ON INTEGRAL REPRESENTATIONS AND CRITICAL VALUES OF CERTAIN TENSOR PRODUCT L-FUNCTIONS

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0. Introduction.

In the case of three holomorphic elliptic cusp forms Garrett [Gar] and Orloff [Orl] (also Piatetski-Shapiro and Rallis, Garrett and Harris) have given an integral representation for the corresponding triple product L-functions. They also obtained the Deligne's period conjecture in this case as a consequence.

In these notes we formulate a conjectural integral representation for fivefold-product L-functions $L(f_1 \otimes ... \otimes f_5, s)$, which from the "motivic point of view" may be considered as a natural generalization of the one in the triple product case. Assuming this identity, we obtain analytic continuation and functional equation for $L(f_1 \otimes ... \otimes f_5, s)$. Next, under the same assumption, we explain how arguments similar to those in Garrett and Orloff give an algebraicity theorem for a part of critical values so that it agrees with the Deligne's period conjecture in this case.

Perhaps it is worth mentioning that here we use the group Sp_{15} and not Sp_5 as it might be expected. We stress here that for tensor product L-functions $L(f_1 \otimes ... \otimes f_m, s)$, $m \geq 7$, odd, we probably cannot expect such type of integral representations (see the last section for a discussion).

After this note had been written, the author informed Prof. S. Böcherer about his conjecture and he pointed out that in case of four cusp forms an integral representation of such type has not taken place (see Remarks on p.3).

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Contents

1. Conjectural integral representation: m=5 case, equal weights.

Let $f_1, ..., f_5$ be five holomorphic elliptic cusp forms of level one and of weight $2k$, which we assume to be normalized eigenforms for the Hecke operators.

Write the Fourier expansions

$$f_i(z) = \sum_{n \geq 1} a(i,n) e(nz)$$

For p prime, we may write $1-a(i,p)X+p^{2k-1}X^2=(1-\alpha(i,p)X)(1-\alpha'(i,p)X)$.

Define the fivefold product L-function $L(f_1\otimes\ldots\otimes f_5,s):=$

$$\prod_p det\left(1_{32}-\begin{pmatrix}\alpha(1,p)&0\\0&\alpha'(1,p)\end{pmatrix}\otimes\ldots\otimes\begin{pmatrix}\alpha(5,p)&0\\0&\alpha'(5,p)\end{pmatrix}p^{-s}\right)^{-1}$$

Let $H_n=\left\{Z\in M(n,\mathbb{C})|Z^t=Z,Im(Z)>0\right\}$ be the usual Siegel upper half-space of genus n. $Sp(n,\mathbb{R})$ acts on H_n as usual. Let $P_{n,0}:=\left\{g\in Sp(n)|g=\begin{pmatrix}\star&\star\\0&\star\end{pmatrix}\right\}$. For $Z\in H_n$ and $g=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in Sp(n,\mathbb{R})$ let $\mu(g,Z):=det(cZ+d)$.

For $s\in\mathbb{C}$, $k\in\mathbb{Z}$ and $Z=X+iY\in H_n$ we define the non-holomorphic Eisenstein series

$$E_{2k}^{(n)}(Z,s):=\sum_{\gamma}det(Im(\gamma Z))^s\mu(\gamma,Z)^{-2k}$$

where γ is summed over $P_{n,0}(Z)\setminus Sp(n,\mathbb{Z})$ and $Im(Z)=(Z-\overline{Z})/2i$.

This series converges for $Re(s)\gg 0$ and can be continued to a meromorphic function in the whole s-plane.

Let

$$\tau:H_1\times\ldots\times H_1\to H_n$$

be defined by

$$\tau(z_1,\ldots,z_n)=\begin{pmatrix}z_1&0&\ldots&0\\0&z_2&\ldots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\ldots&z_n\end{pmatrix}.$$

Let $\Gamma=SL(2,\mathbb{Z})$, $H=H_1$.

