QUILLEN COHOMOLOGY AND BAUES-WIRSCHING COHOMOLOGY OF ALGEBRAIC THEORIES

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Abstract. Les théories algébriques elles-mêmes peuvent être considérées en tant que certaine sorte de structures algébriques, par consquent il est possible de considérer leur cohomologie dans le sens du Quillen.

Dans cette note on le montre que le cohomologie de Quillen d'une théorie algébrique est isomorphe à sa cohomologie de Baues-Wirsching.

Introduction

The aim of this work is to construct cohomology groups of algebraic theories. Our construction follows the general philosophy of Barr&Beck and Quillen. As prescribed by it, the category of coefficients for cohomology of a theory $\mathbb{T}$ is the category $\mathscr{F}(\mathbb{T})$ of internal abelian groups in the comma category $\mathscr{Theories}/\mathbb{T}$ of theories over $\mathbb{T}$. We show that $\mathscr{F}(\mathbb{T})$ is an abelian category with enough projective and injective objects, and we give two more alternative descriptions of it. First, it is equivalent to the full subcategory of the category of natural systems on $\mathbb{T}$ in sense of [6], namely, of the s. c. *cartesian* natural systems (see Section 2); on the other hand, we construct an explicit ringoid valued functor $\mathscr{U}_{\mathbb{T}}$ and prove that $\mathscr{F}(\mathbb{T})$ is also equivalent to the category of modules over $\mathscr{U}_{\mathbb{T}}$.

After establishing these three alternative descriptions of the category $\mathscr{F}(\mathbb{T})$ we accordingly give three different constructions of the cohomology groups of $\mathbb{T}$ with coefficients in an object of $\mathscr{F}(\mathbb{T})$. The first construction follows the Quillen approach and uses simplicial resolutions of $\mathbb{T}$ in $\mathscr{Theories}$ by free theories. The second "Cartan-Eilenberg style" approach defines cohomology groups as suitable Ext-groups in the category $\mathscr{F}(\mathbb{T})$. Finally, the third approach utilizes the Baues-Wirsching cohomology of $\mathbb{T}$ considered as a small category.

Our main result claims that these three approaches give essentially the same result. In particular, we prove that the Baues-Wirsching cohomology of a free theory with coefficients in a cartesian natural system is trivial in dimensions > 1.

Finally, we must note that our constructions and results generalize the work of the authors on this subject [10]. In that paper, coefficients for the cohomology of theories were defined in much more restricted situation — which however was of sufficient generality for the theories of modules over a ring. The difference roughly corresponds to the difference between bifunctors and Cartesian natural systems as coefficients for the Baues-Wirsching cohomology.

1. Recollections

1.1. Cohomology of small categories.

1.1.1. *Basic definitions.* Let $\boldsymbol{C}$ be a category. Then the category $\mathbf{F}\boldsymbol{C}$ of factorizations in $\boldsymbol{C}$ is defined as follows. Objects of $\mathbf{F}\boldsymbol{C}$ are morphisms $f : A \to B$ in $\boldsymbol{C}$ and morphisms $(a, b) : f \to g$ in $\mathbf{F}\boldsymbol{C}$ are commutative diagrams

$$\begin{tikzcd} A \arrow[d, "f"] & A' \arrow[l, "a"'] \arrow[d, "g"] \\ B \arrow[r, "b"'] & B' \end{tikzcd}$$

in the category $\boldsymbol{C}$. A *natural system* on $\boldsymbol{C}$ is a functor $D:\mathbf{F}\boldsymbol{C}\to\mathscr{Ab}$ to the category of abelian groups. We write $D(f)=D_f$. If $a:C\to D$, $f:A\to C$ and $g:D\to B$ are morphisms in $\boldsymbol{C}$, then the induced homomorphism $(1_A,a)_*:D_f\to D_{af}$ will be denoted by $\xi\mapsto a\xi$, for $\xi\in D_f$, while $(a,1_B)_*:D_g\to D_{ga}$ will be denoted by $\eta\mapsto\eta a$, $\eta\in D_g$. We denote by $C^*(\boldsymbol{C};D)$ the following cochain complex:

$$C^n(\boldsymbol{C};D)=\prod_{\left(A_0\xleftarrow{a_1}A_1\leftarrow\cdots\xleftarrow{a_n}A_n\right)\in\boldsymbol{C}}D_{a_1\dots a_n},$$

with the coboundary map given by

$$d(\varphi)(a_1,a_2,...,a_{n+1})=a_1\varphi(a_2,...,a_{n+1})+\\
+\sum_{i=1}^{n}(-1)^i f(a_1,...,a_ia_{i+1},...,a_{n+1})+(-1)^{n+1}\varphi(a_1,...,a_n)a_{n+1}.$$

According to [6] the cohomology $H^*(\boldsymbol{C};D)$ of $\boldsymbol{C}$ with coefficients in D is defined as the homology of the cochain complex $C^*(\boldsymbol{C};D)$.

A morphism of natural systems is just a natural transformation. For a functor $q:\boldsymbol{C}'\to\boldsymbol{C}$, any natural system D on $\boldsymbol{C}$ gives a natural system $D\circ(\mathbf{F}q)$ on $\boldsymbol{C}'$ which we will denote $q^*(D)$. There is a canonical functor $\mathbf{F}\boldsymbol{C}\to\boldsymbol{C}^{\mathrm{op}}\times\boldsymbol{C}$ which assigns the pair (A,B) to $f:A\to B$. This functor allows one to consider any bifunctor $D:\boldsymbol{C}^{\mathrm{op}}\times\boldsymbol{C}\to\mathscr{Ab}$ as a natural system. In what follows bifunctors are considered as natural systems via this correspondence. Similarly, one has a projection $\boldsymbol{C}^{\mathrm{op}}\times\boldsymbol{C}\to\boldsymbol{C}$, which yields the functor $\mathbf{F}\boldsymbol{C}\to\boldsymbol{C}$ given by $(a:A\to B)\mapsto B$. This allows us to consider any functor on $\boldsymbol{C}$ as a natural system on $\boldsymbol{C}$. In particular one can talk about cohomology of a category $\boldsymbol{C}$ with coefficients in bifunctors and in functors as well. One easily sees that for a bifunctor $D:\boldsymbol{C}^{\mathrm{op}}\times\boldsymbol{C}\to\mathscr{Ab}$ the group $H^0(\boldsymbol{C};D)$ coincides with the *end* of the bifunctor D (see [12]), which consists of all families $(x_C)_{C\in\mathrm{Ob}\,\boldsymbol{C}}$, where $x_C\in D_{1_C}$, for each $C\in\mathrm{Ob}\,\boldsymbol{C}$, satisfying the condition $a(x_A)=(x_B)a$ for all $a:A\to B$. In the case of a functor $F:\boldsymbol{C}\to\mathscr{Ab}$ the group $H^0(\boldsymbol{C};F)$ is isomorphic to the limit of the functor F and the groups $H^*(\boldsymbol{C};F)$ are isomorphic to the higher limits (see [6]).

1.1.2. *Linear extensions and second cohomology of categories.* We will need the definition of *linear extensions of categories* and their relationship with the second cohomology following [6]. Let D be a natural system on a small category $\boldsymbol{C}$. A linear extension

$$0\to D\to\boldsymbol{E}\xrightarrow{p}\boldsymbol{C}\to 0$$

of $\boldsymbol{C}$ by D is a category $\boldsymbol{E}$, a full functor p which is identity on objects, and, moreover, for each morphism $f:A\to B$ in $\boldsymbol{C}$, a transitive and effective action of the abelian group D_f on the subset $p^{-1}(f)\subseteq\mathrm{Hom}_{\boldsymbol{E}}(A,B)$,

$$D_f\times p^{-1}(f)\to p^{-1}(f);\quad (a,\tilde f)\mapsto a+\tilde f,$$

such that the following identity holds

$$(a+\tilde f)(b+\tilde g)=fb+ag+\tilde f\tilde g.$$

Here f and g are two composable arrows in $\boldsymbol{C}$, $\tilde f\in p^{-1}(f)$, $\tilde g\in p^{-1}(g)$ and $a\in D_f$, $b\in D_g$. Two linear extensions $\boldsymbol{E}$ and $\boldsymbol{E}'$ are *equivalent* if there is an isomorphism of categories $\epsilon:\boldsymbol{E}\to\boldsymbol{E}'$ with $p'\epsilon=p$ and with $\epsilon(a+\tilde f)=a+\epsilon(\tilde f)$. For example, there is a *trivial* linear extension $D\rtimes\boldsymbol{C}\to\boldsymbol{C}$ with

$$\mathrm{Hom}_{D\rtimes\boldsymbol{C}}(A,B)=\coprod_{f\in\mathrm{Hom}_{\boldsymbol{C}}(A,B)}D_f$$

and composition given by

$$ab=fb+ag$$

for any composable f and g in $\boldsymbol{C}$ and any $a\in D_f$, $b\in D_g$. It is proved in [11, 1.6] that for any natural system D on a category $\boldsymbol{C}$ the trivial linear extension $D\rtimes\boldsymbol{C}\to\boldsymbol{C}$ has the structure of an internal abelian group in the comma category $\mathbf{CAT}/\boldsymbol{C}$ of categories over $\boldsymbol{C}$ and moreover there is a one-to-one correspondence between linear extensions of $\boldsymbol{C}$ by D and $(D \rtimes \boldsymbol{C} \to \boldsymbol{C})$-torsors in $\mathbf{CAT}/\boldsymbol{C}$.

Let $\operatorname{Linext}(\boldsymbol{C}; D)$ be the set of equivalence classes of linear extensions of $\boldsymbol{C}$ by D.

1.1.3. **Theorem.** ([6]) *There is a natural bijection*

$$\operatorname{Linext}(\boldsymbol{C}; D) \approx H^2(\boldsymbol{C}; D).$$

1.2. **Finite product theories.**

1.2.1. *Basic definitions.* A *finite product theory* (simply theory for us) is a small category with finite products. A morphism of theories is a functor preserving finite products. With these morphisms, theories form a category $\mathscr{T}\!\mathit{heories}$. Let $\mathscr{C}$ be a category with finite products. A *model* of a theory $\mathbb{T}$ in the category $\mathscr{C}$, also termed a *$\mathscr{C}$-valued model of* $\mathbb{T}$, or an $\mathbb{T}$-*model in* $\mathscr{C}$, is a functor $\mathbb{T} \to \mathscr{C}$ preserving finite products. Models of $\mathbb{T}$ in $\mathscr{C}$ form a category $\mathbb{T}(\mathscr{C})$, with natural transformations as morphisms. Models in the category $\mathscr{E}\!\mathit{ns}$ of sets will be called simply models, and the category $\mathbb{T}(\mathscr{E}\!\mathit{ns})$ will be also denoted by $\mathbb{T}$-**mod**. It is known that the category $\mathbb{T}$-**mod** is complete and cocomplete for any theory $\mathbb{T}$. Moreover the inclusion $\mathbb{T}\text{-}\mathbf{mod} \hookrightarrow \operatorname{Funct}(\mathbb{T}, \mathscr{E}\!\mathit{ns})$ preserves all limits and has a left adjoint, and the Yoneda embedding $\mathbb{T}^{\mathrm{op}} \to \operatorname{Funct}(\mathbb{T}, \mathscr{E}\!\mathit{ns})$ factors through it, i. e. there is a full embedding $F : \mathbb{T}^{\mathrm{op}} \to \mathbb{T}\text{-}\mathbf{mod}$. Models in the image of F are called *finitely generated free models*, so that $\mathbb{T}$ is equivalent to the opposite of the category of such models. It is easy to see that the functor F preserves coproducts, i. e. $F(X \times Y)$ is a coproduct of $F(X)$ and $F(Y)$ in the category of models. A morphism of theories $f : \mathbb{T} \to \mathbb{T}'$ induces a functor

$$f^* : \mathbb{T}'\text{-}\mathbf{mod} \to \mathbb{T}\text{-}\mathbf{mod},$$

where $f^*(M) = M \circ f$. Clearly this functor preserves all limits. Since moreover the categories of models have small generating subcategories (those of free models), by Freyd's Special Adjoint Functor Theorem the functor f^* has a left adjoint

$$f_! : \mathbb{T}\text{-}\mathbf{mod} \to \mathbb{T}'\text{-}\mathbf{mod}.$$

One can see that the square

$$\begin{tikzcd}
\mathbb{T}^{\mathrm{op}} \arrow[r, "I_{\mathbb{A}}"] \arrow[d, "f^{\mathrm{op}}"'] & \mathbb{T}\text{-}\mathbf{mod} \arrow[d, "f_!"] \\
\mathbb{T}'^{\mathrm{op}} \arrow[r, "I_{\mathbb{B}}"] & \mathbb{T}'\text{-}\mathbf{mod}
\end{tikzcd}$$

commutes. See [2] for details.

