PROJECTIVE STRUCTURES ON A RIEMANN SURFACE, II

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1. Introduction

For a compact Riemann surface X, let Δ be the diagonal in $X \times X$, and p_i, $i = 1, 2$, be the projection of $X \times X$ onto the i-th factor. We denote by Δ_n the n-th order infinitesimal neighborhood of Δ in $X \times X$, defined by the non-reduced divisor $(n+1)\Delta$, and by $\mathcal{L}$ a square-root of the holomorphic tangent bundle T_X of X.

We recall that a *projective structure* on X is an equivalence class of coverings by holomorphic coordinate charts such that all the transition functions are Möbius transformations. Deligne has shown in [De] that a projective structure can also be defined as the extension of a natural embedding of Δ_2 in a certain projective bundle P_{tg} to an embedding of Δ_3. The projective bundle P_{tg} coincides with the projectivized jet bundle $\mathbb{P}(J^1(\mathcal{L}))$. The line bundle $\mathcal{M} := p_1^*\mathcal{L} \otimes p_2^*\mathcal{L} \otimes \mathcal{O}_{X\times X}(-\Delta)$ admits a natural trivialization over Δ_1. In part I [BR] it was shown that a projective structure on X can be viewed as a choice of an extension of the trivialization of the line bundle $\mathcal{M}$ on Δ_1 to a trivialization on Δ_2.

The question that we address here is the interpretation of an embedding of higher order infinitesimal neighborhoods of Δ into P_{tg} and of the trivialization of $\mathcal{M}$ on higher order infinitesimal neighborhoods of Δ.

In Theorem 3.6 (cf. Remark 3.12) we prove that for $n \geq 3$ the space of all trivializations of $\mathcal{M}$ on Δ_n, which restrict to the canonical trivialization over Δ_1, is canonically identified with the space of all projective structures on X together with a differential of order i for each $i \in [3, n]$. In Lemma 4.8 we prove that the latter space is also identified, in a canonical fashion, with the space of all embeddings of Δ_{n+1} in $\mathbb{P}(J^1(\mathcal{L}))$. In other words, the two spaces, of trivializations and embeddings respectively, are canonically identified.

In Theorems 3.11 and 4.9 we establish that for $n \geq 2$ these spaces, of trivializations or embeddings, can be identified with a certain natural class of differential operators on X of order n. The differential operators in question map sections of $\mathcal{L}^{\otimes(n-1)}$ to sections of $\mathcal{L}^{\otimes(-n-1)}$ and have the following form in local coordinates

$$\frac{d^n}{dx^n} + \sum_{i=2}^{n} f_i \frac{d^{n-i}}{dx^{n-i}}$$

The results of Appendices A and B are used in Sections 2 and 3.

One of the motivations of the present work has been from Mathematical Physics, viz. a continuing study of a certain model quantum field theory on a curve, known as the $b-c$ system, which appears in conformal field theory and string theory studies (see [R1] for a review of earlier work). An important aspect of such a quantum field theory model is its so-called *Operator Product Expansion* (OPE). The OPE is basic to the theory of vertex operator algebras [K], but is usually restricted to the genus zero case. We can regard [R2] and [BR] as identifying the "one point functions" of the first and second terms of the holomorphic part of the OPE of the quantum fields b and c as the "one point function" of, respectively, the *current* and the *energy-momentum tensor* of the system. The present paper can be regarded as identifying the "one point function" of the first n terms of the holomorphic part of the OPE as an n-th order differential operator of the above type, for all $n \geq 2$. Indeed, the interpretation can be extended to n=1 as well, since it was shown in [R2] that the current $< j >$ is a one form and arises from the connection $d + < j >$, where d is the de Rham connection on X.

The first named author thanks Tim McCune for his remarks. The other (AKR) gratefully acknowledges the support and hospitality of the Max-Planck-Institut für Mathematik, Bonn, during this work.

2. Projective structures on a Riemann surface

A *projective atlas* on a Riemann surface X is a covering $\{U_\alpha, \phi_\alpha\}_{\alpha\in I}$ of X by holomorphic coordinate charts, where ϕ_α is a biholomorphism from U_α to an open set in $\mathbb{C}$, such that any $\phi_\beta \circ \phi_\alpha^{-1}$, $\alpha, \beta \in I$, is the restriction of a Möbius transformation to $\phi_\alpha(U_\alpha)$. Such an atlas on X gives a one cocycle on X with values in $PSL(2, \mathbb{C})$, the Möbius group of automorphisms of $\mathbb{CP}^1$. A *projective structure* on X is an equivalence class of compatible projective atlases and determines an element in $H^1(X,\ PSL(2,\mathbb{C}))$ [Gu].

Let X be a compact Riemann surface equipped with a projective structure $\mathfrak{P}$. Choosing a line bundle $\mathcal{L}$ with $\mathcal{L}^{\otimes 2} = T_X$, is equivalent to choosing a lift of the element corresponding to $\mathfrak{P}$ in $H^1(X,\ PSL(2,\mathbb{C}))$, to an element in $H^1(X,\ SL(2,\mathbb{C}))$ [Ty]. The map from $H^1(X,\ SL(2,\mathbb{C}))$ to $H^1(X,\ PSL(2,\mathbb{C}))$ in question is the one induced by the natural projection of $SL(2,\mathbb{C})$ onto $PSL(2,\mathbb{C})$.

2a. **A line bundle on the self-product.** We shall first recall an alternative description of a projective structure from [BR].

Let X be a compact connected Riemann surface. Fix a line bundle $\mathcal{L}$ on X, which is a square-root of T_X, along with an isomorphism

$$\Psi \ : \ \mathcal{L}^{\otimes 2} \ \longrightarrow \ T_X \tag{2.1}$$

The space of all projective structures on X has the following description in terms of the line bundle $\mathcal{M} := p_1^*\mathcal{L} \otimes p_2^*\mathcal{L} \otimes \mathcal{O}_{X\times X}(-\Delta)$ on $X \times X$ [BR]:

Theorem 2.2. *Take any nonzero integer n. The restriction of the line bundle $\mathcal{M}^{\otimes n}$ on $X\times X$ to the (nonreduced) divisor 2Δ has a canonical trivialization, which is invariant under the involution of $X \times X$, defined by switching coordinates if n is even, and it is anti-invariant if n is odd. To each projective structure on X there is a naturally associated trivialization of the formal completion of $\mathcal{M}^{\otimes n}$ along Δ, invariant (resp. anti-invariant) under the involution of $X \times X$ if n is even (resp. odd), with the property that the trivialization restricts to the natural trivialization of $\mathcal{M}^{\otimes n}$ over 2Δ. Moreover, this association gives a bijective map between the space of all projective structures on X and the space of all trivializations of $\mathcal{M}^{\otimes n}$ over 3Δ which restrict to the natural trivialization over 2Δ.*

Remark 2.3.

1. A trivialization of the formal completion of $\mathcal{M}^{\otimes n}$ along Δ means compatible trivializations of the line bundle $\mathcal{M}^{\otimes n}$ over infinitesimal neighborhoods of Δ in $X \times X$ of every order([Ha], page 194). This is equivalent to trivializing $\mathcal{M}^{\otimes n}$ over some analytic neighborhood of Δ.

2. The restriction of the line bundle $\mathcal{O}_{X\times X}(-n\Delta)$ to Δ is $K_\Delta^{\otimes n}$, the n-th tensor power of the cotangent bundle, and hence the restriction of $\mathcal{M}^{\otimes n}$ to Δ is the trivial line bundle. If n is even then the trivialization is canonical; if n is odd then there is a natural trivialization up to sign. This indeterminacy of sign can be removed by using the ordering of factors in the Cartesian product $X \times X$. There is a natural trivialization of $\mathcal{M}^{\otimes n}$ over 2Δ which is an extension of the trivialization over Δ. A description of this trivialization using coordinate charts is given in (2.4). This trivialization over 2Δ is determined by the condition that when n is even (resp. odd), then it is invariant (resp. anti-invariant) under the involution of $X \times X$ defined by switching coordinates. Thus, if the involution and the trivialization are denoted by τ and s respectively, then s coincides with $\tau^* s$ (resp. $-\tau^* s$) if n is even (resp. odd).

