DIFFERENTIAL OPERATORS ON HOMOGENEOUS SPACES II: RELATIVE ENVELOPING ALGEBRAS

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(*) partially supported by a N.S.F. grant.

MPI 86-58

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SUMMARY

Our main object here is to analyze the process of induction of (two-sided) ideals in enveloping algebras, in particular the behaviour of their associated varieties, by means of the new concept of relative enveloping algebras. For any algebraic group H, and any H principal fibration $X \longrightarrow Y$ we define the relative enveloping algebra as the sheaf of algebras $\mathcal{U}$ on Y made up by all H invariant differential operators on X. If the principal fibration is trivial, then $\mathcal{U}$ identifies with the enveloping algebra $U(\underline{h})$ of the Lie algebra $\underline{h}$ of H, tensored by the sheaf D_Y of differential operators on the base Y. One of our main constructions is the following: For an ideal J of $U(\underline{h})$ we construct an sheaf of ideals $\tilde{J}$ in $\mathcal{U}$, which is $D_Y \otimes J$ in a local trivialization. In a special case (H a torus, J maximal), the sheaf $\mathcal{U}/\tilde{J}$ is known as "twisted differential operators" in the literature [BeBe].

For our present main purpose, we apply this new concept to the principal fibration $G \rightarrow G/H$ of an algebraic group G by a closed subgroup H. Let $\underline{g} = \text{Lie } G$. The left G action gives rise to an algebra homomorphism $U(\underline{g}) \longrightarrow \Gamma(G/H, \mathcal{U})$, and so an ideal J of $U(\underline{h})$ gives rise to a homomorphism $U(\underline{g}) \longrightarrow \Gamma(G/H, \mathcal{U}/\tilde{J})$. Now the kernel of this homomorphism turns out to be the ideal I induced from J in $U(\underline{g})$, up to a certain "twist" (corollary of proposition 2).

Our main result (theorem 2) gives then the precise relation between the associated varieties of J resp. I in $\underline{h}$ resp. $\underline{g}$, in the case H parabolic. The behaviour of associated varieties of ideals is analogous to that of wave front sets of representations, as studied by Barbasch and Vogan [BV]. We define the characteristic variety of a coherent $\mathcal{U}$-module $\mathcal{M}$ as a subvariety $Ch(\mathcal{M})$ in $(T^*X)/H$, and we prove that its image under the momentum map contains the associated variety of the $U(\underline{g})$-module $\Gamma(G/H, \mathcal{M})$, assuming H parabolic. Application to $\mathcal{M} = \mathcal{U}/\tilde{J}$ then yields our main result. This generalizes some of the results in [BB] I and [BB] III.

We also use this opportunity to report on recent work of V.Ginsburg and B.Kostant concerning shifted cotangent bundles and polarizations, and to relate it to our present context. Finally, some applications to Dixmier sheets and to a conjecture of Gelfand and Kirillov are made.

INTRODUCTION

Our present article is the second in a series of three about differential operators on homogeneous spaces. We note that the first and the third have already appeared in print as [BB]I resp. [BB]III, but that all three may be read separately, because they are essentially logically independent, though closely related. The relation is given not only by the common central object, that is a systematic study of rings and sheaves of differential operators on a homogeneous space, but also by a common motivating problem, which is the computation of associated varieties of ideals in enveloping algebras. In each of the three articles, we first develop some new ideas and results for a systematic theory of differential operators on homogeneous spaces, and then apply them to the determination of associated varieties, as a kind of "test problem".

The theoretical effort made to develop our new methods is justified by such applications, in view of the progress made in each part on open questions raised in the previous literature. For detailed reviews of the contents of parts I and III, we refer to the individual introductions in [BBI] resp. [BBIII]. Before we now turn to a specific report of the contents of the present part II , let us first make our general terminology a little more precise.

By a "homogeneous space" Y, we mean a complex algebraic variety Y, equipped with a transitive action of some linear algebraic group G. Equivalently, $Y = G/H$ is the quotient of G by some closed subgroup H, up to isomorphism. Moreover, we have to assume Y complete, or equivalently H parabolic, at least for most of our major results. Our differential operators are linear, with algebraic coefficients. They make up a sheaf of noetherian rings on Y, denoted $\mathcal{D}_Y$.

Since we make a point, in our whole series, of working entirely in terms of algebraic methods, we have to refer here to the ètale topology on Y. To clarify this point, we recall for the convenience of the reader the definition of the sheaf $\mathcal{D}_Y$, as well as its local description, in § 0.

The main purpose of this article is to develop the concept of a "relative enveloping algebra" for the study of the canonical fibration $G \longrightarrow G/H$. In fact, we introduce this notion in chapter 1 for the more general situation of an arbitrary H-principal fibre bundle $f: X \longrightarrow Y$, that is a smooth morphism of algebraic manifolds with H action such that the fibres are free H orbits. The relative enveloping algebra of such a fibration is defined as the sheaf of rings on Y given by

$$\mathcal{U} = f_*((\mathcal{D}_X)^H),$$

that is we take all differential operators on the total space X which are H invariant, and consider the sheaf that they make up on the base space Y.

In the special case where the base Y reduces to a single point, so that the sheaf $\mathcal{U}$ reduces to a single stalk, observe that we obtain the usual definition of the enveloping algebra $U(\underline{h})$ of the Lie algebra $\underline{h}$ of H, namely as the ring of all differential operators on H invariant under right translations. More generally, the sheaf $\mathcal{U}$ may be described as a tensor product $\mathcal{D}_Y \otimes U(\underline{h})$ whenever the fibration is trivial. In full generality, such a description holds locally, for any local trivialization of the bundle, but not globally. In this sense, the new concept is a proper generalization (not only a scalar extension) of that of an enveloping algebra.

Our original purpose for introducing relative enveloping algebras was to study <u>induced ideals</u>. This is a short term for annihilators of induced modules. The process of induction attaches to each ideal J in $U(\underline{h})$ a certain ideal I in the enveloping algebra of $\underline{g}$ = Lie G, namely the largest one contained in $JU(\underline{g})$. (Here "ideal" means two-sided ideal.) Although this process is so easily defined, its effect had turned out to be surprisingly difficult to understand. We can get here some more control over this process by means of the following more refined construction in the relative enveloping algebra $\mathcal{U}$ of the fibration $G \longrightarrow G/H$: for each ideal J in $U(\underline{h})$, we construct a certain <u>sheaf of ideals</u> $\tilde{J}$ in $\mathcal{U}$ which "locally" (i.e. for each local trivialization of the bundle) coincides with $\mathcal{D}_Y \otimes J$.

In case of the fibration $G \longrightarrow G/H$, the left G action gives rise to a ring homomorphism $U(\underline{g}) \longrightarrow \Gamma(G/H,\mathcal{U})$, since it commutes with the right H action on G. Then a crucial result states that the kernel of the composed homomorphism

$$\mathcal{U}(\underline{g}) \longrightarrow \Gamma(G/H,\mathcal{U}) \longrightarrow \Gamma(G/H,\mathcal{U}/\tilde{J})$$

is the ideal induced by J in $U(\underline{g})$, up to a certain "twist", see the Corollary of proposition 2 for a precise statement. This result generalizes our earlier result in [BB]I, 3.6; it provides the basis for our applications to the study of induced ideals.

We point out here, however, that the construction of $\tilde{J}$ again applies to an arbitrary H principal fibration. Let us mention the following example, which is known from the previous literature. We first note that if H is a <u>torus</u>, hence $U(\underline{h}) = S(\underline{h})$ commutative, and if we take for J a <u>maximal</u> ideal in $U(\underline{h})$, then $\mathcal{U}/\tilde{J}$ is sheaf of "twisted differential operators" on Y.

Then we apply the construction to the principal H fibration

$$X = G/(B,B) \longrightarrow Y = G/B,$$

where H is a maximal torus in a Borel subgroup B of G, to obtain the twisted differential operators considered in [BeBe].

In chapter 2, we introduce the notion of <u>characteristic variety</u> of a coherent $\mathcal{U}$-module. This is just an obvious modification of the usual concept of characteristic variety of a coherent $\mathcal{D}$-module, that is a support of an associated graded module, as familiar from the general theory of $\mathcal{D}$-modules. By definition, the characteristic variety $Ch(M)$ of the coherent $\mathcal{U}$-module M is a closed subvariety in the quotient $(T^*X)/H$ of the cotangent space T^*X over the bundle space X by the free H action. In proposition 3, we give a description of the characteristic variety $Ch(\mathcal{U}/\tilde{J})$, where J is the sheaf of ideals "induced" from an ideal J in $U(\underline{h})$ by the above mentioned construction. This description is in terms of the <u>associated variety</u> of J, that is the zero set in $\underline{h}^*$ of the associated graded ideal in $\mathrm{gr}\, U(\underline{h}) = S(\underline{h})$, usually denoted $V(\mathrm{gr}\, J) = V(U(\underline{h})/J)$ in the literature (see e.g. [BB]I).

Again the special case where H is <u>commutative</u>, and J a <u>maximal</u> ideal of $U(\underline{h}) = S(\underline{h})$, deserves special attention as an example relating our new concepts to the work of other authors. The associated variety of J consists of a single point $\mu \in \underline{h}^*$ in this case, and the characteristic variety of $\mathcal{U}/\tilde{J}$ identifies with the so-called "<u>shifted cotangent bundle</u>" $T^*_\mu Y$ of the base space Y; in particular it carries a natural structure of a symplectic manifold. The notion of shifted cotangent bundles was suggested independently by V. Ginsburg [G1], [G2], and B. Kostant (unpublished). If the "shift" μ is zero, then it coincides with the cotangent bundle T^*Y in the ordinary sense, otherwise it is only locally (i.e. with respect to a local trivialization) isomorphic to T^*Y, but differs from it in general by some global "twist". If we further specialize this example to the case of a one parameter group H, then the sections of the shifted cotangent bundle may be interpreted as connections of our H principal fibration by a result of A. Weinstein (see proposition 5 for details). - In this context, we also communicate some results of B. Kostant about polarizations of coadjoint G orbits in $\underline{g}^*$, and their characteristization in terms of shifted cotangent bundles (theorem 1), complemented by some results on associated varieties in the context of chapter 3 (theorem 3). We are grateful to B. Kostant for his generous kind permisssion to report on these unpublished ideas of his at the present opportunity.

In chapter 3, we finally come to our main purpose which originally motivated our concept of relative enveloping algebras, namely to establish the behaviour of associated varieties of ideals in enveloping algebras under parabolic induction. This is achieved in our present theorem 2 in full generality, while our methods employed in part I resp. III give only partial results, restricted to annihilators of finite dimensional resp. of highest weight modules (cf. [BB]I resp. [BB]III). A completely analogous result had previously been established for the so-called "wave-front-sets" of representations by Barbasch and Vogan [BV]. The behaviour under parabolic induction is very easy to guess, but surprisingly hard to prove for associated varieties (cf. the comments in [BV]). Let us explain our solution of this problem here a little more precisely.

In our previous notation, we consider the principal H fibration $G \to G/H$, and we now assume H a parabolic subgroup. Let J be an ideal in $U(\underline{h})$, and I the induced ideal in $U(\underline{g})$. The claim is that the associated variety of I

is obtained from that of J as

$$V(\mathrm{gr}\ I) = G\rho^{-1}\ V(\mathrm{gr}\ J) \qquad (1)$$

where ρ: $\underline{g}^* \longrightarrow \underline{h}^*$ denotes restriction, and G ... denotes G-saturation (cf. theorem 2). As a first ingredient, we use the sheaf of ideals $\tilde{J}$ in U, and the description of its characteristic variety in $(T^*G)/H \cong Gx^H\ g^*$, as mentioned before in this introduction, to obtain an analogue of (1) on the level of characteristic varieties, viz.

$$Ch(U/\tilde{J}) = Gx^H\ \rho^{-1}\ V(\mathrm{gr}\ J). \qquad (2)$$

Second, we use the canonical map $\pi:(T^*G)/H \longrightarrow \underline{g}^*$ (a modified "momentum map", cf. [BB]I) to relate (2) to (1). In fact this map obviously maps the right-hand sides onto each other, so it suffices to prove the same for the left-hand sides, in order to derive (1) from (2). It actually suffices to prove only one inclusion, namely

$$V(\mathrm{gr}\ I) \subset \pi\ Ch(U/\tilde{J}). \qquad (3)$$

(This is because the other conclusion in (1) is anyway true [Bo1]). Third, we use the above mentioned fact that the induced ideal I appears as the kernel of the canonical homomorphism of $U(\underline{g})$ into the ring of global sections of $U/\tilde{J}$, which gives

$$V(\mathrm{gr}\ I) = V(U(\underline{g})/I) \subset V(\Gamma(G/H, U/\tilde{J})). \qquad (4)$$

Hence, as a fourth and last step to complete our argument, it suffices to know for the U-module $M = U/\tilde{J}$ and its $U(\underline{g})$-module of global sections $M = \Gamma(G/H, M)$ that we have

$$V(M) \subset \pi\ Ch(M). \qquad (5)$$

(Because (5) and (4) imply (3) and hence (1).) In fact, we establish (5) even for an arbitrary coherent U-module M in our proposition 5 (special case i = 0 there).

At this point of our argument, the completeness of G/H (or the parabolicity of H) is crucial, since it implies first properness of π, and then finite type of M. We use similar arguments in [BB]III to prove a result related to (5), which also relates associated to characteristic varieties, see loc.cit., theorem 1.9.

As a final topic in this chapter, we discuss the case of the H principal fibration $X = G/(P,P) \longrightarrow G/P$, where G is semisimple, P parabolic, and $H = P/(P,P)$ hence a torus. It seems to us that the relative enveloping algebra $\mathcal{U}$ for this fibration should provide a usefull setting to study the concept of a "Dixmier map", which can be (well-)defined as in [Bo 3] on the set of G-orbits in the "Dixmier-sheet" [Bo 2], [BK] attached to P. In fact, the image of the "moment map" $\pi:(T^*X)/H \longrightarrow \underline{g}^*$ is the closure of this Dixmier sheet (identify $\underline{g}^* = \underline{g}$), and the map π itself is its famous "generalized Grothendieck simultaneous resolution". Now each G-orbit in the Dixmier sheet for P is represented by an element f polarized by $\underline{p}$ = Lie P (in fact, this defines the Dixmier sheet), and the so-called "orbit method" of Dixmier-Kirillov attaches to this orbit the ideal in $U(\underline{g})$ induced from the maximal ideal of $U(\underline{p})$ given by f. It can be proven, generalizing [Bo3], that this process attaches a well-defined induced ideal to each orbit in the Dixmier sheet. It is a very delicate question, answered positively in [BJ] for $G = SL_n$ only, wether this map is also injective on the orbit space of the fixed sheet . We suggest here to modify the orbit method by attaching to f above the corresponding sheaf of induced ideals in $\mathcal{U}$, since the G-orbit of f can actually be reconstructed from this sheaf, as we show in proposition 7 and its corollaries. So our "modified orbit map" becomes injective. For this purpose, we have to introduce another notion of characteristic varieties, refering to an alternative filtration on the relative enveloping algebra.

However, this circle of ideas has not been developped very far yet and should be further investigated in the future. In particular, we conclude the last chapter by stating some typical open problems arising in this context.

Remarks.