1.2.2. *Single sorted theories.* Let $\mathbb{S}^{\mathrm{op}} \hookrightarrow \mathscr{E}\!\mathit{ns}$ be the full subcategory of $\mathscr{E}\!\mathit{ns}$ with the objects $\boldsymbol{n} = \{1, ..., n\}$ for $n \geqslant 0$. Since the category $\mathbb{S}^{\mathrm{op}}$ has finite coproducts, the category $\mathbb{S}$, opposite of the category $\mathbb{S}^{\mathrm{op}}$ is a theory, which is called the *theory of sets*. To distinguish objects of $\mathbb{S}$ and $\mathbb{S}^{\mathrm{op}}$ we redenote objects of $\mathbb{S}$ by $X^0 = 1$, $X^1 = X$, $X^2, X^3, \cdots$. For any $1 \leqslant i \leqslant n$ we denote by $x_i : X^n \to X$ the morphism of $\mathbb{S}$ corresponding to the map $\{1\} \to \boldsymbol{n}$, which takes 1 to i. It is clear that $\boldsymbol{n}$ is a coproduct of n copies of $\{1\}$ in $\mathbb{S}^{\mathrm{op}}$. It follows that $x_1, ..., x_n : X^n \to X$ is a product diagram in $\mathbb{S}$. One observes that $\mathbb{S}(\mathscr{C})$ is equivalent to $\mathscr{C}$ for any category with finite products $\mathscr{C}$. In particular $\mathbb{S}$-**mod** is equivalent to the category $\mathscr{E}\!\mathit{ns}$.

A single sorted theory is a theory morphism $\mathbb{S} \to \mathbb{T}$ which is identity on objects. The full subcategory of $\mathbb{S}/\mathscr{T}\!\mathit{heories}$ with single sorted theories as objects will be denoted by $\mathscr{T}\!\mathit{h}_1$. Thus objects of single sorted theories are just natural numbers, which are denoted by $X^0 = 1, X^1 = X$, $X^2, X^3, \cdots$. There are projections $x_1, ..., x_n$ from X^n to X. If M is a model of a single sorted theory $\mathbb{T}$, then $M(X)$ is called the *underlying set of* M. It is then equipped with operations $u_M : M(X)^n \to M(X)$ for each element u of $\operatorname{Hom}_{\mathbb{T}}(X^n, X)$, satisfying identities prescribed by category structure of $\mathbb{T}$. By this reason, elements of $\operatorname{Hom}_{\mathbb{T}}(X^n, X)$ will be called *n-ary operations of* $\mathbb{T}$. Thus for any single sorted theory $\mathbb{T}$, the category $\mathbb{T}$-**mod** is a variety of universal algebras. Conversely, for any variety $\mathbf{V}$, the opposite of the category of the algebras freely generated by the sets $\boldsymbol{n} = \{1, ..., n\}$, $n \geqslant 0$, is a single sorted theory, whose category of models is equivalent to $\mathbf{V}$.

1.2.3. *Multisorted theories.* Let I be a set and consider the category $\mathbb{S}^{\mathrm{op}}/I$ of maps $\boldsymbol{n} \to I$ for various sets $\boldsymbol{n} = \{1, ..., n\}$. Morphisms in $\mathbb{S}^{\mathrm{op}}/I$ from $\boldsymbol{n} \to I$ to $\boldsymbol{m} \to I$ are commutative diagrams of sets

$$\begin{tikzcd} \boldsymbol{n} \arrow[rr] \arrow[dr] & & \boldsymbol{m} \arrow[dl] \\ & I & \end{tikzcd}$$

One easily sees that this category has finite coproducts; for example, coproduct of $f_1 : \boldsymbol{n}_1 \to I$ and $f_2 : \boldsymbol{n}_2 \to I$ is $\binom{f_1}{f_2} : \boldsymbol{n}_1 \sqcup \boldsymbol{n}_2 \to I$. in fact, the set of objects of $\mathbb{S}^{\mathrm{op}}/I$ can be identified with the free monoid generated by the set I in such a way that a word $i_1...i_n$ represent the coproduct of the objects $i_\nu : \boldsymbol{1} \to I$, $\nu = 1, ..., n$. So any $f : \boldsymbol{n} \to I$ is the coproduct of the objects $f(1) : \boldsymbol{1} \to I$, ..., $f(n) : \boldsymbol{1} \to I$ in $\mathbb{S}/I$. We let $\mathbb{Fam}_I$ be the opposite of the category $\mathbb{S}^{\mathrm{op}}/I$. Then $\mathbb{Fam}_I$ is a theory called the *theory of I-indexed families.* To distinguish objects of $\mathbb{Fam}_I$ and $\mathbb{S}^{\mathrm{op}}/I$ we denote the object of $\mathbb{Fam}_I$ corresponding to a map $f : \boldsymbol{n} \to I$ by X_f. Hence an object of $\mathbb{Fam}_I$ has the form $X_{i_1} \times ... \times X_{i_n}$ for a unique n-tuple $(i_1, ..., i_n) \in I^n$. It is straightforward to check that the functor

$$\mathbb{Fam}_I(\mathscr{C}) \to \mathscr{C}^I \tag{*}$$

which assigns to a model $M : \mathbb{Fam}_I \to \mathscr{C}$ the family $M(X_i)_{i \in I}$ is an equivalence.

For a set I, an *I-sorted theory* is a theory morphism $\mathbb{Fam}_I \to \mathbb{T}$ which is identity on objects. The full subcategory of $\mathbb{Fam}_I/\mathscr{Theories}$ with I-sorted theories as objects will be denoted by $\mathscr{Th}_I$.

Although I-sorted theories appear to be of very special kind, one has

1.2.4. **Proposition.** *For any theory $\mathbb{T}$ there is a set I and an I-sorted theory $\mathbb{Fam}_I \to \tilde{\mathbb{T}}$ such that the category $\tilde{\mathbb{T}}$ is equivalent to $\mathbb{T}$.*

Proof. Let I be the set $\mathrm{Ob}(\mathbb{T})$ of objects of $\mathbb{T}$. We then are forced to take for the set of objects of $\tilde{\mathbb{T}}$ the free monoid $\sum_{n \geqslant 0} \mathrm{Ob}(\mathbb{T})^n$ on I. There is an obvious map from this monoid to the set of objects of $\mathbb{T}$, $\Pi : \mathrm{Ob}(\tilde{\mathbb{T}}) \to \mathrm{Ob}(\mathbb{T})$ which assigns to an n-tuple $(X_1, ..., X_n)$ of objects of $\mathbb{T}$ its product $X_1 \times ... \times X_n$ in $\mathbb{T}$. We then simply define

$$\mathrm{Hom}_{\tilde{\mathbb{T}}}((X_1, ..., X_n), (Y_1, ..., Y_m)) = \mathrm{Hom}_{\mathbb{T}}(\Pi(X_1, ..., X_n), \Pi(Y_1, ..., Y_m)).$$

This clearly defines the category $\tilde{\mathbb{T}}$ with the same objects as $\mathbb{Fam}_{\mathrm{Ob}(\mathbb{T})}$ and a functor $\tilde{\mathbb{T}} \to \mathbb{T}$ which is full and faithful and surjective on objects, i. e. it is an equivalence. Moreover by (*) above, models of $\mathbb{Fam}_{\mathrm{Ob}(\mathbb{T})}$ in a category with finite products $\mathscr{C}$ are families $(C_X)_{X \in \mathrm{Ob}(\mathbb{T})}$ of objects of $\mathscr{C}$, so the tautological family $(X)_{X \in \mathrm{Ob}(\mathbb{T})}$ gives a finite product preserving functor $\mathbb{Fam}_{\mathrm{Ob}(\mathbb{T})} \to \mathbb{T}$. It is then obvious that this functor lifts to a functor $\mathbb{Fam}_{\mathrm{Ob}(\mathbb{T})} \to \tilde{\mathbb{T}}$ which is identity on objects. $\square$

A model of an I-sorted theory $\mathbb{Fam}_I \to \mathbb{T}$ is just a $\mathbb{T}$-model. For such a model $\mathbb{T} \to \mathscr{C}$ in a category $\mathscr{C}$ its *underlying family* is the object of $\mathscr{C}^I$ corresponding to the composite $\mathbb{Fam}_I \to \mathbb{T} \to \mathscr{C}$. When safe, we will denote images of morphisms $\omega : X_{i_1} \times ... \times X_{i_n} \to X_i$ of $\mathbb{T}$ under a model $\mathbb{T} \to \mathscr{C}$ by ω again. Thus intuitively, models M of an I-sorted theory $\mathbb{Fam}_I \to \mathbb{T}$ in categories with finite products $\mathscr{C}$ are I-tuples of objects $(C_i)_{i \in I}$, $C_i = M(X_i)$, equipped with additional structure, namely various operations of the form

$$\omega : C_{i_1} \times ... \times C_{i_n} \to C_i$$

corresponding to morphisms $\omega : X_{i_1} \times ... \times X_{i_n} \to X_i$ in $\mathbb{T}$. These operations must further satisfy various identities expressing the fact that M is a product preserving functor. In detail, this amounts to the following:

- the morphisms corresponding to the projections $\pi_1 : X_{i_1} \times ... \times X_{i_n} \to X_{i_1}$, ..., $\pi_n : X_{i_1} \times ... \times X_{i_n} \to X_{i_n}$ must be product projections themselves;

- for morphisms $\omega : X_{i_1}\times ...\times X_{i_n} \to X_i$, $\omega' : X_{i'_1}\times ...\times X_{i'_m} \to X_i$ and $\omega_1 : X_{i'_1}\times ...\times X_{i'_m} \to X_{i_1}$, ..., $\omega_n : X_{i'_1}\times ...\times X_{i'_m} \to X_{i_n}$ in $\mathbb{T}$ with $\omega(\omega_1, ..., \omega_n) = \omega'$, the diagram

$$\begin{tikzcd} & C_{i_1}\times ...\times C_{i_n} \arrow[dl, "\omega"'] & \\ C_i & & C_{i'_1}\times ...\times C_{i'_m} \arrow[ul, "(\omega_1,...,\omega_n)"'] \arrow[ll, "\omega'"] \end{tikzcd}$$

must commute.

The "substrate" underlying the structure of an I-sorted theory is a family of sets of the form $(S_{(i_1,...,i_n),i})_{(i_1,...,i_n)\in I^n, i\in I}$ for $n = 0, 1, ...$, namely, the sets $\mathrm{Hom}_{\mathbb{T}}(X_{i_1}\times ...\times X_{i_n}, X_i)$. We thus have a forgetful functor

$$U : \mathscr{T}\!h_I \to \prod_{n\geqslant 0} \mathscr{E}ns^{I^n\times I}.$$

It is proved in [7] that this functor admits a left adjoint F. Theories in the image of this left adjoint are *free* theories. It is more or less obvious that the adjunction counits $FU\mathbb{T} \to \mathbb{T}$ are all full functors, so that in particular one has

1.2.5. **Proposition.** *For any theory $\mathbb{T}$ there exists a morphism $\mathbb{F} \to \mathbb{T}$ from a free theory to $\mathbb{T}$ which is a full functor.*

□

Moreover, since every componentwise surjective map in $\prod_{n\geqslant 0} \mathscr{E}ns^{I^n\times I}$ admits a section, it follows

1.2.6. **Proposition.** *Let $P : \mathbb{T} \to \mathbb{F}$ be a morphism in $\mathscr{T}\!h_I$ which is a full functor. If $\mathbb{F}$ is a free theory, then P has a section, i. e. there is a morphism $S : \mathbb{F} \to \mathbb{T}$ in $\mathscr{T}\!h_I$ with $PS = 1$.*

□

1.3. **Ringoids and modules over them.** Let us here recall some well known facts about ringoids and modules over them. A good reference on this subject is [13].

A *ringoid* is a category enriched in abelian groups. It is thus a small category $\mathscr{R}$ together with the structure of abelian group on its Hom-sets in such a way that composition is biadditive. Morphisms of ringoids are enriched functors, i. e. functors preserving the abelian group structures. These are also called *additive* functors. The category of ringoids will be denoted by $\mathscr{R}ingoids$.

Let $\mathscr{R}$ be a ringoid. We denote by $\mathscr{R}$-**mod** the category of all covariant additive functors from $\mathscr{R}$ to $\mathscr{A}b$, and by **mod**-$\mathscr{R}$ the category of all contravariant additive functors from $\mathscr{R}$ to $\mathscr{A}b$. Objects from $\mathscr{R}$-**mod** are called *left modules* over $\mathscr{R}$, while those from **mod**-$\mathscr{R}$ are called *right modules.*

For any small category $\boldsymbol{I}$, we let $\mathbb{Z}[\boldsymbol{I}]$ be the ringoid with the same objects as $\boldsymbol{I}$, while for any objects i and j the group of homomorphisms from i to j in $\mathbb{Z}[\boldsymbol{I}]$ is the free abelian group generated by $\mathrm{Hom}_{\boldsymbol{I}}(i, j)$:

$$\mathrm{Hom}_{\mathbb{Z}[\boldsymbol{I}]}(i, j) = \mathbb{Z}[\mathrm{Hom}_{\boldsymbol{I}}(i, j)],$$

whereas the composition law is induced by

$$\mathbb{Z}[\mathrm{Hom}_{\boldsymbol{I}}(i, j)] \otimes \mathbb{Z}[\mathrm{Hom}_{\boldsymbol{I}}(j, k)] \cong \mathbb{Z}[\mathrm{Hom}_{\boldsymbol{I}}(i, j)\times \mathrm{Hom}_{\boldsymbol{I}}(j, k)] \to \mathbb{Z}[\mathrm{Hom}_{\boldsymbol{I}}(i, k)].$$

Then clearly one has $\mathbb{Z}[\boldsymbol{I}]$-**mod** $\simeq \mathscr{A}b^{\boldsymbol{I}}$.