Denoting by q_i, $i = 1, 2$, the projection of $\mathbb{CP}^1 \times \mathbb{CP}^1$ onto the i-th factor, the line bundle $q_1^*\mathcal{O}(n) \otimes q_2^*\mathcal{O}(n) \otimes \mathcal{O}_{\mathbb{CP}^1\times\mathbb{CP}^1}(-n\Delta_0)$, where $\Delta_0 \subset \mathbb{CP}^1 \times \mathbb{CP}^1$ is the diagonal divisor, has a canonical trivialization given by the section

$$s(n) \quad := \quad (z_1 - z_2)^n \Big(\frac{\partial}{\partial z_1}\Big)^{n/2} \otimes \Big(\frac{\partial}{\partial z_2}\Big)^{n/2} \tag{2.4}$$

where (z_1, z_2) is the coordinate function on $\mathbb{CP}^1 \times \mathbb{CP}^1$ obtained from the obvious coordinate function z on $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$, and $\big(\frac{\partial}{\partial z}\big)^{1/2}$ denotes a section of $\mathcal{O}(1)$ such that $\Big(\big(\frac{\partial}{\partial z}\big)^{1/2}\Big)^{\otimes 2}$ coincides with the section $\frac{\partial}{\partial z}$ by the natural isomorphism between $\mathcal{O}_{\mathbb{CP}^1}(2)$ and $T_{\mathbb{CP}^1}$. Note that though there are two choices for the section $\big(\frac{\partial}{\partial z}\big)^{1/2}$, the section in (2.4) does not depend upon the choice.

For any biholomorphism $\phi : U \longrightarrow U'$, between open sets of $\mathbb{CP}^1$, the two sections, $s(n)$ and $(\phi,\phi)^* s(n)$ respectively, defined over $U \times U$, actually coincide when they are restricted to $2\Delta_0 \cap (U \times U)$. Thus $s(n)$ gives a trivialization of $\mathcal{M}^{\otimes n}$ over 2Δ for any Riemann surface X. Indeed, the above property of $s(n)$ ensures that for any two coordinate charts on X, the two pullbacks of $s(n)$ using the two coordinate functions actually coincide on the first order infinitesimal neighborhood of the diagonal.

We noted earlier that for a projective structure $\mathfrak{P}$ on X, the element

$$\rho \;\in\; H^1(X,\, PSL(2,\mathbb{C}))$$

corresponding to $\mathfrak{P}$, lifts to a cohomology class

$$\bar{\rho} \;\in\; H^1(X,\, SL(2,\mathbb{C}))$$

using $\mathcal{L}$. The line bundle on X associated to $\bar{\rho}$, for the natural action of $SL(2,\mathbb{C})$ on $\mathcal{O}_{\mathbb{CP}^1}(1)$, is canonically identified with $\mathcal{L}$.

The diagonal action of $SL(2,\mathbb{C})$ on $\mathbb{CP}^1 \times \mathbb{CP}^1$ naturally lifts to the line bundle $q_1^*\mathcal{O}(n) \otimes q_2^*\mathcal{O}(n) \otimes \mathcal{O}(-n\Delta_0)$ preserving its trivialization.

For a covering of X by coordinate charts compatible with a given projective structure, the pullbacks of $\mathcal{O}_{\mathbb{CP}^1}(1)$ by coordinate maps patch together to produce the line bundle $\mathcal{L}$ on X. The trivialization of $q_1^*\mathcal{O}(n) \otimes q_2^*\mathcal{O}(n) \otimes \mathcal{O}(-n\Delta_0)$ gives a trivialization of $\mathcal{M}^{\otimes n}$ over an analytic neighborhood of Δ. The association between projective structures and trivializations, mentioned in Theorem 2.2, is obtained by restricting this trivialization to the completion of $X \times X$ along Δ.

Theorem 2.2 has been proved in [BR] with $n = 2$. The proof in any other case is identical to it.

Restricting $\mathcal{M}^{\otimes n}$ to 2Δ and 3Δ respectively, and using the trivialization of $\mathcal{M}^{\otimes n}$ over Δ, we have the following exact sequence

$$0 \longrightarrow K_\Delta^{\otimes 2} \longrightarrow \left(\mathcal{M}^{\otimes n}\right)\Big|_{3\Delta} \longrightarrow \left(\mathcal{M}^{\otimes n}\right)\Big|_{2\Delta} \longrightarrow 0$$

Hence, if $\overline{\mathfrak{P}}$ and $\overline{\mathfrak{P}'}$ denote the trivializations of $\left(\mathcal{M}^{\otimes n}\right)|_{3\Delta}$ corresponding to any two projective structures $\mathfrak{P}$ and $\mathfrak{P}'$ on X,

$$\overline{\mathfrak{P}'} - \overline{\mathfrak{P}} \;\in\; H^0(X,\, K_X^{\otimes 2})$$

since $\overline{\mathfrak{P}}$ and $\overline{\mathfrak{P}'}$ agree over 2Δ.

On the other hand, the space of all projective structures on X is an affine space for $H^0(X,\, K_X^{\otimes 2})$ [Gu]. Thus

$$\mathfrak{P}' - \mathfrak{P} \;\in\; H^0(X,\, K_X^{\otimes 2})$$

The relationship between these two observations is provided by the following identity ([BR], Lemma 3.6)

$$\overline{\mathfrak{P}'} - \overline{\mathfrak{P}} \;=\; \frac{n}{12}(\mathfrak{P}' - \mathfrak{P}) \tag{2.5}$$

2b. **Decomposition of a differential operator.** For a holomorphic vector bundle E on X and a positive integer n, the n-th order *jet bundle* of E, denoted by $J^n(E)$, is defined to be the following direct image on X:

$$J^n(E) \quad := \quad p_{1*}\Big(\frac{p_2^*E}{p_2^*E\otimes\mathcal{O}_{X\times X}(-(n+1)\Delta)}\Big)$$

where, as before, p_i is the projection of $X\times X$ onto the i-th factor. Since Δ is an effective divisor, $p_2^*E\otimes\mathcal{O}_{X\times X}(-(n+1)\Delta)$ is a subsheaf of p_2^*E. There is a natural exact sequence

$$0 \longrightarrow K_X^{\otimes(n+1)}\otimes E \longrightarrow J^{n+1}(E) \longrightarrow J^n(E) \longrightarrow 0$$

The inclusion $K_X^{\otimes(n+1)}\otimes E \longrightarrow J^{n+1}(E)$ is constructed by using the inclusion

$$K_X^{\otimes(n+1)} \longrightarrow J^{n+1}(\mathcal{O}_X)$$

which is defined at $x\in X$ by $(df)^{\otimes(n+1)} \longmapsto f^{(n+1)}/(n+1)!$, where f is any function with $f(x)=0$.

The sheaf of differential operators $\mathrm{Diff}^n_X(E,F)$ coincides with $\mathrm{Hom}(J^n(E),\, F)$. The homomorphism

$$\sigma\ :\ \mathrm{Diff}^n_X(E,F) \quad\longrightarrow\quad \mathrm{Hom}(K_X^{\otimes n}\otimes E\,,F)$$

obtained by restricting a homomorphism from $J^n(E)$ to F to the subsheaf $K_X^{\otimes n}\otimes E$, is known as the *symbol map*.

Furthermore, for any two integers $r,s\in\mathbb{N}$, the natural projection

$$J^{r+s}(E) \quad\longrightarrow\quad J^s(E)$$

admits a canonical lift

$$\mu\ :\ J^{r+s}(E) \quad\longrightarrow\quad J^r(J^s(E))$$

which is an injective homomorphism of vector bundles.

Now consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{L}^{-n-1}=K_X^{\otimes n}\otimes\mathcal{L}^{\otimes(n-1)} \arrow[r] & J^n(\mathcal{L}^{\otimes(n-1)}) \arrow[r] \arrow[d, "\mu"] & J^{n-1}(\mathcal{L}^{\otimes(n-1)}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K_X\otimes J^{n-1}(\mathcal{L}^{\otimes(n-1)}) \arrow[r] & J^1(J^{n-1}(\mathcal{L}^{\otimes(n-1)})) \arrow[r] & J^{n-1}(\mathcal{L}^{\otimes(n-1)}) \arrow[r] & 0
\end{tikzcd} \tag{2.6}$$

The differential operator $\mathcal{D}_{\mathfrak{P}}(n)$ constructed in (B.4) of Appendix B gives a homomorphism

$$D_n \,:\, J^{n-1}(\mathcal{L}^{\otimes(n-1)}) \longrightarrow J^n(\mathcal{L}^{\otimes(n-1)}) \tag{2.7}$$

which is a splitting of the top exact sequence in (2.6). The composition $\mu \circ D_n$ is a splitting of the bottom exact sequence in (2.6). In other words, $\mu \circ D_n$ is a holomorphic connection on $J^{n-1}(\mathcal{L}^{\otimes(n-1)})$. Thus we have the following lemma [Bi, Theorem 4.1]:

Lemma 2.8. *For a Riemann surface X with a projective structure, for any $n \in \mathbb{N}$, the jet bundle $J^n(\mathcal{L}^{\otimes n})$ is equipped with a natural flat connection.*

Using the splitting D_n in (2.7) as the base point, the space of all splittings of the top exact sequence in (2.6) is identified with

$$H^0(X,\, \mathrm{Diff}^{n-1}_X(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)}))$$

Similarly, the space of all splittings of the bottom exact sequence in (2.6) is identified with

$$H^0(X,\, \mathrm{End}(J^{n-1}(\mathcal{L}^{\otimes(n-1)})) \otimes K_X)$$

The composition of a splitting of the top exact sequence in (2.6) with the inclusion μ (in (2.6)) is a splitting of the bottom exact sequence in (2.6). Thus we have a natural homomorphism

$$\Phi \;:\; H^0(X,\, \mathrm{Diff}^{n-1}_X(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)})) \;\longrightarrow\; H^0(X,\, \mathrm{End}(J^{n-1}(\mathcal{L}^{\otimes(n-1)}))\otimes K_X)$$

Using the $GL(n, \mathbb{C})$ invariant polynomials $A \longmapsto \mathrm{trace}(A^i)$, $1 \leq i \leq n$, on $M(n, \mathbb{C})$ we have a homomorphism

$$H \;:\; H^0(X,\, \mathrm{End}(J^{n-1}(\mathcal{L}^{\otimes(n-1)})) \otimes K_X) \;\longrightarrow\; \bigoplus_{i=1}^{n} H^0(X,\, K_X^{\otimes i}) \tag{2.9}$$

which is known as the *Hitchin map* [Hi].

