Axioms for the equivariant Lefschetz number and for the Reidemeister trace*

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Abstract

In this note we characterize the Reidemeister trace, the equivariant Lefschetz number and the equivariant Reidemeister trace in terms of certain axioms.

Introduction

The Euler characteristic and the Lefschetz number have been systematically used in order to treat fixed point theory, where the former can be regarded as a particular case of the latter. C. Watts in [Wa] characterized the Euler characteristic in an axiomatic way, and recently M. Arkowitz and R. Brown in [AB] characterized the Lefschetz number also in an axiomatic way.

A sharper invariant for the study of fixed point theory is the Reidemeister trace, which was originally defined by K. Reidemeister in [R]. S. Husseini in [Hu] provided an algebraic version of the so called *Generalized Lefschetz*

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number which under certain identifications constitutes an extension of the Reidemeister trace. This can be used to define in a more geometric way an *extended Reidemeister trace* (see section 1). It is defined on a larger category than the Reidemeister trace, and the Reidemeister trace of an endomorphism is equal to the extended Reidemeister trace of the corresponding endomorphism in the larger category.

The first purpose of this work is to show that this extended Reidemeister trace can be characterized in terms of some axioms which are similar to (but not the same as) the ones used in [AB]. For equivariant fixed point theory, we have the notion of an equivariant Lefschetz number (see [LR]) and an equivariant Reidemeister trace (see [We], where it was called generalized equivariant Lefschetz invariant). The second purpose of this work is to show that the equivariant invariants can also be characterized by corresponding axioms.

The paper contains 4 sections. In section 1 we revise a few facts about the Reidemeister trace, where the definition of the Reidemeister trace here extends the one given in [R]. The main ideas are from S. Husseini [Hu]. In section 2 we give a set of axioms and show that these characterize the Reidemeister trace. In section 3 we deal with the axioms characterizing the equivariant Lefschetz number, and in section 4 we treat the axioms for the equivariant Reidemeister trace.

1 Preliminaries about the Reidemeister trace

The original Reidemeister trace was defined in [R] for endomorphisms in a certain category. In this section we extend this definition to endomorphisms of a larger category which will be used in our approach. We simply refer to this as an extension of the Reidemeister trace to this larger category.

Extensions of the Reidemeister trace to larger categories than the one considered by Reidemeister have been studied in a purely algebraic way by S. Husseini in [Hu], and more geometrically by J. S. Fares and E. Hart in [FH], [Ha]. We will refer to them for more details.

We will consider an extension of the Reidemeister trace to a category which seems suitable for the type of problem we want to analyze, namely axiomatization. We start by recalling the Reidemeister trace and discussing the extension in detail.

Let $\mathcal{C}$ be the category where the objects are finite connected CW-complexes and the morphisms are continuous maps. Let $X \in \mathcal{C}$ and denote by $\tilde{X}$ the universal covering space of X. Each module $C_n(\tilde{X})$ of the cellular chain complex $C_*(\tilde{X})$ is a free $\mathbb{Z}[\pi]-$module. As in [R], given a cellular map $f : X \to X$ consider the induced chain homomorphism $\tilde{f}_{n\#} : C_n(\tilde{X}) \to C_n(\tilde{X})$. From [R] one has the definition of the "trace" $tr(\tilde{f}_{n\#})$ of each homomorphism $\tilde{f}_{n\#}$ which is a well defined element in the abelian group $\mathbb{Z}[R(f)]$ where $R(f)$ is the set of the Reidemeister classes of f. The element $\Sigma_{n=0}^{\infty}(-1)^n tr(\tilde{f}_{n\#})$ of the abelian group $\mathbb{Z}[R(f)]$ is called the *Reidemeister trace of the function* f, we denote it by $RT(f)$.

Now consider the category $\mathcal{C}_0$ where the objects are pairs (X, H) where X is a CW-complex and $H \subset \pi_1(X)$ is a normal subgroup. Morphisms are pairs $(f, f_\#)$ where f is a continuous map $f : X \to X$ such that $f_\#(H) \subset H$, with $f_\#$ the induced homomorphism on the fundamental group. There is a natural way to define an extension of the Reidemeister trace for the morphisms of $\mathcal{C}_0$, see [Ha], [FH] and [Hu] for more details. We simply call this extension an extension of the Reidemeister trace to the category $\mathcal{C}_0$.

Finally denote by $\mathcal{C}_1$ the category where the objects are pairs (Y, G) where G is a discrete group, Y is a connected and finite-dimensional CW-complex, G acts cellularly and either freely or semi-freely where the fixed point set of the action is one point, i.e., $Y^G = y_0$, and $C_n(Y)$ (which is a free $\mathbb{Z}[G]$-module) has finite rank. (This is fulfilled if G acts cocompactly.) Denote by $\mathcal{C}_1^0 \subset \mathcal{C}_1$ the objects where the action is free. By a map between two objects (Y_1, G_1) and (Y_2, G_2), we mean a pair (f, ϕ) where $\phi : G_1 \to G_2$ is a group homomorphism and $f : Y_1 \to Y_2$ is a cellular equivariant map with respect to ϕ (meaning $f(g.y) = \phi(g).f(y)$).

The category $\mathcal{C}$ embeds in the category $\mathcal{C}_1$ in the following way: To a finite connected CW-complex X we associate the pair $(\tilde{X}, \pi_1(X))$. To a morphism $f\colon X \to Y$, we associate the pair $(\tilde{f}, f_\#)$, where $f_\#$ is the map induced by f on the fundamental group.

We need to use the category $\mathcal{C}_1$ because of the cofibration axiom: When we have a cofibration $A \to X$ and we want to pass to the universal cover $\tilde{X}$, then we only obtain a cofibration $\tilde{A} \to \tilde{X}$ if $\pi_1(A) = \pi_1(X)$. We also want to treat the case where $\pi_1(A) \neq \pi_1(X)$ by using the corresponding cover $\hat{A}$ of A and then associating $(\hat{A}, \pi_1(X))$ to A.

