The Bloch-Wigner-Ramakrishnan polylogarithm function

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The polylogarithm function

$$Li_m(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^m} \qquad (x \in \mathbf{C}, |x| \le 1, m \in \mathbf{N})$$

appears in many parts of mathematics and has an extensive literature [2]. It can be analytically extended to the cut plane $\mathbf{C} \setminus [1, \infty)$ by defining $Li_m(x)$ inductively as $\int_0^x Li_{m-1}(z)\, z^{-1} dz$ but then has a discontinuity as x crosses the cut. However, for $m = 2$ the modified function

$$D(x) = \Im(Li_2(x)) + \arg(1-x)\log|x|$$

extends (real-) analytically to the entire complex plane except for the points $x = 0$ and $x = 1$ where it is continuous but not analytic. This modified dilogarithm function, introduced by D. Wigner and S. Bloch (cf. [1]), has many beautiful properties. In particular, its values at algebraic arguments suffice to express in closed form the volumes of arbitrary hyperbolic 3-manifolds and the values at $s = 2$ of the Dedekind zeta functions of arbitrary number fields (cf. [6] and the expository article [7]). It is therefore natural to ask for similar real-analytic and single-valued modification of the higher polylogarithm functions Li_m. Such a function D_m was constructed, and shown to satisfy a functional equation relating $D_m(x^{-1})$ and $D_m(x)$, by Dinakar Ramakrishnan [3]. His construction, which involved monodromy arguments for certain nilpotent subgroups of $GL_m(\mathbf{C})$, is completely explicit, but he does not actually give a formula for D_m in terms of the polylogarithm. In this note we write down such a formula and give a direct proof of the one-valuedness and functional equation. We will also:

i) prove a formula (generalizing a formula of Bloch for $m = 2$) expressing certain infinite sums of the D_m as special values of Kronecker double series related to L-series of Hecke characters,

ii) describe a relation between the $D_m(x)$ and certain Green's functions for the unit disc, and

iii) discuss the conjecture that the values at $s = m$ of the Dedekind zeta function $\zeta_F(s)$ for an arbitrary number field F can be expressed in terms of values of $D_m(x)$ with $x \in F$.

The last relationship, which seems to be the most interesting property of the higher polylogarithm functions, is closely connected with algebraic K-theory and in facts leads to a conjectural description of higher K-groups of fields, as will be discussed in more detail in a later paper [9].

1. Definition of the function $D_m(x)$. For $m \in \mathbf{N}$ and $x \in \mathbf{C}$ with $|x| \le 1$ define

$$L_m(x) = \sum_{j=1}^{m} \frac{(-\log|x|)^{m-j}}{(m-j)!} Li_j(x), \qquad D_m(x) = \begin{cases} \Im(L_m(x)) & (m \text{ even}), \\ \Re(L_m(x)) + \dfrac{(\log|x|)^m}{2m!} & (m \text{ odd}). \end{cases}$$

PROPOSITION 1. *$D_m(x)$ can be continued real-analytically to $\mathbf{C} \smallsetminus \{0,1\}$ and satisfies the functional equation $D_m(\frac{1}{x}) = (-1)^{m-1} D_m(x)$.*

REMARKS: Ramakrishnan's D_m is equal to ours for m even but is just $\Re(L_m(x))$ for m odd. We have included the extra term $(\log|x|)^m/2m!$ for m odd in order to make the functional equation as simple as possible (Ramakrishnan's function satisfies $D_m(1/x) = D_m(x) + (\log|x|)^m/m!$ for m odd), but at the cost of making the function discontinuous at 0 in this case. (For m even, D_m extends to a continuous function on the extended plane $\mathbf{C} \cup \{\infty\}$, vanishing on $\mathbf{R} \cup \{\infty\}$.) The definition of D_m here also differs by a factor $(-1)^{[m/2]+1}$ from the normalization given in [7], which was chosen to give a simpler relation between $\partial D_m/\partial z$ and D_{m-1}. The functions $D_1(x)$ and $D_2(x)$ are equal to $\log|x^{\frac{1}{2}} - x^{-\frac{1}{2}}|$ and $D(x)$, respectively.

PROOF: As mentioned in the introduction, we can continue $Li_m(x)$ analytically to the cut plane $\mathbb{C} \smallsetminus [1,\infty)$ by successive integration along, say, radial paths from 0 to x. The two branches just below and just above the cut then continue across the cut. Write Δ for the difference of these two analytic functions in their common region of definition (say, in the range $|\arg(x-1)| < \epsilon$, where ϵ is small). Since $Li_1(x) = \log\frac{1}{1-x}$ for $|x| < 1$, we have $\Delta Li_1 = 2\pi i$, and it then follows from the formula $xLi_m'(x) = Li_{m-1}(x)$ that $\Delta Li_m(x) = 2\pi i(\log x)^{m-1}/(m-1)!$ for each $m \geq 1$. (This is well-defined in the region in question: we take the branch of $\log x$ which vanishes at $x = 1$.) Consequently,

$$\Delta L_m(x) = 2\pi i \sum_{j=1}^{m} \frac{(-\log|x|)^{m-j}}{(m-j)!} \frac{(\log x)^{j-1}}{(j-1)!} = \frac{2\pi i}{(m-1)!} \left(\log\frac{x}{|x|}\right)^{m-1}.$$

Since $\log\frac{x}{|x|}$ is pure imaginary, this is real for m even and pure imaginary for m odd. Hence $\Re\big(i^{m+1}L_m(x)\big)$ is one-valued, proving the first assertion of the proposition.

