TWO LECTURES ON LOCAL RIEMANNIAN GEOMETRY, $Spin^{\mathbb{C}}$ - STRUCTURES AND SEIBERG - WITTEN EQUATION

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Introduction

The mathematical part of Seiberg - Witten theory is now a powerful tool in the framework of smooth geometry of low dimensions. Using the celebrated equations introduced by E. Witten in [1] one could establish a number of new results and reprove known ones. In the original paper [1] the author first of all reproves Donaldson fundamental results which ensure us that there are exist homeomorphic but not diffeomorphic smooth compact 4 - dimensional manifolds. The key point is that for any Kahler manifold one has nontriviality of the invariant (which is called Seiberg - Witten invariant) while for almost all "topomodels" the invariant is trivial (we'll give all the definitions and constructions below). This fact itself destroys a number of physically - philosophical speculations about the structure of Universe; usually they discussed what is the topological structure and what is the manifold which corresponds (and represents) our world. But now if say the famous Poincare smooth conjecture is not true (one can suppose it because despite of the huge attempts to prove it we have not yet any answer on this question) then the discussion would be illuminated by new colors. The problem arose is not only what is the manifold but additionally which smooth structure plays on. For example usual $\mathbb{R}^4$ looks like a bad joke: Donaldson proved that it admits continuum set of different smooth structures. And it doesn't matter what a metric one uses on this $\mathbb{R}^4$, euclidean or lorentzian.

But not only known results were proved using the new invariant. For the Donaldson theory it was usual to compute the Donaldson invariants in the framework of algebraic geometry. Almost all examples in [23] come in this way. It was established by Taubes that Seiberg - Witten invariants are related as well with symplectic geometry — in [3] he proves that for any symplectic manifold with $b_2^+ > 1$ the associated canonical class is the basic one. The proof was so simple and natural that it was conjected that the symplectic geometry is exactly the subject containing all non trivial cases for the Seiberg - Witten invariants. This suggestion was disproved almost immediately by Taubes himself together with D. Kotschik and J. Morgan in [5], but the counterexample is a 4 - dimensional manifold with nontrivial fundamental group. It means that one can correct the suggestion just imposing the condition of simply connectedness.

There are many courses now in the Seiberg - Witten theory (we mention in the References a number of these lectures or surveys) so I hadn't in mind to present a textbook composed of the materials which can be derived from the library. I want just to add some geometrical facts and constructions whose presence simplifying

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the reading of papers on the Seiberg - Witten theory. The most attention will be payed to the notion of $Spin^{\mathbb{C}}$ - structure and to the relation between complex and spinor geometries. The second aim is to ask some natural questions arise during the discussion which are from my point of view meaningful mathematical problems themselfs.

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1. The complex geometry and the spinor geometry

Let X be a smooth orientable compact riemannian manifold of real dimension 4. Topological data, characterized X, are its fundamental group $\pi_1(X)$, cohomology groups $H^i(X, \mathbb{Z})$ whose ranks are usually called Betti numbers $b_i(X)$ of X and the intersection form Q_X which is an integer quadratic form on the lattice $H^2(X, \mathbb{Z})$ whose definition requires the choice of an orientation over X. Thus in simply connected case it remains only the integer lattice and the intersection form encoding the topological information about X. In the gauge theories (in the framework of differential geometry one usually combines the Donaldson and the Seiberg - Witten theories using the term "gauge theories") it's quite normeous to consider only the case when the underlying manifold X has trivial fundamental group. Once again in this case one has only lattice data. The changing (or reversing) of the orientation multiplies the intersection form by -1. If an appropriate orientation is fixed then one can reduce the intersection form Q_X in enlarged space $H^2(X, \mathbb{R})$ to a diagonal form and take the numbers of positive and negative squares. These numbers are denoted as $b_2^+(X)$ and $b_2^-(X)$; the sum of these numbers is equal to the second Betti number $b_2(X)$.

Example. The simplest examples are S^4 and $\mathbb{CP}^2$ — 4 - dimensional real sphere and complex projective plane. The last one being considered as a 4 - dimensional real manifold is endowed with the canonical orientation coming from the complex structure. Both the 4 - dimensional manifolds have trivial fundamental groups. The sphere has trivial second Betti number so the lattice $H^2(S^4, \mathbb{Z})$ and consequently the intersection form are trivial. In the projective plane one has a well known 2 - dimensional submanifold — projective line, which is from the real geometry point of view just a 2 dimensional sphere. We know from the standard algebraic geometry that every two projective lines either coincide or have exactly one mutual point. This intersection point is a "complex" point that is the index of intersection is equal to 1. It follows from the positivity of the complex geometry. Thus the lattice $H^2(\mathbb{CP}^2, Z)$ is isomorphic to $Z[h]$ with the natural intersection form

$$Q(k[h], l[h]) = k \cdot l.$$

This intersection form is usually denoted as 1. In this case $b_2^+ = 1, b_2^- = 0$. Below we'll meet $\overline{\mathbb{CP}}^2$ — the projective plane with reverse orientation. This manifold plays quite important role in the constructions. It's clear that it has intersection form -1.

One has the tangent bundle TX over our real based manifold which as a real bundle has the following characteristic classes: the Shtiffel - Witney classes w_i belong to $H^i(X, \mathbb{Z}_2)$, the Pontryagin class $p_1(X)$ belongs to $H^2(X, \mathbb{Z})$ and the Euler class which gives us the euler characteristic χ of our based manifold X. This euler characteristic is related with the Betti numbers as follows

$$\chi(X) = b_0(X) - b_1(X) + b_2(X) - b_3(X) + b_4(X) = 2(b_0(X) - b_1(X)) + b_2^+(X) + b_2^-(X).$$

Here we consider the connected case so $b_0(X) = 1$. On the other hand the Pontryagin class is related to the signature of the based manifold

$$\sigma(X) = b_2^+ - b_2^- = -3 \int_X p_1(X).$$

Coming further let us suppose that our based manifold admits an almost complex structure I so is a nondegenerated operator

$$I : TX \to TX$$

with square

$$I^2 = -id.$$

Such an operator can exist even if X doesn't admit a complex structure. Here and below we establish that the existence of an almost complex structure over X is pure topological (or arithmetical) fact. Let us now examine the case when such a structure exists. It means that the real bundle TX endowed with I is a complex rank 2 bundle $T^{1,0}X$ which has as a complex rank 2 bundle the first and the second Chern classes c_1 and c_2. The first Chern class is fixed modulo two by the following condition

$$c_1 = w_2(X)(mod 2)$$

being an integer lifting of the second Shtiffel - Whitney class $w_2(X)$. At the same time we can realize the Pontryagin class $p_1(X)$ by the definition as the second Chern class $c_2(T_{\mathbb{C}}X)$ of the complexified tangent bundle $T_{\mathbb{C}}X$ (let us recall that odd Chern classes of any complexified bundle are trivial). So

$$T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X = T^{1,0}X \oplus \overline{T^{1,0}X}$$

and it follows that

$$c_2(T_{\mathbb{C}}X) = 2c_2 - c_1^2.$$

The integration of the last formula over X gives us the following relationship

$$c_1^2 = 2\chi(X) + 3\sigma(X),$$

which is called Wu formula. The arithmetics comes with the statement of Wu theorem which claims that the existence of an almost complex structure over a 4 - dimensional manifold is equivalent to the existence of solutions to the following pure arithmetical conditions:

$$c_1 = w_2(X)(mod 2) \qquad and \qquad c_1^2 = 2\chi + 3\sigma.$$

We've proved the theorem in one direction, using a brilliant result of Hirzebruch, who proved that

$$\int_X p_1(TX) = -3\sigma(X).$$

Example. Really it was unknown in 50th is there any almost complex structure over S^4 or not. Nowadays we can solve this problem almost immediately following F. Hirzebruch. Let us compute the number

$$\kappa = 2\chi + 3\sigma$$

for 4 - dimensional sphere S^4. We have

$$\kappa(S^4) = 2(2 - 0 + 0) + 3(0) = 4.$$

At the same time S^4 has no nontrivial 2 - cohomological classes with nontrivial squares. So the answer to the old problem is dictated by this pure arithmetical fact.

At the same time for $\mathbb{CP}^2$ we have

$$\kappa(\mathbb{CP}^2) = 2(2 - 0 + 1) + 3(1) = 9.$$

Therefore one gets

$$c_1 = \pm 3[h]$$

and this one is not an unexpected answer.

At the same time for the reverse oriented manifold $\overline{\mathbb{CP}}^2$ the computation gives

$$\kappa(\overline{\mathbb{CP}}^2) = 2(2 - 0 + 1) + 3(-1) = 3,$$

but every 2 - cohomological class in $H^2(\overline{\mathbb{CP}}^2, \mathbb{Z})$ has nonpositive square.

Now we want to realize the numbers which come to the signature in a different style. Namely let us fix a riemannian metric g over X. Together with the fixed orientation it gives us the Hodge star operator:

$$* : \Omega^i_X \to \Omega^{4-i}_X;$$
$$< \beta, *\alpha >_g d\mu = \beta \wedge \alpha$$

for any $\beta \in \Omega^{4-i}_X$ where $d\mu$ is the volume form. This operator could be exploited to deform the usual de Rham complex where the role of ordinary d plays the formal adjoint operator d^* which is a conjugation of d by the Hodge star operator. A famous Hodge theorem tells us that for every $i = 0, ..., 4$ one can decompose the space of i - forms into the following direct sum

$$\Omega^i = \mathcal{H}^i \oplus d\Omega^{i-1} \oplus d^*\Omega^{i+1}$$

and this decomposition is orthogonal with respect to the scalar product, induced by the riemannian metric g. The first space $\mathcal{H}^i$ consists of harmonic i - form which satisfy

$$\mathcal{H}^i = \{\rho \in \Omega^i_X \quad | \quad d\rho = d^*\rho = 0\}.$$

The space has finite dimension equals to b_i. So every cohomological class is represented uniqually by a harmonic form.

Further, 2 - forms over 4 - dimensional manifold have a specialty in the framework of the Hodge theory namely it's not hard to see that the Hodge star operator being considered on Ω^2_X is an involution

$$* : \Omega^2_X \to \Omega^2_X, \qquad *^2 = id$$

(it follows from the fact that the wedge product is symmetric on Ω^2_X). Thus one can decompose the space of 2 - forms into the direct sum of eigenspaces of the Hodge star operator. Let Ω^+ be the eigenspace with eigenvalue $+1$ while Ω^- corresponds to -1. It means that for every "positive" 2 - form $\alpha \in \Omega^+$ the wedge product is positive

$$al \wedge \alpha = |\alpha|^2_g d\mu$$

and for every "negative" $\beta \in \Omega^-$ the wedge square is negative

$$\beta \wedge \beta = -|\beta|^2_g d\mu.$$

Moreover one takes the intersections

$$\mathcal{H}^\pm = \mathcal{H}^i \cap \Omega^\pm.$$

The space Ω^+ is called the space of self dual forms while Ω^- is called the space of anti self dual forms. Since the decomposition is pure local it induces the corresponding decomposition of the bundle $\Lambda^2 T^* X$ into the direct sum of two subbundles

$$\Lambda^2 T^* X = \Lambda^+ \oplus \Lambda^-.$$

Let us recall that in dimension 4 all the decompositions and notions are comformally invariant so they depend only on conformal class of the riemannian metric rather then on riemannian metric itself.