It is a pleasure to thank the Institut des Hautes Etudes Scientifiques (IHES) at Bures-sur-Yvette (France), and the Max-Planck-Institut für Mathematik (MPI) at Bonn (Germany) for their generous hospitality and support on various occasions, which provided to a large extent the practical basis for our cooperation. Our collaboration on differential operators on homogeneous spaces and their application to associated varieties of primitive ideals was started in the spring of 1981, while one of us (W.B.) spent a sabbatical term at the IHES. The project of publishing our results in a three part series of papers, according to three different levels of methods, was formulated in detail during a shorter visit to the IHES in the fall of the same year. The contents of all three parts were announced in our introductory remarks to [BB]I, thus fixing a numeration which later turned out to be not a chronological one. A preliminary draft of the core of the present part two was circulated to some experts already in 1981, and a more elaborated version of the present article resulted from our simultaneous visits to the MPI in 1982.
However, we then first turned to our work on part three [BB]III, which seemed to us more urgent in view of certain new applications. In fact, in the sequel,part three opened the way for our joint papers with MacPherson on "Nilpotent orbits, primitive ideals, and characteristic classes". Since part two was not logically necessary for part three, nor for these subsequent projects, it happened that we delayed finalizing and preprinting this part until now. We also have had in mind to further elaborate the applications to Dixmier sheets and Dixmier's orbit method (as sketched now at the end of chapter three), and to work on relations to gauge theories, but in view of the by now very long delay, we finally decided that these points may wait for another opportunity. We apologize for the delay to all those who kept asking us for the missing part two of our series of articles.

Finally, we are pleased to thank V. Ginsburg, B. Kostant, and D. Vogan for useful discussions and correspondence.

§ 0. Differential operators on algebraic varieties: definitions and local expressions

Let X be a smooth algebraic variety over a field k of characteristic 0; we denote by $\mathcal{O}_X$ the sheaf of (germs of) regular functions on X. Inside the sheaf $\mathcal{E}nd_k(\mathcal{O}_X)$ associated to the presheaf $U \longmapsto \mathcal{E}nd_k(\mathcal{O}_U)$, one may, following [BGG], consider the subsheaves $\mathcal{D}_X(m)$ (for $m \in \mathbb{N}$) defined as follows:

(i) $\mathcal{D}_X(-1) = 0$ and $\mathcal{D}_X(0) = \mathcal{O}_X \subset \mathcal{E}nd_k(\mathcal{O}_X)$.

(ii) $\mathcal{D}_X(m)$ is the subsheaf of $\mathcal{E}nd_k(\mathcal{O}_X)$, whose local sections satisfy $[\mathcal{D}_X(m), \mathcal{O}_X] \subset \mathcal{D}_X(m-1)$, for any integer $m \geq 0$.

Then $\mathcal{D}_X = \bigcup_m \mathcal{D}_X(m)$ is a sheaf of algebras on X; one has $\mathcal{D}_X(m).\mathcal{D}_X(k) \subset \mathcal{D}_X(m+k)$ and $[\mathcal{D}_X(m), \mathcal{D}_X(k)] \subset \mathcal{D}_X(m+k-1)$, hence $gr\mathcal{D}_X = \bigoplus_m [\mathcal{D}_X(m)/\mathcal{D}_X(m-1)]$ is a commutative sheaf of rings on X. If $q: T^*(X) \longrightarrow X$ is the projection, $q_*\mathcal{O}_{T^*(X)}$ is isomorphic to $gr\mathcal{D}_X$. By [EGA 2], $q_*\mathcal{O}_{T^*(X)}$ is a coherent sheaf of rings which is even noetherian in the sense of [KK]. Hence the same is true of $\mathcal{D}_X$.

In order to give the traditional description of $\mathcal{D}_X$ in terms of local coordinates, we will need the following

Proposition 0: $\mathcal{D}_X$ is in a natural way a sheaf of rings on X for the êtale topology.

Proof: First we show that $\mathcal{D}_X$ is a presheaf on X for the êtale topology. For $i: U \longrightarrow X$ an êtale morphism, we have a morphism $\Gamma(X,T_X) \longrightarrow \Gamma(U,T_U)$ since any tangent vector field lifts uniquely under an êtale morphism. This defines a map $\Gamma(X, \mathcal{D}_X(1)) \longrightarrow \Gamma(U,\mathcal{D}_U(1))$. Now $\mathcal{D}_X$ is generated by $\mathcal{D}_X(1)$ as a sheaf of algebras, hence it is easy to extend the previous map to a morphism of algebras. This proves our first claim.

Now let $(i_\alpha\colon U_\alpha \longrightarrow X)$ be a covering of X in the êtale topology [Gi], [SGA4], i.e. the i_α are êtale and their images cover X. We have to prove that the sequence

$$0 \longrightarrow \Gamma(X,\mathcal{D}_X) \longrightarrow \bigoplus_\alpha \Gamma(U_\alpha,\mathcal{D}_{U_\alpha}) \longrightarrow \bigoplus_{\alpha,\beta} \Gamma(U_\alpha \times_X U_\beta,\ \mathcal{D}_{U_\alpha \times_X U_\beta})$$

is exact. So let $P_\alpha \in \Gamma(U_\alpha,\mathcal{D}_{U_\alpha})$ be such that P_α and P_β restrict to the same element $P_{\alpha,\beta}$ of $\Gamma(U_\alpha \times_X U_\beta,\ \mathcal{D}_{U_\alpha \times_X U_\beta})$. It is well-known that $\mathcal{O}_X$ is a sheaf for the êtale topology, hence the sequence

$$0 \longrightarrow \Gamma(X,\mathcal{O}_X) \longrightarrow \bigoplus_\alpha \Gamma(U_\alpha,\mathcal{O}_{U_\alpha}) \longrightarrow \bigoplus_{\alpha,\beta} \Gamma(U_\alpha \times_X U_\beta,\ \mathcal{O}_{U_\alpha \times_X U_\beta}).$$

is exact. Therefore the commutative diagram

$$\begin{array}{ccc}
\bigoplus_\alpha \Gamma(U_\alpha,\mathcal{O}_{U_\alpha}) & \longrightarrow & \bigoplus_{\alpha,\beta} \Gamma(U_\alpha \times_X U_\beta,\ \mathcal{O}_{U_\alpha \times_X U_\beta}) \\
\Big\downarrow \bigoplus_\alpha P_\alpha & & \Big\downarrow \bigoplus_{\alpha,\beta} P_{\alpha,\beta} \\
\bigoplus_\alpha \Gamma(U_\alpha,\mathcal{O}_{U_\alpha}) & \longrightarrow & \bigoplus_{\alpha,\beta} \Gamma(U_\alpha \times_X U_\beta,\ \mathcal{O}_{U_\alpha \times_X U_\beta})
\end{array}$$

defines an endomorphism of $\Gamma(X,\mathcal{O}_X)$. We may similarly construct an endomorphism of $\Gamma(V,\mathcal{O}_V)$ for any Zariski-open set of X, and these endomorphisms are compatible. Hence we have a section P of $\mathcal{E}nd_k\,(\mathcal{O}_X)$, which maps to the section P_α of $\mathcal{D}_{U_\alpha} \subset \mathcal{E}nd_k\,(\mathcal{O}_{U_\alpha})$; let us check that P is in fact a section of $\mathcal{D}_X$. Let $x \in X$, and V a neighbourhood of x which is contained in the image of some map i_α. If P_α is a section of $\mathcal{D}_{U_\alpha}(m_\alpha)$, then $P_{|V}$ is a section of $\mathcal{D}_X(m_\alpha)$, which proves the proposition.

Now for any point x of X, there is a neighbourhood U of X and an êtale morphism $\mu\colon U \longrightarrow \mathbb{A}^n$. Recall that for $(z^1,\ldots, z^n)$ linear coordinates on $\mathbb{A}^n$, $\mathcal{D}_{\mathbb{A}^n}$ is a free-left $\mathcal{O}_{\mathbb{A}^n}$-module with basis

$$\frac{\partial}{\partial z^\alpha} = \frac{\partial^{\alpha_1}}{\partial z_1^{\alpha_1}} \cdot \cdots \cdot \frac{\partial^{\alpha_n}}{\partial z_n^{\alpha_n}}$$

indexed by $\alpha \in \mathbb{N}^n$. Since $\mathcal{D}_U = \mu^*\mathcal{D}_{\mathbb{A}^n} = \mathcal{O}_U \otimes_{\mu^{-1}(\mathcal{O}_{\mathbb{A}^n})} \mu^{-1}(\mathcal{D}_{\mathbb{A}^n})$ it follows that

$\mathcal{D}_U$ is a free left $\mathcal{O}_U$-module with basis $\mu^{-1}(\frac{\partial}{\partial z^\alpha})$ or simply $\frac{\partial}{\partial z^\alpha}$. So any element of $\Gamma(U,\mathcal{D}_U)$ has a unique expression as

$$\sum_{|\alpha| \leq m} f_\alpha \frac{\partial}{\partial z^\alpha}$$

where $f_\alpha \in R(U)$. In the same situation, there is a rational morphism $T^*(U) \longrightarrow T^*(\mathbb{A}^n)$. The corresponding morphism $R(T^*(\mathbb{A}^n)) \longrightarrow R(T^*(U))$ coincides with the graded map of $D(\mathbb{A}^n) \longrightarrow D(U)$. In fact we have even $T^*(U) = U \times_{\mathbb{A}^n} T^*(\mathbb{A}^n)$.

<u>Remark</u>: We refer to [EGA4] for a definition of the sheaf of differential operators of order $\leq m$ as elements of $\mathcal{H}om_{\mathcal{O}_X}(p_{1,*}(\mathcal{O}_{X\times X}/\mathcal{J}_{\Delta_X}^{m+1}),\mathcal{O}_X)$ where $\Delta_X \subset X\times X$ is the diagonal, with defining sheaf of ideals $\mathcal{J}_{\Delta_X}$, and $p_1\colon X\times X \longrightarrow X$ is the first projection. Grothendieck's duality theory [H2] then implies the following equality

$$\mathcal{D}_X = p_{1,*} H^{\dim X}_{\Delta_X}(p_2^* \Omega_X^{max})$$

where Ω_X^{max} is the sheaf of differential forms of maximal degree (i.e. of degree dim X); in the complex-analytic context, this is used in [SKK] as a as a definition of $\mathcal{D}_X$.

§ 1. Induced representations and the concept of relative enveloping algebras

Our Lie algebras are finite dimensional over some algebraically closed base field k of characteristic 0. Let $\underline{g}$ be a Lie algebra, and $\underline{h} \subset \underline{g}$ a subalgebra. Let $U(\underline{g})$ resp. $U(\underline{h}) \subset U(\underline{g})$ denote their enveloping algebras. Let us recall the notion of <u>induction</u> of modules (resp. ideals) of $U(\underline{h})$ to modules (resp. ideals) of $U(\underline{g})$:

If M is a (left) $U(\underline{h})$-module, then

$$\mathrm{Ind}_{\underline{h}}^{\underline{g}} M := U(\underline{g}) \otimes_{U(\underline{h})} M$$

is called the (left) $U(\underline{g})$-<u>module induced from</u> M. If I is a (two-sided) ideal of $U(\underline{h})$, then the largest (two-sided) ideal J contained in the left ideal $U(\underline{g})I$ generated by I in $U(\underline{g})$ is called the <u>ideal induced from</u> I in $U(\underline{g})$. The two notions are related as follows, as is well-known (and easy to prove, cf. [Di].

<u>Lemma 1</u>: Let M be a $U(\underline{h})$-module, and $I = \mathrm{Ann}\, M$ its annihilator. Let N be the $U(\underline{g})$-module induced from M, and $J = \mathrm{Ann}\, N$ its annihilator. Then J is the ideal induced from I in $U(\underline{g})$.

<u>Remark</u>: We recall that Dixmier [Di], introduces the notion of "twisted induction". Let χ denote the character of $\underline{h}$ such that

$$\chi(x) = \frac{1}{2} \mathrm{trace}_{\underline{g}/\underline{h}}\, \mathrm{ad}\, x, \qquad \text{for all } x \in \underline{h}.$$

For any $U(\underline{h})$-module M, the "twisted induced module" is given by

$$\mathrm{Ind}_{\underline{h}}^{\underline{g}}\ M \otimes_k k_\chi .$$

We shall meet below instead the "doubly twisted induction", replacing χ by $\lambda = \chi$, which results in the $U(\underline{g})$-module

$$\mathrm{Ind}_{\underline{h}}^{\underline{g}}\ M \otimes_k k_\lambda = \mathrm{Ind}_{\underline{h}}^{\underline{g}}\ M \otimes \Lambda^{max}(\underline{g}/\underline{h}).$$

In the sequel, H is a linear algebraic group over k, acting on a smooth algebraic variety X. We shall study sheaves of modules over $\mathcal{D}_X$, the sheaf of differential operators on X (cf. parts I,III), which are "equivariant" under the action of H in the following weak sense:

Let $q: H \times X \longrightarrow X$ denote the action, and $p: H \times X \longrightarrow X$ the projection. Notice that both inverse image sheaves $p^*\mathcal{M}$ and $q^*\mathcal{M}$ are modules over $\mathcal{D}_{H \times X} \cong \mathcal{D}_H \boxtimes \mathcal{D}_X$ (the external tensor product, notation [Godement]), hence over the subsheaf $\mathcal{O}_H \boxtimes \mathcal{D}_X$. Then we define a weakly H-equivariant $\mathcal{D}_X$-module to be a $\mathcal{D}_X$-module $\mathcal{M}$, equipped with an isomorphism

$$\alpha: q^*\mathcal{M} \longrightarrow p^*\mathcal{M}$$

of $\mathcal{O}_H \boxtimes \mathcal{D}_X$-modules, such that α induces a group action of H on $\mathcal{M}$ (i.e. α satisfies the appropriate cocycle condition ensuring this, cf. [Mu]. The isomorphism α will be refered to as the (weak) equivariance datum. The same weak notion of equivariance is used in [BBM, § 2.12], see also [Mu] for more back-ground.

In the present paper, we shall use this notion to interpret modules over the enveloping algebra as weakly equivariant $\mathcal{D}$-modules on the group, as stated more precisely in the next lemma. We let $\underline{h}$ denote the Lie algebra of H, which we define as the Lie algebra of right-invariant vector-fields on H. Then the enveloping algebra $U(\underline{h})$ identifies with the algebra of right invariant differential operators on H (as usual, cf. e.g. our part I, section 3.3). Hence we can make any $\underline{h}$-module M into a $\mathcal{D}_H$-module by the process

$$M \longrightarrow \mathcal{D}_H \otimes_{U(\underline{h})} M, \qquad (*)$$

which is known as the "localization of M on H" (cf. part I, [BeBe], [BK]). The following facts may be enlightening.

Lemma 2:

a) The localization on H of any $U(\underline{h})$-module is weakly equivariant with respect to the action of H on itself by right multiplications.

b) The localization functor (*) establishes an equivalence of the category of all left $U(\underline{h})$-modules with the category of all weakly H-equivariant quasi-coherent left $\mathcal{D}_H$-modules.

c) A quasi-inverse functor is given by

$$M \longrightarrow \Gamma(H,\mathcal{M})^H, \qquad (**)$$

i.e. by taking the module of those global sections of $\mathcal{M}$ which are invariant under right translations.