To prove the second, it will be convenient to introduce the generating function $\mathcal{L}(x;t) = \sum_{m=1}^{\infty} L_m(x)\, t^{m-1}$. For $|x| < 1$, $|t| < 1$ we have

$$\begin{aligned}
\mathcal{L}(x;t) &= \sum_{j\geq 1,\, k\geq 0} \frac{(-\log|x|)^k}{k!} Li_j(x)\, t^{j+k-1} = |x|^{-t} \sum_{j=1}^{\infty} Li_j(x)\, t^{j-1} \\
&= |x|^{-t} \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \frac{t^{j-1}}{n^j} x^n = |x|^{-t} \sum_{n=1}^{\infty} \frac{x^n}{n-t}
\end{aligned}$$

or

$$\mathcal{L}(re^{i\theta};t) = \sum_{n=1}^{\infty} \frac{r^{n-t}}{n-t} e^{in\theta} = \int_0^r \frac{u^{-t}\, du}{e^{-i\theta} - u} \qquad (0 \leq r < 1),$$

where we have written $\frac{r^{n-t}}{n-t}$ as $\int_0^r u^{n-t-1}\, du$ and summed the geometric series under the integral sign. The integral converges also for $r \geq 1$ and immediately gives the extension to the cut plane $|\arg(1-z)| < \pi$. Since the integrand has a simple pole of residue $-e^{it\theta}$ at

$u = e^{-i\theta}$, we again see that the difference between the two branches of $L_m(re^{i\theta})$ near the cut is $2\pi i^m \theta^{m-1}/(m-1)!$, giving the one-valuedness of D_m as before. In terms of $\mathcal{L}(x;t)$, the functional equation can be stated as the assertion that $\mathcal{L}(re^{i\theta};t) + \mathcal{L}(re^{-i\theta};-t) + \frac{1}{t}r^{-t}$ is unchanged when r is replaced by r^{-1}. But for $0 < t < 1$ we have

$$\begin{aligned}\mathcal{L}(re^{i\theta};t) + \mathcal{L}(re^{-i\theta};-t) + \frac{r^{-t}}{t} &= \int_0^r \frac{u^{-t}\,du}{e^{-i\theta}-u} + \int_0^r \frac{v^t\,dv}{e^{i\theta}-v} + \int_r^\infty u^{-t-1}\,du \\ &= \left(\int_0^\infty - \int_r^\infty - \int_{r^{-1}}^\infty\right) \frac{u^{-t}\,du}{e^{-i\theta}-u} \qquad (v = u^{-1}).\end{aligned}$$

This makes the desired symmetry obvious.

2. The functions $D_{a,b}(x)$ and Kronecker double series. It is clear from the definition that the Bloch-Wigner function $D(x)$ goes to 0 like $|x|\,\log|x|$ as $x \to 0$, and from the functional equation that $D(x) = O(|x|^{-1}\log|x|)$ as $x \to \infty$. Hence, for a complex number q of absolute value strictly less than 1 and any complex number x, the doubly infinite series

$$D(q;x) = \sum_{l=-\infty}^{\infty} D(q^l x)$$

converges with exponential rapidity. Clearly $D(q;x)$ is invariant under $x \mapsto qx$, so it is in fact a function on the elliptic curve $\mathbb{C}^\times/q^{\mathbb{Z}}$. In other words, if we write $q = e^{2\pi i\tau}$ with τ in the complex upper half-plane and $x = e^{2\pi i u}$ with $u \in \mathbb{C}$, then $D(q;x)$ depends only on the image of u in the quotient of $\mathbb{C}$ by the lattice $L = \mathbb{Z}\tau + \mathbb{Z}$. In [1], Bloch computed the Fourier development of this non-holomorphic elliptic function. Actually, he found that $D(x)$ should be supplemented by adding an imaginary part $-iJ(x)$, where

$$J(x) = \log|x|\,\log|1-x| \qquad (x \in \mathbb{C},\ x \neq 0,1).$$

The function $J(x)$ is small as $|x| \to 0$ but large as $|x| \to \infty$, so we cannot form the series $\sum_{l\in\mathbb{Z}} J(q^l x)$ as we did with D. However, using the functional equation $J(x^{-1}) = -J(x) + \log^2|x|$ we find after a short calculation that the function