As it was stated in a famous Hodge theorem every cohomological class is represented by a harmonic form with respect to every riemannian metric. But the case "2 in 4" is special, and one could ask whether or not a given 2 - cohomological class $[al] \in H^2(X, \mathbb{Z})$ can be realized by a self dual (or anti self dual) harmonic 2 - form. A priori we have only one type of 2 - classes which definitely cann't be represented by self or anti self dual forms namely isotropical 2 - classes with zero squares. For any 2 - cohomological class $\eta \in H^2(X, \mathbb{Z})$ with positive square one has an open question in smooth 4 - geometry: is there exist such riemannian metric (more rigouresly - such conformal class of riemannian metrics) that the harmonic with respect to this metric form α_η, representing the class, is self dual. The same question takes place for any 2 - cohomological class with negative square. These two questions are closely related since the reversing of the orientation interchanges the notions of self duality and anti self duality. But we want to formulate an important unsolved problem in 4 - dimensional differential geometry in more general and geometric form. To do this we need to introduce the so called real period map.

Let X be a smooth compact orientable riemannian 4 - manifold with signature

$$\sigma = b_2^+ - b_2^-.$$

The standard slant product (which gives us the intersection form Q_X) extended to the space $H^2(X,\mathbb{R})$ gives us a bilinear pairing on this real vector space. We will call a b_2^+ - dimensional subspace $V \subset H^2(X,\mathbb{R})$ positive if the restriction of this bilinear pairing is strictly positive on V. At the same time one has the notion of negative subspace of dimension b_2^- but for every positive V^+ one has unique negative V^- taking the orthogonal compliment of the first one. Now let us take the positive Grassmann manifold $Gr^+(H^2(X,\mathbb{R})$ consists of all positive b_2^+ subspaces. It is canonically isomorphic to the negative Grassmann $Gr^-(H^2(X,\mathbb{R})$ so one can remove the choice of orientation identifying these Grassmann manifolds. If we choose a riemannian metric (or a conformal class) over X we get two points $p^\pm \in Gr^\pm(H^2(X,\mathbb{R}))$ which correspond to the subspaces of self dual and anti self dual harmonic 2 - forms respectively. Again if we forget about the choice of orientation then we get identifying the Grassmann manifolds and the points a map

$$P : \mathcal{R} \to Gr^+(H^2(X,\mathbb{Z})),$$

where $\mathcal{R}$ is the space of all riemannian metric the given smooth structure on X admits which is called real period map. It is a real analogy of the celebrated period map in algebraic geometry.

S. Donaldson (see [23]) proved that this map is locally surjective. It means that for every point $p \in Gr^+(H^2(X,\mathbb{R}))$ realized by say self dual harmonic subspace in Ω^2_X there exists a small neighborhood consists of points which can be realized as well. But generally one has the following important:

Problem. *Is the real period map globally surjective?*

It would be strange if it is not so but nevertheless nobody knows now is it true or not. I have to say here that this problem is in some sense an analogy of the famous Hodge conjecture in algebraic geometry. These two problems are quite parallel: in the algebraic geometrical setup the condition that a cohomological class has (1, 1) - type is of the same level as in the differential geometrical setup the condition that a cohomological class lies on a positive subspace. So if we thought of these two conditions as analogous conditions (cohomological conditions) then these two problems are about realizations of these classes by real objects: some algebraic subvariety and some harmonic self dual (or anti self dual) form.

Thus we've discussed what a riemannian metric chosen over X brings to the picture. But it is only one part; let us turn to the second. We call this part "twistor and spinor geometry". As we'll see it is extremely close to complex geometry but usually isn't considered in the framework of the last one.

Let one take a riemannian metric g over our based smooth manifold X and fix an appropriate orientation. Then one gets two projective bundles $\mathbb{P}^\pm \to X$ which is called twistor bundles. We would like to give below two description of these bundles having in mind to relate closely spinor and complex geometries. The first description is quite classical: over each point $x \in X$ one takes the set of all complex structures which are compatible with the given riemannian metric g. Identifying the fiber T_xX of the tangent bundle with standard $\mathbb{R}^4$ and reduce the riemannian metric to the standard form one gets the standard description of the set in terms if operators I, J, K (the standard quaternionic generators) so any compatible complex operator can be represented as

$$A = \alpha I + \beta J + \gamma K, \qquad \alpha, \beta, \gamma \in \mathbb{R}$$

such that

$$\alpha^2 + \beta^2 + \gamma^2 = 1.$$

Thus one gets a 2 - dimensional sphere which is topologically isomorphic to projective line. Now one can change the given orientation and consider the complex structures which are compatible with g and with the reverse orientation. This gives us again projective line. Let us distinguish these two lines assigning $+$ to the first one and $-$ to the second. Generalizing this local picture over all X one gets two projective bundles

$$\mathbb{P}^{\pm} \to X,$$

such that each fiber consists of complex operators compatible with our metric and given (or reverse) orientation. This definition is the classical one. The total space of the projective bundle $\mathbb{P}^- \to X$ is called "twistor space" (the choice of the sign is conventional; for this twistor space one has the wonderful relationship between instantons over X and stable holomorphic vector bundles over the twistor space, see [12]). It is a real 6 - dimensional manifold fibered over X on 2 - dimensional sphere. Moreover this 6 - dimensional real manifold is canonically endowed by an almost complex structure. Since we want to formulate a relative problem in a future let us recall how one gets this almost complex structure.

Every fiber has its own complex structure being a complex projective manifold (we will show in a moment why the 2 - sphere has the fixed structure). Our riemannian metric g defines the Levi - Civita connection which induces a special connection on the projective bundle $\mathbb{P}^-$. If we denote as Y the total space of the projective bundle then the tangent space T_yY in arbitrary point $y \in Y$ is decomposed due to the induced connection into two direct summands

$$T_yY = V \oplus W \qquad (*)$$

where V is the tangent space to the fiber in this point and W is isomorphic to T_xX where x is the image of y with respect to the canonical projection

$$\pi : Y \to X.$$

Note that the first summand V has a complex operator coming from the complex structure on the fiber. On the second summand W one takes the complex operator which corresponds to the point y of the fiber (all points in the fibers are complex operators on T_xX). Thus the direct sum of the operators gives us a complex operator on T_yY. Globalizing this picture one gets an almost complex structure over all Y. This is really twisted structure since it isn't constant along the fibers but is twisted along these. It's a well known fact (see f.e. [12]) that the integrability condition for this almost complex structure is satisfied if and only if the started riemannian metric is anti self dual.

Example. There are two mostly used examples of integrable twistor spaces: the first one is for S^4 with the standard conforamally flat riemannian metric (the corresponding twistor space is $\mathbb{CP}^3$) and the second is for $\mathbb{CP}^2$ with the standard Kahler metric (here one gets the flag variety, see [12]).

Now let us discuss the second description of the twistor bundles. For this again consider the riemannian case locally. Let $x \in X$ is a point of X. The tangent space $T_x X$ is a real space endowed with symmetric positive bilinear form induce by fixed riemannian metric g. Let complexify the space $T_x X$ and then projectivize it. During the complexification we extend g_x on $T^{\mathbb{C}}_x X$ as a complex bilinear form. So we get on the complex space $T^{\mathbb{C}}_x X$ not a hermitian form but a complex metric (in the style of [14]). This complex bilinear form $g^{\mathbb{C}}_x$ gives us after the projectivization a nondegenerated quadric

$$Q_x \subset \mathbb{CP}^3 = \mathbb{P}(T^{\mathbb{C}}_x X),$$

consists of isotropical vectors of $g^{\mathbb{C}}_x$. Moreover one has a standard real structure Θ on this $\mathbb{CP}^3$ inherited during the complexification (since our space was real before). It's not hard to see that

- our quadric Q_x is a real quadric since our metric g was real before complexification

and

- it hasn't real points at all since our metric g being riemannian doesn't admit any real isotropic vectors.

Thus locally riemannian geometry in dimension 4 gives at each point the following data: $\mathbb{CP}^3$ with a standard real structure and a real quadric $Q \subset \mathbb{CP}^3$ without real points.

It's well known fact (see f.e. [13]) that every nondegenerated quadric in 3 - dimensional projective space is the direct product of two projective lines

$$Q = \mathbb{P}^+ \times \mathbb{P}^-.$$

One can understand it more geometrically: there are two family of projective lines lie on the quadric and $\mathbb{P}^+$ and $\mathbb{P}^-$ parameterize the families. So points of $\mathbb{P}^+$ one can understand as projective lines lie on Q and belong to the first family while $\mathbb{P}^-$ consists of projective lines from the second one. One can distinguish these two family due to the fixed orientation. As we will see the first family just corresponds to complex operators compatible with the given orientation while the second one is associated with reverse orientation. To see this more explicitly let us study what is a compatible complex structure in this local picture.

Let I be a local almost complex structure, compatible with the given riemannian metric and orientation. When we complexify the tangent space $T_x X$ we can decompose the complexified space into the direct sum of two part with respect to by - types:

$$T^{\mathbb{C}}_x X = T^{1,0}_x \oplus T^{0,1}_x,$$

where the first summand is the eigenspace of I with eigenvalue $\imath$ and the second one is the eigenspace with eigenvalue $-\imath$ as usual. One can see that this decomposition defines our operator I uniquely. After the projectivization this gives us two projective lines in our $\mathbb{CP}^2$ namely

$$\mathbb{P}(T^{1,0}_x), \mathbb{P}(T^{0,1}_x) \subset \mathbb{CP}^3.$$

Denote these lines as l_I and $\bar{l}_I$. We know which one is the first and which one is the conjugation of the first one by our standard real structure Θ:

$$\Theta(l_I) = \bar{l}_I.$$

It's clear that these two projective lines have no mutual points. Therefore local picture for complex geometry in dimension 4 is as follows: the projective space $\mathbb{CP}^3$ together with a standard real structure and a pair of conjugated projective lines without real points.