Given a map (f, ϕ) from a pair (Y, G) to itself we define:

Definition 1.1 *The* extended Reidemeister trace *of a self map* $(f, \phi) : (Y, G) \to (Y, G)$*, denoted by* $RTE(f, \phi)$*, is given by:*
a) If the action is free it is the element $\Sigma_{n=0}^{\infty}(-1)^n tr(f_{n\#})$ *of the abelian group* $\mathbb{Z}[R(\phi)]$*, where* $R(\phi)$ *is the set of the Reidemeister classes of the homomorphism* $\phi : G \to G$*, and* $f_n : C_n(Y) \to C_n(Y)$ *is the* $\mathbb{Z}[G]$*-homomorphism of the chain complex in dimension n, equivariant with respect to* ϕ*.*
b) If the action is semi-free, similarly as in item a), it is the element

$\Sigma_{n=1}^{\infty}(-1)^n tr(f_{n\#}) + tr(\bar{f}_{0\#})$ *of the abelian group* $\mathbb{Z}[R(\phi)]$, *where* $R(\phi)$ *are the Reidemeister classes of the homomorphism* $\phi : G \to G$, $f_n : C_n(Y) \to C_n(Y)$ *is the* $\mathbb{Z}[G]$*-homomorphism of the chain complex in dimension* $n > 0$ *and* $f_0 : C_0(Y)/C_0(y_0) \to C_0(Y)/C_0(y_0)$ *is the* $\mathbb{Z}[G]$*-homomorphism of the quotient of the chain complex in dimension* 0. *Here* y_0 *is the basepoint of* Y.

The difference between a) and b) is that in b) we ignore the summand $\mathbb{Z}$ of the 0-chain which comes from the point fixed by the action.

We briefly comment on how RTE extends the original definition of Reidemeister trace. Given an object $X \in \mathcal{C}$ consider the pair $(\tilde{X}, \pi_1(X))$. It is easy to check that this pair is in $\mathcal{C}_1$. Given a map $f : X \to X$ we consider a lift $\tilde{f} : \tilde{X} \to \tilde{X}$. It is also easy to see that $RT(f) = RTE(\tilde{f}, f_\#)$.

Now we will prove two lemmas about RTE where the first can be used to give an alternative description of a set of axioms which will characterize the Reidemeister trace.

Lemma 1.2 *a) Let* $(Y, G) \in \mathcal{C}_1^0$ *and let* (f, ϕ) *be an endomorphism of* (Y, G), *i.e.,* ϕ *is an endomorphism of* G *and* $f : Y \to Y$ *is a* ϕ*-equivariant map. Let* $\bar{Y} = Y/G$, *and let* $\bar{f} : \bar{Y} \to \bar{Y}$ *be the induced map. Then the natural homomorphism* $\theta : \pi_1(\bar{Y}) \to G$ *satisfies* $\theta(RT(\bar{f})) = RTE(f, \phi)$.

b) Let $(Y_i, G_i) \in \mathcal{C}_1^0$ *for* $i = 1, 2$ *and let* (f_i, ϕ_i) *be endomorphisms of* (Y_i, G_i), *i.e., let* ϕ_i *be endomorphisms of* G_i *and* $f_i : Y_i \to Y_i$ *be* ϕ_i*-equivariant maps for* $i = 1, 2$. *Let* $\theta : G_2 \to G_1$ *be a surjective homomorphism and let* $p : Y_2 \to Y_1$ *be a regular covering, where* p *is* θ*-equivariant and* f_2 *is a lift of* f_1 *with respect to the covering. If the deck tranformations of the regular covering correspond to a subgroup of* G_2 *such that we have a short exact sequence*

$$1 \to \pi_1(Y_1)/\pi_1(Y_2) \to G_2 \to G_1 \to 1,$$

then $\theta(RTE(f_2, \phi_2)) = RTE(f_1, \phi_1)$.

Proof. Part a) Consider the universal covering $\tilde{Y}$ of Y and the covering map $p : \tilde{Y} \to Y$. The pair (p, θ) is a map from $(\tilde{Y}, \pi_1(\bar{Y}))$ to (Y, G) and the result follows straight from the definitions of $RT(\bar{f})$ and $RTE(f, \phi)$.

Part b) Let $\bar{Y}_1 = Y_1/G_1$ and $\bar{Y}_2 = Y_2/G_2$. Because of the hypotheses we have a well defined map $\bar{p} : \bar{Y}_2 \to \bar{Y}_1$ which is indeed a homeomorphism and we denote this space (hoemeomorphic to $\bar{Y}_1$ and $\bar{Y}_2$) by Y. Let $f : Y \to Y$ be the quotient map and let $\tilde{Y}$ be the universal cover of Y which is also the universal cover of Y_i for $i = 1, 2$. The projection $\theta_i : \pi_1(Y) \to G_i$ maps the Reidemeister trace of f to $RTE(f_i, \phi_i)$ for $i = 1, 2$ respectively and the result follows. ∎

For the next lemma, for all $n \geq 1$ let us denote by Y_n a bouquet of spheres S^n which as CW-complex has one 0-cell and a number of n-cells equal to the number of spheres in the bouquet. Suppose that Y_n is a G-CW-complex where G acts semi-freely with respect to the CW structure of Y_n described above.

Lemma 1.3 *Let $\phi : G \to G$ be a homomorphism, and let $f_n : Y_n \to Y_n$ be a ϕ-equivariant homomorphism where the suspension of f_n is homotopic to f_{n+1}, for all $n \geq 1$. Then $RTE(f_1, \phi) = (-1)^{n-1} RTE(f_n, \phi)$.*

Proof. From the definition of RTE, for each space Y_n we have to consider only one free module which corresponds to the chains at level n, denoted by F_n. But these modules are isomorphic and the $\mathbb{Z}[G]$-homomorphisms are the same because of the hypotheses. So the result follows. ∎

Some examples

The following examples show the necessity of axioms slightly more complicated than the ones used to characterize the Lefschetz number. If we work with the Lefschetz number, for a given cofibration $A \to X \to X/A$ and a self map of this cofibration, we know that $L(f) = L(f') + L(\bar{f})$, where f' is the restriction of f to A and $\bar{f}$ is the induced map in the quotient.