It remains to exhibit the equivariance datum for the $\mathcal{D}_H$-module $\mathcal{D}_H \otimes_{U(\underline{h})} M$, to which the lemma refers. For this purpose, let us first recall the following (cf. also [BB III], 1.7).

Lemma 3: Multiplication in $\mathcal{D}_H$ induces an isomorphism

$$\mathcal{O}_H \otimes_k U(\underline{h}) \xrightarrow{\sim} \mathcal{D}_H$$

of sheaves of $(\mathcal{O}_H, U(\underline{h}))$-bimodules.

This may be proven by filtering both sides in the usual way. Then one gets, on the associated graded sheaves, in each degree $n \geq 0$, the morphisms

$$\mathcal{O}_H \otimes_k S^n(\underline{h}) \longrightarrow S^n(\mathcal{T}_H),$$

where $\mathcal{T}_H$ is the tangent sheaf of H. These are isomorphisms, namely the standard trivializations of $S^n(\mathcal{T}_H)$.

Now we conclude from Lemma 3 that

$$\mathcal{D}_H \otimes_{U(\underline{h})} M \cong \mathcal{O}_H \otimes_k M$$

as $\mathcal{O}_H$-modules. On the right hand side, $\underline{h}$ acts by

$$x.(f \otimes m) = x(f) \otimes m + f \otimes (x.m),$$

(where $x \in \underline{h}$, $f \in \mathcal{O}_H$, $m \in M$). Now let us exhibit the equivariance datum α,

$$\alpha\colon q^*(\mathcal{D}_H \otimes_{U(\underline{h})} M) \longrightarrow p_2^*(\mathcal{D}_H \otimes_{U(\underline{h})} M),$$

where $q\colon H \times H \longrightarrow H$ denotes the action, and $p_2\colon H \times H \longrightarrow H$ projection on the second factor. As $\mathcal{O}_{H \times H}$-modules, we have

$$p_2^*(\mathcal{D}_H \otimes_{U(\underline{h})} M) \cong \mathcal{O}_{H \times H} \otimes_{p_2^{-1}(\mathcal{O}_H)} p_2^{-1}(\mathcal{O}_H \otimes_k M) \cong \mathcal{O}_{H \times H} \otimes_k M,$$

and similarly,

$$q^*(\mathcal{D}_H \otimes_{U(\underline{h})} M) \cong \mathcal{O}_{H \times H} \otimes_k M.$$

So both inverse image sheaves of $\mathcal{D}_H \otimes_{U(\underline{h})} M$ with respect to p_2 resp. q identify with the same sheaf $\mathcal{O}_{H \times H} \otimes_k M$. Hence the sought-for equivariance datum α. Using lemma 3 once again, one easily checks that α is both $\mathcal{O}_{H \times H}$-linear and $p_2^{-1}(U(\underline{h}))$-linear, hence $\mathcal{O}_H \boxtimes \mathcal{D}_H$-linear. This proves lemma 2a), that is for each $U(\underline{h})$-module M, the $\mathcal{D}_H$-module $\mathcal{M} := \mathcal{D}_H \otimes_{U(\underline{h})} M$ becomes weakly H-equivariant.

To describe the quasi-inverse functor (2c)), let us first describe the action of H on $\Gamma(H,\mathcal{M})$. By the linearity of our group H, and the equivariance datum α, we obtain a morphism

$$\Gamma(H,\mathcal{M}) \longrightarrow \Gamma(H \times H, q^*\mathcal{M})$$
$$\longrightarrow \Gamma(H \times H, p_2^*\mathcal{M}) = \Gamma(H,\mathcal{O}_H) \otimes_k \Gamma(H,\mathcal{M}),$$

which by the cocycle condition for α gives $\Gamma(H,\mathcal{M})$ the structure of a comodule over $\Gamma(H,\mathcal{O}_H)$, hence an H action (cf. [Mu]. In this comodule, the H-invariants

$$\Gamma(H,\mathcal{M})^H \simeq \Gamma(H,\mathcal{O}_H \otimes_k M)^H \simeq \Gamma(H,\mathcal{O}_H)^H \otimes_k M \simeq M$$

form a module over the algebra

$$\Gamma(H,\mathcal{D}_H)^H \simeq U(\underline{h}).$$

Hence we have reconstructed M from $\mathcal{M}$, as claimed in lemma 2c). - To complete the proof of the lemma, we note that the functor $M \longrightarrow \mathcal{M}$ is obviously fully-faithful, and that its essential surjectivity follows from the fact that a quasi-coherent H-equivariant $\mathcal{O}_H$-module $\mathcal{L}$ is generated by its invariant global sections $L = \Gamma(H,\mathcal{L})^H$.

As suggested by lemma 2, we shall now proceed by describing the process of "induction" from $\underline{h}$ to $\underline{g}$ in terms of $\mathcal{D}$-modules. Our sheaf-theoretic analogue of the induced module will be a sheaf of modules on the homogeneous space G/H, over a certain sheaf of algebras $\mathcal{U}$ on G/H, which we will call a "relative enveloping algebra", and which we want to define now.

We shall introduce the notion of a "relative enveloping algebra" here in the more general situation of any principal homogeneous fibration $f: X \longrightarrow Y$ under a right action of H on X; in the applications, X will be a linear algebraic group G, and $Y = G/H$ its quotient by a closed subgroup $H \subset G$.

So X and Y are smooth algebraic varieties over k, $f: X \longrightarrow Y$ is a smooth morphism, H acts on X on the right, the diagram

$$\begin{array}{ccc} X \times H & \xrightarrow{\ m\ } & X \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{\ f\ } & Y \end{array}$$

commutes, and the H-bundle f is <u>locally trivial</u> for the êtale topology ("iso-triviale" in the terminology of J.-P. Serre [Se]1.): For any $y \in Y$, there exists an êtale morphism $\mathcal{U} \xrightarrow{i} Y$, the image of which contains y, such that there exists

$$X \times_Y \mathcal{U} \xrightarrow{\sim} \mathcal{U} \times H,$$

an H-invariant isomorphism, H acting on $\mathcal{U} \times H$ by right translation on the second factor.

<u>Remark</u>: In practice, it is too strict a condition for a fibration to be locally trivial for the Zariski-topology. Most of the constructions to follow would also work in the context of complex analytic manifolds.

Consider the projection $f_*(\mathcal{D}_X)$ of the sheaf $\mathcal{D}_X$ to Y. This is a sheaf of algebras, on which the group H acts. Let $\mathcal{U}$ be the sheaf of invariants under H in $f_*(\mathcal{D}_X)$: $\mathcal{U} = (f_*(\mathcal{D}_X))^H$.

Definition: $\mathcal{U}$ is called a "relative enveloping algebra".

Let $U \xrightarrow{i} Y$ be an étale open set such that $X \times_Y U \xrightarrow{\sim} U \times H$ as above; then it is easy to identify $i^{-1}(\mathcal{U})$ with $\mathcal{D}_U \otimes_k U(\underline{h})$ (the tensor product of the sheaves of algebras $\mathcal{D}_U$ and $U(\underline{h})$.

Lemma 4: $\mathcal{U}$ is a coherent sheaf of rings. Its fibers are left and right noetherian rings.

This is easy to prove when the H-bundle is trivialized. Hence the statement of the lemma is valid on each open set of an étale open covering, hence it is true on Y (see [EGA 4], 2$^{\text{érre}}$ partie, 2.5).

Choose $x \in X$, and let $y = f(x)$. To each left $U(\underline{h})$-module M, we will associate a quasi-coherent sheaf $\tilde{M}_x$ of left $\mathcal{U}$-modules, concentrated at the point y. First recall the sheaf of left $\mathcal{D}_H$-modules $\mathcal{M} = \mathcal{D}_H \otimes_{U(\underline{h})} M$. Since H is identified with $f^{-1}(y)$ by the choice of x, in such a way that the right action of H on $f^{-1}(y)$ corresponds to the action of H on itself by right translations, one may view $\mathcal{M}$ as a sheaf of left $\mathcal{D}_{f^{-1}(y)}$-modules. One then has a way of constructing a sheaf $\tilde{\mathcal{M}}$ of left $\mathcal{D}_X$-modules, concentrated on $f^{-1}(y)$. To explain this, let $Z = f^{-1}(y)$, and let $j: Z \hookrightarrow X$ be the inclusion. One may define a sheaf $\mathcal{D}_{X \leftarrow Z}$ of $(j\, \mathcal{D}_X, \mathcal{D}_Z)$-bimodules [K1,p.43], [Bj]. Let Ω_X^{max} (resp. Ω_Z^{max}) be the sheaves of differential forms of maximal degree on X (resp. Z). Then Ω_X^{max} is a sheaf of right $\mathcal{D}_X$-modules (with $w.v = -\theta_v(w)$, for w a section of Ω_X^{max}, v a vector field on X, and θ_v denoting Lie derivation, see [K1] or [Bj]. Consider the sheaf $\mathcal{D}_X \otimes_{\mathcal{O}_X} (\Omega_X^{max})^{\otimes -1}$, it is a left $\mathcal{D}_X$-module in two different ways

- the first module structure is just given by

$$P.(Q \otimes w) = PQ \otimes w.$$

- the second one arises as follows: $\mathcal{D}_X$ is obviously a right $\mathcal{D}_X$-module; now for any sheaf N of right $\mathcal{D}_X$-modules,

$$N \otimes_{\mathcal{O}_X} (\Omega_X^{max})^{\otimes -1} = \underline{Hom}_{\mathcal{O}_X} (\Omega_X^{max}, N)$$

is a sheaf of left $\mathcal{D}_X$-modules, a vector field v acting on a section φ of $\underline{Hom}_{\mathcal{O}_X} (\Omega_X^{max}, N)$ by

$$v.\varphi(w) = \varphi(w.v) - \varphi(w).v.$$

This gives the second $\mathcal{D}_X$-module structure on $\mathcal{D}_X \otimes_{\mathcal{O}_X} (\Omega_X)^{\otimes -1}$. If $(z^1,\ldots, z^n)$ are local coordinates, $(dz)^{\otimes -1}$ the section of $(\Omega_X)^{\otimes -1}$ which maps $dz = dz^1 \wedge \ldots \wedge dz^n$ to 1, then the vector field $\frac{\partial}{\partial z^i}$ acts on $\mathcal{D}_X \otimes_{\mathcal{O}_X} (\Omega_X)^{\otimes -1}$ by $\frac{\partial}{\partial z^i} (P \otimes (dz)^{\otimes -1}) = - (P.\frac{\partial}{\partial z^i}) \otimes (dz)^{\otimes -1}$.

Consider now $j^{-1}(\mathcal{D}_X \otimes_{\mathcal{O}_X} (\Omega_X^{max})^{\otimes -1})$, the inverse image to Z of $\mathcal{D}_X \otimes_{\mathcal{O}_X} (\Omega_X^{max})^{\otimes -1}$; the second $\mathcal{D}_X$-module structure on $\mathcal{D}_X \otimes_{\mathcal{O}_X} (\Omega_X^{max})^{\otimes -1}$ allows us to endow $j^{-1}(\mathcal{D}_X \otimes_{\mathcal{O}_X} (\Omega_X^{max})^{\otimes -1})$ with a left $\mathcal{D}_Z$-module structure. We still have an action of $j^{-1}(\mathcal{D}_X)$, coming from the <u>first</u> action of $\mathcal{D}_X$ on $\mathcal{D}_X \otimes_{\mathcal{O}_X} (\Omega_X^{max})^{\otimes -1}$; this action commutes with the action of $\mathcal{D}_Z$. Hence

$$\mathcal{D}_{X \leftarrow Z} = j^{-1}(\mathcal{D}_X \otimes_{\mathcal{O}_X} (\Omega_X^{max})^{\otimes -1}) \otimes_{j^{-1}(\mathcal{O}_X)} (\Omega_Z^{max})$$

is a $(j^{-1}\mathcal{D}_X, \mathcal{D}_Z)$-bimodule.

Given a sheaf M of left $\mathcal{D}_Z$-modules, we then define $i_*(M) = \tilde{M} = j_*(\mathcal{D}_{X \leftarrow Z} \otimes_{\mathcal{D}_Z} M)$. (Here j_* just indicates that we consider this sheaf, concentrated on Z, as a sheaf on X with zero fibres outside Z.) Let us recall that this is an exact functor mapping (quasi-)coherent $\mathcal{D}_Z$-modules to (quasi-) coherent $\mathcal{D}_X$-modules with support in Z (See [Bj, p. 234-5]), Prop. 2.10.1 for exactness, 2.10.2 for coherence; Björk considers only the complex analytic case, but the arguments carry over to the algebraic case considered here, replacing open neighbourhoods by êtale neighbourhoods.)

Local computation: Let $(z^1, \dots, z^n)$ be local coordinates on X such that Z has defining ideal $(z^1, \dots, z^k)$. Then $(dz^{k+1} \wedge \dots \wedge dz^n)^{\otimes -1}$ may be viewed as a section of $j^*(\Omega_X^{max})^{\otimes -1} \otimes (\Omega_Z^{max})$, hence as a section of $\mathcal{D}_{X \leftarrow Z}$. As a $j^{-1}(\mathcal{D}_X)$-module, $\mathcal{D}_{X \leftarrow Z}$ is then generated by $(dz^{k+1} \wedge \dots \wedge dz^n)^{\otimes -1}$, the annihilator of which is the ideal of $j^{-1}(\mathcal{D}_X)$ generated by $(z^{k+1}, \dots, z^n)$. The action of $\frac{\partial}{\partial z^i}$ (for $k + 1 \leq i \leq n$), viewed as a vector field on Z, on $\mathcal{D}_{X \leftarrow Z}$ is given by

$$(P|_Z \otimes (dz^{k+1} \wedge \dots \wedge dz^n)^{\otimes -1}) = -(P.\frac{\partial}{\partial z^i})\Big|_Z \otimes (dz^{k+1} \wedge \dots \wedge dz^n)^{\otimes -1}$$

where P is a section of $\mathcal{D}_X$, $P|_Z$ the corresponding section of $j^{-1}(\mathcal{D}_X)$, and $P.\frac{\partial}{\partial z^i}$ is the product of two sections of $\mathcal{D}_X$. Hence as a $\mathcal{D}_Z$-module, $\mathcal{D}_{X \leftarrow Z}$ is free with basis $\left(\frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \dots \partial z_k^{\alpha_k}}\right)_{\alpha_i \geq 0}$, where $|\alpha| = \sum_1^k \alpha_i$.

In case Z is a point, $\mathcal{D}_{X \leftarrow Z}$ is isomorphic to $H_Z^n(X, \mathcal{O}_X)$, which is described in detail in [BB I, 1.6]. Notice also that $\mathcal{D}_{X \xleftarrow{id} X} = \mathcal{D}_X$.

Note that in our case where $Z = f^{-1}(y)$, $D_{X \leftarrow Z}$ is a weakly H-equivariant sheaf of $(j^{-1}(\mathcal{D}_X), \mathcal{D}_Z)$-bimodules, hence $\tilde{M}$ is a weakly H-equivariant sheaf of $\mathcal{D}_X$-bimodules, supported on $f^{-1}(y)$. Then $\tilde{M}_X := [f_*(\tilde{M})]^H$ is a left $\mathcal{U}$-module concentrated on the point y, which we may identify with its fibre at y. This will be called the $\mathcal{U}_y$-module induced from M (of course, depending on the choice of $x \in f^{-1}(y)$).

