Kleinian Groups Acting On S^3 Which Are Extensions

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1. Introduction.

The main objects of this paper are discontinuous (Kleinian) subgroups of the group $M(n)$ of conformal transformations of $\overline{\mathbf{R}}^n = S^n = \mathbf{R}^n \cup \{\infty\}$.

A group $G \subset M(n)$ is called discontinuous (Kleinian) if there exists a point $x \in S^n$ and a neighbourhood $U(x) \subset S^n$ such that $\{g \in G : gU(x) \cap U(x) \neq \emptyset\}$ is at most finite. The set of all such points forms domain of discontinuity $\Omega(G) \subset S^n$.

We say that a finitely generated Kleinian group $G \subset M(3)$ is a function group if there exists a connected component $\Omega_G \subset S^3$ invariant under the action of G.

One of the most intriguing questions of the present theory is to describe the topological type of the orbifold $M(G) = \Omega(G)/G$ (a manifold in the case when G is torsion-free), in particular, when G is a function group it is important to know in which cases the fundamental group $\pi_1(M_G = \Omega_G/G)$ turns out to be finitely generated.

It was proved in [Ka-P] that the weakest topological version of the well known finiteness theorem of Ahlfors does not hold in higher dimensions. Namely we constructed a function group $F \subset M(3)$ such that the group $\pi_1(\Omega_F/F)$ is not finitely generated.

It has also been shown that there exists a finitely generated Kleinian group with infinitely many conjugacy classes of parabolics [Ka].

On the another hand we constructed a group F_1 without parabolics such that $\pi_1(\Omega_{F_1}/F_1)$ is not finitely generated [P1]. This construction allows one to get such groups as subgroups (of infinite index) of groups arising as boundary points of the deformation space of fundamental groups of hyperbolic 3-manifolds fibering over the circle [P2]. Moreover, such a group can be realized as a subgroup in a discrete co-compact subgroup of $Iso(\mathbf{H}^4)$ [Bo-M].

In this paper we are going to interrupt this chain of negative results on the finiteness problem and prove that under some restrictions on the algebraic structure of a Kleinian group we will have that $\pi_1(M_G)$ is finitely generated.

One can be convinced that all known counterexamples to the finiteness theorem for finitely generated Kleinian groups in higher dimensions arise in the following situation (see

[Ka-P], [Ka], [P1], [P2], [Bo-M]). Suppose we have an exact sequence of finitely generated non-elementary Kleinian groups F and G

$$1 \to F \to G \to \mathbf{Z} \to 1 \tag{1}$$

and let G be geometrically finite, function group satisfying some complementary conditions then $\pi_1(\Omega_F/F)$ is not finitely generated.

Question. Suppose we have the exact sequence (1) of non-elementary function groups acting discontinuously on the main component $\Omega_G = \Omega_F \subset S^3$. Assume that G is geometrically finite then: is the group $\pi_1(M_F = \Omega_F/F)$ finitely generated iff $\pi_1(\Omega_F) \cong 1$?

The aim of this paper is to give a positive answer to this question in the case when G is isomorphic to the fundamental group of a 3-manifold.

We will denote by $X = N\tilde{\times}S^1$ a compact 3-manifold which is a surface bundle over the circle with a compact surface N as fiber. We recall that a normal non-elementary subgroup in a Kleinian group has the same domain of discontinuity.

Theorem 1. *Let $G \subset M(3)$ be a geometrically finite function group any parabolic element of which is of rank two. Suppose that there exists an isomorphism $i : G \to \pi_1(X)$ where $X = N\tilde{\times}S^1$ is a surface bundle over S^1. Then the following assertions are equivalent:*
i) $\pi_1(\Omega_G) \cong \{1\}$
ii) There is a non-trivial normal subgroup $F_0 \triangleleft G$ of infinite index such that $\pi_1(M_{F_0} = \Omega_G/F_0)$ is finitely generated.
iii) For every non-trivial normal subgroup $F \triangleleft G$ of infinite index the group $\pi_1(M_{F_0} = \Omega_G/F_0)$ is finitely generated.

Notice that in general there are infinitely many different fibrations on a given 3-manifold X [T2].

Let us denote by $p : \Omega_G \to M_G = \Omega_G/G$ the natural projection, by $\tau : \pi_1(M_G) \to G$ the epimorphism induced by p and let $i : G \to \pi_1(X = N\tilde{\times}S^1)$ is an isomorphism. Then there is an infinite regular cyclic covering

$$p_1 : M_F \to M_G \quad , \quad F = i^{-1}(\pi_1(N)) , \tag{2}$$

and our next result shows that homologically the manifold M_F is the same as $N \times \mathbf{R}$ which is an infinite cyclic covering of X.

For an incompressible surface $S \subset M_G$ denote by $j_* : H_1(S) \to H_1(M_G)$ the map induced by inclusion $j : S \to M_G$. If also $\tau' = \tau \mid_{\pi_1(S)}: \pi_1(S) \to \pi_1(X)$ then by τ'_* we mean the corresponding map between first homology groups.

Theorem 2. *Suppose that G, F and X are as above and X is a closed 3-manifold fibering over S^1. Then $H_*(M_G, \mathbf{Z}) \cong H_*(X, \mathbf{Z})$. Moreover, there exists an incompressible surface $S \subset M_G$ such that $\tau' = \tau \mid_{\pi_1(S)}: \pi_1(S) \to F$ is onto and the following diagram is commutative*

$$\begin{array}{ccc} H_1(S) & \overset{\tau'}{\to} & H_1(F) \\ j_* \searrow & & \nearrow \varphi \\ & H_1\,(M_F) & \end{array} \tag{3}$$

The map φ is an isomorphism and j_* is an epimorphism.

Remark. Following [Ku] we call (2) a covering of finite type if the group of Deck-transformations of p_1 is of finite cohomological dimension and $H^*(M_F)$ is finitely generated. It was shown in [Ku] that a regular covering of finite type has cohomology either of a point, a circle or a compact surface.

We show in Theorem 2 that in the context of Kleinian groups satisfying the conditions above p_1 is of finite type and the first two possibilities are ruled out.

In the Appendix we will obtain as an application of our method some results concerning hyperbolic 3-manifolds. In particular we give a simplified proof of recent result of M.Boileau and S.Wang containing in ([B-W]) that there is countable set $\mathcal{H}$ of hyperbolic 3-manifolds of finite volume such that for each $M \in \mathcal{H}$ there is an infinite tower of finite coverings

$$...N_i \overset{p_i}{\to} N_{i-1} \overset{p_{i-1}}{\to} ... \overset{p_2}{\to} N_1 \overset{p_1}{\to} M$$

such that all manifolds N_i and M do not fiber over the circle.

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2. Background material and preliminary results.

We start from the finitely generated Kleinian group $G \subset M(3)$ acting discontinuously on $\Omega(G) \subset S^3$ and we will assume that G is function, $G\Omega_G = \Omega_G$ and $M_G = \Omega_G/G$. Suppose also that there exists an isomophism $i : G \to \pi_1(X)$ where X is a compact 3-manifold fibering over the circle.

For the rest of this paper we assume that any parabolic element $g \in G$ (if there is such) is of rank 2.