Although given f' and $\bar{f}$ there are many different homotopy classes of maps $[f]$ of cofiber maps $f : X \to X$ which make the diagram commutative up to homotopy, the Lefschetz number of any one possible f is the same and it is determined by f' and $\bar{f}$ as in the formula above. This is not the case for the Reidemeister trace. Therefore we need some suitable replacement for the cofibration axiom.

Example 1 *Take $A = S^1$, $X = S^1 \vee S^2$ and $X/A = S^2$. Let us define two maps $f_1, f_2 : X \to X$, not homotopic, such that the induced maps on the subspace and on the quotient are homotopic. (There are infinitely many such maps homotopic to some prescribed f' on A and $\bar{f}$ on X/A.) Denote by ι_1, ι_2 the generators of $\pi_1(S^1), \pi_2(S^2)$. Let f_1, f_2 be maps such that the induced homomorphisms on the generators ι_1, ι_2 are defined as follows: for f_1 the homomorphism is the identity and for f_2 we have $\iota_1 \to \iota_1$ and $\iota_2 \to \iota_1 \circ \iota_2$ where the multiplication $\circ$ means the action of π_1 on the group π_2. The Reidemeister trace of f_1 is not the same as the Reidemeister trace of f_2. This tells us that the calculation of the Reidemeister trace cannot be computed simply using f' and $\bar{f}$ as in the Lefschetz number case.*

Example 2 *Take $A = S^1$, let X be the torus and take the inclusion $S^1 \to S^1 \times S^1$ to be $x \mapsto (x, y_0)$ for some base point $y_0 \in S^1$. The space X/A has the* *homotopy type of a wedge of S^1 and S^2. All these spaces have nice models for the universal covering. There are many maps $f : T \to T$ (they are classified by the induced homomorphism in homology) which satisfy $f(A) \subset A$, but they have different Reidemeister trace. In this case the space is not a wedge as in example 1.*

2 Axioms for the Reidemeister trace

Now we formulate the axioms for the Reidemeister trace. Let μ be a function which to a self map (f, ϕ) of an object $(Y, G) \in \mathcal{C}_1$ associates an element of the abelian group $\mathbb{Z}[R(\phi)]$. Suppose that μ satisfies the following axioms:

1. Homotopy property:
   Let $(Y, G) \in \mathcal{C}_1$ and let ϕ be an endomorphism of G. If $f, g : Y \to Y$ are ϕ-homotopic then $\mu(f, \phi) = \mu(g, \phi)$.

2. Commutativity property:
   Let $(Y_1, G_1), (Y_2, G_2) \in \mathcal{C}_1$ and let $\phi_1 : G_1 \to G_2$, $\phi_2 : G_2 \to G_1$ be homomorphisms. If $f : Y_1 \to Y_2$, $g : Y_2 \to Y_1$ are ϕ_1-, ϕ_2-equivariant maps respectively, then $\phi_{1*}\mu(g \circ f, \phi_2\phi_1) = \mu(f \circ g, \phi_1\phi_2)$. Here ϕ_{1*} denotes the map induced by ϕ_1 on the set of Reidemeister classes.

3. Cofibration property:
   Let A be a G-sub-CW-complex of X, $(X, G) \in \mathcal{C}_1$ and $A \to X \xrightarrow{p} X/A$ the cofiber sequence. Consider the induced action of G on A given by the restriction of the action of G on X, and an action on X/A such that the projection $p : X \to X/A$ is a G-map, i.e., $p(g.x) = g \circ p(x)$.

   Suppose that there exists a homotopy commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f'"] & X \arrow[r, "p"] \arrow[d, "f"] & X/A \arrow[d, "\bar{f}"] \\ A \arrow[r] & X \arrow[r] & X/A \end{tikzcd}$$

   where the vertical maps are G-maps with respect to a given homomorphism ϕ, then $\mu(f, \phi) = \mu(f', \phi) + \mu(\bar{f}, \phi)$ as an element of $\mathbb{Z}[R(\phi)]$.

4. Wedge property:
   Given K any one dimensional G-complex where $(K, G) \in \mathcal{C}_1$, and a self-map (f, ϕ) of the object (K, G), then $\mu(f) = RTE(f)$.

We have the following alternative description of the axiom 4 above:

4'. Alternative description of the Wedge property:
4'.1) Let (K, G) be a pair where K is a one-dimensional complex which is a tree and G acts freely on K. If (f, ϕ) is an endomorphism, then $\mu(f, \phi) = RTE(f, \phi)$.
4'.2) Let (K, G) be a pair where K is a bouquet of circles where G acts semi-freely on K and K^G is the distinguished point of the bouquet. If (f, ϕ) is an endomorphism, then $\mu(f, \phi) = RTE(f, \phi)$.
4'.3) Let (X_1, G_1), (X_2, G_2) be pairs and let (p, θ) be a morphism such that $p : X_1 \to X_2$ is a normal covering map, the group of deck tranformations of p, denoted by H, is a subgroup of G_1, and we have a short exact sequence $1 \to H \to G_1 \to G_2 \to 1$. Then given $f_2 : X_2 \to X_2$ and $f_1 : X_1 \to X_1$ where f_1 covers f_2 with respect to p, we have $p_{\#}\mu(f_1) = \mu(f_2)$.

Now we are in a position to prove the main result.

Theorem 2.1 *Let μ be a function which associates to each endomorphism (f, ϕ) of an object $(Y, G) \in \mathcal{C}_1$ an element of the Reidemeister group $\mathbb{Z}[R(\phi)]$ such that it satisfies the 4 axioms above. Then μ coincides with RTE and consequently μ restricted to $\mathcal{C}$ coincides with the Reidemeister trace RT.*

Proof. The proof is by induction on the dimension of the CW-complex.

If the dimension of K is one then the function μ coincides with RTE by the Wedge axiom. It remains to show that this is also true if the Alternative Wedge axiom holds. But this follows from Lemma 1.2.