Conjecture 1. *We have the following integral representation*

$$\int_{(\Gamma\setminus H)^{15}}E_{2k}^{(15)}(\tau(z_1,\ldots,z_{15}),s)\overline{f_1(z_1)f_1(z_2)f_1(z_3)}\cdot\ldots\cdot\overline{f_5(z_{13})f_5(z_{14})f_5(z_{15})}d\mu$$

$$"="(-1)^k2^{-16s-14k+8}\pi^{-s-16k+15}L(f_1\otimes\ldots\otimes f_5,s+6k-6)$$

$$\times\frac{\prod_{i=0}^{2}\Gamma(s+(2k-1)(3-i)-3)^{\binom{5}{i}}}{\Gamma(s+2k)\zeta(2s+2k)\prod_{j=1}^{7}\Gamma(2s+4k-2j)\prod_{j=1}^{7}\zeta(4s+4k-2j)}$$

Here $z_j = x_j + iy_j$, $j = 1, ..., 15$ and $d\mu = \prod_{j=1}^{15} y_j^{2k-2} \prod_{t=1}^{15}(dx_t dy_t)$. "=" means the equality up to the possible multiple $\phi(k,s)$ (which the author can not imagine) so that Theorems 1,2 take place.

Remarks (a) The author has at present no idea how to compute the integral in the above conjecture. The method of [Gar], [Orl] is not generalized in any obvious way. As Prof. S. Böcherer and Dr. B. Heim informed the author, for the case

$$\int_{(\Gamma\backslash H)^4} E_{2k}^{(4)}(\tau(z_1), ..., z_4), s)\overline{f_1(z_1)...f_4(z_4)}d\mu$$

it appears infinite many orbits and in this case the corresponding Dirichlet series has no Euler product decomposition. In any case, such type of integral representations for tensor product L-functions seems to be not a general principle; the conjecture 1 if true, is apart from the results on triple products, rather exception (see Concluding Remarks).

(b) In the case of arbitrary level, we should in principle (similar to the case of triple products [GaH]) multiple the left hand side in conjecture 1 by the bad factors corresponding to level divisors.

(c) In general, let $f_1, ..., f_m$ be cusp forms of weights $k_1, ..., k_m$ respectively; assume $k_1 \leq k_2 \leq ... \leq k_m$.

In a case of triple products we have two, entirely different, cases to treat: $k_1 + k_2 > k_3$ and $k_1 + k_2 < k_3$. The first one was treated in [Gar], [Orl], [GaH],... The second case was considered lately by M. Harris and S. Kudla [HaK]: they established the algebraicity theorem for the central critical point.

For five cusp forms we have more cases, and our conjecture treat the case $k_1 + ... + k_4 > k_5$.

(d) The following observation is due to I. Piatetski-Shapiro [Pia].

Let k be a global field, Σ be a finite set of places, containing all the archimedean ones. Let G be any split reductive group, defined over k. Let $\pi = \otimes_p \pi_p$ be a cuspidal representation of G; assume π_p is unramified for $p \notin \Sigma$. Let ρ be an irreducible finite dimensional representation of ${}^L G$.

Let

$$L^{\Sigma}(\pi, \rho, s) := \prod_{p \notin \Sigma} L_p(\pi_p, \rho, s)$$

be the corresponding L-function.

I. Piatetski-Shapiro constructed a Poincare series $P^{\Sigma}(g, s)$ such that

$$\int_{G_k \backslash G_{\mathbf{A}}} P^{\Sigma}(g, s)\phi(g)dg = c \cdot L^{\Sigma}(\pi, \rho, s)$$

where $c \neq 0$, and ϕ is a cusp form with Whittaker model lying in automorphic representation π such that ϕ is right invariant under K_p, $p \notin \Sigma$.

But it turns out, that in general, the problem of meromorphic continuation of $P^{\Sigma}(g, s)$ is equivalent to the problem of meromorphic continuation of $L^{\Sigma}(\pi, \rho, s)$.

2. The functional equation.

In this section we deduce an explicit functional equation for $L(f_1 \otimes ... \otimes f_5, s)$ from the Conjecture 1. In what follows we need the following explicit form of functional equation for Eisenstein series on $Sp(15,\mathbb{R})$.

Proposition 1. *Let*
$F_{2k}^{(15)}(Z,s)$

$$:= \pi^{-15s}\Gamma(s+2k)\zeta(2s+2k)\prod_{j=1}^{7}\Gamma(2s+4k-2j)\prod_{j=1}^{7}\zeta(4s+4k-2j)E_{2k}^{(15)}(Z,s)$$

Then

$$F_{2k}^{(15)}(Z,8-2k-s) = -F_{2k}^{(15)}(Z,s)$$

Proof: It follows, for instance, from [Boe, Miz] and routine calculations using well known properties of Γ-function.