A group G is called geometrically finite if there exists an ε-neighbourhood $N_\varepsilon\left(M_c^4(G)\right)$ of the convex core of $M_c^4(G)$ in the 4-dimensional hyperbolic manifold $M^4(G) = \mathbf{H}^4/G$ such that $\mathrm{vol}\left(N_\varepsilon\left(M_c^4(G)\right)\right) < +\infty$ (see [Bo], Mo]).

Coming back to the manifold X we notice that there exists unique up to isotopy a system of tori $\mathcal{T} = (T_1, \ldots, T_n)$ in X which are all incompressible, non-parallel and non-isotopic to the boundary such that the components

$$cl(X \backslash \mathcal{T}) = X_1 \cup \cdots \cup X_\ell \tag{2}$$

are either hyperbolic surface bundles over S^1 or Seifert fiber spaces [J-S].

The manifold X is irreducible (each embedded sphere $S \subset X$ bounds a ball) so there exists a map $f : M_G \to X$ compatible with τ. Hence, if $f_* : \pi_1(M(G)) \to \pi_1(X)$ is the map induced by f then the following diagram is commutative

$$\begin{array}{ccc} & G & \\ \tau \nearrow & & \searrow i \\ \pi_1(M_G) & \xrightarrow{f_*} & \pi_1(X) \end{array} \qquad (*)$$

It is easy to see that non-trivial Seifert components never occur in the decomposition (2) if $\pi_1(X)$ is isomorphic to $G \subset M(n)$. Otherwise, if X_i is a Seifert manifold which is not $S^1 \times S^1 \times I$ then there exist non-elementary subgroups $G_i \hookrightarrow G$ having non-trivial center which is impossible [Ma1].

It has been proved in [P3] that we can choose the map f in such a way that $f^{-1}(\mathcal{T})$ is a system $\mathcal{M} = (\Sigma_1, \ldots, \Sigma_r)$ of incompressible surfaces in M_G realizing geometricaly the algebraic splitting

$$G = \left((G_1 *_{H_1} G_2 *_{H_2} \cdots *_{H_{n-1}} G_n) *_{H_n} \cdots \right) \qquad (3)$$

where G_i is isomorphic to a discrete subgroup of $PSL_2\mathbf{C}$ and $H_i \cong \mathbf{Z} \oplus \mathbf{Z}$.

Decomposition (3) follows from the corresponding decomposition of $\pi_1(X)$ which exists due to the well-known torus decomposition of a Haken 3-manifold.

Recall that geometric realization for any expression of the type

$$G = G_1 *_H G_2 \quad \text{or} \quad G = G_1 *_H \qquad (3')$$

means that there exists an incompressible surface $\Sigma \subset M_G$ such that $\tau(\pi_1(\Sigma)) = H$, $\tau(\pi_1(M_i)) = G_i$ for components M_i of $cl(M_G \backslash \Sigma)$ ($cl(\cdot)$ is the closure of a set).

We will use the above mentioned result [P3] to show that the limit set $\Lambda(G)$ of the group G admits some nice description.

Denote by $\operatorname{int} S$ ($\operatorname{ext} S$) the interior of the bounded (unbounded) component of $S^3 \backslash S$ for any surface S embedded in S^3 ($\infty \in \operatorname{ext} S$), $S^3 = \mathbf{R}^3 \cup \{\infty\}$.

Let us consider an infinite collection $L = \bigcup_{i \in I} S_i$ of 2-spheres (topological) satisfying the following conditions

a. $S_i \subset S^3$, $\operatorname{int} S_i \cap \operatorname{int} S_j = \emptyset$, $i \neq j$, $i, j \in I$.

b. There exists at most one point in $S_i \cap S_k$ $(i, j \in I,\ i \neq j)$ which we call the point of tangency. Each point of tangency divides L ($x \in S_i \cap S_k \Rightarrow L \backslash x$ is not connected).

c. For a given $C > 0$ there exists at most a finite number of spheres $S_i \in L$ such that $\operatorname{dia} S_i > C$ (dia *is spherical diameter of the set in S^3), $i \in \{1, 2, \ldots n\}$.*

A finite collection $(S_1, \ldots, S_n)$ of spheres $S_i \in L$ is called a path if $S_i \cap S_k \neq \emptyset$ only for $k \in \{i-1, i+1\}$ where $i \in \{2, \ldots, n-1\}$. An infinite collection Λ_x with the same properties will be called an infinite branch.

d. Every two spheres S_i and S_k can be connected by a path $(S_i, S_{i+1}, \dots, S_k)$.

e. Every branch Λ_x determines the accumulation point $x = \lim_{S_n \in \Lambda_x} S_n$ and for each such point x there is only one branch Λ_x , so if $x \neq y$ are accumulation points of branches Λ_x and Λ_y then Λ_x and Λ_y contains infinitely many different spheres.

We call the point $x = \lim_{S_n \in \Lambda_x} S_n$ the infinite point of the branch Λ_x and denote by T the set of all such points.

Definition. The set $\Lambda = L \cup T$ is a called spherical tree if $\Lambda = cl(L)$ and L satisfies conditions (a - e).

3. Theorem 3.1. *Let G be a group satisfying the assumptions of Theorem 1 then the limit set $\Lambda(G)$ is a spherical tree.*

Proof. Let us prove Theorem 3.1 by induction on the number n of non-isotopic tori in the system $\mathcal{T}$ (see (2)).

When $n = 0$ then G is isomorphic to the fundamental group of a hyperbolic 3-manifold $X = \mathbf{H}^3/\Gamma$, $\Gamma \subset PSL_2\mathbf{C}$ where X is a surface bundle over S^1 [T1].

By assumption the isomorphism $i : G \to \Gamma$ preserves the type of elements of G so by a theorem of Tukia [Tu] $\Lambda(G)$ is homeomorphic to $\Lambda(\Gamma) \cong S^2 = \partial\mathbf{H}^3$ as required.

Suppose now that for $k \leq n$ the assertion of the Theorem is true. We will prove it for $k = n$. Consider the case

$$G = G_k *_H G' \tag{4}$$

where $H \cong Z \oplus Z$, G' is isomorphic to a discrete subgroup of $PSL_2\mathbf{C}$ and $G_k \cong \pi_1(X_k)$. The length of the toral hierarchy in X_k is at most k.

Remark. The case when G is HNN-extension, $G = G_k *_H$, can be given by analogy.

By [P3] we have an incompressible closed surface $\Sigma \subset M_G$ such that $\tau(\pi_1(\Sigma)) = H$, and for components M_i of $cl(M_G \backslash \Sigma)$ we get also $\tau(\pi_1(M_1)) = G_k$, $\tau(\pi_1(M_2)) = G'$ (recall the epimorphism $\tau : \pi_1(M_G) \to G$ comes from the quotient $\pi_1(M_G)/p_*(\pi_1(\Omega_G)) \cong G$).

Let us fix a component $\sigma^0 \in p^{-1}(\Sigma) \subset \Omega_G$ and by choosing basis points in an appropriate way we can assume that $H = \text{Stab}(\sigma^0, G) = \{g \in G : g\sigma^0 = \sigma^0\}$. We know from [P3] that $H \cong \mathbf{Z} \oplus \mathbf{Z}$ and H is generated by two parabolic elements $\alpha, \beta \in G$ and $\Lambda(H) = \text{Fix}(\alpha) = \text{Fix}(\beta) = a$ is the common fixed point.