$$J(q;x) = \sum_{l=0}^{\infty} J(q^l x) - \sum_{l=1}^{\infty} J(q^l x^{-1}) + \frac{\log^3|x|}{3\log|q|} - \frac{\log^2|x|}{2} + \frac{\log|x|\log|q|}{2} \qquad (q, x \in \mathbb{C},\ |q| < 1)$$

is invariant under $x \mapsto qx$, so descends to the elliptic curve $\mathbb{C}^\times/q^{\mathbb{Z}} \simeq \mathbb{C}/L$ as before. Bloch's result can then be written

$$D(q;x) - iJ(q;x) = \frac{i}{\pi}\Im(\tau)^2 \sum_{m,n}{}' \frac{\sin(2\pi(n\xi - m\eta))}{(m\tau+n)^2(m\bar\tau+n)},$$

where $q = e^{2\pi i\tau}$, $x = e^{2\pi i u}$ with $u = \xi\tau + \eta$ ($\xi,\ \eta \in \mathbb{R}/\mathbb{Z}$) and the sum is over all pairs of integers $(m,n) \neq (0,0)$. This is a classical series studied by Kronecker (see for instance Weil's book [5]). The special case when τ is quadratic over $\mathbb{Q}$ and ξ and η are rational numbers occurs in evaluating L-series of Hecke grossencharacters of type A_0 and weight 1 at s=2. To get other weights and other special values, we have to study series of the same type but with other powers of $m\tau + n$ and $m\bar{\tau} + n$ in the denominator. In this section we will prove the analogue of Bloch's formula for such series, the function $D(x) - iJ(x)$ being replaced by a suitable linear combination of the Ramakrishnan functions $D_m(x)$.

To define these combinations, we will need certain combinatorial coefficients, and we begin by defining these. For integers a, m, r with $1 \le a, m \le r$ let $c_{a,m}^{(r)}$ denote the coefficients of x^{a-1} in the polynomial $(1-x)^{m-1}(1+x)^{r-m}$. These coefficients are easily computed by the recursion $c_{a,m}^{(r)} = c_{a,m}^{(r-1)} + c_{a-1,m}^{(r-1)}$ or by the closed formula $c_{a,m}^{(r)} = \sum_{h=1}^{a} (-1)^{h-1}\binom{m-1}{h-1}\binom{r-m}{a-h}$. They have the symmetry properties

$$(1) \qquad c_{a,m}^{(r)} = (-1)^{a-1} c_{a,r+1-m}^{(r)} = (-1)^{m-1} c_{r+1-a,m}^{(r)}, \qquad \binom{r-1}{m-1} c_{a,m}^{(r)} = \binom{r-1}{a-1} c_{m,a}^{(r)},$$

the former being obvious and the latter a consequence of the identity

$$\sum_{a=1}^{r}\sum_{m=1}^{r} \binom{r-1}{m-1} c_{a,m}^{(r)} x^{a-1} y^{m-1} = (1+x+y-xy)^{r-1}.$$

The definition of $c_{a,m}^{(r)}$ is equivalent to saying that the $r \times r$ matrix $C_r = (c_{a,m}^{(r)})_{a,m=1,\ldots,r}$ gives the transition between the bases $\{t^{r-1}, t^{r-2}u, \ldots, tu^{r-2}, u^{r-1}\}$ and $\{(t+u)^{r-1}, (t+u)^{r-2}(t-u), \ldots, (t+u)(t-u)^{r-2}, (t-u)^{r-1}\}$ of the space of homogeneous polynomials of degree $r-1$ in two variables t and u. The fact that the matrix $\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ has square 2 implies that

$$(2) \qquad C_r^{-1} = 2^{-r+1} C_r.$$

We will also need the formulas

$$(3) \qquad \begin{aligned} \sum_{m=k}^{r} \binom{r-k}{m-k} c_{a,m}^{(r)} &= (-1)^{a-1}\binom{k-1}{a-1} 2^{r-k} \\ \sum_{m=k}^{r} (-1)^{m-1}\binom{r-k}{m-k} c_{a,m}^{(r)} &= (-1)^{r-a}\binom{k-1}{r-a} 2^{r-k} \end{aligned} \qquad (1 \le a, k \le r)$$

(the expressions on the right being 0 for $k < a$ or $k < r+1-a$, respectively) and

$$(4) \qquad \sum_{\substack{m=1 \\ m \text{ odd}}} \binom{r}{m} c_{a,m}^{(r)} = 2^{r-1} \qquad (1 \le a \le r).$$

We leave the proofs to the reader (hint: expand $(1-x)^{k-1}\{1+x \pm (1-x)\}^{r-k}$ for $0 \le k \le r$). As numerical examples to illustrate properties (1)–(4) we give the $c_{a,m}^{(r)}$ for $r = 6$ and 7:

$$C_6 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 5 & 3 & 1 & -1 & -3 & -5 \\ 10 & 2 & -2 & -2 & 2 & 10 \\ 10 & -2 & -2 & 2 & 2 & -10 \\ 5 & -3 & 1 & 1 & -3 & 5 \\ 1 & -1 & 1 & -1 & 1 & -1 \end{pmatrix}, \qquad C_7 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 6 & 4 & 2 & 0 & -2 & -4 & -6 \\ 15 & 5 & -1 & -3 & -1 & 5 & 15 \\ 20 & 0 & -4 & 0 & 4 & 0 & -20 \\ 15 & -5 & -1 & 3 & -1 & -5 & 15 \\ 6 & -4 & 2 & 0 & -2 & 4 & -6 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{pmatrix}.$$