The next step is to combine these two local picture: what is a riemannian metric with a compatible almost complex structure? The compatibility condition for g and I is quite simple: riemannian metric g and complex structure I are compatible if and only if at each point of X the projective lines $l_I, \bar{l}_I$ lie on the quadric Q. But since the intersection of l_I and $\bar{l}_I$ is trivial these two projective lines belong to one family. If I is compatible with given orientation then $l_I, \bar{l}_I \in \mathbb{P}^+$. If it induces reverse orientation then $l_I, \bar{l}_I \in \mathbb{P}^-$. Therefore the projective line $\mathbb{P}^+$ parameterizes complex structures compatible with given orientation and $\mathbb{P}^-$ respects for almost complex structures induce reverse orientation. Thus we get here two 2 - spheres which were called above "twistor bundles". But now we get much more important information about these twistor bundles, namely from the local picture in $\mathbb{CP}^3$ we derive that these $\mathbb{P}^\pm$ are endowed with their own complex structures. Moreover, our projective bundles are $PU(2)$ - bundles. They automatically carry not only own complex structures but as well every fiber has a kahler structure! To see this explicitly we have to add to the picture the action of our standard real structure Θ. Since our quadric is real this means that one has two possibilities for the families of projective lines on it: either Θ maps one family to the other or Θ maps each family to itself. The first case is forbidden because of riemannian property of our metric. If Θ maps $\mathbb{P}^+$ to $\mathbb{P}^-$ then it has to be real points on Q! Really, if one takes a line l say from $\mathbb{P}^+$ and if $\Theta(l)$ belongs to $\mathbb{P}^-$ then it would be the intersection point $p = l \cap \Theta(l)$ and it should be real. It implies that

$$\Theta(\mathbb{P}^\pm) = \mathbb{P}^\pm.$$

Further, the action of Θ on $\mathbb{P}^\pm$, preserving the lines, induces a pair of quaternionic real structures $\theta^\pm$ on the lines. Let us recall that a quaternionic real structure as an anti holomorphic involution on $\mathbb{CP}^m$ where m is odd which has the following form in appropriate homogeneous coordinates

$$\theta(z_0 : z_1 : ... : z_{m-1} : z_m) = (-\bar{z}_1 : \bar{z}_0 : -\bar{z}_m : \bar{z}_{m-1})$$

(while a standard real structure exists on any projective space and has the form

$$\Theta(z_0 : ... : z_m) = (\bar{z}_0 : ... : \bar{z}_m)).$$

Of course the points l_I and $\bar{l}_I$ are conjugated by θ^+.

It is an additional ingredient which we didn't use yet: we've considered the bilinear extension of the riemannian metric g to the complexified space $T^{\mathbb{C}}_x X$ but as well one has the sesqueliniar extension makes the complex space $T_x^{\mathbb{C}}X$ into a hermitian space and consequently makes the projective space $\mathbb{CP}^3$ into a kahler manifold endowed with a kahler metric with a kahler form. Of course this kahler metric by the construction is invariant under the action of our standard real structure Θ. Therefore it splits with respect to the decomposition

$$Q = \mathbb{P}^+ \times \mathbb{P}^-$$

and induces two kahler metrics on $\mathbb{P}^{\pm}$. We have to emphasize here that these kahler structures depend on riemannian metric rather then the previously described objects depend only on its conformal class. It is the striking difference between conformal and riemannian geometries over twistor spaces. To illustrate the difference we would like to propose a problem which goes quite parallel with the classical solved problem mentioned above about integrabily of the twistor almost complex structure over a twistor space. Now let us emphasize again that we fix a riemannian metric on our based manifold X. Then on the twistor space $Y \to X$ we have a hermitian triple (G, I, Ω) consists of a riemannian metric G, a compatible almost complex structure I and the corresponding almost kahler form Ω. The second element is the same as in the classical twistor construction and has been described above. To construct the triple one can either describe the first element or the third one because in a hermitian triple every two elements uniquely define the third element. Using again the decomposition (*) induced by the Levi - Civita connection one takes at the point $y \in Y$ the following direct sum of the kahler form on the fiber and the "twistor" 2 - form on the horizontal component which corresponds to the given riemannian metric g and the complex structure operator represented by the point of the fiber. Totally we get a nondegenerated globally defined 2 - form Ω on Y which has type (1,1) with respect to the twistor almost complex structure I and it is a positive form. This means that it together with I defines a riemannian metric G over the twistor space Y and thus the hermitian triple (G, I, Ω) is constructed.

Thus one could say that the difference between conformal and riemannian geometries can be formulated on the twistor space level as the difference between almost complex and almost kahler geometries. It implies the following very natural and important question.

Problem. *When the induced by a riemannian metric g almost kahler form Ω on the twistor space Y is closed so is a symplectic form?*

The solution to this problem would open a way to get the twistor construction for the Seiberg - Witten theory. Let us recall that for instantons the twistor construction gives us an important relationship between anti self dual connections and holomorphic vector bundles over the twistor space. So the paradigm for the Donaldson theory is the geometry of stable holomorphic vector bundles. At the same time there is now a very popular idea which tries to explain the mirror duality comparing holomorphic and symplectic geometries. If one has two mirror manifolds then it should be a map from the holomrphic objects of the first one to a symplectic objects of the second manifold and vice versa. On the other hand there is a miracle duality between the Donaldson theory and the Seiberg - Witten theory which was explained physically by E. Witten and mathematically by A. Tyurin and V. Pidstrigach. But the explanations give us only numerical duality so as the values of the invariants coincide. But if one believes that more strong then pure numerical duality takes place it should be a construction relates the theories on the twistor level, and this duality on the twistor level would be reflected by a duality between holomorphic and symplectic setups. It is just a suggestion now but anyway the problem formulated above is quite interesting and meaningful itself.

Now the time of spinor geometry comes. Since our based manifold is orientable then according to an old classical result of Hirzebruch and Hopf every projective bundle can be lifted to a vector bundle. Such a lifting depends on the choice of the first Chern class. Namely if we choose a class $c \in H^2(X, \mathbb{Z})$ such that

$$c = w_2(X)(mod 2)$$

where $w_2(X)$ is the second Shtiffel - Whitney class $w_2(X) \in H^2(X, \mathbb{Z}_2)$ of X then we automatically get the corresponding lifts of $\mathbb{P}^{\pm}$ to a pair of $U(2)$ - bundles $W^{\pm}$ with the first Chern classes

$$c_1(W^+) = c_1(W^-) = c.$$

This choice is called a choice of $Spin^{\mathbb{C}}$ - structure on X. For every such lifting the pair $W^{\pm}$ are endowed automatically by hermitian structures induced by our riemannian metric g and satisfy the following fundamental for the theory properties:

1) the homomorphism bundle $Hom(W^-, W^+)$ is isomorphic to $T^{\mathbb{C}}X$, moreover, the "real" part $Hom_J(W^-, W^+)$ consists of the maps which preserve both hermitian structures is isomorphic to TX;

2) the adjoint bundles $adW^{\pm}$ are isomorphic to $\Lambda^{\pm}$ modulo diagonal parts where $\Lambda^{\pm}$ are the direct summands in the decomposition of $\Lambda^2 T^* X$ induced by the Hodge star operator.

Let us recall that for a given hermitian vector bundle one can associate the corresponding principle bundle and then represent the last one using the adjoint representation. The resulting adjoint bundle for $W^{\pm}$ is locally defined by skew hermitian endomorphisms. If ϕ is a spinor field so is a section of say $\bigwedge^{2+}$ then one has using the hermitian structure a endomorphism

$$a_{\phi} = \phi \otimes \bar{\phi} \in EndW^+,$$

which preserves the hermitian structure. If our spinor bundle W^+ moreover is $SU(2)$ - bundle so the determinant line bundle $detW^+$ is trivial then adW^+ is exactly isomorphic to Λ^+ since one has a local isomorphism of $SU(2)$ and $SO(3)$. This case is special; this means that our chosen $Spin^{\mathbb{C}}$ - structure $c \in H^2(X, \mathbb{Z})$ is trivial and it's possible if and only if the second Shtiffel - Whitney class $w_2(X)$ is trivial. One calls the manifolds with trivial $w_2(X)$ spin manifolds. The distinguished lifting of the twistor bundles to $SU(2)$ - bundles is called the choice of $Spin$ - structure. And repeating again in this spin - case one has exact isomorphisms

$$adW^{\pm} \equiv \Lambda^{\pm}.$$

We would like to generalize these original relationships to the case when $W^{\pm}$ have non trivial determinant. For this in the framework of the Seiberg - Witten theory it's sufficient to construct the following quadratic maps

$$q : W^{\pm} \to \Lambda^{\pm}$$

based again on complex geometry. Namely as we have seen for our given metric g one can define compatible local almost complex structures in terms of spinor geometry. If $\phi \in \Gamma(W^+)$ is a section which doesn't vanish in a neighborhood $O(x)$ of a chosen point $x \in X$ then it defines a compatible almost complex structure I_ϕ. Namely the projectivization of the section gives us a local section of the twistor bundle $\mathbb{P}^+$ and this is clearly as we have seen above an almost complex structure defined in the neighborhood $O(x)$. Thus one has the corresponding self dual 2 - form ω, reconstructed from g and I_ϕ. But this is a feature of twistor geometry; for spinor geometry all the story is lifted to the vector level therefore one has a correspondence between two multiplications: every spinor field can be multiplied by a local complex function while every self dual form can be multiplied by a local real function. Then if we realize the corresponding section of the twistor bundle as a normalized spinor field with a fixed phase then the multiplication by a complex function z of the spinor field induces the multiplication by $z\bar{z}$ of the distinguished self dual 2 - form. Since $detW^+$ is not trivial then one couldn't choose an apporpirate phase fixing simultaneously over X; nevertheless if one takes the bundle adW^+ as a subbundle of $EndW^+$ consists of skew hermitian endomorphisms then going further one can take inside of $EndW^+$ the projection of adW^+ to the subspace of traceless endomorphisms. This gives us the possibility to compare spinor fields with self dual forms; reversing the orientation one gets the same story for adW^- and Λ^-.

At the same time we can continue our discussion of the Wu theorem which we've proved in one direction. Now to complete the proof let us just mention here that the question of the existence of a compatible almost complex structure for a given riemannian metric g is pure topological. Namely if there exists a lifting of the twistor bundle $\mathbb{P}^+$ to such spinor bundle W^+ that it admits a nonvanishing smooth section ϕ_0 then there exists globally defined compatible almost complex structure. And such a section exists if and only if the second Chern class $c_2(W^+)$ is trivial. Therefore if a numerical condition holds for a $Spin^{\mathbb{C}}$ - structure $c = c_1(W^{\pm}) \in H^2(X, \mathbb{Z})$ then there exists a nonvanishing spinor field and consequently a globally defined almost complex structure.

Using these arguments together with some additional remarks one can establish that if one chooses an appropriate class $c \in H^2(X, \mathbb{Z})$ to lift $\mathbb{P}^{\pm}$ to vector bundles $W^{\pm}$ then determinently

$$\begin{aligned} c_2(W^+) &= \frac{1}{4}(c^2 - 2\chi - 3\sigma), \\ c_2(W^-) &= c_2(W^+) + \chi, \end{aligned}$$

where as usual χ is the euler characteristic of X and σ is the signature. Thus one should see that topologically $Spin^{\mathbb{C}}$ - structure $c \in H^2(X, \mathbb{Z})$ is a generalization of the notion of canonical class.

To finish this lecture let us illustrate the properties of spinor bundles mentioned above on the projective level.