The set $\sigma = cl(\sigma^0) = \sigma^0 \cup \{a\}$ is a closed surface in S^3. Following [Ma2] we will introduce the following notations.

Denote by B_i' the components of $P^{-1}(M_i)$ adjacent along σ^0 and let B_i be open components of $S^3 \backslash \sigma$ such that $\text{int}(B_i') \subset B_i$ (int$(\cdot)$ is interior of a set). Evidently $M_1 = B_1'/G_k$, $M_2 = B_2'/G'$. Due to the fact that $\Sigma \subset M_G$ is embedded we immediately obtain

$$\begin{array}{ll} g\sigma^0 \cap \sigma^0 = \emptyset, & g \in G\backslash H \\ \gamma B_2 \cap B_2 = \emptyset, & \gamma \in G_k\backslash H \\ \gamma B_1 \cap B_1 = \emptyset, & \gamma \in G'\backslash H \\ h\sigma^0 = \sigma^0, & h \in H. \end{array} \tag{5}$$

Moreover we have

Lemma 3.2. $g\sigma \cap \sigma = \emptyset$, $g \in G\backslash H$.

The proof of this Lemma is contained in [P3] but we present it here briefly. Suppose to the contrary that for some $\gamma \in G\backslash H$, $\gamma\sigma \cap \sigma \neq \emptyset$ then by (5) $\gamma a = a$ and $\gamma \notin H$. Hence, there exists a non-trivial extension H_1 of H in G which is also abelian of rank 2. In the manifold X the group $L = i(H)$ is realized by an embedded torus $T \in \mathcal{T}$.

By [Fe], $\pi_1(T)$ is the maximal abelian group of $\pi_1(X)$, $H = H_1$ and the Lemma is proved.

We claim now that each parabolic element in G_k or in G' has rank 2. This follows from the assertion below.

Lemma 3.3. *Let X be a Haken manifold and $T \subset X$ an embedded incompressible torus inducing the decomposition $\pi_1(X) = \Gamma_1 *_{\pi_1(T)} \Gamma_2$ for the dividing torus T and $\pi_1(X) = \Gamma_1 *_{\pi_1(T)}$ otherwise. If there exist $a \in \Gamma_1$, $b \in \pi_1(X) \setminus \Gamma_1$ such that $[a,b] = 1$ then there exists an element $b_1 \in \Gamma_1 \setminus < a >$ such that $[a, b_1] = 1$.*

For the proof see [P3, Lemma 5].

By Lemma 3.3 and our induction assumption it now follows that $\Lambda_k = \Lambda(G_k)$ and $\Lambda' = \Lambda(G')$ are spherical trees.

Let us consider now an arbitrary reduced word $w \in G$

$$w = \gamma_1 \cdot \gamma_2 \cdots \gamma_n, \text{ where } \gamma_i \in G'\backslash H_i, \gamma_{i+1} \in G_k\backslash H \text{ or } \gamma_i \in G_k\backslash H_i, \gamma_{i+1} \in G'\backslash H. \tag{5}$$

For a given word w the number n does not depend on the decomposition (5) and is called the length $|w|$ of w [Ma2].

It is easy to see that $\Lambda_k \subset B_1$, $\Lambda' \subset B_2$ and $\{a\} = \Lambda_k \cap \Lambda' \cap \sigma$. Let us consider the following sets

$$T_n = \bigcup_m (w_{mn}(\Lambda_k) \cup w'_{mn}(\Lambda')) \tag{6}$$

where the union is taken over all words w_{mn} (w'_{mn}) in G of length n where the last letter γ_n in the decomposition (5) of w_{mn} (w'_{mn}) belongs to $G'(G_k)$.

It was shown in [Ma2] that the limit set $\Lambda(G)$ has the following description

$$\Lambda(G) = \bigcup_{n=1}^{\infty} T_n \cup T, \tag{7}$$

where $T = \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} w_{mn}(B_1) \cup \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} w'_{mn}(B_2)$ so every $x \in T$ arises as an intersection of nested domains $x = \bigcap_{n=1}^{\infty} w_{m_n n}(B_1)$ or $x = \bigcap_{n=1}^{\infty} w'_{m_n n}(B_2)$ where the sequence m_n is determined by x.

The set $L = \bigcup_{n=1}^{\infty} T_n$ is a union of spheres satisfying condition a. of the definition of the spherical tree.

Step 1. Condition c. Consider now an infinite sequence of different spheres $\tau_n \subset L$ then $\tau_n \subset w_n(B_i)$ and by our induction hypothesis w_n are different modulo $H = G_k \cap G'$. By Koebe Lemma [Ma2] we get $\lim_{n\to\infty} w_n(B_i) = 0$ and so condition c. now follows.

Step 2. Condition b. Suppose now $\{S_i, S_j\} \subset L$ and $S_i \cap S_j \neq \emptyset$. We want to show that there exists $t \in G$ such that $t(S_i \cap S_j) = \Lambda_k \cap \Lambda' = a$.

Claim 3.4. Let $|g| > 1$ then $g\Lambda_k \cap \Lambda_k \neq \emptyset$ and $g\Lambda' \cap \Lambda' = \emptyset$; moreover if $|g| > 2$ then $g\Lambda' \cap \Lambda_k = \emptyset$.

Proof. Recall g has a decomposition of the form (5). We will prove that $g\Lambda_k \cap \Lambda_k \neq \emptyset$ if $|g| > 1$ (the inequality $g\Lambda' \cap \Lambda' \neq \emptyset$ is similar).

Suppose at first that $\gamma_n \in G_k$ then $\gamma_{n-1}\gamma_n\Lambda_k \subset \gamma_{n-1}B_1 \subset B_2$, $\gamma_{n-1}\gamma_n\Lambda_k \cap \partial B_2 = \emptyset$. Hence, $g\Lambda_k \subset \text{int}(\gamma_1 \cdots \gamma_{n-2}B_2) \subset \text{int}(\gamma_1 B_2)$ if $\gamma_1 \in G_k$ and $g\Lambda_k \subset \text{int}(\gamma_1 B_1)$ if $\gamma_1 \in G'$. In both cases $g\Lambda_k \cap cl(B'_1) = \emptyset$ and $g\Lambda_k \cap \Lambda_k = \emptyset$ because $\Lambda_k \subset cl(B'_1)$.

If now $\gamma_n \in G'$ then the same observations lead us to the conclusion $g\Lambda_k \subset \gamma_1 \cdots \gamma_{n-1}(B_2)$ and also $g(\Lambda_k) \cap cl(B'_1) = \emptyset$ and we are done.

Assume now $g\Lambda' \cap \Lambda_k \neq \emptyset$, $|g| > 2$ and either $\gamma_n \in G_k$ in which case $g\Lambda' \subset \text{int}\, g\gamma_n^{-1}(B_1)$ and thus $g\Lambda' \cap \Lambda_k = \emptyset$ or $\gamma_n \in G'$ and, hence, $g\Lambda' \subset \text{int}(g\gamma_n^{-1}\gamma_{n-1}^{-1})(B_1)$. We obtain also $g\Lambda' \cap \Lambda_k = \emptyset$. Thus $g\Lambda' \cap \Lambda_k \neq \emptyset$ is possible iff $g \in G_k$ or $g = \gamma_1 \cdot \gamma_2$, $\gamma_1 \in G_k$, $\gamma_2 \in G'$. Claim is proved.