We need the following self-evident:

Lemma 5: Let $Z_1 \hookrightarrow X_1$, $Z_2 \hookrightarrow X_2$ be closed imersions of smooth varieties. There is a natural isomorphism

$$\mathcal{D}_{X_1 \times X_2 \leftarrow Z_1 \times Z_2} \cong \mathcal{D}_{X_1 \longleftarrow Z_1} \otimes \mathcal{D}_{X_2 \longleftarrow Z_2}$$

Let U be an (étale) neighbourhood of y such that $U\times_Y X \xrightarrow{\sim} U \times H$ as always. We may then write:

$$\tilde{M}_{f^{-1}(U)} = j_* (\mathcal{D}_{U\times H \leftarrow \{y\}\times H} \underset{\{y\}\times H}{\otimes} M) = \mathcal{D}_{U \leftarrow \{y\}} \underset{k}{\otimes} M.$$

Hence, using lemma 2, we have $\Gamma(f^{-1}(y),\tilde{M}) = \mathcal{D}_{U \leftarrow \{y\}} \otimes_k \Gamma(f^{-1}(y), M)$, therefore the space of H-invariants in $\Gamma(f^{-1}(y), \tilde{M})$ is equal to $\mathcal{D}_{U\leftarrow\{y\}} \otimes_k M$. Here $M = \Gamma(f^{-1}(y),M)^H$ via the equivalence explained in lemma 2. From this we deduce

Lemma 6: The functor $M \longmapsto \tilde{M}_x$ is exact, and transforms $U(\underline{h})$-modules of finite type to $\mathcal{U}_y$-modules of finite type.

We now turn to the induction of ideals.

Proposition 1: Let M be a $U(\underline{h})$-module, J its annihilator. There exists a sheaf $\tilde{J}$ of two-sided ideals of $\mathcal{U}$, such that for any $x \in X$, $y = f(x)$, the fibre $\tilde{J}_y$, is the annihilator of the $\mathcal{U}_y$-module $\tilde{M}_x$. Furthermore, $\tilde{J}$ is a coherent sheaf of left (and right) ideals of $\mathcal{U}$.

Proof.: One first verifies that $\tilde{J}_y := \mathrm{Ann}\, \tilde{M}_x$ is independent on the choice of $x \in f^{-1}(y)$. To this aim, one may replace Y by an étale neighbourhood of y, and then reduce to the case $X \cong Y \times H$. Then for $h \in H$, one has $\tilde{M}_{x\cdot h} = \mathcal{D}_{Y \leftarrow \{y\}} \otimes_k {}^{\mathrm{Ad}\, h}M$ where ${}^{\mathrm{Ad}\, h}M$ is the $U(\underline{h})$-module obtained from M by "twisting" the action of $\underline{h}$ via $\mathrm{Ad}\, h$. The annihilator of $\tilde{M}_{x\cdot h}$ is then $\mathcal{D}_{Y,y} \otimes {}^{\mathrm{Ad}\, h}J = \mathcal{D}_{Y,y} \otimes J \subset \mathcal{U}_y$: it does not depend on h. To prove that the collection of the $\tilde{J}_y$, for $y \in Y$, defines a sheaf $\tilde{J}$, one may verify it on each open set of an étale covering, using the étale descent for quasi-coherent sheaves of $\mathcal{O}_Y$-modules [SGA 1], Exposé VIII. One then reduces to the case $X = Y \times H$, which is obvious from the above description of the $\tilde{J}_y$. The last statement follows also by étale descent. Q.e.d.

We now specialize these considerations to the situation $f: X = G \longrightarrow Y = G/H$, as mentioned earlier. Recall the bimodule-isomorphism $\mathcal{O}_G \otimes_k U(\underline{g}) \xrightarrow{\sim} \mathcal{D}_G$ (Lemma 3). Projecting to G/H and taking the H-invariants, we get

$$\mathcal{U} = [f_*(\mathcal{D}_G)]^H = [f_*(\mathcal{O}_G)\otimes_k U(\underline{g})]^H = [f_*(\mathcal{O}_G)]^H \otimes_k U(\underline{g}) = \mathcal{O}_{G/H}\otimes_k U(\underline{g})$$

as a $(\mathcal{O}_{G/H},U(\underline{g}))$-bimodule. Similarly, $\mathcal{U} = U(\underline{g})\otimes_k \mathcal{O}_{G/H}$, as $(U(\underline{g}),\mathcal{O}_{G/H})$-bimodule.

This is a nice description of $\mathcal{U}$, although it somehow hides the multiplicative structure: Let us mention that for g a section of $\mathcal{O}_{G/H}$, $\xi \in \underline{g}$, the commutator $[\xi,g]$, computed in $\mathcal{U}$, is a function on G/H, such that its pull-back to G is $\xi.(g\circ f)$. On the other hand, the description of $\mathcal{U}$ using a local trivialization of f does not lead to an easy description of the morphism $U(\underline{g}) \longrightarrow \mathcal{U}$.

<u>Proposition 2</u>: <u>Let</u> M <u>be a left</u> $U(\underline{h})$<u>-module</u>. <u>Then the</u> $\mathcal{U}_{f(1)}$<u>-module</u> $\tilde{M}_1$ <u>is canonically isomorphic to</u> $U(\underline{g})\otimes_{U(\underline{h})}[M\otimes k_\lambda]$ as a $U(\underline{g})$-module.

Here we meet the chararcter $\lambda = 2\chi = \mathrm{trace}_{\underline{g}/\underline{h}}$ ad of $\underline{h}$, as mentioned at the beginning of this chapter.

<u>Proof</u>: We first notice that

$$j^*(\Omega_G^{max})^{\otimes -1}\otimes_{\mathcal{O}_H}(\Omega_H^{max}) \cong \mathcal{O}_H\otimes_k \Lambda^{max}(T_{Y,y}) \cong \mathcal{O}_H\otimes_k k_\lambda,$$

where j is the inclusion of $f^{-1}(y)$ into G, $y = f(1)$, as before. Hence, for the localization of M on H,

$$\mathcal{M} := \mathcal{D}_H\otimes_{U(\underline{h})} M,$$

we have a natural map from

$$M\otimes_k k_\lambda = \Gamma(H,\mathcal{D}_H\otimes_{U(\underline{h})}[M\otimes k_\lambda])^H = \Gamma(H,\mathcal{M}\otimes_k k_\lambda)^H$$

to

$$\Gamma(G,\mathcal{D}_{G\leftarrow H}\otimes_{\mathcal{D}_H}\mathcal{M})^H = \Gamma(G,\tilde{\mathcal{M}})^H = \tilde{M}_1.$$

This map is $U(\underline{h})$-linear, and so, by functoriality of induced modules, extends uniquely to a $U(\underline{g})$-linear morphism

$$\mathrm{Ind}_{\underline{h}}^{\underline{g}}\, M \otimes k_\lambda = U(\underline{g}) \otimes_{U(\underline{h})} [M \otimes k_\lambda] \longrightarrow \tilde{M}_1.$$

Our claim is that this map is an isomorphism. To prove this, we use a free presentation of $M \otimes k_\lambda$ as a $U(\underline{h})$-module, and apply lemma 3, to reduce to the case $M \otimes k_\lambda = U(\underline{h})$, that is $M = U(\underline{h}) \otimes k_{-\lambda}$. Then the above induced module is just $U(\underline{g})$, we have $\tilde{M} = j_*(j^*\mathcal{D}_G)$, and we have to verify that the natural map

$$U(\underline{g}) \longrightarrow \Gamma(H, j^*\mathcal{D}_G)^H$$

is an isomorphism.But

$$\Gamma(H, j^*\mathcal{D}_G)^H = \Gamma(H, U(\underline{g}) \otimes_k \mathcal{O}_H)^H = U(\underline{g}) \otimes_k \Gamma(H, \mathcal{O}_H)^H = U(\underline{g}),$$

and the map in question is just the identity map on $U(\underline{g})$. Q.E.D.

Corollary: Let M be any $U(\underline{h})$-module, and let $I = \mathrm{Ann}\, M$ denote its annihilator ideal in $U(\underline{h})$. Then for any point y in G/H the kernel of the map $U(\underline{g}) \longrightarrow [\mathcal{U}/\tilde{I}]_y$ is given by the annihilator of the induced module

(*) $\quad \tilde{\tilde{\mathrm{Ind}}}_{\underline{h}}^{\underline{g}}\, M := U(\underline{g}) \otimes_{U(\underline{h})} [M \otimes k_\lambda].$

Remark: We shall call (*) the "doubly twisted" induced module of M, and its annihilator the "doubly twisted" induced ideal of I (cf. the introductory remarks of this chapter).

The corollary follows from the definition of $\tilde{\mathcal{J}}$ (given in proposition 1), and from proposition 2, in the case $y = f(1)$. By using the G-homogeneity of the situation, it follows for arbitrary y as well.

Let us mention here also the following variant of proposition 2, dealing with a situation which we had to study already in [BBI], (see 3.11, 4.6). Instead of the prinicipal H-fibration $f: G \longrightarrow G/H$, we consider in this variant the principal P/H-fibration $f: G/P \longrightarrow G/H$, where P is another subgroup of G, in which H is contained as a normal subgroup. Let $\underline{p} = \mathrm{Lie}\, P$. Let M be any left $\underline{p}/\underline{h}$-module, and consider it as a $\underline{p}$-module with trivial action of $\underline{h}$. Denoting $x \in G/P$, resp. $y = f(x) \in G/H$ the images of $1 \in G$, let us consider the $\mathcal{U}_y$-module $\tilde{M}_x$ as constructed in a more general situation above. Then we have

(**) $\qquad \tilde{M}_x \cong \tilde{\tilde{\mathrm{I}}}\mathrm{nd}_{\underline{p}}^{\underline{g}}\, M$.

as $U(\underline{g})$-modules. (Here the "doubly twisted" induced module is $U(\underline{g}) \otimes_{U(\underline{h})} [M \otimes k_\lambda]$, $\lambda = \mathrm{trace}_{\underline{g}/\underline{p}}\mathrm{ad}$, notation (*)). As from proposition 2, we may also conclude from its variant (**) a corollary describing the annihilator of the induced module $\tilde{\tilde{\mathrm{I}}}\mathrm{nd}_{\underline{p}}^{\underline{g}}\, M$. Details are left to the reader.

The reader may compare this result with loc. cit., corollary 3.11, where we were dealing with the special case that $H = (P,P)$ is the commutator subgroup of a parabolic subgroup P of a semisimple group G. On this comparison, the reader will notice that we actually encountered "relative enveloping algebras" in this context already in [BBI], although in a slightly disguised form.

Let us now discuss an important class of <u>examples</u> of quotients of relative enveloping algebras. For an arbitrary principal H-fibration $f: X \longrightarrow Y$, we obtain from any ideal I of $U(\underline{h})$ a sheave of algebras $\mathcal{U}/\tilde{I}$ from the relative enveloping algebra $\mathcal{U}$ (notation: proposition 1). Let us consider more specifically the case of the kernel of a character, that is

$$I = I_\mu := (\ker\mu)U(\underline{h}) = \mathrm{Ann}\ k_\mu,$$

where $\mu: \underline{h} \longrightarrow k$ is a character of $\underline{h}$. We observe that <u>the sheaf of algebras</u>

$$\mathcal{U}_\mu := \mathcal{U}/\tilde{I}_\mu,$$

<u>coming from a character</u> μ, <u>is locally isomorphic to</u> $\mathcal{D}_Y$. (Here "locally" refers to the étale topology.) Indeed, for $i: U \longrightarrow Y$ an étale morphism such that $X \times_Y U \xrightarrow{\sim} U \times H$, we have $i^{-1}\mathcal{U}_\mu = \mathcal{D}_U \otimes_k [U(\underline{h})/I_\mu] \cong \mathcal{D}_U$.

For $\mu = 0$ for example, we note that $\mathcal{U}_0 = \mathcal{U}/\tilde{I}_0$ is in fact canonically isomorphic to $\mathcal{D}_Y$. To see this, first notice that the natural morphism of sheaves of vector fields

$$f_* (\mathcal{T}_X)^H \longrightarrow \mathcal{T}_Y$$

extends to a natural morphism of sheaves of algebras of differential operators

$$\mathcal{U} \longrightarrow \mathcal{D}_Y,$$

which is clearly surjective. It remains to determine its kernel. By an easy computation, locally in the êtale topology, we identify the kernel with $\tilde{I}_0$.

Applying this to the case of the fibration of a group by a subgroup, $f: X = G \longrightarrow Y = G/H$, we conclude the following

Corollary: The kernel of the "operator representation" $U(\underline{g}) \longrightarrow \mathcal{D}_Y$ (terminology [BBI]) is the annihilator of the induced module

$$\widetilde{\mathrm{Ind}}_{\underline{h}}^{\underline{g}}\, k_\circ = U(\underline{g}) \otimes_{U(h)} k_\lambda.$$

Comment: This result was established in [BBI], corollary 3.6. It was actually the desire to extend this result to induction of more general (infinite-dimensional) $\underline{h}$-modules, which led us to the concept of "relative enveloping algebras" as developed in the present paper.

Let us return to the sheaves $\mathcal{U}_\mu = \mathcal{U}/\tilde{I}_\mu$, as defined above for $\mu \in X(\underline{h})$, where $X(\underline{h})$ denotes the vector space of characters of $\underline{h}$. We note that the set of equivalence classes of those sheaves of algebras on Y, which are locally isomorphic to $\mathcal{D}_Y$ (in the êtale topology),and contain $\mathcal{O}_Y$ as a subalgebra, is in bijection with $H^1(Y_{et}, \Omega^1_{Y,cl})$, the first cohomology group of the sheaf $\Omega^1_{Y,cl} \subset \Omega^1_Y$ of closed 1-forms on the êtale site Y_{et} of Y. This observation is due to Beilinson. Hence we obtain a map:

$$X(\underline{h}) \longrightarrow H^1(Y_{et}, \Omega^1_{Y,cl}),$$
$$\mu \longrightarrow \text{class of } \mathcal{U}_\mu .$$

The composition of this map with the canonical map into $H^1(Y,\Omega^1_Y)$ is described as follows. By replacing H with H/(H,H), and X with X/(H,H), we may assume H commutative, In that case, we have an exact sequence of locally free sheaves on Y

$$0 \longrightarrow \underline{h} \otimes_k \mathcal{O}_Y \longrightarrow f_*(\mathcal{T}_X)^H \longrightarrow \mathcal{T}_Y \longrightarrow 0.$$

To $\mu \in X(\underline{h})$ is associated another extension of $\mathcal{T}_Y$ by $\mathcal{O}_Y$, which is classified by an element of

$$\mathcal{E}xt^1_{\mathcal{O}_Y}(\mathcal{T}_Y,\mathcal{O}_Y) = \mathcal{E}xt^1_{\mathcal{O}_Y}(\mathcal{O}_Y,\Omega^1_Y) = H^1(Y,\Omega^1_Y),$$

since Y is smooth. This element is the image of the class of $\mathcal{U}_\mu$ in $H^1(Y,\Omega^1_Y)$.

