complete connected hyperkähler tri-Hamiltonian T^n-manifolds of finite topological type and dimension $4n$ are classified, up to Taub-NUT deformations, by arrangements of codimension 3 affine subspaces in $\mathbb{R}^3 \otimes \mathbb{R}^n$ of the form*

$$H_k = \{(x^1, x^2, x^3) \in \mathbb{R}^3 \otimes \mathbb{R}^n;\, \langle x^i, u_k\rangle = \lambda_k^i,\ \ i = 1, 2, 3\}$$

for some finite collection of vectors u_k in $\mathbb{R}^n$ and scalars λ_k^i, $i = 1, 2, 3$, such that, for any $p \in \mathbb{R}^3 \otimes \mathbb{R}^n$, the set $\{u_k; p \in H_k\}$ is part of a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

This arrangement is the image of singular T^n-orbits by the hyperkähler moment map. Taub-NUT deformations of different order are distinguished by the volume growth of the metric. In particular we have:

Theorem 4. *Let M be as in Theorem 1. If M has Euclidean volume growth, then M is isomorphic, as a tri-Hamiltonian hyperkähler T^n-manifold, to a hyperkähler quotient of a flat $\mathbb{H}^d$ by a $(d-n)$-dimensional subtorus N of $Sp(d)$, i.e. to one of the toric hyperkähler manifolds of* [7].

Remark 4 after Theorem 1 holds here as well.

Sometimes one is interested in having an $SU(2)$ action which preserves the metric but not the complex structures. This is the situation for the Taub-NUT and for the metrics which are cones over 3-Sasakian manifolds [8, 7].

Theorem 5. *Let M be as in Theorem 1 and suppose, in addition, that there is an effective isometric action of $SU(2)$ or $SO(3)$ rotating the complex structures and commuting with the action of T^n. Then M is isometric to $\mathbb{H}^n \times (S^1 \times \mathbb{R}^3)^{n-p}$, for some $p \le n$, or its Taub-NUT deformation.*

Our last two results deal with compact Einstein manifolds which admit some sort of quaternionic structure.

We recall that a Riemannian manifold (S, g) is 3-Sasakian if the metric cone $(\mathbb{R}_{>0} \times S, dr^2 + r^2 g)$ is hyperkähler. Such a manifold is Einstein of positive scalar curvature and examples of compact 3-Sasakian T^n-manifolds were given and studied in [8, 10, 7, 5]. These examples were obtained as 3-Sasakian quotients (see [8]) of spheres. We shall prove:

Theorem 6. *Let N^{4n-1} be a connected simply connected compact 3-Sasakian manifold admitting an effective T^n-action by 3-Sasakian isometries. Then N is isomorphic, as a 3-Sasakian T^n-manifold, to a 3-Sasakian quotient of a sphere by a torus.*

In fact, the isomorphism can be required to preserve the 3-Sasakian moment map.

A 3-Sasakian manifold fibers over a quaternion-Kähler (i.e. having holonomy in $Sp(n-1)Sp(1)$) orbifold. Theorem 6 in conjunction with Corollary 3 of [5] gives:

Corollary 7. *A $4n$-dimensional compact connected quaternion-Kähler manifold with positive scalar curvature and an effective action of T^{n+1} by isometries is isometric to $\mathbb{H}P^n$ or $Gr_2(\mathbb{C}^{n+2})$ with their symmetric metrics.*

The idea behind the proof of Theorem 1 is quite transparent. It is essentially sufficient to prove the result in dimension 4, since, were there any new complete T^n-invariant hyperkähler manifolds in higher dimensions, we would get new examples in dimension 4 by taking hyperkähler quotients by subtori. In dimension 4, under the assumption of finiteness of the Betti numbers, the action of S^1 has finitely many fixed points $m_1,\dots,m_d$ and the hyperkähler moment map induces a conformal immersion $\bar{\mu}:\bigl(M-\{m_1,\dots,m_d\}\bigr)/S^1\to\mathbb{R}^3$. Furthermore the local conformal factors are harmonic functions which implies that the metric on $\bigl(M-\{m_1,\dots,m_d\}\bigr)/S^1$ has nonnegative scalar curvature. We modify this metric so that it becomes a complete metric on $\bigl(M-\{m_1,\dots,m_d\}\bigr)/S^1$ with nonnegative scalar curvature. Then the results of Schoen and Yau [21] imply that $\bar{\mu}$ is injective and $\partial\mu(M)$ has Newtonian capacity zero (meaning that $\partial\mu(M)$ is removable for bounded harmonic functions). If $\mu(M)=\mathbb{R}^3$, then we are done, since positive harmonic functions with finitely many isolated singularities at $\mu(m_1),\dots,\mu(m_d)$ are easily classified. To show that the boundary of $\mu(M)$ is indeed empty, we prove an estimate which implies that a metric, given by a harmonic conformal factor on a complement of a set E of Newtonian capacity zero, is complete if and only if E is empty.

1. The Legendre transform and $\mathbb{R}^n$-invariant hyperkähler metrics

The fundamental idea of Lindström and Roček [19] is that hyperkähler manifolds of dimension $4n$ with n commuting Killing vector fields can be constructed from real-valued functions F on an open subset U of $\mathbb{R}^3\otimes\mathbb{R}^n$ which are harmonic on $U\cap L$ for any 3-dimensional affine subspace L of the form $\mathbb{R}^3\otimes\mathbb{R}v$, $v\in\mathbb{R}^n$. If we identify $\mathbb{R}^3\otimes\mathbb{R}^n$ with $\mathbb{R}^n\times\mathbb{C}^{2n}$ with coordinates x_i,z_i, $i=1,\dots,n$, then Lindström and Roček show that the Legendre transform with respect to the x_i gives a Kähler potential K of a hyperkähler metric. The local holomorphic coordinates for this metric are given by the z_i and some u_i and we have

$$K=F-2\sum_{i=1}^{n}(u_i+\bar{u}_i)x_i \tag{1.1}$$

where the x_i are determined by

$$\frac{\partial F}{\partial x_i}=2(u_i+\bar{u}_i). \tag{1.2}$$

The vector fields $\partial/\partial y_i=\sqrt{-1}(\partial/\partial\bar{u}_i-\partial/\partial u_i)$ are triholomorphic isometries and the moment map for their action is simply $(x_i,\Re z_i,\Im z_i)$.

An important observation of Pedersen and Poon [20] is that the metric has the form

$$g=\sum_{i,j}\Bigl(\Phi_{ij}(dx_idx_j+dz_id\bar{z}_j)+(\Phi^{-1})_{ij}(dy_i+A_i)(dy_j+A_j)\Bigr) \tag{1.3}$$

where $\Phi_{ij}=\frac{1}{4}F_{x_ix_j}$ and $A_j=\frac{\sqrt{-1}}{2}\sum_l(F_{x_j\bar{z}_l}d\bar{z}_l-F_{x_jz_l}dz_l)$. In particular, the quotient metric on U is given by

$$\sum_{i,j}\Phi_{ij}(dx_idx_j+dz_id\bar{z}_j). \tag{1.4}$$

The Φ_{ij} are also polyharmonic functions and, in fact, the matrix $[\Phi_{ij}]$ determines (locally) the hyperkähler and tri-Hamiltonian structure of the manifold. Indeed, if we have another function F' with $F'_{x_ix_j}=4\Phi_{ij}$, then $G=F-F'$ is of the form $\sum x_ig_i$, where the g_i are polyharmonic functions of $z_j,\bar{z}_j$ only. Computing the difference of the two connection 1-forms A and A', we see that it is given by

$$\frac{\sqrt{-1}}{2}\sum_{i,j}\bigl(\frac{\partial g_i}{\partial\bar{z}_j}d\bar{z}_j-\frac{\partial g_i}{\partial z_j}dz_j\bigr).$$

Since the functions g_i are polyharmonic, this form is closed and so it gives rise a local equivariant isometry ϕ. Moreover ϕ respects the Kähler potentials given by (1.1) and so the complex structure I_1. It also respects the coordinates x_i, z_i (hence the moment map) and so, by the formula (2.8) in [15], ϕ respects I_2 and I_3.