We now define for integers $a, b \geq 1$ and $x \in \mathbb{C}$

$$D_{a,b}(x) = 2\sum_{m=1}^{r} c_{a,m}^{(r)} D_m^*(x) \frac{(-\log|x|)^{r-m}}{(r-m)!} + \frac{(-2\log|x|)^r}{2r!} \qquad (r = a+b-1),$$

where $D_m^*(x) = D_m(x)$ for m odd, $D_m^*(x) = iD_m(x) = \frac{1}{2}[L_m(x) - \overline{L_m(x)}]$ for m even.

PROPOSITION 2. *(i) $D_{a,b}$ is a one-valued real-analytic function on $\mathbb{C} \smallsetminus [1,\infty)$ and satisfies the functional equation*

$$D_{a,b}(\frac{1}{x}) = (-1)^{r-1} D_{a,b}(x) + \frac{(2\log|x|)^r}{r!}.$$

(ii) $D_{a,b}$ is given in terms of the polylogarithm by

$$D_{a,b}(x) = (-1)^{a-1}\sum_{k=a}^{r} 2^{r-k}\binom{k-1}{a-1}\frac{(-\log|x|)^{r-k}}{(r-k)!} Li_k(x)$$
$$+ (-1)^{b-1}\sum_{k=b}^{r} 2^{r-k}\binom{k-1}{b-1}\frac{(-\log|x|)^{r-k}}{(r-k)!}\overline{Li_k(x)}.$$

(iii) The function defined for $q, x \in \mathbb{C}$ with $|q| < 1$ by

$$D_{a,b}(q;x) = \sum_{l=0}^{\infty} D_{a,b}(q^l x) + (-1)^{r-1}\sum_{l=1}^{\infty} D_{a,b}(q^l x^{-1}) + \frac{(2\log|q|)^r}{(r+1)!} B_{r+1}\left(\frac{\log|x|}{\log|q|}\right)$$

($B_{r+1}(x)$ = $(r+1)$st Bernoulli polynomial) is invariant under $q \mapsto qx$.

PROOF: Statement (i) follows immediately from Proposition 1 and statement (ii) from equations (3) and (4). For (iii), we note first that the infinite sum converges absolutely for any x, because $D_{a,b}(x) = O(|x|\log^{a+b}|x|)$ as $|x| \to 0$. Hence $D_{a,b}(q;x)$ makes sense. Using (i) and the property $B_{r+1}(x+1) - B_{r+1}(x) = (r+1)x^r$, we find

$$D_{a,b}(q;x) - D_{a,b}(q;qx) = D_{a,b}(x) - (-1)^{r-1}(x) D_{a,b}(x^{-1}) - \frac{(-2\log|q|)^r}{(r+1)!}(r+1)\left(\frac{\log|x|}{\log|q|}\right)^r = 0.$$

This completes the proof of the proposition.

Notice that we can use the inversion formula (2) to write

$$D_m^*(x)\frac{(-\log|x|)^n}{n!} = \sum_{\substack{a,b\geq 1\\ a+b=r+1}} c_{m,a}^r \left\{2^{-r} D_{a,b}(x) - \frac{(-\log|x|)^r}{2r!}\right\} \qquad (m \geq 1,\ n \geq 0, r = m+n);$$

in particular, the Ramakrishnan functions D_m are linear combinations of the $D_{a,b}$. We could therefore have equally well defined the functions $D_{a,b}$ directly by the formula in (ii) and taken them rather than the functions D_m as the primitive objects of study. The proof of the analytic continuation can be given directly from (ii) by the same method as in the proof of Proposition 1: using $\Delta Li_k(x) = 2\pi i(\log x)^{k-1}/(k-1)!$ and the binomial theorem, one finds easily that $\Delta D_{a,b} = 0$.

Part (iii) of the proposition says that the function $D_{a,b}(q; e^{2\pi i u})$ is a (non-holomorphic) elliptic function of u. Our goal is to compute the Fourier development of this function.