Suppose that we have proved the result for all CW-complexes of dimension less than or equal to n and let Y have dimension $n+1$ with $(Y, G) \in \mathcal{C}_1$. We consider two cases. In the first case G acts freely, in the second case it acts semi-freely.

In the first case consider the cofibration $Y^n \to Y \to Y/Y^n \cong \vee S^{n+1}$. Since the action of G is cellular, it can be restricted to Y^n, and we obtain a free G-action on Y^n and a semi-free action on Y/Y^n. The map f induces a self-map of the cofibration above, i.e., there is a commutative diagram

$$\begin{tikzcd} Y^n \arrow[r] \arrow[d, "f'"] & Y \arrow[r] \arrow[d, "f"] & Y/Y^n \arrow[d, "\bar{f}"] \\ Y^n \arrow[r] & Y \arrow[r] & Y/Y^n \end{tikzcd}$$

where f' and $\bar{f}$ are ϕ-equivariant maps. The hypotheses of the "Cofibration property" axiom are satisfied. So we obtain $\mu(f, \phi) = \mu(f', \phi) + \mu(\bar{f}, \phi) =$

$RTE(f', \phi) + \mu(\bar{f}, \phi)$. In order to show that $\mu(f, \phi) = RTE(f, \phi)$ it suffices to show that $\mu(\bar{f}, \phi) = RTE(\bar{f}, \phi)$.

Now we proceed as follows: Let us consider the bouquet of $(n+1)$-spheres indexed by a set J. Consider the bouquet of n-spheres indexed by the same set and the map $\bar{f}_{des}$ a desuspension of the given map $\bar{f}$. Then consider the bouquet of discs of dimension $n+1$ indexed by the same set and extend the map $\bar{f}$ to a map $\bar{f}_{discs}$. From the homotopy property and the cofibration property (the action defined on the spaces satisfies the hypotheses of the "cofibration property" axiom) it follows that $\mu(\bar{f}_{discs}, \phi) = 0$ and that $\mu(\bar{f}_{des}, \phi) = -\mu(\bar{f}, \phi)$. From Lemma 1.3 we know that $RTE(\bar{f}_{des}, \phi) = -RTE(\bar{f}, \phi)$, so by the induction hypothesis we have $\mu(\bar{f}, \phi) = RTE(\bar{f}, \phi)$ and the result follows for a free action. If the action is semi-free the proof is similar and is left to the reader. ∎

3 Axioms for the equivariant Lefschetz number

In this section we give a set of axioms for the equivariant Lefschetz number with values in the Burnside ring, corresponding to the axioms given in [AB].

Let Γ be a finite group. The *Burnside ring* $A(\Gamma)$ of Γ is defined to be the Grothendieck ring of finite Γ-sets with the additive structure coming from disjoint union and the multiplicative structure coming from the Cartesian product. Additively,

$$A(\Gamma) = \bigoplus_{(H) \in \mathrm{c}(\Gamma)} \mathbb{Z} \cdot [\Gamma/H],$$

where $\mathrm{c}(\Gamma)$ denotes the set of conjugacy classes of subgroups of Γ. Let X be a finite Γ-CW-complex. Given a Γ-equivariant cellular endomorphism $f \colon X \to X$, one defines the *equivariant Lefschetz number with values in the Burnside ring* of f [LR, Equation 4.1] by

$$L_\Gamma(f) := \sum_{(H) \in \mathrm{c}(\Gamma)} L^{\mathbb{Z}WH}(f^H, f^{>H}) \cdot [\Gamma/H] \in A(\Gamma).$$

Here, for $H \leq \Gamma$, we have the fixed point set $X^H := \{x \in X \mid hx = x \text{ for all } h \in H\}$ and set $f^H := f|_{X^H}$. We set $X^{>H} := \{x \in X^H \mid \Gamma_x \neq H\}$, where Γ_x denotes the isotropy group of x, and $f^{>H} := f|_{X^{>H}}$.

Using [LR, Lemma 1.9], we have

$$L^{\mathbb{Z}WH}(f^H, f^{>H}) = \sum_{p \geq 0} (-1)^p \sum_{WH \cdot e \in WH \backslash I_p(X^H, X^{>H})} \mathrm{inc}(f^H, e).$$

Here $I_p(X^H, X^{>H})$ denotes the set of p-cells of $X^H \setminus X^{>H}$ and $\mathrm{inc}(f^H, e)$ is the *incidence number*, the degree of the composition

$$\overline{e}/\partial e \xrightarrow{i_e} \bigvee S^p \xrightarrow{k^{-1} \circ f^H \circ k} \bigvee S^p \xrightarrow{\mathrm{pr}_e} \overline{e}/\partial e,$$

where k is the appropriate homotopy equivalence. The general definition is given in [LR, Definition 1.4]. If f is not cellular, one takes any Γ-equivariant cellular approximation [tD, II.2] $\hat{f}$ of f and defines $L(f) := L(\hat{f})$.

Now let X be a based Γ-CW-complex, where the isotropy group of the basepoint is Γ. Let f be an endomorphism fixing the basepoint. We define the *reduced equivariant Lefschetz number* $\widetilde{L_\Gamma}$ to be $\widetilde{L_\Gamma}(f) := L_\Gamma(f) - 1$. We characterize the reduced equivariant Lefschetz number as follows.

Theorem 3.1 *Let λ_Γ be a function from the set of Γ-equivariant endomorphisms of finite based Γ-CW-complexes to the Burnside ring $A(\Gamma)$ that satisfies the following conditions:*

1. *Γ-Homotopy Axiom:*
   If $f, g\colon X \to X$ are Γ-homotopic, then $\lambda_\Gamma(f) = \lambda_\Gamma(g)$.

2. *Cofibration Axiom:*
   If A is a sub-Γ-CW-complex of X and $A \to X \to X/A$ is the resulting cofiber sequence, and if there exists a commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & X & \longrightarrow & X/A \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \overline{f}} \\ A & \longrightarrow & X & \longrightarrow & X/A, \end{array}$$

   then $\lambda_\Gamma(f) = \lambda_\Gamma(f') + \lambda_\Gamma(\overline{f})$.