The two basic examples of this construction are flat S^1-invariant metrics on $S^1 \times \mathbb{R}^3$ and on $\mathbb{H}$. In the first case we have

$$F(x, z, \bar{z}) = 2x^2 - z\bar{z} \tag{1.5}$$

and consequently $\Phi \equiv 1$, while in the second case

$$F(x, z, \bar{z}) = x \ln(x + r) - r \tag{1.6}$$

where $r^2 = x^2 + z\bar{z}$. This time $\Phi = 1/4r$. More general forms are given [12] and [7]. In the latter, the functions F and the metrics for hyperkähler quotients of flat vector spaces are computed. They are essentially obtained by taking linear combinations and compositions with linear maps of the solution (1.6) (see also the proof of Theorem 1 in section 5). Our aim is to show that, in the case of a complete metric, the only other possibility is adding a linear combination of (1.5), which corresponds to a Taub-NUT deformation (see definition 2).

For a metric of the form (1.3) taking hyperkähler quotients by subtori is very simple. Indeed, the moment map equations are now linear (in x_i, z_i), and the hyperkähler quotient corresponds to restricting the function F to an appropriate affine subspace of $\mathbb{R}^3 \otimes \mathbb{R}^n$. In fact, the requirement that F be polyharmonic is a consequence of the fact that we must be able to take hyperkähler quotients by any subtorus.

An explanation of this construction in terms of twistors was given by Hitchin, Karlhede, Lindström and Roček [15]. In particular, they have shown that any hyperkähler $4n$-manifold with a free effective tri-Hamiltonian $\mathbb{R}^n$-action which extends to a $\mathbb{C}^n$-action with respect to each complex structure and such that the moment map is surjective is given by the Legendre transform. In the next section we shall show that this statement is always true for local tri-Hamiltonian $\mathbb{R}^n$-actions.

2. Potentials for hyperkähler metrics

Let X^{2n} be a Kähler manifold with the Kähler form ϕ. It is well known that ϕ can be always written in a local holomorphic chart as

$$\phi = -\frac{i}{2} \sum_{i,j} \frac{\partial^2 K}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j \tag{2.1}$$

for a real-valued function K.

We now wish to show that a hyperkähler structure is also locally determined by a single real-valued function K. Before proceeding we remark that the situation when there is a simultaneous Kähler potential for all three Kähler forms is very rigid [24] - it is equivalent to the hyperkähler manifold being locally a cone over a 3-Sasakian manifold.

Let M^{4n} be an arbitrary hyperkähler manifold and let $\omega_1, \omega_2, \omega_3$ be the three Kähler forms. Then $\omega = \omega_2 + i\omega_3$ is holomorphic for the complex structure I_1. The Darboux theorem holds for complex symplectic forms and we can find a local

I_1-holomorphic chart u_i, z_i, $i = 1, \dots, n$ such that

$$\omega = \sum_{i=1}^{n} du_i \wedge dz_i. \tag{2.2}$$

In this local chart ω_1 can be written as in (2.1):

$$\omega_1 = -\frac{i}{2} \sum_{i,j} \left(K_{u_i \bar{u}_j} du_i \wedge d\bar{u}_j + K_{u_i \bar{z}_j} du_i \wedge d\bar{z}_j + K_{z_i \bar{u}_j} dz_i \wedge d\bar{u}_j + K_{z_i \bar{z}_j} dz_i \wedge d\bar{z}_j \right), \tag{2.3}$$

for a real-valued function K. We see that the complex structure I_2 is given by:

$$\begin{aligned} I_2 \left(\frac{\partial}{\partial u_i} \right) &= \sum_{j=1}^{n} \left(K_{z_i \bar{u}_j} \frac{\partial}{\partial \bar{u}_j} + K_{z_i \bar{z}_j} \frac{\partial}{\partial \bar{z}_j} \right) \\ I_2 \left(\frac{\partial}{\partial z_i} \right) &= \sum_{j=1}^{n} \left(-K_{u_i \bar{z}_j} \frac{\partial}{\partial \bar{z}_j} - K_{u_i \bar{u}_j} \frac{\partial}{\partial \bar{u}_j} \right). \end{aligned} \tag{2.4}$$

Thus the condition $(I_2)^2 = -1$ gives a system of nonlinear PDE's for K. This system is equivalent to the following condition:

$$\begin{pmatrix} K_{u_i \bar{u}_j} & K_{u_i \bar{z}_j} \\ K_{z_i \bar{u}_j} & K_{z_i \bar{z}_j} \end{pmatrix} \in Sp(2n, \mathbb{C}), \tag{2.5}$$

where the symplectic group is defined with respect to the form (2.2).

Conversely, suppose that in some local coordinate system u_i, z_i we have a Kähler form ω_1 given by a Kähler potential K such that this system of PDE's is satisfied. Then, if we define $\omega_2 + i\omega_3$ by the formula (2.2), we obtain a hyperhermitian structure. However ω_2 and ω_3 are closed, and so, by Lemma 4.1 in [1], I_2 and $I_3 = I_1 I_2$ are integrable and we have locally a hyperkähler structure. Therefore there is 1-1 correspondence between Kähler potentials satisfying the above system of PDE's and local hyperkähler structures.

For $n = 1$ the condition (2.5) reduces to the Monge-Ampère equation

$$K_{u\bar{u}} K_{z\bar{z}} - K_{u\bar{z}} K_{z\bar{u}} = 1. \tag{2.6}$$

In higher dimensions, narrowing the situation from Ricci-flat Kähler to hyperkähler is equivalent to replacing a single equation with high-order nonlinearities by a system of equations with quadratic nonlinearities.