For any ringoid $\mathscr{R}$ and an object $c \in \mathscr{R}$ we define $\mathrm{h}_c : \mathscr{R} \to \mathscr{A}b$ and $\mathrm{h}^c : \mathscr{R}^{\mathrm{op}} \to \mathscr{A}b$ by

$$\mathrm{h}_c(x) = \mathrm{Hom}_{\mathscr{R}}(c, x)$$

and

$$\mathrm{h}^c(x) = \mathrm{Hom}_{\mathscr{R}}(x, c).$$

Then one has natural isomorphisms

$$\mathrm{Hom}_{\mathscr{R}\textbf{-mod}}(\mathrm{h}_c, M) \cong M(c)$$

and

$$\mathrm{Hom}_{\mathbf{mod}\text{-}\mathscr{R}}(\mathrm{h}^c, N) \cong N(c).$$

Therefore, the family of objects $(\mathrm{h}_c)_{c\in\mathrm{Ob}(\mathscr{R})}$ (resp. $(\mathrm{h}^c)_{c\in\mathrm{Ob}(\mathscr{R})}$) forms a family of small projective generators in $\mathscr{R}$-**mod** (resp. in **mod**-$\mathscr{R}$). The functor h_c is called *the standard free left $\mathscr{R}$-module concentrated at c.*

Let $f : \mathscr{R} \to \mathscr{S}$ be a morphism of ringoids. Composition with f induces a functor

$$f^* : \mathscr{S}\text{-}\mathbf{mod} \to \mathscr{R}\text{-}\mathbf{mod}.$$

It is well known that f^* has right and left adjoint functors f_* and $f_!$ respectively (the so-called right and left Kan extensions).

There is a generalization to ringoids of the fact that to any ring R corresponds the theory $\mathbb{M}_R$ of (left) R-modules, which obviously is the category opposite to that of free finitely generated left R-modules and their homomorphisms. Note that equivalently we may take for $\mathbb{M}_R$ the category of free finitely generated *right* R-modules.

In fact there is a functor $\mathbb{M} : \mathscr{Ringoids} \to \mathscr{Theories}$. It assigns to a ringoid $\mathscr{R}$ the theory $\mathbb{M}_{\mathscr{R}}$ of $\mathscr{R}$-modules. $\mathbb{M}_{\mathscr{R}}$ is the additive category freely generated by $\mathscr{R}$, i. e. it is an additive category equipped with a homomorphism of ringoids $I_{\mathscr{R}} : \mathscr{R} \to \mathbb{M}_{\mathscr{R}}$ which has the following universal property: for any additive category $\mathscr{A}$, precomposition with $I_{\mathscr{R}}$ induces an equivalence of categories

$$\mathrm{Add}(\mathbb{M}_{\mathscr{R}}, \mathscr{A}) \cong \mathrm{Hom}_{\mathscr{Ringoids}}(\mathscr{R}, \mathscr{A}).$$

There exists an explicit description of $\mathbb{M}_{\mathscr{R}}$ as the category of *matrices* over $\mathscr{R}$: $\mathbb{M}_{\mathscr{R}}$ can be chosen to be an $\mathrm{Ob}(\mathscr{R})$-sorted theory, so that its objects are finite families of objects of $\mathscr{R}$, pictured as $a_1 \oplus ... \oplus a_n$, for any $a_1, ..., a_n \in \mathscr{R}$, $n \geqslant 0$. Moreover $\mathrm{Hom}_{\mathbb{M}_{\mathscr{R}}}(a_1 \oplus ... \oplus a_n, b_1 \oplus ... \oplus b_m)$ is defined as

$$\prod_{\substack{i=1,\dots,m\\ j=1,\dots,n}} \mathrm{Hom}_{\mathscr{R}}(a_j, b_i),$$

with composition defined via matrix multiplication, i. e. $(f \circ g)_{ik} = \sum_j f_{ij}g_{jk}$ for $f_{ij} : b_j \to c_i$, $g_{jk} : a_k \to b_j$.

1.3.1. *Enveloping ringoids.* There is a functor in the opposite direction, from theories to ringoids.

1.3.2. **Proposition.** *For any I-sorted theory $\mathbb{T}$ there exists a ringoid $U(\mathbb{T})$, depending functorially on $\mathbb{T}$, such that $\mathbb{A}\mathrm{b}(\mathbb{T}$-**mod**$)$ is equivalent to the category of $U(\mathbb{T})$-modules.*

Proof. The key observation here is that in the presence of an abelian group structure any operation like $\omega : X_1 \times ... \times X_n \to X$ must be an abelian group homomorphism, hence have the form $\omega(x_1, ..., x_n) = \omega_1(x_1) + ... + \omega_n(x_n)$ for some unary operations $\omega_i : X_i \to X$.

Let the set of objects of $U(\mathbb{T})$ be I, and present morphisms of $U(\mathbb{T})$ by generators and relations as follows. For each $\omega : X_{i_1} \times ... \times X_{i_n} \to X_i$ in $\mathbb{T}$ we pick n generators $\partial_1(\omega) : X_{i_1} \to X_i$, ..., $\partial_n(\omega) : X_{i_n} \to X_i$. And for each such ω and any $\omega_1 : X_{i'_1} \times ... \times X_{i'_m} \to X_{i_1}$, ..., $\omega_n : X_{i'_1} \times ... \times X_{i'_m} \to X_{i_n}$ we impose the relations

$$\partial_\mu\left(\omega(\omega_1, ..., \omega_n)\right) = \sum_{\nu=1}^{n} \partial_\nu(\omega) \circ \partial_\mu(\omega_\nu)$$

for $\mu = 1, ..., m$. Furthermore we impose the relations

$$\partial_\mu(x_\nu) = \delta_{\mu\nu},$$

with δ the Kronecker symbol, meaning the zero morphism for $\mu \neq \nu$ and the identity morphism for $\mu = \nu$. Here, $x_\nu : X_{i_1} \times ... \times X_{i_n} \to X_{i_\nu}$ are the projections, $\mu, \nu = 1, ..., n$.

Thus a $U(\mathbb{T})$-module is a collection of abelian groups $(A_i)_{i\in I}$ and homomorphisms $\partial_\nu(\omega) : A_{i_\nu} \to A_i$, $\omega \in \mathrm{Hom}_{\mathbb{T}}(X_{i_1} \times ... \times X_{i_n}, X_i)$, $\nu = 1, ..., n$ satisfying the above relations. Then from any such module we obtain an object of $\mathbb{A}\mathrm{b}(\mathbb{T}$-**mod**$)$ by defining

$$\omega(a_1, ..., a_n) = \sum_{\nu=1}^{n} \partial_\nu(\omega)a_\nu$$

for ω as above and $(a_1, ..., a_n) \in A_{i_1} \times ... \times A_{i_n}$. Conversely, if $(A_i)_{i\in I}$ is given the structure of an object from $\mathbb{Ab}(\mathbb{T}\text{-}\mathbf{mod})$, then we define

$$\partial_\nu(\omega)a = \omega(0, ..., 0, \underset{\nu\text{-th}\uparrow\text{position}}{a}, 0, ..., 0).$$

It is easy to see that these procedures determine mutually inverse equivalences between the category of $U(\mathbb{T})$-modules and $\mathbb{Ab}(\mathbb{T}\text{-}\mathbf{mod})$. □

2. Cartesian natural systems

2.1. **The notion.** Let $\mathbb{T}$ be a theory and let D be a natural system on $\mathbb{T}$. We will say that the natural system D is *cartesian* (or *compatible with products* — cf. [5]) if for any product diagram $p_k : X_1 \times ... \times X_n \to X_k$, $k = 1, ..., n$ and any morphism $f : X \to X_1 \times ... \times X_n$ the homomorphism

$$D_f \to D_{p_1 f} \times ... \times D_{p_n f}$$

given by $a \mapsto (p_1 a, ..., p_n a)$ is an isomorphism. Obviously D is cartesian if and only if it satisfies the above condition with $n = 0$ and $n = 2$, i. e.

- $D_{!_X} = 0$ for the unique morphism $!_X : X \to 1$ to the terminal object;
- $D_f \to D_{p_1 f} \times D_{p_2 f}$ is an isomorphism for any $f : X \to X_1 \times X_2$.

One observes that if a bifunctor $D : \mathbb{T}^{\mathrm{op}} \times \mathbb{T} \to \mathscr{Ab}$ preserves products in the second variable, then the natural system induced by D is cartesian. We denote by $\mathscr{F}(\mathbb{T})$ the category of cartesian natural systems on $\mathbb{T}$.

2.2. **Motivation and properties.** The following fact goes back to [10].

2.2.1. **Lemma.** *Let*

$$0 \to D \to \mathbb{E} \xrightarrow{P} \mathbb{T} \to 0$$

be a linear extension of a theory $\mathbb{T}$ by a natural system D. Then D is cartesian iff $\mathbb{E}$ is a theory and P is a theory morphism.

Proof. Take a product diagram $p_i : X_1 \times ... \times X_n \to X_i$, $i = 1, ..., n$, and choose arbitrarily $\tilde{p}_i$ in $\mathbb{E}$ with $P(\tilde{p}_i) = p_i$. This then gives a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{E}}(X, X_1 \times ... \times X_n) \arrow[r, "\tilde{f}\mapsto(\tilde{p}_1\tilde{f},...,\tilde{p}_n\tilde{f})"] \arrow[d, "P"'] & \mathrm{Hom}_{\mathbb{E}}(X, X_1) \times ... \times \mathrm{Hom}_{\mathbb{E}}(X, X_n) \arrow[d, "P"] \\
\mathrm{Hom}_{\mathbb{T}}(X, X_1 \times ... \times X_n) \arrow[r, "\approx"] & \mathrm{Hom}_{\mathbb{T}}(X, X_1) \times ... \times \mathrm{Hom}_{\mathbb{T}}(X, X_n)
\end{tikzcd}$$

which shows that $\mathbb{E}$ has and P preserves finite products iff all the maps

$$P^{-1}(f) \to P^{-1}(p_1 f) \times ... \times P^{-1}(p_n f),$$

given by $\tilde{f} \mapsto (\tilde{p}_1\tilde{f}, ..., \tilde{p}_n\tilde{f})$ are bijective.

On the other hand the above maps are equivariant with respect to the group homomorphisms

$$D_f \to D_{p_1 f} \times ... \times D_{p_n f}$$

and the actions given by the linear extension structure. Our proposition then follows from the following easy lemma. □

2.2.2. **Lemma.** *Suppose given a group homomorphism $f : G_1 \to G_2$ and an f-equivariant map $x : X_1 \to X_2$ between sets X_i with transitive and effective G_i-actions. Then x is bijective iff f is an isomorphism.*

Proof. See e. g. [10, Lemma 3.5] □

2.2.3. **Theorem.** *For any I-sorted theory $\mathbb{T}$ there is an equivalence of categories*

$$\Xi : \mathscr{F}(\mathbb{T}) \xrightarrow{\simeq} \mathbb{Ab}(\mathscr{Th}_I/\mathbb{T})$$

of the category $\mathscr{F}(\mathbb{T})$ and the category of internal abelian groups in $\mathscr{Th}_I/\mathbb{T}$.

Proof. It is easy to see from 1.1.2 above that for any natural system D on $\mathbb{T}$, the trivial linear extension $D \rtimes \mathbb{T} \to \mathbb{T}$ of $\mathbb{T}$ by D is an internal abelian group in categories over $\mathbb{T}$. If D is moreover cartesian, then by 2.2.1 $D \rtimes \mathbb{T}$ is actually a theory and the projection is a morphism of theories. Furthermore the group structure functors $+ : D \rtimes \mathbb{T} \times_{\mathbb{T}} D \rtimes \mathbb{T} \to D \rtimes \mathbb{T}$, $- : D \rtimes \mathbb{T} \to D \rtimes \mathbb{T}$ and $0 : \mathbb{T} \to D \rtimes \mathbb{T}$ over $\mathbb{T}$ are evidently morphisms of theories, i. e. preserve product projections, so that one obtains an internal abelian group $\Xi(D)$ in $\mathscr{T\!h}_I/\mathbb{T}$. The aforementioned correspondence between natural systems and internal abelian groups is in fact functorial and it is equally easy to see that under it morphisms of cartesian natural systems are carried to product preserving functors.

Conversely, given an internal abelian group structure on an object $p : \mathbb{E} \to \mathbb{T}$ of $\mathscr{T\!h}_I/\mathbb{T}$, put $D(p)_f = p^{-1}(f)$ for a morphism f in $\mathbb{T}$ and define for any composable f, g the actions $D(p)_g \to D(p)_{fg}$, $D(p)_f \to D(p)_{fg}$ by

$$f\tilde{g} = 0(f)\tilde{g},$$
$$\tilde{f}g = \tilde{f}0(g)$$

for any $p\tilde{f} = f$, $p\tilde{g} = g$, where $0 : \mathbb{T} \to \mathbb{E}$ is the functor defining zero of the internal abelian group structure. This clearly defines a natural system $D(p)$ on $\mathbb{T}$. It is easy to see that $D(p)$ is cartesian if (and only if) p is a morphism of theories, i. e. preserves products. Moreover any morphism of theories $f : \mathbb{E} \to \mathbb{E}'$ over $\mathbb{T}$ clearly defines a natural transformation of the corresponding natural systems.

