Graphs, Strings and Actions

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Summary. In this paper, we revisit the formalism of graphs, trees and surfaces which allows one to build cell models for operads of algebraic interest and represent them in terms of a dynamical picture of moving strings — hence relating string dynamics to algebra and geometry. In particular, we give a common framework for solving the original version of Deligne's conjecture, its cyclic, A_∞ and cyclic–A_∞ version. We furthermore study a question raised by Kontsevich and Soibelman about models of the little discs operad. On one hand we give a new smooth model and the other hand a minimally small cell model for the A_∞ case. Further geometric results these models provide are novel decompositions and realizations of cyclohedra as well as explicit simple cell representatives for Dyer-Lashof-Cohen operations. We also briefly discuss the generalizations to moduli space actions and applications to string topology as well as further directions.

Introduction

As often happens in pure mathematics a dynamical physical point of view can be very helpful in solving complex problems. One instance of these dynamics which has been particularly useful is string theory. There are many incarnations of this theory given by highly developed mathematical tools, such as Gromov–Witten theory or Singularity Theory. We will take a less algebraic and more geometric point of view in the following. Surprisingly this approach turns out to have far reaching algebraic and topological implications. The basic idea is to treat a string as an interval or a circle with a measure. As these types of strings move, split and recombine, they give rise to a surface *with a partially measured foliation.* These ideas are completely described in [KP], where actually we are considering strings that move on an oriented surface with boundary. We will only consider the closed case here and furthermore restrict ourselves to surfaces with no internal punctures.

The first step in obtaining applications to algebra and topology is to represent these surfaces by certain types of ribbon graphs. The measure of the foliation translates to weights on the edges of these graphs. To be precise there are two types of graphs. One is called the *arc graph* which is obtained by replacing each band of parallel leaves of the foliation with one edge called an arc. There is a dual picture provided the foliation sweeps out the surface. The condition for this to occur is that the complementary regions of the arcs are polygonal. This condition is called quasi–filling. In this case there is the natural notion of a dual graph. This dual graph is again a ribbon graph with weights on its edges and furthermore the surface it defines is precisely of the same topological type as the underlying surface. We stress that in general this need not be the case. Usually we call this dual graph if it exists the associated ribbon graph or simply the ribbon graph. In these considerations we take the closed strings to be pointed, which induces marked points on the boundary and marked points on the cycles of the ribbon graph.

Now it is striking that with this picture one obtains several well–known algebraic and topological objects in one big swoop. The first object is an operad [KLP] which is defined when all the boundaries are hit by arcs. This contains the moduli space of genus g curves with n marked points and a tangent vector at each of its points as a rational suboperad. Here rational means densely defined. Furthermore taking a different route and using $\mathbb{R}_{\geq 0}$ graded operads instead [KP] one can even induce a modular operad structure on cohomology. We will forgo this option and concentrate on the cell level instead. This cell level is described by graphs, one ribbon graph of the above type for each cell. Focussing our attention on different types of graphs, we obtain natural operads, cyclic operads, PROPs and other algebraic structures[1].

Moreover, we are naturally lead to cell models for various known and important operads such as *the little discs, the framed little discs*. Extending the graphs we are lead to the definition of a ribbon graph operad for a cell model of moduli space and a model for a PROP which can be called the Sullivan PROP[2].

In the current note, we wish to present the results as a reverse engineering of sorts, starting with the combinatorics and building spaces out of them. This is contrary to the historical genesis and the dynamic approach mentioned above, but it is a purely algebraic/combinatorial formulation which matches up beautifully with natural operations on the Hochschild complex of various algebras. We will treat the associative, A_∞, Frobenius and Frobenius A_∞ algebra cases. The latter has sometimes been called [Ko2] a cyclic A_∞ algebra. The classical case has been solved in [Ko3, T, MS1, Vo1, KS, MS2, BF, K2], the cyclic case was first established in [K3] and then extended in [TZ] (see also [MS4] for an announcement of a different proof), the A_∞ case has been treated in [KS]. The plethora of proofs goes back to the possibility to choose suitable chain models. In our approach the chain models are all CW–models which are minimal in a sense we explain below. Moreover they all appear

[1] See e.g. [MSS] for a review of these notions and the operads mentioned below.

[2] There are actually several versions of this PROP on the topological and chain level see e.g. [CS, S1, S2, CG, TZ], our version is that of [K4, K5].

naturally in a geometric picture painted out by string dynamics. The desire to have such operations has three main sources. String topology [CS, Vo2, CJ, CG, Ch, Me, S1, S2], Deligne's conjecture and D-brane considerations [KR, KLi1, KLi2]— see [K4, K5] for details.

Taking this approach, there are algebraic questions and obstacles, but it turns out that each time the geometry tells us how to overcome them. Along the way, we introduce new cell models for the little discs and framed little discs, some of which are smooth. This partially answers a question of Kontsevich and Soibelman on this subject. Finally, our cell models also cast light on the Dyer-Lashof Araki-Kudo [AK, Co, DL] operations on the Hochschild cohomology, which thanks to the affirmative answer to Deligne's conjecture formally has the structure of a double loop space. Here we give the explicit cells that are responsible for the operations, naturally reproducing the results of [We, Tou].

Finally, we comment on a new natural geometric stabilization for our surface operad. This lends itself to exploring all of the above constructions in a stable limit.

The paper is organized as follows:

§1 contains all the necessary details about graphs. §2 contains the construction of various cell models of the little discs and framed little discs using trees and graphs. In this paragraph, in particular, we also give a new smooth cellular model for the little discs and the framed little discs and a cell model for the minimal operad of [KS]. We furthermore identify the cells responsible for the Dyer-Lashof operations. To illustrate our approach to operations using trees §3 contains a full self-contained proof of the cyclic version of Deligne's conjecture for a Frobenius algebra. In §3 we also go on to treat the A_∞ and cyclic A_∞ versions. §4 contains the extensions to Moduli space and the Sullivan PROP, hence string topology. It also contains the important new notion of operadic correlation functions. We close the discussion in §5 with an outlook and complementary results on the higher loop spaces and stabilization.

Acknowledgments

This paper is dedicated to my teacher Yuri Ivanovich Manin, who inspired me by sharing his deep insight into the inner structure of mathematics, the mathematical structure of physics and the beautiful results which one can obtain by combining them.

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Conventions

We fix k to be a field of arbitrary characteristic. We let $\bar{n}$ be the set $\{0, \ldots, n\}$. I will denote the interval $[0, 1]$ and Δ^n the standard n–simplex. Furthermore

K_n is the n–th Stasheff polytope or associahedron and W_n is the n–the Cyclohedron or Bott-Taubes polytope, see e.g. [MSS] for the definitions of these polytopes.

1 Graphs, spaces of Graphs and cell models

1.1 Classes of Graphs

In this section, we formally introduce the graphs and the operations on graphs which we will use in our analysis.

We will use several types of trees and ribbon graphs.

Graphs

A graph Γ is a tuple $(V_\Gamma, F_\Gamma, \imath_\Gamma : F_\Gamma \to F_\Gamma, \partial_\Gamma : F_\Gamma \to V_\Gamma)$ where $\imath_\Gamma$ is an involution $\imath_\Gamma^2 = id$ without fixed points. We call V_Γ the vertices of Γ and F_Γ the flags of Γ. The edges E_Γ of Γ are the orbits of the flags under the involution $\imath_\Gamma$. A directed edge is an edge together with an order of the two flags which define it. In case there is no risk of confusion, we will drop the subscripts Γ. Notice that $f \mapsto (f, \imath(f))$ gives a bijection between flags and directed edges.

We also call $F_{\Gamma,v} := \partial^{-1}(v) \subset F_\Gamma$ the set of flags of the vertex v. If Γ is clear from the context, we will just write F_v, and we also call $|F_v|$ the valence of v and denote it by $val(v)$. We also let $E(v) = \{\{f, \imath(f)\} | f \in F(v)\}$ and call these edges the edges incident to v.

The geometric realization of a graph is given by considering each flag as a half-edge and gluing the half-edges together using the involution $\imath$. This yields a one-dimensional CW complex whose realization we call the realization of the graph.

As usual a tree is a graph whose image is contractible. A black and white graph, b/w for short, is a graph with a map $V_\Gamma \to \{0, 1\}$. The inverse image of 1 are called the white vertices and denoted by V_w while the inverse image of 0 are called the black vertices denoted by V_b.

Ribbon graphs

A ribbon graph with tails is a connected graph together with a cyclic order of the set of flags $F_\Gamma(v)$ of the vertex v for every vertex v. A ribbon graph with tails that satisfies $val(v) \geq 2$ for all vertices v will simply be called a ribbon graph. Notice that we do *not* fix $val(v) \geq 3$. We will call a ribbon graph stable if it does satisfy this condition.

For a ribbon graph with tails, the tail vertices are $V_{tail} = \{v \in V_\Gamma | val(v) = 1\}$, the tail edges $E_{tail}(\Gamma)$ are the edges incident to the tail vertices and the tail flags $F_{tail}(\Gamma)$ are those flags of the tail edges which are *not* incident to the tail vertices.

A tree that is a ribbon graph with tails is called a planar tree.

A graph with a cyclic order of the flags at each vertex gives rise to bijections $\mathrm{Cyc}_v : F_v \to F_v$ where $\mathrm{Cyc}_v(f)$ is the next flag in the cyclic order. Since $F = \amalg F_v$ one obtains a map $\mathrm{Cyc} : F \to F$. The orbits of the map $N := \mathrm{Cyc} \circ \imath$ are called the cycles or the boundaries of the graph. These sets have the induced cyclic order.

Notice that each boundary can be seen as a cyclic sequence of directed edges. The directions are as follows. Start with any flag f in the orbit. In the geometric realization go along this half-edge starting from the vertex $\partial(f)$, continue along the second half-edge $\imath(f)$ until you reach the vertex $\partial(\imath(f))$ then continue starting along the flag $\mathrm{Cyc}(\imath(f))$ and repeat.

An angle is a pair of flags $(f, \mathrm{Cyc}(f))$, we denote the set of angles by $\angle_\Gamma$. It is clear that $f \mapsto (f, \mathrm{Cyc}(f))$ yields a bijection between F_Γ and $\angle_\Gamma$. It is however convenient to keep both notions.

By an angle marking we mean a map $mk^\angle : \angle_\Gamma \to \mathbb{Z}/2\mathbb{Z}$.

The genus of a ribbon graph and its surface

The genus $g(\Gamma)$ of a ribbon graph Γ is given by $2 - 2g(\Gamma) = |V_\Gamma| - |E_\Gamma| + Cyc(\Gamma) = \chi(\Gamma) + Cyc(\Gamma)$ where $Cyc(\Gamma) = \#cycles$.

The surface $\Sigma(\Gamma)$ of a ribbon graph Γ is the surface obtained from the realization of Γ by thickening the edges to ribbons. I.e. replace each 0-simplex v by a closed oriented disc $D(v)$ and each 1-simplex e by $e \times I$ oriented in the standard fashion. Now glue the boundaries of $e \times I$ to the appropriate discs in their cyclic order according to the orientations. This is a surface whose boundary components are given by the cycles of Γ. The graph Γ is naturally embedded as the spine of this surface $\Gamma \subset \Sigma(\Gamma)$. Let $\bar{\Sigma}(\Gamma)$ be the surface obtained from $\Sigma(\Gamma)$ by filling in the boundaries with discs. Notice that the genus of the $\bar{\Sigma}(\Gamma)$ is $g(\Gamma)$ and $\chi(\Gamma) = 2 - 2g(\Sigma(\Gamma))$.

Treelike, normalized Marked ribbon graphs

Definition 1.1. A ribbon graph together with a distinguished cycle c_0 is called *treelike* if

i) the graph is of genus 0 and
ii) for all flags either $f \in c_0$ or $\imath(f) \in c_0$ (and not both).

In other words each edge is traversed exactly once by the cycle c_0. Therefore there is a cyclic order on all (non–directed) edges, namely the cyclic order of c_0.

The data above are called *almost treelike* if the condition i) holds and in condition ii) the exclusive “or” is replaced by the logical “or”. This means that there might be edges both of whose flags belong to c_0 . We call these edges the black edges of the graph.

Definition 1.2. A *marked ribbon graph* is a ribbon graph together with a map $mk : \{cycles\} \to F_\Gamma$ satisfying the conditions

i) For every cycle c the directed edge $mk(c)$ belongs to the cycle.
ii) All vertices of valence two are in the image of mk, that is $\forall v, val(v) = 2$ implies $v \in Im(\partial \circ mk)$.

Notice that on a marked treelike ribbon graph there is a linear order on each of the cycles c_i. This order is defined by upgrading the cyclic order to the linear order $\prec_i$ in which $mk(c_i)$ is the smallest element.

The intersection tree of an almost treelike ribbon graph

Notice that an almost treelike ribbon graph need not be a tree. Indeed if it has more than two cycles it won't be. But the following construction yields a black and white tree. The following definition of a dual tree is indeed a duality, as one can recover the ribbon graph from its dual tree. For the gory combinatorial details, see the appendix of [K2].

Dual b/w tree of a marked ribbon graph

Given a marked almost treelike ribbon graph Γ, we define its dual tree to be the colored graph whose black vertices are given by V_Γ and whose set of white vertices is the set of cycles c_i of Γ. The set of flags at c_i are the flags f with $f \in c_i$ and the set of flags at v are the flags $\{f : f \in c_0, \partial(f) = v\}$. The involution is given by $\imath_\tau(f) = N(f)$ if $f \in c_0$ and $\imath_\tau(f) = N^{-1}(f)$ else.

This graph is a tree and is b/w and bipartite by construction. It is also planar, since the c_i and the sets $F(v)$ have a cyclic order and therefore also $F_v \cap c_0$. It is furthermore rooted by declaring $\partial(mk(c_0))$ to be the root vertex. Declaring $mk(c_0)$ to be the smallest element makes it into a planted tree.

An equivalent definition is given by defining that there is an edge between a pair of a black and a white vertex if and only if the vertex corresponding to b is on the boundary of the cycle c_i, i.e. $v \in \partial(c_i) := \{\partial(f) : f \in c_i\}$ and two black vertices are connected if there was an black edge between them.

Spineless marked ribbon graphs

A marked almost treelike ribbon graph is called *spineless*, if

i) There is at most one vertex of valence 2. If there is such a vertex v_0 then $\partial(mk(c_0)) = v_0$.
ii) The induced linear orders on the c_i are (anti–)compatible with that of c_0, i.e. $f \prec_i f'$ if and only if $\imath(f') \prec_0 \imath(f)$.

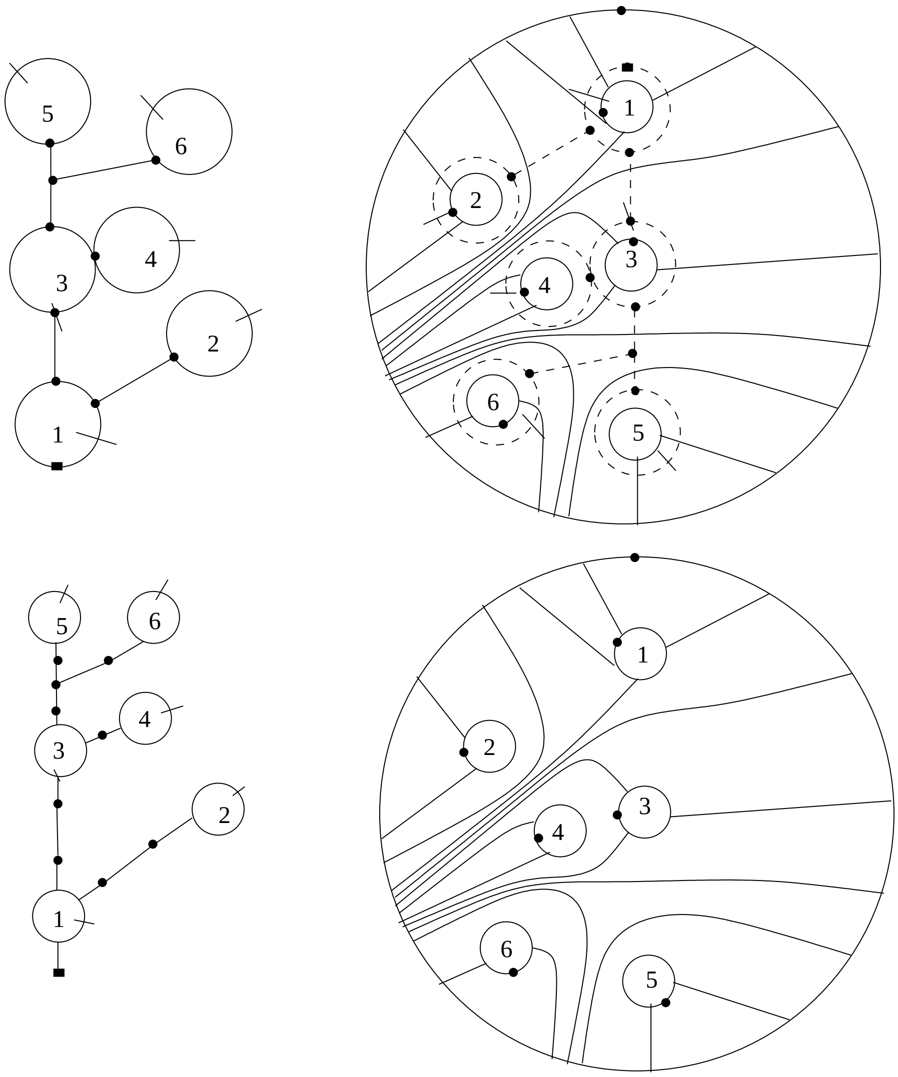

Fig. 1. An example of an element in $\mathcal{C}acti_\infty$, its arc graph and tree.

