# COMBINATORIAL STRUCTURE OF EXCEPTIONAL SETS IN RESOLUTIONS OF SINGULARITIES

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Abstract. The dual complex can be associated to any resolution of singularities whose exceptional set is a divisor with simple normal crossings. It generalizes to higher dimensions the notion of the dual graph of a resolution of surface singularity. In this preprint we show that the dual complex is homotopy trivial for resolutions of 3-dimensional terminal singularities and for resolutions of Brieskorn singularities. We also review our earlier results on resolutions of rational and hypersurface singularities.

## 1. Introduction

In this preprint we continue our study of the dual complex associated to a resolution of singularities started in [20] and [21]. The dual complex is defined in the following way. Let $(X, S)$ be a germ of an algebraic variety or an analytic space $X$, $S = \operatorname{Sing}(X)$, and let $f\colon Y \to X$ be a good resolution of singularities. By this we mean that the exceptional set $Z = f^{-1}(S)$ is a divisor with simple normal crossings on $Y$. *The dual complex associated to the resolution* $f$ is just the incidence complex $\Delta(Z)$ of the divisor $Z$, i. e., if $Z = \sum Z_i$ is the decomposition of $Z$ into its prime components, then 0-simplexes (vertices) $\Delta_i$ of $\Delta(Z)$ correspond to the divisors $Z_i$, 1-simplexes (edges) $\Delta_{ij}^{(k)}$ correspond to the irreducible components $Z_{ij}^{(k)}$ of all intersections $Z_i \cap Z_j = \cup_k Z_{ij}^{(k)}$ and the edges $\Delta_{ij}^{(k)}$ join the vertices $\Delta_i$ and $\Delta_j$, 2-simplexes (triangles) correspond to the irreducible components of triple intersections $Z_i \cap Z_j \cap Z_k$ and are glued to the 1-skeleton of $\Delta(Z)$ in a natural way, and so on. In the case $X$ to be 2-dimensional we have the usual definition for the dual graph of resolution $f$.

*Example* 1.1. Consider a 3-dimensional singularity

$$(\{g(x, y, z, t) = x^5 + y^5 + z^5 + t^5 + xyzt = 0\}, 0) \subset (\mathbb{C}^4, 0)\,.$$

Blowing up the origin produces a good resolution. The exceptional divisor $Z$ of this blow-up is defined in the projective space $\mathbb{P}^3$ by the homogeneous part $g_4 = xyzt$ of $g$. Thus $Z$ consists of 4 planes in general position. It follows that $\Delta(Z)$ is the surface of tetrahedron. This example can easily be

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generalised to arbitrary dimension $n \geq 2$. This gives complexes $\Delta(Z)$ which are borders of standard simplexes $\Delta^{n-1}$. We see that the dual complex associated to a resolution can be homeomorphic to the sphere $S^{n-1}$.

Note that in general $\Delta(Z)$ is a triangulated topological space but not a simplicial complex. For example, let $\dim X = 2$ and $Z$ contain 2 curves meeting transversally at 2 points (see Fig. 1 *a*). The corresponding fragment of the dual complex is shown in Fig. 1 *b*.

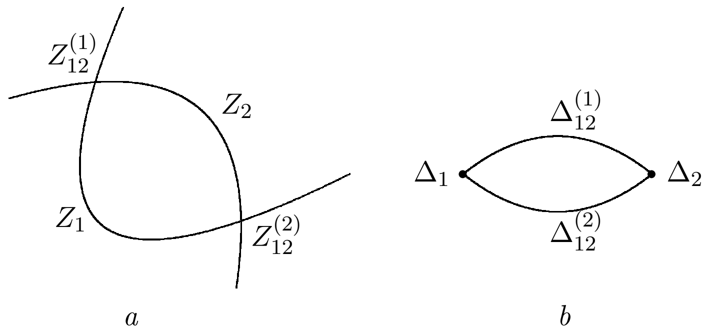


FIGURE 1. $\Delta(Z)$ is not a simplicial complex.

The complex $\Delta(Z)$ is simplicial if and only if all the intersections $Z_{i_1} \cap Z_{i_2} \cap \dots \cap Z_{i_k}$ are irreducible. This can be achieved on some resolution of $X$. In this case $\Delta(Z)$ coincides with the topological nerve of the covering of $Z$ by subsets $Z_i$. Also note that if $\dim X = n$, then the condition $Z$ to have normal crossings implies $\dim \Delta(Z) \leq n-1$.