More generally. we may consider any <u>finite-dimensional</u> irreducible H-module E with annihilator $I = \mathrm{Ann}_{U(\underline{h})} E$, and still obtain a similarly natural description of the sheaf of algebras $\mathcal{U}/\tilde{I}$ as follows: We have $U(\underline{h})/I \xrightarrow{\sim} \mathcal{E}nd_k\, E$. Let $E^\sim$ denote the vector-bundle on Y determined by E. Then $\mathcal{U}/\tilde{I}$ <u>is naturally isomorphic to the sheaf of algebras</u> $\mathcal{D}_Y(E^\sim)$ of differential operators on Y, acting on the sections of $E^\sim$.

<u>Remark on direct and inverse images of $\mathcal{U}$-modules</u>. Given the classical notion of morphisms of H-principal fibrations, one may develop a formalism of <u>direct</u> and <u>inverse image</u> for sheaves of $\mathcal{U}$-modules, entirely analogous to the corresponding formalism of $\mathcal{D}$-modules [K1,4.3]. For example, let us give a few indications on how to define direct images, with respect to a morphism of H-principal fibrations, given by a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow[h]{} & Y' \end{array}$$

We first define a sheaf $\mathcal{U}_{Y' \leftarrow Y}$ of $(h^{-1}\mathcal{U}_{Y'}, \mathcal{U}_Y)$-bimodules by

$$\mathcal{U}_{Y' \leftarrow Y} := h^{-1}(\mathcal{U}_{Y'} \otimes_{\mathcal{O}_{Y'}} ([f'_* \Omega^{max}_{X'}]^H)^{\otimes -1} \otimes_{h^{-1}(\mathcal{O}_{Y'})} [f_* \Omega^{max}_X]^H.$$

Then the direct image, or "integration along the fibres", of a complex $M^\cdot$ of $\mathcal{U}_Y$-modules may be defined by (cf. loc. cit.)

$$\int_h M^\cdot = Rh_* (\mathcal{U}_{Y' \leftarrow Y} \otimes^L_{\mathcal{U}_Y} M^\cdot).$$

Similarly, in the <u>duality for</u> $\mathcal{U}_Y$-<u>modules</u>, one uses the $\mathcal{U}_Y$-module $[f_* \Omega^{max}_X]^H$ instead of Ω^{max}_Y. However, since we have no use for this formalism here, let us leave the details to the reader.

§ 2. Characteristic varieties of $\mathcal{U}$-modules and shifted cotangent bundles

As usual, H is a linear algebraic group, and $f: X \longrightarrow Y$ is a H-principal fibration. The sheaf of relative enveloping algebras, $\mathcal{U} := [f_*(\mathcal{D}_X)]^H$ admits a filtration given by the degrees m of differential operators as

$$\mathcal{U}(m) = [f_*(\mathcal{D}_X(m))]^H.$$

It is easy to determine the associated graded sheaf $\operatorname{gr} \mathcal{U} = \bigoplus_{m \geq 0} \mathcal{U}(m)/\mathcal{U}(m-1)$: The quotient $T^*(X)/H$ of the cotangent bundle $T^*(X)$ by the action of H is a vector bundle over Y; then

$$\operatorname{gr} \mathcal{U} = p_*(\mathcal{O}_{T^*(X)/H}),$$

where $p: T^*(X)/H \longrightarrow Y$ denotes the structural morphism. For instance, $\mathcal{U}(1)/\mathcal{U}(0) = [f_* \mathcal{T}_X]^H$ identifies with the subsheaf of $p_*(\mathcal{O}_{T^*(X)/H})$ of functions which are linear on each fibre of p. It is easily seen that any coherent sheaf $\mathcal{M}$ of $\mathcal{U}$-modules admits locally a good filtration $\{\mathcal{M}_n\}_{n \in \mathbb{Z}}$, which may be used to define the characteristic variety $Ch(\mathcal{M})$: We define $Ch(\mathcal{M})$ as the support in $T^*(X)/H$ of the coherent sheaf $\operatorname{gr}(\mathcal{M})$ of $\operatorname{gr}(\mathcal{U})$-modules. Of course, this is not really a new notion: $f^{-1}(\mathcal{M})$ is a coherent sheaf of $\mathcal{D}_X$-modules, and $Ch\, f^{-1}(\mathcal{M})$ is the inverse image of the characteristic variety $Ch\, \mathcal{M}$ in the usual sense under the quotient map $T^*(X) \longrightarrow T^*(X)/H$.

There is a natural morphism $i: T^*(Y) \longrightarrow T^*(X)/H$ of vector bundles on Y. If $\mathcal{M}$ is a coherent sheaf of $\mathcal{D}_Y$-modules, then $i\,(Ch\, \mathcal{M})$ is the characteristic variety of the $\mathcal{U}$-module $\mathcal{M}$. (Recall the natural surjection $\mathcal{U} \twoheadrightarrow \mathcal{D}_Y$ of §1.)

We shall mainly be interested in the description of the characteristic variety $Ch(\mathcal{U}/\tilde{\mathcal{J}})$, where $\tilde{\mathcal{J}}$ is the sheaf of ideals in $\mathcal{U}$ induced from an ideal

J in $U(\underline{h})$ as defined in §1. It is clear that $Ch(U/\tilde{J})$ contains $T^*(Y) \subset T^*(X)/H$ (we assume $J \neq U(\underline{h})$). Evenmore, let T_f^* be the vector bundle on Y which is the cokernel of i (T_f^* is dual to the tangent bundle T_f to the fibres of f), and let $q: T^*(X)/H \longrightarrow T_f^*$ be the quotient morphism. Then $Ch(U/\tilde{J})$ is the q-inverse image of a subvariety of T_f^*, which we now determine.

Lemma 7: Let $V \subset \underline{h}^*$ be a subvariety invariant under the coadjoint action of H. There exists a unique subvariety $\tilde{V}$ of T_f^* such that, for any êtale map j: $U \longrightarrow Y$ with $U \times_Y X \xrightarrow{\sim} U \times H$ (hence $j^{-1} T_f^* \cong U \times \underline{h}^*$), the inverse image $j^{-1}(V)$ gets identified with $U \times V$.

Proof: Uniqueness is clear. It then must be shown that any automorphism of $U \times H \xrightarrow{p_1} U$, commuting with the right action of H, induces an automorphism of $T_{p_1}^* = U \times \underline{h}^*$ which stabilizes $U \times V$. But such an automorphism is of the following type

$$U \times H \ni (x,y) \longmapsto (x,a(x)y),$$

where $a: U \longrightarrow H$ is a regular map. This induces the automorphism

$$(x,h^*) \longrightarrow (x,Ad^*\, a(x) \cdot h^*) \quad \text{of} \quad U \times \underline{h}^*,$$

which fixes $U \times V$, since V is stable under the coadjoint action of H. Hence the lemma.

For any $U(\underline{h})$-module M, let $V(M)$ or $V_{\underline{h}}(M)$ denote its associated variety (in the sense of Bernstein, cf. our part I, 4.1).

Proposition 3: For a (two-sided) ideal I of $U(\underline{h})$, the characteristic variety of $U/\tilde{I}$, considered as a sheaf of U-modules, may be described in terms of the associated variety of $U(\underline{h})/I$, considered as a (left) $U(\underline{h})$-module, as follows:

$$Ch(U/\tilde{I}) = q^{-1}(\tilde{V}(U(\underline{h})/I)),$$

the preimage under the quotient morphism $q: T^*X/H \longrightarrow T_f^*$ (= Coker i) of the variety defined in the previous lemma.

This proposition is proven by an easy local computation, which is left to the reader.

Shifted cotangent bundles. It seems appropriate at this point to mention some recent work of Ginsburg [Gi] and Kostant (unpublished), explaining in particular how their new notion of "shifted cotangent bundles" fits into our present context. Here we assume the group H commutative, and $f: X \longrightarrow Y$ is a principal H-fibration as before. Given any $\mu \in \underline{h}^*$, we consider the subvariety determined in T^*X/H by the one element set $V = \{\mu\}$ as $q^{-1}(\tilde{V})$ (notation as in proposition 3).

Definition: This subvariety, made into a sympletic manifold by the proposition below, is called a "shifted cotangent bundle" of Y. It is denoted by $T_\mu^*(Y) = q^{-1}(\widetilde{\{\mu\}})$.

Proposition 4: (Ginsburg-Kostant): a) For each $\mu \in \underline{h}^*$, the variety $T_\mu^*(Y)$ carries a natural structure of a sympletic manifold.

b) This "shifted cotangent bundle" $T_\mu^*(Y)$, considered as a scheme over Y, and as a sympletic manifold, is locally (with respect to the êtale topology) isomorphic to T^*Y, the "ordinary" cotangent bundle.

c) If the "shift" μ is zero, then the above construction reproduces the ordinary cotangent bundle, that is $T_o^*(Y)$ is naturally isomorphic to T^*Y.

Remark: In the subsequent sketch of a proof of the proposition, we apply the so-called method of Hamiltonian reduction, which is explained more generally, and in more detail in a paper by Kazhdan, Kostant and Sternberg [KKS].

Proof: Since the natural action of G on T^*X is Hamiltonian, we have a momentum map $\pi: T^*X \longrightarrow \underline{h}^*$ (see [BBI], 2.3 for a definition in this case, or loc. cit., and [Kostant] for more general terminology). In the present case, the momentum map is smooth, since we are dealing with a principal H-fibration (consider a local trivialization to see this). It follows that the fibres of π are smooth subvarieties of T^*X. Furthermore, we claim that they are involutive:

Lemma 8: The fibre $\pi^{-1}\mu$ is an involutive subvariety of T^*X.

This means (by definition of "involutivity") that the tangent space $T_p\,\pi^{-1}\mu$ at any point p of the fibre, is co-isotropic (contained in its orthogonal) in $T_p\,T^*(X)$. - As usual, we may assume $X = Y \times H$, hence

$$T^*X = T^*Y \times T^*H \cong T^*Y \times H \times \underline{h}^*.$$

Then $\pi^{-1}\mu = T^*Y \times H \times \{\mu\}$, so let $p = (z, h, \mu)$ with $z \in T^*Y$, $h \in H$. Then

$$T_p\,\pi^{-1}\mu = T_zT^*Y \oplus T_hH = T_zT^*Y \oplus \underline{h} \oplus 0 \subset T_zT^*Y \oplus \underline{h} \oplus \underline{h}^*.$$

In this description, the orthogonal of the tangent space $T_p\,\pi^{-1}\mu$ is $0 \oplus \underline{h} \oplus 0$, which is, indeed, contained in $T_z\pi^{-1}(\mu)$, as was to be shown. This simultaneouly shows that this orthogonal is the image of $\underline{h}$ in $T_p\,\pi^{-1}\mu$ under the map $\underline{h} \longrightarrow T_p\pi^{-1}\mu$ coming from the H-action on $\pi^{-1}\mu$.

Hence the mapping $\pi^{-1}\mu \longrightarrow \pi^{-1}\mu/H$ has for fibres the leaves of the "foliation" defined by the kernels of $\omega|_{T_p\pi^{-1}\mu}$, the restrictions of the canonical sympletic form ω on T^*X to the tangent spaces of $\pi^{-1}\mu$ at its various points $p \in \pi^{-1}\mu$. This is the so-called "null-foliation". It follows now that the restriction of ω to the tangent bundle of $\pi^{-1}\mu$ is the inverse image of a 2-form ω' on $\pi^{-1}\mu/H$, which is non-degenerate. We have exhibited now the sympletic structure of the "shifted cotangent bundle" $T^*_\mu(Y) \cong \pi^{-1}\mu/H$, and the additional claims of proposition 4 are now easily seen from the description of this structure as given above. Q.e.d.

Affine structure on shifted cotangent bundles. Notice that we have a natural morphism

$$T^*_\mu(Y) \times_Y T^*_\nu(Y) \longrightarrow T^*_{\mu+\nu}(Y)$$

for any $\mu,\nu \in \underline{h}^*$. This satisfies an obvious associativity law. In paricular, taking $\nu = 0$, we see that $T^*Y = T^*_0(Y)$, as a group scheme over Y, acts on the Y-scheme $T^*_\mu(Y)$. Thus the shifted cotangent bundle $T^*_\mu(Y)$, for any $\mu \in \underline{h}^*$, becomes a principal homogeneous T^*Y-space. In the terminology of Ginsburg (loc. cit.), the scheme $T^*_\mu Y \longrightarrow Y$ is endowed with an affine structure. In more concrete terms, the fibers of the projection map $T^*_\mu(Y) \longrightarrow Y$ have the structure of affine spaces. It is then clear that the sheaf $\mathcal{O}_{T^*_\mu(Y)}$ of functions on the shifted cotangent bundle carries a natural filtration, given by degrees of restrictions of the functions to fibers, these restrictions being considered as polynomials. The symplectic structure of $T^*_\mu(Y)$ defines a Poisson bracket structure on $\mathcal{O}_{T^*_\mu(Y)}$. If g and h are two germs of sections of $\mathcal{O}_{T^*_\mu(Y)}$, which are affine on each fibre i.e. of degree ≤ 1, then their Poisson bracket $\{g,h\}$ is also affine.

An interpretation in terms of connections. Let us now assume the group H one-dimensional, so its Lie algebra is $\underline{h} = k$. In this case, the shifted cotangent bundle $T^*_1(Y)$ has the following nice interpretation due to A.Weinstein, in terms of connections of the H-principal fibration $f: X \longrightarrow Y$ under consideration. Let us recall that a connection on this fibration is given by a morphism

$$\gamma: f^*(T_Y) \longrightarrow T_X$$

of coherent sheaves on X satisfying the following two conditions (see [KN]): To state the conditions, let $T_{X|Y}$ denote the subsheaf in T_X of those vector fields on X tangent to the fibres of f, hence an exact sequence of locally free coherent sheaves on X:

$$0 \longrightarrow T_{X|Y} \xrightarrow{r} T_X \xrightarrow{df} f^*(T_Y)$$

Then the defining conditions of a connection are:

a) γ is H-equivariant.

b) γ splits the above exact sequence.

Alternatively, we may describe the connection in terms of a 1-form β on X satisfying certain properties. To do this, we associate to γ given as above the morphism of coherent sheaves on X, say

$$\beta: T_X \longrightarrow T_{X|Y},$$

such that

$$r \circ \beta = \mathrm{Id} - \gamma \circ df.$$

This definition makes sense, since $\mathrm{Id} - \gamma \circ df$ has image contained in $T_{X|Y}$, and β and γ determine each other uniquely. We may view β as a 1-form on X. In conclusion, a connection on the H-principal fibration $f: X \longrightarrow Y$ is the same thing as the datum of a 1-form β on X satisfying:

a') β is invariant under H.

b') $\beta \circ r = \mathrm{Id}_{T_{X|Y}}$.

The latter condition means that, if v denotes the canonical image of $1 \in \underline{h}$ ($= k$) in $T_{X|Y} \subset T_X$ (coming from the H action on X), then $\langle \beta, v \rangle = 1$.

Now observe that a section of the fibration $\pi^{-1}(1) \longrightarrow X$ determines a differential form β on X which satisfies b'). And β will satisfy a') if and only if β is induced from a section of the quotient fibration $\pi^{-1}(1)/H = T_1^*(Y) \longrightarrow Y$. Hence we have proved:

Proposition 5 (Weinstein): For a one-parameter group H, the connections of an H principal fibration $f: X \longrightarrow Y$ are naturally in bijective correspondence with the sections of the shifted cotangent bundle $T_1^*(Y) \longrightarrow Y$.