We now go back to the main subject of this paper and we assume that there is a local tri-Hamiltonian action of $\mathbb{R}^n$ on M. This action will extend to a local $\mathbb{C}^n$-action with respect to I_1 (or any other complex structure). Since there is a moment map for this $\mathbb{C}^n$-action, we can locally identify M with a neighbourhood of 0 in $\mathbb{C}^n \times \mathbb{C}^n$, where the first factor corresponds to the action and the second one is given by the moment map. Since we have now an equivariant holomorphic retraction onto $\mathbb{C}^n \times \{0\}$, the proof of the equivariant Darboux theorem, as given in [14], goes through for $\omega = \omega_2 + i\omega_3$. Therefore we identify locally M, as a complex $\mathbb{C}^n$-Hamiltonian manifold, with $T^*\mathbb{C}^n \simeq \mathbb{C}^n \times \mathbb{C}^n$ with coordinates u_i, z_i in such a way that the u_i correspond to the action of $\mathbb{C}^n$ and the z_i give the moment map for this action. Furthermore ω has the form (2.2). The Kähler form ω_1 is given by a Kähler potential K which we can assume to be $\mathbb{R}^n$-invariant, i.e. independent of

$u_i - \bar{u}_i$, $i = 1, \ldots, n$. Let x_i denote the moment map for the action of $\sqrt{-1}(\bar{u}_i - u_i)$ with respect to ω_1. From the formula (2.3) (using the fact that $K_{u_j} = K_{\bar{u}_j}$):

$$x_i = -\frac{1}{2} K_{u_i}. \tag{2.7}$$

We define a function F by

$$F = K + \sum_{i=1}^{n} 2x_i(u_i + \bar{u}_i). \tag{2.8}$$

In particular F is independent of $u_i - \bar{u}_i$, $i = 1, \ldots, n$. Differentiating (2.8) with respect to u_i, $i = 1, \ldots, n$, yields the system of equations:

$$\sum_{j=1}^{n} \left(\frac{\partial F}{\partial x_j} - 2u_j - 2\bar{u}_j \right) \frac{\partial x_j}{\partial u_i} = 0, \qquad i = 1, \ldots, n.$$

We claim that the matrix $[\partial x_j / \partial u_i]$ is everywhere nonsingular. Indeed suppose that at some point $(\underline{x}, \underline{z})$ we have $\sum a_j \partial x_j / \partial u_i = 0$, $i = 1, \ldots, n$, for some scalars $a_1, \ldots, a_n$. The expression $\sum a_j x_j$ is the moment map for a Hamiltonian vector field Y_a. If we restrict to the $\mathbb{C}^n$-orbit O of $(\underline{x}, \underline{z})$ we conclude that the moment map for Y_a has a critical point in the Kähler manifold O. This implies that Y_a vanishes at $(\underline{x}, \underline{z})$ and so $a_1 = \cdots = a_n = 0$. Therefore the above equations imply equations (1.2). Thus we have shown that the Kähler potential of a hyperkähler metric with n nonvanishing tri-Hamiltonian vector fields can be written in the form (1.1) for a function F satisfying (1.2). We wish to show that F is polyharmonic. Let us take locally any hyperkähler quotient of M by an $(n-1)$-dimensional subgroup of $\mathbb{R}^n$. This corresponds to restricting F to a 3-dimensional affine subspace in $\mathbb{R}^3 \otimes \mathbb{R}^n$ of the form $a + \mathbb{R}^3 \otimes \mathbb{R}v$. Similarily, by identifying the hyperkähler and complex-symplectic quotient, we restrict K to an appropriate subspace. The restricted K is still of the form (1.1) for the restricted F. Therefore to show that, in the general case, the function F is polyharmonic it is enough to show that for $n = 1$ such a function is harmonic. In this case, formula (3.143) of [15], gives:

$$\bar{\partial}\partial K = \bar{\partial}(F_z) \wedge dz + 2du \wedge \bar{\partial}(x).$$

This yields

$$K_{z\bar{z}} = F_{z\bar{z}} + F_{zx} \frac{\partial x}{\partial \bar{z}}.$$

We also have, from (2.7),

$$K_{u\bar{u}} = -2\frac{\partial x}{\partial \bar{u}}, \;\; K_{u\bar{z}} = -2\frac{\partial x}{\partial \bar{z}}.$$

Now, from (1.2),

$$0 = \frac{\partial(u + \bar{u})}{\partial \bar{z}} = F_{xx} \frac{\partial x}{\partial \bar{z}} + F_{x\bar{z}} \quad \text{and} \quad 1 = \frac{\partial(u + \bar{u})}{\partial \bar{u}} = \frac{1}{2} F_{xx} \frac{\partial x}{\partial \bar{u}}.$$

Therefore

$$K_{u\bar{u}} = -4(F_{xx})^{-1}, \;\; K_{u\bar{z}} = 2(F_{xx})^{-1} F_{x\bar{z}}, \;\; K_{z\bar{z}} = F_{z\bar{z}} - (F_{xx})^{-1} F_{x\bar{z}} F_{xz}$$

and we see that the equation (2.6) is equivalent to $4F_{z\bar{z}} + F_{xx} = 0$ which is the same as $\Delta F = 0$. Thus we have proved

Proposition 2.1. *Let M^{4n} be a hyperkähler manifold with n non-vanishing commuting tri-Hamiltonian vector fields. Then M is locally given by the Legendre transform of Lindström and Roček.* □

Remark 2.2. It follows from the above arguments that if (M^{4n}, I, ω_1) is a Kähler manifold which also has a complex-symplectic form ω and n non-vanishing commuting vector fields which are Hamiltonian for both ω_1 and ω, then M is locally given by the Legendre transform for some function F. This function is polyharmonic if and only if M is hyperkähler.

3. Local structure of the moment map

We shall prove some facts about Hamiltonian torus actions on hyperkähler manifolds which we shall need in the sequel. We start with a simple

Lemma 3.1. *Let M^{4n} be a hyperkähler manifold equipped with an effective tri-Hamiltonian action of T^m. Then $m \leq n$.*

Proof. At any point $p \in M$ let V denote the subspace of the tangent space spanned by the vector fields generated by the action of T^n. It is simple to check, using the three moment maps, that the subspaces V, IV, JV, KV are mutually orthogonal. □

The following fact was proved in [7] for hyperkähler manifolds which are toral quotients of flat quaternionic vector spaces (compare also [11] for the symplectic case).

Proposition 3.2. *Let M^{4n} be a hyperkähler manifold equipped with an effective tri-Hamiltonian action of T^n with a hyperkähler moment map $\mu = (\mu_1, \mu_2, \mu_3) : M \to \mathbb{R}^3 \otimes \mathbb{R}^n$. The image $\mu(M^s)$ of the union M^s of singular T^n-orbits (i.e. $M^s = \{m; \dim \mathrm{Stab}_{T^n}(m) \geq 1\}$) is of the form $\mu(M) \cap \bigcup_S H_k$ where*

$$H_k = \{(x^1, x^2, x^3) \in \mathbb{R}^3 \otimes \mathbb{R}^n;\ \langle x^i, u_k \rangle = \lambda_k^i,\ \ i = 1, 2, 3\} \tag{3.1}$$

for some countable collection S of vectors u_k in $\mathbb{R}^n$ and scalars λ_k^i, $i = 1, 2, 3$. Furthermore, for any p in M:

(i) *the isotropy group T_p of p is the connected subtorus of T^n whose Lie algebra is spanned by $\{u_k \in S; \mu(p) \in H_k\}$;*
(ii) *After rescaling, the set $\{u_k \in S; \mu(p) \in H_k\}$ is part of a $\mathbb{Z}$-basis of $\mathbb{Z}^n$ and the isotropy representation of T_p is determined by this set.*

Proof. Let $\rho \in \mathbb{R}^n$ and let M_ρ be any component of the fixed point set of the 1-parameter subgroup of T^n generated by ρ, i.e. the corresponding vector field X_ρ vanishes on M_ρ. Then M_ρ is a hyperkähler submanifold of M. It follows that, for $p \in M_\rho$ and $v \in T_pM$, we have $\langle d\mu_i(v), \rho \rangle = 0$, $i = 1, 2, 3$. Therefore $\langle \mu_i, \rho \rangle$ is constant on M_ρ, $i = 1, 2, 3$, which proves that $\mu(M^s)$ is as stated. This also proves that the Lie algebra of T_p is spanned by $\{u_k \in S; \mu(p) \in H_k\}$.