1.2 Operations on graphs

In this section, we will give the basic definitions of the operations on graphs which we will need.

Contracting edges

The contraction $\Gamma/e = (\bar{V}_\Gamma, \bar{F}_\Gamma, \bar{\imath}, \bar{\partial})$ of a graph $\Gamma = (V_\Gamma, F_\Gamma, \imath, \partial)$ with respect to an edge $e = \{f, \imath(f)\}$ is defined as follows. Let $\sim$ be the equivalence relation induced by $\partial(f) \sim \partial(\imath(f))$. Then let $\bar{V}_\Gamma := V_\Gamma / \sim$, $\bar{F}_\Gamma = F_\Gamma \setminus \{f, \imath(f)\}$ and $\bar{\imath} : \bar{F}_\Gamma \to \bar{F}_\Gamma, \bar{\partial} : \bar{F}_\Gamma \to \bar{V}_\Gamma$ be the induced maps.

For a ribbon graph the cyclic order is the one which descends naturally.

For a marked ribbon graph, we define the marking of $(\bar{V}_\Gamma, \bar{F}_\Gamma, \bar{\imath}, \bar{\partial})$ to be $\overline{mk}(\bar{c}) = \overline{mk(c)}$ if $mk(c) \notin \{f, \imath(f)\}$ and $\overline{mk}(\bar{c}) = \overline{N \circ \imath(mk(c))}$ if $mk(c) \in \{f, \imath(f)\}$, viz. the image of the next flag in the cycle.

If there is an angle marking, set $f' = N^{-1}(f), f'' = \mathrm{Cyc}(f), g' = N^{-1}(\imath(f))$ and $g'' = \mathrm{Cyc}(\imath(f))$, let $mk^\angle(f', f) = a, mk^\angle(f, f'') = b, mk^\angle(g', \imath(f)) = c$ and $mk^\angle(\imath(f), g'') = d$, after the contraction we set $mk^\angle(f', g'') = \overline{\bar{a}\bar{d}}$ and $mk^\angle(g', f'') = \overline{\bar{b}\bar{c}}$, where we use the notation $\bar{a} = 1 - a \in \mathbb{Z}/2\mathbb{Z}$.

1.3 Spaces of graphs with metrics

Notation 1.3. We will write $\mathcal{R}ib_{n,g}$ for the set of marked ribbon graphs of genus g with n cycles and, by abuse of notation, also for the free Abelian group generated by this set.

We set $\mathcal{R}ib := \amalg_{n,g}\mathcal{R}ib_{n,g}$, and we will again not distinguish in notation between the set $\mathcal{R}ib$, the free Abelian group generated by it, and the set $\{\amalg_g \mathcal{R}ib_{n,g} : n \in \mathbb{N}\}$ to avoid unnecessary clutter. We also write $\mathcal{R}ib(n)$ for the set of marked ribbon graphs with $n+1$ cycles together with a labelling by $\{0, \dots, n\}$ of these cycles. Again we also denote the free Abelian group generated by this set as $\mathcal{R}ib(n)$. Finally, to streamline the notation, we will denote the collection $\{\mathcal{R}ib(n) | n \in \mathbb{N}\}$ simply by $\mathcal{R}ib$.

The meaning of the symbols will always be clear from the context.

Graphs with a metric

A metric w_Γ for a graph is a map $E_\Gamma \to \mathbb{R}_{>0}$. The (global) re-scaling of a metric w by λ is the metric $\lambda w : \lambda w(e) = \lambda w(e)$. The length of a cycle c is the sum of the lengths of its edges $length(c) = \sum_{f \in c} w(\{f, \imath(f)\})$. A metric for a treelike ribbon graph is called normalized if the length of each non–distinguished cycle is 1. We will write $\mathcal{MR}ib_{n,g}$ for the set of metric marked ribbon graphs of genus g with n boundary cycles.

Projective metrics

Notice that there is an $\mathbb{R}_{>0}$–action on $\mathcal{MR}ib$ which scales the metric μ by an overall factor. This action of course preserves the genus and number of boundaries. We set $\mathbb{P}\mathcal{R}ib := \mathcal{MR}ib/\mathbb{R} > 0$. The elements of $\mathbb{P}\mathcal{R}ib$ are called graphs with a projective metric. Notice that one can always choose a normalized representative for any projective metric. We set $\mathbb{P}\mathcal{R}ib_{n,g} = \mathcal{MR}ib_{n,g}/\mathbb{R}_{>0}$.

Remark 1.4. Now and in the following, we do not wish to dwell on distinguishing projective and non-projective metrics.

Definition 1.5. By a local scaling at a cycle i, we mean that the metric is scaled only on the edges belonging to the cycle i.

The space of metric ribbon graphs

We endow these above sets with a topology by constructing $\mathbb{P}\mathcal{R}ib_{n,g}$ in the standard fashion. That is we realize them as a subspace of the quotient of the disjoint union of simplices by an equivalence relation. For each graph $\Gamma \in \mathcal{R}ib_{n,g}$ with $|E(\Gamma)| = k+1$ we fix a k-simplex Δ_Γ. Using barycentric coordinates for this simplex, a point of this simplex can be identified with a choice of projective weights on the edges. The points of $\mathbb{P}\mathcal{R}ib_{n,g}$ can thus be identified with the interior of the disjoint union over all $\Delta_\Gamma : \Gamma \in \mathcal{R}ib_{n,g}$. Furthermore the faces of Δ_Γ correspond to the edges of Γ. Now, we use the following identifications: A face of Δ_Γ is identified with $\Delta_{\Gamma/e}$ if $\Gamma/e \in \mathcal{R}ib_{n,g}$. We give the resulting space the quotient topology (this is actually a CW complex) and identify $\mathbb{P}\mathcal{R}ib$ with the image of the interiors of the Δ_Γ. Then we give $\mathcal{M}\mathcal{R}ib := \mathbb{P}\mathcal{R}ib \times \mathbb{R}_{>0}$ the product topology.

Cacti and spineless cacti and thickened Cacti

Definition 1.6. We let $\mathcal{C}acti(n)$ the subspace of the treelike ribbon graphs with n labeled cycles (that is excluding the distinguished cycle c_0). Furthermore we let $\mathcal{C}act(n) \subset \mathcal{C}acti(n)$ be the subset of spineless cacti.

We let $\mathcal{C}acti_\infty$ be the almost treelike ribbon graphs and $\mathcal{C}act_\infty$ be the almost treelike spineless ribbon graphs.

Marked ribbon graphs with metric and maps of circles.

For a marked ribbon graph with a metric, let c_i be its cycles, let $|c_i|$ be their image in the realization and let r_i be the length of c_i. Then there are natural maps $\phi_i : S^1 \to |c_i|$ which map S^1 onto the cycle by starting at the vertex $v_i := \partial(mk(c_i))$ and going around the cycle mapping each point $\theta \in S^1$ to the point at distance $\frac{\theta}{2\pi} r_i$ from v_i along the cycle c_i. This observation connects the current constructions to those involving a more geometric definition of $\mathcal{C}acti$ in terms of configurations of circles [Vo2, K1] and other geometric constructions involving such configurations like the map $\mathcal{L}oop$ used for the $\mathcal{A}rc$ operad [KLP]. In particular the treelike ribbon graphs correspond to $\mathcal{C}acti$ and the spineless treelike ribbon graphs correspond to $\mathcal{C}act$.

This observation is also useful in order to describe the gluing operations.

$$\circ_i : \mathcal{C}acti_\infty(n) \times \mathcal{C}acti_\infty(m) \to \mathcal{C}acti_\infty(n+m-1) \tag{1}$$

$$(\Gamma_1, \Gamma_2) \mapsto \Gamma_1 \circ_i \Gamma_2 \tag{2}$$

which are given by scaling Γ_2 to the size of the length of the ith cycle of Γ_1 and then gluing together the graphs by using the identification given by the corresponding maps of S^1s parameterizing the scaled cycle c_0 of Γ_2 and the cycle c_i of Γ_1.

For a purely combinatorial version of this construction we refer to the appendix of [K2]. The version presented there *loc. cit.* which pertains to $\mathcal{C}acti$ can easily be adapted to the current case of $\mathcal{C}acti_\infty$.

Proposition 1.7. *The spaces $\mathcal{C}acti_\infty(n)$ together with the $\mathbb{S}_n$ action permuting the labels and the gluing operations $\circ_i$ of eq. (1) form a topological operad, and the subspaces $\mathcal{C}act_\infty(n), \mathcal{C}acti(n), \mathcal{C}act(n)$ form sub-operads.*

Proof. Straightforward.

Recall that two operads are equivalent as operads if there is a chain of quasi–isomorphisms connecting them.

Theorem 1.8. *$\mathcal{C}acti_\infty(n)$ and $\mathcal{C}acti(n)$ are equivalent to the framed little discs operad and $\mathcal{C}act_\infty(n)$ and $\mathcal{C}act(n)$ are equivalent to the little discs operad.*

Proof. The statements about $\mathcal{C}acti$ and $\mathcal{C}act$ are contained in [K1]. The corresponding statements about $\mathcal{C}acti_\infty$ and $\mathcal{C}act_\infty$ follow from the observation that these spaces are homotopic to $\mathcal{C}acti$ and $\mathcal{C}act$ by the homotopy which contracts all the edges both of who's flags are elements of the distinguished cycle c_0.

Cactus terminology

The edges of a cactus are traditionally called arcs or segments and the cycles of a cactus are traditionally called lobes. The vertices are sometimes called the marked or special points. Furthermore the distinguished cycle c_0 is called the outside circle or the perimeter and the vertex $\partial(mk(c_0))$ is called the global zero. And the vertices $\partial(mk(c_i)), i \neq 0$ are called the local zeros. In pictures these are represented by lines rather than fat dots.

Normalized treelike and almost treelike ribbon graphs and their cell complexes

Definition 1.9. An element of $\mathcal{C}acti_\infty$ is called normalized, if the length of all the cycles except for possibly the distinguished cycle are 1 and the length of all of the black edges is less or equal to 1. We use the superscript 1 on the spaces above to indicate the subset of normalized elements, e.g. $\mathcal{C}acti^1_\infty$.

Notation 1.10. We will call an element of the set $\{\mathcal{C}acti, \mathcal{C}act, \mathcal{C}act_\infty, \mathcal{C}acti_\infty\}$ simply a species of cactus.

Lemma 1.11. *Every species of cactus is homotopy equivalent to its subspace of normalized elements.*

Proof. The homotopy is given by locally scaling each lobe to size 1. Notice this is possible, because the graphs are almost treelike.

The normalized versions have their good side and their bad side. On the bad side, we see that they are not stable under gluing, but we can modify the gluing as follows to obtain a topological quasi–operad, that is an operad which is associative only up to homotopy.

$$\circ_i : \mathcal{C}acti^1_\infty(n) \times \mathcal{C}acti^1_\infty(m) \to \mathcal{C}acti_\infty(n+m-1) \tag{3}$$

$$(\Gamma_1, \Gamma_2) \mapsto \Gamma_1 \circ_i \Gamma_2 \tag{4}$$

Here the composition is given by first locally scaling the lobe i of Γ_1 to the length of the distinguished cycle of Γ_2 and then glue.

Proposition 1.12. *The normalized elements of any species of cactus together with the $\mathbb{S}_n$ action of re-labeling and the glueings above form a topological quasi–operad.*

Proof. Tedious but straightforward. See [K1] for $\mathcal{C}act$ and $\mathcal{C}acti$, the more general version is covered under the Sullivan PROP in [K4], see also §4.

The relations between the species are as follows:

Theorem 1.13. *[K1] The operad of cacti is the bi–crossed product of the operad $\mathcal{C}act$ of spineless cacti with the operad $\mathcal{S}^1$ based on the monoid S^1. Furthermore this bi–crossed product is homotopic to the semi–direct product of the operad of cacti without spines with the circle group S^1.*

$$\mathcal{C}acti \cong \mathcal{C}act \bowtie \mathcal{S}^1 \simeq \mathcal{C}act \rtimes \mathcal{S}^1 \tag{5}$$

The same holds true for the thickened versions

$$\mathcal{C}acti_\infty \cong \mathcal{C}act_\infty \bowtie \mathcal{S}^1 \simeq \mathcal{C}act_\infty \rtimes \mathcal{S}^1 \tag{6}$$

The details of the semi-direct products and bi–crossed products are given below.

Proof. The proof of the first statement is given by verifying that the two operad structures coincide. For the second statement one notices that the homotopy diagonal is homotopy equivalent to the usual one and that one can find homotopies to the diagonal which continuously depend on the cactus. The third statement follows from contracting the factors $\mathbb{R}^n_{>0}$ and using Theorem 1.15. Full details are given in [K1] for the non–thickened species. They go over *mutatis mutandis* for the thickened species.

Corollary 1.14. *The homology operad of $\mathcal{C}acti$ is the semi-direct product of $\mathcal{C}act$ and the homology of the operad $\mathcal{S}^1$ built on the monoid S^1. The same holds true for $\mathcal{C}acti_\infty$.*

Theorem 1.15. *Every species of cacti is homotopy equivalent through quasi-operads to its normalized version.*

Proof. The statement for regular cacti is contained in [K1] and the argument carries over *mutatis mutandis* to the thickened versions.

Corollary 1.16. *Every species of cactus is quasi-isomorphic as quasi-operads to its normalized version and in particular the induced homology quasi-operads are operads and are isomorphic as operads.*

Details of the bi–crossed product structure for Cacti

In this section we recall the construction of the bi-crossed product as it was given in [K1] to which we refer the reader for more details.

First notice that there is an action of S^1 on $\mathcal{C}act(n)$ given by rotating the base point *clockwise* (i.e. in the orientation opposite the usual one of c_0) around the perimeter. We denote this action by

$$\rho^{S^1}: S^1 \times \mathcal{C}act(n) \to \mathcal{C}act(n)$$

With this action we can define the twisted gluing

$$\begin{aligned} \circ_i^{S^1}: \mathcal{C}act(n) \times S^1(n) \times \mathcal{C}act(m) &\to \mathcal{C}act(n+m-1) \\ (C, \theta, C') &\mapsto C \circ \rho^{S^1}(\theta_i, C') =: C \circ_i^{\theta_i} C' \end{aligned} \tag{7}$$

Given a cactus without spines $C \in \mathcal{C}act(n)$ the orientation reversed perimeter (i.e. going around the outer circle *clockwise* i.e. reversing the orientation of the source of ϕ_0) gives a map $\Delta_C: S^1 \to (S^1)^n$.

As one goes around the perimeter the map goes around each circle once and thus the map Δ_C is homotopic to the diagonal $\Delta_C(S^1) \sim \Delta(S^1)$.

We can use the map Δ_C to give an action of S^1 and $(S^1)^{\times n}$.

$$\rho^C: S^1 \times (S^1)^{\times n} \stackrel{\Delta_C}{\to} (S^1)^{\times n} \times (S^1)^{\times n} \stackrel{\mu^n}{\to} (S^1)^{\times n} \tag{8}$$

here μ_n is the diagonal multiplication in $(S^1)^{\times n}$ and $\bar{\circ}_i$ is the operation which forgets the i-th factor and shuffles the last m factors to the i-th, ..., $i+m-1$st places. Set

$$\begin{aligned} \circ_i^C: (S^1)^{\times n} \times (S^1)^{\times m} &\stackrel{(id \times \pi_i)(\Delta) \times id}{\longrightarrow} (S^1)^{\times n} \times S^1 \times (S^1)^{\times m} \\ &\stackrel{id \times \rho^C}{\longrightarrow} (S^1)^{\times n} \times (S^1)^{\times m} \stackrel{\bar{\circ}_i}{\longrightarrow} (S^1)^{\times n+m-1} \end{aligned} \tag{9}$$

These maps are to be understood as perturbations of the usual maps

$$\begin{aligned} \circ_i: (S^1)^{\times n} \times (S^1)^{\times m} &\stackrel{(id \times \pi_i)(\Delta) \times id}{\longrightarrow} (S^1)^{\times n} \times S^1 \times (S^1)^{\times m} \\ &\stackrel{id \times \rho}{\longrightarrow} (S^1)^{\times n} \times (S^1)^{\times m} \stackrel{\bar{\circ}_i}{\longrightarrow} (S^1)^{\times n+m-1} \end{aligned} \tag{10}$$

where now ρ is the diagonal action of S^1 on $(S^1)^{\times n}$. The maps $\circ_i$ and the permutation action on the factors give the collection $\{\mathcal{S}^1(n)\} = (S^1)^{\times n}$ the structure of an operad. In fact this is exactly the usual construction of an operad built on a monoid.

The multiplication in the bi-crossed product is given by

$$(C,\theta)\circ_i(C',\theta')=(C\circ_i^{\theta_i}C',\theta\circ_i^{C'}\theta') \tag{11}$$

The multiplication in the semi-direct product is given by

$$(C,\theta)\circ_i(C',\theta')=(C\circ_i^{\theta_i}C',\theta\circ_i\theta') \tag{12}$$

Also, normalized cacti are homotopy equivalent to cacti which are homotopy equivalent to the bi-crossed product of normalized cacti with $\mathcal{S}^1$ and the semi-direct product with $\mathcal{S}^1$, where all equivalences are as quasi-operads

$$\mathcal{C}acti^1\sim\mathcal{C}acti\cong\mathcal{C}act\bowtie\mathcal{S}^1\sim\mathcal{C}act^1\bowtie\mathcal{S}^1\sim\mathcal{C}act^1\rtimes\mathcal{S}^1 \tag{13}$$

2 The tree level: Cell models for (framed) little discs and their operations

The virtue of the normalized species is that they provide cellular models. In order to give the cell model, we will use the dualized trees.