By (6 - 7) there exists a pair of elements $\{g_i, g_j\} \in G$ such that $g_iS_i = S'_i$, $g_jS_j = S'_j$ for which $\{S'_i, S'_j\} \subset \Lambda' \cup \Lambda_k$.

We will prove condition b. in the assumption that $S'_i \in \Lambda_k$, the case $S'_i = \Lambda'$ can be done by analogy.

We have $S'_i \cap g_ig_j^{-1}(S'_j) \neq \emptyset$ and notice that if $S'_j \in \Lambda_k$ then by Claim 3.4 $g_ig_j^{-1} \in G_k$ and $S''_j = g_ig_j^{-1}(S'_j) \in \Lambda_k$. Thus, by induction, the point $x = S'_i \cap S''_j$ divides Λ_k. On the other hand the point $a = \Lambda(H) = \Lambda_k \cap \Lambda'$ divides L. Otherwise there exists a simple loop $\alpha \subset L$ such that $\alpha \cap \Lambda_k \neq \emptyset$, $\alpha \cap \Lambda' \neq \emptyset$, $a \in \alpha$. Here the intersection index between homology classes of α and surface σ is non-trivial ($\sigma \cap \Lambda(G) = \{a\}$) which is impossible because σ divides S^3. Thus, $L \backslash \{a, x\}$ contains 3 components and $L \backslash \{x\}$ is not connected. It follows now that $g_i^{-1}(x) = S_i \cap S_j$ divides L.

It remains to consider the case when $S'_j = \Lambda'$. Notice that $S'_i \cap g_ig_j^{-1}(\Lambda') \neq \emptyset$ and $\Lambda_k \cap g'(\Lambda') \neq \emptyset$ for $g' = g_ig_j^{-1} \in G$. By Claim 3.4, $g' = \gamma_1 \cdot \gamma_2$ or $g' = \gamma_1$, $\gamma_1 \in G_k$, $\gamma_2 \in G'$. Hence, $\Lambda_k \cap g'(\Lambda') = \gamma_1(\Lambda_k \cap \Lambda') = \gamma_1(a)$ because $\gamma_1\Lambda_k = \Lambda_k$.

Finally, we have proved that if $S_i \cap S_j \neq \emptyset$ then there always exists $t \in G$ such that $t(S_i \cap S_j) = \Lambda_k \cap \Lambda' = a$. As we have shown the point a divides L so is $t^{-1}(a)$. Condition b. is proved.

Step 3. Condition d. We will prove that for $S \subset L$ there exists a finite path $(S_1, \ldots, S)$ connecting S with $S_1 = \Lambda'$ or with $S_1 \in \Lambda_k$ and $S = g(S_1)$, $g \in G$.

Let us prove this by induction on the length of g. If $|g| = 1$ then we are given four possibilities.

I. $S_1 \in \Lambda_k$, $g \in G_k$.

II. $S_1 \in \Lambda_k$, $g \in G'$.

III. $S_1 \in \Lambda'$, $g \in G_k$.

IV. $S_1 \in \Lambda'$, $g \in G'$.

Consider first II. We have that S_1 can be connected by a path ℓ with the sphere $\tau \in \Lambda_k$ such that $\tau \cap \Lambda' = \Lambda_k \cap \Lambda' = a$ due to our assumption that Λ_k is a spherical tree. Hence $g(\tau) \cap \Lambda' \neq \emptyset$ and we get a path $\ell \cup g(\ell) \cup \Lambda' = (S_1, \ldots, \tau, \Lambda', g(\tau), \ldots, S)$.

Case III is analogous to II and cases I and IV are trivial.

Suppose we have proved our assertion for all words of length less than n and let $S = \gamma_1 \gamma_2 \cdots \gamma_n(S_1)$, $\gamma_i \in (G_k \cup G')\backslash H$. Thus, $S' = \gamma_2 \cdots \gamma_n(S_1)$ can be connected by a path ℓ' with S_1. Again there are two possibilities: $\gamma_1 \in G_k$ or $\gamma_1 \in G'$. Considering the first we obtain $S'' = \gamma_1(S_1)$ which can be connected with S_1 by a path ℓ''. By the considerations above the path $\ell = \gamma_1 \ell' \cup \ell'' \cup \ell'$ is required.

Step 4. Consider an infinite branch $\Lambda_x \subset \Lambda(G)$ consisting of spheres $\{S_1, \ldots, S_n, S_{n+1}, \ldots\}$, $S_m \cap S_n \neq \emptyset$ only for $m \in \{n+1, n-1\}$.

Condition e. will now follow from

Lemma 3.5. *The embedding $\Lambda(G) \hookrightarrow S^3$ has the property that if $\Lambda_x \neq \Lambda_y$ then there exists a constant C (depending on Λ's) such that* $\mathrm{dia}(\Lambda_x, \Lambda_y) \geq C > 0$.

Proof. First let us show that for every point $x \in T$ of formula (7) we can find the appropriate infinite branch Λ_x. Suppose for concreteness $x = \bigcap_{n=1}^{\infty} w_{m_n n}(B_1) = \lim_{n \to \infty} w_{m_n n}(\tau)$, with $\tau \in \Lambda_k$, $\tau \cap \Lambda' = a$. Connecting each sphere $w_{mn}(\tau)$ with τ by a path ℓ_n; by property d. we get the infinite branch $\Lambda_x = \bigcup_{n=1}^{\infty} \ell_n$ with x as infinite point.

Suppose now we are given two branches Λ_x and Λ_y such that $\Lambda_x \neq \Lambda_y$. By step 3, up to a finite collection of spheres, we can assume that the branches $\Lambda_x = (\mu_1, \mu_2, \ldots)$, $\Lambda_y = (\xi_1, \xi_2, \ldots)$ have common first sphere $\mu_1 = \xi_1$ and all the rest distinct. Let $t_1 = \mu_2 \cap \mu_2$, $t_2 = \mu_1 \cap \xi_2$ and $e_1 = \Lambda_x \backslash \mu_1$, $e_2 = \Lambda_y \backslash \mu_1$. By step 2 we get $w_i \in G$ such that $w_i(t_i) = a = \sigma \cap \Lambda(G)$. Take surfaces $\sigma_i = w_i^{-1}(\sigma)$ invariant under groups $w_i H w_i^{-1}$, $\sigma_i \cap \Lambda(G) = t_i$.

We claim now that there exist two components of the set $cl(S^3 \backslash \sigma_i)$ such that $e_i \subset \sigma_i^-$ and $\sigma_1^- \cap \sigma_2^- = \emptyset$. Indeed, the surface σ_1 (respectively σ_2) separates the sphere μ_1 from

μ_2 (respectively μ_1 from ξ_2), it means that μ_1 and μ_2 (respectively μ_1 and ξ_2) lie in different components of $cl(S^3\backslash\sigma_1)$ $(cl(S^3\backslash\sigma_2))$. By connectedness of e_i and the fact that $\sigma_i \cap \Lambda(G) = t_i$ we get that $e_i \subset \sigma_i^-$ for components $\sigma_i^- \subset S^3\backslash\sigma_i$ not containing μ_1.

Now σ_i are closed in S^3 and $\sigma_1 \cap \sigma_2 = \emptyset$. Hence, $\text{dia}(\sigma_1, \sigma_2) > 0$. Lemma 3.5. and Theorem 3.1 are proved.