We shall now prove the statement (ii) and the connectedness of T_p. Let $\dim T_p = i$ and let H_0 be the identity component of T_p. There is an $(n-i)$-dimensional subtorus H_1 of T^n such that $T^n = H_0 \times H_1$. Let $\Gamma = T_p / H_0 = H_1 \cap T_p$, so that the T^n-orbit of p is isomorphic to H_1/Γ. The fixed point set of T_p is a totally geodesic hyperkähler submanifold X of M and we can identify a T^n-invariant neighbourhood of p with a T^n-invariant neighbourhood of X in the normal bundle $N = T_XM/T_XX$. Since the torus H_1/Γ acts freely on X, we can identify X (via moment maps) with a neighbourhood of H_1/Γ in $H_1/\Gamma \times \mathbb{R}^{3n-3i}$. Since M and X both have a hypercomplex structure, so does the fiber of N, which we then identify with $\mathbb{H}^i$. The isotropy group T_p must act on $\mathbb{H}^i$ via a homomorphism $\rho : T_p \to Sp(i)$. Since the action of T^n is effective, ρ must be injective and hence $\Gamma = 1$.

By the existence of normal forms for Hamiltonian actions (see section 41 in [14]) even the moment map μ_1 can be identified with a symplectic moment map for the action of T^n on $T^*H_1 \times \mathbb{R}^{2i} \times \mathbb{C}^{2i}$ (the action is trivial on the middle factor). Let ϕ_1 denote the moment map for the action of $T_p = H_0$ on $\mathbb{C}^{2i}$. If the weights of the representation are $\alpha_1, \ldots, \alpha_i$, then we have (cf. [7]):

$$\phi_1(z_1, \ldots, z_i, w_1, \ldots, w_i) = \frac{1}{2} \sum_{k=1}^{i} \big(|z_k|^2 - |w_k|^2\big)\alpha_k. \tag{3.2}$$

Now it is to see that the vector u_k is orthogonal to α_j for $i \neq k$. Since the representation is an isomorphism, the α_i form a $\mathbb{Z}$-basis of $\mathbb{Z}^i$, and therefore the u_k, after rescaling, also form a $\mathbb{Z}$-basis of their span. This finishes the proof. □

Finally, we need:

Proposition 3.3. *Let M^{4n} be a hyperkähler manifold equipped with an effective tri-Hamiltonian action of T^n. Then the hyperkähler moment map $\mu = (\mu_1, \mu_2, \mu_3) : M \to \mathbb{R}^{3n}$ induces a local homeomorphism from the orbit space M/T^n into $\mathbb{R}^{3n}$.*

Proof. First we consider a T^n-invariant neighbourhood U of a T^n-fixed point m, which by the existence of normal forms we can identify with a neighbourhood of the origin in $\mathbb{C}^{2n}$ and the moment map μ_1 with the standard moment map

$$(z, w) = (z_1, \ldots, z_n, w_1, \ldots, w_n) \mapsto \frac{1}{2}\big(|z_1|^2 - |w_1|^2, \ldots, |z_n|^2 - |w_n|^2\big).$$

Furthermore, we can require that at the origin the complex structures I_1, I_2, I_3 coincide with the standard complex structures I_1^0, I_2^0, I_3^0 of $\mathbb{H}^n$. Let $\mu_1^0, \mu_2^0, \mu_3^0$ be the corresponding standard moment maps so that $\mu_1^0 = \mu_1, (\mu_2^0 + i\mu_3^0)(z, w) = (z_1 w_1, \ldots, z_n w_n)$. In a neighbourhood of 0 we have an expansion $I_2 = I_2^0 +$(*terms of degree* ≥ 1 *in* $z_i, \bar{z}_i, w_i, \bar{w}_i$), and similarly for I_3. Now, for any vector v tangent to U, $d\mu_2(v) = -d\mu_1(I_3 v)$ and $d\mu_3(v) = d\mu_1(I_2 v)$. Therefore

$$\mu_2 + i\mu_3 = \mu_2^0 + i\mu_3^0 + (\textit{terms of degree} \geq 3 \textit{ in } z_i, \bar{z}_i, w_i, \bar{w}_i). \tag{3.3}$$

The ring of (real) T^n-invariant polynomials on $\mathbb{C}^{2n}$ is generated by $y_1^i = |z_i|^2, y_2^i = |w_i|^2, y_3^i = \Re(z_i w_i), y_4^i = \Im(z_i w_i)$, $i = 1, \ldots, n$, with relations

$$y_1^i y_2^i = (y_3^i)^2 + (y_4^i)^2. \tag{3.4}$$

In particular, as the invariant polynomials are homogeneous of degree 2, the terms of degree 3 in (3.3) vanish. We conclude that the moment map $\mu_2 + i\mu_3$ can be written, on a sufficiently small neighbourhood of 0, as

$$(\mu_2+i\mu_3)(z, w) = f\big(|z_1|^2, |w_1|^2, \Re(z_1 w_1), \Im(z_1 w_1), \ldots, |z_n|^2, |w_n|^2, \Re(z_n w_n), \Im(z_n w_n)\big),$$

where $f : \mathbb{R}^{4n} \to \mathbb{C}^n$ is a C^∞ function of the form

$$f(y_1^1, y_2^1, y_3^1, y_4^1, \ldots, y_1^n, y_2^n, y_3^n, y_4^n) = \big(y_3^1 + iy_4^1, \ldots, y_3^n + iy_4^n\big) + O(r^2)$$

with r denoting the distance from the origin in $\mathbb{R}^{4n}$. The image S of the neighbourhood of m in the orbit space M/T^n is identified with a neighbourhood of 0 in the variety given by the equations (3.4) with additional constraints $y_1^i \geq 0$, $y_2^i \geq 0$. The map $\bar{\mu} : M/T^n \to \mathbb{R}^{3n}$ is of the form $\bar{\mu}^0 + O(r^2)$, with

$$\bar{\mu}^0(y_1^1, y_2^1, y_3^1, y_4^1, \ldots, y_1^n, y_2^n, y_3^n, y_4^n) = \big((y_1^1 - y_2^1)/2, y_3^1, y_4^1, \ldots, (y_1^n - y_2^n)/2, y_3^n, y_4^n\big).$$

One easily checks that the function $g : \mathbb{R}^{3n} \mapsto \mathbb{R}^{4n}$ defined by

$$g(p_1^1, p_2^1, p_3^1, \ldots, p_1^n, p_2^n, p_3^n) = \Big(p_1^1, -p_1^1 + \sqrt{(p_1^1)^2 + (p_2^1)^2 + (p_3^1)^2}, p_2^1, p_3^1, \ldots, \\ p_1^n, -p_1^n + \sqrt{(p_1^n)^2 + (p_2^n)^2 + (p_3^n)^2}, p_2^n, p_3^n\Big) \quad (3.5)$$