2.1 A first cell model for the little discs: $\mathcal{C}act^1$

In this section we will give a cell model for $\mathcal{C}act^1$. It will be indexed by the dual trees of the ribbon graphs. The specific type of trees we need are given by the sets $\mathcal{T}^{bp}(n)$, that is planar planted bipartite black and white trees with only white leaves. Here as usual a leaf is a vertex of valence one that is not the root. Since the tree is rooted the edges have a natural direction towards the root and we call the edges which are incoming to white vertices the white edges and denote the set they form by E_w.

Notice that the differential on the ribbon graphs induces a differential on the dual trees.

Definition 2.1. We define $\mathcal{T}^{bp}(n)^k$ to be the elements of $\mathcal{T}^{bp}(n)$ with $|E_w|=k$.

Definition 2.2. For $\tau\in\mathcal{T}^{bp}$ we define $\Delta(\tau):=\times_{v\in V_w(\tau)}\Delta^{|v|}$. We define $C(\tau)=|\Delta(\tau)|$. Notice that $\dim(C(\tau))=|E_w(\tau)|$.

Given $\Delta(\tau)$ and a vertex x of any of the constituting simplices of $\Delta(\tau)$ we define the x-th face of $C(\tau)$ to be the subset of $|\Delta(\tau)|$ whose points have the x-th coordinate equal to zero.

Definition 2.3. We let $K(n)$ be the CW complex whose k-cells are indexed by $\tau \in \mathcal{T}^{bp}(n)^k$ with the cell $C(\tau) = |\Delta(\tau)|$ and the attaching maps e_τ defined as follows. We identify the x-th face of $C(\tau)$ with $C(\tau')$ where $\tau' = \partial_x(\tau)$ is the local contribution of the differential contracting the corresponding white edge. This corresponds to contracting an edge of the cactus if its weight goes to zero so that $\Delta(\partial\tau)$ is identified with $\partial(\Delta(\tau))$.

Lemma 2.4. *$K(n)$ is a CW composition for $\mathcal{C}act$.*

Proof. It is straightforward to see that the differential on the graphs which contracts an edge on the tree side collapses an angle.

Proposition 2.5. *$K(n)$ is a cellular chain model for the little discs.*

Proof. The claim is that already on the cell level the induced quasi-operad is an operad. This is indeed the case, since in a cell all possible positions of the lobes are possible and the composition again gives all possible positions, see [K1] for details.

2.2 A CW decomposition for $\mathcal{C}acti^1$ and a cellular chain model for the framed little discs

Definition 2.6. A $\mathbb{Z}/2\mathbb{Z}$ decoration for a black and white bipartite tree is a map $dec^{\pm} : V_w \to \mathbb{Z}/2\mathbb{Z}$.

Proposition 2.7. *The quasi–operad of normalized cacti $\mathcal{C}acti^1$ has a CW–decomposition which is given by cells indexed by planar planted bi–partite trees with a $\mathbb{Z}/2\mathbb{Z}$ decoration. The k–cells are indexed by trees with $k-i$ white edges and i vertices marked by 1.*

Moreover cellular chains are a chain model for the framed little discs operad and form an operad. This operad is isomorphic to the semi–direct product of the chain model of the little discs operad given by $CC_*(\mathcal{C}act)$ of [K2] and the cellular chains of the operad built on the monoid S^1.

Proof. For the CW decomposition we note that as spaces $\mathcal{C}acti^1(n) = \mathcal{C}act^1(n) \times (S^1)^{\times n}$. Now viewing $S^1 = [0,1]/0 \sim 1$ as a 1-cell together with the 0-cell given by $0 \in S^1$ the first part of the proposition follows immediately, by viewing the decoration by 1 as indicating the presence of the 1-cell of S^1 for that labeled component in the product of cells.

To show that the cellular chains indeed form an operad, we use the fact that the bi–crossed product is homotopy equivalent to the semi–direct product in such a way that the action of a cell S^1 in the bi–crossed product is homotopic to the diagonal action. This is just the observation that the diagonal and the diagonal defined by a cactus are homotopic. Since a semi-direct product of a monoid with an operad is an operad the statement follows. Alternatively one could just remark that there is also an obvious functorial map induced by the diagonal for these cells.

The chains are a chain model for the framed little discs operad since $\mathcal{C}acti^1(n)$ and $\mathcal{C}acti(n)$ are homotopy equivalent and the latter is equivalent to the framed little discs operad.

Although the above chain model is the one one would expect to use for framed little discs, it does not have enough cells for our purposes. In order to translate the proofs in the arc complex given in [KLP] into statements about the Hochschild complex, we will need a slightly finer cell structure than the one above. After having used the larger structure one can reduce to the cell model with less cells as they are obviously equivalent.

Definition 2.8. A spine decoration dec' for a planted planar bi–partite tree is a $\mathbb{Z}/2\mathbb{Z}$ decoration together with the marking of one angle at each vertex labeled by one and a flag at each vertex labeled by zero. We call the set of such trees which are n-labeled by $\mathcal{T}^{bp,dec'}(n)$ and again use this notation as well for the free Abelian group and the k vector space generated by these sets. We let $\mathcal{T}^{bp,dec'}$ be their union respectively direct sum. In pictures we show the angle marking as a line emanating from the vertex which lies between the marked edges and an edge marking by a line through the respective edge. For an example see Figure 2 VI. We sometimes omit the edge marking if the marked edge is the outgoing edge, e.g. in Figure 3.

The realization $\hat{\tau}$ of a planar planted bi–partite tree τ with a spine decoration is the realization of τ as a planar planted tree (the root is fixed to be black) together with one additional edge inserted into each marked angle connecting to a new vertex. We call the set of these edges spine edges and denote them by E_{spine}. Likewise set V_{spine} to be the set of new vertices called the spine vertices which are defined to be black. The spine edges are then white edges. Like for tails, we will only consider the flags of E_{spine}, which are not incident to the spine vertices. We call the set of these flags F_{spine}. Notice that this tree is the dual tree of a cactus with an explicit marking of the flags $mk(c_i)$. Given a cactus, we call its dual tree with explicit markings its topological type. If τ had tails, we will split the set of tails of the realization into spines and free tails which are the images of the original tails: $E_{tails}(\hat{\tau}) = E_{ftails}(\hat{\tau}) \amalg E_{spine}(\hat{\tau})$; and we proceed likewise for the respective flags.

A spine decoration induces a new linear order on the flags incident to the white vertices of its realization. This order $\prec'_v$ is given by the cyclic order at v and declaring the smallest element to be the spine flag in case $dec^{\pm}(v) = 1$ and the marked flag in case $dec^{\pm}(v) = 0$. This gives a canonical identification of $F_{\prec'_v} : F_v \to \{0, \dots, |v|\}$.

Proposition 2.9. *The spaces $\mathcal{C}acti^1(n)$ of the quasi–operad of normalized cacti $\mathcal{C}acti^1$ have CW–decompositions $K'(n)$ whose cells are indexed by spine decorated planar planted bi–partite trees $(\tau, dec') \in \mathcal{T}^{bp,dec'}$ corresponding to the topological type of the cacti. The k–cells are indexed by n-labeled trees with $k-i$ white edges and i markings by* 1.

Moreover cellular chains of the complex above are a chain model for the framed little discs operad and form an operad.

Proof. The decomposition is almost as in the preceding proposition except that in the product $\mathcal{C}act^1(n) \times (S^1)^{\times n}$ we decompose each factor S^1 as indicated by the lobe it presents. I.e. for the S^1 associated to the n–th lobe we chose the 0–cells to be corresponding to the marked points and 1–cells corresponding to the arcs with gluing given by attaching the 1–cells to the 0–cells representing the endpoints of the arcs. (E.g. 4 0-cells and 4 1-cells for the lobe 1 in Figure 2 VIa). In terms of trees, the arcs correspond to the angles and thus we take a marking of an arc to be the inclusion of the corresponding 1-cell in the tensor product of the cell complexes. Likewise the edges correspond to the marked points and we take a marking of an edge to be the inclusion of the corresponding 0-cell in the tensor product of the cell complexes.

For the operadic properties, we remark that moving the spine along an arc and then gluing, which is what is parameterized by marking an angle on the lobe i of c when calculating $c \circ_i c'$, has the effect of moving the base point of c' along a complete sequence of arcs until it coincides with a marked point in the composition of the two cacti. This is one side of the bi-crossed product. The effect on the local zeros of c' of the movement of the base point is to move them corresponding to structure maps of the bi-crossed product above. The local zeros thus move through a full arc if the global zero passes through the arc on which they lie. Therefore the $\circ_i$ product of two cells results in sums of cells. Marking an arc of c' obviously gives rise to a sum of cells. Alternatively, one can again just remark that there is a functorial map for the diagonal for this cell model, since there is such a map on the first factor by [K2] and its existence is obvious on the second factor.

The associativity follows from the associativity of cacti. Let $C(\tau)$, $\tau \in \mathcal{T}^{bp,dec'}(n)$ be the cells in the CW-complex and $\dot{C}(\tau)$ their interior. Then $P(\tau) = \dot{C}(\tau) \times \mathbb{R}^n_{>0}, \tau \in \mathcal{T}^{bp,dec'}$ give a pseudo-cell decomposition $Cacti(n) = \amalg_\tau P(\tau)$. It is easy to see that $Im(P(\tau) \circ_i P(\tau')) = \amalg_k P(\tau_k)$ for some τ_k and $\circ_i$ is a bijection onto its image. Let $\circ_i^{comb}$ be the quasi-operad structure pulled back from K' to $\mathcal{T}^{bp,dec'}$ and $\circ_i^{+}$ be the operad structure pulled back from the pseudo-cell decomposition of $\mathcal{C}acti$ to $\mathcal{T}^{bp,dec'}$. Then these two operad structures coincide over $\mathbb{Z}/2\mathbb{Z}$ thus yielding associativity up to signs. The signs are just given by shuffles, c.f. §3.1, and are associative as well.

Remark 2.10. Pulling back the operadic compositions, the differential and the grading yields a dg-operad structure on $\mathcal{T}^{bp,dec'}$ which is isomorphic to that of $CC_*(Cacti^1) := \bigoplus_n CC_*(K'(n))$, where $CC_*(K'(n))$ are the cellular chains of the CW–model $K'(n)$ of $\mathcal{C}act^1(n)$.

The operation is briefly as follows: given two trees $\tau, \tau' \in \mathcal{T}^{bp,dec'}$ the product is $\tau \circ_i^{comb} \tau' = \sum \pm \tau_k$ where the τ_k are the trees obtained by the following procedure. Delete v_i to obtain an ordered collection of trees $(\tau_l^c, \prec'_v)$ then graft these trees to τ' keeping their order by first identifying the spine edge or marked edge of v_i with the root edge of τ' and then grafting the rest of the branches to τ' so that their original order is compatible with that of τ'. Lastly contract the image of the root edge of τ' and declare the image of the root of τ to be the new root. The sign is as explained in §3.1. Due to the isomorphism between $CC_*(Cacti^1)$ and $\mathcal{T}^{bp,dec'}$ we will drop the superscript *comb*.

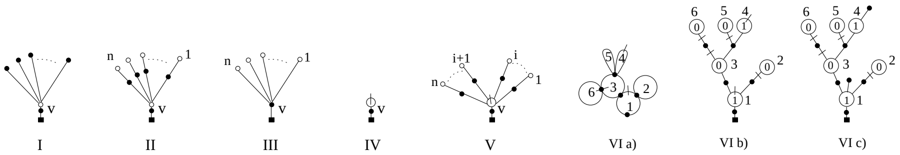

Fig. 2. I. the tree l_n II. the tree τ_n III. the tree τ_n^b IV. the tree O' V. the tree $\tau'_{n,i}$ VI.a) a marked tree-like ribbon graph b) the corresponding decorated tree c) its realization

2.3 The GBV structure

The picture for the GBV structure is essentially that of [KLP] and goes back to [CS]. It appears here in another guise, however, since we are now dealing with cells in $CC_*(Cacti^1)$.

First notice that there is a product on the chain level induced by the spineless cactus given by the rooted tree τ_n depicted in Figure 2. Explicitly: $a \cdot b \mapsto \gamma(\tau_2^b; a, b)$ where γ is the usual operadic composition. This product gives $CC_*(Cacti^1)$ the structure of an associative algebra with unit. Moreover the product is commutative up to homotopy. The homotopy is given by the usual operation which is induced by $\gamma(\tau_1; a, b)$. This also induces a bracket which is Gerstenhaber up to homotopy. This can be seen by translating the statements from [KLP, K2], but it also follows from the BV description of the bracket below (Figure 5).

To give the BV structure, let O' be the tree with one white vertex, no additional black edges, no free tails and a spine. Notice that the operation δ induced by $a \mapsto \gamma(O', a)$ on $CC_*(Cacti^1)$ breaks up on products of chains as follows, see Figure 3

$$\begin{aligned} \delta(ab) &\sim \delta(a,b) + (-1)^{|a||b|}\delta(b,a) \\ \delta(abc) &\sim \delta(a,b,c) + (-1)^{|a|(|b|+|c|)}\delta(b,c,a) \\ &\quad + (-1)^{|c|(|a|+|b|)}\delta(c,a,b) \end{aligned} \tag{1}$$

$$\delta(a_1 a_2 \cdots a_n) \sim \sum_{i=0}^{n-1} (-1)^{\sigma(c^i,a)} \delta(a_{c^i(1)}, \ldots, a_{c^i(n)}) \tag{2}$$

where c is the cyclic permutation and $\sigma(c^i, a)$ is the sign of the cyclic permutations of the graded elements a_i.

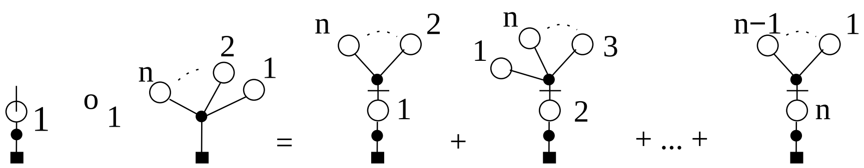

Fig. 3. The decomposition of the BV operator

Lemma 2.11.

$$\delta(a,b,c) \sim (-1)^{(|a|+1)|b|} b\delta(a,c) + \delta(a,b)c - \delta(a)bc \tag{3}$$

Proof. The proof is contained in Figure 4.

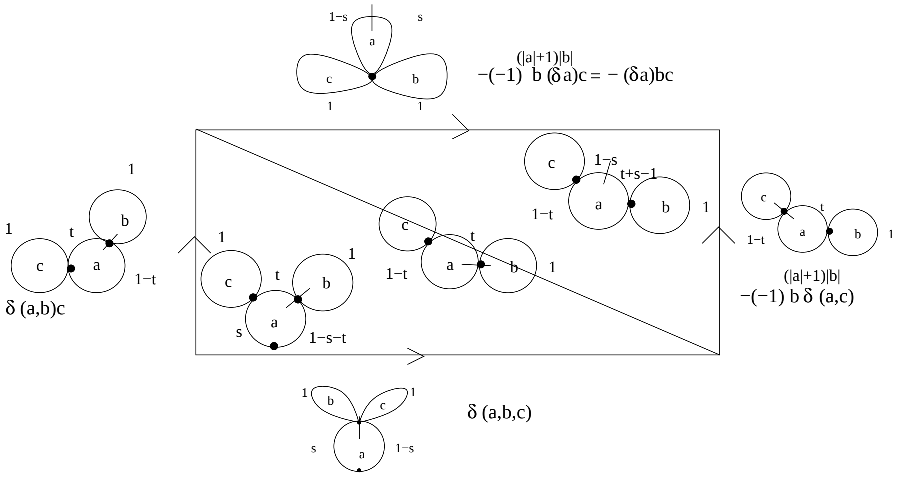

Fig. 4. The basic chain homotopy responsible for BV

Proposition 2.12. *The chains $CC_*(Cacti^1)$ are a GBV algebra up to homotopy. That is there is a bracket and a BV operator that satisfy the usual equations up to homotopy. Taking coefficients in k when k is of characteristic zero, the homology of $\mathcal{C}acti$ hence becomes a GBV algebra.*

Proof. The BV structure follows from the Lemma 2.11 via the calculation:

$$\begin{aligned}\delta(abc) &\sim \delta(a,b,c) + (-1)^{|a|(|b|+|c|)}\delta(b,c,a) + (-1)^{|c|(|a|+|b|)}\delta(c,b,a)\\ &\sim (-1)^{(|a|+1)|b|}b\delta(a,c) + \delta(a,b)c - \delta(a)bc + (-1)^{|a|}a\delta(b,c)\\ \\ &\quad +(-1)^{|a||b|}\delta(b,a)c - (-1)^{|a|}a\delta(b)c + (-1)^{(|a|+|b|)|c|}a\delta(b,c)\\ &\quad +(-1)^{|b|(|a|+1|)+|a||c|}b\delta(c,a) - (-1)^{|a|+|b|}ab\delta(c)\\ &\sim \delta(ab)c + (-1)^{|a|}a\delta(bc) + (-1)^{|a+1||b|}b\delta(ac) - \delta(a)bc\\ &\quad -(-1)^{|a|}a\delta(b)c - (-1)^{|a|+|b|}ab\delta(c) \qquad (4)\end{aligned}$$

Figure 5 contains the homotopy relating the BV operator to the bracket.