4. Suppose that Λ is a spherical tree and T is the set of all infinite points of Λ then our main goal now is

Theorem 4.1. $H_1^o(\Lambda, T) \cong \widetilde{H}_0^o(T)$ (open, reduced, with integer coefficients).

Proof. Consider an arbitrary cycle $z \in Z_1^o(\Lambda, T)$. Let $\partial_* z = \Sigma x_i$ where $x_i \in T$ and ∂_* is the boundary z in $C_1^o(\Lambda)$.

As we have shown before there exists a unique infinite branch Λ_{x_i} of the tree Λ the infinite point of which is x_i ($\Lambda_{x_i} = \bigcup\limits_{n=1}^{\infty} S_n^i$, $\lim\limits_{n\to\infty} S_n^i = x_i$).

It is easy to see that z intersects Λ_{x_i} in an infinite number of spheres $S_n \in \Lambda_{x_i}$. Now if $z \cap S_r \neq \emptyset$ and $z \cap S_m \neq \emptyset$, $m > r$ then $z \cap S_{r+j} \neq \emptyset$ because each S_{r+j} divides Λ , here $S_{r+j} \in \Lambda_{x_i}$ $(j = 1, \ldots, m-r)$.

Hence,

$$z \cap \Lambda_{x_i} = \sum_j z_{x_j}^j,$$

where $z_{x_i}^j$ are half-infinite connected 1-chains from $C_1^o(\Lambda)$ such that $z_{x_i}^j \subset \Lambda$ and the infinite point of $z_{x_i}^j$ is exactly $x_i \in T$. The last formula follows from the following simple observation:

"If z leaves Λ_{x_i} in a point b then z can return to Λ_{x_i} only through b".

This slogan is true because b divides Λ. So, the chain $w_j = z_{x_i}^j + z_{x_i}^{j+1}$ $(j \geq 2)$ is a closed path in Λ_{x_i}. On the other hand $w_j = \sum\limits_n w_n^j$, where w_n^j is a closed curve in sphere $S_n \in \Lambda_{x_i}$, which again follows from the fact that the points $S_n \cap S_{n+1}$ divides Λ.

Thus, our cycle z is homologous to the cycle

$$\widehat{z} = \sum_{\mu \in M} z_\mu,$$

where $z_\mu \in C_1^o(\Lambda, T)$, $\partial_* z_\mu = \{x_\mu, x_0\} \in T$, $\text{int}\, z_\mu \cap T = \emptyset$, $x_\lambda \cap x_\mu = \emptyset$, $\lambda, \mu \in M$, $\lambda \neq \mu \neq 0$.

Indeed, we just fix one point $x_0 \in \partial_* z$ and consider all infinite 1-chains $z_\mu \subset z$ connecting x_0 with any point $x_\mu \in \partial_* z$, by previous considerations z_μ does not contain any other point from T and moreover x_μ is a boundary of unique 1-chain $z_\mu \in C_1^o(\Lambda, T)$.

Say that $\widehat{z}$ is a standard representative of the class $[z]$ in $H_1^o(\Lambda, T)$. Suppose that we have now two elements $[z]$ and $[w]$ from this group such that $\partial_*[z] = \partial_*[w]$. Let us take standard representatives

$$\widehat{z} = \sum_{\mu} z_{\mu},$$

$$\widehat{w} = \sum_{\mu} w_{\mu}.$$

We have $\partial_* z_\mu = \partial_* w_\mu$ and the cycle $u_\mu = z_\mu - w_\mu$ is homologous to zero. Indeed, $u_\mu \cap T = \partial_* z_\mu = \partial_* w_\mu$ and by using again the fact that each point $x_\mu \in \partial_* z_\mu$ is determined by a unique branch Λ_x we get

$$u_\mu = \sum_j u_\mu^i, u_\mu^j \subset S_j.$$

Thus, $[z] = [w]$ in $H_1^o(\Lambda, T)$. Lemma is proved.

Corollary 4.2. $H_1(\Lambda) = 0$ and $H_1(\Omega_G) = 0$

Proof. Indeed from the exact sequence of pair we have

$$0 \to H_1^o(\Lambda) \to H_1^o(\Lambda, T) \to \widetilde{H}_0^o(T) \to 0$$

and by Theorem 4.1 we obtain $H_1^o(\Lambda) = 0$ but Λ is closed so, $H_1^o(\Lambda) = H_1(\Lambda) = 0$.

From Theorem 3.1 and the formula above it follows that $H_1(\Lambda(G)) \cong 0$, so by Alexander duality we immediately obtain

$$H^1\left(S^3 \backslash \Lambda(G)\right) = H_1\left(\Omega_G\right) = H_1(\Lambda(G)) \cong 0. \tag{8}$$

The Corollary is proved. *QED.*

6. Proof of Theorem 1.

Given Corollary 4.2 the proof of Theorem 1 which we restate below is now fairly easy.

Theorem 1. *Let $G \subset M(3)$ be a geometrically finite function group any parabolic element of which is of rank two. Suppose that there exists an isomorphism $i : G \to \pi_1(X)$ where $X = N \widetilde{\times} S^1$ is a surface bundle over S^1. Then the following assertions are equivalent:*

i) $\pi_1(\Omega_G) \cong \{1\}$

ii) There is a non-trivial normal subgroup $F_0 \triangleleft G$ of infinite index such that $\pi_1(M_{F_0} = \Omega_G/F_0)$ is finitely generated.

iii) For every non-trivial normal subgroup $F \triangleleft G$ of infinite index the group $\pi_1(M_{F_0} = \Omega_G/F_0)$ is finitely generated.

Proof of Theorem 1. Obviously the condition ii) is the weakest one, so to prove the Theorem it is enough to show the equivalence: $i) \Longleftrightarrow ii)$.

The implication $i) \Longrightarrow ii)$ is obvious. Let us prove the converse statement.

The group F_0 is normal in G and non-elementary, so they both have the same limit set and act on the same component Ω_G [Ma1]. The maps $\Omega_G \xrightarrow{p_2} M_{F_0} \xrightarrow{p_1} M_G$ are regular infinite coverings, where $p = p_1 \circ p_2 : \Omega \longrightarrow M_G = \Omega_G/G$ is the natural projection.

If $\pi_1(M_{F_0})$ is finitely generated then by Stallings fibration theorem [He, Th. 11.1] the group $\pi_1(M(F_0))$ is isomorphic to either the fundamental group of a closed surface or to a free group.

Hence, for $\pi_1(\Omega_G)$ one has the following trihotomy: either it is isomorphic to the fundamental group of a closed surface, or it is isomorphic to a free group, or is trivial group.

Notice that the first two possibilities are ruled out since $\pi_1(\Omega_G) = [\pi_1\Omega_G, \pi_1\Omega_G]$ by Corollary 4.2. The remaining possibility is the desired one. The Theorem is proved. *QED.*

6. Proof of Theorem 2.

We will assume that $i : G \to \pi_1(X)$ is an isomorphism and the manifold X is a surface bundle over a closed surface N. Let $p : \Omega_G \to M_G = \Omega_G/G$ be a regular covering and $G \subset M(3)$ be a function group acting on an invariant component $\Omega_G \subset S^3$.