Theorem 2.10. *Let X be a compact Riemann surface equipped with a projective structure. The homomorphism*

$$H \circ \Phi \;:\; H^0(X,\, \mathrm{Diff}^{n-1}_X(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)})) \;\longrightarrow\; \bigoplus_{i=1}^{n} H^0(X,\, K_X^{\otimes i})$$

is an isomorphism.

Proof. We shall first consider the case where the genus of X is at least 2. The remaining cases will be treated separately.

Our first step will be to establish the injectivity of the homomorphism $H \circ \Phi$.

For $D \in H^0(X, \mathrm{Diff}_X^{n-1}(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)}))$, let $\sigma \in H^0(X, K_X^{\otimes i})$ be the component of $H \circ \Phi(D)$ of lowest degree. In other words, $\sigma \neq 0$, and furthermore, if $j < i$, then the component of $H \circ \Phi(D)$ in $H^0(X, K_X^{\otimes j})$ vanishes identically. It is a simple computation to check that the symbol of the differential operator D coincides with σ. (This implies that D is actually of order $n-i$.) Thus $H \circ \Phi$ must be injective.

We shall complete the proof by showing that the dimensions of the domain and the target of $H \circ \Phi$ coincide.

Using the long exact sequence of cohomologies for the following exact sequence of vector bundles on X

$$0 \longrightarrow \mathrm{Diff}_X^{i}(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)}) \longrightarrow \mathrm{Diff}_X^{i+1}(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)}) \longrightarrow K_X^{\otimes(n-i-1)} \longrightarrow 0$$

it can be deduced that

$$H^1(X, \mathrm{Diff}_X^{j}(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)})) = 0 \tag{2.11}$$

for all $j \leq n-2$. Indeed, if $n \geq 2$, then

$$H^1(X, \mathrm{Diff}_X^{0}(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)})) = H^1(X, K_X^{\otimes n}) = 0$$

since the genus of X is at least two. Now (2.11) follows from the exact sequence

$$\begin{aligned} H^1(X, \mathrm{Diff}_X^{j}(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)})) &\longrightarrow H^1(X, \mathrm{Diff}_X^{j+1}(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)})) \\ &\longrightarrow H^1(X, K_X^{\otimes(n-j-1)}) \longrightarrow 0 \end{aligned}$$

by induction on j.

Finally, (2.11) implies that

$$\dim H^0(X, \mathrm{Diff}_X^{j}(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)})) = \sum_{i=n-j}^{n} \dim H^0(X, K_X^{\otimes i})$$

for all $j \leq n-1$. This can be seen by considering the long exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(X, \mathrm{Diff}_X^{j}(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)})) &\longrightarrow H^0(X, \mathrm{Diff}_X^{j+1}(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)})) \\ \longrightarrow H^0(X, K_X^{\otimes(n-j-1)}) &\longrightarrow H^1(X, \mathrm{Diff}_X^{j}(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)})) \end{aligned}$$

and using induction on j. This completes the proof of the theorem in the situation where the genus of X is at least 2.

If $X = \mathbb{CP}^1$, then the dimension of both the domain and the target is 0. To see that the domain has dimension 0, observe that $J^{n-1}(\mathcal{L}^{\otimes(n-1)})$ is a trivial bundle, as it has a flat connection [Lemma 2.8].

If X is an elliptic curve, then $\mathrm{Diff}_X^{n-1}(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)})$ is actually isomorphic to $\bigoplus_{i=1}^n H^0(X,\, K_X^{\otimes i})$. This follows from the observation that $J^{n-1}(\mathcal{L}^{\otimes(n-1)})$ is isomorphic to the symmetric power $S^{n-1}(J^1(\mathcal{L}))$ [Bi]. This completes the proof of the theorem. $\square$

The isomorphism $H \circ \Phi$ depends upon the projective structure that was used in its construction. The homomorphisms Φ or H are not individually isomorphisms. In fact if $g > 1$ then,

$$\dim H^0(X,\, \mathrm{End}(J^{n-1}(\mathcal{L}^{\otimes(n-1)})) \otimes K_X) \quad > \quad \sum_{i=1}^{n} \dim H^0(X,\, K_X^{\otimes i})$$

Now, any element $D \in H^0(X,\, \mathrm{Diff}_X^n(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)}))$ decomposes uniquely as

$$D \;= c\mathcal{D}_{\mathfrak{P}}(n) \,+\, D_0$$

where $c \in \mathbb{C}$ and $D_0 \in H^0(X,\, \mathrm{Diff}_X^{n-1}(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)}))$. Hence Theorem 2.10 has the following corollary:

Corollary 2.12. *For a Riemann surface X with a projective structure, the following decomposition*

$$H^0(X,\, \mathrm{Diff}_X^n(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)})) \quad = \quad \bigoplus_{i=0}^{n} H^0(X,\, K_X^{\otimes i})$$

is valid.

The differentials appearing on the right hand side may be viewed as the *Laguerre-Forsyth invariants* of the differential operator [W].

In the next section we shall establish an identification between a certain subspace of the space of differential operators $H^0(X,\, \mathrm{Diff}_X^n(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)}))$ and the space of trivializations of $\mathcal{M}$ over an infinitesimal neighborhood of the diagonal.

3. Differential operators on a Riemann surface

We continue with the notation of the previous section, but in the present section we shall not assume that the Riemann surface X is equipped with a projective structure.

For a nonnegative integer n and an integer $d \geq 3$, let

$$S(n,d) \;\in\; H^0\Big(X\times X,\, \frac{\mathcal{M}^{\otimes n}}{\mathcal{M}^{\otimes n}\otimes \mathcal{O}_{X\times X}(-d\Delta)}\Big) \tag{3.1}$$

be a trivialization of $\mathcal{M}^{\otimes n}$ over $d\Delta$, such that its restriction to Δ is the constant function 1. Furthermore, denoting the involution of $X\times X$ by τ, the section $S(n,d)$ is required to coincide with $\tau^*S(n,d)$ (resp. $-\tau^*S(n,d)$) if n is even (resp. odd). Hence the restriction of $S(n,d)$ to 2Δ is the canonical trivialization referred to in Theorem 2.2.

Consider the restriction of the section $S(n,d)$ to 3Δ. We shall denote by $\mathfrak{P}_{S(n,d)}$ the natural projective structure on X associated to it by Theorem 2.2. Let

$$S'(n,d) \;\in\; H^0\Big(X\times X,\, \frac{\mathcal{M}^{\otimes n}}{\mathcal{M}^{\otimes n}\otimes \mathcal{O}_{X\times X}(-d\Delta)}\Big) \tag{3.2}$$

be the trivialization associated to the projective structure $\mathfrak{P}_{S(n,d)}$ using Theorem 2.2.

Let ξ denote the direct image

$$\xi \;:=\; p_{1*}\Big(\frac{S(n,d)}{S'(n,d)}\Big) \;\in\; H^0(X,\, J^{d-1}(\mathcal{O}_X))$$

on X of the function over $d\Delta$ given by the quotient of the two trivializations defined in (3.1) and (3.2) respectively. The image of ξ in $J^2(\mathcal{O}_X)$ coincides with the image of the constant function 1.

Setting $d = n+2$, let

$$M_\xi \;:\; J^{n+1}(\mathcal{L}^{\otimes n}) \;\longrightarrow\; J^{n+1}(\mathcal{L}^{\otimes n})$$

be the homomorphism defined by multiplication with ξ. The multiplication in question is the natural surjective homomorphism

$$J^n(L)\bigotimes J^n(L') \;\longrightarrow\; J^n(L\bigotimes L')\,,$$

where L and L' are any two line bundles on X, which sends any two local sections s and s' of L and L' respectively, to $s\otimes s'$.