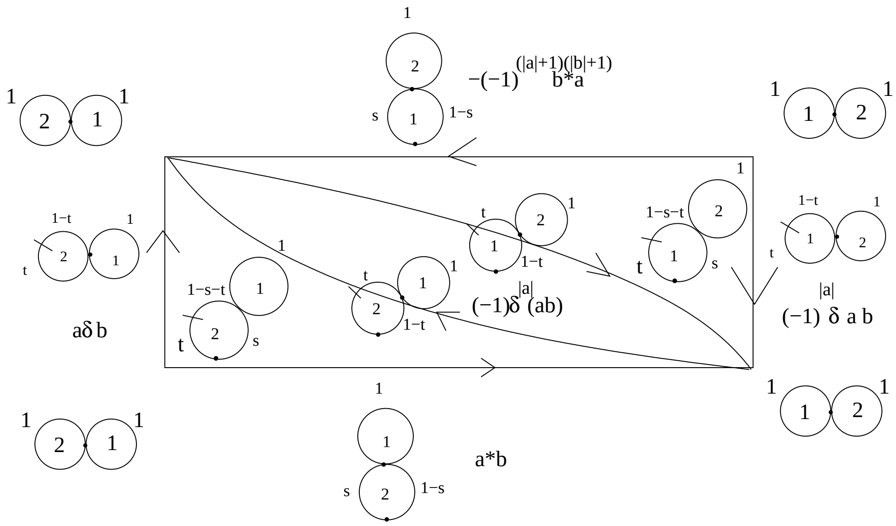

Fig. 5. The compatibility of the BV operator and the bracket

2.4 Cells for the Araki–Kudo–Cohen, Dyer–Lashof operations

By the general theory, see e.g. [Tou] we need to find elements

$$\xi_1 \in H_{p-1}(\mathcal{C}act^1(p)/\mathbb{S}_p, \pm\mathbb{Z}/p\mathbb{Z})$$

that is homology classes with values in the sign representation.

Now taking co-invariants on $\mathcal{C}act^1$ we see that the iteration of the product $*$ that is the operation given by ${}^n* := \gamma(\gamma(\dots(\gamma(\tau_1),\tau_1),\dots,\tau_1),\tau_1)$ gives a class that is the sum over all trees of the highest dimension where the partial order on the labeled vertices when considered in the usual tree partial order is compatible with the linear order on $\bar{n}$.

Proposition 2.13. ${}^n*$ *is the cohomology class* ξ_1 *in* $H_{p-1}(\mathcal{C}act^1(p)/\mathbb{S}_p, \pm\mathbb{Z}/p\mathbb{Z})$.

Proof. First we could re-engineer the result from the proof of Tourtchine [Tou], but it also follows from a straightforward calculation of the boundary of said cell.

The first example for $p = 2$ is given by the operation of τ_1 which has boundaries in the multiplication and its opposite, cf. Figure 5, and the example for $p = 3$ is the hexagon of Figure 6 with $i = 1$.

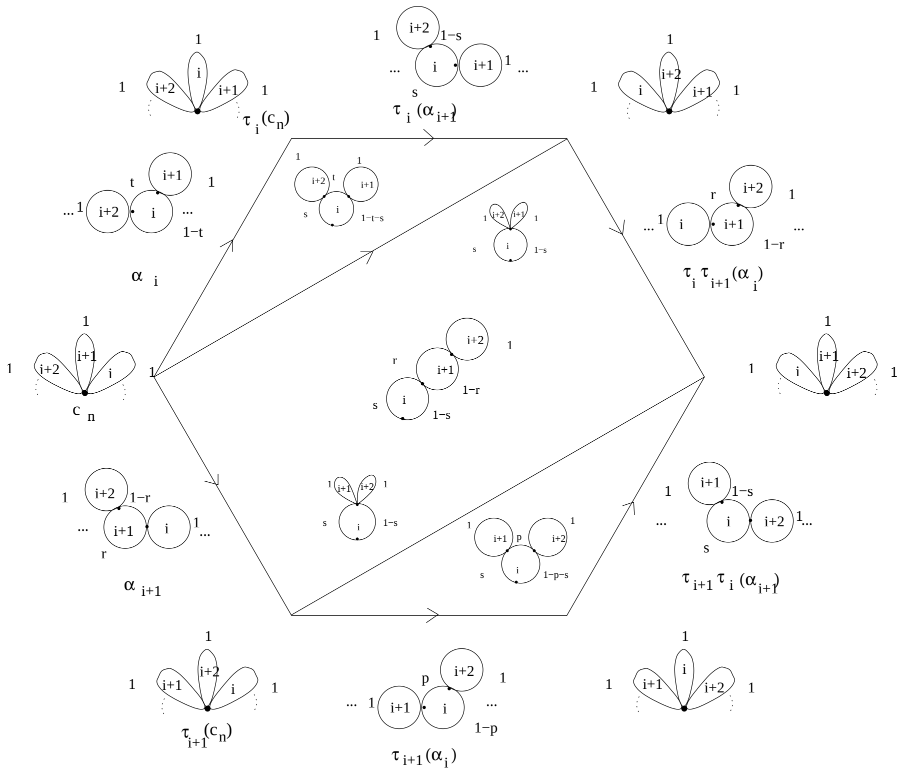

Fig. 6. The hexagon that gives the Dyer–Lashof operation, establishes that $\mathcal{C}acti$ is a braid operad and shows the associativity up to homotopy of the multiplication.

Remark 2.14. We wish to point out two interesting facts. First, the class is solely induced by an operation for $p = 2$ and secondly, the resulting cell description is just the left iteration of $*$, whereas the right iteration of $*$ is the simple class given by a cube.

2.5 A smooth cellular model for the framed little discs: $\mathcal{C}acti_\infty$

The above CW model for the little discs is actually the smallest model which solves Deligne's conjecture and has enough cells to support the brace and the multiplication operations. However the model is easily seen to be non–smooth starting at $n = 3$. We can read this off Figure 6, since two of these hexagons glue to give a "cylinder with wings".

There is, however, a surprisingly small CW model that is smooth. It is given by considering $\mathcal{C}act_\infty$. This is not the minimal model that yields a solution for the A_∞–Deligne conjecture, which will occupy the next paragraph.

The relevant trees

Again the cells of this model will be indexed by certain types of trees which are the dual trees of the ribbon graphs that are elements of $\mathcal{C}act^1_\infty$. These are planar planted b/w trees with heights.

It will be convenient to use the convention that these trees have a black root with valence one and call the unique incident edge the root edge. We will call the edges which are *incident to a white vertex* the *white edges* –E_w– the other edges are considered to be black –E_b–. The exception is the root edge which is not considered to be black in case it is not white. We will also fix the stability condition that there are not black vertices of valence 2 with two black edges[3]. We well call such trees $\mathcal{T}_{b/w}$.

Definition 2.15. We let $\mathcal{HT}(n)$ be pairs (τ, ht) of a planar planted b/w tree with white leaves only and a black root (τ) and a function $\mathrm{ht} : E_b \to \{1, var\}$. We will let E_{var} be the inverse image of var, and call them variable edges. Likewise let $\mathcal{HT}^{top}(n)$ be pairs $(\tau, \mathrm{ht}^{top})$ with τ as above and $\mathrm{ht}^{top} : E \to [0, 1]$ such that the sum of the weights of the edges adjacent to a white vertex is 1.

Remark 2.16. Notice that there is a natural differential on the underlying ribbon graphs which can also be considered to have white and black edges. The latter which are labeled by $1, var$. The differential is given by summing (with the appropriate sign) over contractions of the white edges, contractions of the black edges labeled by var and re-labelings of these edges by 1.

Definition 2.17. We define $\mathcal{HT}(n)^k$ to be the elements of $\mathcal{HT}(n)$ with $|E_{var}| + |E_w| - |V_w| = k$.

Definition 2.18. For $\tau \in \mathcal{HT}$ we define $\Delta(\tau) := \times_{v \in V_w(\tau)} \Delta^{|v|} \times \times_{e \in E_{var}(\tau)} I$. We define $C(\tau) = |\Delta(\tau)|$. Notice that $\dim(C(\tau)) = |E_w(\tau)| + |E_b(\tau)|$.

Given $\Delta(\tau)$ and a vertex x of any of the constituting simplices of $\Delta(\tau)$ we define the x-th face of $C(\tau)$ to be the subset of $|\Delta(\tau)|$ whose points have the x-th coordinate equal to zero. The boundaries of the intervals are taken to be 0 and 1.

[3]This means no parallel arcs in the dual picture.

Definition 2.19. We let $K_\infty(n)$ be the CW complex whose k-cells are indexed by $\tau \in \mathcal{HT}(n)^k$ with the cell $C(\tau) = |\Delta(\tau)|$ and the attaching maps e_τ defined as follows. We identify the x-th face of $C(\tau)$ with $C(\tau')$ where $\tau' = \partial_x(\tau)$. This corresponds to contracting a white edge of the cactus as its weight goes to zero so that $\Delta(\partial\tau)$ is identified with $\partial(\Delta(\tau))$ for these edges. For the black edges, passing to the boundaries of the intervals corresponds to letting the weight of the edge go to 1 or 0, and the latter is taken to mean that the relevant edge is contracted.

Lemma 2.20. *$K_\infty(n)$ is a CW composition for $\mathcal{C}act^1_\infty$.*

Proof. For this it suffices to remark that the dual ribbon graph of the tree indexing a cell and an element of this cell has a natural metric on the corresponding graph given by the barycentric coordinates on the simplices for the white edges and the natural coordinates the intervals between 0 and 1 on the black edges. Conversely using the dual tree construction turns any element of $\mathcal{C}acti_\infty$ into a tree of the given type and the metric determines a unique point in the open cell..

Proposition 2.21. *$K_\infty(n)$ is a cellular chain model for the little discs.*

Proof. The proof is analogous to the one for normalized spineless cacti.

Theorem 2.22. *The space $\mathcal{C}act^1_\infty$ is smooth, that is it is a manifold with corners.*

Proof. The easiest way to see this is to use the dual description in terms of arc graph. The arc graph to be the dual graph on the surface $\Sigma(\Gamma)$ to Γ, where Γ is imbedded as the spine of this surface (More details are contained in §4.1). Now fix an element $p \in \mathcal{C}acti^1_\infty(n)$. Either it has the maximal number of edges, that is the complementary regions of the arc graph are triangles[4], then we can vary the weights of the white edges freely and the weights of the black variable edges as well, while the ones for the black edges with weight 1 can only decrease. So for the interior of the maximal cells we are done. If we are in the interior of a cell of lower dimension, some of the complementary regions are not triangles, but rather polygons. Now not all the diagonals are allowed, since we have to take care that the resulting ribbon graph is still tree-like. To be precise the vertices of the polygons are labeled by $i \neq 0$ or by 0 and the diagonals are not allowed to connect two vertices with non–zero labels. But, the vertices adjacent to a vertex with a non–zero label have to be labeled by zero. See Figure 8 for an example. The relevant space is a subspace of the product of the spaces of the diagonals of all of the polygons. Now the space of diagonals of a polygon near the point without diagonals is homeomorphic to a neighborhood of zero in the corresponding Stasheff polytope. There is

[4] We contract the edges of the polygon which lie on the boundary and label them by the corresponding boundary component

a subpolygon given by connecting the non–zero labeled vertices. Removing these points corresponds to collapsing cubes in the cubical decomposition of the Stasheff polytope in such a fashion that the result is again a polytope. See Figure 7 for an example. The image of 0 can however now lie on a face of the polytope. Nevertheless we again have found a neighborhood which is homeomorphic to a neighborhood of 0 in $\mathbb{R}^n \times \mathbb{R}^k_{\geq 0}$.

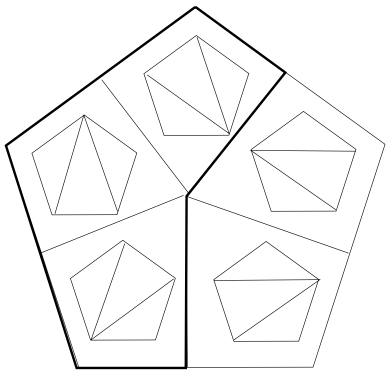

Fig. 7. The Stasheff polytope K_4 its cubical decomposition and the polytope of the cells avoiding one diagonal of the underlying polygon

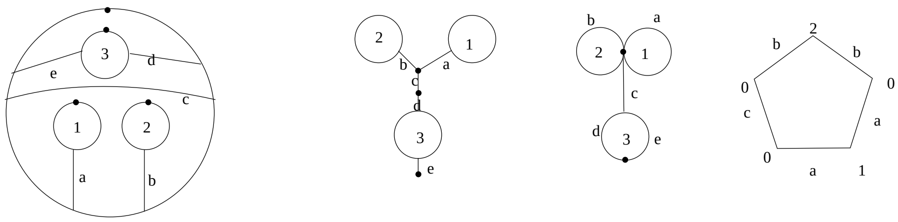

Fig. 8. An arc graph, its tree, cactus representation and one of its polygons

Remark 2.23. This cell model almost answers a question of Kontsevich and Soibelman in [KS]. Namely the existence of a certain smooth CW model for the Fulton MacPherson configuration spaces. In fact this is a minimal thickening of a minimal cell model of the little discs which is minimal in the sense that it contains all the cells for the A_∞ multiplications and the brace operations, that is a cell incarnation of the minimal operad $\mathcal{M}$ of [KS] which we construct in the next section. See also Remark 2.32.

2.6 The KW cell model for the little discs

In this section, we construct two more cell models for the little discs operad. The first will be a cell incarnation of the minimal operad of [KS] and the second will be a cacti based model which is a contraction of the model $\mathcal{C}act^1_\infty$ above. We need this second model only as a mediator, to establish the equivalence to the little discs operad.

Trees

The relevant trees are the stable b/w planar planted trees of [KS] with white leaves $\mathcal{T}_\infty$. Here stability means that there are no black vertices of valence 2.

The minimal A_∞ complex

Let K_n denote the n–th Stasheff polytope (Associahedron) of dimension $n-2$ and let W_n denote the n–the Bott–Taubes polytope (Cyclohedron) of dimension $n-1$,

We will now construct the following CW complex $\mathcal{KW}$. The cells are indexed by $\tau \in \mathcal{T}_\infty$ the cell for τ is given by

$$\Delta(\tau) := \times_{v \in V_{\text{white}}} W_{|v|} \times \times_{v \in V_{\text{black}}} K_{|v|-1} \tag{5}$$

The boundary of this cell is given by

$$\begin{aligned}\partial(\Delta(\tau)) &= \sum_{v' \in V_{\text{white}}} \pm \partial W_{|v|} \times \times_{v \neq v' \in V_{\text{white}}} W_{|v|} \times \times_{v \in V_{\text{black}}} K_{|v|} \\ &\quad + \sum_{v' \in V_{\text{black}}} \pm \times_{v \in V_{\text{white}}} W_{|v|} \times \partial K_{|v'|} \times \times_{v \neq v' \in V_{\text{black}}} K_{|v|} \\ &= \Delta(\partial(\tau)) \end{aligned} \tag{6}$$

Fixing n we inductively glue the cells corresponding to $\tau \in \mathcal{T}_\infty(n)$ to the existing skeleton by identifying the boundary pieces with the cells of lower dimension. For this we have to remark that indeed the cell differential given above agrees with the differential on $\mathcal{T}_\infty$ which is straightforward.

We call the resulting CW complex $\mathcal{KW}(n)$.

Lemma 2.24. *The collection $CC_*(\mathcal{KW}(n))$ form an operad isomorphic to the minimal operad of [KS].*

Proof. Since $\mathcal{T}_\infty$ is an operad, we just pull back the operad structure, since as Abelian groups $CC_*(\mathcal{KW}(n)) \simeq \mathcal{T}_\infty(n)$.

2.7 A finer cell model, the generalized Boardman-Vogt decomposition

In order to connect the above cell model with the little discs, we need to transform it slightly by subdividing the cells. We will call the corresponding model $\mathcal{KS}$. First, we identify the spaces of the two CW models and then afterwards, we can contract to the model $\mathcal{C}act^1$. The full details are in [KSch].

Decomposing the Stasheff polytope

For this we need two basic decompositions. First we decompose the associahedron in its Boardman-Vogt decomposition (see e.g [MSS]). We will actually need a topological realization, which is given by trees with heights. In this case, we consider a planar planted tree as used in this construction as a b/w trees with white leaves and topologically realize the cubical cells by using a height function on the black edges. This means a point in this cubical model of K_n is an element of $\mathcal{HT}^{top}$.

Decomposing the cyclohedra

We actually decompose the cyclohedra as a blow–up of the simplex. For this we again use b/w trees in $\mathcal{HT}^{top}$ as above. The basic simplex is given by taking a tree with one internal vertex. Now we glue to this simplex the cells that allow black edges to appear. This is again easiest to describe in the arc graph. We consider all arc graphs corresponding to at most one internal white vertex, but we allow diagonals -that is edges between 0 and 0– that do not form a triangle of which two sides are identified. An example of such a complementary region is given in Figure 8.

Trees and their cell complex

In other words, we consider trees of $\mathcal{T}_{b/w}$ with the following restrictions. There are no black vertices of valence two such that one edge is a leaf edge and the other is black[5].