We recall that $\tau : \pi_1(M_G) \to G$ is a natural isomorphism induced by p, $F = i^{-1}(\pi_1(N))$ and $p_1 : M_F = \Omega_G/F \to \Omega_G/G$ is an infinite regular cyclic covering.

Theorem 2. *Suppose that G, F and X are as above and X is a closed 3-manifold fibering over S^1. Then $H_*(M_G, \mathbf{Z}) \cong H_*(X, \mathbf{Z})$. Moreover, there exists an incompressible surface $S \subset M_G$ such that $\tau' = \tau\mid_{\pi_1(S)}: \pi_1(S) \to F$ is onto and the following diagram is commutative*

$$\begin{array}{ccc} H_1(S) & \stackrel{\tau'}{\to} & H_1(F) \\ j_* \searrow & & \nearrow \varphi \\ & H_1(M_F) & \end{array} \tag{3}$$

The map φ is an isomorphism and j_* is an epimorphism.

Let us consider again the commutative diagram $(*)$ in §2. By [He, Lemma 6.5], we have up to small isotopy that $f^{-1}(N) = \mathcal{M} = \{S_1, \cdots, S_m\}$ is a collection of incompressible surfaces in M_G. We can choose a loop $\beta \subset \pi_1(X)$ the intersection number of which with N is one.

Then for any loop $\alpha \subset M_G$ which meets $f^{-1}(N)$ transversely $f_*([\alpha]) = [\beta]^n$ where n is the sum of "signed" intersection numbers of α with $\mathcal{M}$. Choosing α in such a way that $n = 1$ we get that $M_G \backslash S_i$ is not connected for some $S_i \in \mathcal{M}$. Put $S = S_i$

Step 1. *Let $F_0 = \tau(\pi_1(S))$, then*

$$G = \Gamma *_{F_0} \tag{10}$$

Proof. Consider one preimage $\Sigma_1 \in p^{-1}(S)$. All images of Σ_1 are disjoint, so

$$g\Sigma_1 \cap \Sigma_1 = \emptyset, g \in G\backslash F_0 \tag{11}$$

$$f\Sigma_1 = \Sigma_1, f \in F_0,$$

where $F_0 = \mathrm{Stab}(\Sigma_1, G)$.

By Alexander duality $H^2(\Omega_G) = H_1^o(\Omega_G)$ (open, with integer coefficients), and $H_1^o(\Omega_G) = H_1(S^3, \Lambda(G)) \cong \widetilde{H}_0(\Lambda(G))$. By Theorem 3.1 $\Lambda(G)$ is connected, so $H_2(\Omega_G) \cong 0$. It follows that Σ_1 divides Ω_G.

Take a component $\widetilde{M} \in p^{-1}(M)$ of the manifold $M = cl(M_G \backslash S)$ such that $\Sigma_1 \in \partial\widetilde{M}$ and let $\Gamma = \mathrm{Stab}(\widetilde{M}, G) = \{g \in G : g\widetilde{M} = \widetilde{M}\}$. Obviously $M = \widetilde{M}/\Gamma$. The conclusion of Step 1 now follows from [Ma 2]. For completeness we present here these considerations.

Let $\gamma = \tau([\alpha])$ then $\gamma(\mathrm{int}\,\widetilde{M}) \cap (\widetilde{M}) = \emptyset$ and, if $\Sigma_2 = \gamma\Sigma_1$ we denote by B_i open components of $\Omega_G \backslash \Sigma_i$ not containing $\widetilde{M}$ and it is not hard to see that up to choice of notations $\gamma(B_1) = \mathrm{int}(\Omega_G \backslash B_2)$. Let also $H_i = \mathrm{Stab}(\Sigma_i, G)$ and $H_2 = \gamma H_1 \gamma^{-1}$.

The group G is generated by Γ and γ because $\pi_1(M_G) = \langle \pi_1(M), [\alpha] \rangle$. Now for an arbitrary word w we get
$w = \gamma^{\alpha_n} \cdot g_n \cdot \gamma^{\alpha_{n-1}} \cdot g_{n-1} \cdots \gamma^{\alpha_1} \cdot g_1$
$g_j \neq 1, \alpha_i \neq 0, \quad i \in (1, \cdots, n-1), j \in (2, \cdots, n)$.

Following [Ma 2, §5] we prove that $w(x) \neq x$ for any $x \in \mathrm{int}\,\widetilde{M}$. Indeed, $\gamma^{\alpha_1} \cdot g_1(x) \in B_1 \cup B_2$ (because, $\gamma\widetilde{M} \subset B_2, \gamma^{-1}\widetilde{M} \subset B_1$). Further $g_2 \cdot \gamma^{\alpha_1} \cdot g_1(x) \in \mathrm{int}(g_2(B_2 \cup B_1)) \subset S^3 \backslash \widetilde{M}$ and finally $w(x) \notin \mathrm{int}\,\widetilde{M}$ so every relation in G is a consequence of relations in Γ or the relation $H_2 = \gamma H_1 \gamma^{-1}$ as required.

Step 2. F_0 *contains a subgroup* K_0 *isomorphic to the fundamental group of a closed surface.*

Proof. Consider the decomposition of $\pi_1(X)$ arising from (10).

$$\pi_1(X) = \Gamma'_{*_{F_0'}}, \quad \Gamma' = i(\Gamma), \quad F_0' = i(F_0) \qquad (10')$$

Let us construct 2-dimensional CW-complexes Y_1 and Y_2 such that $\pi_1(Y_1) \cong \Gamma'$, $\pi_1(Y_2) \cong F_0'$ and a complex $Y = Y_1 \cup (Y_2 \times I)$ for which $\pi_1(Y) \cong \pi_1(X)$ and let j this isomorphism. There is a map $\psi : X \to Y$ inducing j, $\psi_* = j$. Again by small isotopy $\psi^{-1}(Y_2 \times \{1/2\})$ is a collection of incompressible closed surfaces $\mathcal{R} = (R_1, \cdots, R_d)$.

By construction, $\psi_*(\pi_1(R_i)) \subset \pi_1(Y_2)$. QED

Step 3. $F_0 = F$.

Proof. We have $f_*(\pi_1(S)) = i \cdot \tau(\pi_1(S)) = i(F_0) \subset \pi_1(N)$ (see diagram (*) in item 2.) and $F_0 \subset F = i^{-1}(\pi_1(N))$. By step 2 we get an incompressible surface $R \subset X$ such that $\pi_1(R) = K_0' \subset F_0' \subset \pi_1(N)$, so index $|\pi_1(N) : K_0'|$ is finite. The group K_0' is realized by the fundamental group of the closed embedded surface R, so K_0' is the maximal surface group in $\pi_1(X)$ [Fe], hence, $K_0' = \pi_1(N)$ and we immediately obtain $F_0' = \pi_1(N), F_0 = F$. QED

Consider now the commutative diagram

$$\begin{array}{ccc} \pi_1(M_G) & \xrightarrow{\tau} & G \\ \mu_1 \downarrow & & \downarrow \mu_2 \\ H_1(M_G) & \xrightarrow{\tau_*} & H_1(G) \end{array} \qquad (11)$$

where μ_i are "abelianization" maps and τ_* is induced by τ. It is easy to see that τ_* is onto. Suppose now $\tau_*(g) = 0$, $g \in H_1(M_G)$ and $g' \in \pi_1(M_G)$ for which $\mu_1(g') = g$. If $g'' = \tau(g')$ then $g'' \in [G, G]$ (the commutator subgroup) and $\mu_2(g'') = 0$. So, we find that $g' = q \cdot w$ where $q \in [\pi_1(M_G), \pi_1(M_G)]$ and $w \in p_*(\pi_1(\Omega))$, hence, $\mu_1(g') = \mu_1(w) = 0$, because each preimage in $p^{-1}(w)$ is a cycle homologous to zero by (8).