Notation 3.3 : *For any integer $n \geq 1$, let $\mathcal{T}(n)$ denote the space of all trivializations $S(n, n+2)$ of $\mathcal{M}^{\otimes n}$ over $(n+2)\Delta$ which restrict to the canonical trivialization over 2Δ.*

Note that there is a one-to-one correspondence between $\mathcal{T}(n)$ and the space of all trivializations of $\mathcal{M}$ over $(n+2)\Delta$ which restrict to the canonical trivialization over 2Δ. The map is defined by simply taking the n-th tensor power of a section of $\mathcal{M}$ over $(n+2)\Delta$.

For any $S(n, n+2) \in \mathcal{T}(n)$, let $\mathcal{D} = \mathcal{D}_{\mathfrak{P}_{S(n,n+2)}}(n+1)$ be the differential operator defined in (B.4) for the projective structure $\mathfrak{P}_{S(n,n+2)}$ constructed earlier. Since $\mathcal{D}$ is a homomorphism from $J^{n+1}(\mathcal{L}^{\otimes n})$ to $\mathcal{L}^{\otimes(-n-2)}$, the composition

$$\mathcal{D}_S \;:=\; \mathcal{D} \circ M_\xi \tag{3.4}$$

is an element of $H^0(X, \mathrm{Diff}_X^{n+1}(\mathcal{L}^{\otimes n}, \mathcal{L}^{\otimes(-n-2)}))$. The earlier observation that the image of ξ in $J^2(\mathcal{O}_X)$ coincides with the image of the constant function 1, implies that the difference

$$\mathcal{D}_S - \mathcal{D} \;\in\; H^0(X, \mathrm{Diff}_X^{n-2}(\mathcal{L}^{\otimes n}, \mathcal{L}^{\otimes(-n-2)}))$$

Let $\mathfrak{P}(X)$ denote the space of all projective structures on the Riemann surface X. The following theorem identifies $H^0(X, \mathrm{Diff}_X^{n-2}(\mathcal{L}^{\otimes n}, \mathcal{L}^{\otimes(-n-2)}))$ with the space of trivializations.

Theorem 3.5. *For a compact Riemann surface X, the map*

$$W \;:\; \mathcal{T}(n) \;\longrightarrow\; \mathfrak{P}(X) \times H^0(X, \mathrm{Diff}_X^{n-2}(\mathcal{L}^{\otimes n}, \mathcal{L}^{\otimes(-n-2)}))$$

defined by $S(n, n+2) \longmapsto (\mathfrak{P}_{S(n,n+2)}\,,\, \mathcal{D}_S - \mathcal{D})$, where $\mathfrak{P}_{S(n,n+2)}$ was constructed earlier and $\mathcal{D}_S$ was defined in (3.4), is bijective.

Proof. To construct the inverse of W, for any

$$(\mathfrak{P}\,,\, D) \;\in\; \mathfrak{P}(X) \times H^0(X, \mathrm{Diff}_X^{n-2}(\mathcal{L}^{\otimes n}, \mathcal{L}^{\otimes(-n-2)}))$$

consider the differential operator $\mathcal{D}' := \mathcal{D}_{\mathfrak{P}}(n+1) + D$, where $\mathcal{D}_{\mathfrak{P}}(n+1)$ is the operator constructed in (B.4). Let

$$M \;:\; J^{n+1}(\mathcal{L}^{\otimes n}) \;\longrightarrow\; J^{n+1}(\mathcal{L}^{\otimes n})$$

be the homomorphism defined by the identity $\mathcal{D}' = \mathcal{D}_{\mathfrak{P}}(n+1) \circ M$.

Let $S \in \mathcal{T}(n)$ be the trivialization associated to the projective structure $\mathfrak{P}$ by Theorem 2.2. Since $M \in H^0((n+2)\Delta, \mathcal{O})$, the product $M.S \in \mathcal{T}(n)$.

Associating $M.S$ to $(\mathfrak{P}, D)$, the inverse of the map W is obtained. It is rather straight-forward to check that this map is indeed the inverse of W. $\square$

From Theorem 2.10, together with the fact used in the proof of Theorem 2.10 that for any

$$D' \in H^0(X, \operatorname{Diff}^{n-1}_X(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)}))$$

the lowest order term of $H \circ \Phi(D')$ is the symbol of D', it immediately follows that if X is equipped with a projective structure then

$$H^0(X, \operatorname{Diff}^{n-2}_X(\mathcal{L}^{\otimes n}, \mathcal{L}^{\otimes(-n-2)})) = \bigoplus_{i=3}^{n+1} H^0(X, K_X^{\otimes i})$$

Now the following theorem is a consequence of Theorems 2.10 and 3.5.

Theorem 3.6. *For a compact Riemann surface X, the space of trivializations $\mathcal{T}(n)$ is identified, in a canonical fashion, with the Cartesian product*

$$\mathfrak{P}(X) \times \Big(\bigoplus_{i=3}^{n+1} H^0(X, K_X^{\otimes i})\Big)$$

Let

$$\Sigma(n+1) \subset H^0(X, \operatorname{Diff}^{n+1}_X(\mathcal{L}^{\otimes n}, \mathcal{L}^{\otimes(-n-2)})) \tag{3.7}$$

be the subset consisting of all differential operators with symbol 1. The vector space

$$\mathcal{V} := H^0(X, \operatorname{Diff}^{n-1}_X(\mathcal{L}^{\otimes n}, \mathcal{L}^{\otimes(-n-2)}))$$

acts on $\Sigma(n+1)$ simply by addition.

It is somewhat surprising (at least to the authors) that the action of $\mathcal{V}$ on $\Sigma(n+1)$ has a distinguished orbit. This is the orbit containing the differential operator $\mathcal{D}_{\mathfrak{P}}(n+1)$, defined in (B.4), where $\mathfrak{P}$ is a projective structure on X. The observation noted down in (B.6) that the difference of two such operators differ by an operator of degree $n-1$, implies that the above defined orbit does not depend upon the choice of the projective structure.

Let

$$\Sigma_0(n+1) \quad \subset \quad \Sigma(n+1) \tag{3.8}$$

denote the distinguished orbit for the action of $\mathcal{V}$.

Remark 3.9. Using the exact sequence

$$0 \longrightarrow K_X^{\otimes(i+1)} \otimes \mathcal{L}^{\otimes(n-1)} \longrightarrow J^{i+1}(\mathcal{L}^{\otimes(n-1)}) \longrightarrow J^i(\mathcal{L}^{\otimes(n-1)}) \longrightarrow 0$$

and the isomorphism between $\mathcal{L}^{\otimes 2}$ and T_X, it readily follows that

$$\bigwedge^n J^{n-1}(\mathcal{L}^{\otimes(n-1)}) \quad = \quad \mathcal{O}_X \tag{3.10}$$

Hence, the top exterior product $\bigwedge^n J^{n-1}(\mathcal{L}^{\otimes(n-1)})$ has a canonical trivialization.

We shall describe a property of the class of differential operators $\Sigma_0(n)$ defined in (3.8), in terms of the above trivializations. For a $D \in \Sigma_0(n)$ let ∇ be the holomorphic connection on $J^{n-1}(\mathcal{L}^{\otimes(n-1)})$ constructed using the map μ in (2.6) and the splitting given by D. It is straight-forward to check that the connection on $\bigwedge^n J^{n-1}(\mathcal{L}^{\otimes(n-1)})$ induced by ∇ coincides with the natural connection on $\mathcal{O}_X$ using the isomorphism (3.10). $\square$

Consider the map

$$\mathfrak{P}(X) \times H^0(X,\, \mathrm{Diff}_X^{n-2}(\mathcal{L}^{\otimes n}, \mathcal{L}^{\otimes(-n-2)})) \quad \longrightarrow \quad \Sigma_0(n+1)$$

defined by $(\mathfrak{P}\,,\, D) \longmapsto \mathcal{D}_{\mathfrak{P}}(n+1) \,+\, D$. The Lemma B.9 implies that this map is an isomorphism.

Now the following theorem is easily derived from Theorem 3.5.

Theorem 3.11. *For any compact Riemann surface X, the space of trivializations $\mathcal{T}(n)$ is canonically identified with the space of differential operators $\Sigma_0(n+1)$.*

For any a trivialization $t \in \mathcal{T}(n)$, its restriction to 3Δ gives a projective structure $\mathfrak{P}$ on X. If D is the differential operator that corresponds to t by Theorem 3.11, then $\mathfrak{P}$ is the unique projective structure such that

$$D \,-\, \mathcal{D}_{\mathfrak{P}}(n+1) \quad \in \quad H^0(X,\, \mathrm{Diff}_X^{n-2}(\mathcal{L}^{\otimes n}, \mathcal{L}^{\otimes(-n-2)}))$$

Remark 3.12. Following Notation 3.3 we noted that the space $\mathcal{T}(n)$ is identified in a canonical fashion with the space of all trivializations of $\mathcal{M}$ over $(n+2)\Delta$ which restrict to the canonical trivialization of $\mathcal{M}$ over 2Δ. Thus, denoting the space of such trivializations of $\mathcal{M}$ by $\mathcal{T}_n$, Theorems 3.6 and 3.11 remain valid if $\mathcal{T}(n)$ is replaced with $\mathcal{T}_n$ in their statements. $\square$

Let

$$\mathbb{P}_X(n) \ := \ \mathbb{P}(J^{n+1}(\mathcal{L}^{\otimes n})) \longrightarrow X$$

denote the projective bundle over X of dimension $n+1$ consisting of all lines in the vector bundle $J^{n+1}(\mathcal{L}^{\otimes n})$. Let

$$\mathbb{P}_X \ := \ \mathbb{P}(J^n(\mathcal{L}^{\otimes n})) \ \subset \ \mathbb{P}_X(n)$$

denote the projective bundle over X for the natural surjection of $J^{n+1}(\mathcal{L}^{\otimes n})$ onto $J^n(\mathcal{L}^{\otimes n})$.