Interpretation of the symplectic form on $T_1^*(Y)$ in terms of a curvature form. Let us now discuss a little bit further the case $H = \mathbb{G}_m$ of the multiplicative group. In this case, we may always realize our principal fibration $f: X \longrightarrow Y$ in the following way: X is the complement of the zero-section $Y \hookrightarrow L$ in the total space of a line bundle L over Y, with $H = \mathbb{G}_m$ acting by homotheties. Given a 1-form β on X satisfying a') and b') above, we may construct a connection ∇ on the line bundle L, that is a morphism of sheaves

$$\nabla: L \longrightarrow \Omega_Y^1 \otimes_{\mathcal{O}_Y} L,$$

such that

$$\nabla(fs) = f.\nabla(s) + df \otimes s$$

for f,s any local sections of $\mathcal{O}_Y$ resp. L.

The construction proceeds as follows. First assume s takes values in $X \subset L$. Then define a morphism of sheaves $\alpha: \mathcal{T}_Y \longrightarrow \mathcal{O}_Y$ by the following diagram

$$\begin{array}{ccc} T_Y & \xrightarrow{ds} & s_* s^{-1}\mathcal{T}_X \\ \alpha \downarrow & & \downarrow \beta \\ \mathcal{O}_Y & = & s_* s^{-1}\mathcal{O}_X \end{array}$$

as $\alpha = \beta \circ ds$. Then ∇s will be characterized by the equality

$$\langle \nabla s, w\rangle = \alpha(w)\cdot s$$

for any local section w of T_Y. To verify that this defines indeed a connection ∇, we may as well assume that $X = Y \times \mathbb{G}_m \subset L = Y \times \mathbb{G}_a$ and that $\beta = z^{-1}dz$, where z is the coordinate on G_a; in fact, any other β' satisfying a') and b') belongs to $\beta + f^*(\Omega^1_Y) \subset \Omega^1_X$. Considering now s as a function on Y, it is now immediate that

$$\alpha(w)|_y = \frac{1}{s(y)} \langle w, ds\rangle|_y,$$

where $|_y$ means evaluation at $y \in Y$. We conclude

$$\langle \nabla w\rangle = \langle w, ds\rangle,$$

which proves that ∇ is in fact a connection. Now consider the <u>curvature</u> of the connection ∇, which is defined as a 2-form on Y. By an easy computation in local coordinates, we see that its inverse image to X coincides whith $d\beta$.

It follows from proposition 5 that the H-principal fibration

$$X \times_Y T^*_1(Y) \longrightarrow T^*_1(Y)$$

admits a connection (which is canonical). Hence there is a canonical 1-form $\tilde{\beta}$ on $X \times_Y T^*_1(Y)$ satisfying the conditions analogous to a') and b'). Now $X \times_Y T^*_1(Y)$ is obviously isomorphic to $\pi^{-1}(1) \subset T^*X$; in this way $\tilde{\beta}$ becomes the restriction to $\pi^{-1}(1)$ of the cotangent 1-form η on T^*X.

By the previous considerations, there is a natural connection on the pull-back of the line bundle L to the shifted cotangent bundle $T^*_1(Y)$. Let R denote the curvature form of this connection. Then the pull-back of R to $X \times_Y T^*_1(Y)$ is equal to

$$d\tilde{\beta} = d(\eta|_{\pi^{-1}(1)}).$$

Now we recall the sympletic form ω' on $T^*_1(Y)$ constructed in the proof of proposition 4, and we observe that the 2-forms R and ω' have the same pull-back $d\tilde{\beta}$ to $X \times_Y T^*_1(Y)$. We conclude that the <u>curvature</u> R <u>and the sym-</u>

plectic form ω' on $T_1^*(Y)$ are equal. This observation is due to V. Ginsburg (loc cit.).

Shifted cotangent bundles and polarizations. Let us now report on some recent results of B. Kostant (unpublished), relating polarizations of linear forms on a Lie algebra to affine bundles of the type $T_1^*(Y)$ considered above. Here we consider a principal $\mathbb{G}_m$-bundle $X \longrightarrow Y$ on a homogeneous space $Y = G/M$ coming from an equivariant line bundle $L \longrightarrow Y$ by removing the zero section. In more detail, let G be a linear algebraic group over k, $M \subset G$ a closed subgroup. Then an equivariant line bundle L on $Y = G/M$ is given by a character $\mu: M \longrightarrow \mathbb{G}_m$ as the associated bundle $L = G \times^M k_\mu \longrightarrow G/M$, and our principal $\mathbb{G}_m$-bundle is thus described as $X = G \times^M k_\mu^* \longrightarrow G/M$. Furthermore, let $\underline{m} \subset \underline{g}$ denote the Lie algebras of $M \subset G$, and $d_\mu: \underline{m} \longrightarrow k$ the differential of the character μ.

We recall that a polarization of some linear form $f \in \underline{g}^*$ is a subalgebra $\underline{h} \subset \underline{g}$ such that (1) f restricts to a character of $\underline{h}$, and (2) the codimension of $\underline{h}$ in $\underline{g}$ equals half the dimension of the coadjoint orbit generated by f (cf. e.g. [Di], [Ko2], [Gi] for more back-ground).

Theorem 1 (Kostant): (i) With notations as above, the following statements are equivalent:

a) $\underline{m}$ is a polarization of some linear form $f \in \underline{g}^*$ which extends $d\mu \in \underline{m}^*$.

b) The shifted cotangent bundle $T_1^*(Y)$ has an open G-orbit.

(ii) Moreover, if such is the case, then , for any $a \in k$, the induced module $\widetilde{\mathrm{Ind}}_{\underline{m}}^{\underline{g}}\, k_{a.d\mu} = U(\underline{g}) \otimes_{U(\underline{m})} k_{a.d\mu+\lambda}$ (notation § 1) admits a central character.

Proof: Let G_f resp. $\underline{g}_f = \text{Lie } G_f$ denote the isotropy group resp. subalgebra for $f \in \underline{g}^*$. Then g_f is the kernel of the (Kirillov-) alternating bilinear form B_f defined on $\underline{g}$ by

$$B_f(x,y) = f([x,y]),$$

for $x,y \in \underline{g}$. Now consider $d\mu + m^{\perp} \subset \underline{g}^*$, that is the set of those linear forms on $\underline{g}$ which restrict to $d\mu$ on $\underline{m}$, and note that this set is M-stable. Then property (1) of a polarization is satisfied by $\underline{m}$ automatically for any $f \in d\mu + m^{\perp}$, and it says that $f \in [\underline{m},\underline{m}]^{\perp}$, so that $\underline{m}$ is isotropic for f; in particular we have then $\underline{g}_f \subset \underline{m} \subset \underline{g}$. But since B_f is non-degenerate on $\underline{g}/\underline{g}_f$, which identifies with the tangent space of the coadjoint orbit $\mathcal{O}_f \cong G/G_f$, we see that then porperty (2) of a polarization requires that $\underline{m}$ be a maximally isotropic subspace of $\underline{g}$ with respect to B_f, or equivalently that

$$\dim \underline{m}/\underline{g}_f = \dim \underline{g}/\underline{m} = \dim \underline{m}^{\perp}.$$

But $\underline{m}/\underline{g}_f$ is the tangent space of the M-orbit Mf generated by f. Since $d\mu + m^{\perp}$ is M-stable we have $Mf \subset d\mu + m^{\perp}$, and the above equation requires that

$$\dim Mf = \dim \underline{m}/\underline{g}_f = \dim \mu + m^{\perp}.$$

We conclude from this discussion that statement a) is equivalent to the following:

a') M has an open orbit in $d\mu + \underline{m}^{\perp}$.

In order to see that this is equivalent to statement b), we notice that the shifted cotangent bundle $T_1^*(Y) \longrightarrow Y$ is a G-equivariant affine bundle over the homogeneous G-space $Y = G/M$ with fiber $d\mu + \underline{m}^{\perp}$ at the base point; it follows that $T_1^*(Y) \cong G\times^M(d\mu + \underline{m}^{\perp})$, and that this associated bundle containes a dense G-orbit if and only if the fibre $d\mu + \underline{m}^{\perp}$ contains a dense M-orbit.

This proves the equivalence of assertions a) and b). To prove the second part of the theorem, let us now assume that a) and hence b) hold. We first claim the following:

Lemma 9: Every G-invariant section of $\mathcal{U}$ belongs to the center of $\mathcal{U}$.

Here $\mathcal{U}$ is the relative enveloping algebra for our $\mathbb{G}_m$-principal fibration $X \longrightarrow Y$. To prove the lemma, we proceed by induction on the degree j of a G-invariant operator $P \in \mathcal{U}$, beginning with $j = -1$, where the claim is empty, since $\mathcal{U}(-1) = 0$. So let $P \in \mathcal{U}(j)$ for some $j \geq 0$, and let $0 \neq \overline{P} \in \mathcal{U}(j)/\mathcal{U}(j-1)$ denote its "principal symbol", considered as a function on $T^*X/\mathbb{G}_m$. Since P is assumed G-invariant, $\overline{P}$ is a G-invariant function. By our assumption b), G has a dense orbit on $T_1^*(Y)$. It follows then that the same is true on $T_\alpha^*(Y)$ for all scalars $0 \neq \alpha \in k$. Hence $\overline{P}$ is constant on each $T_\alpha^*(Y)$. But the $T_\alpha^*(Y) = \pi^{-1}\alpha$ are the fibers of the momentum map $\pi: T^*X/\mathbb{G}_m \longrightarrow \mathbb{G}_a$, hence we conclude that $\overline{P}$ comes from a function on $\mathbb{G}_a$. Hence $\overline{P} = \overline{Q}$ equals the principal symbol of some $Q \in \mathcal{U}(j)$ which belongs to the enveloping algebra of $\mathrm{Lie}(\mathbb{G}_m) = k$, which is central in $\mathcal{U}$. Now $P - Q \in \mathcal{U}(j-1)$ is central by induction hypothesis, hence P is central, which proves the lemma.

To prove statement (ii) of the theorem, let us apply proposition 2 in order to interpret the induced module

$$\widetilde{\widetilde{\mathrm{I}}}\mathrm{nd}_{\underline{m}}^{\underline{g}}\, k_{a.d\mu} = U(\underline{g}) \otimes_{U(\underline{m})} k_{a d\mu + \lambda}$$

in terms of $\mathcal{U}$-modules: Denoting $\overline{1}$ the base point of $Y = G/M$, and N the one-dimensional module for $\mathrm{Lie} = (\mathbb{G}_m) = k$ given by multiplication with $a \in k$, the above induced module, in the notation of proposition 2, is equal to $\widetilde{N}_{\overline{1}}$, made into a $U(\underline{g})$-module by means of the algebra-homomorphism $U(\underline{g}) \longrightarrow \mathcal{U}_{\overline{1}}$. By lemma 9, the center $Z(\underline{g})$ of $U(\underline{g})$ maps into $U(\text{"k"}) \subset \mathcal{U}_{\overline{1}}$, where "k" stands for the Lie algebra of $\mathbb{G}_m$. But $U(\text{"k"})$ acts on $\widetilde{N}_{\overline{1}}$ by the character corresponding to N (resp. a), hence by scalars. This proves that the center of $U(\underline{g})$ acts by scalars. Now the proof of the theorem is complete. Q.e.d.

Remark 1: Assuming statement b) in the theorem to be valid, let $\widetilde{\mathcal{O}}$ denote the open G-orbit in $T_1^*(Y)$. Then the momentum map

$$\pi : T^*(X)/\mathbb{G}_m \longrightarrow \underline{g}^*$$

maps $\tilde{\mathcal{O}}$ G-equivariantly onto a coadjoint orbit $\mathcal{O} \subset \underline{g}^*$. This is the orbit generated by any f as in statement a) of the theorem. We note that the mapping $\tilde{\mathcal{O}} \longrightarrow \mathcal{O}$ is a finite covering: In fact, if $\tilde{f} \in \tilde{\mathcal{O}}$ is a preimage of f, then $G_{\tilde{f}} = G_f \cap M$, and since $\underline{g}_f \subset \underline{m}$, we have $G_f^o \subset M$, hence $\tilde{\mathcal{O}} = G/G_{\tilde{f}} \longrightarrow G/G_f = \mathcal{O}$ is a covering of degree $[G_f : G_{\tilde{f}}] = [G_f : (G_f \cap M)] \leq [G_f : G_f^o] < \infty$. Note that the covering $\tilde{\mathcal{O}} \longrightarrow \mathcal{O}$ is an <u>isomorphism</u>, if and only if $G_f \subset M$. It is easy to verify that the inverse image of the Kostant-Kirillov symplectic form on $\mathcal{O}$ (see [Ko1], §5) to $\tilde{\mathcal{O}}$ coincides with the restriction of the symplectic form on $T_1^*(Y)$, as described in proposition 4. For a systematic discussion of finite coverings of coadjoint orbits, the reader may consult the work of Kostant, loc.cit..

<u>Remark 2</u>: Let us only mention at this point, that Ginsburg has weakened the notion of a polarization in such a way that all linear forms on $\underline{g}$ admit a polarization in this weak sense. For this point, and, more generally, for a very careful analysis of the geometry of the diagram

$$\underline{g}^* \longleftarrow T^*(Y) \longrightarrow Y$$

the reader is refered to Ginsburg's papers [G1], [G2].

§ 3. Application to the description of the associated variety of an induced ideal

In this chapter, G is a linear algebraic group with Lie algebra $\underline{g}$, P is a parabolic subgroup of G, with Lie algebra $\underline{p}$, and we consider the P-principal fibration $f: G \longrightarrow G/P$ as in §1. The assumption of parabolicity on P amounts to the assumption of completeness of the base space G/P (see [Sp] for such standard background on algebraic groups), and some of the arguments below are heavily based on this extra assumption.

Proposition 6: Let $\mathcal{M}$ be a coherent sheaf of left $\mathcal{U}$-modules which admits a global good filtration on G/P. Then:

a) The sheaf cohomology groups $H^i(G/P,\mathcal{M})$, for any integer i, are $U(\underline{g})$-modules of finite type.

b) The associated (Bernstein-) varieties in $\underline{g}^*$ of these $U(\underline{g})$-modules (terminology [BBI], 4.1) are contained in the image of the characteristic variety $Ch(\mathcal{M})$ (as defined in §2) under the momentum map $\pi:(T^*G)/P \longrightarrow \underline{g}^*$. That is to say

$$V(H^i(G/P,\mathcal{M})) \subset \pi(Ch(\mathcal{M}))$$

for all $i \in \mathbb{Z}$.

Here the assumption on $\mathcal{M}$ means that $\mathcal{M}$ is represented as a union of coherent $\underline{\mathcal{O}}_{G/P}$-submodules $\mathcal{M}(r)$, $r \in \mathbb{Z}$, such that

(i) $\mathcal{U}(m)\,\mathcal{M}(r) \subset \mathcal{M}(m+r)$, for all $m,r \in \mathbb{Z}$,

(ii) the associated graded sheaf $\operatorname{gr}\mathcal{M} := \bigoplus_r \mathcal{M}(r)/\mathcal{M}(r-1)$ is coherent as a $\operatorname{gr}\mathcal{U}$-module.