First of all, we have $TX = Hom_J(W^-, W^+)$. Returning to our $\mathbb{CP}^3$ endowed with a standard real structure Θ and containing a nondegenerated real quadric Q without real points we have to represent each point of $\mathbb{CP}^3$ as a map from $\mathbb{P}^-$ to $\mathbb{P}^+$ where $\mathbb{P}^\pm$ as usual parameterize two families of projective lines on Q. The map ψ_p is defined as follows: for every projective line $l \in \mathbb{P}^-$ you take the projective plane $\pi(p, l)$ which is spanned by the point p and the line l. This plane intersects the quadric Q in a plane curve of dimension 2 as it is predicted by Bezout theorem. But this intersection curve has a special type since l itself lies on Q. This means that the intersection consists of exactly two projective lines: line $l \in \mathbb{P}^-$ and a projective line $\psi_p(l)$ which intersects l and lies on Q therefore it contains in $\mathbb{P}^+$. Thus we defined the map

$$\psi_p : \mathbb{P}^- \to \mathbb{P}^+$$

for each point $p \in \mathbb{CP}^3$ say away from the quadric Q. Really we want to construct these maps only for real points of $\mathbb{CP}^3$ which correspond to TX rather then to $T^{\mathbb{C}}X$. It means that we can leave the question about the maps for the points of Q turning to the reality question for ψ_p if p is real. But this question is almost trivial: if p is a real point of $\mathbb{CP}^3$ then it remains unchanged during the action of our standard real structure Θ. But as we mentioned above the action induces two quaternionic real structures on $\mathbb{P}^\pm$. Then we get applying Θ to ψ_p the following commutation relation

$$\psi_p(\theta^-(l)) = \theta^+(\psi_p(l))$$

for real p and this gives the reality condition which declare that ψ_p for real p preserves both the quaternionic structure.

To compare $adW^\pm$ and $\Lambda^\pm$ we can consider another projective space coming from local consideration. Namely let one takes the fiber of $\Lambda^2 T^* X$ over the point $x \in X$, complexify it and then projectivize. Since $\Lambda^2 T^*_x X$ has real dimension 6 then one gets $\mathbb{CP}^5$. In this projective space we have the Grassmann quadric $Gr \subset \mathbb{CP}^5$ which paramertizes all projective lines in our $\mathbb{CP}^3$. Really for every complex 4 - dimensional space V the set of 2 - dimensional subspaces can be identified with a special class of 2 - dimensional form up to constant from $\Lambda^2 V^*$ which have rank 2. This condition is equivalent to the equality

$$\omega \wedge \omega = 0,$$

and the last one is a quadratic equation whose solutions form Plucker quadric Gr in $\mathbb{CP}^5$. Thus the quadric Gr represents all projective lines in our $\mathbb{CP}^3$. Therefore one has two curves $l^\pm \subset Gr$ correspond to families $\mathbb{P}^\pm$ defined by our riemannian metric g. On the other hand the Hodge decomposition

$$\Lambda^2 T^* X = \Lambda^+ \oplus \Lambda^-$$

gives us two projective planes

$$\pi^+, \pi^- \subset \mathbb{CP}^5,$$

and it's not hard to see that

$$l^{\pm} = Gr \cap \pi^{\pm}.$$

From this one sees that $l^{\pm}$ are two plane conics with involutions without fixed points induced by quaternionic real structures $\theta^{\pm}$. Now for every point $\phi \in l^{+}$ we can construct a class of self dual 2 - forms as follows. For the point ϕ one takes the conjugated point $\bar{\phi} \in l^{+}$ and constructs two tangent to the conic l^{+} projective lines $r_{\phi}, r_{\bar{\phi}} \subset \pi^{+}$ in the points. There are no bi - tangent lines to any conic hence for any ϕ the lines $r_{\phi}, r_{\bar{\phi}}$ are distinct and define uniquely the intersection point

$$\omega = r_{\phi} \cap r_{\bar{\phi}}.$$

This point ω represents a class of 2 -forms which are defined by the point up to a scalar multiple. So on the porjectiviztion level one gets a quadratic map from spinor fields to self dual 2 - forms compatible with hermitian structures.

To finish we add that the same one takes place for spinor fields from W^{-} and anti self dual forms.

2. The Seiberg - Witten equations

The background idea of the previous lecture was that spinor and complex geometries are very closed and even coincide in same points. A background idea of the present one is that the nature of the Seiberg - Witten invariants is complex to.

Again let X be a real 4 - dimensional compact orientable smooth riemannian manifold and g is a fixed riemannian metric on it. Additionally we fix an orientation and choose a $Spin^{C}$ - structure $c \in H^{2}(X, \mathbb{Z})$ which is a lifting of projective twistor bundles $\mathbb{P}^{\pm}$ to a pair of spinor $U(2)$ - bundles $W^{\pm}$ with the same determinant line bundle $detW^{\pm}$ and hence with the same first Chern class $c_1(W^{\pm}) = c$. These spinor bundles $W^{\pm}$ satisfy as usual the properties listed in the previous lecture.

The configuration space for the system of equations which were called the Seiberg - Witten equations is the following: it is the direct product of $\mathcal{A}_h(detW^{+})$ — the space of hermitian connections on the determinant line bundle — and $\Gamma(W^{+})$ — the space of all spinor fields that is smooth sections of W^{+}. According to the physical tradition we call the first component bosonic while the second fermionic parts of the configuration space. For a point

$$(a, \phi) \in \mathcal{A}_h(detW^{+}) \times \Gamma(W^{+})$$

of the configuration space one defines two equations. First of all the Levi - Civita connection defined by our riemannian metric induces a hermitian connection on adW^{+}; at the same time one has the following usual decomposition

$$\mathcal{A}_h(W^{+}) = \mathcal{A}_h(detW^{+}) \times \mathcal{A}_h(adW^{+})$$

hence for any hermitian connection $a \in \mathcal{A}_h(detW^{+})$ one automatically has the corresponding hermitian connection on whole bundle W^{+} with covariant derivative ∇_a:

$$\nabla_a : \Gamma(W^{+}) \to \Gamma(W^{+} \otimes T^{*}X).$$

But for T^*X we have the conjugated identification

$$T^*X = Hom(W^+, W^-)$$

coming from the first property of $Spin^{\mathbb{C}}$ - structure. Then one gets the following combination

$$\begin{aligned}\nabla_a : \Gamma(W^+) \to \Gamma(W^+ \otimes T^*X) = \Gamma(W^+ \otimes Hom(W^+, W^-)) = \\ \Gamma(W^+ \otimes (W^+)^* \otimes W^+) \to \Gamma(W^-)\end{aligned}$$

where at the last step we use the natural cancellation which is called in this case Clifford multiplication. This combination gives us an operator

$$D_a : \Gamma(W^+) \to \Gamma(W^-)$$

which is called Dirac operator coupled by the abelian connection a. In a simple case when the based manifold is a spin manifold so admits a canonical lifting of twistor bundles for a riemannian metric g to spinor bundles with trivial determinants one has a preffered Dirac operator on the spinor bundle W_0^+. Namely on the trivial determinant bundle one has the trivial connection which corresponds to ordinary differential and hence one takes the Dirac operator defined by this trivial connection. This Dirac operator is a distinguished operator but in our general $Spin^{\mathbb{C}}$ - case there is no any preference to choose an operator. Originally Dirac got it taking a "square root" from the Laplacian defined by riemannian metric. The Laplace operator is a real operator but complex geometry comes when one tries to define the square root. As we will see the Dirac operator is exactly the square root in the flat case; in a curved case with $Spin^{\mathbb{C}}$ - structure one has to add the curvature of the abelian coupling connection a and the scalar curvature of the riemannian metric.

Thus the first equation of the Seiberg - Witten system reads as

$$D_a(\phi) = 0$$

and can be understood as the harmonicity condition for spinor field with respect to coupled Dirac operator.

The second equation of the system is based on the second property of $Spin^{\mathbb{C}}$ - structure. Following it one can compare the self dual part of the curvature tensor of our abelian connection a and a skew symmetric traceless endomorphism of the spinor bundle W^+ defined by spinor field ϕ which we denote as $(\phi \otimes \bar{\phi})_0$ having in mind the projective story of the previous lecture. So the last equation reads as

$$F_a^+ = -(\phi \otimes \bar{\phi})_0$$

and now we can collect these two together getting

$$\begin{aligned} D_a(\phi) &= 0 \\ F_a^+ &= -(\phi \otimes \bar{\phi})_0 \end{aligned}$$

which is the celebrated Seiberg - Witten system of equations. These equations are called the equations of abelian monopoles. Although one defines a hermitian connection of whole W^+ the bosonic variable is just abelian. On the other hand the gauge group of the theory is abelian. It means as usual that one has a natural group action on the configuration space such that the equations are invariant under this action. In the case we are discussing now such a group comes if we take

$$\mathcal{G} = Aut_h(detW^+)$$

the group of fiberwise automorphisms of the determinant line bundle which preserve its hermitian structure. So over each point on has $U(1)$ as the automorphism group thus locally the gauge group $\mathcal{G}$ is modeled by maps

$$\eta : N_x \to U(1)$$

where N_x is a small neighborhood of a point $x \in X$. On the first component of the configuration space it acts in the usual way while in the second case it acts only on the central part (so locally if η is the multiplication by a complex function e^ψ then $\eta(\psi) = e^\psi \phi$ where $\phi \in \Gamma(W^+)$). Thus one defines two quotient spaces: the first one comes with factorization of the configuration space by the gauge group action

$$\mathcal{C} = \mathcal{A}_h(detW^+) \times \Gamma(W^+)/\mathcal{G};$$

at the same time it's convenient to consider the subspace of irreducible pairs (a, ϕ) which have maximal stabilizer in the gauge group. In our theory it just means that (a, ϕ) is reducible if and only if the spinor component ϕ vanishes identically. We define the smooth part of the quotient space

$$\mathcal{C}^* = (\mathcal{A}_h(detW^+) \times \Gamma(W^+))^*/\mathcal{G}$$

where $(\mathcal{A}_h(detW^+) \times \Gamma(W^+))^*$ consists of irreducible pairs.

The key point is that the Seiberg - Witten equations are invariant under the gauge group action. Therefore this invariance ensures that one can consider the moduli space of solutions. "Moduli" means that it is the set of solutions *modulo* gauge transformations. We denote it as $\mathcal{M}(g, c)$ and call it the moduli space of abelian monopoles. The notation reflects the dependence on riemannian metric g and $Spin^{\mathbb{C}}$ - structure c.

The first question we would like to discuss is the problem of reducible solutions. We will see that such a solution exists if and only if the fixed riemannian metric g satisfies a special property.

Namely let (a, ϕ) be a reducible solution

$$(a, \phi) = (a, 0).$$

Then the first equation of the systems is satisfied automatically while the second gives us the condition

$$F_a^+ = 0$$

which is the equation of abelian instanton. Such an instanton exists if and only if the fixed riemannian metric is of a special type: the harmonic form representing the first Chern class $c_1(detW^+) = c$ is anti self dual. Really if F_a^+ is trivial then

$$F_a = F_a^-,$$

but any abelian curvature is a pure imaginary closed 2 - form thus $dF_a^- = 0 = d^*F_a^-$ since every closed (anti) self dual 2 - form is automatically co - closed and therefore harmonic. It means that

$$\frac{i}{2\pi} F_a$$

is a real harmonic anti self dual 2 - form, but according to the Chern - Weyl theory is represents the first Chern class of the determinant line bundle $detW^+$ since our abelian connection lives on this bundle. But this property is very special on riemannian metrics (see discussion on real period maps from the previous lecture). For example if $c^2 > 0$ there is no such riemannian metric at all!Really for the existence of such a representation by an appropriate riemannian metric it's necessary that the square of the class is non positive because there are no anti self dual forms with positive squares at all. Moreover if $c^2 = 0$ and c is non trivial element in the lattice $H^2(X, \mathbb{Z})$ then there are no anti self dual representations of this class because any anti self dual class with zero square has to be trivial. Further even if there are exist riemannian metrics admitting for c anti self dual representations they form sufficiently "thin" subset in the space of riemannian metrics. Whole the space is isomorphic to an infinite dimensional ball and the metrics admitting such representations form a subset of codimension $b_2^+(X)$ (it follows from the local surjectivity theorem for the real period map proved by S. Donaldson). From these notes one can derive that:

if $b_2^+(X)$ is bigger then zero then for a generic riemannian metric there are no reducible solutions at all;

if $b_2^+(X) > 1$ then for every pair of generic metrics g_0, g_1 there exists a smooth path γ in the space of riemannian metrics with ends at g_0 and g_1 such that every riemannian metric which is parameterized by the path the harmonic 2 - form representing the class c is not anti self dual. So in these cases the moduli spaces are smooth orbifolds.