A differential operator $D \in H^0(X, \mathrm{Diff}_X^{n+1}(\mathcal{L}^{\otimes n}, \mathcal{L}^{\otimes(-n-2)}))$, with the constant function 1 as the symbol, naturally gives a section of the fiber bundle

$$\mathcal{A}_X(n) \ := \ \mathbb{P}_X(n) - \mathbb{P}_X$$

by simply considering D as a homomorphism of $J^{n+1}(\mathcal{L}^{\otimes n})$ onto $\mathcal{L}^{\otimes(-n-2)}$.

Conversely, for any section of $\mathcal{A}_X(n)$ there is a differential operator such that the section arises from it in the above fashion. Indeed, any section s of $\mathbb{P}_X(n)$ is given by a surjective homomorphism, say H, of $J^{n+1}(\mathcal{L}^{\otimes n})$ onto a line bundle ζ over X. If s is a section of $\mathcal{A}_X(n)$ then the restriction of H to $\mathcal{L}^{\otimes(-n-2)}$ is nowhere zero. In other words, ζ is isomorphic to $\mathcal{L}^{\otimes(-n-2)}$ by H.

Thus we have a one-to-one bijective correspondence between the space of sections of $\mathcal{A}_X(n)$ and the subspace of of the space of differential operators

$$H^0(X, \mathrm{Diff}_X^{n+1}(\mathcal{L}^{\otimes n}, \mathcal{L}^{\otimes(-n-2)}))$$

consisting of operators with symbol 1.

Using the above one-to-one correspondence, the class of operators $\Sigma_0(n+1)$ defined in (3.8) gives a class of sections of $\mathcal{A}_X(n)$. Let $\mathcal{S}$ denote this class of sections. It will be interesting to be able to characterize geometrically the class of sections $\mathcal{S}$. Also, it will be interesting to be able to directly find the element in $\mathcal{T}(n)$, given by Theorem 3.11, corresponding to any given element of $\mathcal{S}$.

4. Embeddings of the higher order infinitesimal neighborhoods of the diagonal in a projective bundle

The structure sheaf of the subscheme $n\Delta \subset X \times X$ is isomorphic to the ringed space

$$(X\,,\, p_{1*}\mathcal{O}_{n\Delta})$$

where p_1 is the projection of $X \times X$ onto the first factor. The $\mathcal{O}_X$-algebra $p_{1*}\mathcal{O}_{n\Delta}$ on X coincides with the jet bundle $J^{n-1}(\mathcal{O}_X)$ with its natural algebra structure.

Another natural embedding of X in a fiber bundle over X is obtained from the exact sequence

$$0 \longrightarrow K_X \otimes \mathcal{L} \longrightarrow J^1(\mathcal{L}) \longrightarrow \mathcal{L} \longrightarrow 0$$

In this case X is embedded in the projective bundle $\mathbb{P}(X) := \mathbb{P}(J^1(\mathcal{L}))$. Denoting the image of X by $\bar{X}$, the structure sheaf of $n\bar{X}$ is the ringed space

$$(X\,,\, q_*\mathcal{O}_{n\bar{X}}) \tag{4.1}$$

where q is the projection of $\mathbb{P}(X)$ onto X.

Lemma 4.2. *For any $n \geq 0$, the vector bundle $q_*\mathcal{O}_{(n+1)\bar{X}}$ on X, defined in (4.1), is canonically isomorphic to*

$$\mathcal{O}_X \,\oplus\, J^{n-1}(\mathcal{L}^{\otimes(n-1)}) \otimes \mathcal{L}^{\otimes(-n-1)}$$

Proof. For $x \in X$, let $\mathbb{P}_x := q^{-1}(x)$ be the projective line. The $(n-1)$-th order infinitesimal neighborhood of the point

$$\bar{x} \;:=\; \mathbb{P}_x \cap \bar{X}$$

in $\mathbb{P}_x$ coincides with the fiber $(q_*\mathcal{O}_{n\bar{X}})_{\bar{x}}$.

In order to determine the infinitesimal neighborhood of $\bar{x}$, consider the following exact sequence on $\mathbb{P}_x$:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_x}(-(n+1)\bar{x}) \longrightarrow \mathcal{O}_{\mathbb{P}_x} \longrightarrow J^n(\mathcal{O}_{\mathbb{P}_x})_{\bar{x}} \longrightarrow 0$$

Since $H^1(\mathbb{P}_x,\, \mathcal{O}_{\mathbb{P}_x}) = 0$, the n-th order infinitesimal neighborhood of $\bar{x}$, namely $J^n(\mathcal{O}_{\mathbb{P}_x})_{\bar{x}}$, coincides with

$$\mathbb{C} \,\oplus\, H^1(\mathbb{P}_x,\, \mathcal{O}_{\mathbb{P}_x}(-(n+1)\bar{x})) \tag{4.3}$$

Using Serre duality, and the canonical isomorphisms $\mathcal{O}_{\mathbb{P}_x}(-\bar{x})_{\bar{x}} = (K_X)_x$ and $\mathcal{L}^{\otimes 2} = T_X$, and the natural self-duality of $H^0(\mathbb{P}_x, \mathcal{O}_{\mathbb{P}_x}(n-1))$ induced by the natural trivialization of the line $\bigwedge^2 J^1(\mathcal{L})_x$, the vector space in (4.3) gets identified with the fiber at x of the vector bundle

$$\mathcal{O}_X \oplus J^{n-1}(\mathcal{L}^{\otimes(n-1)}) \otimes \mathcal{L}^{\otimes(-n-1)}$$

The details of the last part of argument can be found in Section 3 of [Bi]. This completes the proof of the lemma. $\square$

Given a projective structure on X, for any $n \in \mathbb{N}$, there is a natural *compatible embedding* of the n-th order infinitesimal neighborhood of Δ into the projective bundle $\mathbb{P}(X)$ [De]. The compatibility of embeddings means that the embedding of the $(n-1)$-th order infinitesimal neighborhood is obtained by restricting the embedding of the n-th order infinitesimal neighborhood. To obtain these embeddings, first observe that if X is $\mathbb{CP}^1$, then $\mathbb{P}(X)$ is $X \times X$ and so the embedding is the (tautological) diagonal embedding. Then observe that this tautological embedding commutes with the diagonal action of $\mathrm{Aut}(\mathbb{CP}^1)$ on $\mathbb{CP}^1 \times \mathbb{CP}^1$. This immediately implies the earlier statement on the existence of canonical compatible embeddings for any Riemann surface with a projective structure.

The embedding of the second order infinitesimal neighborhood 3Δ of Δ into $\mathbb{P}(X)$ actually does not depend upon the choice of the projective structure. A direct construction of this embedding of the second order infinitesimal neighborhood of Δ into $\mathbb{P}(X)$ is given below.

In view of Lemma 4.2, giving an identification of the second order infinitesimal neighborhoods of the above type is equivalent to giving an isomorphism

$$J^2_0(\mathcal{O}_X) \longrightarrow J^1(\mathcal{L}) \otimes \mathcal{L}^{\otimes -3} \tag{4.4}$$

where the fiber $J^2_0(\mathcal{O}_X)_x \subset J^2(\mathcal{O}_X)_x$ at any $x \in X$ is given by the jets of functions vanishing at x. Since the map defined by $f \longmapsto df$ identifies $J^2_0(\mathcal{O}_X)$ with $J^1(K_X)$, we have a natural isomorphism of the type (4.4) as soon as we have established the following statement:

For any nonzero integer i, there is a nondegenerate pairing between $J^1(T_X)$ and $J^1(\mathcal{L}^{\otimes i})$ with values in $\mathcal{L}^{\otimes i}$.

Now, the operation of taking the Lie derivative with respect to a vector field is exactly such a pairing. Indeed, the Lie derivative

$$(V\,,\,s)\ \longmapsto\ L_V s$$

where s (resp. V) is a local section of $\mathcal{L}^{\otimes i}$ (resp. T_X), defines an isomorphism of $J^1(T_X)$ with $J^1(\mathcal{L}^{\otimes i})^* \bigotimes \mathcal{L}^{\otimes i}$. Using the isomorphism Ψ in (2.1) and the Leibniz identity

$$L_V(s \otimes t) \ = \ (L_V s) \otimes t \, + \, s \otimes L_V t$$

the Lie derivative of sections of any tensor power of $\mathcal{L}$ is defined. The canonical isomorphism of the type (4.4) that we are seeking is obtained by setting $i \, = \, 1$ and -2.