THEOREM 1. *Write $q = e^{2\pi i\tau}$, $x = e^{2\pi iu}$ with τ in the complex upper half-plane and $u = \xi\tau + \eta \in \mathbb{C}$, $\xi, \eta \in \mathbb{R}/\mathbb{Z}$. Then*

$$D_{a,b}(q;x) = \frac{(\tau-\overline{\tau})^r}{2\pi i} \sum_{m,n}{}' \frac{e^{2\pi i(n\xi - m\eta)}}{(m\tau+n)^a(m\overline{\tau}+n)^b}.$$

PROOF: Since $D_{a,b}(e^{2\pi i\tau}, e^{2\pi i(\xi\tau+\eta)})$ is invariant under $\xi \mapsto \xi+1$, we can develop it into a Fourier series $\sum_{n\in\mathbb{Z}} \lambda_n e^{2\pi in\xi}$ with

$$\begin{aligned}\lambda_n &= \int_0^1 e^{-2\pi in\xi} D_{a,b}(e^{2\pi i\tau}, e^{2\pi i(\xi\tau+\eta)})\,d\xi \\ &= \int_0^\infty e^{-2\pi in\xi} D_{a,b}(e^{2\pi i(\xi\tau+\eta)})\,d\xi + (-1)^{r-1}\int_0^\infty e^{2\pi in\xi} D_{a,b}(e^{2\pi i(\xi\tau-\eta)})\,d\xi \\ &\qquad + \frac{(4\pi\Im(\tau))^r}{(r+1)!}\int_0^1 e^{-2\pi in\xi} B_{r+1}(\xi)\,d\xi,\end{aligned}$$

where we have substituted for $D_{a,b}$ the expression defining it and then in the first two terms combined the sum over l and the integral from 0 to 1 into a single integral from 0 to ∞ by the substitution $l \pm \xi \to \xi$. It is well-known (and easily shown by repeated integration by parts, using $B_j' = jB_{j-1}$ and $B_j(1) = B_j(0)$ for $j \neq 1$) that the last integral is equal to 0 for $n = 0$ and to $-(r+1)!/(2\pi in)^{r+1}$ for $n \neq 0$. Substituting for $D_{a,b}(x)$ from part (ii) of the proposition, we find

$$\begin{aligned}\lambda_n = (-1)^{a-1}\sum_{k=a}^{r} 2^{r-k}\binom{k-1}{a-1}\frac{(2\pi\Im(\tau))^{r-k}}{(r-k)!}\int_0^\infty \xi^{r-k}\big[Li_k(e^{2\pi i(\xi\tau+\eta)})e^{-2\pi in\xi} \\ + (-1)^{r-1}Li_k(e^{2\pi i(\xi\tau-\eta)})e^{2\pi in\xi}\big]\,d\xi + \begin{pmatrix} a\leftrightarrow b \\ \tau \leftrightarrow -\overline{\tau}\end{pmatrix} - \frac{(-2i\Im(\tau))^r}{2\pi i}n^{-r-1},\end{aligned}$$

where the second term denotes the result of interchanging a and b and replacing τ by $-\overline{\tau}$ in the first term and the last term is to be omitted if $n = 0$. The two arguments of Li_k in the integrand are less than 1 in absolute value, so we can replace Li_k by its definition as a power series, obtaining

$$\begin{aligned}&\int_0^\infty \xi^{r-k}\big[Li_k(e^{2\pi i(\xi\tau+\eta)})e^{-2\pi in\xi} + (-1)^{r-1}Li_k(e^{2\pi i(\xi\tau-\eta)})e^{2\pi in\xi}\big]\,d\xi \\ &\quad = \sum_{m=1}^{\infty}\frac{1}{m^k}\int_0^\infty \big\{e^{2\pi i[(m\tau-n)\xi+m\eta]} + (-1)^{r-1}e^{2\pi i[(m\tau+n)\xi-m\eta]}\big\}\xi^{r-k}\,d\xi \\ &\quad = \sum_{m=1}^{k}\frac{1}{m^k}\frac{(r-k)!}{(-2\pi i)^{r+1-k}}\left\{\frac{e^{2\pi im\eta}}{(m\tau-n)^{r+1-k}} + (-1)^{r-1}\frac{e^{-2\pi im\eta}}{(m\tau+n)^{r+1-k}}\right\} \\ &\quad = \frac{(-1)^k(r-k)!}{(2\pi i)^{r+1-k}}\sum_{m\neq 0}\frac{e^{-2\pi im\eta}}{m^k(m\tau+n)^{r+1-k}},\end{aligned}$$

where we have used the formula $\int_0^\infty e^{-\lambda\xi}\xi^l\,d\xi = l!\,\lambda^{-l-1}$ for $\Re(\lambda) > 0$. Hence

$$2\pi i\lambda_n = (-1)^b \sum_{k=a}^{r} \binom{k-1}{a-1} (2i\Im(\tau))^{r-k} \sum_{m\neq 0} \frac{e^{-2\pi i m\eta}}{m^k(m\tau+n)^{r+1-k}} + \begin{pmatrix} a \leftrightarrow b \\ \tau \leftrightarrow -\overline{\tau} \end{pmatrix} - \frac{2(-2i\Im(\tau))^r}{n^{r+1}}.$$

Applying the easily checked identity

$$(-1)^a \sum_{k=a}^{r} \binom{k-1}{a-1} \frac{(X-Y)^{r-k}}{X^{r+1-k}} + \sum_{k=b}^{r} \binom{k-1}{b-1} \frac{(X-Y)^{r-k}}{Y^{r+1-k}} = \frac{(X-Y)^r}{X^b Y^a} \qquad (r = a+b-1)$$

to $X = m\tau + n$, $Y = m\overline{\tau} + n$, we find

$$2\pi i\lambda_n = (2i\Im(\tau))^r \sum_{\substack{m\in\mathbf{Z} \\ (m,n)\neq(0,0)}} \frac{e^{-2\pi i m\eta}}{(m\tau+n)^a(m\overline{\tau}+n)^b}.$$

This proves the theorem.