The next question is what are the dimensions of moduli spaces in these regular cases? To establish the numerical characterization one has to study the linearization of the system. The first equation

$$D_a(\phi) = 0$$

is linear itself; from this we take the index of the coupled Dirac operator. It is not very hard to compute this number applying the well known Atiyah - Hirzebruch index theorem which says that every $\mathbb{C}$ - linear elliptic operator over a real compact smooth manifold has the index which equals to a combination of Chern classes of the "source" and "target" vector bundles and A - genus of the based manifold. Every algebraic geometer knows this index theorem under the name of Riemann - Roch formula — and ommiting the calculations we claim that in our case

$$indD_a = \frac{1}{4}(c^2 - \sigma(X))$$

(here we multiply the complex index by 2 to get the real one). The second equation can be reduced to the infinitesimal level up to gauge transformations with the following operator

$$\delta_a = d_a^* \oplus d_a^+ : \Omega^1_X(i\mathbb{R}) \to \Omega^0_X(i\mathbb{R}) \oplus \Omega^+_X(i\mathbb{R})$$

which is very well known in the Donaldson theory (see [23]). Its index coincides with the index of the basic operator

$$\delta = d^* \oplus d^+ : \Omega^1_X \to \Omega^0_X \oplus \Omega^+_X$$

which defines "a half" of the standard de Rham complex. It gives

$$ind\delta_a = ind\delta = -(1 - b_1(X) + b_2^+(X)) = -\tfrac{1}{2}(\chi(X) + \sigma(X)).$$

Thus the total index is

$$indSW = \frac{1}{4}(c^2 - \sigma) - \tfrac{1}{2}(\chi + \sigma) = \frac{1}{4}(c^2 - 2\chi - 3\sigma),$$

therefore the virtual dimension of the moduli space $\mathcal{M}(g, c)$ equals to

$$v.dim\mathcal{M}(g, c) = \frac{1}{4}(c^2 - 2\chi - 3\sigma).$$

This number is very crucial for the theory at all: we have seen in the previous lecture that this number is the second Chern number of the positive spinor bundle

$$v.dim\mathcal{M}(g, c) = c_2(W^+);$$

if this number vanishes then c is the canonical (or anti canonical) class of some compatible almost complex structure. For a generic metric g if $b_2^+(X) > 0$ the moduli space is a smooth real manifold of this dimension.

Until the time the story comes quite parallel to the consideration in the framework of the Donaldson theory. As in the Donaldson case we take a configuration space, consider a gauge invariant equation and then construct the moduli space of solutions. The same question of reducible solutions arose. But the moduli space of $SU(2)$ - instantons, constructed by S. Donladson, has two type of singularities. The first type comes with reducible solutions (and we avoid this point taking sufficiently generic metric) while the second type comes from the conformal invariance of the instanton theory: the so - called bubbling instantons exist due to the conformal invariance. So in the Donaldson theory one deals with geniunly non compact moduli spaces. In the Seiberg - Witten theory the situation is much more convenient for investigations: the theory is not conformally invariant (so the solutions change when we take another riemannian metric in the same conformal class as our given g) but the moduli spaces are compact! Due to the fruitful computation firstly proposed in [2] one gets a bound on norms of solutions and thus one establishes that the moduli spaces are compact.

The computation is so simple that every person can make it herself. For this one uses the Weitzenbock formula mentioned above

$$D_a^* D_a \phi = \nabla_a^* \nabla_a \phi + \frac{1}{2} F_a^+ \phi + \frac{s}{4} \phi$$

where F_a^+ is considered as an endomorphism of W^+ and s is the scalar curvature of the fixed riemannian metric g. Let (a, ϕ) is a solution of the Seiberg - Witten system, then we take the pointwise norm $|\phi|^2$ and at the point where it is maximal one gets

$$\begin{aligned} 0 \leq \Delta |\phi|^2 = 2 < \nabla_a^* \nabla_a \phi, \phi > -2 < \nabla_a \phi, \nabla_a \phi > \\ \leq 2 < \nabla_a^* \nabla_a \phi, \phi >= -\frac{s}{2} |\phi|^2 \\ - < (\phi \otimes \bar{\phi})_0 \phi, \phi >= -\frac{s}{2} |\phi|^2 - \tfrac{1}{2} |\phi|^4 \end{aligned}$$

and if $|\phi|^2$ is non trivial at the maximal point one can obtain dividing by $|\phi|^2$ the bound on the norm $|\phi|^2$:

$$max_{x \in X} |\phi|^2 = max(0, -s)$$

(see [2]). This means additionally that if g has non negative scalar curvature then there are no irreducible solutions for the Seiberg - Witten system of equations. Therefore in this case the moduli space for a generic riemannian metric is empty.

Now we know that for a generic metric the moduli space is a smooth compact real manifold. But what about the orientability? The operators which represent the linearization of the Seiberg - Witten equations are D_a and δ_a and the first one defines the canonical orientation being originally a complex operator. The second can be endowed with an orientation by the choice of an orientation of the trivial determinant line $det H^0(X, \mathbb{R}) \otimes det H^1(X, \mathbb{R}) \otimes H^+(X, \mathbb{R})$. This data is the same as in the Donaldson theory so the moduli spaces of instantons and monopoles can be oriented simultaneously.

The definition of the Seiberg - Witten invariant requires that the based manifold X has $b_2^+(X) > 1$. First of all one considers the case when the $Spin^{\mathbb{C}}$ - structure c induces zero dimensional moduli spaces for generic metrics. Then for the generic metric g the moduli space $\mathcal{M}(g, c)$ is a compact oriented real manifold of dimension 0 so it is a finite set of points. One can count the number of these points with signs defined by the orientation getting an integer number $N_{SW}(g, c)$. But since $b_2^+(X) > 1$ this number doesn't depend on the choice of generic riemannian metric: if g_0, g_1 are two generic metric with the numbers $N_{SW}(g_0, c)$ and $N_{SW}(g_1, c)$ which can be jointed by a smooth path γ (see above) and the function $N_{SW}(\gamma, c)$ has to be continuous under the deformation along γ. But any integer continuous function has to be constant. This implies that two integer numbers $N_{SW}(g_0, c)$ and $N_{SW}(g_1, c)$ are the same. It means that if $b_2^+(X) > 1$ the number N_{SW} doesn't depend on the choice of generic riemannian metric and is an invariant of the smooth structure. Thus in this case we understand N_{SW} as a function on the set of appropriate $Spin^{\mathbb{C}}$ - structures with integer values. The simple case with zero dimensional moduli spaces just corresponds to complex geometry — we've reproved Wu theorem and we know that the case when

$$v.dim\mathcal{M} = 0 = \frac{1}{4}(c^2 - 2\chi - 3\sigma)$$

is exactly the complex case. Further let us define the function N_{SW} on the subset of $Spin^{\mathbb{C}}$ - structures with negative $c_2(W^+)$ just as zero. If for $Spin^{\mathbb{C}}$ structure c this number is positive and even then there exists a procedure which again gives us an integer number which we understand as the value of N_{SW} on this class. Namely the ambient space $\mathcal{C}^*$ of irreducible pairs modulo gauge equivalence always has a distinguished 2 - cohomological class which can be derived as follows. Let us fix a point $x \in X$ and consider the subgroup $\mathcal{G}_0$ of the gauge group $\mathcal{G}$ consists of all gauge transformations identical on the fiber over x. Consider the quotient space

$$\mathcal{C}_0^* = (\mathcal{A}_h(detW^+) \times \Gamma(W^-))^*/\mathcal{G}_0$$

which is a principle $U(1)$ - bundle over $\mathcal{C}^*$. The canonical action is induced by $U(1)$ transformations of the fiber over x. This natural principle $U(1)$ - bundle has a topological characteristic — the first Chern class $\eta \in H^2(\mathcal{C}, \mathbb{Z})$. On the other hand since $v.dim\mathcal{M}$ is even it means that one has the fundamental class $[\mathcal{M}] \in H_{2d}(\mathcal{C}^*, \mathbb{Z})$ (perhaps after a small deformation) where

$$2d = \frac{1}{4}(c^2 - 2\chi - 3\sigma)$$

is even. Then one takes the natural pairing

$$< \eta^d, [\mathcal{M}] > \in \mathbb{Z}$$

and it is an integer number which again depends (if $b_2^+(X) > 1$) only on smooth structure. So this number is the value of N_{SW} if $\frac{1}{4}(c^2 - 2\chi - 3\sigma)$ is even. In the odd case we again say that the number is trivial.

Thus the Seiberg - Witten invariant for a smooth compact orientable real 4 - dimensional manifold X is an integer function

$$N_{SW} : Spin \to \mathbb{Z}$$

where $Spin$ is the set of possible $Spin^{\mathbb{C}}$ - structures over X. This function is invariant under the action of $Diff^+X$ and one knows how it changes under the action of whole $DiffX$. At the same time there is a natural symmetry namely if one takes two $Spin^{\mathbb{C}}$ - structures $\pm c$ where c is a fixed class then one gets a symmetry between the solutions. But since the question of compatibility of the orientations depends on numerical characteristics of X one gets

$$N_{SW}(-c) = (-1)^k N_{SW}(c)$$

where

$$k = \frac{1}{4}(\chi(X) + \sigma(X)).$$

It's easy to see that if a $Spin^{\mathbb{C}}$ - structure with even second Chern class of the spinor bundle exists then k belongs to integer numbers. If this number is even then the Seiberg - Witten function N_{SW} is symmetric while in the odd case it is skew symmetric.