We call this subset $\mathcal{T}^{rig}_{b/w}$. For the height functions, we have one more restriction. A height function for $\mathcal{T}^{rig}_{b/w}$ is compatible if the height of a black edge, both of whose vertices are valence 2 has to be 1. This restriction is needed, but in a sense is un-geometric. Omitting it, one is led to the thickened model above. It is necessary to make the incidence relations of the cells match.

It is clear that we can again glue a cell complex from these trees. This time

$$\Delta(\tau) := \times_{v\in V_w(\tau)}\Delta^{|v|} \times \times_{e\in E_{var}I} \qquad (7)$$

[5]This means that there is no triangle with two sides given by the same arc in the polygon picture.

In particular, there is a new subdivision of cyclohedra, that is not the Boardman–Vogt subdivision. The cells are products of cubes *and* simplices. This also allows for a partially linear realization in terms of trees with heights. Here the restriction for the cyclohedron is that there is only one non–leaf white vertex. See Figure 9 for an example in the language of arc graphs, for further details we refer to [KSch].

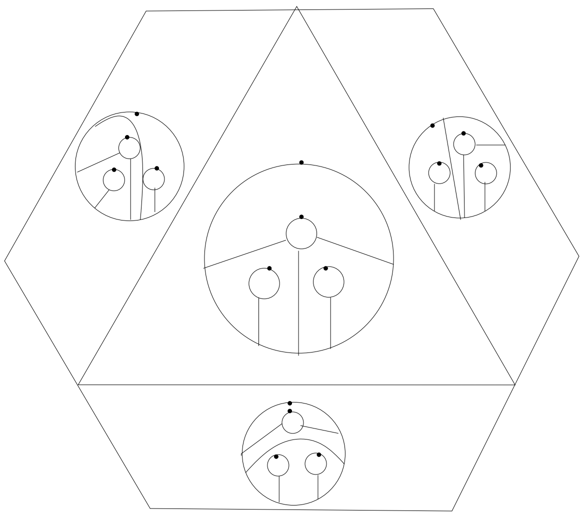

Fig. 9. The subdivision of W_2 into a simplex and cubes.

Proposition 2.25. *Each element of $\mathcal{KW}$ corresponds to a pair (τ, ht) with $\tau \in \mathcal{T}^{rig}_{b/w}$ and* ht *a compatible height function. That is the $\mathcal{KS}$ and $\mathcal{KW}$ are cell models for the same space. In the description $\mathcal{KS}$ an element is given by the tree of its cell and a compatible topological height function.*

Proof. Any element of $\mathcal{KW}$ lies in a unique maximal cell. This corresponds to a tree $\tilde{\tau} \in \mathcal{T}_\infty$. Now each cyclohedron and associahedron of the product making up $|D(\tau)|$ has a decomposition as above and our element inside the cell $\Delta(\tilde{\tau})$ lies inside one of these finer cells. Inside this product the element is given by a tree with height satisfying the given conditions. Moreover given a pair (τ, ht) satisfying the above conditions it is easy to see that this element in the above description belongs to the cell $\Delta(\tilde{\tau})$ where $\tilde{\tau}$ is the tree in which all the black edges with $\mathrm{ht} < 1$ are contracted.

The homotopy from $\mathcal{KS}$ to $\mathcal{C}act^1$

Definition 2.26. We define the flow $\Psi : I \times \mathcal{KS} \to \mathcal{C}act^1$ by $1 \geq t > 0$: $\Psi(t)((\tau, \mathrm{ht})) = (\tau, \psi(t)(\mathrm{ht}))$ where

$$\psi(t)(\mathrm{ht})(e) = \begin{cases} \mathrm{ht}(e) & \text{if } e \notin E_{\mathrm{black}} \\ t\,\mathrm{ht}(e) & \text{if } e \in E_{\mathrm{black}} \end{cases}$$

and $\Psi(0)(\tau, \mathrm{ht}) = \tilde{\tau}, \tilde{\mathrm{ht}})$ where $\tilde{\tau}$ is the tree τ with all black edges contracted and $\tilde{\mathrm{ht}}$ is ht descended to $\tilde{\tau}$.

Definition 2.27. We define $i_\infty^{\mathrm{top}} : \mathcal{C}act^1(n) \to \mathcal{KW}(n)$ by mapping (τ, ht) to itself.

Proposition 2.28. *The spaces $\mathcal{C}act^1(n)$ and $\mathcal{K}S(n)$ are homotopy equivalent and hence $\mathcal{KW}$ is too.*

Proof. Using the flow Ψ and the maps i_∞^{top} the statement is straightforward.

The cell level: maps π_∞ and i_∞

On the cell level this induces the following maps. There are maps $\pi_\infty : \mathcal{T}_\infty \to \mathcal{T}^{bp}$ and $i_\infty : \mathcal{T}^{bp} \to \mathcal{T}_\infty$.

The first π_∞ is given as follows. If there is a black vertex of valence > 3, then the image is set to be 0. If all black vertices are of valence 3, then contract all black edges and then insert a black vertex into each white edge. It is clear that the leaves will stay white. The global marking is defined to be the image under the contraction.

The second map i_∞ is given as follows. Remove all black vertices with valence $= 2$ and replace each black vertex of valence > 2 by the binary tree, with all branches to the left.

It is clear that π_∞ is surjective and $\pi_\infty \circ i_\infty = id$.

Lemma 2.29. *These maps behave well with respect to the differential. $\pi_\infty(\partial(\tau)) = \partial\pi_\infty(\tau)$ and the same for i_∞. And π_∞ is an operadic map.*

Proof. It is straightforward to check.

We now come to the main statement of the paragraph:

Theorem 2.30. *The topological spaces $\mathcal{KW}(n)$ and $\mathcal{C}act^1(n)$ are homotopy equivalent. Moreover the homotopy is given by an explicit contraction Ψ which descends to the chain level operadic map $\pi_\infty : CC_*(\mathcal{KW}) \to CC_*(Cact^1)$ where we used the isomorphisms of operads $CC_*(\mathcal{KW}) \simeq \mathcal{T}_\infty$ and $CC_*(Cact^1) \simeq \mathcal{T}^{bp}$ to pull back the map π_∞.*

Proof. First it is clear that Φ contracts onto the image of i_∞^{top} which give the desired statement about homotopies. We see that any cell of $\mathcal{T}_\infty$ is contracted to a cell of lower dimension as soon as there is a black vertex whose valence is greater than 3, so that these cells are sent to zero. If the vertices only have valence 3 then the black subtrees are contracted onto the image of i_∞ which yields a cell of the same dimension indexed by the tree $\pi_\infty(\tau)$. Finally since π_∞ is an operadic map and $CC_*(Cact^1)$ is an operadic chain model for the little discs, we deduce that $CC_*(\mathcal{KW})$ also has this property.

Corollary 2.31. *$\mathcal{KW}$ is a cell model for the little discs operad whose cells are indexed by $\mathcal{T}_\infty$.*

Remark 2.32. This remark should be seen in conjunction with Remark 2.23. We have identified a natural cell model for the minimal operad of [KS]. This is however not smooth. We can thicken it by the procedure above to obtain the smooth model $\mathcal{C}act_\infty$. Its dimension is however too large and some cells will have to operate as 0. We do wish to point out that there is an inclusion of the cells of $\mathcal{KW}$ into $\mathcal{C}act_\infty$ and indeed there are cells which correspond to $\mathcal{T}_\infty$. So it seems that finding a smooth *and* minimal cell model for the A_∞ Deligne conjecture is not possible.

The versions for the framed little discs

We do not wish to go through all of the details again. Going over to the framed versions means taking a bi–crossed product on the topological level, which on the cell level can be realized by inducing $\mathbb{Z}/2\mathbb{Z}$ decorations as in §2.2.

3 Operations of the cell models on Hochschild complexes.

In this paragraph we use the tree-language in order to naturally obtain operations on the Hochschild complex.

3.1 The cyclic Deligne conjecture:

In this subsection we give the full details of an action of the model $\mathcal{C}acti$ of the framed little discs on the Hochschild complex.

Assumption

Now we fix A to be a finite–dimensional associative algebra with unit 1 together with an inner product $\eta : A \otimes A \to k$ which is non-degenerate and both i) invariant: $\eta(ab, c) = \eta(a, bc)$ and ii) symmetric: $\eta(a, b) = \eta(b, a)$. Such an algebra is called a Frobenius algebra.

We will use CH to stand for Hochschild cochains $CH^n(A, A) := Hom(A^{\otimes n}, A)$.

Actually, it would be enough to have a non-degenerate inner-product η on $A \simeq CH^0(A, A)$ for which i) holds on $HH^0(A, A)$, that is up to homotopy for A. The condition ii) will then hold automatically up to homotopy since $CH^0(A, A)$ is commutative up to homotopy [G].

If one wishes to furthermore relax the other conditions “up to homotopy”, one can fix that η needs to be non-degenerate only on $HH^0(A, A)$ and only require that $HH^0(A, A)$ has to be finite–dimensional. In this case, the operadic operations defined below will give operations $f : A^{\otimes n} \to HH^0(A, A)$ and will thus give actions only up to homotopy. This is enough to get the BV structure on $CH^*(A, A)$, but not quite enough to lift the action to the chain level.

Notation

Let (e_i) be a basis for A and let $C := e_i \eta^{ij} \otimes e_j$ be the Casimir element, i.e. η^{ij} is the inverse to $\eta_{ij} = \eta(e_i, e_j)$.

With the help of the non–degenerate bilinear form, we identify

$$CH^n(A, A) = Hom(A^{\otimes n}, A) \cong A \otimes A^{*\otimes n} \cong A^{*\otimes n+1} \tag{1}$$

We would like to stress the order of the tensor products we choose. This is the order from right to left, which works in such a way that one does not need to permute tensor factors in order to contract.

If $f \in Hom(A^{\otimes n}, A)$, we denote by $\tilde{f}$ its image in $A^{*\otimes n+1}$, explicitly $\tilde{f}(a_0, \dots, a_n) = \eta(a_0, f(a_1, \dots, a_n))$.

With the help of (1) we can pull back the Connes' operators b and B (see e.g. [L]) on the spaces $A^{\otimes n}$ to their duals and to $Hom(A^{\otimes n}, A)$.

Also let $t : A^{\otimes n} \to A^{\otimes n}$ be the operator given by performing a cyclic permutation $(a_1, \dots, a_n) \mapsto (-1)^{n-1}(a_n, a_1, \dots a_{n-1})$ and $N := 1 + t + \dots + t^{n-1} : A^{\otimes n} \to A^{\otimes n}$.

It is easy to check that the operator induced by b is exactly the Hochschild differential; we will denote this operator by ∂. We write Δ for the operator induced by B. It follows that $\Delta^2 = 0$ and $\Delta\partial + \partial\Delta = 0$.

Assumption

To make the formulas simpler we will restrict to normalized Hochschild cochains $\overline{CH}^n(A, A)$ which are the $f \in CH^n(A, A)$ which vanish when evaluated on any tensor containing $1 \in A$ as a tensor factor (see e.g. [L]). On the normalized chains the operator Δ is explicitly defined as follows: for $f \in \overline{CH}^n(A, A)$

$$\eta(a_0, (\Delta f)(a_1, \dots a_{n-1})) := \eta(1, f \circ N(a_0, \dots a_n)) \tag{2}$$

Correlators from decorated trees

We will use the notation of tensor products indexed by arbitrary sets, see e.g. [D]. For a linearly ordered set I denote by $\bigcup_I a_i$ the product of the a_i in the order dictated by I.

Definition 3.1. Let τ be the realization of a spine decorated planted planar b/w tree, $v \in V_w$, and $f \in \overline{CH}^{|v|}(A, A)$. We define $Y(v, f) : A^{F_v(\tau)} \to k$ by

$$Y(v, f)(\bigotimes_{i \in F_v(\tau)} a_i) := \eta(a_{F^{-1}_{\prec'_v}(0)}, f(a_{F^{-1}_{\prec'_v}(1)} \otimes \dots \otimes a_{F^{-1}_{\prec'_v}(|v|)}))$$

Set $V_{b-int} := V_b(\tau) \setminus (V_{tail} \cup \{v_{root}\} \cup V_{spine})$. For $v \in V_{b-int}$ we define $Y(v) := A^{F_v(\tau)} \to k$ by

$$Y(v)(\bigoplus_{i\in F_v(\tau)} a_i) = \eta(1, \bigcup_{i\in F_v} a_i)$$

Definition 3.2. Let τ be the realization of a planar planted b/w tree with n free tails and k labels and $f_i \in \overline{CH}^{n_i}(A, A)$. For such a tree there is a canonical identification $\{v_{root}\} \cup V_{ftail} \to \{0, 1, \ldots, |V_{ftail}|\}$ which is given by sending v_{root} to 0 and enumerating the tails in the linear order induced by the planted planar tree. Set $E_{int}(\tau) := E(\tau) \setminus (E_{tail} \cup E_{root} \cup E_{spine})$ and for $(a_0, \ldots, a_n) \in A^{\otimes(\{v_{root}\}\cup V_{ftail})}$ set

$$Y(\tau)(f_1, \ldots, f_k)(a_0, \ldots, a_n) :=$$
$$\left(\bigotimes_{v\in V_w(\tau)} Y(v, f_{Lab(v)}) \bigotimes_{v\in V_{b-int}} Y_v\right)\left(\Big(\bigotimes_{i\in F_{ftail}(\tau)\cup\{F_{root}\}} a_i\Big)\Big(\bigotimes_{j\in F_{spine}} 1\Big) \otimes C^{\otimes E_{int}(\tau)}\right) \quad (3)$$

In other words, decorate the root flag by a_0, the free tail flags by $a_1, \ldots, a_n$, the spines by 1 and the edges by C and then contract tensors according to the decoration at the white vertices while using the product at the black vertices.

Definition 3.3. We extend the definition above by

$$Y(\tau)(f_1, \ldots, f_k)(a_0, \ldots, a_n) = 0 \text{ if } |v_{Lab^{-1}(i)}| \neq n_i =: |f_i| \quad (4)$$

The foliage operator

Let F be the foliage operator of [K2] applied to trees. This means that $F(\tau)$ is the formal sum over all trees obtained from τ by gluing an arbitrary number of free tails to the white vertices. The extra edges are called free tail edges E_{ftail} and the extra vertices V_{ftail} are defined to be black and are called free tail vertices.

Using the trees defined in Figure 2 this corresponds to the formal sum $F(\tau) := \sum_n l_n \circ_v \tau$ where the operadic composition is the one for b/w trees which are not necessarily bi-partite (see [K2]). In our current setup we should first form $\tilde{F}(\tau) := \sum_n \tau_n \circ_v \tau$ and then delete the images of all leaf edges together with their white vertices of the τ_n to obtain $F(\tau)$.

Signs

The best way to fix signs of course is to work with tensors indexed by edges like in [K2, KS]. For this one fixes a free object L (free $\mathbb{Z}$-module or k-vector space) generated by one element of degree ± 1 and calculates signs using $L^{\otimes E_w(\tau)}$ before applying the foliage operator while using $L^{\otimes E_{weight}}$ after applying the foliage operator, where $E_{weight} = E_w \cup E_{root} \cup E_{ftail} \cup E_{spine}$.

Explicitly, we fix the signs to be given as follows. For any tree τ' in the linear combination above, we take the sign of τ' to be the sign of the permutation which permutes the set E_{weight} in the order induced by $\prec$ to the order where at each vertex one first has the root if applicable, then all non–tail edges, then all the free tails, and if there is a spine edge, the spine.

The explicit signs above coincide with usual signs [L] for the operations and the operators b and B and also coincide with the signs of [G] for the $\circ_i$ and hence for the brace operations [Ge, Kad, GV]. The signs for the operations corresponding to operations on the Hochschild side are fixed by declaring the symbols "," and "{" to have degree one.

Definition 3.4. For $\tau \in \mathcal{T}^{bp,dec'}$ let $\hat{\tau}$ be its realization. We define the operation of τ on $\overline{CH}(A,A)$ by

$$\eta(a_0, \tau(f_1,\dots,f_n)(a_1,\dots,a_N)) := Y(F(\hat{\tau}))(f_1,\dots,f_n)(a_0,\dots,a_N) \qquad (5)$$

Notice that due to Definition 3.3 the right hand side is finite.

Examples

We will first regard the tree O' with one white vertex, no additional black edges, no free tails and a spine, see Figure 2. For a function $f \in \overline{CH}^n$ we obtain:

$$\begin{aligned} Y(F(O'))(f)(a_0,\dots,a_{n-1}) &= \eta(1, f(a_0,\dots a_{n-1}) + (-1)^{n-1} f(a_{n-1},a_0,\dots,a_{n-2}) + \dots) \\ &= \eta(a_0, \Delta(f)(a_1,\dots,a_{n-1})) \end{aligned}$$

Let $\tau'_{n,i}$ be the tree of Figure 2. Then the operation corresponds to

$$Y(F(\tau'_{n,i}))(f;g_1,\dots,g_n)(a_0,\dots,a_N) = \eta(1, f\{'g_{i+1},\dots,g_n,g_1,\dots,g_i\}(a_{(2)},a_0,a_{(1)}))$$

where $N = |f| + \sum |g_i| - n - 1$ and we used the short hand notation

$$\begin{aligned} &f\{'g_{j+1},\dots,g_n,g_1,\dots,g_j\}(a_{(2)},a_0,a_{(1)}) = \sum \pm f(a_{k+1},\dots,a_{i_{j+1}-1}, \\ &g_{j+1}(a_{i_{j+1}},\dots,a_{i_{j+1}+|g_{j+1}|}),\dots,a_{i_n-1},g_n(a_{i_n},\dots,a_{i_n+|g_n|}),\dots,a_N,a_0, \\ &a_1,\dots,a_{i_1-1},g_1(a_{i_1},\dots,a_{i_1+|g_1|}),\dots,a_{i_j-1},g_j(a_{i_j},\dots,a_{i_j+|g_j|}),\dots,a_k) \end{aligned}$$

where the sum runs over $1 \le i_1 \le \dots \le i_j \le \dots \le k \le \dots \le i_{j+1} \le \dots \le i_n \le N : i_l + |g_l| \le i_{l+1}, i_j + |g_j| \le k$ and the signs are as explained above.