3. *Commutativity Axiom:*
   If $f\colon X \to Y$ and $g\colon Y \to X$ are equivariant maps, then $\lambda_\Gamma(gf) = \lambda_\Gamma(fg)$.

4. *Wedge of Circles Axiom:*
   If $f\colon \bigvee^k \Gamma/H_i \times S^1 \to \bigvee^k \Gamma/H_i \times S^1$ is an equivariant map, then

$$\lambda_\Gamma(f) = -(\mathrm{inc}(f, e_1) \cdot [\Gamma/H_1] + \ldots + \mathrm{inc}(f, e_k) \cdot [\Gamma/H_k]).$$

Then λ_Γ coincides with the reduced equivariant Lefschetz number $\widetilde{L_\Gamma}$.

This theorem is the equivariant generalization of [AB, Theorem 1.1]. We use several lemmas in the proof, corresponding to [AB, Lemma 3.1 – 3.4].

Lemma 3.2 *If $f\colon X \to X$ is an equivariant map and $h\colon X \to Y$ is a Γ-homotopy equivalence with Γ-homotopy inverse $k\colon Y \to X$, then $\lambda_\Gamma(f) = \lambda_\Gamma(hfk)$.*

Lemma 3.3 *If $f\colon X \to X$ is Γ-homotopic to a constant map, then $\lambda_\Gamma(f) = 0$.*

Lemma 3.4 *If X is a based Γ-space, $f\colon X \to X$ is a based equivariant map and $\Sigma f\colon \Sigma X \to \Sigma X$ is the suspension of f, then $\lambda_\Gamma(\Sigma f) = -\lambda_\Gamma(f)$.*

Lemma 3.5 *For any $k \geq 0$ and $n \geq 0$, if $f\colon \bigvee^k \Gamma/H_i \times S^n \to \bigvee^k \Gamma/H_i \times S^n$ is a Γ-equivariant map, then*

$$\lambda(f) = (-1)^n(\mathrm{inc}(f, e_1) \cdot [\Gamma/H_1] + \ldots + \mathrm{inc}(f, e_k) \cdot [\Gamma/H_k]).$$

Proof.[Sketch] The proofs are only slight modifications of those given in [AB]. The only point to remark here is that Lemma 3.5 also holds for the case $n = 0$. This is derived from Lemma 3.4, using the fact that the suspension is a wedge of one-dimensional spheres and thus the value of $\lambda_\Gamma(f) = -\lambda_\Gamma(\Sigma f)$ is given by Axiom (4). ∎

Proof. of Theorem. We easily check that the reduced equivariant Lefschetz number $\widetilde{L_\Gamma}$ fulfills the above axioms. Axiom (1) follows directly from [LR, Lemma 1.6 (a)] and Axiom (3) from [LR, Lemma 1.6 (b)]. Axiom (2) follows from additivity, as stated for example in [We, Definition 2.3], which holds because of [We, Theorem 5.14]. Using the Γ-pushout-diagram

$$\begin{tikzcd}
(A, f') \arrow[r] \arrow[d] & (*, \mathrm{id}_*) \arrow[d] \\
(X, f) \arrow[r] & (X/A, \overline{f})
\end{tikzcd}$$

one obtains

$$L_\Gamma(f) + L_\Gamma(\mathrm{id}_*) = L_\Gamma(f') + L_\Gamma(\overline{f}).$$

We have $L_\Gamma(\mathrm{id}_*) = 1$, so

$$\widetilde{L_\Gamma(f)} = \widetilde{L_\Gamma(f')} + \widetilde{L_\Gamma(\overline{f})}.$$

Axiom (4) follows from [LR, Lemma 1.9], using the fact that the WH-cells of $X^H \setminus X^{>H}$ are in a one-to-one correspondence with the Γ/H-cells of X. The zero-cell gives a contribution $+1$ to $L_\Gamma(f)$, this disappears since we consider $\widetilde{L_\Gamma}(f)$.

We now have to check that any invariant satisfying the axioms is equal to $\widetilde{L_\Gamma}$. This can be done analogously to [AB].

Lemma 3.5 gives the value for based 0-dimensional spaces. If $f\colon X \to X$ is an endomorphism of a 0-dimensional Γ-CW-complex without basepoint, adding a disjoint basepoint and looking at the cofibration sequence

$$(X, f) \to (X_+, f_+) \to (S^0, \mathrm{id}_{S^0})$$

gives by the Cofibration Axiom

$$\lambda_\Gamma(f_+) = \lambda_\Gamma(f) + \lambda_\Gamma(\mathrm{id}_{S^0}),$$

and since $\lambda_\Gamma(\mathrm{id}_{S^0}) = \mathrm{inc}(f, e) = 1$ we have

$$\lambda_\Gamma(f) = \lambda_\Gamma(f_+) - 1.$$

Using the above lemmas, we can now prove the desired result by induction. Using Γ-equivariant cellular approximation [tD, II.2] and the Γ-homotopy axiom, we can suppose without loss of generality that f is cellular.

Let $X = \amalg_{i=1}^k \Gamma/H_i$ be a 0-dimensional Γ-CW-complex, and let $f\colon X \to X$ be an endomorphism. By definition we have $\widetilde{L_\Gamma}(f) = (\sum_i \mathrm{inc}(f, e_i) \cdot [\Gamma/H_i]) - 1$. (Note that $\mathrm{inc}(f, e_i) = 1$ if f induces the identity on this 0-cell, and $\mathrm{inc}(f, e_i) = 0$ otherwise.) We have the same value for $\lambda_\Gamma(f)$ by Lemma 3.5.