We have thus defined functors in both directions between $\mathscr{F}(\mathbb{T})$ and $\mathbb{Ab}(\mathscr{T\!h}_I/\mathbb{T})$. It is straightforward to check that the composite $\mathscr{F}(\mathbb{T}) \to \mathscr{F}(\mathbb{T})$ is identity. To show that the other composite is isomorphic to the identity of $\mathbb{Ab}(\mathscr{T\!h}_I/\mathbb{T})$, note that for an internal abelian group $(p : \mathbb{E} \to \mathbb{T}, 0 : \mathbb{T} \to \mathbb{E}, - : \mathbb{E} \to \mathbb{E}, + : \mathbb{E} \times_{\mathbb{T}} \mathbb{E} \to \mathbb{E})$ in $\mathscr{T\!h}_I/\mathbb{T}$ functoriality of $+ : \mathbb{E} \times_{\mathbb{T}} \mathbb{E} \to \mathbb{E}$ implies

$$\tilde{f}\tilde{g} = (0(f) + \tilde{f})(\tilde{g} + 0(g)) = 0(f)\tilde{g} + \tilde{f}0(g)$$

for any composable f, g in $\mathbb{T}$ and any $\tilde{f} \in p^{-1}f$, $\tilde{g} \in p^{-1}g$. It follows that $\mathbb{E}$ is isomorphic to $D(p) \rtimes \mathbb{T}$ over $\mathbb{T}$. □

2.2.4. *Remark.* The above theorem can be also deduced from general results of [4, 1.5 and 4.11]. We omit the details.

The terminal object of $\mathscr{T\!h}_I/\mathbb{T}$ is obviously the identity functor $1_{\mathbb{T}} : \mathbb{T} \to \mathbb{T}$. The global sections functor

$$\Gamma = \mathrm{Hom}_{\mathscr{T\!h}_I/\mathbb{T}}(1_{\mathbb{T}}, \_) : \mathbb{Ab}(\mathscr{T\!h}_I/\mathbb{T}) \to \mathscr{Ab}$$

composed with the above equivalence $\mathscr{F}(\mathbb{T}) \to \mathbb{Ab}(\mathscr{T\!h}_I/\mathbb{T})$ yields the functor

$$\mathbb{Der}(\mathbb{T}; \_) : \mathscr{F}(\mathbb{T}) \to \mathscr{Ab}.$$

It is easy to identify the functor $\mathbb{Der}(\mathbb{T}; \_)$ explicitly. Given a Cartesian natural system D on $\mathbb{T}$, the abelian group $\mathbb{Der}(\mathbb{T}; D)$ is by definition the group of global sections of the projection $D \rtimes \mathbb{T} \to \mathbb{T}$. It is then straightforward to calculate that this amounts to

2.2.5. **Proposition.** *For any theory $\mathbb{T}$ and any Cartesian natural system $D \in \mathscr{F}(\mathbb{T})$ there is an isomorphism*

$$\mathbb{Der}(\mathbb{T}; D) \cong \left\{ d \in \prod_{\omega \in \mathrm{Hom}_{\mathbb{T}}(X,Y)} D_\omega \;\middle|\; \begin{array}{l} \forall X \xrightarrow{\omega} Y,\ Y \xrightarrow{\omega'} Z \\ d(\omega'\omega) = d(\omega')\omega + \omega' d(\omega) \end{array} \right\}.$$

Proof. A section $\mathbb{T} \to D \rtimes \mathbb{T}$ must assign to each morphism $\omega \in \mathrm{Hom}_{\mathbb{T}}(X, Y)$ a morphism $(d(\omega), \omega) \in \mathrm{Hom}_{D \rtimes \mathbb{T}}(X, Y)$; preservation of composition amounts precisely to the above equality. The latter also implies that $d(\text{identity}) = 0$, so identities are preserved too. □

3. Enveloping ringoids and modules over ringoid valued functors

3.1. **Enveloping ringoids.**

3.1.1. *The Grothendieck construction.* Our next goal is to prove that the category $\mathscr{F}(\mathbb{T})$ is an abelian category with enough projectives and injectives. To do that, we are going to generalize the notion of module over a ringoid to that of one over a ringoid valued functor on a small category. We will then realize $\mathscr{F}(\mathbb{T})$ as the category of modules over certain ringoid valued functor.

Suppose given a functor $\boldsymbol{F} : \boldsymbol{I} \to \mathbf{CAT}$ from a small category $\boldsymbol{I}$ to the category of categories, denoted $(\varphi : i \to j) \mapsto (F_\varphi : \boldsymbol{F}_i \to \boldsymbol{F}_j)$. Then the Grothendieck construction $\int_{\boldsymbol{I}} \boldsymbol{F}$ of $\boldsymbol{F}$ is defined as the lax colimit of $\boldsymbol{F}$. Explicitly, it is a category with objects of the form (i, X), with $i \in \mathrm{Ob}(\boldsymbol{I})$ and $X \in \mathrm{Ob}(\boldsymbol{F}_i)$; morphisms $(i, X) \to (i', X')$ are defined to be pairs (φ, f), with $\varphi : i \to i'$ and $f : F_\varphi(X) \to X'$. Identity morphism for (i, X) is $(\mathrm{id}_i, \mathrm{id}_X)$, and composition of $(\varphi' : i' \to i'', f' : F_{\varphi'}(X') \to X'')$ with (φ, f) as above is defined to be the pair $(\varphi'\varphi, f'F_{\varphi'}(f))$. There is a canonical functor $P_{\boldsymbol{F}} : \int_{\boldsymbol{I}} \boldsymbol{F} \to \boldsymbol{I}$ given by projection onto the first coordinate, i. e. sending (i, X) to i and (φ, f) to φ.

3.1.2. *Comma category as models.* As an application of previous discussion we prove that the comma category of a category of models of a theory is still a category of models for a theory.

3.1.3. **Proposition.** *For an I-sorted theory $\mathbb{T}$ and any model M in $\mathbb{T}$-**mod**, the category $\int_{\mathbb{T}} M$ is a $\left(\coprod_{i\in I} M_i\right)$-sorted theory and moreover the comma category $\mathbb{T}$-**mod**$/M$ is equivalent to the category of models $\left(\int_{\mathbb{T}} M\right)$-**mod**.*

Proof. Any object N of $\mathbb{T}$-**mod** equipped with a morphism $f : N \to M$ can be considered as a collection of sets

$$\left(N_x = f_A^{-1}(x) \subseteq N(A)\right)_{x\in\coprod_{A\in\mathrm{Ob}(\mathbb{T})} M(A)}$$

and maps $N_{x_1} \times ... \times N_{x_n} \to N_{\omega(x_1,...,x_n)}$, for all $(x_1, ..., x_n) \in M(X_{i_1}) \times ... \times M(X_{i_n})$ and $\omega : X_{i_1} \times ... \times X_{i_n} \to X_i$ in $\mathbb{T}$, fitting into certain commutative diagrams.

Then regarding M as an object of $\mathscr{E}ns^{\mathbb{T}}$, and defining $N(x) = N_{M(p_1)x} \times ... \times N_{M(p_n)x}$, for $x \in M(X_{i_1} \times ... \times X_{i_n})$, we can consider the above data as a functor $\tilde{N} : \int_{\mathbb{T}} M \to \mathscr{E}ns$, which sends the object $x \in M(X_{i_1} \times ... \times X_{i_n})$ of the latter category to the product of the objects $\tilde{N}(X_{i_\nu})$, $\nu = 1, ..., n$. Now the proof follows from the subsequent lemma. $\square$

3.1.4. **Lemma.** *A functor $M : \mathbb{T} \to \mathscr{E}ns$ preserves finite products if and only if the category $\int_{\mathbb{T}} M$ has finite products and the canonical functor $P : \int_{\mathbb{T}} M \to \mathbb{T}$, sending $m \in M(X)$ to X, preserves them.*

Proof. Let us first recall that functors of the form $P : \int_{\mathbb{T}} M \to \mathbb{T}$ for any functor $M : \mathbb{T} \to \mathscr{E}ns$ are characterized by a property called *discrete opfibration*:

for any $x \in \int_{\mathbb{T}} M$ and any $\varphi : Px \to a$, there is a unique $\psi : x \to y$ with $P\psi = \varphi$.

Using this property it is easy to prove that a pullback of a product preserving discrete fibration between categories with products along a product preserving functor is again a product preserving functor between categories with products.

The "only if" part then follows because of the following pullback diagram in the category of categories

$$\begin{tikzcd}
\int_{\mathbb{T}} M \arrow[r] \arrow[d, "P"'] & \mathscr{E}ns_\bullet \arrow[d, "U"] \\
\mathbb{T} \arrow[r, "M"] & \mathscr{E}ns
\end{tikzcd}$$

in which $\mathscr{E}ns_\bullet$ denotes the category of pointed sets and U the forgetful functor: since the latter is a discrete opfibration and preserves products, it follows that $\int_{\mathbb{T}} M$ will have and $P : \int_{\mathbb{T}} M \to \mathbb{T}$ preserve them too.

For the "if" part, we again use the discrete fibration property to prove

a) $M(1)$ has single element: the particular case of the above discrete opfibration condition with $Px = a = 1$ implies that for any $x \in P^{-1}(1)$ one has $\left(x \xrightarrow{\mathrm{id}_x} x\right) = \left(x \xrightarrow{!_x} 1\right)$, since $P(\mathrm{id}_x) = P(!_x) = \mathrm{id}_1$.

b) $M(a_1 \times a_2) \xrightarrow{(M\pi_1, M\pi_2)} Ma_1 \times Ma_2$ is bijective: this follows from another two particular cases of the discrete opfibration condition — with $x = x_1 \times x_2$ for some $x_i \in P^{-1}(a_i)$ and $\varphi = \pi_i$, $i = 1, 2$; indeed these cases give that there are unique ψ_i starting out of x with $P(\psi_i) = \pi_i$, hence x is a unique element of $M(a_1 \times a_2)$ satisfying $M\pi_i(x) = x_i$, $i = 1, 2$.

□

3.1.5. **Corollary.** *For any model M of a theory $\mathbb{T}$, there exists a ringoid $\mathscr{U}(M)$, the* enveloping ringoid *of M, depending functorially on M, such that the category $\mathbb{Ab}(\mathbb{T}\text{-}\mathbf{mod}/M)$ is equivalent to the category of $\mathscr{U}(M)$-modules.*

Proof. Of course this is just a particular case of 1.3.2 in view of 3.1.3. Let us, however, give explicit presentation of $\mathscr{U}(M) = U(\textstyle\int_{\mathbb{T}} M)$ in this case, assuming for simplicity that $\mathbb{T}$ is an I-sorted theory. The set of objects of $\mathscr{U}(M)$ is then $\coprod_{i \in I} M(X_i)$, and the morphisms are generated by ones of the form $\partial_\nu(\omega)(c_1, ..., c_n) : c_\nu \to \omega(c_1, ..., c_n)$, for each $\omega \in \mathrm{Hom}_{\mathbb{T}}(X_{i_1} \times ... \times X_{i_n}, X_i)$, $(c_1, ..., c_n) \in M(X_{i_1}) \times ... \times M(X_{i_n})$ and $\nu \in \{1, ..., n\}$. The defining relations are indexed by data $\omega \in \mathrm{Hom}_{\mathbb{T}}(X_{i_1} \times ... \times X_{i_n}, X_i)$, $\omega_1 \in \mathrm{Hom}_{\mathbb{T}}(X_{i'_1} \times ... \times X_{i'_m}, X_{i_1})$, ..., $\omega_n \in \mathrm{Hom}_{\mathbb{T}}(X_{i'_1} \times ... \times X_{i'_m}, X_{i_n})$, $(c_1, ..., c_m) \in M(X_{i'_1}) \times ... \times M(X_{i'_m})$, and $\mu \in \{1, ..., m\}$ and have the form

$$\partial_\mu(\omega(\omega_1, ..., \omega_n))(c_1, ..., c_m) = \sum_{\nu=1}^{n} \partial_\nu(\omega)(\omega_1(c_1, ..., c_m), ..., \omega_n(c_1, ..., c_m)) \circ \partial_\mu(\omega_\nu)(c_1, ..., c_m)$$

and

$$\partial_\mu(x_\nu)(c_1, ..., c_n) = \begin{cases} \mathrm{id}_{c_\nu}, & \mu = \nu, \\ 0, & \mu \neq \nu. \end{cases}$$

Once again, functoriality is obvious from this presentation. □

Occasionally we will write $\mathscr{U}_{\mathbb{T}}(M)$ to make explicit dependence on $\mathbb{T}$. This construction is known under various names in the literature — see e. g. [3] or [16].