is the the inverse of $\bar{\mu}^0$ restricted to S. Since g is locally Lipschitz, it follows that $|\bar{\mu}(u) - \bar{\mu}(v)| \geq C|u - v|$ for $u, v \in S$, providing that we restrict ourselves to a sufficiently small neighbourhood of 0 in $\mathbb{R}^{4n}$. Thus we have shown that $\bar{\mu}$ is open and locally $1 - 1$ in a neighbourhood of a fixed point. Consider now an arbitrary point $m \in M$ and let $l = \dim \mathrm{Stab}_{T^n}(m)$. From Proposition 3.2 $\mu(m)$ lies on exactly l flats H_k. The vectors u_k defining these H_k are a part of $\mathbb{Z}$-basis and we can complete it to a full $\mathbb{Z}$-basis. Let A be the subtorus of T^n generated by the $n-l$ vectors used to complete the basis and let $\mu_A : M \to \mathbb{R}^{3n-3l}$ be the moment map for A. According to Proposition 3.2, A acts freely in a neighbourhood of m and so any hyperkähler quotient of M by A with the level set of the moment map lying close enough to $\mu_A(m)$ is a manifold (in a neighbourhood of the point induced by m). For any such hyperkähler quotient $\mu_A^{-1}(\lambda)/A$, the point induced by m is a fixed point for T^n/A and so, by the previous argument, the moment map for the action of T^n/A induces a locally $1-1$ mapping from $\mu_A^{-1}(\lambda)/T^n$ to $\mathbb{R}^{3l}$. Now, $\mu_A^{-1}(\lambda)$ can be viewed as $\mu^{-1}(V_\lambda)$, where V_λ is a $3l$-dimensional affine subspace of $\mathbb{R}^3 \otimes \mathbb{R}^n$ of the form $\mathbb{R}^3 \otimes W_\lambda$ with W_λ generated by the vectors u_k such that m lies on H_k and passing through the point $(0, \lambda)$ in the chosen $\mathbb{Z}$-basis of $\mathbb{Z}^n$. It follows that $\bar{\mu}$ is $1 - 1$ in a neighbourhood of the orbit of m. Since the spaces are locally compact and $\bar{\mu}$ is continuous, $\bar{\mu}$ is a homeomorphism of this neighbourhood onto its image. The orbit space is a topological manifold (this follows from (3.5)) and so $\bar{\mu}$ is a local homeomorphism into $\mathbb{R}^{3n}$. □

4. Metrics with harmonic local conformal factors

We wish to consider the following situation which generalizes the 4-dimensional case of Theorem 1. In this section Y is a complete Riemannian $(n+3)$-dimensional manifold with an effective isometric action of T^n which is not free only at a finite number of points $y_1, \ldots, y_d$. Furthermore there is a T^n-invariant map $\mu : Y \to \mathbb{R}^3$ which induces a local homeomorphism $\bar{\mu} : Y/T^n \to \mathbb{R}^3$ whose restriction to $N = (Y - \{y_1, \ldots, y_d\})/T^n$ (with the quotient metric) is a conformal immersion with all local conformal factors harmonic. In other words N can be covered by open sets V_α, on which $\bar{\mu}$ is $1 - 1$ and the quotient metric g on V_α satisfies

$$(\bar{\mu}^{-1})^* g = \Phi_\alpha g_e$$

where g_e is the flat metric on $\mathbb{R}^3$ and Φ_α is a positive harmonic function on an open subset $\bar{\mu}(V_\alpha)$ of $\mathbb{R}^3$. We shall also assume, to simplify the proof, that the Φ_α do not extend to any y_i, $i = 1, \ldots, d$. We aim to show

Proposition 4.1. *Under the above assumptions, $\bar{\mu}$ is a homeomorphism onto $\mathbb{R}^3$ and g is globally conformally flat with the conformal factor $\Phi(x) = \sum_{i=1}^{d} \frac{a_i}{|x-x_i|} + b$, where $x_i = \mu(y_i)$ and $a_i > 0, b \geq 0$, $i = 1, \ldots, d$, are constants.*

We remark that a completely analogous result holds (and has a simpler proof) if we replace $\mathbb{R}^3$ with any $\mathbb{R}^p$, $p > 3$.

We start the proof by observing that, since $\bar{\mu}$ is a homeomorphism in a neighbourhood of each y_i, we have one (positive) harmonic function Φ in a neighbourhood of each point y_i. We have the following simple lemma about such functions.

Lemma 4.2. *Let Φ be a nonnegative harmonic function in $U - \{0\}$ where U is a neighbourhood of 0 in $\mathbb{R}^3$. Then $\Phi = c/r + \phi$ where c is a nonnegative constant, $r(x) = |x|$ and ϕ is harmonic in U.*

Proof. Choose $\epsilon > 0$ so that $B(0, 2\epsilon) \subset U$. Let x be a point of the sphere $S(0, \epsilon)$. The Harnack inequality applied to the ball $B(x, \epsilon - t/2)$ implies that $\Phi(tx) \leq Ct^{-2}\Phi(x)$ for all $t > 0$, where C depends only on ϵ. Thus $\Phi(x) = O(|x|^{-2})$ in $B(0, \epsilon)$. Then it follows that $\Phi = \Phi_1 + c/r + \phi$ where c, r, ϕ are as in the statement and Φ_1 is a linear combination (with constant coefficients) of the first order partial derivatives of the fundamental solution $1/r$. Since Φ is nonnegative, $\Phi_1 \equiv 0$. □

We now deform the metric on N.

Proposition 4.3. *There exists on N a complete metric of nonnegative scalar curvature for which $\bar{\mu}$ is a conformal immersion.*

Proof. A metric of the form Φg_e on an open subset of $\mathbb{R}^3$ has nonnegative scalar curvature if and only if $\Phi^{1/4}$ is superharmonic (see, for example, [22], Chapter V). Since, all powers Φ^α, $0 < \alpha < 1$, of a harmonic (or superharmonic) function Φ are superharmonic, we know that the quotient metric g on N has a nonnegative scalar curvature. This metric is incomplete, but the incompleteness occurs only at the points $y_1, \ldots y_d$ (as the metric on M is complete). Hence, if we can modify metric g so that it is complete at these points and still has a nonnegative scalar curvature, we will be done. Therefore we work in a neighbourhood U of the origin in $\mathbb{R}^3$ where, by Lemma 4.2, the metric g is given by $(c/r + \phi)g_e$. By the last assumption made about Y, $c > 0$. For any function ψ and any $\alpha \in (0,1)$ we have

$$\Delta(\psi^\alpha) < 0 \Longleftrightarrow \psi \Delta \psi + (\alpha - 1)|\nabla \psi|^2 < 0. \tag{4.1}$$

Assume now that ψ is of the form $f(r) + \phi$, where ϕ is harmonic in U and f approaches $+\infty$ as $r \to 0$. Then we estimate the right-hand side of (4.1) as

$$\psi \Delta \psi + (\alpha - 1)|\nabla \psi|^2 \leq (f + \phi)\Delta f + (\alpha - 1)\big(\nabla f \cdot \nabla (f + 2\phi)\big).$$

For any $\epsilon > 0$ and r small enough, $f + \phi \leq (1 + \epsilon) f$ and $2|\phi_i| \leq \epsilon |f_i|$, $i = 1, 2, 3$, where the subscript i denotes the i-th partial derivative. Therefore the last estimate can be replaced by

$$\psi \Delta \psi + (\alpha - 1)|\nabla \psi|^2 \leq (1 + \epsilon)\big(f \Delta f + (\alpha - 1)\frac{1 - \epsilon}{1 + \epsilon}|\nabla f|^2\big).$$

In our case, $\alpha = 1/4$, and if we choose ϵ so that $(\alpha - 1)\frac{1-\epsilon}{1+\epsilon} = -1/2$, we see from (4.1) that if $f^{1/2}$ is superharmonic, then $\psi^{1/4}$ is superharmonic (in a small neighbourhood of the origin).