Theorem 3.5. *[K3]* (The cyclic Deligne conjecture) *The Hochschild cochains of a finite-dimensional associative algebra with a non–degenerate, symmetric, invariant, bilinear form are an algebra over the chains of the framed little discs operad. This operation is compatible with the differentials.*

Proof. We will use the cellular chains $CC_*(Cacti^1)$ as a model for the chains of the framed little discs operad. It is clear that 3.4 defines an action. On the Hochschild side, the $\circ_i$ operations are substitutions of the type $f_i = \psi(g_1, \dots, g_n)$. For $CC_*(Cacti^1)$ the $\tau \circ_i \tau'$ operations are the pull-back via the foliage operator of all possible substitutions of elements of $F(\tau), \tau \in CC_*(Cacti^1)$ into the position i of $F(\tau')$. The action Y then projects onto the substitution $f_i = \psi(g_1, \dots, g_n)$ so that the action is operadic. Explicitly the substitution $t \circ_i^s t'$ for planted planar bi-partite trees with a decoration dec' and additional free tails is given as follows: Say the number of tails of t' coincides with $|F(v_i)|$. In this case replace the vertex v_i of t, its edges and the black vertices corresponding to the edges with the tree t' matching the flags of v_i with the tails of t' by first matching the root edge with the marked flag of v_i and then using the linear order. Lastly contract the image of the root flag. Otherwise set $t \circ_i^s t' = 0$. With this definition it is easy to see that $F(\tau \circ \tau') = F(\tau) \circ_i^s F(\tau')$.

The compatibility of the Hochschild differential with the differential of the cell complex follows from the relevant statements for τ_n and τ_n^b, which are straightforward but lengthy calculations (see e.g. [K2, G]), together with the calculations above §3.1 which are easily modified to show that $(\partial O')(f) = \Delta(\partial(f))$ and that $(\partial\tau'_{n,i})(f, g_1, \dots, g_n) = (\partial\tau'_{n,i})(f, g_1, \dots, g_n) \pm (\tau'_{n,i})(\partial f, g_1, \dots, g_n) + \sum_i \pm(\tau'_{n,i})(f, g_1, \dots, \partial(g_i), \dots, g_n)$ via an even more lengthy but still straightforward calculation. This then verifies the claim in view of the compatibility of the differentials and the respective operad structures.

Alternatively, in view of the operation of the foliage operator, the compatibilities follow from a straightforward translation of trees with tails into operations on the Hochschild complex. The compatibility of the differential then follows from the almost identical definition of the differential for trees with tails of [K2] and that in the Hochschild complex as $\partial(f) = f \circ \cup - (-1)^{|f|} \cup \circ f$.

Corollary 3.6. *The normalized Hochschild cochains of an algebra as above are a GBV algebra up to homotopy in the sense of Proposition 2.12.*

This could of course have been checked directly without recourse to the operation of a chain model, but we do not know of any source for this result. It also seems to be difficult to guess the right homotopies as Gerstenhaber did in the non-cyclic case [G].

Corollary 3.7. *Over a field of characteristic zero, the Hochschild cohomology of an algebra as above is a BV algebra, such that the induced bracket is the Gerstenhaber bracket.*

Lastly, since our second version of cellular chains of Proposition 2.9 are a subdivision of the cell decomposition of Proposition 2.7, we can also use the latter cell decomposition.

Corollary 3.8. *The normalized Hochschild cochains of an algebra as above are an algebra over the semi–direct product over a chain model of the little discs operad and a chain model for the operad $\mathcal{S}$ built on the monoid S^1.*

Remark 3.9. The operation of the little discs operad by braces, viz. the original Deligne conjecture as discussed in [K2] for Frobenius algebras, corresponds to the decorations in which $dec^{\pm} \equiv 0$ and the decorated edge is always the outgoing edge.

Remark 3.10. In the Theorem 3.5 we can relax the conditions and implications as explained in §3.1.

3.2 The Araki–Kudo–Cohen, Dyer–Lashof operations on the Hochschild complex

By the positive answer to Deligne's conjecture, the Hochschild complex behaves as if it were a double loop space. So we should expect operation ξ_1 and ζ_1 on it. Indeed they were found by Westerland [We] for $p = 2$ and by Tourtchine [Tou] for general p. We wish to point out that the cells of §2.4 naturally induce these operations. It is easy to see that ξ_1 is just the iterated $\circ$ product and ζ_1 is the product of such iterations. That is

$$\xi_1(x) = x \circ (x \circ (\cdots \circ x) \dots) \tag{6}$$

Note the result is not novel, only the cells of §2.4 are. This description however simplifies matters very much.

3.3 The A_∞-Deligne conjecture

Theorem 3.11. *There is an action of the cellular chains model $CC_*(\mathcal{KW})$ on the Hochschild co-chain complex of an A_∞–algebra.*

Proof. This follows from the Theorem above in conjunction with the Theorem of [KS] that the operad $\mathcal{T}_\infty$ acts in a dg–fashion on $C^*(A, A)$.

Remark 3.12. We recall that the action is given by viewing the tree as a flow chart. Given functions $f_1, \dots, f_n$ the action of $\tau \in \mathcal{T}_\infty(n)$ is defined as follows. First "insert" the functions f_i into the vertex labeled by i and then view the tree as a flow chart using the operations μ_n of the A_∞ algebra at each black vertex of arity n and the brace operation $h\{g_1, \dots, g_k\}$ at each white vertex marked by h of arity k to concatenate the function. Here the brace operation [Ge, Kad, GV] is given by

$$\begin{aligned} h\{g_1, \dots, g_n\}(x_1, \dots, x_N) := \\ \sum_{\substack{1 \le i_1 \le \cdots \le i_n \le |h| : \\ i_j + |g_j| \le i_{j+1}}} \pm h(x_1, \dots, x_{i_1 - 1}, g_1(x_{i_1}, \dots, x_{i_1 + |g_1|}), \dots, \\ \dots, x_{i_n - 1}, g_n(x_{i_n}, \dots, x_{i_n + |g_n|}), \dots, x_N) \end{aligned} \tag{7}$$

3.4 The cyclic A_∞ case

We assume that we have an A_∞ algebra A which is Frobenius in the sense that there is a non–degenerate symmetric inner product such that the higher multiplications μ_n are all cyclic with respect to the inner product. These are sometimes called cyclic A_∞ algebras, see [Ko2].

Theorem 3.13. *The cyclic A_∞ conjecture holds.*

Sketch of proof. For the proof of this statement use spine decorated stable trees. That is trees in $\mathcal{T}_\infty$ together with a spine decoration. First they give compatible operations and secondly they index a cell model of the framed little discs. Both these claims follow from constructions completely analogous to the ones above. □

4 The moduli space vs. the Sullivan PROP

There are two generalizations of interest for the construction of the previous paragraph. The first is given by generalizing the restriction *In* to *Out* to the case of several "Out"s and the second is given by going to the full moduli space. Surprisingly these lead to slightly different results. The first route leads one into the realm of Penner's combinatorial compactification and it fits perfectly with the algebra of the Hochschild complex. However it does not exhaust moduli space. Alternatively one can expand to moduli space and even omit invoking the compactification, but the price one pays is in terms of further construction on the Hochschild side to make things match.

4.1 Ribbon graphs and Arc graphs

A short introduction to the arc operad

In this section, we start by giving a brief review of the salient features of the $\mathcal{A}rc$ operad of [KLP] which is reasonably self-contained. The presentation of the material closely follows the Appendix B of [K1]. For full details, we refer to [KLP]. In addition to this review, we furthermore introduce an equivalent combinatorial language which will be key for the following, in particular for [K5]. Simultaneously, we introduce new cell level structures and then go on to define new cell level operads and extensions of the $\mathcal{A}rc$ operad structure.

4.2 Spaces of graphs on surfaces

Fix an oriented surface $F^s_{g,r}$ of genus g with s punctures and r boundary components which are labeled from 0 to $r-1$, together with marked points on the boundary, one for each boundary component. We call this data F for short if no confusion can arise.

The piece of the $\mathcal{A}rc$ operad supported on F will be an open subspace of a space $\mathcal{A}^s_{g,r}$. The latter space is a CW complex whose cells are indexed by graphs on the surface $F^s_{g,r}$ up to the action of the pure mapping class group PMC which is the group of orientation preserving homeomorphisms of $F^s_{g,r}$ modulo homotopies that pointwise fix the set which is the union of the set of the marked points on the boundary and the set of punctures. A quick review in terms of graphs is as follows.

Embedded Graphs

By an embedding of a graph Γ into a surface F, we mean an embedding $i : |\Gamma| \to F$ with the conditions

i) Γ has at least one edge.
ii) The vertices map bijectively to the marked points on the boundaries.
iii) No images of two edges are homotopic to each other, by homotopies fixing the endpoints.
iv) No image of an edge is homotopic to a part of the boundary, again by homotopies fixing the endpoints.

Two embeddings are equivalent if there is a homotopy of embeddings of the above type from one to the other. Note that such a homotopy is necessarily constant on the vertices.

The images of the edges are called arcs. And the set of connected components of $F \setminus i(\Gamma)$ are called complementary regions.

Changing representatives in a class yields natural bijections of the sets of arcs and connected components of $F \setminus i(\Gamma)$ corresponding to the different representatives. We can therefore associate to each equivalence class of embeddings its sets of arcs together with their incidence conditions and connected components — strictly speaking of course the equivalence classes of these objects.

Definition 4.1. By a graph γ on a surface we mean a triple $(F, \Gamma, [i])$ where $[i]$ is an equivalence class of embeddings of Γ into that surface. We will denote the isomorphism class of complementary regions by $Comp(\gamma)$. We will also set $|\gamma| = |E_\Gamma|$. Fixing the surface F, we will call the set of graphs on a surface $\mathcal{G}(F)$.

A linear order on arcs

Notice that due to the orientation of the surface the graph inherits an induced linear order of all the flags at every vertex $F(v)$ from the embedding. Furthermore there is even a linear order on all flags by enumerating the flags first according to the boundary components on which their vertex lies and then according to the linear order at that vertex. This induces a linear order on all edges by enumerating the edges by the first appearance of a flag of that edge.

The poset structure

The set of such graphs on a fixed surface F is a poset. The partial order is given by calling $(F, \Gamma', [i']) \prec (F, \Gamma, [i])$ if Γ' is a subgraph of Γ with the same vertices and $[i']$ is the restriction of $[i]$ to Γ'. In other words, the first graph is obtained from the second by deleting some arcs.

We associate a simplex $\Delta(F, \Gamma, [i])$ to each such graph. Δ is the simplex whose vertices are given by the set of arcs/edges enumerated in their linear order. The face maps are then given by deleting the respective arcs. This allows us to construct a CW complex out of this poset.

Definition 4.2. Fix $F = F^s_{g,n}$. The space $\mathcal{A}'^s_{g,n}$ is the space obtained by gluing the simplices $\Delta(F, \Gamma', [i'])$ for all graphs on the surface according to the face maps.

The pure mapping class group naturally acts on $\mathcal{A}'^s_{g,n}$ and has finite isotropy [KLP].

Definition 4.3. The space $\mathcal{A}^s_{g,r} := \mathcal{A}'^s_{g,r}/PMC$.

CW structure of $\mathcal{A}^s_{g,r}$

Definition 4.4. Given a graph on a surface, we call its PMC orbit its arc graph. If γ is a graph on a surface, we denote by $\bar{\gamma}$ its arc graph or PMC orbit. We denote the set of all arc graphs of a fixed surface F by $\overline{\mathcal{G}}(F)$. A graph is called exhaustive if there are no vertices v with $val(v) = 0$. This condition is invariant under PMC and hence we can speak about exhaustive arc graphs. The set of all exhaustive arc graphs on F is denoted by $\overline{\mathcal{G}}^e(F)$.

Notice that since the incidence conditions are preserved, we can set $|\bar{\gamma}| = |\gamma|$ where γ is any representative and likewise define $Comp(\bar{\gamma})$. We call an arc graph exhaustive if and only if it contains no isolated vertices, that is vertices with $val(v) = 0$.

Now by construction it is clear that $\mathcal{A}^s_{g,r}$ is realized as a CW complex which has one cell for each arc graph $\bar{\gamma}$ of dimension $|\gamma| - 1$. Moreover the cell for a given class of graphs is actually a map of a simplex whose vertices correspond to the arcs in the order discussed above. The attaching maps are given by deleting edges and identifying the resulting face with its image. Due to the action of PMC some of the faces might become identified by these maps, so that the image will not necessarily be a simplex. The open part of the cell will however be an open simplex. Let $C(\bar{\alpha})$ be the image of the cell and $\dot{C}(\bar{\alpha})$ be its interior, then

$$\mathcal{A}^s_{g,r} = \cup_{\bar{\alpha} \in \overline{\mathcal{G}}(F^s_{g,r})} C(\bar{\alpha}), \quad \mathcal{A}^s_{g,r} = \amalg_{\bar{\alpha} \in \overline{\mathcal{G}}(F^s_{g,r})} \dot{C}(\bar{\alpha}) \tag{1}$$

Let Δ^n denote the standard n–simplex and $\dot{\Delta}$ its interior then $\dot{C}(\gamma) = \mathbb{R}^{|E_\Gamma|}_{>0}/\mathbb{R}_{>0} = \dot{\Delta}^{|E_\Gamma|-1} =: C(\Gamma)$ which only depends on the underlying graph Γ of γ.

This also means that the space $\mathcal{A}^s_{g,r}$ is filtered by the cells of dimension less than or equal to k. We will use the notation $(\mathcal{A}^s_{g,r})^{\leq k}$ for the pieces of this filtration.

Open-cell cell complex

It is clear by construction that the $\mathcal{A}rc$ operad again has a decomposition into open cells.

$$\mathcal{A}rc^s_g(n) = \amalg_{\gamma \in \overline{\mathcal{G}}^e(F^s_{g,n+1})} \dot{C}(\gamma) \tag{2}$$

again $\dot{C}(\gamma) = \mathbb{R}^{|E_\Gamma|}_{>0}/\mathbb{R}_{>0} = \dot{\Delta}^{|E_\Gamma|-1} := \dot{C}(\Gamma)$ only depends on the underlying graph Γ of γ.

We will denote the free Abelian group generated by the $C(\alpha)$ as above by $\mathcal{C}^*_o(\mathcal{A}rc)^s_g(n)$. We will write $\mathcal{C}^*_o(\mathcal{A}rc)(n) = \amalg_{g,s}\mathcal{C}^*_o(\mathcal{A}rc)^s_g(n)$ and $\mathcal{C}^*_o(\mathcal{A}rc) = \amalg_n \mathcal{C}^*_o(\mathcal{A}rc)(n)$. We choose the notation to reflect the fact that we are strictly speaking not dealing with cellular chains, however see [K4].

$\mathcal{C}^*_o(\mathcal{A}rc)(n)$ is also graded by the dimension of the cells, we will write $\mathcal{C}^*_o(\mathcal{A}rc)(n)^k$ for the subgroup generated by cells of dimension k and we will also write $\mathcal{C}^*_o(\mathcal{A}rc)(n)^{\leq k}$ for the subgroup of cells of dimension $\leq k$. It is clear that $\mathcal{C}^*_o(\mathcal{A}rc)(n)^{\leq k}$ induces a filtration on $\mathcal{C}^*_o(\mathcal{A}rc)(n)$ and that the associated graded is isomorphic to the direct sum of the $\mathcal{C}^*_o(\mathcal{A}rc)(n)^k$

$$Gr\mathcal{C}^*_o(\mathcal{A}rc)) := Gr(\mathcal{C}^*_o(\mathcal{A}rc)(n), \leq) \simeq \bigoplus_k \mathcal{C}^*_o(\mathcal{A}rc)^k(n) \tag{3}$$

The differential ∂ of $\mathcal{A}^s_{g,r}$ also descends to $\mathcal{C}^*_o(\mathcal{A}rc)$ and $Gr\mathcal{C}^*_o(\mathcal{A}rc))$ by simply omitting the cells which are not in $\mathcal{A}rc$. Applying the differential twice will kill two arcs, each original summand will either be twice treated as zero or appear with opposite sign as in $\mathcal{A}^s_{g,r}$. Hence the differential squares to zero.

Relative cells

The complex $\mathcal{C}^*_o(\mathcal{A}rc)^s_g(n)$ and the isomorphic complex $Gr\mathcal{C}^*_o(\mathcal{A}rc))^s_G(n)$ can be identified with the complex of relative cells $CC_*(A, A \setminus \mathcal{A}rc)$.

Elements of the $\mathcal{A}^s_{g,r}$ as projectively weighted graphs

Using barycentric coordinates for the open part of the cells the elements of $\mathcal{A}^s_{g,r}$ are given by specifying an arc graph together with a map w from the edges of the graph E_Γ to $\mathbb{R}_{>0}$ assigning a weight to each edge s.t. the sum of all weights is 1.