We proved that τ_* is isomorphism.

By duality, $H^2(M_G) \cong H_1(M_G) \cong H^2(X)$, and $H_2(M_G) \cong H_2(X)$ because $H_1(M_G) \cong H_1(X)$.

If now $p_1 : M_F \to M_G$ is a regular cyclic covering then the considerations above give an isomorphism $\varphi : H_1(M_F) \to H_1(F)$ and if $\tau'_*(s) = 0$ for $s \in H_1(S)$ then $s \in \mu_1((p_1)_*(\pi_1(\Omega_G)))$ and $j_*(s) = 0$ (see diagram (3)), here $(p_1)_* : \pi_1(M_F) \to \pi_1(M_G)$ is monomorphism induced by p_1. Theorem 2 is proved.

7. Appendix. Some applications.

As application of the above method we will give (our Corollary 7.2) a of the following.

Theorem [Boileau-Wang, B-W]. *There is countable set $\mathcal{H}$ of hyperbolic 3-manifolds of finite volume $\mathcal{H} = \{M_1, ..., M_n ...\}$ such that for each $M \in \mathcal{H}$ there is an infinite tower of finite coverings p_i*

$$...N_i \xrightarrow{p_i} N_{i-1} \xrightarrow{p_{i-1}} ... \xrightarrow{p_2} N_1 \xrightarrow{p_1} M \qquad (12)$$

and all manifolds N_i and M do not fiber over the circle.

This related somehow to the well-know Thurston's conjecture that is any finite volume hyperbolic 3-manifold finitely covered by a circle bundle?

Obviously, we can not garantee that (12) gives the complete list of all finite coverings over M. Our proof does not use degree one maps which was a crucial tool of [B-W] and seems to be more elementary. Although we also use essentially the notion of totally null-homotopic knots due to [BDM]. Our proof is based on the following simple fact which was used somehow in the previous chapter.

Key Observation. *Suppose that M is a compact 3-manifold which admits an infinite covering $p : \Omega \to M$ such that $H_1(\Omega, \mathbf{Z}) = 0$ (with compact support) and Ω is not simply connected then M does not fiber over the circle.*

Proof. Suppose to the contrary that M fibers over the circle. Then there is an infinite cyclic covering $p_1 : M(F) \to M$ corresponding to a normal surface subgroup $f \triangleleft \pi_1(M)$ and we have the diagram of coverings

$$\begin{array}{ccc} \Omega & & M(F) \\ \searrow p & & \swarrow p_1 \\ & M & \end{array} \tag{13}$$

Our goal is to show that the diagram (13) can be made commutative or else that the p_1 can be lifted to a covering $p_2 : \Omega \to M(F)$.

Indeed, we have $p_*([\pi_1\Omega, \pi_1\Omega] = \pi_1\Omega) \subset [\pi_1 M, \pi_1 M] \subset \pi_1 M$, where we denote $[G, G]$ the commutator subgroup of G. On the other hand $[\pi_1 M, \pi_1 M] \subset (p_1)_*(\pi_1 M(F))$, hence $p_*([\pi_1\Omega, \pi_1\Omega]) \subset (p_1)_*(\pi_1 M(F))$.

We have proved that there is a covering $p_2 : \Omega \to M(F)$ such that $p = p_1 \circ p_2$. The group $\pi_1(M(F))$ is isomorphic either to the fundamental group of a closed surface or to a free group of finite rank, so for $\pi_1\Omega$ we have three possibilities: a) $\pi_1\Omega$ is trivial group, b) $\pi_1\Omega$ is the fundamental group of a closed surface or c) $\pi_1\Omega$ is a free group (possibly of infinite rank).

The first possibility is ruled out by the hypothesis and the last two are impossible since $[\pi_1\Omega, \pi_1\Omega] = \pi_1\Omega$. The result follows. *QED*.

Let in our Key Observation $M = \Omega/G$ where $G = Deck(p)$.

Corollary 7.1. *For any subgroup of finite index $H \subset G, |G : H| \leq \infty$ the manifold $M(H) = \Omega/H$ does not fiber over the circle.*

Proof. Repeat the proof of the Observation for $M(H)$. *QED*.

Remark. One can think of Ω as an invariant component of torsion-free Kleinian group $G \subset M(3)$ acting on S^3. In particular, an example of such a covering is given by B.Apanasov and A.Tetenov [A-T] where G is isomorphic to the fundamental group of a hyperbolic 3-manifold $N = \mathbf{H}^3/\Gamma$, $G \cong \Gamma$, the group G is itself Kleinian whose limit set $\Lambda(G)$ is a wild sphere embedded in S^3. By the Alexander duality one has here $H_1(\Lambda(G), \mathbf{Z}) = H_1(\Omega(G), \mathbf{Z}) = 0$, ($\Lambda(G)$ is a closed set, so all homologies coincide).

The construction insures that one of the components $S^3 \setminus \Lambda(G)$, say Ω, is not simply connected. So, by the above Observation the manifold $M(G) = \Omega/G$ does not fiber over the circle as well as all of its finite coverings of the type $M(G_n) = \Omega/G_n$ where G_n is a finite-index subgroup of G. Note that by the construction $M(G)$ is not itself hyperbolic since contains an incompressible torus obtained by the projection of an incompressible torus in discontinuity domain. *End of the Remark*.

The rest of the Chapter is devoted to the proof that there exists a countable set $\mathcal{H}$ of finite volume hyperbolic 3-manifolds such that for every $M \in \mathcal{H}$ there is a regular infinite covering $\Omega \to M$ for which $H_1(\Omega, \mathbf{Z}) = 0$ and $\pi_1\Omega \ncong 1$.

We need to introduce some terminology (see [BDM] and [B-W]). Say that the simple closed curve $c \subset M$ is totally null-homotopic knot in M if c bounds a singular disk $D \subset M$ whose regular neighborhood $N(D)$ embeds trivially which means $i_*(\pi_1(N(D)) = 1$ where $i : N(D) \to M$ is the natural embedding.

Let $c \subset M$ is totally null-homotopic knot, by $M(n, m)$ we denote the result of (n, m)-surgery on c (i.e. we first delete a regular neighborhood of c and then glue back the solid torus such that the new meridian will have (n, m)-slope in the surgered manifold). If c is totally null-homotopic one defines in a standard way the longitude-meridian system (a, b) where b is chosen to be homologically trivial in $\overline{M \setminus N(c)}$ [BDM].

Let $p : \tilde{M} \to M$ be a regular covering then the restriction of p to each component of $\tilde{C} = p^{-1}(c)$ ($\tilde{C}$ possibly infinite) is a homeomorphism since c is homotopically trivial in M. Moreover since c is totally null-homotopic, all components of both $p^{-1}(N(D))$ and $p^{-1}(N(c))$ are disjoint and the map p restricted to each of them is a homeomorphism too. Thus, we can choose a component $\tilde{a}_i$ of $p^{-1}(a)$ and $\tilde{b}_i$ of $p^{-1}(b)$ which give the meridian-longitude system on the boundary of some component $\tilde{N}_i$ of $\tilde{N}(D) = p^{-1}(N(D)$. In fact, $\tilde{b}_i \in p^{-1}(b)$ is the longitude since it is homologically trivial in $\tilde{N}_i$ and all $\tilde{N}_k$ are disjoint, so it is homologically trivial in the whole manifold $\tilde{M} \setminus p^{-1}(N(D))$.