Put

$$M(s) := (2\pi)^{-16s-50k+25}\prod_{i=0}^{2}\Gamma(s-i(2k-1))^{\binom{5}{i}}L(f_1\otimes ...\otimes f_5, s)$$

Theorem 1. *Assume Conjecture 1 takes place. Then*

$$M(s) = -M(10k-4-s)$$

Proof: Multiplying Conjecture 1 by

$$\pi^{-15s}\Gamma(s+2k)\zeta(2s+2k)\prod_{j=1}^{7}\Gamma(2s+4k-2j)\prod_{j=1}^{7}\zeta(4s+4k-2j)$$

and applying Proposition 1, we obtain

$$2^{-16s-14k+8}\pi^{-16s-16k+15}\prod_{i=0}^{2}\Gamma(s+(2k-1)(3-i)-3)^{\binom{5}{i}}L(f_1\otimes ...\otimes f_5, s+6k-6)$$

$$= -2^{16s+18k-120}\pi^{16s+16k-113}\prod_{i=0}^{2}\Gamma((2k-1)(2-i)+4-s)^{\binom{5}{i}}L(f_1\otimes ...\otimes f_5, 4k+2-s)$$

Now the linear substitution $s \longmapsto -s+4k+2$ gives the desired functional equation.

3. Special values.

Blasius [Bla] has used Deligne's period conjecture [Del] and some period calculations to compute the expected critical values for tensor product L-functions of arbitrary many cusp forms. In the case of five forms of equal weight $2k$ we have the following

Conjecture 2.

$$\frac{L(f_1 \otimes \ldots \otimes f_5, n)}{\pi^{16n-50k+25} \prod_{i=1}^{5} < f_i, f_i >^3} \in \mathbb{Q}(f_1, \ldots, f_5)$$

for $4k - 1 \leq n \leq 6k - 3$.

Theorem 2. *Conjecture 1* $\Rightarrow$ *Conjecture 2 for* $4k + 2 \leq n \leq 6k - 6$.

The proof will occupy the remainder of this section.

Let $X + iY = Z = (z_{ij})$ be a variable on H_{15}.

Let

$$\partial_{ij} = \begin{cases} 2\frac{\partial}{\partial z_{ij}}, & \text{for } i = j \\ \frac{\partial}{\partial z_{ij}}, & \text{for } i \neq j \end{cases}$$

and

$$\triangle = i det(\partial_{ij})$$

For $k \in \mathbb{Z}_{\geq 0}$, the Maass operator

$$D_{2k} := det(Y)^{7-2k} \circ \triangle \circ det(Y)^{2k-7}$$

maps C^∞-modular forms (with respect to $Sp(15, \mathbb{Z})$) of weight k to those of weight $k + 2$.

Let

$$\epsilon_m(s) := \prod_{i=0}^{m} \left(s + \frac{i-1}{2}\right)$$

Proposition 2.

$$D_{2k} E_{2k}^{(15)}(Z, s) = \epsilon_{15}(s + 2k - 7) E_{2k+1}^{(15)}(Z, s - 1)$$

Proof: See [Sh2].

It is well known (by the work of Siegel, Feit,...) that the series $E_{2k}^{(m)}(Z, 0)$ is a holomorphic function with rational Fourier coefficients for $k \geq 1$.

For an integer $r > 0$ define $D_{2k}^{(r)} := D_{2k+2r-2} \circ \ldots \circ D_{2k+2} \circ D_{2k}$, $D_{2k}^{(0)} = 1$.

The following result follows immediately from Proposition 2.

Corollary 1. *For integers $k \geq 1$, $r \geq 2$, we have*

$$D_{2k}^{(r)} E_{2k}^{(15)}(Z,0) = c \cdot E_{2k+2r}^{(15)}(Z,-r)$$

with a non-zero rational constant c.

Also immediately from the Corollary 1 and the definition of $\triangle$, we obtain

Corollary 2. *For $k \geq 1$ and $0 \leq r \leq k-1$*

$$E_{2k}^{(15)}(\tau(z_1,...,z_{15}),-r) = \pi^{15r} \sum_{j=0}^{15r} Q_j \left(\frac{1}{y_1},...,\frac{1}{y_{15}}\right) \pi^{-j} h_j(z_1,...,z_{15})$$

where Q_j is a homogeneous polynomial with rational coefficients and $h_j(z_1,...,z_{15})$ is a holomorphic function with rational Fourier coefficients. Moreover $degQ_j = j$ and no $\frac{1}{y_j}$ appears to a power higher than r.