If $b_2^+(X)$ is equal to 1 then it remains a possibility to define the invariant using the chamber structure on the space of all riemannian metrics. Often this variant is quite useful (see [2]). The space of riemannian metric is divided by so called walls into a set of disjoint chambers. Every wall consists of such riemannian metrics that harmonic 2 - forms representing the $Spin^{\mathbb{C}}$ - structure class c are anti self dual. But the number derived from the Seiberg - Witten moduli space doesn't change inside of any chamber; if two riemannian metrics g_0 and g_1 lie in the same chamber we can join them by a smooth path which doesn't intersect the walls therefore the value of the integer function has to be constant along the path. Moreover if $c^2 > 0$ then as we've mentioned above there are no reducible solutions at all for any riemannian metric. Therefore even if $b_2^+ = 1$ in this case one can define the invariant integer function without any references to the chamber structure. But for the main interest of the smooth 4 - geometry which is focused on complex projective plane considered as a real 4- dimensional manifold this remarks gives almost nothing. The point is that as we've seen from the Kronheimer and Mrowka calculations there are no irreducible solutions for the Seiberg - Witten system if the defining riemannian metric has non negative scalar curvature. But it's the case of the "canonical" kahler metric on $\mathbb{CP}^2$ thus despite of the fact that for every $Spin^{\mathbb{C}}$ - structure c over $\mathbb{CP}^2$ there are no metrics admitting anti self dual realization for the class (the intersection form is positive definite so every nontrivial class has positive square) and consequently there is no the chamber structure on the space of all metrics the invariant has to be trivial. Really since there are no walls it means that one can join every riemannian metric with the canonical kahler metric but for the last one the integer number equals to zero hence it is trivial for all riemannian metrics. This type of arguments ensures that every 4 - dimensional manifold which admits riemannian metric with non negative scalar curvature has trivial Seiberg - Witten invariant.

On the other hand we have the original case which was the starting point of the theory. We mean the spin case when one has over X a distinguished $Spin^{\mathbb{C}}$ - structure $c = 0$. This is possible if and only if the second Shtiffel - Whitney class of X vanishes. In this case it's easy to see that any riemannian metric admits anti self dual representation for this class since the trivial class is represented by zero harmonic form which of course is anti self dual (as well as self dual). This feature leads one to apply some additional gears for the definition and calculations of the invariant.

First of all what is the difference between nontriviality of the invariant and the existence of solutions? The point is that even if the invariant is trivial then solutions may exist. We will illustrate this fact using some examples below. But if the invariant is nontrivial then the existence of solutions is stable; for example for any riemannian metric a solution exists (may be it is a reducible solution but nevertheless) if the invariant is non trivial. But there is another kind of deformation for the system, namely on can perturb the second equation by a sufficiently small self dual form getting the following system

$$D_a(\phi) = 0$$
$$F_a^+ = -(\phi \otimes \bar{\phi})_0 + \eta$$

where η is a self dual form. The point is that using this deformation we can define the invariant in the cases mentioned above. For the trivial $Spin^{\mathbb{C}}$ - structure $c = 0$ let us take an appropriate deformation of the original system with a 2 - form η such that this perturbed system doesn't admit reducible solutions. If our riemannian metric g has non negative scalar curvature then the adding of an appropriate η kills the vanishing argument for spinor fields above (see [2]).

Now let us come to the examples which we promised to present as the illustration of the difference. To deal with first of all let us recall two basic facts which were established in the starting work [1]. Using these facts Witten reproved in [1] the Donaldson result which claims that some algebraic manifolds aren't diffeomorphic to their topomodels. So if X is a kahler manifold with $b_2^+(X) > 1$ endowed as usual with an integrable complex structure I and kahler metric with kahler form ω then the Seiberg - Witten invariant of canonical class K_I as $Spin^{\mathbb{C}}$ - structure equals to ± 1. On the other hand if one takes the connected sum

$$Y = X_1 \sharp X_2$$

where X_1 and X_2 both have non trivial $b_2^+(X_i)$ then all the invariants vanish for Y. These means that almost all topomodels have trivial Seiberg - Witten invariants.

We'll prove the first statement in a time in much more wider context following Taubes results about symplectic manifolds. The author in [1] argues as follows to prove the second statement. Studying the gluing procedure one gets that for a glued metric on Y there is a non trivial circle action on the moduli space of solutions. For example if the index formula predicts that the moduli space has dimension zero then it happens that the moduli space is either 1 - dimensional for glued metrics or is empty for a generic one. Thus for glued metrics solution exist while the invariant is trivial and for generic metric there are no solutions at all. On the other hand it is quite fruitful technique in the theory to consider some connected sums instead of a given manifold, but usually one takes the connected sum with a number of $\overline{\mathbb{CP}}^2$s which is the simplest manifold with trivial b_2^+. We will return to these connected sums at the end of this lecture.

The results established in the theory can be divided into two sets: the set of so called vanishing results and the set of "positive" results. We've presented two from the first set (non negative scalar curvature and the connected sum) while the second set has been represented by the fact that every kahler manifold has non trivial Seiberg - Witten invariant. This result is quite natural and expected since the theory looks like a complex theory. Really in the definition of the invariants the first case is when one has an almost complex structures over X compatible with g. In this case the moduli spaces are zero dimensional and arithmetically it corresponds to the condition

$$c^2 = 2\chi + 3\sigma$$

so our $Spin^{\mathbb{C}}$ - structure c is a canonical class over X. If the based manifold is simply connected then there is a geometric procedure which relates the invariant coming from positive dimensional moduli space and zero dimensional moduli space over a connected sum. As a pattern we use the following remark from [5]. Let X be a based manifold as usual in the theory with $b_2^+ > 1$ and let for a $Spin^{\mathbb{C}}$ - structure $c \in H^2(X, \mathbb{Z})$ and a riemannian metric g one has the corresponding moduli space $\mathcal{M}_X(g, c)$. Glue together our X with $\overline{\mathbb{CP}}^2$ getting another smooth manifold

$$Y = X \sharp \overline{\mathbb{CP}}^2$$

endowed with $Spin^{\mathbb{C}}$ - structure

$$c + h \in H^2(Y, \mathbb{Z}) = H^2(X, \mathbb{Z}) \oplus \mathbb{Z}h,$$

where h is the standard generator of $H^2(\overline{\mathbb{CP}}^2, \mathbb{Z})$ with square -1 and a riemannian metric which is the gluing of our given g and the standard metric on reverse oriented projective plane. For this $Spin^{\mathbb{C}}$ - structure $\tilde{c}$ and riemannian metric $\tilde{g}$ one has the corresponding moduli space $\mathcal{M}_Y(\tilde{g}, \tilde{c})$. First of all let us compute the virtual dimension

$$v.dim \mathcal{M}_Y(\tilde{g}, \tilde{c}) = \frac{1}{4}(\tilde{c}^2 - 2\chi(Y) - 3\sigma(Y)) =$$
$$\frac{1}{4}(c^2 - 1 - 2(\chi(X) + 1) - 3(\sigma(X) - 1)) = \frac{1}{4}(c^2 - 2\chi(X) - 3\sigma(X))$$

thus the moduli spaces have the same virtual dimensions. This gives the hint that they are isomorphic. The comparing of solutions form these moduli spaces uses some special technique which is well known from the Donaldson theory (shrinking the neck etc). Namely changing the canforamlly flat metric on the neck joining two manifolds one can divide any solution from $\mathcal{M}_Y$ into three parts: the part which lives over X, the part living over $\overline{C\mathbb{P}}^2$ and the remainder over the conformally flat neck. The first part can be deformed to a solution over X itself while the part over $\overline{C\mathbb{P}}^2$ has to be reducible with trivial spinor field (since the standard metric has positive scalar curvature) but this reducible solution is unique up to gauge transformations. It remains to check that one can arrange the smoothing procedure over the neck to glue the classes from the moduli space $\mathcal{M}_X$ and this unique reducible solution. Therefore the moduli spaces are isomorphic and consequently the invariants for X and Y are the same.

Now let us modify this construction. In [10] one finds the following geometrical remark which is very closed to this from [5]. Let $c \in H^2(X, \mathbb{Z})$ be a $Spin^{\mathbb{C}}$ - structure over a simply connected based manifold X with an even positive virtual dimension

$$\frac{1}{4}(c^2 - 2\chi - 3\sigma) = 2d$$

where $d \in \mathbb{N}$. Then the $Spin^{\mathbb{C}}$ - structure can be extended to a canonical class over connected sum.

$$Y = X \sharp d \overline{\mathbb{CP}}^2.$$

Really take

$$K_c = c + \sum_{i=1}^{d} 3h_i$$

where h_i are the generators over each $\overline{\mathbb{CP}}^2$ and compute

$$\begin{aligned} K_c^2 &= c^2 - 9d = 8d + 2\chi(X) + 3\sigma(X) - 9d \\ &= 2\chi(X) + 3\sigma(X) - d = 2\chi(Y) + 3\sigma(Y) \end{aligned}$$

since

$$\begin{aligned} \chi(Y) &= \chi(X) + d \\ \sigma(Y) &= \sigma(X) - d. \end{aligned}$$

Moreover the extension of c to a canonical class exists if and only if the number $\frac{1}{4}(c^2 - 2\chi - 3\sigma)$ is even. Why we search this extension on the connected sums of this type? The point is that one has to take connected sum with manifolds whose intersection forms are negative definite. In other cases one kills all the invariants (see the vanishing result above). To simplify the construction we would like to take simply connected manifolds but due to the Donaldson result every simply connected 4 - manifold with definite intersection form is homeomorphic to d copies of projective planes. Since we don't know until now are there some other smooth structures on $\mathbb{CP}^2$ unlike to the standard one we could not explain why we take this concrete smooth $\mathbb{CP}^2$ but we believe that the famous smooth Poincare conjecture is true consequently it would be only one manifold which can be taken as the summand.

Now the point is that for simply connected based manifold the invariants computed using $\mathcal{M}_X(g, c)$ over X and using $\mathcal{M}_Y(\tilde{g}, K_c)$ over Y coincide. It means that one can reduce the computation of the invariants to an almost complex case. It means that the theory is complex (or almost complex) itself.

Returning to the definition we recall that for a smooth 4 - manifold X with $b_2^+(X) > 1$ one constructs an integer function

$$N_{SW} : Spin \to \mathbb{Z}$$

(where $Spin$ is the set of all possible $Spin^{\mathbb{C}}$ - structures) which is invariant under the $Diff^+X$ action. Every class $c \in Spin$ with nontrivial image is called basic class. The set of basic classes is a finite subset $B_X \subset H^2(X, \mathbb{Z})$ in general case (for the finiteness see [1]) and this set is an invariant of smooth structure ones more. If all the basic classes are canonical for some almost complex structures then one calls the smooth manifold X as a manifold of simple type. In the simply connected case one can always reduce the story to the simple type: one chooses the basic class with maximal virtual dimension of the corresponding moduli space and then one takes the connected sum of X with half of the dimension copies of $\overline{\mathbb{CP}}^2$.

Thus we will discuss mostly the cases when one has basic canonical classes. So we are in the framework of hermitian geometry.

The list of the first results which were established in [1] can be continued exactly in this way. We left the fact that for every kahler manifold the canonical class is a basic class without proof because of the following result which is much more wider.

In [3] Taubes proved that the same is true for any symplectic manifold (with as usual $b_2^+ > 1$). First of all let us recall how these two cases can be related.

Let X be as usual in the theory real 4 - manifold. And let it admit almost complex structures. Then one can take any pair (g, J) over X where g is a riemannian metric and J is a compatible almost complex structure. Together they define automatically non degenerated 2 - form ω in the usual way. This 2 - form is self dual with respect to g and to the orientation given by J and it has bi - type (1,1) with respect to J. So together we get a hermitian triple (g, J, ω) where each element can be reconstructed from the others. Then in this terms one has the following hierarchy

$$AG \subset KG \subset SG \subset GG \subset DG$$

where AG is algebraic geometry, KG is kahler geometry, SG is symplectic geometry, GG is gauge geometry so the geometry of manifolds with non trivial Seiberg - Witten invariants and the last one of course is the most general differential geometry of 4 - manifolds. The top subject AG means that one has a hermitian triple (g, J, ω) over X which satisfies the following conditions:

i) J is integrable;

ii) ω is closed;

iii) the corresponding 2 - cohomology class $[\omega]$ belongs to integer lattice $H^2(X, \mathbb{Z})$ in $H^2(X, \mathbb{R})$.