3.2. **Derivations.** Given a theory $\mathbb{T}$, its model $M \in \mathbb{T}\text{-}\mathbf{mod}$, and an object $p : A \to M$ of the category $\mathbb{Ab}(\mathbb{T}\text{-}\mathbf{mod}/M) \simeq \mathscr{U}_{\mathbb{T}}(M)\text{-}\mathbf{mod}$, we will denote by $\mathrm{Der}(M; A)$ the abelian group of all sections of $A \to M$, i. e. the set of all morphisms $s : M \to A$ of $\mathbb{T}$-models with $ps = 1_M$. Elements of $\mathrm{Der}(M; A)$ will be called *derivations* of M in A. $\mathrm{Der}(M; A)$ is contravariantly functorial in M, in the following sense. For a morphism $f : M' \to M$ of models we get the induced homomorphism $f^* : \mathrm{Der}(M; A) \to \mathrm{Der}(M'; f^*A)$, where f^*A denotes the pullback of $p : A \to M$ along f. Equivalently, one might interpret $\mathrm{Der}(M'; f^*A)$ as the abelian group of all $\mathbb{T}$-model morphisms $M' \to A$ over M, i. e. fitting in the commutative diagram

$$\begin{tikzcd} & A \arrow[d, "p"] \\ M' \arrow[ur] \arrow[r, "f"] & M \end{tikzcd} . \tag{1}$$

Clearly also $\mathrm{Der}(M; A)$ is covariantly functorial in A and so defines a functor $\mathrm{Der}(M; \_)$ on $\mathscr{U}_{\mathbb{T}}(M)$-$\mathbf{mod}$. We then have

3.2.1. **Proposition.** *The functor $\mathrm{Der}(M; \_)$ is representable. That is, there exists an $\mathscr{U}_{\mathbb{T}}(M)$-module Ω^1_M with natural isomorphism $\mathrm{Der}(M; A) \cong \mathrm{Hom}_{\mathscr{U}_{\mathbb{T}}(M)}(\Omega^1_M, A)$ for all A. Moreover Ω^1_M depends functorially on M. When M is a finitely generated free $\mathbb{T}$-model, then Ω^1_M is a projective object of $\mathscr{U}(M)$-$\mathbf{mod}$.*

Proof. Following the equivalence from 3.1.5, for an $\mathscr{U}(M)$-module A the corresponding object of $\mathbb{Ab}(\mathbb{T}\text{-}\mathbf{mod}/M)$ is the $\mathbb{T}$-model with $X_i \mapsto \coprod_{x \in M(X_i)} A(x)$, with the $\mathbb{T}$-model structure assigning to $\omega : X_{i_1} \times ... \times X_{i_n} \to X_i$ the operation

$$\omega : \left(\coprod_{x_1 \in M(X_{i_1})} A(x_1)\right) \times ... \times \left(\coprod_{x_n \in M(X_{i_n})} A(x_n)\right) \to \coprod_{x \in M(X_i)} A(x)$$

given by

$$\omega(a_1, ..., a_n) = \sum_{\nu=1}^{n} \partial_\nu(\omega)(x_1, ..., x_n)a_\nu.$$

Then

$$\mathrm{Der}(M; A) \subset \prod_{\substack{i \in I \\ x \in M(X_i)}} A(x)$$

consists of those families $(d(x) \in A(x))_{x \in \coprod_i M(X_i)}$ which respect all these operations. That is, $\mathrm{Der}(M; A)$ consists of assignments, to each $x \in M(X_i)$, of an element $d(x) \in A(x)$, in such a way that for any $\omega : X_{i_1} \times ... \times X_{i_n} \to X_i$ and any $x_\nu \in M(X_{i_\nu})$, $\nu = 1, ..., n$, one has

$$(*) \qquad d(\omega(x_1, ..., x_n)) = \sum_{\nu=1}^{n} \partial_\nu(\omega)(x_1, ..., x_n)d(x_\nu).$$

Because of this expression it is natural to call such assignments *derivations*.

We then present Ω^1_M by generators and relations as a $\mathscr{U}(M)$-module as follows: it has generators $d(x) \in \Omega^1_M(x)$ for each $x \in M(X_i)$ and each $i \in I$; and the defining relations are (*) above. It is then clear that Ω^1_M carries a generic derivation d, so that one has a natural isomorphism

$$\mathrm{Hom}_{\mathscr{U}(M)}(\Omega^1_M, A) \xrightarrow{\cong} \mathrm{Der}(M; A)$$

given by $f \mapsto fd$. That Ω^1_M is functorial in M is also clear from the construction.

Now suppose M is a finitely generated free model $F(X)$, i. e. there is an $X \in \mathbb{T}$ with $M = \mathrm{Hom}_{\mathbb{T}}(X, \_)$. Then it is straightforward to check using Yoneda lemma that for an object of $\mathbb{Ab}(\mathbb{T}\text{-}\mathbf{mod}/M)$ corresponding to a $\mathscr{U}(M)$-module A we will have $\mathrm{Der}(F(X); A) \cong A(\mathrm{id}_X)$. It follows that $\mathrm{Hom}_{\mathscr{U}(F(X))}(\Omega^1_{F(X)}, A)$ is an exact functor of A, i. e. $\Omega^1_{F(X)}$ is projective. In fact of course this actually means that $\Omega^1_{F(X)} = \mathrm{h}_{\mathrm{id}_X}$. □

3.3. **Ringoid valued functors.** Let us consider now a small category $\boldsymbol{I}$ and a covariant functor

$$\mathscr{R} : \boldsymbol{I} \to \mathscr{Ringoids}.$$

It is easy to see that the category $\int_{\boldsymbol{I}} \mathscr{R}$ is a ringoid in a canonical way.

We will say that M is a *left $\mathscr{R}$-module* if the following data are given:

i) a left $\mathscr{R}_i$-module M_i for each object $i \in \boldsymbol{I}$;
ii) a homomorphism $M_\alpha : M_i \to \mathscr{R}^*_\alpha M_j$ of $\mathscr{R}_i$-modules for each arrow $\alpha : i \to j$ of $\boldsymbol{I}$.

Moreover it is required that for any composable morphisms α and β one has $M_{\alpha\beta} = M_\alpha M_\beta$.

If M is a left $\mathscr{R}$-module, i is an object of $\boldsymbol{I}$, and x is an object of the ringoid $\mathscr{R}_i$, then we denote by $M_{(i,x)}$ the value $M_i(x)$ of M_i on x. Having this in mind it is clear that a left $\mathscr{R}$-module is nothing else but a functor $M : \int \mathscr{R} \to \mathscr{Ab}$ such that each composition $M \circ \xi_i : \mathscr{R}_i \to \mathscr{Ab}$, $i \in \boldsymbol{I}$, is an additive functor. The category of all left $\mathscr{R}$-modules will be denoted by $\mathscr{R}$-**mod**.

3.3.1. *Example.* As an example, we can take any small subcategory $\boldsymbol{I}$ of the category of commutative rings and let $\mathscr{O}$ be the inclusion $\boldsymbol{I} \hookrightarrow \mathscr{Rings}$. Thus $\mathscr{O}$ is a ring valued functor, hence can be regarded as a functor with values in ringoids with a single object. For any ring $S \in \boldsymbol{I}$ the absolute Kähler differentials Ω^*_S is a module over S. Since Ω^*_S functorially depends on S we obtain that $\Omega^* \in \mathscr{O}$-**mod**. Another example comes from topology. Let $\boldsymbol{I}$ be a small subcategory of the category of topological spaces. Then for any ring R, the ordinary (singular) cohomology of spaces with coefficients in R defines a ring valued functor $H^*(\_; R)$, and for any R-module M the functor $H^*(\_; M)$ is a module over $H^*(\_; R)$ in the above sense. Similarly $X \mapsto \mathbb{Z}[\pi_1 X]$ is a ring valued functor defined on any small subcategory of the category of pointed topological spaces, while $X \mapsto \pi_i X$ is a module over it, for any $i \geqslant 2$.

An alternative description of the category $\mathscr{R}$-**mod** is possible, showing that it is equivalent to the category of modules over a single ringoid. Given a functor $\mathscr{R} : \boldsymbol{I} \to \mathscr{R}\!\mathit{ingoids}$ as above, we define its *total ringoid* $\mathscr{R}[\boldsymbol{I}]$ in the following way: the set $\operatorname{Ob}(\mathscr{R}[\boldsymbol{I}])$ of objects of the ringoid $\mathscr{R}[\boldsymbol{I}]$ is the disjoint union $\coprod_{i\in\operatorname{Ob}(\boldsymbol{I})}\operatorname{Ob}(\mathscr{R}_i)$ — or else again the set of pairs (i,x), just as for $\int\mathscr{R}$. Morphisms of the ringoid $\mathscr{R}[\boldsymbol{I}]$ are given by

$$\operatorname{Hom}_{\mathscr{R}[\boldsymbol{I}]}((i,x),(j,y)) = \bigoplus_{i\xrightarrow{\alpha}j}\operatorname{Hom}_{\mathscr{R}_j}(\mathscr{R}_\alpha(x),y).$$

Composition homomorphisms are given by

$$\begin{aligned}\left(\bigoplus_{i\xrightarrow{\alpha}j}\operatorname{Hom}_{\mathscr{R}_j}(\mathscr{R}_\alpha(x),y)\right)\otimes\left(\bigoplus_{j\xrightarrow{\beta}k}\operatorname{Hom}_{\mathscr{R}_k}(\mathscr{R}_\beta(y),z)\right)&\\ \xrightarrow{\cong}\bigoplus_{i\xrightarrow{\alpha}j\xrightarrow{\beta}k}\operatorname{Hom}_{\mathscr{R}_j}(\mathscr{R}_\alpha(x),y)\otimes\operatorname{Hom}_{\mathscr{R}_k}(\mathscr{R}_\beta(y),z)&\\ \xrightarrow{\bigoplus_{\alpha,\beta}\mathscr{R}_\beta\otimes 1}\bigoplus_{i\xrightarrow{\alpha}j\xrightarrow{\beta}k}\operatorname{Hom}_{\mathscr{R}_k}(\mathscr{R}_\beta\mathscr{R}_\alpha(x),\mathscr{R}_\beta y)\otimes\operatorname{Hom}_{\mathscr{R}_k}(\mathscr{R}_\beta(y),z)&\\ \xrightarrow{\bigoplus_{\alpha,\beta}\circ}\bigoplus_{i\xrightarrow{\alpha}j\xrightarrow{\beta}k}\operatorname{Hom}_{\mathscr{R}_k}(\mathscr{R}_\beta\mathscr{R}_\alpha(x),z)\to\bigoplus_{i\xrightarrow{\gamma}k}\operatorname{Hom}_{\mathscr{R}_k}(\mathscr{R}_\gamma(x),z),&\end{aligned}$$

and the identity of $x\in\operatorname{Ob}(\mathscr{R}_i)$ is the element of $\bigoplus_{i\xrightarrow{\varepsilon}i}\operatorname{Hom}_{\mathscr{R}_i}(\mathscr{R}_\varepsilon(x),x)$ given by the identity of x in $\mathscr{R}_i$, situated in the id_i-th summand. It is straightforward to check that this construction indeed yields a ringoid. One then has

3.3.2. **Proposition.** *For any ringoid-valued functor $\mathscr{R} : \boldsymbol{I} \to \mathscr{R}\!\mathit{ingoids}$, the category of left $\mathscr{R}$-modules is equivalent to $\mathscr{R}[\boldsymbol{I}]$-**mod**.*

Proof. An $\mathscr{R}[\boldsymbol{I}]$-module M is a family of abelian groups $(M_{(i,x)})_{x\in\coprod_i\operatorname{Ob}(\mathscr{R}_i)}$ and a family of abelian group homomorphisms

$$\left(\bigoplus_{i\xrightarrow{\alpha}j}\operatorname{Hom}_{\mathscr{R}_j}(\mathscr{R}_\alpha(x),y)\xrightarrow{M_{(i,x),(j,y)}}\operatorname{Hom}_{\mathscr{Ab}}(M_{(i,x)},M_{(j,y)})\right)_{x\in\operatorname{Ob}(\mathscr{R}_i),y\in\operatorname{Ob}(\mathscr{R}_j)},$$

satisfying certain conditions. Just by universality of sums then, specifying the above homomorphisms $M_{(i,x),(j,y)}$ is equivalent to specifying families

$$\left(\operatorname{Hom}_{\mathscr{R}_j}(\mathscr{R}_\alpha(x),y)\xrightarrow{M_\alpha}\operatorname{Hom}_{\mathscr{Ab}}(M_{(i,x)},M_{(j,y)})\right)_{\alpha\in\operatorname{Hom}_{\int\mathscr{R}}((i,x),(j,y))}.$$

It is then straightforward to check that the conditions on the $M_{(i,x),(j,y)}$ to form an $\mathscr{R}[\boldsymbol{I}]$-module give precisely the conditions on the M_α to form an $\mathscr{R}$-module. □

It is thus clear that $\mathscr{R}$-**mod** is an abelian category with enough projective and injective objects. Let us give the explicit description of the projective generators and injective cogenerators corresponding to the standard ones from $\mathscr{R}[\boldsymbol{I}]$.