3. D_m and the Green's function of the unit disc. Let $\mathfrak{H} = \{z = x+iy \in \mathbf{C} \mid y > 0\}$ denote the upper half-plane and for each positive integer k define a function $G_k^{\mathfrak{H}} : \mathfrak{H} \times \mathfrak{H} \smallsetminus (\text{diagonal}) \to \mathbf{R}$ by

$$G_k^{\mathfrak{H}}(z,z') = -2\,Q_{k-1}\Big(1 + \frac{|z-z'|^2}{2yy'}\Big) \qquad (z = x+iy,\ z' = x'+iy' \in \mathfrak{H}).$$

Here $Q_n(t)$ $(n \geq 0)$ is the n^{th} Legendre function of the second kind:

$$Q_0(t) = \frac{1}{2}\log\frac{t+1}{t-1}, \quad Q_1(t) = \frac{t}{2}\log\frac{t+1}{t-1} - 1, \quad Q_2(t) = \frac{3t^2-1}{4}\log\frac{t+1}{t-1} - \frac{3}{2}t$$

and in general $Q_n(t) = P_n(t)Q_0(t) - R_n(t)$ where $P_n(t)$ and $R_n(t)$ are the unique polynomials of degree n and $n-1$, respectively, making $Q_n(t) \sim \dfrac{2^n n!^2}{(2n+1)!} t^{-n-1}$ for $t \to \infty$. The function $G_k^{\mathfrak{H}}$ is real-analytic on $\mathfrak{H} \times \mathfrak{H} \smallsetminus (\text{diagonal})$, has a singularity of type

$$G_k^{\mathfrak{H}}(z,z') = \log|z-z'|^2 + \text{ continuous} \qquad (z' \to z)$$

along the diagonal, and satisfies the partial differential equation $\Delta_z G_k^{\mathfrak{H}} = \Delta_{z'} G_k^{\mathfrak{H}} = k(1-k)G_k^{\mathfrak{H}}$, where $\Delta_z = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ denotes the hyperbolic Laplace operator. Moreover, by virtue of the defining property of Q_{k-1}, it is small enough at infinity that the series

$$G_k^{\mathfrak{H}/\mathbf{Z}}(z,z') = \sum_{n=-\infty}^{\infty} G_k^{\mathfrak{H}}(z, z'+n)$$

converges and has properties similar to those of $G_k^{\mathfrak{H}}$, but now with z and z' in $\mathfrak{H}/\mathbf{Z}$. This "Green's function" is studied (in connection with the analogously defined functions $G_k^{\mathfrak{H}/\Gamma}$, where Γ is a subgroup of finite index in $PSL(2,\mathbf{Z})$) in [8] and is shown there to be closely related to Ramakrishnan's modified polylogarithm function. We content ourselves with stating the result, referring to [8] for the proof.

THEOREM 2. *Let* $k \in \mathbb{N}$, $z = x + iy$, $z' = x' + iy' \in \mathfrak{H}$. *Then*

$$G_k^{\mathfrak{H}/\mathbb{Z}}(z,z') = \sum_{n=1}^{k} f_{k,n}(2\pi y, 2\pi y')\left[D_{2n-1}(q/q') - D_{2n-1}(q\overline{q'})\right],$$

where $q = e^{2\pi i z}$, $q' = e^{2\pi i z'}$ *and*

$$f_{k,n}(u,v) = 2^{1-2k}(uv)^{1-k} \sum_{\substack{r,s\geq 0 \\ r+s=k-n}} \frac{(2k-2-2r)!}{r!(k-1-r)!}\,\frac{(2k-2-2s)!}{s!(k-1-s)!}\,u^{2r}v^{2s}.$$

Note that the symmetry of $G_k^{\mathfrak{H}/\mathbb{Z}}$ in its two arguments is reflected by the two symmetry properties $D_{2n-1}(x) = D_{2n-1}(x^{-1}) = D_{2n-1}(\overline{x})$. The map $z \to q$ identifies $\mathfrak{H}/\mathbb{Z}$ with the punctured unit disc $\{q \in \mathbb{C} \mid 0 < |q| < 1\}$, but the right-hand side of the formula in the theorem now makes sense for any $q, q' \in \mathbb{C}^\times$ (with $2\pi y$, $2\pi y'$ replaced by $-\log|z|$, $-\log|z'|$) and represents some kind of Green's function on $\mathbb{C}^\times \times \mathbb{C}^\times$.