Let X be an n-dimensional Γ-CW-complex. Inclusion of the $n-1$-skeleton gives a cofibration sequence

$$(X_{n-1}, f_{n-1}) \to (X, f) \to (X/X_{n-1}, \overline{f}) = (\bigvee^k \Gamma/H_i \times S^n, \overline{f}),$$

whence $\lambda_\Gamma(f) = \lambda_\Gamma(f_{n-1}) + \lambda_\Gamma(\overline{f})$. Using the induction hypothesis and Lemma 3.5 we obtain $\lambda_\Gamma(f) = \widetilde{L_\Gamma}(f_{n-1}) + \widetilde{L_\Gamma}(\overline{f}) = \widetilde{L_\Gamma}(f)$. ∎

Remark 3.6 *It might also be possible to introduce an induction structure into the axioms, this is fulfilled by the equivariant Lefschetz number [LR, Lemma 1.6, assertions (c) and (d)]. Using the induction structure, we would only need the wedge axiom for S^1 with trivial group action.*

Remark 3.7 *The commutativity axiom does not seem to be necessary in its full generality, it is only needed to know that the value on the one-point-space is trivial. We keep the commutative axiom because this axiom has been used more frequently in calculations.*

4 Axioms for the equivariant Reidemeister trace

We generalize Section 2 to the equivariant case, using the equivariant Reidemeister trace RT_Γ which was developed under the name *generalized equivariant Lefschetz invariant* λ_Γ in [We]. (We call it RT_Γ here to avoid confusion with λ_Γ introduced in Section 3.) We define an equivariant extended Reidemeister trace RTE_Γ and we give axioms characterizing this invariant, and thus also the equivariant Reidemeister trace.

We start by reviewing the definition of the equivariant Reidemeister trace. We adopt a geometric approach, a thorough treatment of the abstract setup can be found in [We].

Let Γ be a finite group. Let X be a finite Γ-CW-complex, and let $f\colon X \to X$ be an equivariant endomorphism. The equivariant Reidemeister trace of f is an element in the abelian group

$$\bigoplus_{\substack{(H)\in c(\Gamma)\\ i\in I_{X,H}}} \mathbb{Z}\mathcal{R}(\phi_{H,i}).$$

Here the indexing set $I_{X,H}$ is used for the connected components of the fixed point set X^H, i.e., $X^H = \amalg_{i\in I_{X,H}} X_i^H$ with X_i^H connected. The Reidemeister set $\mathcal{R}(\phi_{H,i})$ is given by

$$\mathcal{R}(\phi_{H,i}) := \pi_1(X_i^H)/\phi_{H,i}(\gamma)\alpha\gamma^{-1} \sim \alpha$$

if $f(X_i^H) \subseteq X_i^H$, and by the empty set $\emptyset$ otherwise.

Here $\alpha \in \pi_1(X_i^H)$ and $\gamma \in A_{H,i}$, where $A_{H,i}$ is given by a group extension $1 \to \pi_1(X_i^H) \to A_{H,i} \to WH_i \to 1$ with $WH_i \leq WH$ being the subgroup that fixes X_i^H. The map $\phi_{H,i}$ is induced by the restriction of $f_\#$ on $\pi_1(X_i^H)$ and by the identity on WH_i.

There is a trace map mapping into this group, and the equivariant Reidemeister trace $RT_\Gamma(f)$ is the image of a certain universal invariant under this trace map [We, Definition 5.13].

Definition 4.1 *Let $f\colon X \to X$ be an equivariant endomorphism of a finite Γ-CW-complex X. The* equivariant Reidemeister trace *of f is*

$$RT_\Gamma(f) := \sum_{H,i} \mathrm{tr}_{\mathbb{Z}A_{H,i}}(\widetilde{f_i^H}, \widetilde{f_i^{>H}}) \in \bigoplus_{\substack{(H)\in c(\Gamma)\\ i\in I_{X,H}}} \mathbb{Z}\mathcal{R}(\phi_{H,i}).$$

Here $X_i^{>H} := \{x \in X_i^H \mid \Gamma_x \neq H\} \subseteq X_i^H$ denotes the singular set, the set of points in X_i^H having isotropy group greater than H. The maps f_i^H and

$f_i^{>H}$ are the restrictions to those spaces, and $\widetilde{f_i^H}$ and $\widetilde{f_i^{>H}}$ are the lifts to the universal cover of X_i^H and to the corresponding cover of $X_i^{>H}$.

The trace map $\mathrm{tr}_{\mathbb{Z}A_{H,i}}$ is the trace map for discrete group extensions introduced in [We, Definition 5.4].

We also want to treat spaces with a basepoint. If X is a based Γ-CW-complex, with the basepoint $x_0 \in X$ having isotropy group Γ and lying in X_1^Γ, then we define the *reduced equivariant Reidemeister trace* $\widetilde{RT_\Gamma}$ to be $\widetilde{RT_\Gamma}(f) := RT_\Gamma(f) - 1$, where $1 \in \mathbb{Z}\mathcal{R}(X_1^\Gamma)$.

We now enlarge the category on which we want to define an equivariant Reidemeister trace.

Definition 4.2 *Let $\Gamma\mathcal{C}$ be the category consisting of finite Γ-CW-complexes and equivariant maps.*

Let $\Gamma\mathcal{C}_1$ be the category consisting of triples $(Z, \mathcal{X}, \mathcal{G})$. Here Z is a finite proper Γ-CW-complex and $\mathcal{X}$ and $\mathcal{G}$ are families indexed by $\{(H) \in \mathrm{c}(\Gamma), i \in I_{Z,H}\}$. At (H, i), the group $G_{H,i}$ is given by a group extension

$$1 \to \pi_1(Z_i^H) \to G_{H,i} \to WH_i \to 1.$$

The family $\mathcal{X}$ consists of pairs of finite-dimensional spaces $(X_{H,i}, X'_{H,i})$, and the group $G_{H,i}$ is required to act freely and cocompactly on $X_{H,i} \setminus X'_{H,i}$.

If Z has a basepoint, then we also have a basepoint in $X_{\Gamma,1}$, and maps are basepoint-preserving.