For surfaces, it is usual to consider weighted resolution graphs. The weights assigned to vertices are the intersection numbers $Z_i \cdot Z_i$. Also the genuses of curves $Z_i$ and the intersection matrix $(Z_i \cdot Z_j)$ are taken into account. We do not know what could be a generalization of this weighting to higher dimension, so we work with the purely combinatorial complex $\Delta(Z)$. However, it is clear that $\Delta(Z)$ somehow reflects the complexity of the given resolution $f$.

The complex $\Delta(Z)$ can be constructed for any divisor with simple normal crossings on some variety $Y$. If $Y$ is a Kähler manifold, cohomologies of $\Delta(Z)$ with coefficients in $\mathbb{Q}$ are interpreted as the weight 0 components of the mixed Hodge structure on cohomologies of $Z$ (see [14], Chapter 4, §2). In [13], D. Mumford introduced *the compact polyhedral complex* associated to a toroidal embedding $U \subset Y$. If $Z = Y \setminus U$ is a divisor with simple normal crossings this is precisely the dual complex $\Delta(Z)$, but endowed with some additional structure. In [13] it is used in the proof of Semi-stable Reduction Theorem. An important feature of the toroidal embedding $U \subset Y$ is that toroidal birational morphisms

$$\begin{tikzcd} & Y' \arrow[dd] \\ U \arrow[ur, hook] \arrow[dr, hook] & \\ & Y \end{tikzcd}$$

are in 1-to-1 correspondence with subdivisions of the conical polyhedral complex associated to $\Delta(Z)$. We make use of this fact in Sections 4 and 5.

As far as we know, the first work in which the dual complex is studied in connection to resolution of singularities (in arbitrary dimension) is [8]. There G. L. Gordon considers the incidence complex of a hypersurface $V_0 \subset W$ which is the singular fiber of a map $\pi\colon W \to D$ onto the unit disc $D \subset \mathbb{C}$. The hypersurface $V_0$ is supposed to have simple normal crossings. In particular, $V_0 \subset W$ can be obtained as an embedded resolution of some singular hypersurface $H_0 \subset V$, $V_0$ being the the strict transform $H_0'$ of $H_0$ plus all exceptional divisors $Z_i$ of the embedded resolution $W \to V$. G. L. Gordon shows that homologies of $\Delta(V_0)$ give some information on the monodromy around $V_0$. His article contains also several interesting examples of the dual complex. But he considers $\Delta(V_0)$ for $V_0 = H_0' + \sum Z_i$ only. According to our definition, the dual complex associated to the resolution $H_0'$ of $H_0$ is $\Delta(\sum Z_i|_{H_0'})$.

In [19] V. V. Shokurov studies (among other things) some complex associated to a resolution of a log-canonical singularity $(X, o)$. It is constructed in exactly the same way as described above but only those prime exceptional divisors are taken into account which have discrepancy $-1$ over $(X, o)$. This complex has a significant property that it is uniquely determined as a topological space, i. e., it depends only on the singularity $(X, o)$ but not on the resolution.

The starting point of our work is the fact that homotopy type of the dual complex $\Delta(Z)$ does not depend on the choice of a resolution $f$. Thus it is an invariant of a singularity. This was observed by the author in [20] for isolated singularities defined over a field of characteristic 0. The proof is an easy consequence of Abramovich-Karu-Matsuki-Włodarczyk Weak Factorization Theorem in the Logarithmic Category ([1]). A. Thuillier in [22] establishes a much more general result:

**Theorem 1.2.** *Let $X$ be an algebraic scheme over a perfect field $k$ and $Y$ be a subscheme of $X$. If $f_1\colon X_1 \to X$ and $f_2\colon X_2 \to X$ are two proper morphisms such that $f_i^{-1}(Y)$ are divisors with simple normal crossings and $f_i$ induce isomorphisms between $X_i \setminus f_i^{-1}(Y)$ and $X \setminus Y$, $i = 1, 2$, then the topological spaces $\Delta(f_1^{-1})$ and $\Delta(f_2^{-1})$ are canonically homotopy equivalent.*

A. Thuillier's proof does not use the Weak Factorization Theorem. Having in mind these results, in the sequel we sometimes simply say that $\Delta(Z)$ is the dual complex of a singularity $(X, S)$ not indicating explicitly which good resolution is considered.

These observations motivate the following task: determine the homotopy type of the dual complex for different classes of singularities. This homotopy type is not always trivial as is shown by Example 1.1. However, it is trivial (i. e., $\Delta(Z)$ is homotopy equivalent to a point) for many important types of singularities. This holds, for example, if $(X, o)$ is an isolated toric singularity (see [20]), if $(X, o)$ is a rational surface singularity (see [3]), or if $(X, o)$ is a 3-dimensional Gorenstein terminal singularity, defined over $\mathbb{C}$ ([21]). As to a general rational singularity, we can only prove that the highest homologies of $\Delta(Z)$ vanish: $H_{n-1}(\Delta(Z)) = 0$ ([21]). Another partial result from [21] is that if $(X, o)$ is an isolated hypersurface singularity over $\mathbb{C}$, $(Y \supset Z) \to (X \ni o)$ its good resolution, then the fundamental group $\pi_1(\Delta(Z))$ is trivial.