The next subject KG is defined by a hermitian triple which satisfies the first two conditions i) and ii). Turning to symplectic geometry one can see that in this case if X has symplectic form ω then there exists a cone of compatible almost complex structures thus taking a structure J from the cone and reconstructing the corresponding riemannian metric one gets a hermitian triple which satisfies only the second condition. All these compatible structures are of the same canonical class which is called associated canonical class of the symplectic structure. These almost complex structures can be exploited to define invariants of smooth structure which are the Gromov invariants counting pseudoholomorphic curves (we'll discuss this construction below). Thus every kahler manifold is a symplectic manifold and it was conjucted long times ago that the reverse implication takes place as well before Thurston constructed examples of symplectic manifolds which do not admit integrable complex structures. Now we get the implication $SG \subset GG$ due to the Taubes result: for symplectic manifold (with $b_2^+ > 1$) the canonical class of associated almost complex structures is a basic class with the invariant equals to ± 1. The proof is so elegant e and simple that we recall it here briefly.

First of all if X is a symplectic manifold with symplectic 2 - form ω then for every compatible riemannian metric g (which induces the corresponding almost complex structure J) there is a preffered $Spin^{\mathbb{C}}$ - structure which is

$$W^+ = I \oplus \Lambda^{0,2}$$
$$W^- = \Lambda^{0,1}$$

with $c = -K_J$ as is clear from the formula. Here I is trivial complex line bundle, $\Lambda^{0,l}$ is the bundle on (0, l) - forms with respect to our almost complex structure J.

The properties of $Spin^{\mathbb{C}}$ - structure can be illustrated by exact formula in this case: if ρ is a real 1 - form and $\alpha \oplus \beta$ belongs to $\Gamma(W^+)$ then

$$\rho(\alpha \oplus \beta) = \alpha \rho^{0,1} \oplus \Lambda(\rho \wedge \beta)$$

where Λ is the adjoint operator to the wedge multiplication by ω:

$$\Lambda : \Omega^{p,q} \to \Omega^{p-1,q-1}.$$

The summand Λ^+ in the Hodge decomposition takes the form

$$(\Lambda^{2,0} \oplus \Lambda^{0,2})_{\mathbb{R}} \oplus \mathbb{R} < \omega >;$$

it acts on W^+ as follows. The symplectic form ω acts as the diagonal operator with eigenspaces I and $\Lambda^{0,2}$ with eigenvalues $-i$ and i respectively. Each other 2 - form $\rho \in (\Omega^{2,0} oplus \Omega^{0,2})_{\mathbb{R}}$ acts as

$$\rho(\alpha \oplus \beta) = \alpha \rho^{0,2} \oplus \Lambda(\rho \wedge \beta);$$

from this it's clear that if X is topological K3 surface then normalized non vanishing 2 - form $\theta \oplus \bar{\theta}$ from $(\Omega^{2,0} \oplus \Omega^{0,2})_{\mathbb{R}}$ defines the second complex structure on W^+.

Moreover, the Dirac operator coupled by $a \in \mathcal{A}(K^{-1})$ has the form

$$D_a = \partial_a \oplus \bar{\partial}_a : I \oplus \Lambda^{0,2} \to \Lambda^{0,1}$$

so is a coupling of the standard operator $\delta = \bar{\partial} \oplus \bar{\partial}^*$. Thus the system in these "coordinates" reads as

$$\bar{\partial}_a \alpha + \bar{\partial}^* \beta = 0$$
$$F_a^{0,2} = \bar{\alpha}\beta$$
$$iF_a^{1,1}\dot{\omega} = (|\beta|^2 - |\alpha|^2)\omega \wedge \omega.$$

On the other hand (see [3]) there exists such hermitian connection $a_0 \in \mathcal{A}_h(detW^+) = \mathcal{A}_h(K^{-1})$ that the corresponding full covariant derivative ∇_{a_0} on whole $W^+ = I \oplus K^{-1}$ projects to the first summand as the ordinary d and such a_0 is unique up to gauge transformations. This is true for absolutely general almost complex situation. Now if one takes a constant function and define the following spinor field

$$\phi = \alpha \oplus \beta = c \oplus 0$$

then the action of the covariant derivative gives us

$$\nabla_{a_0}(c \oplus 0) = 0 \oplus b$$

where b is a section of $\Lambda^{0,2} \otimes T^*X$. Indeed, the projection of the resulting section to the first summand $I \otimes T^*X$ has to be trivial by the definition of a_0 hence we get this b. This b is essentially the torsion of our almost complex structure; one can define the Nijenhuis tensor $N \in Hom(\Lambda^{0,1}, \Lambda^{2,0})$ using this b. It means that this b is trivial if and only if our complex structure J is integrable. Formally here is the end point of joint running for the ways of kahler geometry and symplectic geometry in the theory. But it is only formally because of a hidden additional argument which was found by Taubes. Namely despite of the nontriviality of $\nabla_{a_0}(c \oplus 0)$ the corresponding coupled Dirac operator D_{a_0} vanishes on the spinor field $c \oplus 0$ if and only if the almost kahler form ω is closed so in the framework of symplectic geometry. This means that despite of the fact that for a symplectic manifold the torsion b is nontrivial while for a kahler one it vanishes the coupled Dirac operator vanishes on this preffered spinor field in symplectic case as well as in kahler case endowing us with the preffered solution for the first equation of the system. Let us check this fundamental fact directly. For this one acts by the almost kahler form ω on the preffered spinor field getting

$$\omega(c \oplus 0) = -i(c \oplus 0).$$

Differentiating this equality by the covariant derivative ∇_{a_0} and then using Clifford multiplication we establish that

$$\begin{aligned} \nabla_{a_0}(\omega) * (c \oplus 0) + \omega(\nabla_{a_0}(c+0)) &= -i\nabla_{a_0}(c+0) \\ &\Longrightarrow \\ Cd^*\omega(c \oplus 0) + iD_{a_0}(c+0) &= -iD_{a_0}(c+0) \end{aligned}$$

where C is a constant and $d^*\omega$ acts as a real 1 - form on the spinor field and transform $c \oplus 0$ to a section of $W^- = \Lambda^{0,1}$. Thus we get the following expression

$$D_{a_0}(c \oplus 0) = \frac{iC}{2} d^*\omega(c \oplus 0).$$

The map from W^+ to W^- defined by 1 - form $d^*\omega$ is nontrivial of full rank unless the case when $d^*\omega$ vanishes. Since ω is self dual than the condition

$$d^*\omega = 0$$

coincides with the closeness condition

$$d\omega = 0$$

therefore

$$D_{a_0}(c+0) = 0 \qquad \mathrm{i}f and only if \qquad d\omega = 0$$

so in the symplectic case (see [3]) one has a distinguished solution for the first equation as well as in the kahler case. To compute the invariant Taubes uses an appropriate perturbation of the original system namely the following one

$$\begin{aligned} D_a(\alpha \oplus \beta) &= 0 \\ F_a^{0,2} &= \bar{\alpha}\beta \\ iF_a^{1,1}\dot{\omega} &= (|\beta|^2 - |\alpha|^2 + \rho^2) + C \end{aligned}$$

where ρ is the perturbation parameter and C is a topological constant which is the slant product of two 2 - cohomological classes:

$$C = < 2\pi c_1(K^{-1})\dot{[}\omega], [X] > \in \mathbb{R}.$$

Direct computation (see f.e. [22]) shows that when ρ comes to be sufficiently large then only one solution (up to gauge transformations) survives and this solution is the obvious one

$$a = a_0, \alpha = \rho, \beta = 0.$$

This gives the Taubes result. On the other hand one can establish coming in this way another important Taubes result which says that if K is the canonical class associated with symplectic form ω over a symplectic manifold with $b_2^+ > 1$ then $K \cdot [\omega] \geq 0$ and for any basic class k one has inequality

$$|k \cdot [\omega]| \leq K \cdot [\omega].$$

Note that while the first result is quite known in the kahler case (if $b_2^+ > 1$ it means that $p_g > 0$ and consequently the canonical bundle has to have non negative pairing with the given polarization) the second restriction is new.

After this fundamental result was established one could suppose that symplectic geometry covers gauge geometry. It was disproved almost immediately in [5]. The construction is again quite simple. Let us take a symplectic manifold X with $b_2^+(X) > 1$ as usual in the theory and consider the following connected sum

$$Y = X \sharp N$$

where N has $b_1(N) = b_2^+(N) = 0$ but non trivial fundamental group. An example used in [5] is the direct product of a homological 3 - sphere with non trivial fundamental group and S^1 when one kills then by the surgery the generator of $\pi_1(S^1)$ in the product. Then one gets a 4 - dimensional smooth manifold with finite non trivial fundamental group and with trivial Betti numbers except b_0 and b_4. Due to the fact proved in [5] and discussed above the invariants for X and for Y are the same and it means that N has nontrivial invariants. On the other hand it couldn't be symplectic. Really if it is then its universal covering $\tilde{Y} \to Y$ has to be symplectic as well as Y. Thus $\tilde{Y}$ by the Taubes theorem has to have non trivial invariants. But this smooth 4 - manifold is represented by the connected sum

$$\tilde{Y} = X \sharp ... \sharp X$$

being the universal covering where the number of summands in this connected sum equals to the order of $\pi_1(N)$. Therefore the invariants of $\tilde{Y}$ all are trivial since it is a connected sum of manifolds with positive b_2^+s. Hence one gets the contradiction with the hypoteze that Y is symplectic (see [5]). It means that the inclusion

$$SG \subset GG$$

is proper at least for simply connected based manifolds.

Now we would like to formulate a natural problem which arises in hermitian or almost kahler 4 - dimensional geometry. As we've seen above the Seiberg - Witten theory is complex. It can be reformulated in terms of almost complex structures and it has a good completely computed illustration in the framework of symplectic geometry (we mean the geometry of pseudo holomorphic curves which we discuss at the end of this lecture).

So the problem is

Problem. *Find a geometrical condition on hermitian triple (g, J, ω) over a smooth almost complex based manifold which would be equivalent to the nontriviality condition for the Seiberg - Witten invariant of the canonical class K_J.*

Let us repeat again that may be the problem in the simply connected case has been solved by the Taubes result.