4. D_m **and special values of Dedekind zeta functions.** The Bloch-Wigner dilogarithm function $D(x)$ is related in a very beautiful way to special values of Dedekind zeta functions. Specifically, we have the following theorem.

THEOREM 3. *Let* F *be an arbitrary algebraic number field,* d_F *the discriminant of* F, r_1 *and* r_2 *the numbers of real and complex places* ($r_1 + 2r_2 = [F : \mathbb{Q}]$), *and* $\zeta_F(s)$ *the Dedekind zeta function* $\zeta_F(s)$. *Then* $\zeta_F(2)$ *is equal to* $\pi^{2(r_1+r_2)}|d_F|^{-\frac{1}{2}}$ *times a rational linear combination of* r_2*-fold products* $D(x^{(r_1+1)})\cdots D(x^{(r_1+r_2)})$ *with* $x \in F$.
(Here $x^{(1)},\ldots,x^{(r_1)},x^{(r_1+1)},\ldots,x^{(r_1+r_2)}, \overline{x^{(r_1+1)}},\ldots,\overline{x^{(r_1+r_2)}}$ are the images of x under the various embeddings $F \hookrightarrow \mathbb{C}$.)

This result was proved in [5] in a somewhat weaker form (it was asserted only that the x could be chosen of degree ≤ 4 over F, rather than in F itself) by a geometric method: the value of $\zeta_F(2)$ was related to the volume of a hyperbolic $3r_2$-dimensional manifold (more precisely, a manifold locally isometric to $\mathfrak{H}_3^{r_2}$, where $\mathfrak{H}_3$ denotes hyperbolic 3-space) and this volume was then computed by triangulating the manifold into a union of r_2-fold products of hyperbolic tetrahedra whose volumes could be expressed in terms of the function $D(x)$. The more precise statement above comes from algebraic K-theory: the value of $\zeta_F(2)$ is related by a result of Borel to a certain "regulator" attached to $K_3(F)$, and this is calculated using results of Bloch, Levine, Suslin and Mercuriev in terms of the Bloch-Wigner function. For details and references, see [4] or [7]. The K-theoretical proof in fact gives a somewhat stronger statement than the above theorem: the value of $|d_F|^{\frac{1}{2}}\zeta_F(2)/\pi^{2r_1+2r_2}$ is equal to an $r_2 \times r_2$ determinant of rational linear combinations of values $D(x)$, rather than merely to a rational linear combination of r_2-fold combinations of such values.

As examples of Theorem 3, we have for $F = \mathbb{Q}(\sqrt{-7})$ $(d_F = -7, r_1 = 0, r_2 = 1)$

$$\zeta_F(2) = \frac{2^2\pi^2}{3\cdot 7^{3/2}}\left(2D(\frac{1+\sqrt{-7}}{2}) + D(\frac{-1+\sqrt{-7}}{4})\right)$$

and for $F = \mathbb{Q}(\theta)$ with $\theta^3 - \theta - 1 = 0$ ($d_F = -23$, $r_1 = r_2 = 1$)

$$\zeta_F(2) = \frac{2^3\pi^4}{3 \cdot 23^{3/2}} D(\theta') = -\frac{2^2\pi^4}{3 \cdot 23^{3/2}} D(-\theta'),$$

where θ' ($= \frac{\theta}{2}(-1 + \frac{i\sqrt{23}}{2\theta + 3})$, if θ is the real root) denotes the conjugate of θ with $\Im(\theta') > 0$.

We can now formulate

CONJECTURE 1. *Theorem 3 holds true for $\zeta_F(m)$ for all positive even m with $\pi^{2(r_1+r_2)}$ replaced by $\pi^{m(r_1+r_2)}$ and with the function D replaced by the function D_m. For m odd a similar statement is true but with π^{mr_2} instead of $\pi^{m(r_1+r_2)}$ and $D_m(x^{(1)}) \cdots D_m(x^{(r_1+r_2)})$ instead of $D_m(x^{(r_1+1)}) \cdots D_m(x^{(r_1+r_2)})$.*

The difference between the two cases m even and m odd is, on the one hand, that D_m satisfies $D_m(\overline{x}) = (-1)^{m-1} D_m(x)$ (so in particular $D_m(x) = 0$ for x real and m even) and, on the other hand, that the order of vanishing of $\zeta_F(s)$ at $s = 1-m$ for $m > 1$ equals r_2 for m even but $r_1 + r_2$ for m odd. Again we can make a more precise conjecture with an $r \times r$ determinant ($r = r_2$ or $r_1 + r_2$) instead of simply a linear combination of r-fold products. Moreover, one can make a more general conjecture with Artin L-functions in place of Dedekind zeta functions. In particular, $\zeta_F(s)/\zeta(s)$ ($\zeta = \zeta_{\mathbb{Q}}$), which is a product of such L-series, should be a sum of $(r-1)$-fold products of values D_m. This statement makes sense also for $m = 1$ and is true by the Dirichlet regulator formula (recall that D_1 is essentially the logarithm-of-the-absolute-value function), but even when $m = 1$ the general conjecture for Artin L-series is unknown (Stark conjectures).