In this preprint we show that $\Delta(Z)$ is homotopy trivial for resolutions of all 3-dimensional terminal singularities over $\mathbb{C}$ (see Theorem 4.1). Applying the Varchenko-Hovanskiĭ embedded toric resolution we prove that if

$$(Y \supset Z) \to (X \ni o)$$

is a good resolution of a non-degenerate isolated hypersurface singularity $(X, o)$ satisfying some additional technical property, then the complex $\Delta(Z)$ does not have intermediate homologies, i. e.,

$$H_k(\Delta(Z), \mathbb{Z}) = 0 \text{ for } 0 < k < n-1,\ n = \dim X$$

(see Corollary 5.7). We describe how $\Delta(Z)$ can be found for such singularities. In particular, it follows that if $X$ is a Brieskorn singularity, then $\Delta(Z)$ is homotopy trivial (Corollary 5.6).

It turns out that condition $Z$ to be the exceptional divisor of some resolution of singularities poses strong restrictions on the homotopy type of $\Delta(Z)$. At the same time if we ask which complex $K$ can be realized as the dual complex of some divisor $Z \subset X$ (not necessarily contractible), then the answer is: any finite simplicial complex. Indeed, suppose we are given a finite simplicial complex $K$. Let $N$ be the number of its vertices and $d$ be its dimension. First consider the divisor $H = \sum_{i=1}^{N} H_i \subset \mathbb{P}^{d+1}$ consisting of $N$ hyperplanes $H_i$ in general position in $\mathbb{P}^{d+1}$. Its dual complex is maximal in the sense that any $k$ vertices of $\Delta(H)$, $1 \le k \le d+1$, form a $(k-1)$-simplex of $\Delta(H)$. Thus $K$ can be identified with some subcomplex of $\Delta(H)$. Now suppose that we blow up the space $\mathbb{P}^{d+1}$ with the center $H_{i_0} \cap H_{i_1} \cap \cdots \cap H_{i_k}$. Let $H'$ be the strict transform of $H$ under this blow-up. Then $\Delta(H')$ is obtained from $\Delta(H)$ by deleting the $k$-simplex $\Delta_{i_0 i_1 \dots i_k}$ and all the simplexes to which $\Delta_{i_0 i_1 \dots i_k}$ belongs as a face. Now it is clear that after a finite sequence of appropriate blow-ups we get a strict transform $Z$ of $H$ such that $\Delta(Z)$ is homeomorphic to $K$. We learned this construction from [19].

The preprint is organized as follows. In Section 2 we show that the homotopy type of $\Delta(Z)$ is independent of a resolution. The proof is reproduced from [20]; this is done for reader's convenience. We deal mostly with isolated singularities over $\mathbb{C}$, so the proof from [20] based on the Weak Factorization Theorem is sufficient for us. In Section 3 we recall some results and sketch proofs from [21] on the dual complex for rational and hypersurface singularities. In Section 4 we consider resolutions of 3-dimensional terminal singularities. Section 5 is devoted to non-degenerate hypersurface singularities.

At the beginning of it we recall the construction of the Varchenko-Hovanskiĭ embedded toric resolution.

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## 2. Invariance of the homotopy type of $\Delta(Z)$

Let $(X, o)$ be a germ of an isolated singularity $o$ of an algebraic variety or an analytic space $X$. In the algebraic case the ground field is supposed to be of characteristic 0. In this section we prove the following

**Theorem 2.1.** *([20]) Let $f\colon Y \to X$ and $f'\colon Y' \to X$ be two good resolutions of $X$ with exceptional divisors $Z$ and $Z'$ respectively. Then the dual complexes $\Delta(Z)$ and $\Delta(Z')$ are homotopy equivalent.*

The proof is based on the following theorem due to Abramovich-Karu-Matsuki-Włodarczyk (see [1] and [15], Theorem 5-4-1).