Morphisms are triples (g, F, Φ), where $g\colon Z_1 \to Z_2$ is an equivariant map which induces a map on the indexing sets. The family Φ consists of maps induced by g, and the family F consists of maps compatible with the map on the indexing set and such that $f_{H,i}$ is $\phi_{H,i}$-equivariant.

We embed $\mathcal{C}$ into $\mathcal{C}_1$ by associating to a finite proper Γ-CW-complex X the triple $(X, \{(\widetilde{X_i^H}, \widetilde{X_i^{>H}})\}, \mathcal{G})$, where $\mathcal{G}$ is given as above.

To a morphism $g\colon X \to Y$, we associate $(g, \{(\widetilde{g_i^H}, \widetilde{g_i^{>H}})\}, \Phi)$, where Φ is induced by g.

The dimension of a fimily $\mathcal{X}$ is defined to be the maximum of the dimensions of the spaces it contains.

Remark 4.3 *A more abstract approach can be made by considering spaces over categories, defined in the spirit of [L]. Using this approach, one can see why the above definitions make sense and work. In this setting, we would define:*

Let $\mathcal{C}_{\Gamma 1}$ be the category consisting of pairs $(\mathcal{X}, \mathcal{G})$, where $\mathcal{X}$ is a finite-dimensional proper $\mathcal{G}$-CW-complex. Here $\mathcal{G}$ is an EI-category (a small category where every endomorphism is an automorphism) which is the fundamental category [L, Definition 8.15] of some Γ-CW-complex, i.e., there exists a Γ-CW-complex Z with $\mathcal{G} \cong \Pi(\Gamma, Z)$.

A morphism from $(\mathcal{X}_1, \mathcal{G}_1)$ to $(\mathcal{X}_2, \mathcal{G}_2)$ in $\mathcal{C}_{\Gamma 1}$ is a pair (f, Φ), where $\Phi \colon \mathcal{G}_1 \to \mathcal{G}_2$ is a functor induced by a morphism $g \colon Z_1 \to Z_2$ between the corresponding Γ-spaces and $f \colon \mathcal{X}_1 \to \mathcal{X}_2$ is a Φ-equivariant map.

We embed $\Gamma\mathcal{C}$ in $\mathcal{C}_{\Gamma 1}$ by associating to a finite proper Γ-CW-complex X the pair $(\widetilde{X}, \Pi(\Gamma, X))$. Here $\widetilde{X}$ is the universal covering space of X in the equivariant sense [L, Definition 8.22], a $\Pi(\Gamma, X)$-CW-complex.

In order to characterize the reduced equivariant Reidemeister trace, we define an equivariant extended Reidemeister trace (RTE_Γ) as a generalization of RTE.

Definition 4.4 *Let Γ be a finite group. The* equivariant extended Reidemeister trace *RTE_Γ is defined on endomorphisms in the category $\Gamma\mathcal{C}_1$. Given an endomorphism (g, F, Φ) of an object $(Z, \mathcal{X}, \mathcal{G})$, we have*

$$RTE_\Gamma(g, F, \Phi) := \sum_{H,i} \operatorname{tr}_{\mathbb{Z}G_{H,i}}(f_{H,i}) \in \bigoplus_{\substack{(H) \in c(\Gamma) \\ i \in I_{Z,H}}} \mathbb{Z}\mathcal{R}(\phi_{H,i}).$$

Here $\operatorname{tr}_{\mathbb{Z}G_{H,i}}$ is the trace map for group extensions [We, Definition 5.4].

We need to prove a preliminary lemma, analogous to Lemma 1.3 in the non-equivariant case.

Lemma 4.5 *For all $n \geq 1$, let $(Z, \mathcal{Y}_n, \mathcal{G}) \in \Gamma\mathcal{C}_1$ with $\mathcal{Y}_n$ consisting of bouquets of n-spheres, seen as pairs $(\vee S^n, \mathrm{pt})$, and where $\mathcal{Y}_{n+1}$ is the family consisting of the suspensions of the spheres in $\mathcal{Y}$.*

Let (g, F_n, Φ) be endomorphisms of $(Z, \mathcal{Y}_n, \mathcal{G})$, with (g, F_{n+1}, Φ) homotopic to the suspension of (g, F_n, Φ). Then

$$RTE_\Gamma(g, F_{n+1}, \Phi) = -RTE_\Gamma(g, F_n, \Phi).$$

Proof. For all $n \geq 1$, the only maps appearing in the definition of $RTE_\Gamma(g, F_n, \Phi)$ are endomorphisms of free $G_{H,i}$-modules of dimension n, and the modules appearing for n and $n+1$ are isomorphic. Since the map f_{n+1} is homotopic to the suspension of f_n, by homotopy invariance of the trace, $\operatorname{tr}_{\mathbb{Z}G_{H,i}}(f_{n+1\,H,i}) = \operatorname{tr}_{\mathbb{Z}G_{H,i}}(f_{n\,H,i})$ for all (H, i), and the result follows. ∎

Now we can state and prove the theorem.

Theorem 4.6 *Let Γ be a discrete group. Let μ_Γ be a function defined on the set of equivariant endomorphisms of objects in $\Gamma\mathcal{C}_1$, mapping an endomorphism $(g,F,\Phi)\colon (Z,\mathcal{X},\mathcal{G}) \to (Z,\mathcal{X},\mathcal{G})$ into $\bigoplus_{\substack{(H)\in c(\Gamma)\\ i\in I_{Z,H}}} \mathbb{Z}\mathcal{R}(\phi_{H,i})$, that satisfies the following conditions:*

1. *Γ-Homotopy Axiom:*
   Let $(Z,\mathcal{X},\mathcal{G}) \in \Gamma\mathcal{C}_1$. If endomorphisms (g_1,F_1,Φ_1) and (g_1,F_2,Φ_2) of $(Z,\mathcal{X},\mathcal{G})$ are homotopic in $\Gamma\mathcal{C}_1$, then
   $$\mu_\Gamma(g_1,F_1,\Phi_1) = \mu_\Gamma(g_2,F_2,\Phi_2).$$