Take $i\in\operatorname{Ob}(\boldsymbol{I})$ and let x be an object of the ringoid $\mathscr{R}_i$. Then, in accord with the above 3.3.2, associated to the standard free $\mathscr{R}[\boldsymbol{I}]$-module concentrated at (i,x) there is a left $\mathscr{R}$-module $\mathrm{h}^{\mathscr{R}}_{i,x}$ given by

$$\left(\mathrm{h}^{\mathscr{R}}_{i,x}\right)_j(y) = \bigoplus_{i\xrightarrow{\alpha}j}\operatorname{Hom}_{\mathscr{R}_j}(\mathscr{R}_\alpha(x),y).$$

In other words $(\mathrm{h}^{\mathscr{R}}_{i,x})_j$ is the direct sum of standard free $\mathscr{R}_j$-modules:

$$\left(\mathrm{h}^{\mathscr{R}}_{i,x}\right)_j = \bigoplus_{i\xrightarrow{\alpha}j} \mathrm{h}_{\mathscr{R}_\alpha(x)}\,.$$

It follows that for any $\mathscr{R}_j$-module X one has isomorphisms

$$\mathrm{Hom}_{\mathscr{R}_j}((\mathrm{h}^{\mathscr{R}}_{i,x})_j, X) \cong \prod_{i\xrightarrow{\alpha}j} X(\mathscr{R}_\alpha(x)).$$

Thus for any $\mathscr{R}$-module M one has a natural isomorphism

$$\mathrm{Hom}_{\mathscr{R}}(\mathrm{h}^{\mathscr{R}}_{i,x}, M) \cong M_i(x).$$

Let now k be an object of $\boldsymbol{I}$ and let A be an $\mathscr{R}_k$-module. We denote by $k_*(A)$ the $\mathscr{R}$-module, whose value at i is given by

$$(k_*A)_i = \prod_{i\xrightarrow{\alpha}k} \mathscr{R}^*_\alpha A.$$

The α-component of $(k_*A)_i$ has an $\mathscr{R}_i$-module structure given by restriction of scalars along the ringoid homomorphism $\mathscr{R}_\alpha : \mathscr{R}_i \to \mathscr{R}_k$. Hence $(k_*A)_i$ is an $\mathscr{R}_i$-module and now it is clear that k_*A is an $\mathscr{R}$-module. Moreover the functor $k_* : \mathscr{R}_k\textbf{-mod} \to \mathscr{R}\textbf{-mod}$ is right adjoint to the evaluation functor $\mathrm{ev}_k : \mathscr{R}\textbf{-mod} \to \mathscr{R}_k\textbf{-mod}$, which is given by $\mathrm{ev}_k(M) = M_k$. In particular, if A is an injective $\mathscr{R}_k$-module then k_*A is an injective $\mathscr{R}$-module. Hence the family $(k_*Q)_{k,Q}$, is a family of injective cogenerators for the category of $\mathscr{R}$-modules. Here k runs over the set of objects of $\boldsymbol{I}$, and then Q over the set of injective cogenerators of the category of $\mathscr{R}_k$-modules.

3.4. **The equivalence.** Our main example of a ringoid valued functor stems from 3.1.5. To any theory $\mathbb{T}$ one can assign a ringoid valued functor $\mathscr{U}_{\mathbb{T}}$ on $\mathbb{T}$ considered as a small category, by sending an object X of $\mathbb{T}$ to the enveloping ringoid $\mathscr{U}_{\mathbb{T}}(F(X))$ of the corresponding free $\mathbb{T}$-model $F(X)$.

For any two objects A, B of $\mathscr{U}_{\mathbb{T}}\textbf{-mod}$ there is a natural system $\mathscr{H}om_{\mathscr{U}}(A,B)$ on $\mathbb{T}$ given by

$$\mathscr{H}om_{\mathscr{U}}(A,B)_{X\xrightarrow{f}Y} = \mathrm{Hom}_{\mathscr{U}_{\mathbb{T}}(F(Y))}(A_Y, F(f)^*B_X),$$

where the ringoid morphism $F(f) : \mathscr{U}_{\mathbb{T}}(F(Y)) \to \mathscr{U}_{\mathbb{T}}(F(X))$ is induced by $F(f) : F(Y) \to F(X)$, i. e. by $(g \mapsto gf) : \mathrm{Hom}_{\mathbb{T}}(Y,\_) \to \mathrm{Hom}_{\mathbb{T}}(X,\_)$. Let us find out when is this natural system cartesian. For this it will be convenient to rewrite the above in the following way:

$$\mathscr{H}om_{\mathscr{U}}(A,B)_{X\xrightarrow{f}Y} = \mathrm{Hom}_{\mathscr{U}_{\mathbb{T}}(F(X))}(F(f)_!A_Y, B_X).$$

Indeed as we saw in 1.2.1 all the functors $F(f)^*$ have left adjoints. The above conditions then show that this natural system is cartesian if and only if

- $\mathrm{Hom}_{\mathscr{U}(F(X))}(F(!_X)_!A_1, B_X) = 0$ for all X;
- the canonical morphism
  $$\mathrm{Hom}_{\mathscr{U}(F(X))}(F(f)_!A_{X_1\times X_2}, B_X) \to \mathrm{Hom}_{\mathscr{U}(F(X))}(F(p_1f)_!A_{X_1} \oplus F(p_2f)_!A_{X_2}, B_X)$$
  is an isomorphism for any $f : X \to X_1 \times X_2$.

In particular $\mathscr{H}om_{\mathscr{U}}(A,B)$ is cartesian for *all* B if and only if A satisfies

- $A_1 = 0$;
- $F(p_1)_!A_{X_1} \oplus F(p_2)_!A_{X_2} \to A_{X_1\times X_2}$ is an isomorphism for any X_1, X_2.

It is natural to call such an A a *cartesian* $\mathscr{U}_{\mathbb{T}}$*-module.*

We next discuss our main example Ω^1 of such an $\mathscr{U}_{\mathbb{T}}$-module, obtained from 3.2.1.

3.4.1. *Example.* Any $\mathscr{U}_{\mathbb{T}}$-module B determines a natural system $\mathscr{D}er(\_;B)$ on $\mathbb{T}$ in the following way: for a morphism $f : X \to Y$ of $\mathbb{T}$, put

$$\mathscr{D}er(\_;B)_f = \mathrm{Der}(F(Y); f^*(B_X)).$$

Here $p_X : B_X \to F(X)$ is the object of $\mathbb{Ab}(\mathbb{T}\textbf{-mod}/F(X))$ corresponding to $B(X)$ under the equivalence $\mathscr{U}_{\mathbb{T}}(F(X))\textbf{-mod} \simeq \mathbb{Ab}(\mathbb{T}\textbf{-mod}/F(X))$. That this is indeed a natural system, follows from the functorial properties of Der. Moreover this natural system is cartesian. Indeed, $\mathbb{T}$-models of the form $F(X)$ are the representable ones, $F(X)(Y) = \mathrm{Hom}_{\mathbb{T}}(X, Y)$. Then considering the diagram (1) we see that $\mathrm{Der}(F(Y); f^*(B_X))$ can be identified with the set of all elements $b \in B_X(Y)$ with $p_X(b) = f \in F(X)(Y) = \mathrm{Hom}_{\mathbb{T}}(X, Y)$. Then given $f_i : X \to X_i$, $i = 1, ..., n$, one has

$$\begin{aligned}\mathscr{D}\!er(\_; B)_{(f_1,...,f_n)} &= \mathrm{Der}(F(X_1 \times ... \times X_n); (f_1, ..., f_n)^*(B_X))\\ &\approx \{b \in B_X(X_1 \times ... \times X_n) \mid p_X(b) = (f_1, ..., f_n)\}\\ &\approx \{(b_1, ..., b_n) \in B_X(X_1) \times ... \times B_X(X_n) \mid p_X(b_i) = f_i, i = 1, ..., n\}\\ &\approx \mathscr{D}\!er(\_; B)_{f_1} \times ... \times \mathscr{D}\!er(\_; B)_{f_n}.\end{aligned}$$

But it is immediate from 3.2.1 that there is an $\mathscr{U}_{\mathbb{T}}$-module Ω^1 such that the natural system $\mathscr{D}\!er(\_; B)$ is actually isomorphic to $\mathscr{H}\!om_{\mathscr{U}}(\Omega^1_{F(\_)}, B)$. Namely, Ω^1 is just given by $X \mapsto \Omega^1_{F(X)}$. It is then a cartesian $\mathscr{U}_{\mathbb{T}}$-module in the above sense, i. e. one has

- $\Omega^1_{F(1)} = 0$;
- $F(p_1)_!\Omega^1_{F(X_1)} \oplus F(p_2)_!\Omega^1_{F(X_2)} \to \Omega^1_{F(X_1 \times X_2)}$ is an isomorphism for any X_1, X_2.

3.4.2. **Theorem.** *There is an equivalence of categories*

$$\Phi : \mathscr{F}(\mathbb{T}) \to \mathscr{U}_{\mathbb{T}}\text{-}\mathbf{mod};$$

*in particular, $\mathscr{F}(\mathbb{T})$ is an abelian category with enough projectives and injectives. Moreover the quasi-inverse of this equivalence assigns to an object A of $\mathscr{U}_{\mathbb{T}}$-**mod** the cartesian natural system $\mathscr{D}\!er(\_; A) \cong \mathscr{H}\!om_{\mathscr{U}}(\Omega^1, A)$ from 3.4.1.*

Proof. As always, we can assume here that $\mathbb{T}$ is an I-sorted theory. Then for a cartesian natural system D on $\mathbb{T}$, to define $\Phi(D)$ we must first name for each $X \in \mathrm{Ob}(\mathbb{T})$ a $\mathscr{U}_{\mathbb{T}}(F(X))$-module $\Phi(D)_X$. The set of objects of $\mathscr{U}_{\mathbb{T}}(F(X))$ is $\coprod_{i \in I} F(X)(X_i)$ (see 3.1.5), i. e. $\coprod_{i \in I} \mathrm{Hom}_{\mathbb{T}}(X, X_i)$. We then define values of $\Phi(D)_X$ on these objects by

$$\Phi(D)_X(X \xrightarrow{x} X_i) = D_x.$$

Next action of morphisms of $\mathscr{U}_{\mathbb{T}}(F(X))$ is uniquely determined by requiring, for $(x_1, ..., x_n) : X \to X_{i_1} \times ... \times X_{i_n}$ and $\omega : X_{i_1} \times ... \times X_{i_n} \to X_i$, commutativity of the diagrams

$$\begin{tikzcd} & D_{(x_1,...,x_n)} \arrow[dl, "\omega\_"'] & D_{x_1} \times ... \times D_{x_n} \arrow[l, "\cong"'] & \\ D_{\omega(x_1,...,x_n)} & & & D_{x_\nu}, \arrow[ul, "\iota_\nu"'] \arrow[lll, "\partial_\nu(\omega)(x_1,...,x_n)"'] \end{tikzcd}$$

where the isomorphism is the inverse of the canonical map that is required by cartesianness of D, and ι_ν is the ν-th embedding into $\oplus = \times$ of abelian groups.

We also have to define action on $\Phi(D)$ of morphisms $f : X \to Y$ in $\mathbb{T}$, which must be $\mathscr{U}_{\mathbb{T}}(F(Y))$-module morphisms $\Phi(D)_Y \to F(f)^*(\Phi(D)_X)$, where the functor $F(f)^* : \mathscr{U}_{\mathbb{T}}(F(X))$-**mod** $\to$ $\mathscr{U}_{\mathbb{T}}(F(Y))$-**mod** is the restriction of scalars along the ringoid morphism $\mathscr{U}_{\mathbb{T}}(F(Y)) \to \mathscr{U}_{\mathbb{T}}(F(X))$ induced by the morphism of $\mathbb{T}$-models $F(f) : F(Y) \to F(X)$. Now $F(f)^*(\Phi(D)_X)$ is easily seen to be given by $(y : Y \to X_i) \mapsto D_{yf}$, so what we must choose is a suitably compatible family of abelian group homomorphisms

$$\Phi(D)_f(Y \xrightarrow{y} X_i) : D_y \to D_{yf},$$

and these we declare to be the action of $\_f$ on D. It is then straightforward that all of the above indeed gives a functor $\Phi : \mathscr{F}(\mathbb{T}) \to \mathscr{U}_{\mathbb{T}}$-**mod**.

Next note that, as we have seen in 3.2.1, one has $\mathrm{Der}(F(X); A) \cong A(\mathrm{id}_X)$ for any $\mathscr{U}_{\mathbb{T}}(F(X))$-module A, so in particular for any $f : X \to Y$ in $\mathbb{T}$ we have by 3.4.1

$$\mathscr{D}\!er(\_; \Phi(D))_f = \mathrm{Der}(F(Y); F(f)^*(\Phi(D)_X)) \cong F(f)^*(\Phi(D)_X)(\mathrm{id}_Y) = D_{\mathrm{id}_Y f} = D_f.$$

Conversely, given a $\mathscr{U}_{\mathbb{T}}$-module A, by definition

$$\Phi(\mathscr{D}er(\_;A))_X(X \xrightarrow{x} X_i) = \mathscr{D}er(\_;A)_x = \mathrm{Der}(F(X_i);F(x)^*(A_X)) \cong F(x)^*(A_X)(\mathrm{id}_{X_i}) = A_X(x).$$

(Of course one should also check these on morphisms, but this is straightforward too). □

3.4.3. **Corollary.** *A natural system on a theory $\mathbb{T}$ is cartesian if and only if it is isomorphic to one of the form*

$$\mathscr{H}om_{\mathscr{U}}(\Omega^1, B)$$

for some $\mathscr{U}_{\mathbb{T}}$-module B.

□

As another corollary we obtain a generalization of [11, 2.4].

3.4.4. **Corollary.** *For any additive theory $\mathbb{T}$ (i. e. a theory which is additive as a category) the functor*

$$\mathbb{T}\text{-}\mathbf{mod}^{\mathbb{T}^{\mathrm{op}}} \to \mathscr{F}(\mathbb{T})$$

which assigns to a functor

$$T : \mathbb{T}^{\mathrm{op}} \to \mathbb{T}\text{-}\mathbf{mod}$$

the cartesian natural system $\tilde{T}$ on $\mathbb{T}$ given by

$$\tilde{T}_{X \xrightarrow{f} Y} = \mathrm{Hom}_{\mathbb{T}\text{-}\mathbf{mod}}(F(Y), T(X))$$

is an equivalence of categories.