**Theorem 2.2** (Weak Factorization Theorem in the Logarithmic Category)**.** *Let $(U_{X_1}, X_1)$ and $(U_{X_2}, X_2)$ be complete nonsingular toroidal embeddings over an algebraically closed field of characteristic zero. Let*

$$\varphi\colon (U_{X_1}, X_1) \dashrightarrow (U_{X_2}, X_2)$$

*be a birational map which is an isomorphism over $U_{X_1} = U_{X_2}$. Then the map $\varphi$ can be factored into a sequence of blow-ups and blow-downs with smooth admissible and irreducible centers disjoint from $U_{X_1} = U_{X_2}$. That is to say, there exists a sequence of birational maps between complete nonsingular toroidal embeddings*

$$\begin{gathered}(U_{X_1}, X_1) = (U_{V_1}, V_1) \overset{\psi_1}{\dashrightarrow} (U_{V_2}, V_2) \overset{\psi_2}{\dashrightarrow} \\ \dots \overset{\psi_{i-1}}{\dashrightarrow} (U_{V_i}, V_i) \overset{\psi_i}{\dashrightarrow} (U_{V_{i+1}}, V_{i+1}) \overset{\psi_{i+1}}{\dashrightarrow} \dots \\ \overset{\psi_{l-2}}{\dashrightarrow} (U_{V_{l-1}}, V_{l-1}) \overset{\psi_{l-1}}{\dashrightarrow} (U_{V_l}, V_l) = (U_{X_2}, X_2)\,,\end{gathered}$$

*where*

*(i) $\varphi = \psi_{l-1} \circ \psi_{l-2} \circ \cdots \circ \psi_1$,*

*(ii) $\psi_i$ are isomorphisms over $U_{V_i}$, and*

*(iii) either $\psi_i$ or $\psi_i^{-1}$ is a morphism obtained by blowing up a smooth irreducible center $C_i$ (or $C_{i+1}$) disjoint from $U_{V_i} = U_{V_{i+1}}$ and transversal to the boundary $D_{V_i} = V_i \setminus U_{V_i}$ (or $D_{V_{i+1}} = V_{i+1} \setminus U_{V_{i+1}}$), i. e., at each point $p \in V_i$ (or $p \in V_{i+1}$) there exists a regular coordinate system $\{x_1, \dots, x_n\}$ in a neighborhood $p \in U_p$ such that*

$$D_{V_i} \cap U_p \ (or\ D_{V_{i+1}} \cap U_p) = \{\prod_{j \in J} x_j = 0\}$$

*and*

$$C_i \cap U_p \ (or\ C_{i+1} \cap U_p) = \{\prod_{j \in J} x_j = 0\,,\ x_{j'} = 0\ \forall j' \in J'\}\,,$$

*where $J, J' \subseteq \{1, \dots, n\}$.*

Here *toroidal embedding* $U \subset X$ means that $U$ is an open dense set in $X$ and $X \setminus U$ is a divisor with simple normal crossings. Theorem 2.2 has also an analytic version. To get it one needs only to replace "birational" with "bimeromorphic" etc.

In order to prove Theorem 2.1, let us take the resolutions $(Y \setminus Z, Y)$ and $(Y' \setminus Z', Y')$ as toroidal embeddings and compactify $Y$ and $Y'$ to smooth varieties (here we use the fact that the given singularity $(X, o)$ is isolated). Now Theorem 2.1 follows from Theorem 2.2 and

**Lemma 2.3.** *Let* $\sigma\colon (X'\setminus Z', X') \to (X\setminus Z, X)$ *be a blow-up of an admissible center* $C \subset Z$ *in a nonsingular toroidal embedding* $(X\setminus Z, X)$*,* $X'\setminus Z' \simeq X\setminus Z$*. Then the topological spaces* $\Delta(Z')$ *and* $\Delta(Z)$ *have the same homotopy type.*

*Proof.* Let $Z = \sum_{i=1}^{N} Z_i$ be the decomposition of $Z$ into its prime components, and let $C \subset Z_i$ for $1 \le i \le l$ and $C \not\subseteq Z_i$ for $l < i \le N$. Assume that $C$ has nonempty intersections also with $Z_{l+1}, \dots, Z_r$, $l < r \le N$. There are two possibilities.
1) $\dim C = n - l$ ($n = \dim X$), i. e., $C$ coincides with one of the irreducible components of the intersection $Z_1 \cap \dots \cap Z_l$: $C = Z^{(1)}_{1\dots l}$. Then after the blow-up the intersection of the proper transforms $Z'_1, \dots, Z'_l$ of the divisors $Z_1, \dots, Z_l$ has $J - 1$ irreducible components (if $J$ is the number of components of $Z_1 \cap \dots \cap Z_l$), but all these proper transforms intersect the exceptional divisor $F$ of the blow-up $\sigma$. Furthermore, $F$ intersects proper transforms $Z'_{l+1}, \dots, Z'_r$ of the divisors $Z_{l+1}, \dots, Z_r$. Now it is clear that the complex $\Delta(Z')$ is obtained from $\Delta(Z)$ by the barycentric subdivision of the simplex $\Delta^{(1)}_{1\dots l}$ with the center at the point corresponding to the divisor $F$. Thus the complexes $\Delta(Z')$ and $\Delta(Z)$ are even homeomorphic.
2) $\dim C < n - l$, let $C \subset Z^{1}_{1\dots l}$. In this case divisors $Z_{i_1}, \dots, Z_{i_s}$ have nonempty intersection if and only if their proper transforms $Z'_{i_1}, \dots, Z'_{i_s}$ have nonempty intersection. Therefore the complex $\Delta(Z')$ is obtained from the complex $\Delta(Z)$ in the following way. Add to $\Delta(Z)$ a new vertex corresponding to the exceptional divisor $F$ of the blow-up $\sigma$ and construct cones with vertex at $F$ over all the maximal cells $\Delta^{(j)}_{i_1\dots i_s}$ of the complex $\Delta(Z)$ possessing the property