As a special case, we make the very specific

CONJECTURE 2. *Let F be a real quadratic field. Then $|d_F|^{\frac{1}{2}}\zeta_F(3)/\zeta(3)$ is a rational linear combination of differences $D_3(x) - D_3(x')$, $x \in F$.*

Here x' denotes the conjugate of x over $\mathbb{Q}$. Note that $\zeta(3) = D_3(1)$, so this is a strengthening of Conjecture 1 in this case. As numerical examples, we give

$$\frac{\zeta_{\mathbb{Q}(\sqrt{5})}(3)}{\zeta(3)} \stackrel{?}{=} \frac{2^5}{3 \cdot 5^{5/2}}\left(3[D_3(\frac{1+\sqrt{5}}{2}) - D_3(\frac{1-\sqrt{5}}{2})] - [D_3(2+\sqrt{5}) - D_3(2-\sqrt{5})]\right)$$

and

$$\frac{\zeta_{\mathbb{Q}(\sqrt{2})}(3)}{\zeta(3)} \stackrel{?}{=} \frac{3}{5 \cdot 2^{5/2}}\bigg([D_3(4+2\sqrt{2}) - D_3(4-2\sqrt{2})] - 9[D_3(2+\sqrt{2}) - D_3(2-\sqrt{2})]$$
$$- 6[D_3(1+\sqrt{2}) - D_3(1-\sqrt{2})] + 9[D_3(\sqrt{2}) - D_3(-\sqrt{2})]\bigg),$$

both true to at least 25 decimals. (These relations were found empirically by using the Lenstra-Lenstra-Lovasz lattice reduction algorithm to search numerically for linear relations between $|d_F|^{\frac{1}{2}}\zeta_F(3)/\zeta(3)$ and selected values of $D_3(x) - D_3(x')$, $x \in F$.)

That the *quotient* $\zeta_F/\zeta_{\mathbb{Q}}$ should be connected with the *differences* $D_m(x) - D_m(x')$ is a special case of a "Galois descent" property which we expect to hold in general, and which is known for the case $m=2$ by the K-theoretical work already cited (cf. [4] for details). Roughly speaking, this property implies that the $\mathbf{Q}$-vector space spanned by the $x \in F$ occurring in the conjecture should be invariant under the group of automorphisms of F over $\mathbf{Q}$ and that the value of an (abelian or Artin) L-function factor of ζ_F at $s=m$ should be the determinant of a matrix of combinations of $D_m(x)$ with x in the corresponding subspace. An example of how this works is provided by the case when F is abelian over $\mathbf{Q}$. Here the assertion of Conjecture 1 is easy if we allow the arguments x to be in the abelian closure $N=\mathbf{Q}(\zeta_f)$ (f = conductor of F), rather than in F itself: ζ_F factors into a product of Dirichlet L-series $L(s,\chi)$ with r_1+r_2 even and r_2 odd Dirichlet characters χ modulo f (of course, either r_1 or r_2 is zero), and the value of $L(m,\chi)$ is an algebraic multiple of π^m if $\chi(-1)=(-1)^m$ and an algebraic linear combination of values of $D_m(x)$, $x^f=1$ in the opposite case. This gives the statement with an algebraic rather than rational combination of products of D-values, but a little more work shows that the algebraic multiples occurring combine correctly to give a rational multiple of $|d_F|^{\frac{1}{2}}$. The point is now that the set of x occurring, and the coefficients with which they occur, are invariant under the action of $\mathrm{Gal}(N/F)$. For instance, in the above case F real quadratic, $m=3$, $f=d_F$, we have

$$d_F^{\frac{1}{2}}\zeta_F(3)/\zeta(3)=f^{\frac{1}{2}}L(3,(\frac{d_F}{\cdot}))=\sum_{n=1}^{f-1}(\frac{d_F}{n})Li_3(e^{2\pi i n/f})=\sum_{n=1}^{f-1}(\frac{d_F}{n})D_3(e^{2\pi i n/f}),$$

and the conjugates of $e^{2\pi i n/f}\in N$ over F are exactly the $e^{2\pi i n'/f}$ with $(\frac{d_F}{n})=(\frac{d_F}{n'})$.

By analyzing the structure of the numerical examples of Conjectures 1 and 2, one can get a more precise conjecture which actually predicts which linear combinations of products of polylogarithm values must be used in order to get zeta-values. Using it, it is easy to produce as many (conjectural) formulas involving polylogarithms and zeta-values as desired. In many cases, these seem to be new even for $F=\mathbf{Q}$, e.g.

$$\frac{67}{24}\zeta(3)\stackrel{?}{=}6D_3(\frac{2}{3})+3D_3(\frac{3}{4})-3D_3(\frac{1}{2})-D_3(\frac{8}{9})-2D_3(\frac{1}{3})+D_3(-\frac{1}{3}).$$

We will discuss the various versions of this conjecture, and its relation to algebraic K-theory, in a later paper [9].

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