Alternatively, we can regard the map $w : E_\Gamma \to \mathbb{R}_{>0}$ as an equivalence class under the equivalence relation of, i.e. $w \sim w'$ if $\exists \lambda \in \mathbb{R}_{>0} \forall e \in E_\Gamma\; w(e) = \lambda w'(e)$. That is w is a projective metric. We call the set of $w(e)$ the projective weights of the edges. In the limit, when the projective weight of an edge goes to zero, the edge/arc is deleted, see [KLP] for more details. For an example see Figure 10, which is discussed below.

An element $\alpha \in \mathcal{A}^s_{g,r}$ can be described by a tuple $\alpha = (F, \Gamma, \overline{[i]}, w)$ where F and Γ are as above, $\overline{[i]}$ is a PMC orbit of an equivalence class of embeddings and w is a projective metric for Γ. Alternatively it can be described by a tuple $(\bar{\gamma}, w)$ where $\bar{\gamma} \in \overline{\mathcal{G}}(F)$ and w is a projective metric for the underlying abstract graph Γ.

Example 4.5. $\mathcal{A}^0_{0,2} = S^1$. Up to PMC there is a unique graph with one edge and a unique graph with two edges. The former gives a zero–cell and the latter gives a one–cell whose source is a 1–simplex. Its two subgraphs with one edge that correspond to the boundary lie in the same orbit of the action of PMC and thus are identified to yield S^1. The fundamental cycle is given by Δ of Figure 10.

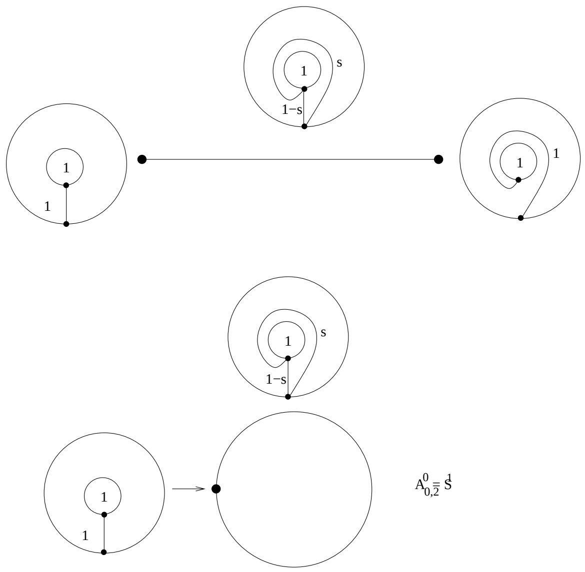

Fig. 10. The space $\mathcal{A}^0_{0,2}$ is given as the CW decomposition of S^1 with one 0–cell and one 1–cell. It can be thought of as the quotient of the interval in which the endpoints are identified by the action of the pure mapping class group. The generator of $CC_*(S^1)$ is called Δ.

4.3 Topological operad structure

The spaces $\mathcal{A}rc(n)$

We begin by reviewing the construction of [KLP]. We then recast it into a purely combinatorial way. This will allow us to define the actions of [K5] more simply, but also allow us to show that although $\mathcal{A}rc_{\#}$ is not an operad on the topological level, it is a rational operad and gives rise to a cellular operad.

Definition 4.6. We define $\mathcal{A}rc_g^s(n) \subset \mathcal{A}_{g,n+1}^s$ to be the subset of those weighted arc graphs whose arc graph is exhaustive. We define $\mathcal{A}rc(n) := \coprod_{s,g\in\mathbb{N}} \mathcal{A}rc_g^s(n)$.[6]

Notice that the space $\mathcal{A}rc(n)$ carries a natural operation of $\mathbb{S}_n$ which permutes the labels $\{1,\dots,n\}$ and one of $\mathbb{S}_{n+1}$ which permutes the labels $\{0,\dots,n\}$. Also notice that the spaces $\mathcal{A}rc_g^s(n)$ inherit the grading and filtration from $\mathcal{A}_g^s(n)$. This is also true for their unions $\mathcal{A}rc(n)$ and we will write $\mathcal{A}rc(n)^{\leq k}$ for these pieces. That is if $\alpha \in \mathcal{A}rc(n)^{\leq k}$ then $|E(\Gamma(\alpha))| \leq k+1$.

Topological description of the gluing [KLP]

To give the composite $\alpha \circ_i \alpha'$ for two arc families $\alpha = (F, \Gamma, \overline{[i]}, w) \in \mathcal{A}rc(m)$ and $\alpha' = (F', \Gamma', \overline{[i']}, w') \in \mathcal{A}rc(n)$ one most conveniently chooses metrics on F and F'. The construction does not depend on the choice. With this metric, one produces a partially measured foliation in which the arcs are replaced by bands of parallel leaves (parallel to the original arc) of width given by the weight of the arc. For this we choose the window representation and also make the window tight in the sense that there is no space between the bands and between the end-points of the window and the bands. Finally, we put in the separatrices. The normalization we choose is that the sum of the weights at boundary i of α coincides with the sum of the weights at the boundary 0, we can also fix them both to be one. Now when gluing the boundaries, we match up the windows, which have the same width, and then just glue the foliations. This basically means that we glue two leaves of the two foliations if they end on the same point. We then delete the separatrices. Afterwards, we collect together all parallel leaves into one band. In this procedure, some of the original bands might be split or "cut" by the separatrices. We assign to each band one arc with weight given by the width of the consolidated band. If arcs occur, which do not hit the boundaries, then we simply delete these arcs. We call these arcs or bands "closed loops" and say that "closed loops appear in the gluing".

Theorem 4.7. *[KLP] Together with the gluing operations above, the spaces $\mathcal{A}rc$ form a cyclic operad.*

[6] Unfortunately there is a typo in the definition of $\mathcal{A}rc(n)$ in [KLP] where $\coprod$ was inadvertently replaced by the direct limit.

In [KLP] we furthermore found that

Theorem 4.8. *[KLP] The chains of the $\mathcal{A}rc$ operad carry the structure of a GBV algebra up to homotopy. That is it has a natural Gerstenhaber algebra structure up to homotopy and a BV operator up to homotopy and they are compatible.*

The dual graph.

Informally the dual graph of an element in $\mathcal{A}rc_{\#}$ is given as follows. The vertices are the complementary regions. Two vertices are joined by an edge if the complementary regions border the same arc. Due to the orientation of the surface this graph is actually a ribbon graph via the induced cyclic order. Moreover the marked points on the boundary make this graph into a marked ribbon graph. A more precise formal definition is given in [K4].

4.4 $\mathcal{DA}rc$

The whole theory of arc graphs can be looked at in two flavors either with projective metrics as we did, or with metrics proper, that is without modding out by the overall scaling. This results in a completely equivalent theory. Here the operad $\mathcal{A}rc$ is replaced by the operad $\mathcal{DA}rc$ where the "D" stands for "de-projectivized".

The relation to Moduli Space

An interesting subspace of $\mathcal{DA}rc$ is the space $\mathcal{DA}rc_{\#}$ which consists of the arc graphs whose complementary regions are all polygons.

Theorem 4.9. *[K4] The space $\mathcal{DA}rc_{\#}$ is equivalent to $M_{g,n+1}^{1^{n+1}}$, that is the moduli space of curves of genus g with n marked points and a tangent vector at each of these points. The gluing operations on $\mathcal{DA}rc$ induce the structure of a rational operad on $M_{g,n+1}^{1^{n+1}}$.*

4.5 Cells

There are several cell models hidden in this construction. First $\mathcal{A}$ is a cell complex from the start.

Secondly we wish to point out that the arc–graphs actually index cells of a relative cell complex. This is in complete analogy to the graph complex that describes the moduli space $M_{g,n}$[Ko1, CV], with the addition, that we are not dealing with a projectivized version, since the tangent vectors have real lengths.

4.6 Di-graphs and Sullivan Chord diagrams

Ribbon Di-graphs

A ribbon graph is a di-graph is a ribbon graph Γ together with a $\mathbb{Z}/2\mathbb{Z}$ labelling of the cycles of Γ: i/o : {cycles of Γ} $\to \mathbb{Z}/2\mathbb{Z}$. We call the cycles $i/o^{-1}(0) =: Out_\Gamma$ the outgoing ones and $i/o^{-1}(1) =: In_\Gamma$ the incoming ones. A di-graph is said to be of type (n,m) if $|In_\Gamma| = n$ and $|Out_\Gamma| = m$. We will denote the set of these graphs by $\mathcal{R}ib^{i/o}$.

A ribbon di-graph is called perfectly partitioned, if $i/o(\imath(f)) = 1 - i/o(f)$ for every flag f. That is each edge is part of one input and one output cycle. We will call the set of these graphs $\mathcal{R}ib^{i\leftrightarrow o}$.

A (S_1, S_2)-labeled ribbon di-graph is a ribbon di-graph together with bijective maps : $In \to S_1$ and $Out \to S_2$. We denote the induced map on $In \amalg Out$ by *Lab*. If (S_1, S_2) is not mentioned, we will use $S_1 = \overline{n}$ and $S_2 = \overline{m}$ as the default indexing sets for a graph of type (n,m).

Sullivan Chord and Ribbon Diagrams

There are many definitions of Sullivan chord diagrams in the literature, we will use the following conventions.

Definition 4.10. A Sullivan chord diagram is a marked labeled ribbon di-graph which satisfies the following condition:

i) after deleting the edges of the incoming cycles one is left with a forest, i.e. a possibly disconnected set of contractible graphs.

Remark 4.11. In terms of the dual arc picture this means that there is a partition of the boundary components of the surface into In and Out and arcs only run from In to Out and Out to Out. A complete list of all versions of Sullivan Chord diagrams and their dual $\mathcal{A}rc$ pictures can be found in [K4].

The most important candidate for us will be a homotopically equivalent version of contracted diagrams.

Definition 4.12. We let $\overline{\mathcal{A}rc}_1^{i\leftrightarrow o}$ be weighted arc graphs on surfaces with marked inputs and outputs such that

1. All arcs run from In to Out
2. The sum of the weights on each In boundary is 1.

The importance of this space is that it is the analog of the normalized cacti, that is it gives a cell model for the Sullivan PROP.

Theorem 4.13. *[K4] The subspaces $\overline{\mathcal{A}rc}_1^{i\leftrightarrow o}$ when bi–graded by the number of In and Out boundaries and endowed with the symmetric group actions permuting the labels form a topological quasi–PROP, i.e. a PROP up to homotopy. It is naturally a CW complex whose cells are indexed by the corresponding graphs and the induced quasi–PROP structure on the cell level is already a PROP structure.*

4.7 Graph actions, Feynman rules and correlation functions

Operadic Correlation Functions

In this section, we introduce operadic correlation functions, which can be thought of as the generalization of an algebra over a cyclic operad to the dg–setting. In order to get to the main definition, we first set up some notation.

Given a pair (A, C) where A is a vector space and $C = \sum c^{(1)} \otimes c^{(2)} \in A \otimes A$ we define the following operations

$$\circ_i : Hom(A^{\otimes n+1}, k) \otimes Hom(A^{\otimes m+1}, k) \to Hom(A^{\otimes n+m}, k) \tag{4}$$

where for $\phi \in Hom(A^{\otimes n+1}, k)$ and $\psi \in Hom(A^{\otimes m+1}, k)$

$$\begin{aligned} &\phi \circ_i \psi(a_1 \otimes \cdots \otimes a_{n+m}) = \\ &\sum \phi(a_1 \otimes \cdots \otimes a_{i-1} \otimes c^{(1)} \otimes a_{i+m} \otimes \cdots \otimes a_{m+n}) \psi(c^{(2)} \otimes a_i \otimes \cdots \otimes a_{i+m-1}) \end{aligned} \tag{5}$$

Definition 4.14. A set of operadic correlation function for a cyclic linear operad $\mathcal{O}$ is a tuple $(A, C, \{Y_n\})$ where A is a vector space, $C = \sum c^{(1)} \otimes c^{(2)} \in A \otimes A$ is a fixed element and $Y_{n+1} : \mathcal{O}(n) \to Hom(A^{\otimes n+1}, k)$ is a set of multi–linear maps. The maps $\{Y_n\}$ should be $\mathbb{S}_{n+1}$ equivariant and for $op_n \in \mathcal{O}(n), op_m \in \mathcal{O}(m)$

$$Y_{n+m}(op_n \circ_i op_m) = Y_{n+1}(op_n) \circ_i Y_{m+1}(op_m) \tag{6}$$

where the $\circ_i$ on the left is the multiplication of equation (4) for the pair (A, C).

We call the data $(A, \{Y_n\})$ of an algebra and the $\mathbb{S}_{n+1}$ equivariant maps correlation functions or simply correlators for $\mathcal{O}$.

Example 4.15. *Correlators for algebras over cyclic operads* An example is given by an algebra over a cyclic operad. Recall that this a triple $(A, \langle\ ,\ \rangle, \{\rho_n\})$ where A is a vector space, $\langle\ ,\ \rangle$ is a non–degenerate bi–linear pairing and $\rho_n : \mathcal{O}(n) \to Hom(A^{\otimes n}, A)$ are multilinear maps also called correlators that satisfy

i) $\rho(op_n \circ_i op_m) = \rho(op_n) \circ_i \rho(op_m)$ where $\circ_i$ is the substitution in the i-th variable.
ii) The induced maps $Y_{n+1} : \mathcal{O}(n) \to Hom(A^{\otimes n+1}, k)$ given by

$$Y_{n+1}(op_n)(a_0 \otimes \cdots \otimes a_n) := \langle a_0, \rho(op_n)(a_1 \otimes \cdots \otimes a_n) \rangle \tag{7}$$

are $\mathbb{S}_{n+1}$ equivariant.

Notation 4.16. Given a finite dimensional vector space A with a non-degenerate pairing $\langle\ ,\ \rangle = \eta \in \check{A} \otimes \check{A}$, let $C \in A \otimes A$ be dual to η under the isomorphism induced by the pairing and call it the Casimir element. It has the following explicit expression: Let e_i be a basis of V, let $\eta_{ij} := \langle e_i, e_j \rangle$ be the matrix of the metric and let η^{ij} be the inverse matrix. Then $C = \sum_{ij} e_i \eta^{ij} \otimes e_j$.

4.8 Operadic correlations functions with values in a twisted $\mathcal{H}om$ operad

Definition 4.17. Let $(A, \langle\ ,\ \rangle, \{Y_n\})$ be as above. And let $\mathcal{H} = \{\mathcal{H}(n)\}$ with $\mathcal{H}(n) \subset \mathrm{Hom}(A^{\otimes n}, A)$ as k–modules be an operad where the $\mathbb{S}_n$ action is the usual action, but the operad structure is *not necessarily* the induced operad structure. Furthermore assume that $\rho_{Y_{n+1}} \in \mathcal{H}(n)$. We say that the $\{Y_n\}$ are operadic correlation functions for $\mathcal{O}$ with values in $\mathcal{H}$ if the maps ρ are operadic maps from $\mathcal{O}$ to $\mathcal{H}$. We will also say that we get an action of $\mathcal{O}$ with values in $\mathcal{H}$.

Signs

As in the case of the Deligne conjecture one twist which we have to use is dictated by picking sign rules. In the case of Deligne's conjecture this could be done by mapping to the brace operad $\mathcal{B}race$ (see e.g. [K2]) or by twisting the operad $\mathcal{H}om$ by lines of degree 1 (see e.g. [KS]). In what follows, our actions will take values on operads that are naturally graded *and moreover* we will identify the grading with the geometric grading by e.g. the number of edges or the number of angles etc.. The signs will then automatically match up, if we use the procedure at the same time for *both* the graph side and the $\mathcal{H}om$ side, i.e. for the operad $\mathcal{H}$. In fact, this approach unifies the two sign conventions mentioned above on the subspace of operations corresponding to $\mathcal{L}Tree_{cp}$.

Definition 4.18. A quasi–Frobenius algebra is a triple $(A, d, \langle\ ,\ \rangle)$ where (A, d) is a unital dg–algebra whose homology algebra $H := H(A, d)$ is finite dimensional and has a non–degenerate pairing $\langle\ ,\ \rangle$ and is a Frobenius algebra for this pairing. A quasi–Frobenius algebra with an integral is a triple $(A, d, \int)$ where $\int : A \to k$ is a linear map such that

i) $\forall a \in A : \int da = 0$
ii) $(A, d, \langle\ ,\ \rangle)$ is a quasi–Frobenius algebra,where $\langle a, b\rangle := \int ab$. The cocycles of a quasi–Frobenius algebra with an integral are the subalgebra $Z = \ker(d) \subset A$ of the algebra above.

4.9 $\mathcal{A}rc^{\angle}$ correlation functions

In order to present the correlation functions we need to partition the arc graphs and endow them with angle markings. Given an arc graph α it gives rise to a formal sum of arc graphs $\mathcal{P}(\alpha)$ where each summand is obtained from α by inserting finitely many parallel edges. See Figure 11 for one such summand. This operation is the analog of the foliage operator. An angle marking is an angle marking of the arc graph. The corresponding space is called $\mathcal{A}^{\angle}$. In keeping with the notation already in place $\mathcal{A}rc^{\angle}$ is the subspace of graphs that hit all boundaries and elements of $\mathcal{A}rc^{\angle}_{\#}$ are also quasi–filling. Given an arc graph there are two standard angle markings. The first marks all angles by 0 except the angles spanned by the smallest and biggest element at each boundary. The second marking marks all angles by 1. When partitioning an angle marked graph, we mark all new angles by 1.