Recall that $\operatorname{gr}\mathcal{U} \cong q_* \mathcal{O}_{(T^*G)/P}$, if $q: (T^*G)/P \longrightarrow G/P$ denotes the canonical projection.

Proof: Let us filter $H^i(G/P,M)$ by the following subspaces

$$H^i(G/P,M)(r) := \mathrm{Im}[H^i(G/P,M(r)) \longrightarrow H^i(G/P,M)].$$

Then

$$gr_r\, H^i(G/P,M) := H^i(G/P,\, M)(r) \,/\, H^i(G/P,M)(r-1)$$

is a quotient of

$$H^i(G/P,M(r)) \,/\, \mathrm{Image}\, H^i(G/P,M(r-1)),$$

which in turn injects into

$$H^i(G/P,gr_r\, M), \text{ where } gr_r\, M := M(r)/M(r-1).$$

We conclude that, considered as modules over $gr\, U(\underline{g}) = S(\underline{g})$, the associated graded module

$$gr\, H^i(G/P,M) = \oplus_r\, gr_r\, H^i(G/P,M)$$

is a subquotient of

$$H^i(G/P,gr\, M).$$

(Cf. also [BB III], lemma 1.6, for a similar kind of argument.) To prove our proposition, it therefore suffices to prove the following:

Lemma: a) $H^i(G/P,gr\, M)$ is a $S(\underline{g})$-module of finite type.

b) Its support is contained in $\pi\, Ch(M)$.

We may interpret $gr\, M$ as a direct image $q_*(N)$ of some coherent sheaf N of $O_{(T^*G)/P}$-modules. Since the morphism q is affine, we have:

(1) $H^i(G/P,gr\, M) = H^i((T^*G)/P,N)$.

Now since $\underline{g}^*$ is affine, we conclude by Serre's vanishing theorem that the cohomology $H^j(\underline{g}^*,R^i\, \pi_*\, N)$ vanishes in degrees $j > 0$, and so the Leray spectral sequence for the sheaf N, and the morphism π degenerates. This gives

(2) $H^i((T^*G)/P,N) = H^0(\underline{g},\, R^i\, \pi_*\, N)$.

Now we use the completeness of G/P to conclude that π is proper, so that Grauert's theorem applies (see [EGA]III, Théorème 3.2.1, or [Hartshorne] III, Theorem 8.8), to give that $R^i\, \pi_*\, N$ is a coherent $O_{\underline{g}^*}$-module. Hence

$H^0(\underline{g}^*, R^i \pi_* N)$ is of finite type. Now equations (2) and (1) above imply a).

To prove b), consider any polynomial function on $\underline{g}^*$, say $F \in S^m(\underline{g})$, whose pull-back to $(T^*G)/P$ vanishes on the characteristic variety $Ch(M)$. Then by the definition of $Ch(M)$, there is a covering $(U_\alpha)_\alpha$ of G/P by Zariski open sets U_α, and there are integers $s_\alpha > o$, such that F^{s_α} annihilates gr M on U_α. By the quasi-compactness of G/P, we may assume this covering to be finite, so there exists an integer $s > o$ such that F^S annihilates gr M, and hence all its cohomology groups $H^i(G/P, \text{gr } M)$. Hence the function F vanishes on the support of $H^i(G/P, \text{gr } M)$. This proves b) of the lemma. Q.e.d.

By the <u>associated variety of an ideal</u> I in $U(\underline{g})$, we mean the zero set in $\underline{g}^*$ of its associated graded ideal gr $I \subset S(\underline{g})$ with respect to the canonical filtration of $U(\underline{g})$, notation: $V(\text{gr}I) = V(U(\underline{g})/I)$ (as in [BBI], 4.2). We are now ready to prove our main result, on the behaviour of associated varieties under "parabolic induction":

<u>Theorem 2</u>: <u>Let</u> I <u>be any (two-sided) ideal of</u> $U(\underline{p})$, <u>and let</u> J <u>be the ideal induced from</u> I <u>in</u> $U(\underline{g})$. <u>Then the associated variety of</u> J <u>is obtained from that of</u> I <u>by taking the</u> G-<u>saturation of the inverse image with respect to the restriction map</u> $\rho: \underline{g}^* \longrightarrow \underline{p}^*$, <u>that is to say</u>:

$$V(\text{gr } J) = G\, \rho^{-1} V(\text{gr } I).$$

<u>Remark</u>: The same statement is - equivalently - true for "twisted" or "doubly twisted" induction. In fact, we have for a $U(\underline{p})$-module M:

$$V(\text{gr Ann } M) = V(\text{gr Ann } M \otimes k_\mu),$$

even for an arbitrary character μ of $\underline{p}$, as is very easy to see.

Proof: It is clear that $V(\mathrm{gr}\ J)$ contains $\rho^{-1}\, V(\mathrm{gr}\ I)$, and that it must be stable under the coadjoint action of G, since J is a two-sided ideal. Hence the inclusion

$$V(\mathrm{gr}\ J) \supset G\, \rho^{-1}\, V(\mathrm{gr}\ I)$$

is easy (same arguments as used already in [Bo1], 2.3). So the point is to prove the opposite inclusion. We prove instead:

$$V(\mathrm{gr}\ J') \subset G\, \rho^{-1}\, V(\mathrm{gr}\ I),$$

for J' the "doubly twisted" induced ideal. (By the previous remark, the "twist" does not matter for establishing the theorem.)

As explained in §1, the ideal I defines a sheaf of ideals $\tilde{I}$ (notation §1) in the relative enveloping algebra $\mathcal{U}$ of the fibration $G \longrightarrow G/P$. Notice that $\mathcal{U}/\tilde{I}$ admits a globally defined good filtration, given by the images of the $\mathcal{U}(m)$, $m \in \mathbb{N}$ (notation §2). So by proposition 6, its ring of global sections $\Gamma(G/P, \mathcal{U}/\tilde{I})$ is of finite type as a left module over $U(\underline{g})$, and its associated variety is contained in $\pi\, Ch(\mathcal{U}/\tilde{I})$.

On the other hand, we know that this $U(\underline{g})$-module contains $U(\underline{g})/J'$ as a submodule (see proposition 2 and its corollary). So we have shown that

$$V(\mathrm{gr}\ J') \subset \pi\, Ch(\mathcal{U}/\tilde{I}).$$

Now we make use of the description of the characteristic variety $Ch(\mathcal{U}/\tilde{I})$ given in proposition 3 and lemma 7. Using the associated fibre bundle descriptions of

$$T^*Y = T^*(G/P) = G \times^P (\underline{g}/\underline{p})^* = G \times^P \underline{p}^{\perp},$$

$$(T^*X)/P = G \times^P \underline{g}^*, \text{ and } T^*P = G \times^P \underline{p}^*,$$

we obtain

$$Ch(\mathcal{U}/\tilde{I}) = G \times^P \rho^{-1}\, V(U(\underline{p})/I).$$

Since $\pi : (T^*X)/P \longrightarrow \underline{g}^*$ is given by "collapsing" the vector-bundle $G \times^P \underline{g}^*$, we finally conclude that

$$V(\mathrm{gr}\, J') \subset \pi\, Ch(U/\tilde{I}) = G\, \rho^{-1}\, V(\mathrm{gr}\, I),$$

which is what we wanted to prove. Q.e.d.

Remarks.

1.) Our result above on the behaviour of associated varieties under induction from a parabolic subalgebra (theorem 2) is analogous to a result of Barbasch and Vogan on the behaviour of wave front sets of representations under parabolic induction [BV].

2.) We do not know about possible extensions of this result to the case of a non-parabolic subgroup P.

3.) The special case of parabolic induction from a finite-dimensional representation is particularly important, since it suffices for most of the applications. For this special case of theorem 2, we have given a proof in more elementary terms in [BB]I (see theorem 4.6 and corollary 4.7). Note that in this previous proof, we used the algebra of those differential operators on $G/(P,P)$ which are invariant under the right action of $P/(P,P)$, hence we were in effect using already there a special case of the method fully developed in the present paper. Later in this chapter, we shall return to this situation from [BB], loc.cit., and discuss it a little bit further in the new light of our present methods.

A related result of Kostant. Let us mention here without proof an unpublished result of B. Kostant (cf. also [Ko2] in this context), which bears some resemblance to our results here, and which he kindly communicated to us in a letter in 1981, shortly after we had established theorem 2. We use the same notation as in theorem 1 (on the shifted cotangent bundle characterization of polarizations). In particular, a character of μ of the closed subgroup $M \subset G$ is given, and its differential $d\mu \in \underline{m}^*$ is extended to a linear form $f \in \underline{g}^*$; the momentum map $\pi: T^*Y \longrightarrow \underline{g}^*$ refers to the homogeneous space $Y = G/M$.

Assumption a) of the theorem below coincides with a) of theorem 1.

Theorem 3 (Kostant): Assume that

a) $\underline{m}$ is a polarization of f,

b) the coadjoint orbit $\mathcal{O} = Gf$ is closed, with centralizer G_f connected, and

c) $\dim \mathcal{O} = \dim \pi(T^*Y)$.

Consider the annihilator ideal $I = \operatorname{Ann} U(\underline{g}) \otimes_{U(\underline{m})} k_{ad\mu+\lambda}$ in $U(\underline{g})$, and compare this to the ideal J in $S(\underline{g}) = \mathcal{O}(\underline{g}^*)$ of all functions vanishing on $\mathcal{O}$. Then, with respect to the canonical filtrations on $U(\underline{g})$ resp. $S(\underline{g})$, we have:

(1) $\operatorname{gr} I$ is prime if and only if $\operatorname{gr} J$ is prime.

(2) If (1) holds, then $\operatorname{gr} I = \operatorname{gr} J$, and the associated variety of I is the closure of the image of the momentum map: $V(\operatorname{gr} I) = \overline{\pi(T^*Y)}$.

Basic filtration of relative enveloping algebras. Let us return once more to the case of a principal H-fibration $f: X \longrightarrow Y$ with respect to a commutative algebraic group H, as considered in §2. As a complement to the theory of the relative enveloping algebra $\mathcal{U}$ as developed so far, we want to suggest now an alternative concept of "characteristic variety", which refers to the following new filtration of $\mathcal{U}$ by subsheaves $\mathcal{U}_{bas}(m)$ $(m \in \mathbb{Z})$. Roughly speaking, $\mathcal{U}_{bas}(m)$ is the subsheaf of $\mathcal{U}$ made of those differential operators on X invariant under H, which have locally order at most m in the base-direction. More precisely, the definition is given by part a) of the next lemma.

Lemma 10:

a) There exists a unique filtration $(\mathcal{U}_{bas}(m))$ $m \in \mathbb{Z}$ of $\mathcal{U}$ such that for any êtale map $U \xrightarrow{i} Y$ with $U \times_Y X \cong U \times H$ we have

$$i^*(\mathcal{U}_{bas}(m)) = \mathcal{D}_U(m) \otimes_k U(\underline{h}). \qquad (*)$$

b) The associated graded sheaf with respect to this filtration,

$$gr_{bas}\,\mathcal{U} := \bigoplus_{m\in\mathbb{Z}} \mathcal{U}_{bas}(m)/\mathcal{U}_{bas}(m-1)$$

is a coherent sheaf of commutative rings on Y, with noetherian stalks.

Proof:

a) To establish the existence of this filtration, it suffices to show that an automorphism α of $U \times H$ of the form $\alpha:(x,h) \longrightarrow (x,a(x)h)$ preserves the right-hand side of (*), for all degrees $m \geq 1$. It actually suffices to consider the case $m = 1$, because $\mathcal{D}_U(m) \otimes_k U(\underline{h})$ is generated on U by the product of m copies of $\mathcal{D}_U(1) \otimes_k U(\underline{h})$. Now $\mathcal{D}_U(1) \otimes_k U(\underline{h})$ is generated by $U(\underline{h})$ and by $\mathcal{T}_U \otimes_k U(\underline{h})$ as an $\mathcal{O}_U$-module, and α preserves $U(\underline{h})$, and maps a vector field on U into an element of $\mathcal{T}_U \oplus (\mathcal{O}_U \otimes_k \underline{h})$. Hence α preserves $\mathcal{D}_U(1) \otimes_k U(\underline{h})$.

b) The claim is clear locally, i.e. over U for any ètale map $i:U \longrightarrow Y$ as in a). Then the same holds already on Y by ètale descent theory. Q.e.d.

Alternative description of $gr_{bas}\mathcal{U}$. Recalling the exact sequence of vector-bundles on Y:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T^*Y & \longrightarrow & (T^*X)/H & \longrightarrow & T^*_f & \longrightarrow & 0 \\ & & & & \big\downarrow q & & & & \\ & & & & Y & & & & \end{array}$$

let us now also define a "basic filtration" of the sheaf

$$F := q_*\mathcal{O}_{(T^*X)/H}$$

by the subsheaves $F_{bas}(m)$ which consist of those germs of functions, which restrict to polynomials of degree $\leq m$ on each plane parallel to $(T^*Y)_y$ in the fibre $q^{-1}(y)$ for any $y \in Y$.

Lemma 11: There is a canonical isomorphism

$$\mathrm{gr}_{bas}\, \mathcal{U} \cong \mathrm{gr}_{bas}\, q_* \mathcal{O}_{(T^*X)/H}.$$

Here the right-hand side is the associated graded sheaf

$$\mathrm{gr}_{bas}\, F := \bigoplus_{m \in \mathbb{Z}} F_{bas}(m)/F_{bas}(m-1)$$

in the above notation.

Proof: Let us define a map $\mathcal{U}_{bas}(1) \longrightarrow F_{bas}(1)$ as follows: A local section s of $\mathcal{U}_{bas}(1)$ may be written as

$$s = v \otimes A + w \otimes B$$

where $A, B \in U(\underline{h})$, and v resp. w are local sections of $\mathcal{O}_Y$ resp. $f_*(T_X)^H$. Our map is defined by sending

$$s \longmapsto B\, \sigma_1(w),$$

where $\sigma_1(w)$ denotes the ordinary principal symbol of w, considered as a function on $(T^*X)/H$, and $B \in U(\underline{h}) = S(\underline{h})$ is viewed as a function on T^*_f, pulled back to $(T^*X)/H$. By taking products, this definition extends to define maps $\mathcal{U}_{bas}(m) \longrightarrow F_{bas}(m)$ in all degrees $m \geq 1$, or a filtered map $\mathcal{U} \longrightarrow F$. Hence we obtain an associated graded map $\mathrm{gr}_{bas}\,\mathcal{U} \longrightarrow \mathrm{gr}_{bas}\, F$. The verification that this is an isomorphism may be done locally for the étale topology, and then it presents no difficulty. Q.e.d.

Comment. We notice that the "basic filtration" of $q_* \mathcal{O}_{T^*X/H}$ does not come from any natural grading, because one can not define the notion of "homogeneous functions" on an affine space without making a choice of an origin.

The geometrical meaning of the "basic filtration" is further clarified by the following lemma.

Lemma 12: Let q' denote the canonical projection map

$$q' : (T^*Y) \underset{Y}{\times} T^*_f \longrightarrow Y.$$

Then we have canonically

$$q_* \mathcal{O}_{(T^*X)/H} \cong q'_* \mathcal{O}_{(T^*Y) \underset{Y}{\times} T^*_f}$$

Proof: Obviously, a local trivialization of f gives rise to such a natural isomorphism. Since this isomorphism is easily seen to be independent of the choice of a trivialization, the lemma follows. Q.e.d.