$$Z^{(j)}_{i_1\dots i_s} \cap C \neq \varnothing\,.$$

Note that the simplex $\Delta_{F,1\dots l}$ corresponding to the intersection $F \cap Z^{(1)}_{1\dots l}$ is regarded as a common simplex for all constructed cones. Now we can define the homotopy equivalence between $\Delta(Z')$ and $\Delta(Z)$ as a contraction of the constructed cones: it sends the vertex $F$ of the complex $\Delta(Z')$ to any of the vertices $Z_1, \dots, Z_l$ of the cell $\Delta^{(1)}_{1\dots l}$ of the complex $\Delta(Z)$ and it is identity on other vertices of $\Delta(Z')$ ($\Delta(Z)$). Then the induced simplicial map is our homotopy equivalence. $\square$

Figure 2 illustrates part 2) of the proof. Here we suppose that $\dim X = 3$ and $Z$ consists of 4 prime components. We denote the corresponding vertices of $\Delta(Z)$ by the same letters $Z_i$. Let $\Delta(Z)$ be as shown in Figure 2, *a*. Then let us blow up a smooth irreducible curve $C \subset Z_1$ which intersect transversally the curves $Z_1 \cap Z_i$, $i = 2, 3, 4$. For the simplicity of drawing we assume that $C$ intersects every $Z_1 \cap Z_i$ at a single point. Then the obtained $\Delta(Z')$ is shown in Figure 2, *b*. Here all the triangles belong to $\Delta(Z')$ together with their interiors.

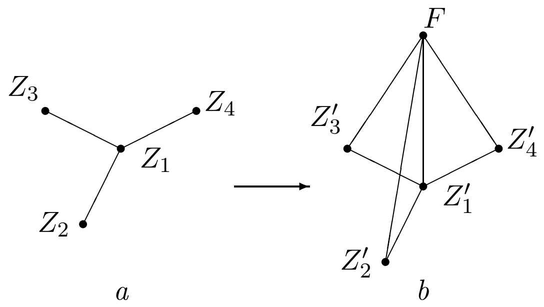


FIGURE 2. Transformation of $\Delta(Z)$ into $\Delta(Z')$

## 3. Dual complex for rational and hypersurface singularities

In this section and in the rest of the paper we consider only varieties (and analytic spaces) over $\mathbb{C}$. Also when we speak about the dual complex $\Delta(Z)$ associated to a resolution of a given singularity $(X, o)$, we shall always assume that $\Delta(Z)$ is a simplicial complex. This can be achieved on some resolution of $X$ and we already established the invariance of the homotopy type of $\Delta(Z)$ in section 2.

3.1. **Rational singularities.** First recall that an algebraic variety (or an analytic space) $X$ has *rational singularities* if $X$ is normal and for any resolution $f\colon Y \to X$ all the sheaves $R^i f_* \mathcal{O}_Y$ vanish, $i > 0$.

It is well known that the exceptional divisor in a resolution of a rational surface singularity is a tree of rational curves. This follows, e. g., from M. Artin's considerations in [3]. Thus in the surface case the dual graph for a rational singularity is homotopy trivial. In [21] we partially generalized this statement to higher dimensions and proved the following result.

**Theorem 3.1.** *Let $o \in X$ be an isolated rational singularity of a variety (or an analytic space) $X$ of dimension $n \geq 2$, and let $f\colon Y \to X$ be a good resolution with the exceptional divisor $Z$. Then the highest homologies of the complex $\Delta(Z)$ vanish:*

$$H_{n-1}(\Delta(Z), \mathbb{Z}) = 0\,.$$

In the first step of the *proof* we follow M. Artin's argument from [3]. Let $Z = \sum_{i=1}^{N} Z_i$ be the decomposition of the divisor $Z$ into its prime components $Z_i$. We can assume that $X$ is projective (since the given singularity is isolated) and $f$ is obtained by a sequence of smooth blow-ups (Hironaka's resolution [12]). Thus all $Z_i$ and $Y$ are Kähler manifolds.