Now we can finish the proof of Theorem 2:

From Conjecture 1 and Corollary 2 we obtain for $0 \leq r \leq k-1$:

$L(f_1 \otimes ... \otimes f_5, -r+6k-6)$

$$= a\pi^{46k-16r-71} \int_{(\Gamma\backslash H)^{15}} \sum_{j=0}^{15r} Q_j \left(\frac{1}{y_1},...,\frac{1}{y_{15}}\right) \pi^{-j} h_j(z_1,...,z_{15})$$

$$\times \quad \overline{f_1(z_1)f_1(z_2)f_1(z_3)} \cdot ... \cdot \overline{f_5(z_{13})f_5(z_{14})f_5(z_{15})} d\mu$$

with a non-zero rational number a.

Write $n = -r + 6k - 6$. Then $5k-5 \leq n \leq 6k-6$ (i.e. "shifted half the critical strip").

Now, in order to obtain Conjecture 2 for this part of the critical strip, we can apply the method of Shimura [Sh1]. The remaining special values (excluding the values $4k-1$, $4k$, $4k+1$, $6k-5$, $6k-4$, $6k-3$) may be obtained by using the functional equation in Theorem 1.

4. Concluding remarks.

(i) First we explain why in general one probably cannot expect such type of integral representation for $L(f_1 \otimes ... \otimes f_m, s)$, $m \geq 7$ odd.

In fact, if one require that from such an integral representation one can deduce functional equation for the corresponding L-function and calculate a part of critical values, then we must have

$$\int_{(\Gamma\backslash H)^M} E_{2k}^{(M)}(\tau(z_1,...,z_M),s)\overline{f_1(z_1)...f_1(z_N)}...\overline{f_m(z_{M-N+1})...f_m(z_M)}d\mu$$

$$= rational \times 2^X \pi^Y L(f_1 \otimes ... \otimes f_m, s+\alpha) \times \Gamma - factor$$

Here $N = \frac{1}{2}\binom{m-1}{\frac{m-1}{2}}$, $M = mN$. But $\alpha = \frac{1}{2}\left((2k-1)(m+1) - \frac{M-3}{2}\right) \in \frac{1}{4}\mathbb{Z}$ doesn't belong to $\mathbb{Z}$ for $m \geq 7$. As in general we have no rationality theorem for $E_{2k}^{(m)}(Z,\beta)$, for $\beta \in \frac{1}{4}\mathbb{Z}$, we cannot apply the Shimura method.

(ii) Here we would like to say a few words about possible Γ-factors. On the one hand we have Conjecture 1 (and Γ-factor occuring there). On the other hand, we may try to define Γ-factor directly, generalizing Garrett's construction (see [Gar; Proposition 5.1]).

More precisely, for $s, x \in \mathbb{C}$, $k \in \mathbb{Z}$ put

$$\eta_{2k}^{(M)}(s) = \int_0^\infty ... \int_0^\infty (y_1...y_M)^{s+2k-2}(y_1+...+y_M)^{1-2k-2s}$$
$$\times q_{2k,s}^{\star}(y_1+...+y_M)exp(-2\pi(y_1+...+y_M))dy_1...dy_M$$

where

$$q_{2k,s}(x) = |x+i|^{-2s}(x+i)^{-2k}$$

and

$$q_{2k,s}^{\star}(y) = \int_{-\infty}^{\infty} q_{2k,s}(x)exp(-2\pi ixy)dx$$

Proposition 3. *We have*

$$\eta_{2k}^{(15)}(s) = (-1)^k 2^{30-28s-58k}\pi^{15-13s-28k}$$
$$\times \frac{\Gamma(s+2k-1)^{15}\Gamma(15s+30k-15)\Gamma(s+4k-2)}{\Gamma(3s+6k-3)\Gamma(14s+28-14)\Gamma(s+2k)}$$

Proof: Under the same lines as in proof of [Gar; Prop.5.1].

Therefore $\eta_{2k}^{(15)}(s)$ is, in fact, far from being the one in Conjecture 1. However, as we have no analogue of decomposition result [Gar; Prop.3.4] proved, we can say nothing more on that matter at the moment.

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