The idea of how to obtain the correlation functions for the tensor algebra is very nice in the $\mathcal{A}rc$ picture where it is based on the polygon picture. This polygon picture can be thought of as an IRF (interaction 'round a face) picture for a grid on a surface which is dual to the ribbon picture. For this we would modify the arc graph by moving the arcs a little bit apart as described. Then the complementary regions of partitioned quasi–filling arc–graphs, denoted by $\mathcal{P}\overline{\mathcal{G}}_{\#}$, are $2k$-gons whose sides alternatingly correspond to arcs and pieces of the boundary. The pieces of the boundary correspond to the angles of the graph and of course any polygonal region corresponds to a cycle of the arc graph. If the graph α^p has an angle marking, then the sides of the polygons corresponding to the boundaries will also be marked. We fix the following notation. For an angle marked partitioned arc graph α^p that is quasi-filling let $Poly(\alpha^p)$ be the set of polygons given by the complementary regions of α^p when treated as above. See Figure 8 for an example. For $\pi \in Poly(\alpha^p)$, let $Sides'$ be the sides corresponding to the angles which are marked by 1 and $Sides'(\alpha^p)$ be the union of all of these sides. If we denote $\angle^+(\Gamma) = (mk^{\angle})^{-1}(1)$ there is a natural bijection between $\angle^+(\alpha^p)$ and $Sides'(\alpha^p)$.

For some purposes it is convenient to contract the edges of the $2k$-gon that belong to pieces of the boundary and label the resulting vertex by the corresponding boundary label.

Correlation functions on the tensor algebra of an algebra

Fix an algebra A with a cyclic trace, i.e. a map $\int : A \to k$ which satisfies $\int a_1 \dots a_n = \pm \int a_n a_1 \dots a_{n-1}$ where $\pm$ is the standard sign.

Now for $\pi \in Poly(\alpha^p)$ set

$$Y(\pi)(\bigotimes_{s \in Sides'(\pi)} a_s) = \int \prod_{s \in Sides'(\pi)} a_s \tag{8}$$

Notice that we only have a cyclic order for the sides of the polygon, but $\int$ is (super)-invariant under cyclic permutations, so that if we think of the tensor product and the product as indexed by sets (8) it is well defined.

For an angle marked partitioned arc family α^p set

$$Y(\alpha^p)(\bigotimes_{s \in (mk^{\angle})^{-1}(1)} a_s) = \bigotimes_{\pi \in Poly(\alpha^p)} Y(\pi)(\bigotimes_{s \in Sides'(\pi)} a_s) \tag{9}$$

where we used the identification of the set $Sides'(\alpha^p) = \amalg_{\pi \in Poly(\alpha^p)} Sides'(\pi)$ with $\angle^+(\alpha^p)$. Since for each $\alpha^p \in \mathcal{P}^{\angle}\overline{\mathcal{G}}^e(n)$ the set of all flags has a linear order, we can think of $Y(\alpha^p)$ as a map $A^{\otimes |F(\alpha^p)|} = \bigotimes_{i=1}^n A^{\otimes |F(v_i)|} \to k$ and furthermore as a map to $TA^{\otimes n} \to k$ by letting it be equal to equation (9) as a map from $\bigotimes_{i=1}^{n} A^{\otimes|F(v_i)|} \subset TA^{\otimes n}$ and setting it to zero outside of this subspace.

Extending linearly, for an angle marked arc family $\alpha \in \mathcal{A}rc^{\angle}$, we finally define

$$Y(\alpha) := Y(\mathcal{P}(\alpha)) \tag{10}$$

Correlators for the Hochschild co-chains of a Frobenius algebra

Let A be an algebra and let $C^n(A,A) = Hom(A^{\otimes n}, A)$ be the Hochschild cochain complex of A. We denote the cyclic cochain complex by $CC^n(A,k) = Hom(A^{\otimes n+1}, k)$. Then one has a canonical isomorphism of $CC^*(A) \cong C^*(A, \check{A})$ as complexes and hence also $HC^*(A) \cong H^*(A,A)$ where HC is Connes' cyclic cohomology and H is the Hochschild cohomology.

Lemma 4.19. *For any Frobenius algebra $(A, \langle\ ,\ \rangle)$, we have canonical isomorphisms $CC^*(A) \cong C^*(A,\check{A}) \cong C^*(A,A)$ and $HC^*(A) \cong H^*(A,A) \cong H^*(A,\check{A})$ induced by the isomorphism of A and $\check{A}$ which is defined by the non-degenerate pairing of A.*

Proof. The only statement to prove is the last isomorphism. As mentioned the map on the chain level is induced by the isomorphism of A and $\check{A}$ defined by the non-degenerate pairing of A. The fact that the complexes are isomorphic follows from the well known fact that the invariance of the pairing $\langle ab, c\rangle = \langle a, bc\rangle$ implies that the isomorphism between A and $\check{A}$ is an isomorphism of A bi–modules, where the bi–module structure of functions $f \in \check{A}$ is given by $a'fa''(c) = f(a''ca')$, see e.g. [L].

For any $f \in C^n(A,A)$ let $\tilde{f} \in \check{A}^{\otimes n}$ be its image under the isomorphism of $\check{A}$ with A defined by the Frobenius structure of A.

Given pure tensors $f_i = f^{0i} \otimes f_{1i} \otimes \cdots \otimes f_{in_i} \in C^{n_i}(A,A), i \in \{0,\dots,n\}$ we write $\tilde{f}_i = f_{0i} \otimes \cdots \otimes f_{in_i}$ for their image in $CC^{n_i}(A)$. Fix $\alpha \in \mathcal{A}rc^{\angle}(n)$. Now decorate the sides $s \in Sides'(\alpha) := (mk^{\angle})^{-1}(1)$ of the complementary regions, which correspond to pieces of the boundary, by elements of A as follows: for a side $s \in Sides'$ let j its position in its cycle c_i counting only the sides of c_i in $Sides'$ starting at the side corresponding to the unique outside angle at the boundary given by the cycle. If the number of such sides at the boundary i is $n_i + 1$ then set $f_s := f_{ij}$.

Now we set

$$Y(\alpha)(f_1,\dots,f_n) := Y(\mathcal{P}(\alpha))(\bigotimes_{s \in \angle^+(\alpha^p)} f_s) \tag{11}$$

We extend this definition by linearity if $f_i \in C^{n_i}(A,A), i \in \bar{n}$. If the condition that $n_i + 1$ equals the number of $Sides'$ at the boundary i is not met, we set $Y(\alpha)(f_0,\dots,f_n) = 0$. An example of a decorated partitioned surface and its polygons is given in Figure 11.

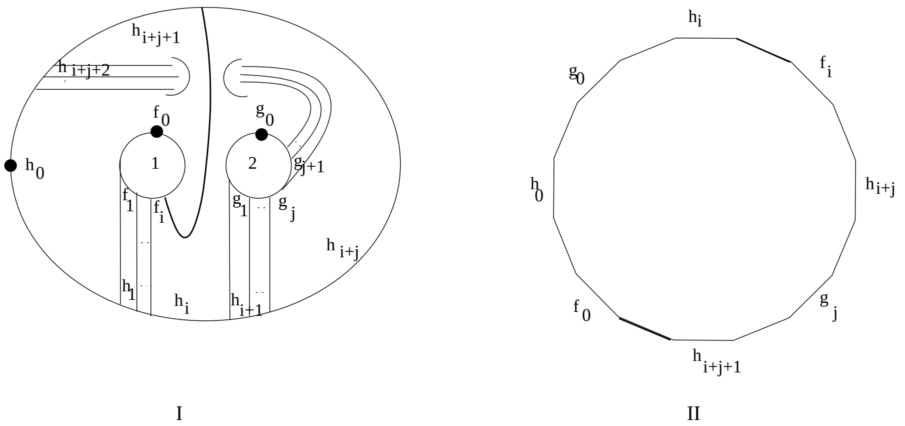

Fig. 11. A partitioned arc graph with decorations by elements of A and one of its decorated polygons. The bold line corresponds to the bold edges.

Theorem 4.20. *[K5] Let A be a Frobenius algebra and let $C(A,A)$ be the Hochschild complex of the Frobenius algebra, then the cyclic chain operad of the open cells of $\mathcal{A}rc^{\angle}$ act on $C(A,A)$ via correlation functions. Hence so do all the suboperads, sub-dioperads and PROPs of [K4] mentioned in the introduction. In particular the graph complex of $M_{g,n+1}^{1^{n+1}}$, the Moduli space of pointed curves with fixed tangent vectors at each point act on $CH(A,A)$ by its two embeddings into $\mathcal{A}rc^{\angle}_{\#}$. Furthermore, there is a natural operad structure on the corresponding partitioned graphs $\mathcal{P}^{\angle}\mathcal{A}rc_{\#}$ and for this operad structure the correlation functions are operadic correlation functions with values in $Gr\mathcal{CM}$. Moreover, the operations of the suboperad $\mathcal{T}ree_{cp}$ correspond to the operations $\sqcup$ and $\square_i$ induced by Ξ_2 as defined in [MS3].*

The same formulism also yields operadic correlation functions for the tensor algebra of the co-cycles of a differential algebra (A,d) over k with a cyclically invariant trace $\int : A \to k$ which satisfies $\int da = 0$ and whose induced pairing on $H = H(A,d)$ turns H into a Frobenius algebra, i.e. they are chain level operadic correlation functions with values in $Gr\mathcal{CM}$.

Here $Gr\mathcal{CM}$ is a the associated graded of a filtered suboperad of Hom which is essentially generated by products, co–products and shuffles.

Remark 4.21. We wish to point out that strictly speaking Deligne's original conjecture also only yields correlations functions with values in the $\mathcal{B}race$ suboperad. This is due to the necessary fixing of signs.

The Sullivan-Chord diagram case

ASSUMPTION: For the rest of the discussion of this subsection let A be a commutative Frobenius algebra.

4.10 Correlators for $\mathcal{A}^\angle$

In general we extend the action as follows. Notice that given an arc graph α each complementary region $S \in Comp(G)$ has the following structure: it is a surface of some genus g with $r \geq 1$ boundary components whose boundaries are identified with a 2k-gons. Alternating sides belong to arcs and boundaries as above and the sides come marked with 1 or 0 by identifying them with the angles of the underling arc graph. Now let $Sides'(S)$ be the sides which have an angle marking by 1 and let χ be the Euler characteristic of S. We set

$$Y(S)(\bigotimes_{s \in Sides'(S)} a) := \int (\prod_{s \in Sides'(S)} a_s) e^{-\chi+1} \tag{12}$$

where $e := \mu(\Delta(1))$ is the Euler element. For an angle marked partitioned arc graph α^p we set

$$Y(\alpha^p)(\bigotimes_{S \in Comp(\alpha_i)} (\bigotimes_{s \in Sides'(S)} a_s)) = \bigotimes_{S \in Comp(\alpha_i)} Y(S)(\bigotimes_{s \in Sides'(S)} a_s) \tag{13}$$

Again, for $\alpha \in CC_*(\mathcal{A}^\angle)$ we simply set

$$Y(\alpha) = Y(\mathcal{P}(\alpha)). \tag{14}$$

Theorem 4.22. *The $Y(\alpha)$ defined in equation (14) give operadic correlation functions for $CC_*(\overline{\mathcal{A}rc}_1^{i\leftrightarrow o})$ and induce a dg–action of the dg-PROP $CC_*(\overline{\mathcal{A}rc}_1^{i\leftrightarrow o})$ on the dg–algebra $\overline{CH}^*(A, A)$ of reduced Hochschild co-chains for a commutative Frobenius algebra A.*

The $Y(\alpha)$ also yield correlation functions on the tensor algebra of the cocycles of a differential algebra (A, d) over k with a cyclically invariant trace $\int : A \to k$ that satisfies $\int da = 0$ and whose induced pairing on $H = H(A, d)$ turns H into a Frobenius algebra. These correlations functions are operadic chain level correlation functions.

Corollary 4.23. *The operadic correlation functions descend to give a PROP action of $H_*(\overline{\mathcal{A}rc}_1^{i\leftrightarrow o})$ on $H^*(A)$ for a commutative Frobenius algebra A.*

4.11 Application to String-topology

Let M be a simply connected compact manifold M and denote the free loop space by $\mathcal{L}M$ and let $C_*(M)$ and $C^*(M)$ be the singular chains and (co)-chains of M. We know from [J, CJ] that $C_*(\mathcal{L}M) = C^*(C^*(M, C_*(M))$ and $H_*(\mathcal{L}M) \simeq H^*(C^*(M), C_*(M))$. Moreover $C^*(M)$ is an associative *dg* algebra with unit, differential d and an integral (M was taken to be a compact manifold) $\int : C^*(M) \to k$ such that $\int d\omega = 0$. By using the spectral sequence and taking field coefficients we obtain operadic correlation functions Y for

$\mathcal{T}ree$ on $E^1 = C^*(H,H)$ which converges to $H^*(C^*(M))$ and which induces an operadic action on the level of (co)-homology. Except for the last remark, this was established in [K3].

Theorem 4.24. *When taking field coefficients, the above action gives a dg action of a dg–PROP of Sullivan Chord diagrams on the E^1–term of a spectral sequence converging to $H_*(LM)$, that is the homology of the loop space a simply connected compact manifold and hence induces operations on this loop space.*

Proof. Recall from [CJ] the isomorphism $C_*(\mathcal{L}M) = C^*(C^*(M, C_*(M))$ comes from dualizing the isomorphism $C_*(\mathcal{L}M) = C_*(C^*(M))$[J]. Calculating the latter with the usual bi-complex [L] then we see that the E^1-term is given by $CH_*(H^*(M))$ and dualizing the corresponding E^1 spectral sequence, we get $CH^*(H^*(M), H_*(M))$, so we get an operation of the E^1 level. Since the operation of $\mathcal{T}ree$ was *dg*, it is compatible with the E^1 differential and hence gives an action on the convergent spectral sequence computing $H_*(\mathcal{L}M)$ and hence on its abutment.

5 Stabilization and Outlook

We have shown that the above methods are well suited to treat the double loop space nature of the Hochschild complex, string topology and a moduli space generalization. The $\mathcal{A}rc$ operad is manifestly BV and since it describes String Topology it should not go beyond the double loop space. To go to higher loop spaces we need a stabilization of the arc operad. In the following, we will give an outlook of the results we aim to prove in the higher loop case.

In this paragraph $s = 0$.

Definition 5.1. The elements in the complement of $\mathcal{A}rc_{\#}$ are called non-effective. Let $\mathcal{A}rc^{ctd}$ be the suboperad of connected arc families.

Definition 5.2. We define $\mathcal{S}t\mathcal{A}rc_0(n) := \varinjlim \mathcal{A}rc^{ctd}$ where the limit is taken with respect to the system $\alpha \to \alpha \circ_i Op_g, \alpha \to Op_g \circ_i \alpha$ where $Op_g \in \mathcal{A}rc_1^{ctd}(2)$ is non-effective.

Claim. The spaces $\mathcal{S}t\mathcal{A}rc_0(n)$ form an operad.

Claim. The operad StArc0(n) detects infinite loop spaces, i.e. if X admits an operadic action of StArc0(n) then it has the homotopy type of an infinite loop space.

Sketch of proof. We can give a hemispherical construction a la Fiedorowich by using the arc graphs for the $\cup_i$ products as given in Figure 12.

Corollary 5.3. *$\mathcal{S}t\mathcal{A}rc_0(n)$ has the homotopy type of an infinite loop space.*

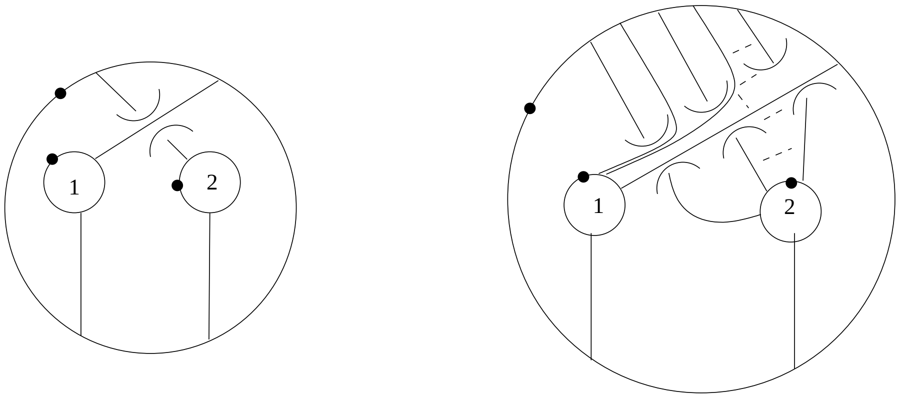

Fig. 12. The $\cup_2$ and the $\cup_i$ operations.

This can be to be compared to the theorems of Tillmann and Madsen on infinite loop spaces and Segal's approach to CFT.

Notice that the construction above only uses the tree part and indeed:

Claim. The suboperad of stabilized linear Chinese trees (cf. [KLP]) has an operadic filtration $\mathcal{S}t\mathcal{G}Tree_g$ in terms of effective genus. The operad linear $\mathcal{S}t\mathcal{G}Tree_g$ is isomorphic to the little $2g$ cubes operad. That is we get cells for the $\cup_i$-operations. A finer filtration gives all k-cubes.

This fits well with the slogan that strings yield all higher dimensional objects. It also gives tools to describe the cells for the higher Dyer-Lashof-Cohen operations.

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