2. *Cofibration Axiom:*
   Let $(Z,\mathcal{X},\mathcal{G}) \in \Gamma\mathcal{C}_1$. Let $(Z,\mathcal{A},\mathcal{G}) \in \Gamma\mathcal{C}_1$ with $(A_{H,i},A'_{H,i}) \to (X_{H,i},X'_{H,i})$ a cofibration for all (H,i). Let $\mathcal{X}/\mathcal{A} = \{(X_{H,i}, A_{H,i}\cup X'_{H,i})\}$. Let (g,F,Φ) be an endomorphism of $(Z,\mathcal{X},\mathcal{G})$ which can be restricted to an endomorphism (g,F',Φ) of $(Z,\mathcal{A},\mathcal{G})$. If there exists an endomorphism $(g,\overline{F},\Phi)$ of $(Z,\mathcal{X}/\mathcal{A},\mathcal{G})$ lying in a homotopy commutative diagram
   $$\begin{tikzcd} \mathcal{A} \arrow[r, "i"] \arrow[d, "F'"] & \mathcal{X} \arrow[r, "p"] \arrow[d, "F"] & \mathcal{X}/\mathcal{A} \arrow[d, "\overline{F}"] \\ \mathcal{A} \arrow[r] & \mathcal{X} \arrow[r] & \mathcal{X}/\mathcal{A}, \end{tikzcd}$$
   then $\mu_\Gamma(g,F,\Phi) = \mu_\Gamma(f,F',\Phi) + \mu_\Gamma(f,\overline{F},\Phi)$.

3. *Commutativity Axiom:*
   Given a morphism $(g_1,F_1,\Phi_1)\colon (Z_1,\mathcal{X},\mathcal{G}_1) \to (Z_2,\mathcal{Y},\mathcal{G}_2)$ and a morphism $(g_2,F_2,\Phi_2)\colon (Z_2,\mathcal{Y},\mathcal{G}_2) \to (Z_1,\mathcal{X},\mathcal{G}_1)$, we have
   $$\Phi_{1*}\mu_\Gamma(gf,\Phi_2\Phi_1) = \mu_\Gamma(fg,\Phi_1\Phi_2)$$
   and
   $$\Phi_{2*}\mu_\Gamma(g_1g_2,F_1F_2,\Phi_1\Phi_2) = \mu_\Gamma(g_2g_1,F_2F_1,\Phi_2\Phi_1).$$

4. *Wedge Axiom:*
   Given an endomorphism (g,F,Φ) of $(Z,\mathcal{K},\mathcal{G}) \in \Gamma\mathcal{C}_1$, where $\mathcal{K}$ is one-dimensional, we have $\mu_\Gamma(g,F,\Phi) = RTE_\Gamma(g,F,\Phi)$.

Then μ_Γ coincides with RTE_Γ, and consequently μ_Γ restricted to $\Gamma\mathcal{C}$ coincides with the equivariant Reidemeister trace RT_Γ.

Proof. The proof is analogous to the non-equivariant case. We proceed by induction on the dimension of the family $\mathcal{X}$.

If the dimension is one, then the function μ_Γ coincides with RTE_Γ by the wedge axiom.

Let $(Z, \mathcal{X}, \mathcal{G}) \in \Gamma\mathcal{C}_1$ with $\mathcal{X}$ having dimension $n+1$. Suppose that we have proved the result for all objects with families of dimension less than or equal to n.

Again, we only consider the basepoint-free case, the case of spaces with basepoint is analogous.

Consider the family of cofibrations $(X^n_{H,i}, X^n_{H,i}{}') \to (X_{H,i}, X'_{H,i}) \to (X_{H,i}, X^n_{H,i} \cup X'_{H,i})$ which we abbreviate by $\mathcal{X}^n_{H,i} \to \mathcal{X}_{H,i} \to \mathcal{X}_{H,i}/\mathcal{X}^n_{H,i}$. Since the action of $G_{H,i}$ is cellular, it can be restricted to $X^n_{H,i}$, and we have a free $G_{H,i}$-action on $X^n_{H,i} \setminus X^n_{H,i}{}'$ and a semi-free action on the quotient. Since the maps are cellular, every map $f_{H,i}$ can be restricted to a $\phi_{H,i}$-equivariant endomorphism on $X^n_{H,i}$, and a $\phi_{H,i}$-equivariant endomorphism $\bar{f}_{H,i}$ on the quotient is induced. We obtain a commutative diagram

$$\begin{array}{ccccc}
\mathcal{X}^n & \xrightarrow{i} & \mathcal{X} & \xrightarrow{p} & \mathcal{X}/\mathcal{X}^n \\
\downarrow{\scriptstyle F'} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle \overline{F}} \\
\mathcal{X}^n & \longrightarrow & \mathcal{X} & \longrightarrow & \mathcal{X}/\mathcal{X}^n,
\end{array}$$

and thus we have $\mu_\Gamma(g, F, \Phi) = \mu_\Gamma(g, F', \Phi) + \mu_\Gamma(g, \overline{F}, \Phi)$.

It remains to show that $\mu_\Gamma(g, \overline{F}, \Phi) = RTE_\Gamma(g, \overline{F}, \Phi)$. We know that $\mathcal{X}/\mathcal{X}^n$ is a family of bouquets of $(n+1)$-spheres with semi-free cellular group actions. As in the non-equivariant case, we can desuspend the endomorphisms and see that $\mu_\Gamma(g, \bar{F}_{des}, \Phi) = -\mu_\Gamma(g, \bar{F}, \Phi)$. From Lemma 4.5 we know that $RTE_\Gamma(g, \bar{F}_{des}, \Phi) = -RTE_\Gamma(g, \bar{F}, \Phi)$, so by the induction hypothesis we obtain $\mu_\Gamma(g, \bar{F}, \Phi) = RTE_\Gamma(g, \bar{F}, \Phi)$ and the result follows. ∎

Remark 4.7 *It might also be possible to introduce an induction structure into the axioms, this is fulfilled by the equivariant generalized Lefschetz invariant [We]. The induction structure would connect the invariants for different groups* Γ*.*

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