At the same time there were some attempts to find such kind of constructive condition in terms of a special map which exists over each almost complex 4 - dimensional manifold. Recall from [9] the definition: let X be an almost complex 4 - manifold (so the manifold which admits almost complex structures) and $\mathcal{M}_X$ is the space of all possible hermitian triples over X. Then one has a map

$$\tau : \mathcal{M}_X \to H^2(X, \mathbb{R})$$

which is defined absolutely canonically. Namely if (g, J, ω) is a triple belongs to $\mathcal{M}_X$ then the first element g and the second J defines a Hodge star operator (one takes the conformal class of g and the orientation coming with J) hence one can decompose the third element with respect to $*_{(g,J)}$ using the Hodge result

$$\omega = \omega_H + d\rho_1 + d^*\rho_2.$$

The first summand ω_H is a harmonic form (moreover due to self duality of ω it is self dual harmonic form) hence it represents a 2 - cohomological class $[\omega_H] \in H^2(X, \mathbb{R})$. This gives us

$$\tau(g, J, \omega) = [\omega] \in H^2(X, \mathbb{R})$$

and totally the map

$$\tau : \mathcal{M}_X \to H^2(X, \mathbb{R}).$$

This map is called canonical map because it doesn't require for the definition any additional choices. This naturality implies the fact that this map is equivariant with respect to the action of $Diff X$.

If X is algebraic or kahler or symplectic then it's clear that there exists a triple over X with nontrivial image in $H^2(X, \mathbb{R})$ namely if ω is closed then

$$\omega = \omega_H$$

and the image $\tau(g, J, \omega)$ is exactly $[\omega] \in H^2(X, \mathbb{R})$. On the other hand it was shown in [9] that the manifold Y constructed in [5] which doesn't admit symplectic structure at the same time admits a hermitian triple with non trivial image in $H^2(X, \mathbb{R})$ (moreover this class is exactly the image of $[\omega]$ with respect to the isomorphism

$$H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z}),$$

where ω is the symplectic form on X, see above). The next step was done in [11] where one decoded the Seiberg - Witten system in spirit of this canonical construction in almost complex geometry. Namely one has the fact that if riemannian metric g doesn't admit reducible solution for the Seiberg - Witten system then every solution (irreducible of course) gives us a hermitian triple with nontrivial image with respect to τ. One have some additional results in this way (see [11]).

On the other hand dealing with this canonical map τ one can generalize Kahler - Einstein equation which is equivalent to Einstein equation for Kahler metrics (see [13]). Namely let us find an appropriate $Diff^+X$ - invariant equation on the space of hermitian triples over an almost complex 4 - dimensional X. Such an equation could be interesting in absolutely general setup of classical gravity theory. After a momoment we see that one has an equation which has the form

$$\tau(g, J, \omega) = K_J \qquad (**)$$

where K_J is the canonical class of J. Equivariance of τ with respect to $Diff^+X$ gives us that this equation is invariant under the action. On the other hand if X is a Kahler manifold and (g, J, ω) is the Kahler triple then this equation coincides (up to constant) with the Kahler - Einstein equation. Let us note that a priori solutions to (**) exist if $K_J^2 \geq 0$ which gives us well known Hitchin - Thorpe inequality for Einstein manifolds (see [15]). So if we consider a gauge theory with $Diff^+X$ as the gauge group using the equation (**) then we get the following result (see [11]) relating this gauge theory with a "classical" gauge theory which is the Seiberg - Witten theory. Namely solutions to (**) exist if K_J is a basic canonical class over X and K_J can be realized by a self dual harmonic form with respect to a riemannian metric g. One expects that pure cohomological equation (**) is related to some "real" equation on forms, that the last one could be related with the classical Einstein equation and together these would imply some new facts about Einstein metrics in dimension 4.

We shall recall here two other constructions which relate the theory with geometrical topics.

First, according to results of R. Fintushel and R. Stern the Seiberg - Witten theory is closely related to the knot theory. The relationship is presented in [7]; there one defines a special polynomial expression on basic classes weighted by the Seiberg - Witten numbers (values of the integer valued Seiberg - Witten function) and the relationship is that if one changes the based manifold applying surgery along a knot γ then the Seiberg - Witten polynomial is multiplied by the Alexander polynomial of the knot.

Second, in the symplectic case there is a beautiful correspondence between Seiberg - Witten and Gromov invariants. Let us recall the construction of the last one. For a symplectic manifold X with symplectic form ω one fixes sufficiently generic almost complex structure J which is compatible with ω. Then for any 2 - homological class $\eta \in H_2(X, \mathbb{Z})$ one takes pseudoholomorphic realizations of this class that is such smooth submanifolds $\Sigma \subset X$ representing η that at each point $s \in \Sigma$ the tangent space $T_s\Sigma$ is a complex subspace with respect to J in T_sX. Such a submanifold was called by M. Gromov "pseudoholomorphic curve" since in the case when X, J is a complex surface this definition gives just holomorphic curves which lie on the surface. Thus for a fixed J one defines a set

$$\mathcal{H}_\eta = \{\Sigma \subset X | [\Sigma] = \eta \in H_2(X, \mathbb{Z}) and \forall s \in \Sigma J(T_s\Sigma) = T_s\Sigma\}$$

and one can prove (see f.e. [6]) that for sufficiently generic J this set is a real analityc orientable manifold of real dimension

$$\mu(\mu - K_J)$$

where $\mu \in H^2(X, \mathbb{Z})$ is Poincare dual to η cohomological class. One can compare this formula with the classical Riemann - Roch formula for divisors on complex surface. Comparing these two formula one has to remember that the first one is real dimension while Riemann - Roch gives complex dimension of the corresponding say complete linear system. The difference however is quite striking: one can see that canonical divisor has in symplectic setup unique representation by pseudoholomorphic curves while in algebraic geometry on has the number p_g which can be different from 1. To define the Gromov invariant one takes a set of $\frac{1}{2}\mu(\mu - K_J)$ points in general position and derive from $\mathcal{H}_\eta$ subspace consists of such pseudoholomorphic curves which contain all of these chosen points. This subspace consists of isolated points which can be counted with signs with respect to a natural orientation and this counting gives us an integer number which is the Gromov invariant of the given symplectic manifold. Again we extend this definition varying homological class η inside of the lattice $H_2(X, \mathbb{Z})$ which gives us an integer valued function on the lattice which is invariant under $Diff X$ - action.

The definition of this invariants is very geometrical and illustrative. Here we deal with natural geometrical objects which take place in almost complex geometry as well as in symplectic geometry. Really for any hermitian triple (g, J, ω) one can take the set of pseudoholomorphic curves representing a fixed homological class but in this general case there are no results which would support the extension of the definition. Returning to the symplectic case we recall that Taubes proved that in this case the Seiberg - Witten invariants are equivalent to the Gromov invariants. As it was mentioned above for symplectic manifold X, ω there is a preffered $Spin^{\mathbb{C}}$ - structure $c = -K_\omega$ (see above); then each other $Spin^{\mathbb{C}}$ - structures (we suppose that $H^2(X, \mathbb{Z})$ has no 2 - torsion) can be labeled by 2 - homological classes namely on the level of spinor bundles we have relationship

$$W^\pm_\eta = L_\eta \otimes W^\pm_0$$

where L_η is the line bundle whose first Chern class is Poincare dual to homological class η. It's clear that homological class η corresponds to $Spin^{\mathbb{C}}$ - structure $2\eta^* - K_\omega$. Then the Taubes result looks like the exact correspondence

$$N_{SW}(2\eta^* - K_\omega) = \pm Gr(\eta)$$

where Gr is the integer Gromov invariant of η.

Thus while the result of Fintashel and Stern relates the 4 - dimensional invariants with the geometry of 1 - dimensional submanifolds the Taubes correspondence $SW = Gr$ defines a link with the geometry of 2 - dimensional submanifolds.

The correspondence $SW = Gr$ can be exploited in the studying of the last problem which we mention in these lectures. This problem was stated and partially solved by Taubes (see [20]). This is the problem of non symplectic blow down of symplectic manifolds.

Let X be a symplectic manifold. Then if we take the connected sum

$$Y = X \sharp \overline{\mathbb{CP}}^2$$

then it is symplectic: the construction was called "symplectic blow up procedure", it corresponds exactly to algebro geometric blown up procedure. Moreover Taubes proved that if Y is a symplectic manifold which contains a smoothly embedded 2 - sphere with self intersection -1 then there exists the following decomposition

$$Y = X \sharp \overline{C\mathbb{P}}^2$$

and the procedure which gives us this X can be called symplectic blow down procedure (being analogously to well known process in algebraic geometry). The conjecture which was proposed by Taubes says that

Problem. *For each symplectic 4 - manifold Y there are no non symplectic blow downs so Y can be decomposed as $X \sharp N$ where $b_2^+(N) = 0$ if and only if Y admits smoothly embedded 2 - sphere with self intersection -1 and N is $\overline{\mathbb{CP}}^2$.*

This problem has many colors and reflects almost all the problems listed above. We end this notes with the problem which is from our view point quite important and intrinsic for the theory itself. As we mentioned above in the first lecture one expects that a "twistor space" for the Seiberg - Witten theory can be constructed in symplectic category and it implies the interest to each fact related to symplectic geometry. On the other hand one can see that all fundamental questions in 4 - geometry form some background of the problem. For example why one takes these $\overline{\mathbb{CP}}^2$s as summands glued to a given manifold? The reason is that every simply connected smooth 4 - manifold with negative definite intersection form is homeomorphic to $k\overline{\mathbb{CP}}^2$ and if one proves that there exists unique smooth structure on $\mathbb{CP}^2$ it would imply that every such manifold is not only homeomorphic to this connected sum but is diffeomorphic to the last one. This would imply that there is only one operation in the simply connected case which doesn't kill the invariants. Thus this problem is closely related to the uniqueness problem for the known 4 - manifold. Other things were studied in [10]: let Y be a symplectic manifold which can be decomposed in connected sum

$$X\overline{\mathbb{CP}}^2$$

such that the associated canonical class K_Y splits with respect to the usual representation

$$H^2(Y, \mathbb{Z}) = H^2(X, \mathbb{Z}) \oplus \mathbb{Z}h$$

where h is the generator of 2 - cohomology group of $\overline{\mathbb{CP}}^2$ as follows

$$K_Y = c_X + kh$$

where c_X is a $Spin^{\mathbb{C}}$ - structure over X with positive dimensional moduli space $\mathcal{M}_X$. Then it can take place if and only if either c_X is a canonical class (but then $\mathcal{M}_X$ has zero virtual dimension) or

$$v.dim\mathcal{M}_X = 2$$

and $k = 3$. So arithmetically this situation (when one has non symplectic blow down) can happen. But operating with moduli spaces of Seiberg - Witten solutions and with moduli spaces of Gromov pseudoholomorphic curves one gets the contradiction with the condition that Y is a symplectic manifold (see [10]). Thus again the problem is projected on the question is there some other smooth manifold which has the same topological properties as $\overline{\mathbb{CP}}^2$ but admits different smooth structure?

Let us say again that these notes do not pretend to be a complete survey of the Seiberg - Witten theory at all. We just remark here some points of the theory which have clear geometrical description or definition. And we try to illuminate a geometrical point of view on the theory at all. We skip a lot of technical details and quite important results since they can be found in the literature. One can divide the references to the subject into three groups. The first one contains original papers in which all the most important results were established. The second part ([16] - [21]) consists of textbooks and courses on the Seiberg - Witten theory. The number of such courses increases all the time so we can list below just the top of the aciberg. The third part is exactly the survey [22] which has to be recommended for every reader. But this paper itself requires something to be considered so at the same part we add a textbook on 4 - dimensional geometry contains all what one needs to study the subject in the widest context.

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