The sheaves $R^i f_* \mathcal{O}_Y$ are concentrated at the point $o$. Via Grothendieck's theorem on formal functions (see [11], (4.2.1), and [10], Ch. 4, Theorem 4.5

for the analytic case) the completion of the stalk of the sheaf $R^i f_* \mathcal{O}_Y$ at the point $o$ is

$$\lim_{\substack{\longleftarrow \\ (r)\to(\infty)}} H^i(Z, \mathcal{O}_{Z_{(r)}}), \tag{1}$$

where $(r) = (r_1, \ldots, r_N)$ and $Z_{(r)} = \sum_{i=1}^{N} r_i Z_i$. If $(r) \geq (s)$, i. e., $r_i \geq s_i$ $\forall i$, there is a natural surjective map $g$ of sheaves on $Z$:

$$g \colon \mathcal{O}_{Z_{(r)}} \to \mathcal{O}_{Z_{(s)}}.$$

Since dimension of $Z$ is $n-1$, the map $g$ induces a surjective map of cohomologies

$$H^{n-1}(Z, \mathcal{O}_{Z_{(r)}}) \to H^{n-1}(Z, \mathcal{O}_{Z_{(s)}}).$$

Recall that the given singularity $o \in X$ is rational, and thus the projective limit (1) is 0. Therefore the cohomology group $H^{n-1}(Z, \mathcal{O}_Z)$ vanishes too (because the projective system in (1) is surjective). For surfaces, $n = 2$, thus $H^1(Z, \mathcal{O}_Z) = 0$. It easy follows, e. g., from exponential exact sequence, that $H^1(Z, \mathbb{Z}) = 0$ and this implies

$$H^1(\Delta(Z), \mathbb{Z}) = H_1(\Delta(Z), \mathbb{Z}) = 0.$$

For an arbitrary $n \geq 2$ we need a more sophisticated argument.

**Lemma 3.2.** *Let $Z = \sum Z_i$ be a reduced divisor with simple normal crossings on a compact Kähler manifold $Y$, $\dim Y = n \geq 2$, and assume that $H^k(Z, \mathcal{O}_Z) = 0$ for some $k$, $1 \leq k \leq n-1$. Then the $k$-th cohomologies with coefficients in $\mathbb{C}$ of the complex $\Delta(Z)$ vanish too:*

$$H^k(\Delta(Z), \mathbb{C}) = 0.$$

For the proof see [21]. The idea is to introduce a kind of Mayer-Vietoris spectral sequence for $Z = \cup Z_i$ and to show that it degenerates in $E_2$. Compare also [9] and [14] where very closed results are stated.

Lemma 3.2 implies $H^{n-1}(\Delta(Z), \mathbb{C}) = 0$; hence $H_{n-1}(\Delta(Z), \mathbb{Z}) = 0$. This completes the proof of Theorem 3.1.

So, the highest homologies of $\Delta(Z)$ vanish for rational singularities. In general we do not know anything about intermediate homologies. If we could prove that they vanish together with the fundamental group $\pi_1(\Delta(Z))$, then it would follow that $\Delta(Z)$ is homotopy equivalent to a point. We can prove this only in the partial cases of 3-dimensional hypersurface singularities (see the next subsection) and for some hypersurfaces which are non-degenerate in Varchenko-Hovanskiĭ sense (see section 5). Moreover, we do not know any example of an isolated $n$-dimensional (not necessarily rational) singularity such that $H_k(\Delta(Z)) \neq 0$ for $0 < k < n-1$.

3.2. **Hypersurface singularities.** If $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$ is a germ of an isolated hypersurface singularity, $n \geq 3$, then its link (the intersection of $X$ with a sufficiently small sphere around $0 \in \mathbb{C}^{n+1}$) is simply connected (J. Milnor [16]). This implies the following result ([21]):

**Theorem 3.3.** *Let $o \in X$ be an isolated hypersurface singularity of an algebraic variety (or an analytic space) $X$ of dimension at least 3 defined over the field $\mathbb{C}$ of complex numbers. If $f \colon Y \to X$ is a good resolution of*

*$o \in X$, $Z$ its exceptional divisor, then the fundamental group of $\Delta(Z)$ is trivial:*

$$\pi(\Delta(Z)) = 0\,.$$

For the *proof*, it suffices to notice that there is a continuous map with connected fibers from the link $M$ of $(X, o)$ onto the exceptional divisor $Z$ (see [2]). This gives $\pi_1(Z) = 0$. But we also can construct a map $\psi$ from $Z$ to $\Delta(Z)$ with the same properties. It is defined as follows.