Our original conformal factor was of the form $c/r + \phi$. In order to make the metric complete we wish the growth to be $1/r^2$. Therefore we have to find a positive function $f(r)$ such that $f(r) = c/r$ for $r \geq \delta$, $f(r) \sim 1/r^2$ near 0, and $\sqrt{f}$ is superharmonic. Let $u = \sqrt{f}$. The condition $\Delta u \leq 0$ is equivalent, for a function $u = u(r)$, to

$$\ddot{u} + \frac{2}{r}\dot{u} \leq 0,$$

and so, for $z = \frac{d}{dr}\ln(-du/dr)$, it is equivalent to $z \leq 2/r$. Let $z(r)$ be any smooth (on $(0,\infty)$) decreasing function satisfying this inequality, equal to $3/2r$ for $r \geq \delta$ and equal to $2/r$ for $r \leq \delta/2$. We obtain a function u by choosing the two free constants (i.e. $u(\delta)$ and $\dot{u}(\delta)$) so that $u(r)$ is tangent to $\sqrt{c/r}$ at $r = \delta$. We define a new conformal factor by $\tilde{\Phi} = u^2 + \phi$. It coincides with $\Phi = c/r + \phi$ for $r \geq \delta$ and is positive as $u(r) \geq \sqrt{c/r}$. Furthermore $\tilde{\Phi}^{1/4}$ is superharmonic and, therefore, $\tilde{\Phi}g_e$ has nonnegative scalar curvature. Finally, this metric is complete at 0 as $\tilde{\Phi}$ has $(1/r^2)$-growth. □

We now appeal to results of Schoen and Yau [21] (Propositions 4.2, 4.3, 4.4, and 4.4'; see also Theorem VI.3.5 in [22]) which give us:

Corollary 4.4. *The map $\bar{\mu} : N \to \mathbb{R}^3$ is injective and the boundary of $\bar{\mu}(N)$ has Newtonian capacity zero.* □

We recall that a G_δ subset E of $\mathbb{R}^n$ has Newtonian capacity zero (or is polar) precisely when the removable singularity theorem holds for E. Another equivalent condition is that there is a tempered positive measure μ_E in $\mathbb{R}^n$ such that the convolution of the Green function $G(x,y) = |x-y|^{2-n}$ with μ is infinite exactly on the set E. In particular the Hausdorff dimension of E is at most $n-2$.

Since the map $\bar{\mu} : Y/T^n \to \mathbb{R}^3$ is a local homeomorphism in a neighbourhood of each y_i and since a set $E \cup \{x_1, \ldots, x_d\}$, $x_i \notin \bar{E}$, is polar if and only if E is polar, the conclusion of the above corollary holds also for Y/T^n, instead of N. Therefore we can assume that Y/T^n is an open subset U of $\mathbb{R}^3$, and that the quotient metric on $U - \{x_1, \ldots, x_d\}$, where $x_i = \mu(y_i)$, is of the form

$$g = \left(\sum_{i=1}^{d} \frac{a_i}{|x - x_i|} + \phi \right) g_e$$

for positive constants a_i and a function ϕ, harmonic in U.

We now wish to show that $\bar{\mu} : Y/T^n \to \mathbb{R}^3$ is onto. This will imply that ϕ is constant and will end the proof of Proposition 4.1.

We have:

Lemma 4.5. *Suppose that the metric on Y is complete. Then ϕ is bounded below and for any C^1-curve $\gamma : [0,1] \to \bar{U}$, $\gamma\big([0,1)\big) \subset U$, $\gamma(1) \in \partial U$, we have*

$$\int_0^1 \big(\phi(\gamma(t)) + C\big)^{1/2} |\gamma'(t)| dt = +\infty$$

for any $C \geq 0$ such that $\phi + C \geq 0$.

In other words, for any $C \geq 0$ such that $\phi + C > 0$, $(\phi + C)g_e$ is a complete Riemannian metric on U.

Proof. The function ϕ is bounded in a compact neighborhood K_i of each of the x_i and outside of $\bigcup \operatorname{Int} K_i$ the function $f = \Phi - \phi$ is also bounded. Since $\phi + f > 0$, ϕ must be bounded below.

As in Proposition 4.3, the only incompleteness of the quotient metric g occurs at the points x_i and so a curve γ with the assumed properties must have infinite length in g. We can assume that γ avoids the points x_i. Let Ψ be the restriction of $\sum a_k/|x - x_k|$ to U. This is finite on γ. For any $C \geq 0$ such that $\phi + C \geq 0$ we have

$$\begin{aligned} +\infty &= \int_0^1 \left((\Psi + \phi)(\gamma(t))\right)^{1/2} |\gamma'(t)| dt \\ &\leq \int_0^1 \left(\Psi(\gamma(t)\right)^{1/2} |\gamma'(t)| dt + \int_0^1 \left(\phi(\gamma(t)) + C\right)^{1/2} |\gamma'(t)| dt + \int_0^1 C^{1/2} |\gamma'(t)| dt. \end{aligned}$$

The first and the last term are finite, and so the second one is infinite. □

Proposition 4.1 will follow from the following fact (compare with Lemma 4.2):

Proposition 4.6. *Let E be a closed subset of Newtonian capacity zero in $\mathbb{R}^3$ and let Ψ be a nonnegative harmonic function on $\mathbb{R}^3 - E$. There is a constant $K \geq 0$ such that for all $x \in \mathbb{R}^3$,*

$$\Psi(x) \leq K + \frac{K}{dist_{S^3}(x, E)}, \tag{4.2}$$

where the distance is measured in the standard metric on S^3.

Proof. To prove the estimate (4.2) for closed sets in $\mathbb{R}^3$ in the distance measured in S^3 is the same as to prove it for compact subsets of $\mathbb{R}^3$ in the Euclidean distance. Thus we can assume that E is compact. Let now μ_E be the positive measure in $\mathbb{R}^3$ whose convolution with $|x - y|^{-1}$ gives the nonnegative superharmonic function ψ_E on $\mathbb{R}^3$ with $\psi_E^{-1}(+\infty) = E$. Then the function

$$f(x) = \begin{cases} \Psi(x) + \psi_E(x) & \text{if } x \notin E \\ +\infty & \text{if } x \in E \end{cases}$$

is nonnegative and superharmonic on $\mathbb{R}^3$. It follows (Corollary 2 of Theorem 1.23 in [17]) that $f = c + G * \mu'$, where c is a nonnegative constant, $G(x,y) = |x - y|^{-1}$ and μ' is a nonnegative measure in $\mathbb{R}^3$. Let $\nu = \mu' - \mu_E$. Then $G * \nu$ coincides with $\Psi - c$ on $\mathbb{R}^3 - E$. Since Ψ solves the Laplace equation on $\mathbb{R}^3 - E$ and $G * \nu$ is the solution of the Poisson equation $\Delta(G * \nu) = -\nu$ in the sense of distributions, we conclude that $\operatorname{supp} \nu \subset E$. Let us write $\nu = \nu^+ - \nu^-$ where ν^+, ν^- are the positive and negative variations of ν. We have

$$\Psi(x) - c = \int_E \frac{d\nu(y)}{|x - y|} \leq \int_E \frac{d\nu^+(y)}{|x - y|} + \int_E \frac{d\nu^-(y)}{|x - y|} \leq \frac{\nu^+(E) + \nu^-(E)}{\operatorname{dist}(x, E)}.$$

This concludes the proof. □

We can now finish the proof of Proposition 4.1.