To equip X with a projective structure is equivalent to extending the above identification, of the second order infinitesimal neighborhoods of Δ and $\bar{X}$, to an identification of their third order infinitesimal neighborhoods ([De], Definition 5.6 bis).

The following lemma will be needed in our study of embeddings of the higher order infinitesimal neighborhoods of Δ.

Lemma 4.5. *For a Riemann surface with a projective structure the space of all differential operators of order $n-1$ from $\mathcal{L}^{\otimes(n-1)}$ to $\mathcal{L}^{\otimes(1-n)}$ admits a natural decomposition of the following form*

$$H^0(X,\,\mathrm{Hom}(J^{n-1}(\mathcal{L}^{\otimes(n-1)}),\mathcal{L}^{\otimes(-n+1)})) \ = \ \bigoplus_{i=0}^{n-1} H^0(X,\,K_X^{\otimes i})$$

Proof. Let $V/\mathbb{C}$ be a vector space of dimension 2 equipped with a symplectic form ω. Denoting by L the line bundle $\mathcal{O}_{\mathbb{P}}(1)$ on the projective line $\mathbb{P} \, = \, \mathbb{P}(V)$, the vector bundle $J^n(L^{\otimes k})$ on $\mathbb{P}$ is canonically isomorphic to $J^n(L^{\otimes n}) \bigotimes L^{\otimes(k-n)}$ in the following two cases:

1. when $k \, < \, 0$,
2. when $k \, \geq \, n$.

The first step to construct the isomorphism is to consider the long exact sequence of cohomologies for the exact sequence

$$0 \longrightarrow L^{\otimes k} \otimes \mathcal{O}_{\mathbb{P}}(-(n+1)x) \longrightarrow L^{\otimes k} \longrightarrow J^n(L^{\otimes k})_x \longrightarrow 0$$

where $x \in \mathbb{P}$. Note that in the first case $H^0(\mathbb{P},\,L^{\otimes k}) = 0$ and in the second case $H^1(\mathbb{P},\,L^{\otimes k} \otimes \mathcal{O}_{\mathbb{P}}(-(n+1)x)) = 0$. A choice of $0 \neq w \in V$

representing x provides an isomorphism between $\mathcal{O}_{\mathbb{P}}(x)$ and L. Now after noting that $H^0(\mathbb{P},\, L^{\otimes i}) = S^i(V^*) = S^i(V)$ (using ω), and also using Serre duality when $k < 0$, the key point in the construction is that for any $0 \neq v \in V$, the following two vector spaces

1. the kernel of the homomorphism $S^{n-1-k}(V) \longrightarrow S^{-k-2}(V)$ defined using the contraction with $(v^*)^{\otimes(n+1)}$, where $k < 0$ and $v^* \in V^*$ is the dual of v with respect to ω,
2. the cokernel of $S^{k-n-1}(V) \longrightarrow S^k(V)$ defined using the multiplication with $v^{\otimes(n+1)}$, where $k \geq n$,

are both canonically isomorphic to $S^n(V)$. Indeed, in the first case the isomorphism is defined using the multiplication with $v^{\otimes(-k-1)}$ on $S^n(V)$, and in the second case it is defined using the contraction with $(v^*)^{\otimes(k-n)}$.

Since the isomorphism is equivariant for the actions of $SL(V)$ on $\mathbb{P}$ and L, it induces an isomorphism

$$\psi(n,k) \,:\, J^n(\mathcal{L}^{\otimes k}) \quad\longrightarrow\quad J^n(\mathcal{L}^{\otimes n}) \otimes \mathcal{L}^{\otimes(k-n)}$$

where X is a Riemann surface equipped with a projective structure together with a choice of a square-root of the canonical bundle, whenever n and k satisfy either of the above two conditions. (The choice of a *theta characteristic* provides a lift of the transition functions to $SL(V)$.)

Now consider

$$\begin{aligned}\psi(i,n-1)\otimes\psi(i,-2(n-1-i)) \quad &: \quad J^i(\mathcal{L}^{\otimes(n-1)}) \otimes J^i(K_X^{\otimes(n-1-i)}) \\ &\longrightarrow \quad J^i(\mathcal{L}^{\otimes i}) \otimes J^i(\mathcal{L}^{\otimes i}) \otimes \mathcal{L}^{\otimes(-n+1)} \qquad (4.6)\end{aligned}$$

where $i \leq n-1$. The symplectic form ω on V induces a nondegenerate form on any $S^i(V)$ (which is symmetric if i is even and skew-symmetric if i is odd), and hence on $J^i(L^{\otimes i})$ over $\mathbb{P}$. This is because any fiber of $J^i(L^{\otimes i})$ is canonically isomorphic to $S^i(V)$ [Bi]. Since this form on $J^i(L^{\otimes i})$ is equivariant under the action of $SL(V)$, we have a nondegenerate form on $J^i(\mathcal{L}^{\otimes i})$ using the coordinate charts on X compatible with the projective structure. Thus (4.6) gives the pairing

$$\bar{\psi}(i,n-1) \,:\, J^i(\mathcal{L}^{\otimes(n-1)}) \otimes J^i(K_X^{\otimes(n-1-i)}) \quad\longrightarrow\quad \mathcal{L}^{\otimes(-n+1)}$$

Now for a section $s \in H^0(X,\, K_X^{\otimes(n-1-i)})$, consider the homomorphism $\bar{\psi}(i, n-1)(-, s)$ which sends any vector $v \in J^i(\mathcal{L}^{\otimes(n-1)})_y$ to $\bar{\psi}(i, n-1)(y)(v, s(y))$. The differential operator

$$\bar{s} \in H^0(X,\, \mathrm{Diff}^i_X(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n+1)}))$$

defined by this homomorphism has $(-1)^i s \in H^0(X,\, K_X^{\otimes(n-1-i)})$ itself as its symbol. This lifting of the symbol of a differential operator that maps any s to $(-1)^i \bar{s}$ immediately yields the decomposition (4.5) of differential operators. This completes the proof of the lemma. $\square$

We note that the decomposition in Lemma 4.5 depends on the choice of projective structure.

For an integer $n \geq 3$, let

$$\rho \,:\, J^n_0(\mathcal{O}_X) \;\longrightarrow\; J^{n-1}(\mathcal{L}^{\otimes(n-1)}) \otimes \mathcal{L}^{\otimes(-n-1)}$$

be an isomorphism of algebra bundles giving an embedding of the n-th order infinitesimal neighborhood of Δ into $\mathbb{P}(X)$ such that the restriction of ρ to the second order infinitesimal neighborhood 3Δ of Δ is the canonical one defined above. Restricting ρ to the third order infinitesimal neighborhood 4Δ of Δ we get a projective structure $\mathfrak{P}$ on X [De].

Let

$$\rho(\mathfrak{P}) \,:\, J^n_0(\mathcal{O}_X) \;\longrightarrow\; J^{n-1}(\mathcal{L}^{\otimes(n-1)}) \otimes \mathcal{L}^{\otimes(-n-1)}$$

be the isomorphism giving the embedding corresponding to the projective structure $\mathfrak{P}$.

It can be checked that any automorphism of $J^{n-1}(\mathcal{L}^{\otimes(n-1)})$, which as an automorphism of $J^{n-1}(\mathcal{L}^{\otimes(n-1)}) \otimes \mathcal{L}^{\otimes(-n-1)}$ preserves its algebra structure, is actually conjugate to a unique automorphism of the form $c.Id + A$, where $c \in \mathbb{C}^*$ and $A \in H^0(X,\, \mathrm{Hom}(J^{n-1}(\mathcal{L}^{\otimes(n-1)}), \mathcal{L}^{\otimes(-n+1)}))$. The inclusion

$$\mathcal{L}^{\otimes(n-1)} \otimes K_X^{\otimes(n-1)} \;=\; \mathcal{L}^{\otimes(-n+1)} \;\longrightarrow\; J^{n-1}(\mathcal{L}^{\otimes(n-1)})$$

enables $H^0(X,\, \mathrm{Hom}(J^{n-1}(\mathcal{L}^{\otimes(n-1)}), \mathcal{L}^{\otimes(-n+1)}))$ to be realized as a subspace of the space of endomorphisms of $J^{n-1}(\mathcal{L}^{\otimes(n-1)})$.