Let us take a triangulation $\Sigma'$ of $Z$ such that all the intersections $Z_{i_0 \dots i_p}$ are subcomplexes. Next we make the barycentric subdivision $\Sigma$ of $\Sigma'$ and the barycentric subdivision of the complex $\Delta(Z)$. Now let $v$ be a vertex of $\Sigma$ belonging to the subcomplex $Z_{i_0 \dots i_p}$ but not to any smaller subcomplex $Z_{i_0 \dots i_p i_{p+1}}$:

$$v \in Z_{i_0 \dots i_p}\,, \quad v \notin Z_{i_0 \dots i_p i_{p+1}}\ \forall\, i_{p+1}\,.$$

Then let

$$\psi(v) = \text{ the center of the simplex } \Delta_{i_0 \dots i_p}\,.$$

This determines the map $\psi$ completely as a simplicial map (depending on the triangulation $\Sigma'$).

It easily follows that $\psi$ is continuous, surjective, and has connected fibers. Thus $\pi_1(\Delta(Z)) = 0$.

Combining Theorems 3.1 and 3.3 we obtain

**Corollary 3.4.** *Let $o \in X$ be an isolated rational hypersurface singularity of dimension 3. If $f\colon Y \to X$ is a good resolution with the exceptional divisor $Z$, then the dual complex $\Delta(Z)$ associated to the resolution $f$ has the homotopy type of a point.*

*Proof.* We know from Theorems 3.1 and 3.3 that $\Delta(Z)$ is simply connected and $H_2(\Delta(Z), \mathbb{Z}) = 0$. Since $\dim X = 3$, we have $\dim(\Delta(Z)) \leq 2$. Now Corollary 3.4 follows from the Inverse Hurevicz and Whitehead Theorems. $\square$

## 4. Dual complex for 3-dimensional terminal singularities

Terminal singularities arose in the framework of Mori theory as singularities which can appear on minimal models of algebraic varieties of dimension $\geq 3$. In dimension 3 terminal singularities are completely classified up to an analytic equivalence by M. Reid, D. Morrison, G. Stevens, S. Mori and N. Shepherd-Barron. Classification looks as follows. Gorenstein (or *index* 1) terminal singularities are exactly isolated compound Du Val (cDV) points. A *cDV-point* is a germ $(X, o)$ of singularity analytically isomorphic to the germ

$$(\{f(x, y, z) + tg(x, y, z, t) = 0\}, 0) \subset (\mathbb{C}^4, 0)\,,$$

where $f$ is one of the following Klein polynomials

$$x^2 + y^2 + z^{n+1},\ n \geq 1\,,\ x^2 + y^2 z + z^{n-1},\ n \geq 4\,,$$
$$x^2 + y^3 + z^4,\ x^2 + y^3 + yz^3,\ x^2 + y^3 + z^5\,.$$

In other words, a cDV-point is a germ of singularity such that its general hyperplane section is a Du Val point. Non-Gorenstein (index $\geq 2$) terminal singularities are quotients of isolated cDV-points by some cyclic group actions, see [18] or [17] for a precise statement.

**Theorem 4.1.** *Let $f\colon (Y,Z) \to (X,o)$ be a good resolution of a three-dimensional terminal singularity $(X,o)$. Then the dual complex $\Delta(Z)$ of $f$ is homotopy trivial.*

*Proof.* First suppose that the singularity $(X,o)$ is Gorenstein. Then, according to the classification, it is an isolated hypersurface singularity. On the other hand, all terminal (and, moreover, canonical) singularities are rational (R. Elkik [5]). Thus Corollary 3.4 applies and we get the homotopy triviality of $\Delta(Z)$.

Now assume that $(X,o)$ is a non-Gorenstein terminal singularity of index $m$. Let $(V,o') \to (X,o)$ be its Gorenstein cover, $X = V/\mathbb{Z}_m$, and consider the diagram

$$\begin{tikzcd} Z' \subset W \arrow[r] \arrow[d, "g'"'] & \widetilde{X} \supset \widetilde{Z} \arrow[d, "g"] \\ o' \in V \arrow[r] & X \ni o \end{tikzcd}$$

where $g'\colon W \to V$ is an equivariant Hironaka resolution of $V$ (see, e. g., [7]), $\widetilde{X} = W/\mathbb{Z}_m$, horizontal arrows stand for natural projections, and $g\colon \widetilde{X} \to X$ is the induced birational morphism. By $Z'$ and $\widetilde{Z}$ we denote the exceptional divisors of $g'$ and $g$ respectively.