Proof of Proposition 4.1. Suppose that $U = \mu(Y) \neq \mathbb{R}^{3n}$. We consider the metric $(\phi + C)g_e$ on U for some large C. This metric is complete by Lemma 4.5. Let $z \in \mathbb{R}^3$ be such that $\operatorname{dist}(z, \partial U) = 1$ and let $y \in \partial U$ be a point where this distance is achieved. Then, by Corollary 4.4, the estimate (4.2) holds for all points x of the segment yz. Therefore the length of this segment is finite in the metric $(\phi + C)g_e$ and we obtain a contradiction. Hence $U = \mathbb{R}^3$ and ϕ must be a constant. □

We finish the section by observing that, if we write (x, z) for the coordinates of $\mathbb{R}^3$ as in section 1, then the function Φ of Proposition 4.1 can be written as $\Phi = F_{xx}$ where (up to a term linear in x) by

$$F(x, z, \bar{z}) = \sum_{k=1}^{d} a_k \big(s_k \ln(s_k + r_k) - r_k\big) + b\big(2x^2 - z\bar{z}\big) \tag{4.3}$$

where $s_k = x - x_k$ and r_k is the distance between $(x, z, \bar{z})$ and $\mu(y_k) = (x_k, z_k, \bar{z}_k)$.

5. Proofs

We shall now prove all results stated in the introduction.

Proof of Theorem 1. Let M satisfy the assumptions of Theorem 1. Let H_k, $k = 1, \dots, d$, be the codimension 3 affine subspaces in $\mathbb{R}^3 \otimes \mathbb{R}^n$ given by Proposition 3.2. Their number is finite because M has finite topological type (this follows from the localization theorem [3], if we notice that were there infinitely many H_k, we could find a circle in T^n with infinitely many fixed points). Consider a small open subset U of $\mathbb{R}^3 \otimes \mathbb{R}^n$ not intersecting any H_k. In particular the T^n-action is free on $\mu^{-1}(U)$ and, if U is small enough, the hyperkähler moment map is a diffeomorphism between $\mu^{-1}(U)/T^n$ and U. Therefore, as in section 1, there is a function $F = F(x_i, z_i, \bar{z}_i)$ defined on U which completely determines a neighbourhood V of a section of $\mu^{-1}(U) \to U$ as a hyperkähler tri-Hamiltonian T^n-manifold. This function is unique up to irrelevant terms linear in the x_i.

For any rational 3-dimensional affine subspace $L = a + \mathbb{R}^3 \otimes \mathbb{R}v$ of $\mathbb{R}^3 \otimes \mathbb{R}^n$ such that $L \cap U \neq \emptyset$ we can consider the $(n-1)$-dimensional subtorus N of T^n whose Lie algebra is generated by vectors orthogonal to v. The level set of the moment map μ_N for N is chosen to be L. According to Proposition 3.2, as long as L does not meet any point at which two of the H_k intersect, N will act locally freely on $\mu_N^{-1}(L)$ and so $Y = \mu_N^{-1}(L)$ is a complete Riemannian manifold which we claim satisfies all assumptions of the previous section. The map $\mu : Y \to L \simeq \mathbb{R}^3$ is simply the restriction of $\mu : M \to \mathbb{R}^3 \otimes \mathbb{R}^n$. By Proposition 3.3, $\bar{\mu}$ is a local homeomorphism. The points y_i map, under μ, to the intersections of L with the H_k. Therefore $Y = \mu_N^{-1}(L)$ satisfies all assumptions of the previous section and, by Proposition 4.1, we know, up to irrelevant linear factors, the restriction of F to $L \cap U$. It is given by the formula (4.3) where the points (x^k, z^k) are intersection points of L with the H_k. Thus we know the restriction of F to a generic rational 3-dimensional subspace of U. This obviously determines F. We claim that on U, F is given by the formula

$$F(x_i, z_i, \bar{z}_i) = \sum_{k=1}^{d} a_k \big(s_k \ln(s_k + r_k) - r_k\big) + \sum_{i,j} b_{ij}\big(2x_i x_j - z_i \bar{z}_j\big) \tag{5.1}$$

where a_k are positive constants, $[b_{ij}]$ is a positive-definite constant matrix and s_k and r_k are defined by (cf. (3.1))

$$s_k(x, z) = \langle x, u_k\rangle - \lambda_k^1, \;\; v_k(x, z) = \langle z, u_k\rangle - \lambda_k^2 - \sqrt{-1}\lambda_k^2, \;\; r_k^2 = s_k^2 + v_k\bar{v}_k.$$

In other words r_k is the distance of a point from H_k, and $|s_k|$ is the distance between the projections of a point and of H_k onto the first factor in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \simeq$

$\mathbb{R}^3 \otimes \mathbb{R}^n$. Indeed, the second sum in (5.1) is the most general form of a function which restricted to any (or generic rational) 3-dimensional subspace L gives us the second term of (4.3). For the first sum, notice that (4.3) implies that F must be of the form

$$\sum_{k=1}^{p} a_k\big(s_k \ln(s_k + r_k) - r_k\big) + \sum_{i,j} b_{ij}\big(2x_i x_j - z_i \bar{z}_j\big)$$

where s_k are linear in the x_i and r_k^2 are quadratic in the $x_i, \Re z_i, \Im z_i$. If, however, any of these was not of the form stated above, then for some 3-dimensional subspace L we would have obtained singular points of Y_L different than those corresponding to the intersections of L with the H_k. Therefore F is of the form (5.1).

The first sum in (5.1) describes the function F of a hyperkähler quotient M' of a flat $\mathbb{H}^d$ by a torus determined by the H_k. This follows from [7], given that the H_k satisfy the $\mathbb{Z}$-basis condition of Proposition 3.2. The metric on $\mathbb{H}^d$ is not the standard one but each factor is rescaled by a_k.

Suppose that $[b_{ij}]$ has rank m. Since it is symmetric, we can write $[b_{ij}] = AA^T$ for an $n \times m$ matrix A of rank m. This A defines an embedding $\mathbb{R}^m \to \mathbb{R}^n$. The function (5.1) is the the function F of the Taub-NUT modification (see Definition 2) of order m of M'. Let us denote M' with this modified metric by M''. We conclude that M and M'' are locally (on connected subsets mapping onto U) isomorphic as tri-Hamiltonian hyperkähler T^n-manifolds. Let ϕ be this local isomorphism. In particular ϕ is an isometry and, as M and M'' are Einstein, hence real-analytic, and complete, ϕ extends to a (unique) isometry between the universal covers $\tilde{M}$ and $\tilde{M}''$ ([16], Corollary VI.6.4). If there are n flats H_k intersecting in a point, then, by [7], M'' is simply connected and so we obtain an isometry between M and M''. This isometry is locally a tri-Hamiltonian isomorphism and so it is such an isomorphism globally (as everything is real-analytic). This proves case (i) of Theorem 1.