Now consider the automorphism

$$\rho \circ (\rho(\mathfrak{P}))^{-1} \,:\, J^{n-1}(\mathcal{L}^{\otimes(n-1)}) \;\longrightarrow\; J^{n-1}(\mathcal{L}^{\otimes(n-1)})$$

where ρ and $(\rho(\mathfrak{P}))^{-1}$ are defined above. Let

$$\bar{\rho} \;\in\; H^0(X,\, \mathrm{Hom}(J^{n-1}(\mathcal{L}^{\otimes(n-1)}), \mathcal{L}^{\otimes(-n+1)}))$$

correspond to $\rho \circ (\rho(\mathfrak{P}))^{-1}$ by the above decomposition.

Consider the decomposition of the differential operator $\bar{\rho}$ according to Lemma 4.5 for the projective structure $\mathcal{P}$ on X. Since, by definition, $\rho(\mathfrak{P})$ and ρ agree on the third order infinitesimal neighborhood of Δ, it follows immediately that the components of $\bar{\rho}$ in $H^0(X,\, K_X^{\otimes i})$ vanish for all $i \leq 2$.

Let

$$F(\rho) \;\in\; \bigoplus_{i=3}^{n-1} H^0(X,\, K_X^{\otimes i}) \tag{4.7}$$

be the element corresponding to $\bar{\rho}$.

Let $\mathcal{G}(n)$ denote the space of all embeddings of the n-th order infinitesimal neighborhood of Δ into $\mathbb{P}(X)$ whose restriction to the second order infinitesimal neighborhood of Δ is the canonical one. Let $\mathfrak{P}(X)$ denote the space of all equivalence classes of projective structures on X.

Lemma 4.8. *The map from the space of embeddings $\mathcal{G}(n)$ to*

$$\mathfrak{P}(X) \times \bigoplus_{i=3}^{n-1} H^0(X,\, K_X^{\otimes i})$$

which sends any $\rho \in \mathcal{G}(n)$ to the pair $(\mathfrak{P}\,,\, F(\rho))$, constructed in (4.7), is a bijective identification.

Proof. The above lemma follows from Theorem 2.10. We omit the details. □

The following theorem is an immediate consequence of Theorem 3.6.

Theorem 4.9. *For any compact Riemann surface X, the space of trivializations $\mathcal{T}(n)$, defined in (3.3), is canonically identified with $\mathcal{G}(n+2)$, the space of embeddings of the $(n+2)$-th order infinitesimal neighborhood of Δ into $\mathbb{P}(X)$ restricting to the canonical embedding on the second order infinitesimal neighborhood* 3Δ.

In the rest of the paper we shall investigate some natural differential operators on Riemann surfaces with a projective structure.

Appendix A : A DIFFERENTIAL OPERATOR

For a holomorphic function g on an open set of $\mathbb{C}$, its n-th derivative, namely $\frac{\partial^n g}{\partial z^n}$, will be denoted by $g^{(n)}$. Also, often g' will be used instead of $g^{(1)}$. Let

$$\gamma \ : \ U \ \longrightarrow \ U_1 \tag{A.1}$$

be a biholomorphism between two simply connected analytic open sets of $\mathbb{C}$. Fix on U a square-root $(\gamma')^{1/2}$ of the function $\gamma^{(1)}$. For an integer $n \geq 1$ and a holomorphic function $f \in \Gamma(U_1,\, \mathcal{O})$ on U_1, define $D(\gamma, n)f \in \Gamma(U,\, \mathcal{O})$ by

$$(D(\gamma, n)f)(z) \ = \ (f^{(n)} \circ \gamma)(z)(\gamma'(z))^{(n+1)/2} - \Big(\frac{f \circ \gamma}{(\gamma')^{(n-1)/2}}\Big)^{(n)}(z) \tag{A.2}$$

where $z \in U$.

Evidently the map $f \longmapsto D(\gamma, n)f$ is a differential operator of order n from $\Gamma(U_1, \mathcal{O})$ to $\Gamma(U,\, \mathcal{O})$. Alternatively, the $\mathbb{C}$-linear homomorphism

$$\psi \ \longmapsto \ D(\gamma, n)(\psi \circ \gamma^{-1})\,, \tag{A.3}$$

where $\psi \in \Gamma(U,\, \mathcal{O})$, is a section over U of the sheaf of differential operators of order n on the trivial line bundle. Since the tangent bundle TU has a natural trivialization, the symbol of a differential operator of order i from $\Gamma(U,\, \mathcal{O})$ to $\Gamma(U,\, \mathcal{O})$ is an element of $\Gamma(U,\, \mathcal{O})$. It is a simple calculation to check that the symbol of this order n differential operator $D(\gamma, n)$ vanishes identically. It involves a slightly more nontrivial computation to check that the symbol of the order $n-1$ differential operator also vanishes identically. The conclusion is that the operator $D(\gamma, n)$ is actually of order $n-2$.

Another simple computation yields that if M is Möbius transformation, which means that

$$M(z) \ = \ \frac{az+b}{cz+d}$$

where $ad - bc = 1$, then

$$D(M, n) \ = \ 0 \tag{A.4}$$

for all $n \geq 1$. In fact, more generally, for a Möbius transformation M as above, denoting the function $w \longmapsto (cw+d)^k$ by $M^{[k]}$, where $k \in \mathbb{Z}$, the following identity is valid:

$$D(M\circ\gamma,n)(f)(z) \;=\; D(\gamma,n)(M^{[n-1]}f\circ M)(z) \qquad (A.5)$$

where $f\in\Gamma(M(U_1),\,\mathcal{O})$ and $z\in U$. Since $D(Id,k)=0$ for any $k\geq 1$, the equality (A.5) immediately gives (A.4). Similarly, the identity

$$D(\gamma\circ M,n)(f)(z) \;=\; D(\gamma,n)(f)(M(z))M^{[-n-1]}(z)\,, \qquad (A.6)$$

where $z\in M^{-1}(U)$, is valid.

Clearly $D(\gamma,1)=0$. To calculate $D(\gamma,2)$, for a holomorphic function g on an open subset of $\mathbb{C}$, let

$$\mathcal{SD}(g) \;:=\; \frac{2g'(z)g'''(z)\,-\,3(g''(z))^2}{2(g'(z))^2}$$

denote the *Schwarzian derivative* of g (page 164 of [Gu]). The following identity relating $D(\gamma,2)$ with $\mathcal{SD}(\gamma)$

$$(D(\gamma,2)f)(z) \;=\; \frac{f(\gamma(z))\mathcal{SD}(\gamma)(z)}{2(\gamma')^{1/2}(z)} \qquad (A.7)$$

is valid. To calculate the symbol of the higher order operators, let

$$\overline{D(\gamma,n)} \;\in\; \Gamma(U,\,\mathrm{Diff}^{n-2}_U(\mathcal{O},\mathcal{O}))$$

be the differential operator of order $n-2$ defined in (A.3), which operates on the sheaf $\Gamma(U,\,\mathcal{O})$. The symbol of this operator $\overline{D(\gamma,n)}$ is calculated to be

$$\mathrm{symbol}\,(\overline{D(\gamma,n)}) \;=\; \frac{n(n^2-1)(\gamma')^{(n-3)/2}}{12}\mathcal{SD}(\gamma) \qquad (A.8)$$

The equality (A.7) follows from (A.8).

For any simply connected open set $W\subseteq\mathbb{C}$, let (L_W,Ψ_W) be a pair consisting of a line bundle L_W on W and a section

$$\Psi_W \;\in\; \Gamma(W,\,L_W)\,.$$

Furthermore, there is a given isomorphism $A_W : L_W^{\otimes 2} \longrightarrow T_W$, such that $A^{\otimes 2}(\Psi_W^{\otimes 2})$ coincides with the section $\frac{\partial}{\partial z}$, of T_W.

Let

$$\mathcal{D}_W(n) \;\in\; \Gamma(W,\,\mathrm{Diff}^n_W(L_W^{\otimes(n-1)},L_W^{\otimes(-n-1)})) \qquad (A.9)$$

be the differential operator of order n on W defined by

$$\mathcal{D}_W(n)(f\Psi_W^{\otimes(n-1)}) \quad = \quad \frac{\partial^n f}{\partial z^n}\Psi_W^{\otimes(-n-1)}$$

where f is a holomorphic function on W. Note that the operator $\mathcal{D}_W(n)$ remains unchanged if the isomorphism Ψ_W is replaced by $-\Psi_W$.

Let γ be as in (A.1), and let (L_U, Ψ_U) and (L_{U_1}, Ψ_{U_1}) be as defined above. Fix an isomorphism

$$F_\gamma \quad : \quad \gamma^* L_{U_1} \quad \longrightarrow \quad L_U$$

such that $(A_U^{\otimes 2}) \circ F_\gamma^{\otimes 2} \circ (A_{U_1}^{\otimes 2})^{-1}(z) \; = \; d\gamma^{-1}(\gamma(z))$, for all $z \in U$, where A_U and A_{U_1} were defined before (A.9); the homomorphism $d\gamma$ is the differential of γ mapping T_U to $\gamma^* T_{U_1}$. The above condition determines F_γ up-to the sign. This ambiguity of sign is removed by imposing the condition that

$$(\gamma')^{1/2} F_\gamma(\gamma^*\Psi_{U_1}) \quad = \quad \Psi_U$$

The isomorphism between $(\gamma^* L_{U_1})^{\otimes i}$ and $L_U^{\otimes i}$, induced by F_γ, will be denoted by $F_\gamma^{\otimes i}$. Note, using the identity

$$(\gamma')^{1/2} F_\gamma(\Psi_{U_1}) \quad = \quad \Psi_U,$$

that F_γ automatically fixes a square-root $(\gamma')^{1/2}$.