The significance of the notion of totally null-homotopic knots is that any surgery on $c \subset M$ can be lifted to a simultaneous surgery with the same coefficients on $\tilde{C} \subset \tilde{M}$. Specifically, we remove the lift $\tilde{N}(D)$ from $\tilde{M}$ and obtain $\tilde{M}(n, m)$ by doing the (n, m)-surgery simultaneously on all components of $\partial\tilde{N}_i$. The following lemma is a simple extension of [B-W, Lemma 5.3] to the case of infinite covering p.

Lemma 7.2. Let $p : \tilde{M} \to M$ any regular covering and $c \in M$ is a totally null-homotopic knot. Then for $H_1(\tilde{M}(p^{-1}(c), (1, m), \mathbf{Z}) = H_1(\tilde{M}, \mathbf{Z})$

Proof. Consider the manifold $\tilde{N} = \overline{\tilde{M} \setminus \tilde{N}(D)}$. It can be easiely seen that for a fixed component $\tilde{c}_i \in p^{-1}(c)$ and for its regular neighborhood $\tilde{N}_i \in p^{-1}(N(c))$ we obtain

$$H_1(\overline{\tilde{M} \setminus \tilde{N}}, (1, m), \mathbf{Z}) = H_1(\overline{\tilde{M} \setminus \tilde{N}_i}, \mathbf{Z}) / < m_i > ,$$

where $< m_i >$ is the cyclic group generated by new surgery meridian m_i with the slope $(1, m)$. Thus, $m_i = \tilde{a}_i + m\tilde{b}_i$ where $\{\tilde{a}_i, \tilde{b}_i\} = p^{-1}(\{a, b\}) \cap \partial\tilde{N}_i$. But $\tilde{b}_i$ is trivial in $H_1(\overline{\tilde{M} \setminus \tilde{N}_i}, \mathbf{Z})$ so $H_1(\overline{\tilde{M} \setminus \tilde{N}_i}, (1, m), \mathbf{Z}) \cong H_1(\tilde{M}, \mathbf{Z}) \cong H_1(\overline{\tilde{M} \setminus \tilde{N}_i}, \mathbf{Z}) / < \tilde{a} >$. the Lemma now follows by induction on the number $\#(p^{-1}(c))$. *QED.*

It was shown in [B-W] by using Myers' result that any compact orientable 3-manifold M contains a totally null-homotopic hyperbolic knot c (i.e. $int(M \setminus c)$ posseds a complete hyperbolic structure of finite volume).

Theorem 7.3. *Let $M = \mathbf{H}^3/\Gamma$ be a hyperbolic 3-manifold of finite volume and $c \subset M$ a totally null-homotopic knot. Then there exists n_0 such that for any $n \geq n_0$ the hyperbolic manifold $M(1, n)$ posseds a regular covering $p : \Omega \to M(1, n)$ such that $H_1(\Omega, \mathbf{Z}) = 0$ and Ω is not simply connected.*

Put $G = Deck(p)$.

Corollary 7.4. *The manifold $M(1, n) = \Omega/G$ does not fiber over S^1 as well as all of its finite coverings of the type $\tilde{M}_n = \Omega/G_n$, where G_n is a subgroup of G of finite index.*

The proof of the Corollary immediately follows from the assertion of the Theorem and our Key Observation.

Proof of Theorem 7.3. Let us first consider the covering $\mathbf{H}^3 \xrightarrow{\pi} \mathbf{H}^3/\Gamma = M$ and the lift $\pi^{-1}(c) = \tilde{C}$. By hyperbolic Dehn surgery theorem [T3] there exists n_0 such that for

$n \geq n_0$ the manifold $M(1,n)$ is hyperbolic. Consider the manifold $\tilde{M} = \overline{\mathbf{H}^3 \setminus \pi^{-1}(N(c))}$ and let $\Omega = \mathbf{H}^3(1,n)$ (recall that $\mathbf{H}^3(1,n)$ is obtained from $\mathbf{H}^3$ by doing (1,n)-Dehn surgery simultaneously on all components of $\tilde{C}$).

Claim. *The covering π induces the regular covering $p : \Omega \to M(1,n) = \Omega/G$ where the group $G = Deck\ p$ is isomorphic to Γ.*

Consider the induced covering $p : \tilde{M} \to \overline{M \setminus N(C)}$ where $p = \pi|_{\tilde{M}}$ and take $\gamma \in \Gamma$. Since the restriction of π to each component $\tilde{N}_i \in \pi^{-1}(N(c))$ is a homeomorphism the element γ sends the meridian-longitude system $(\tilde{a}_i, \tilde{b}_i)$ to that of on $\gamma(\partial\tilde{N}_i)$. As we do our $(1,n)$-surgery on $\tilde{M}$ simultaneously, γ preserves the $(1,n)$-surgery slope of $\partial(\pi^{-1}(N(c)))$, and so extends to some homeomorphism γ^* of Ω. By repeating this contruction for all generators of Γ we obtain the group Γ^* acting discontinuously on Ω. Indeed, the action of Γ^* on $\tilde{M}$ coincide with that of Γ, and so is discontinuous there; further each solid torus component $\tilde{N}_i$ is strictly invariant in Γ^* under the identity: $\forall\gamma^* \in \Gamma^* \setminus \{id\} : \ \gamma^*(\tilde{N}_i) \cap \tilde{N}_i = \emptyset$, so the action is discontinuous everywhere in Ω.

The map $\phi : \Gamma \to \Gamma^*$ is well-defined and, obviously, isomorphism since each $\gamma^* \in \Gamma^*$ is uniquely determined by its action on $\tilde{M}$. The Claim is proved. *QED.*

Put $G = \Gamma^* = \phi(\Gamma)$. and notice that Lemma 7.2 implies that $H_1(\Omega, \mathbf{Z}) = 0$. To finish the proof of our Theorem we need only to show that Ω is not simply connected. In fact if it were not so, then the manifold $M(1,n) = \Omega/G$ is hyperbolic and has the fundamental group isomorphic to Γ, so M and $M(1,n)$ would be isometric by Mostow rigidity theorem. The last is impossible since $M(1,n)$ and M have different volume by hyperbolic Dehn Surgery Theorem[T3]. The Theorem is proved. *QED.*

Remark. The above construction of Ω looks like a sort of "conformal Dehn surgery". We have in this connection the following.

Question. Suppose that Γ, Ω and G as above. Is it possible to embed Ω to S^3 and to realize G to be a Kleinian group having Ω as invariant component?

The positive answer to this question would provide a standard way to produce all known examples of "patological" of Kleinian groups acting in S^3 (e.g. [A-T], [Ka-P], [P]). One can also point out if such G was Kleinian then Γ and G would belong to different components of the deformation variety $Def(\Gamma, Iso_+(\mathbf{H}^4))$ which follows from Marden-Sullivan stability theorem.

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