Modified notion of characteristic variety. Based on the notion of basic filtrations introduced above, we may now define for a coherent sheaf M of left U-modules a new notion of characteristic variety $Ch_{bas}(M)$. This is, by definition, the support of the associated graded gr_{bas} U-modules obtained from M by using locally a "basic" good filtration of M. In view of the canonical identifications made by lemmas 10 and 11, this defines a subvariety

$$Ch_{bas}(M) \subset (T^*Y) \times_Y T^*_f.$$

This definition is not particularly useful in itself. However, we may use it - under a certain assumption - to define yet another notion of characteristic variety, which is contained in $(T^*X)/H$, and which is no longer necessarily homogeneous. We first state the extra assumption, and then the new definition.

Assumption: The characteristic variety $Ch_{bas}(M)$ is a union of fibres of the projection map $p_2 : T^*Y \times_Y T^*_f \longrightarrow T^*_f$, that is

(*) $$Ch_{bas}(M) = p_2^{-1}\, p_2\, Ch_{bas}(M).$$

Definition: If (*) holds, then we define

$$Ch_{str}(M) := t^{-1} p_2\, Ch_{bas}(M),$$

where t is the projection map $(T^*X)/H \longrightarrow T^*_f = \underline{h}^* \times Y$.

We note that the two varieties $Ch_{\mathrm{str}}(M)$ and $Ch_{\mathrm{bas}}(M)$ have the same dimension.

This definition applies for instance to the study of induced ideals as follows: Let J be an ideal in $U(\underline{h}) = S(\underline{h})$, and let $\check{J}$ denote the sheaf of "induced" ideals in U as considered in §§1 and 2.

Lemma 13: The module $M = U/\tilde{J}$ satisfies (*), so $Ch_{\mathrm{str}}(U/\tilde{J})$ is defined. Moreover, it is described by the support $\mathrm{Supp}(S(\underline{h})/J) \subset \underline{h}^*$ as

$$Ch_{\mathrm{str}}(U/\tilde{J}) = t^{-1}(\mathrm{Supp}(S(\underline{h})/J) \times Y).$$

The case of parabolic induction and relation to Dixmier sheets. We shall illustrate and clarify now the purpose of our new notion of characteristic variety by specializing the previous discussion to the study of parabolic induction as begun in [BB]I,4.6 ff.. So let P be a parabolic subgroup of a semisimple group G. Now we take $X = G/(P,P)$, $Y = G/P$, and $f: X \longrightarrow Y$ the canonical fibration; this is an A-principal fibration with respect to the commutative group $A = P/(P,P)$, a torus. Let us describe the quotient map $T^*X \longrightarrow (T^*X)/A$ and the moment map $\pi: (T^*X)/A \longrightarrow \underline{g}^*$ for this case more explicitly. For convenience, we may identify $\underline{g}^*$ with $\underline{g}$ by the Killing form, and we let $\underline{p} = \mathrm{Lie}\, P$, $\underline{a} = \mathrm{Lie}\, A$ etc. the Lie algebras as usual. We note that the orthogonal in $\underline{g}^* = \underline{g}$ of the commutator $[\underline{p},\underline{p}] = \mathrm{Lie}\,(P,P)$ is equal to the solvable radical $\underline{r}_{\underline{p}}$ of $\underline{p}$. Then the canonical maps $T^*X \longrightarrow (T^*X)/A \xrightarrow{\pi} \underline{g}^*$ have the following description in terms of associated fibre bundles:

$$\begin{array}{ccccc}
T^*X & = & G\times^{(P,P)}[\underline{p},\underline{p}]^{\perp} & = & G\times^{(P,P)}\underline{r}_{\underline{p}} \\
\downarrow & & \downarrow & & \downarrow \\
(T^*X)/A & = & G\times^{P}[\underline{p},\underline{p}]^{\perp} & = & G\times^{P}\underline{r}_{\underline{p}} \\
\downarrow \pi & & \downarrow & & \downarrow \\
\underline{g}^* & \supset & G[\underline{p},\underline{p}]^{\perp} & = & G\underline{r}_{\underline{p}}
\end{array}$$

(The bottom right arrow is known as a case of "collapsing" a vector-bundle, or in this particular case it is also known as a "generalized Grothendieck simultaneous resolution"; this situation has been extensively studied in various contexts by many authors; for more back-ground, we refer to [BK], [Bo2], [BM], [BB]I, 4.6, [S1].) In particular, the image of the moment map π identifies with

$$\mathrm{Im}\ \pi = \underline{Gr}_P = \overline{S}_P,$$

the closure of the so-called "Dixmier sheet" $S_P \subset \underline{g}$ attached to the parabolic subgroup P (see [BK]).

Let $\rho:\underline{g}^* \longrightarrow \underline{p}^*$ denote the restriction map, as in theorem 2. Note that its kernel $\underline{p}^{\perp}$ identifies with the nilradical $\underline{n}_P$ of $\underline{p}$ under the Killing form identification $\underline{g}^* = \underline{g}$. Let $\underline{p} = \underline{n}_P + \underline{s}_P$ be the Levi decomposition, and $\underline{n}_P^-$ the complement of $\underline{p}$ in $\underline{g}$ which is stable for the reductive subalgebra $\underline{s}_P$, so that $\underline{p}^- := \underline{s}_P + \underline{n}_P^-$ is the parabolic subalgebra "opposite" to $\underline{p}$. Then we may identify $\rho:\underline{g}^* \longrightarrow \underline{p}^*$ with the projection map of $\underline{g} = \underline{n}_P \oplus \underline{p}^-$ onto $\underline{p}^-$. We further observe that the center of the Levi subalgebra $\underline{s}_P$, which equals $\underline{s}_P \cap \underline{r}_P$, maps isomorphically onto $\underline{a} = \underline{p}/[\underline{p},\underline{p}]$ ($\cong \underline{a}^*$ by Killing form); by transposition, we get a linear map of $\underline{a}^*$ into $\underline{g}^* = \underline{g}$, $\mu \longmapsto \alpha(\mu)$, where $\alpha(\mu)$ is μ on $\underline{a}$, and 0 on the complement $[\underline{p},\underline{p}] + \underline{n}_P^-$ of $\underline{a}$. With this notation, we shall have for any $\mu \in \underline{a}^*(\subset \underline{p}^*)$:

$$\rho^{-1}(\mu) = \alpha(\mu) + \underline{n}_P.$$

Now let $I_\mu := \mathrm{Ann}_{U(\underline{p})}\, k_\mu$ denote the annihilator of the one-dimensional $\underline{p}$-module of weight $\mu \in \underline{a}^*$, and let $\tilde{I}_\mu$ denote the sheaf of "induced" ideals in $\mathcal{U}$, as introduced in §1. Let us first note that the "ordinary" characteristic variety $Ch\ (\mathcal{U}/\tilde{I}_\mu)$ is not of much interest to us, because it is independent of μ. In fact, it is determined by the associated variety of I_μ (by proposition 3), and this is always the zero point only. More precisely, proposition 3 gives that:

$$Ch(U/\tilde{I}_\mu) = Gx^P\rho^{-1}(0) = Gx^P\underline{n}_P,$$

which is a vector-bundle (isomorphic to T^*Y) independent of μ.

Next, let us determine the "new" characteristic variety for this case; from the previous discussion, we conclude:

Proposition 7: $Ch_{str}(U/\tilde{I}_\mu) = Gx^P\rho^{-1}(\mu) = Gx^P(\alpha(\mu) + \underline{n}_P) \subset Gx^P\underline{r}_P$.

This is an affine bundle depending on μ, and not a vector-bundle unless $\mu = 0$. Not only does this characteristic variety depend on μ, but even its image under the moment map does, for we have:

Corollary: $\pi(Ch_{str}(U/\tilde{I}_\mu)) = G(\alpha(\mu) + \underline{n}_P)$, i.e. the image under the moment map equals the G-saturation of $\alpha(\mu) + \underline{n}_P$.

Now let us recall a few facts from the theory of sheets [Bo2] (see also [BK], [BJ]): (1) For each $\mu \in \underline{a}^*$, the above G-stable subset $G(\alpha(\mu) + \underline{n}_P)$ is the closure of a single G-orbit, denoted $G(\alpha(\mu) + \underline{n}_P)^{reg}$ (notation of loc.cit.). (2) These G-orbits have all dimension equal to $2 \dim G/P = 2 \dim \underline{n}_P$. (3) The union of these orbits, for all $\mu \in \underline{a}^*$ is a maximal irreducible G-stable subset of $\underline{g}$ containing only orbits of this dimension. (4) This union is the Dixmier sheet

$$S_P = \bigcup_{\mu \in \underline{a}^*} G(\alpha(\mu) + \underline{n}_P)^{reg} = G\underline{r}^{reg}.$$

(5) The map $\mu \longmapsto G(\alpha(\mu) + \underline{n}_P)^{reg}$ induces a bijective correspondence

$$\underline{a}^*/W_P \xrightarrow{\sim} S_P/G$$

between the set of G-orbits in the sheet S_P, and the set of W_P-orbits in $\underline{a}^*$, where W_P is the normalizer of $\underline{a}$ in the Weyl group (see loc.cit.). The last statement is one version of the "parametrization theorem" for the orbits in a sheet, as proved in [BO2], 5.6.

Corollary: The moment map $\pi:(T^*X)/A \longrightarrow \underline{g}^*$ maps $Ch_{str}(U/I_\mu)$ onto the closure of the orbit in the Dixmier sheet S_P determined by the parameter μ (according to the parametrization theorem (5)).

In conclusion, we obtain the remarkable fact that the orbit determined by μ in the Dixmier sheet S_P is intrinsically attached to the sheaf of ideals $\tilde{I}_\mu$ in U.

Some complementary remarks and open questions. The basic filtration of U, as defined before lemma 11, induces an interesting filtration on the ring $R := U(\underline{g})/\mathrm{Ann}\ \mathrm{Ind}_{[\underline{p},\underline{p}]}^{\underline{g}}(k_o)$ (the quotient of $U(\underline{g})$ considered already in [BBI], theorem 4.6), which is in general not the one induced from the natural filtration of $U(\underline{g})$.

Problem 1: What is the "associated variety" of $U(\underline{g})/\mathrm{Ann}\ \widetilde{\widetilde{\mathrm{Ind}}}_{\underline{p}}^{\underline{g}}\ k_\mu$ with respect to this non-standard filtration of R?

We conjecture that this associated variety should be again that closure of the orbit determined by μ in the Dixmier sheet S_P.

Problem 2: Work out the precise relation between the primitive ideals of the ring R above, and those of $\Gamma(G/P,U)$.

We expect that this relation would essentially amount to a Galois theory of central ring extensions, provided that one has solved the following:

Problem 3: Is $\Gamma(G/P,U)$ equal to the central extension $R \otimes_{Z(\underline{g})} S(\underline{a})$?

This has been proved - up to a central localization - by Gelfand and Kirillov for the special case where $P = B$ is a Borel subgroup [GK], Corollary 12.1.

In this case, $\underline{a} = \underline{h}$ is a Cartan subalgebra, the central extension refers to the Harish-Chandra isomorphism of the center $Z(\underline{g})$ of $U(\underline{g})$ onto the Weyl group invariants $S(\underline{a})^W$, and the Galois group of this extension is the Weyl group W.

We prove the following result for P=B, conjectured in [GK], remark 10.4.

<u>Proposition 8</u>: <u>The ring of global sections</u> $\Gamma(G/B,\mathcal{U})$ <u>is equal to the central extension</u> $U(\underline{g}) \otimes_{Z(\underline{g})} S(\underline{h})$.

<u>Proof</u>: Both $U(\underline{g})$ and $S(\underline{h})$ have natural filtrations by order; these filtrations have the same restriction to $Z(\underline{g})$, as follows by a careful examination of Harish-Chandra's isomorphism $\omega: Z(\underline{g}) \xrightarrow{\sim} S(\underline{h})^W$ used for identification. Now the left action of G, and the right action of $H = B/(B,B)$ on G/B provide us with natural ring homomorphisms $U(\underline{g}) \longrightarrow \Gamma(G/B,\mathcal{U})$ resp. $S(\underline{h}) = U(\underline{h}) \longrightarrow \Gamma(G/B,\mathcal{U})$ (the"operator representations" in the terminology of [BB]I) which respect filtrations. These two homomorphisms coincide on $Z(\underline{g}) = S(\underline{h})^W$, as can be seen from the following characterization of Harish-Chandra's isomorphism ω:

$$\omega(P) - P \in \underline{n}_B\, U(\underline{g}) \quad \text{for all} \quad P \in Z(\underline{g}).$$

Since furthermore the left G-action commutes with the right H-action, we get a morphism

$$\phi: U(\underline{g}) \otimes_{Z(\underline{g})} S(\underline{h}) \longrightarrow \Gamma(G/B,\mathcal{U})$$

of filtered rings. In order to prove that ϕ is an <u>isomorphism</u>, it is therefor <u>enough to prove that the associated graded homomorphism</u> gr ϕ is an isomorphism, where

$$\text{gr}\ \phi: S(\underline{g}) \otimes_{S(\underline{h})^W} S(\underline{h}) \longrightarrow \mathcal{O}((T^*X)/H) = \mathcal{O}(T^*X)^H$$

(recall that $X = G/(B,B)$) coincides with the ring homomorphism given by the canonical map

$$\varphi: (T^*X)/H \longrightarrow \underline{g}^* \times_{\underline{h}^*/W} \underline{h}^*.$$

This latter map φ is the composition of the momentum map of the $G \times H$ action on X, and the projection map $\underline{g}^* \times \underline{h}^* \longrightarrow \underline{g}^* \times_{\underline{h}^*/W} \underline{h}^*$. Now it is easily seen that the map φ is proper and birational. It follows that $\mathcal{O}((T^*X)/H)$ is an integral extension of $S(\underline{g}) \otimes_{S(\underline{h})^W} S(\underline{h})$, and $\varphi^* = \mathrm{gr}\,\Phi$ induces an isomorphism of the fields of fractions. Therefor the following lemma implies that φ^* must be an isomorphism and hence completes the proof of proposition 3. Q.e.d.

Lemma 14: The variety $\underline{g}^* \times_{\underline{h}^*/W} \underline{h}^*$ is normal.

To prove this, we use the normality criterion of Serre [Se2]. Observe that $g^* \times_{\underline{h}^*/W} \underline{h}^*$ is finite over $\underline{g}^*$, hence affine. Its singular locus has codimension ≥ 2, since this singular locus is contained in the inverse image of the singular locus of the discriminant of $\underline{h}^* \longrightarrow h^*/W$. Since $\mathcal{O}(\underline{h}^*) = S(\underline{h})$ is free over $\mathcal{O}(\underline{h}^*/W) = S(\underline{h})^W$ by Chevalley's theorem, it follows that $\mathcal{O}(\underline{g}^* \times_{\underline{h}^*/W} \underline{h}^*)$ is free as a module over $\mathcal{O}(\underline{g}^*) = S(\underline{g})$, hence is a Cohen-Macanlay ring. Now Serre's criterion applies and gives the lemma.

Remark: This proof will fail for a general G/P, since the corresponding variety $\overline{S_P} \times_{\underline{a}^*/W_P} \underline{a}^*$ will in general not be normal.

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