Proof. Following the proof of 3.4.2 in this case, we see that any cartesian natural system D on $\mathbb{T}$ can be given by

$$D_{X \xrightarrow{f} Y} = \mathrm{Der}(F(Y); f^*(B_X))$$

where B is some $\mathscr{U}_{\mathbb{T}}$-module and $B_X \to F(X)$ is the object of $\mathbb{Ab}(\mathbb{T}\text{-}\mathbf{mod}/F(X))$ corresponding to $B(X)$ under the equivalence

$$\mathscr{U}_{\mathbb{T}}(F(X))\text{-}\mathbf{mod} \simeq \mathbb{Ab}(\mathbb{T}\text{-}\mathbf{mod}/F(X)).$$

But since the category $\mathbb{T}$-**mod** is additive (even abelian), there is a canonical equivalence of categories

$$\mathbb{Ab}(\mathbb{T}\text{-}\mathbf{mod}/M) \simeq \mathbb{T}\text{-}\mathbf{mod}$$

for any $\mathbb{T}$-model M. Composing these two equivalences we obtain that there is a functor $T : \mathbb{T}^{\mathrm{op}} \to \mathbb{T}\text{-}\mathbf{mod}$ such that for each object X of $\mathbb{T}$ the above internal abelian group in $\mathbb{T}\text{-}\mathbf{mod}/F(X)$ represented by $B_X \to F(X)$ is naturally isomorphic to the constant one given by the direct sum projection $F(X) \oplus T(X) \to F(X)$. Moreover this isomorphism gives compatible isomorphisms

$$\mathrm{Der}(F(Y); f^*(B_X)) \cong \mathrm{Hom}(F(Y), T(X))$$

for any $f : X \to Y$. □

3.4.5. *Example.* As an example of essentially non-additive situation, let us take the case when $\mathbb{T}$ is the theory of groups $\mathbb{Gr}$. The corresponding coefficient systems according to [10] were functors from the category of finitely generated free groups to the category of abelian groups.

The category $\mathscr{F}(\mathbb{Gr})$ is equivalent to a larger category whose objects are assignments M of an F-module M_F to each finitely generated free group F, in a way which is functorial in F. Then coefficients in the sense of [10] correspond to those objects M for which the F-module structure on M_F is trivial for all F.

The enveloping ringoid $\mathscr{U}_{\mathbb{Gr}}(G)$, for any group G, has the set of objects equal to G. From the relations given in 3.1.5 it is clear that all morphisms of $\mathscr{U}_{\mathbb{Gr}}(G)$ are linear combinations of composites of the ones of the form

$$\partial_\nu(x_1x_2)(g_1, g_2) : g_\nu \to g_1g_2, \quad \nu = 1, 2,$$

for $g_1, g_2 \in G$. Moreover these relations imply that $\partial_\nu(x_1x_2)(g_1,g_2)$ are isomorphisms, with the inverses given by

$$\partial_1(x_1x_2)(g_1,g_2)^{-1} = \partial_1(x_1x_2)(g_1g_2, g_2^{-1})$$

and

$$\partial_2(x_1x_2)(g_1,g_2)^{-1} = \partial_2(x_1x_2)(g_1^{-1}, g_1g_2).$$

Indeed the relations from 3.1.5 easily imply that for any $1 \leqslant \nu \leqslant n$ an operation ω of the form

$$X_{i_1} \times ... \times X_{i_n} \xrightarrow{\text{projection}} X_{i_1} \times ... \times X_{i_{\nu-1}} \times X_{i_{\nu+1}} \times ... \times X_{i_n} \to X_i$$

in any theory $\mathbb{T}$ one has for any model M of $\mathbb{T}$ and any $(c_1,...,c_n) \in M(X_{i_1}) \times ... \times M(X_{i_n})$

$$\partial_\nu(\omega)(c_1,...,c_n) = 0 : c_\nu \to \omega(c_1,...,c_n);$$

taking this into account readily gives

$$\begin{aligned}\mathrm{id}_{g_1} &= \partial_1(x_1)(g_1g_2g_2^{-1})\\ &= \partial_1(x_1x_2)(g_1g_2, g_2^{-1})\partial_1(x_1x_2)(g_1,g_2) + \partial_2(x_1x_2)(g_1g_2,g_2^{-1})\partial_1(x_2^{-1})(g_1,g_2)\\ &= \partial_1(x_1x_2)(g_1g_2,g_2^{-1})\partial_1(x_1x_2)(g_1,g_2)\end{aligned}$$

and

$$\begin{aligned}\mathrm{id}_{g_1g_2} &= \partial_1(x_1)(g_1g_2, g_2^{-1}) = \partial_1(x_1x_2x_2^{-1})(g_1g_2,g_2^{-1})\\ &= \partial_1(x_1x_2)(g_1,g_2)\partial_1(x_1x_2)(g_1g_2,g_2^{-1}) + \partial_2(x_1x_2)\partial_1(x_2^{-1})(g_1g_2,g_2^{-1})\\ &= \partial_1(x_1x_2)(g_1,g_2)\partial_1(x_1x_2)(g_1g_2,g_2^{-1}),\end{aligned}$$

and similarly for the inverse of $\partial_2(x_1x_2)(g_1,g_2)$.

We thus see that all objects of $\mathscr{U}_{\mathbb{Gr}}(G)$ are isomorphic to each other, so that $\mathscr{U}_{\mathbb{Gr}}(G)$ is equivalent to the ringoid with single object whose endomorphism ring is that of the unit of G in $\mathscr{U}_{\mathbb{Gr}}(G)$. It is easy to show that this ring is isomorphic to the group ring $\mathbb{Z}[G]$ of G. Indeed this is also clear already from the statement of 3.1.5 since it is well known that the category $\mathbb{Ab}(\mathscr{Groups}/G)$ is equivalent to the category of $\mathbb{Z}[G]$-modules for any group G.

Moreover it is easy to see that under this equivalence the functor Der corresponds to taking derivations of G with values in G-modules, hence the $\mathscr{U}_{\mathbb{Gr}}(G)$-module Ω^1_G described in 3.2.1 corresponds to the $\mathbb{Z}[G]$ module equal to its augmentation ideal. It follows that the $\mathscr{U}_{\mathbb{Gr}}$-module Ω^1 assigns to the group F the $\mathscr{U}_{\mathbb{Gr}}(F)$-module uniquely determined by the fact that its value on the unit object is the augmentation ideal of the group ring $\mathbb{Z}[F]$, with actions of morphisms of $\mathscr{U}_{\mathbb{Gr}}(F)$ prescribed by the structure of F-submodule of $\mathbb{Z}[F]$.

3.4.6. *Example.* Let us give another example in which the notation Ω^1 has its "usual" meaning. For a commutative ring k, let $\mathbb{A}_k$ be the theory of commutative k-algebras. Finitely generated free k-algebras are the polynomial algebras $k[x_1,...,x_n]$, so as a category $\mathbb{A}_k$ is equivalent to the full subcategory of the category of affine k-schemes whose objects are the affine spaces $\mathbb{A}^n_k = \mathrm{Spec}(k[x_1,...,x_n])$.

Similarly to the above example, it is easy to see that for any k-algebra A the ringoid $\mathscr{U}_{\mathbb{A}_k}(A)$ with the set of objects A is equivalent to the ringoid with a single object whose endomorphism ring is A. Moreover this equivalence indentifies the functor Der with usual k-derivations of A with values in A-modules, so the $\mathscr{U}(A)$-module Ω^1_A corresponds to the classical module $\Omega^1_{A/k}$ of Kähler differentials. It follows that the values $\Omega^1(\mathbb{A}^n_k)$ of the $\mathscr{U}_{\mathbb{A}_k}$-module Ω^1 are determined by assigning to the zero object $0 \in A$ of the ringoid $\mathscr{U}_{\mathbb{A}_k}(\mathbb{A}^n_k)$ the module $\Omega^1_{k[x_1,...,x_n]/k} = k[x_1,...,x_n]\langle dx_1,...,dx_n\rangle$, with the action of morphisms determined by the free module structure on the latter.

4. The local-global spectral sequence

The aim of this section is to construct our main technical tool — a spectral sequence computing the Ext groups in the category of modules over a ringoid valued functor, using some local data.

4.1. **Construction.** Let $\boldsymbol{I}$ be a small category and let $\mathscr{R} : \boldsymbol{I} \to \mathscr{Ringoids}$ be a ringoid valued functor on $\boldsymbol{I}$. As we have seen in 3.3.2, the category $\mathscr{R}$**-mod** is an abelian category with enough projective and injective objects. One can generalize the construction in 3.4.1 and define for any $\mathscr{R}$-modules M and N the natural systems $\mathscr{Hom}_{\mathscr{R}}(M,N)$ and $\mathscr{Ext}^n_{\mathscr{R}}(M,N)$ on $\boldsymbol{I}$ by

$$\mathscr{Hom}_{\mathscr{R}}(M,N)_{i\xrightarrow{\chi}j} = \operatorname{Hom}_{\mathscr{R}_i}(M_i,N_j)$$

and

$$\mathscr{Ext}^n_{\mathscr{R}}(M,N)_{i\xrightarrow{\chi}j} = \operatorname{Ext}^n_{\mathscr{R}_i}(M_i,N_j)$$

respectively, where the actions of $\mathscr{R}_i$ on N_j are given via restriction of scalars along $\mathscr{R}_\chi : \mathscr{R}_i \to \mathscr{R}_j$. We call the natural systems $\mathscr{Hom}_{\mathscr{R}}(M,N)$ and $\mathscr{Ext}^n_{\mathscr{R}}(M,N)$ *local Hom and local Ext groups.* One observes that in the case when $\mathscr{R}$ is a constant functor, these natural systems actually come from bifunctors. The following theorem, which is the main result of this section, was proved for the particular case of such constant $\mathscr{R}$ with values in rings in [10].

4.1.1. **Theorem** (the local-to-global spectral sequence)**.** *Let $\boldsymbol{I}$ be a small category and let $\mathscr{R}$: $\boldsymbol{I} \to \mathscr{Ringoids}$ be a functor to the category of ringoids. For any $\mathscr{R}$-modules M and N there exists a spectral sequence with*

$$E_2^{pq} = H^p(\boldsymbol{I};\ \mathscr{Ext}^q_{\mathscr{R}}(M,N)) \Longrightarrow \operatorname{Ext}^{p+q}_{\mathscr{R}\textbf{-mod}}(M,N).$$

4.1.2. **Corollary.** *Let $\boldsymbol{I}$ be a small category and let M, N be $\mathscr{R}$-modules, where*

$$\mathscr{R} : \boldsymbol{I} \to \mathscr{Ringoids}$$

is a functor. Then one has a five-term exact sequence

$$\begin{aligned}0 \to H^1(\boldsymbol{I}; \mathscr{Hom}_{\mathscr{R}}(M,N)) &\to \operatorname{Ext}^1_{\mathscr{R}\textbf{-mod}}(M,N)\\ &\to H^0(\boldsymbol{I};\ \mathscr{Ext}^1_{\mathscr{R}}(M,N)) \to H^2(\boldsymbol{I}; \mathscr{Hom}_{\mathscr{R}}(M,N)) \to \operatorname{Ext}^2_{\mathscr{R}\textbf{-mod}}(M,N).\end{aligned}$$

Moreover, if $\operatorname{gl.dim} \mathscr{R}_i \leqslant 1$ *for each object i, then one has an exact sequence*

$$\begin{aligned}0 \to H^1(\boldsymbol{I}; \mathscr{Hom}_{\mathscr{R}}(M,N)) &\to \cdots \to H^n(\boldsymbol{I}; \mathscr{Hom}_{\mathscr{R}}(M,N))\\ &\to \operatorname{Ext}^n_{\mathscr{R}\textbf{-mod}}(M,N) \to H^{n-1}(\boldsymbol{I};\ \mathscr{Ext}^1_{\mathscr{R}}(M,N)) \to H^{n+1}(\boldsymbol{I}; \mathscr{Hom}_{\mathscr{R}}(M,N)) \to \cdots\end{aligned}$$

4.1.3. **Corollary.** *Suppose M_i is a projective $\mathscr{R}_i$-module for each $i \in \operatorname{Ob}(\boldsymbol{I})$. Then there is an isomorphism*

$$H^*(\boldsymbol{I}; \mathscr{Hom}_{\mathscr{R}}(M,N)) \cong \operatorname{Ext}^*_{\mathscr{R}\textbf{-mod}}(M,N).$$

4.2. **Proof of Theorem 4.1.1.** We fix a left $\mathscr{R}$-module N. We claim that for any left $\mathscr{R}$-module X one has an isomorphism:

$$H^0(\boldsymbol{I}; \mathscr{Hom}_{\mathscr{R}}(X,N)) \cong \operatorname{Hom}_{\mathscr{R}\textbf{-mod}}(X,N).$$

Indeed, it follows from the definition of cohomology that $H^0(\boldsymbol{I}; \mathscr{Hom}_{\mathscr{R}}(X,N))$ is isomorphic to the kernel

$$\operatorname{Ker}\left(\prod_{i\in\operatorname{Ob}(\boldsymbol{I})} \operatorname{Hom}_{\mathscr{R}_i\textbf{-mod}}(X_i,N_i) \to \prod_{i\xrightarrow{\alpha}j} \operatorname{Hom}_{\mathscr{R}_i\textbf{-mod}}(X_i,N_j)\right).$$