If there are at most $n-l$, $l>0$, flats H_k intersecting at any given point, then M'' is isomorphic to a product of a simply connected X and $(S^1 \times \mathbb{R}^3)^l$. Hence the universal covers are isometric to $X \times (\mathbb{R}^4)^l$. Once more the isometry between $\tilde{M}$ and $X \times (\mathbb{R}^4)^l$ respects the tri-Hamiltonian structure for the action of $T^{n-l} \times \mathbb{R}^l$ ($\mathbb{R}^l$ acts by translations on the first factor of $(\mathbb{R}^l)^4$). It follows that $\pi_1(M)$ is a subgroup of $T^{n-l} \times \mathbb{R}^l$. Since T^{n-l} has a fixed point in X, any element of $\pi_1(M)$ has a nontrivial component in $\mathbb{R}^l$ and, hence, the projection onto the second factor of $T^{n-l} \times \mathbb{R}^l$ gives us an embedding of $\pi_1(M)$ into $\mathbb{R}^l$. Since we do have a T^n-action on M, $\pi_1(M)$ must be a full lattice of $\mathbb{R}^l$ which acts on X via a rational homomorphism $\lambda : \mathbb{R}^l \to \mathbb{R}^{n-l} \simeq \mathrm{Lie}\,(T^{n-l})$. We can choose a basis of $\mathbb{R}^l$ so that $\pi_1(M)$ is identified with $\mathbb{Z}^l$ and the i-th generator of $\mathbb{Z}^l$ acts on X as $t_i \in T^{n-l}$. Then the map $X \times \mathbb{R}^{4l}$ to itself given by $(x, y) \mapsto (t_1^{-1}x, y)$ induces an equivariant isomorphism between $\big(X \times \mathbb{R}^{4l}\big)/\mathbb{Z}$ and $X \times (S^1 \times \mathbb{R}^3) \times \mathbb{R}^{4(l-1)}$. We can continue with successive generators of $\mathbb{Z}^l$ and conclude that M is isomorphic to some $X \times (S^1 \times \mathbb{R}^3)^l$. This proves case (ii) of Theorem 1. □

Proof of Corollary 2. This follows from the classification of 4-dimensional quotients of vector spaces by tori, see for example [7]. □

Proof of Theorem 3. This follows from Theorem 1, Proposition 3.2 and the results of [7]. □

Proof of Theorem 4. We have to compute the volume growth of a hyperkähler quotient of a vector space by a torus or its Taub-NUT deformation. The metric on a hyperkähler quotient of a vector space by a torus is asymptotic to a cone metric on a 3-Sasakian space and therefore it has Euclidean volume growth. Now suppose that M is a Taub-NUT deformation of such a quotient of order m. If we diagonalize the matrix $[b_{ij}]$, then $\Phi_{ij} = F_{x_i x_j}$ is of the form $\lambda_{ij} + \sum_{k=1}^{d} a_{ijk}/r_k$ with $\lambda_{ij} = 0$ if $i \neq j$ or $i > m$. The volume growth is then comparable to the volume growth of $(S^1 \times \mathbb{R}^3)^m \times$ (*metric cone of dimension* $4n - 4m$), which is $4n - m$. □

Proof of Theorem 5. Such an action of $SU(2)$ or $SO(3)$ induces the standard $SO(3)$-action on the first factor of $\mathbb{R}^3 \otimes \mathbb{R}^n$. The affine subspaces H_k must be preserved by this action, and so they all pass through the origin. By part (ii) of Proposition 3.2 we have at most n of H_k's. Now the result follows from Theorem 3. □

Proof of Theorem 6. Let $C^*(N)$ denote the punctured metric cone over N. This is a hyperkähler manifold which is complete except for the puncture. The moment map always exists on a 3-Sasakian manifold [8] and so it exists on $C^*(N)$ (where it commutes with dilatations). The flats H_k corresponding to $C^*(N)$ satisfy the condition (ii) of Proposition 3.2 everywhere except the origin. We can therefore, as in the proof of Theorem 1, consider moment map level sets for a generic $(n-1)$-dimensional subtorus. This shows that $C^*(N)$ is locally equivariantly isometric to a Taub-NUT modification of $C^*(N')$ where N' is a 3-Sasakian quotient of a sphere determined by the H_k (existence of N' follows from Theorem 4.1 in [7]). Such a modification, however, can be locally 3-Sasakian if and only if it is of order zero (this follows, for example, from the fact that a higher order modification does not posess a hyperkähler potential in the sense of [24]). Therefore $C^*(N)$ is locally equivariantly isometric to $C^*(N')$. This local isometry f respects the hyperkähler structures and commutes with the action of $\mathbb{R}$ by dilatations. Hence f restricted to N induces the 3-Sasakian structure on its image, which must therefore be contained in N'. Thus we have obtained an equivariant 3-Sasakian isometry from a connected open subset of N to a connected open subset of N'. As in the proof of Theorem 1, real-analycity, compactness and simple-connectedness of S yield the result. □

Proof of Corollary 7. This follows now from Corollary 3 in [5]. □

6. Possible generalizations

Theorem 1 can be generalized in several directions. First of all, one can consider proper $\mathbb{R}^n$-actions instead of T^n-actions. Secondly, we have to ask what happens if M has infinite topological type? Can M be obtained as a hyperkähler quotient of a quaternionic Hilbert space by a Hilbert group (cf. [13])?

Thirdly, we observe that the proof hardly uses the fact that M is a manifold at points lying on singular T^n-orbits. Thus it should be possible to generalize Theorem 1 to orbifolds or even stratified manifolds of certain type. We remark that the Delzant theorem mentioned in the introduction has been generalized to orbifolds by Lerman and Tolman [18]. The statement is that compact symplectic orbifolds of dimension $2n$ with a Hamiltonian T^n-action are classified by labelled simple rational polytopes. In the hyperkähler case the analogous statement would be that complete hyperkähler $4n$-dimensional orbifolds of finite topological type with a tri-Hamiltonian T^n-action are classified, up to Taub-NUT deformations, by labelled simple rational arrangements of codimension 3 affine subspaces in $\mathbb{R}^3 \otimes \mathbb{R}^n$ of the form (3.1) (cf. Theorem 3 and [7]).

As the final generalization, there is a complex-symplectic analogue of Delzant's theorem. Namely, using the slice theorem of Sjamaar [23] and the sheaf-theoretic methods of [18], one can show the following (see also Remark 2.2):

Theorem 6.1. *Let (M^{4n}, I, ω) be a complex-symplectic manifold which also has a Kähler form ω_1. Suppose that there is a holomorphic action of $(\mathbb{C}^*)^n$ on M which is Hamiltonian for ω and such that the induced T^n-action is Hamiltonian for ω_1. Let $\mu : M \to \mathbb{C}^n$ be the moment map for ω and suppose that μ has connected fibers and $\mu(M)$ is Stein and 2-connected. Then (M^{4n}, I, ω) is determined, as a complex-Hamiltonian $(\mathbb{C}^*)^n$-manifold, by $\mu(M)$ and the image of singular $(\mathbb{C}^*)^n$-orbits in $\mu(M)$.* □

This image is a simple arrangement of complex hyperplanes in $\mathbb{C}^n$ satisfying the $\mathbb{Z}$-basis condition of Proposition 3.2(ii). We observe that this result determines the diffeomorphism type of a class of (incomplete) hyperkähler $4n$-dimensional manifolds (not necessarily of finite topological type) with a tri-Hamiltonian T^n-action. In particular, the assumption that μ has connected fibers is satisfied for generic complex structure of such hyperkähler manifold if the hyperkähler moment map is proper [6].

An interesting question in relation to Theorem 6.1 is what happens when M itself is Stein. For example, if M also connected, simply connected and of finite topological type, then is M a complex-symplectic quotient of $\mathbb{C}^{2d}$ by $(\mathbb{C}^*)^{d-n}$?

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