The following lemma is follows from a simple computation.

Lemma A.10. *Let γ be as in (A.1), and let (L_U, Ψ_U) and (L_{U_1}, Ψ_{U_1}) be as defined above. Then for any holomorphic function $f \in \Gamma(U_1, \mathcal{O})$, the identity*

$$F_\gamma^{\otimes(-n-1)}\mathcal{D}_{U_1}(n)(f\Psi_{U_1}^{\otimes(n-1)}) - \mathcal{D}_U(n)((F_\gamma(\Psi_{U_1}))^{\otimes(n-1)} f\circ\gamma) \quad = \quad \Psi_U^{\otimes(-n-1)} D(\gamma, n) f$$

is valid, where $D(\gamma, n)f$ as in (A.2).

Note that the differential operator, which maps $f\Psi_{U_1}^{\otimes(n-1)}$ to $\Psi_U^{\otimes(-n-1)} D(\gamma, n)f$, does not depend upon the choice of the square-root $(\gamma')^{1/2}$. Indeed, if $(\gamma')^{1/2}$ is replaced by $-(\gamma')^{1/2}$ then F_γ changes to $-F_\gamma$.

In Appendix B we shall derive some implications of the operator $D(\gamma, n)$ in the context of a Riemann surface equipped with a projective structure.

Appendix B : A Differential operator on a Riemann surface with a projective structure

Let X be a Riemann surface equipped with a projective structure, which we shall denote by $\mathfrak{P}$. Fix a pair $(\mathcal{L}\,,\Psi)$, where

$$\Psi\,:\,\mathcal{L}^{\otimes 2}\,\longrightarrow\,T_X \tag{B.1}$$

is an isomorphism of line bundles.

Take a coordinate chart (U_α,ϕ_α) compatible with $\mathfrak{P}$. Recall the pair (L_W,Ψ_W) constructed in Section 2 for any open set W of $\mathbb{C}$. Substituting $\phi_\alpha(U_\alpha)$ for W, fix an isomorphism

$$\theta_\alpha\,:\,\mathcal{L}|_{U_\alpha}\quad\longrightarrow\quad \phi_\alpha^* L_{\phi_\alpha(U_\alpha)} \tag{B.2}$$

such that $\theta_\alpha^{\otimes 2}\circ\Psi^{-1}=d\phi_\alpha$. The isomorphism from $\mathcal{L}^{\otimes i}|_{U_\alpha}$ to $(\phi_\alpha^* L_{\phi_\alpha(U_\alpha)})^{\otimes i}$, induced by θ_α, is denoted by $\theta_\alpha^{\otimes i}$. Note that there is a natural isomorphism between $L_W^{\otimes 2}$ and T_W. Let $\mathcal{D}_\alpha(n)$ be the differential operator on U_α obtained from the operator $\mathcal{D}_W(n)$ defined in (A.9), using the isomorphism θ_α, namely

$$\mathcal{D}_\alpha(n)\quad:=\quad(\theta_\alpha^{\otimes(-n-1)})^{-1}\circ\mathcal{D}_{\phi_\alpha(U_\alpha)}(n)\circ\theta_\alpha^{\otimes(n-1)}\quad\in\quad\Gamma(U_\alpha,\,\mathrm{Diff}^n_{U_\alpha}(\mathcal{L}^{\otimes(n-1)},\mathcal{L}^{\otimes(-n-1)})) \tag{B.3}$$

which maps sections of $\mathcal{L}^{\otimes(n-1)}$ to sections of $\mathcal{L}^{\otimes(-n-1)}$. Note that although there are two choices of the homomorphism θ_α, differing by multiplication with -1, the operator $\mathcal{D}_\alpha(n)$ does not depend upon the choice of θ_α.

Let (U_β,ϕ_β) be another coordinate chart compatible with the projective structure $\mathfrak{P}$. Since $\phi_\beta\circ\phi_\alpha^{-1}$ is a Möbius transformation, Lemma A.10 and the identity (A.4) combine together to imply that the restrictions of the two differential operators $\mathcal{D}_\alpha(n)$ and $\mathcal{D}_\beta(n)$ to the open set $U_\alpha\cap U_\beta$ actually coincide. Thus patching these operators we have a global differential operator

$$\mathcal{D}_{\mathfrak{P}}(n)\quad\in\quad H^0(X,\,\mathrm{Diff}^n_X(\mathcal{L}^{\otimes(n-1)},\mathcal{L}^{\otimes(-n-1)})) \tag{B.4}$$

on X which depends on the projective structure $\mathfrak{P}$. This differential operator is called the *Bol's operator* [Bo]. (An alternative construction of this operator can be found in [Bi].)

Take another projective structure $\mathfrak{P}_1$ on X. Let ϕ and ϕ_1 be two coordinate functions on $U_0\subseteq X$ compatible with $\mathfrak{P}$ and $\mathfrak{P}_1$ respectively. As in (B.2), let θ and θ_1 be two isomorphisms from $\mathcal{L}|_{U_0}$ to

$\phi^* L_{\phi(U_0)}$ and $\phi_1^* L_{\phi_1(U_0)}$ respectively. With a slight abuse of notation, the isomorphisms of tensor powers induced by θ (and θ_1) will also be denoted by θ (and θ_1).

Consider the differential operator $D(\gamma, n)$ defined in (A.2), after substituting $U = \phi(U_0)$ and $\gamma = \phi_1 \circ \phi^{-1}$. Finally, let

$$\mathcal{D}^n_{U_0}(\mathfrak{P}, \mathfrak{P}_1) \quad \in \quad \Gamma(U_0, \, \mathrm{Diff}^{n-2}_{U_0}(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)})) \qquad (B.5)$$

be the differential operator of order $n-2$ defined by

$$\mathcal{D}^n_{U_0}(\mathfrak{P}, \mathfrak{P}_1)(\theta_1^{-1}(f \Psi^{\otimes(n-1)}_{\phi_1(U_1)})) \quad = \quad \theta^{-1}(\Psi^{\otimes(-n-1)}_{\phi(U)} D(\gamma, n) f)$$

where $f \in \Gamma(\phi_1(U_0), \mathcal{O})$ and Ψ_U as in Appendix A. The identities (A.4), (A.5) and (A.6) together imply that for any two open sets U_0 and U_1 of X, the two corresponding differential operators, defined in (B.5), actually coincide over $U_0 \cap U_1$. Thus, patching up these locally defined operators we get a global differential operator

$$\mathcal{D}^n_X(\mathfrak{P}, \mathfrak{P}_1) \quad \in \quad H^0(X, \, \mathrm{Diff}^{n-2}_X(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)})) \qquad (B.6)$$

of order $n-2$ on X.

The following lemma is immediate from Lemma A.10.

Lemma B.7. *Let $\mathcal{D}_{\mathfrak{P}}(n)$ and $\mathcal{D}_{\mathfrak{P}_1}(n)$ be the two differential operators for $\mathfrak{P}$ and $\mathfrak{P}_1$ respectively, as constructed in (B.4). Then the identity*

$$\mathcal{D}_{\mathfrak{P}_1}(n) \, - \, \mathcal{D}_{\mathfrak{P}}(n) \quad = \quad \mathcal{D}^n_X(\mathfrak{P}, \mathfrak{P}_1)$$

is valid.

The space of all projective structures on the Riemann surface X is an affine space for $H^0(X, K_X^{\otimes 2})$ [Gu]. Let $\omega \in H^0(X, K_X^{\otimes 2})$ be such that

$$\mathfrak{P}_1 \quad = \quad \mathfrak{P} + \omega \qquad (B.8)$$

Note that $\mathcal{D}_{\mathfrak{P}_1}(2) - \mathcal{D}_{\mathfrak{P}}(2) \in H^0(X, K_X^{\otimes 2})$. The following equality

$$\mathcal{D}_{\mathfrak{P}_1}(2) \quad = \quad \mathcal{D}_{\mathfrak{P}}(2) + \frac{\omega}{2}$$

is valid [HS, CMZ].

Lemma B.7 and the equality (A.8) together immediately imply the following lemma:

Lemma B.9.

$$\operatorname{symbol}(\mathcal{D}_{\mathfrak{P}_1}(n) - \mathcal{D}_{\mathfrak{P}}(n)) = \operatorname{symbol}(\mathcal{D}^n_X(\mathfrak{P}, \mathfrak{P}_1)) = \frac{n(n^2-1)\omega}{12}$$

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