Thus $H^0(\boldsymbol{I}; \mathscr{Hom}_{\mathscr{R}}(X,N))$ consists of families $(f_i : X_i \to N_i)$ of R_i-homomorphisms, such that for any $\alpha : i \to j$ the diagram

$$\begin{array}{ccc} X_i & \xrightarrow{f_i} & N_i \\ \downarrow{\scriptstyle X_\alpha} & & \downarrow{\scriptstyle N_\alpha} \\ X_j & \xrightarrow{f_j} & N_j \end{array}$$

commutes, and the claim is proved. One observes that the diagram

$$\begin{tikzcd}
 & \mathrm{Nat}(\boldsymbol{I}) \arrow[dr, "H^0(\boldsymbol{I};-)"] & \\
\mathscr{R}\text{-}\mathbf{mod}^{\mathrm{op}} \arrow[ur, "\mathscr{H}\!om_{\mathscr{R}}(-,N)"] \arrow[rr, "\mathrm{Hom}_{\mathscr{R}\text{-}\mathbf{mod}}(-,N)"'] & & \mathscr{A}b
\end{tikzcd}$$

commutes and the Theorem is a consequence of the Grothendieck spectral sequence for composite functors. Of course in order to apply the Grothendieck theorem we first have to show that $H^n(\boldsymbol{I};\mathscr{H}\!om_{\mathscr{R}}(M,N)) = 0$ as soon as $n > 0$ and M is projective. To this end we can assume without loss of generality that $M = h^{\mathscr{R}}_{i,x}$, for some $i \in \boldsymbol{I}$ and $x \in \mathscr{R}_i$. In this case

$$\mathscr{H}\!om_{\mathscr{R}}(M,N)_{c\xrightarrow{\varphi}d} \cong \prod_{i\xrightarrow{\alpha}c} N(d,\mathscr{R}_{\varphi\alpha}(x))$$

and therefore we can use the following Lemma to finish the proof. □

4.2.1. **Lemma.** *Let us fix $i \in \boldsymbol{I}$ and $x \in \mathrm{Ob}(\mathscr{R}_i)$. For any functor $N : \int_{\boldsymbol{I}} \mathscr{R} \to \mathscr{A}b$ consider the natural system D on $\boldsymbol{I}$ given by*

$$D_{c\xrightarrow{\varphi}d} := \prod_{i\xrightarrow{\alpha}c} N(d,\mathscr{R}_{\varphi\alpha}(x)).$$

Then

$$H^0(\boldsymbol{I};D) = N(i,x).$$

and

$$H^n(\boldsymbol{I};D) = 0 \textit{ for } n > 0.$$

Proof. One easily checks that

$$C^*(\boldsymbol{I};D) \cong C^*(i/\boldsymbol{I};T),$$

where $i/\boldsymbol{I}$ is the comma category under the object i and $T : i/\boldsymbol{I} \to \mathscr{A}b$ is given by

$$T\left(i \xrightarrow{\alpha} c\right) = N(c,\mathscr{R}_\alpha(x)).$$

Hence the cohomology of $\boldsymbol{I}$ with coefficients in D coincides with the cohomology of the category $i/\boldsymbol{I}$ with coefficients in the functor T.

Now cohomology groups of a category with coefficients in functors are isomorphic to the right derived functors of the inverse limit on that category. Since 1_i is the initial object in the category $i/\boldsymbol{I}$, inverse limit of a functor on it is given by the value of this functor on 1_i. So the inverse limit is exact and its right derived functors vanish. This gives the Lemma. □

5. Cohomology of algebraic theories

5.1. **Definitions.** There are several possible approaches to define cohomology of an I-sorted theory $\mathbb{T}$. First, there is a general approach of Quillen to define cohomology of an object $\mathbb{T}$ in $\mathscr{T}\!h_I$ with coefficients in an object of $\mathbb{A}\mathrm{b}(\mathscr{T}\!h_I/\mathbb{T})$, which by 2.2.3 we know to be equivalent to $\mathscr{F}(\mathbb{T})$. The main ingredient needed to construct this cohomology is availability of simplicial resolutions in $\mathscr{T}\!h_I$ by degreewise free objects. In our case this is possible due to 1.2.5. Namely, for any $\mathbb{T}$ we choose a simplicial object $\mathbb{F}_\bullet$ in the full subcategory of $\mathscr{T}\!h_I$ on free theories and an augmentation $\varepsilon : \mathbb{F}_0 \to \mathbb{T}$. This is called a *free resolution* of $\mathbb{T}$ if for any sorts $i_1,...,i_n,i \in I$ the augmentation ε induces a weak equivalence from the simplicial set $\mathrm{Hom}_{\mathbb{F}_\bullet}(X_{i_1}\times ... \times X_{i_n},X_i)$ given by

$$\mathrm{Hom}_{\mathbb{F}_\bullet}(X_{i_1}\times ... \times X_{i_n},X_i)_k = \mathrm{Hom}_{\mathbb{F}_k}(X_{i_1}\times ... \times X_{i_n},X_i)$$

to the discrete simplicial set on the set of 0-simplices $\mathrm{Hom}_{\mathbb{T}}(X_{i_1}\times ... \times X_{i_n},X_i)$. Existence of such a free resolution is a consequence of the work of Quillen [14]. Namely, it is straightforward to check that the category $\mathscr{T}\!h_I$ satisfies condition $(**)$ of Theorem 4 in §4 of Chapter II of [14] (page 4.2). This allows one to apply the whole machinery of Quillen (simplicial) closed model category theory to $\mathscr{Th}_I$. In particular, our resolution is a cofibrant replacement of $\mathbb{T}$ considered as a constant simplicial object of $\mathscr{Th}_I$.

Having this, we then define for any $\mathbb{T}$ in $\mathscr{Th}_I$ and any $A \in \mathscr{F}(\mathbb{T})$ the Quillen cohomology groups of $\mathbb{T}$ with coefficients in A by the equality

$$H^*_{\mathrm{Q}}(\mathbb{T}; A) := H^*(\mathbb{D}\mathrm{er}(\mathbb{F}_\bullet; A)),$$

where A is considered as an object of each of the categories $\mathscr{F}(\mathbb{F}_n)$, $n \geqslant 0$, via pullback along the unique morphism of theories $\mathbb{F}_n \to \mathbb{T}$ given by the resolution.

For a theory $\mathbb{T}$ and an object $A \in \mathscr{F}(\mathbb{T})$, we next define the Cartan-Eilenberg type cohomology

$$H^*_{\mathrm{CE}}(\mathbb{T}; A)$$

by the equality

$$H^*_{\mathrm{CE}}(\mathbb{T}; A) := \mathrm{Ext}^*_{\mathscr{U}_{\mathbb{T}}\text{-}\mathbf{mod}}(\Omega^1, \Phi(A))$$

Here Ω^1 is from 3.4.1 above.

Finally, there is yet a third approach to constructing cohomology. Given a theory $\mathbb{T}$ and an object $A \in \mathscr{F}(\mathbb{T})$, one can form the Baues-Wirsching cohomology

$$H^*(\mathbb{T}; A)$$

of the category $\mathbb{T}$ with coefficients in the cartesian natural system A as in 1.1.1.

5.2. **Equivalence.** We will show that these three approaches actually give isomorphic results. More precisely, for any I-sorted theory $\mathbb{T}$ and any Cartesian natural system $A \in \mathscr{F}(\mathbb{T})$ there are natural isomorphisms

$$H^*_{\mathrm{CE}}(\mathbb{T}; A) \cong H^*(\mathbb{T}; A) \cong H^{*-1}_{\mathrm{Q}}(\mathbb{T}; A).$$

5.2.1. **Theorem.** *Let $\mathbb{T}$ be an I-sorted theory and let $A \in \mathscr{F}(\mathbb{T})$ be any Cartesian natural system on $\mathbb{T}$. Then there are isomorphisms*

$$H^*_{\mathrm{CE}}(\mathbb{T}; A) \cong H^*(\mathbb{T}; A).$$

Proof. Since by 3.4.2 Φ and $\mathscr{Der}(\_; -)$ are mutually inverse equivalences, by Proposition 3.2.1 one has an isomorphism of natural systems:

$$A \cong \mathscr{Hom}(\Omega^1, \Phi(A)).$$

Hence the isomorphism to be proved is a consequence of Corollary 4.1.3. The fact that the condition of Corollary 4.1.3 holds here follows from Proposition 3.2.1. □

5.2.2. **Lemma.** *If $\mathbb{F}$ is a free I-sorted theory and A is a cartesian natural system on $\mathbb{F}$, then*

$$H^i(\mathbb{F}; A) = 0, \quad i > 1.$$

Proof. First consider the case $i = 2$; thanks to Theorem 1.1.3 it suffices to show that any linear extension of $\mathbb{F}$ by A splits. By Lemma 2.2.1 any such extension is an extension in $\mathscr{Theories}$ and we can use Proposition 1.2.6 to conclude that it really splits. If $i \geqslant 3$ we can use the isomorphism of Theorem 5.2.1 above to identify $H^i(\mathbb{F}; A)$ with $H^i_{\mathrm{CE}}(\mathbb{F}; A)$. These are Ext-groups in appropriate abelian categories vanishing on injective objects. As we showed, they also are identically zero in dimension two. Standard homological algebra argument shows that derived functors identically vanishing in some dimension are zero in all higher dimensions too. This finishes the proof. □

This result in the case when A is a bifunctor over a single sorted theory was proved in [10] (see Proposition 4.22 of loc. cit.).

5.2.3. **Theorem.** *There is an isomorphism*

$$H^n_{\mathrm{Q}}(\mathbb{T}; A) \cong \begin{cases} \mathbb{D}\mathrm{er}(\mathbb{T}; A), & n = 0, \\ H^{n+1}(\mathbb{T}; A), & n > 0 \end{cases}$$

for any theory $\mathbb{T}$ and any $A \in \mathscr{F}(\mathbb{T})$.

Proof. Let $C^{*+1}(\mathbb{T};A)$ be the downshift by one of the cochain complex from 1.1.1. That is, it is the cochain complex with $C^{n+1}(\mathbb{T};A)$ in degrees $n\geqslant 0$ and zero in all negative degrees. Thus we have

$$H^n(C^{*+1}(\mathbb{T};A))=\begin{cases}\mathbb{Der}(\mathbb{T};A), & n=0,\\ H^{n+1}(\mathbb{T};A), & n>0.\end{cases}$$

Next suppose given a free resolution $\varepsilon:\mathbb{F}_\bullet\to\mathbb{T}$ of $\mathbb{T}$ in $\mathscr{T}\!\!h_I/\mathbb{T}$. We then similarly obtain a cosimplicial cochain complex $C^{*+1}(\mathbb{F}_\bullet;A)$, with two spectral sequences converging to the cohomology of the total complex of the associated bicomplex.

The spectral sequence with

$${}'E_1^{pq}=H^q(C^{*+1}(\mathbb{F}_p;A))$$

has

$${}'E_1^{pq}=\begin{cases}\mathbb{Der}(\mathbb{F}_p;A), & q=0,\\ 0, & q>0,\end{cases}$$

by Lemma 5.2.2 above, so the common abutment is isomorphic to $H^*_{\mathrm{Q}}(\mathbb{T};A)$ by definition.

The second spectral sequence has

$${}''E_1^{p*}=H^*(C^{p+1}(\mathbb{F}_\bullet;A)).$$

By definition of the resolutions, for any objects Y, Z of $\mathbb{T}$ the augmentation ε induces a weak equivalence from the simplicial set $\mathrm{Hom}_{\mathbb{F}_\bullet}(Y,Z)$ to the discrete set $\mathrm{Hom}_{\mathbb{T}}(Y,Z)$. In particular, the latter is in one-to-one correspondence with the set of connected components of the former.

Now, looking at the explicit formula for the cochain complex C^* in 1.1.1, we see that there are isomorphisms

$$C^{p+1}(\mathbb{F}_\bullet;A)\xrightarrow{\cong}\prod_{Y_0,\dots,Y_{p+1}}C^*(\mathrm{Hom}_{\mathbb{F}_\bullet}(Y_1,Y_0)\times\dots\times\mathrm{Hom}_{\mathbb{F}_\bullet}(Y_{p+1},Y_p);A_{(-)})$$

to the product of cochain complexes of simplicial sets $\mathrm{Hom}_{\mathbb{F}_\bullet}(Y_1,Y_0)\times\dots\times\mathrm{Hom}_{\mathbb{F}_\bullet}(Y_{p+1},Y_p)$ with coefficients in abelian groups equal to $A_{f_1\cdots f_{p+1}}$ on the connected component of the former corresponding to $(f_1,\dots,f_{p+1})\in\mathrm{Hom}_{\mathbb{T}}(Y_1,Y_0)\times\dots\times\mathrm{Hom}_{\mathbb{T}}(Y_{p+1},Y_p)$. Since these simplicial sets have trivial cohomology in positive dimensions, we obtain

$${}''E_1^{pq}=0 \text{ for } q>0.$$

Moreover obviously

$${}''E_1^{p0}=\prod_{Y_0,\dots,Y_{p+1}}H^0(\mathrm{Hom}_{\mathbb{F}_\bullet}(Y_1,Y_0)\times\dots\times\mathrm{Hom}_{\mathbb{F}_\bullet}(Y_{p+1},Y_p);A_{(-)})\cong C^{p+1}(\mathbb{T};A),$$

so on the other hand the abutment is isomorphic to $\mathbb{Der}(\mathbb{T};A)$ in dimension zero and to $H^{n+1}(\mathbb{T};A)$ in dimensions $n>0$.

Comparing these two descriptions of the abutment gives the theorem. □

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