Since $\widetilde{X}$ has only cyclic quotient singularities, $\widetilde{X} \setminus \widetilde{Z} \subset \widetilde{X}$ is a toroidal embedding in the sense of [13], Chapter II, §1, Definition 1. It is clear that the definition of the dual complex applies also to the partial resolution $g$. It is also possible to interpret $\Delta(\widetilde{Z})$ as the underlying CW-complex of Mumford's compact polyhedral complex of the toroidal embedding $\widetilde{X} \setminus \widetilde{Z} \subset \widetilde{X}$ (see [13], pp. 69–71). On the other hand, the action of $\mathbb{Z}_m$ on $W$ naturally induces an action of $\mathbb{Z}_m$ on $\Delta(Z')$, so that $\Delta(\widetilde{Z}) = \Delta(Z')/\mathbb{Z}_m$.

We already know that $\Delta(Z')$ is homotopy trivial. It is a topological fact that the quotient of a finite homotopy trivial CW-complex by a finite group is again homotopy trivial ([23], p. 222, Theorem 6.15). Thus we get homotopy triviality of $\Delta(\widetilde{Z})$.

Let $\Delta'(\widetilde{Z})$ be the conical polyhedral complex associated to $\Delta(\widetilde{Z})$. Any subdivision of $\Delta'(\widetilde{Z})$ gives rise to a new toroidal embedding $Y \setminus Z \subset Y$ and a birational toroidal morphism $(Y \setminus Z, Y) \to (\widetilde{X} \setminus \widetilde{Z}, \widetilde{X})$ ([13], Chapter II, §2, Theorem 6*). The corresponding dual complex $\Delta(Z)$ is a subdivision of $\Delta(\widetilde{Z})$. In particular, we can take a subdivision of $\Delta'(\widetilde{Z})$ which gives a good resolution $(Y,Z)$ of $\widetilde{X}$. Since $\Delta(Z)$ is just a subdivision of $\Delta(\widetilde{Z})$, it is also homotopy trivial. □

## 5. Non-degenerate hypersurface singularities

5.1. **Varchenko-Hovanskiĭ embedded toric resolution.** We shall connect the dual complex of a non-degenerate hypersurface singularity to its Newton diagram. For this we need to recall the definition of non-degeneracy and the construction of Varchenko-Hovanskiĭ embedded toric resolution. We mainly follow [24].

Let $f \in \mathbb{C}\{x_0, \ldots, x_n\}$ be a convergent power series,

$$f(x) = \sum_{m \in \mathbb{Z}^{n+1}_{\geq 0}} a_m x^m ,$$

where $x^m = x_0^{m_0} \ldots x_n^{m_n}$. *The Newton polyhedron* $\Gamma_+(f)$ of $f$ is the convex hull of the set $\bigcup\limits_{a_m \neq 0} (m + \mathbb{R}^{n+1}_{\geq 0})$ in $\mathbb{R}^{n+1}$. The union of all compact faces of $\Gamma_+(f)$ is *the Newton diagram* $\Gamma(f)$ of $f$. If $\gamma$ is a face of $\Gamma(f)$, then we denote by $f_\gamma$ the polynomial consisting of all terms $a_m x^m$ of $f$ such that $m \in \gamma$, i. e.,

$$f_\gamma(x) = \sum_{m \in \gamma} a_m x^m .$$

A series $f \in \mathbb{C}\{x_0, \ldots, x_n\}$ is called *non-degenerate* if for any face $\gamma$ of $\Gamma(f)$ the polynomial $f_\gamma(x_0, \ldots, x_n)$ defines a non-singular hypersurface in $(\mathbb{C}^*)^{n+1}$.

For notation and general background in toric geometry see, e. g., [4] or [6]. Suppose that

$$(X, 0) = (\{f(x_0, \ldots, x_n) = 0\}, 0) \subset (\mathbb{C}^{n+1}, 0)$$

is an isolated hypersurface singularity given by a non-degenerate series $f$. Let us consider the ambient space $\mathbb{C}^{n+1}$ as a toric variety $V(\sigma_+)$, where $\sigma_+$ is the non-negative cone $\mathbb{R}^{n+1}_{\geq 0}$ in $\mathbb{R}^{n+1}$. A. N. Varchenko in [24] constructs a subdivision $\Sigma$ of the cone $\sigma_+$ such that the corresponding toric morphism $\pi\colon V(\Sigma) \to V(\sigma_+)$ is an embedded resolution of $(X, 0)$. Here this means that (i) $V(\Sigma)$ and the strict transform $Y$ of $X$ are smooth, and (ii) the union of $Y$ and the exceptional set $Z$ of $\pi$ is a divisor with simple normal crossings.

Denote by $W$ the space $\mathbb{R}^{n+1}$ where the cone $\sigma_+$ lies and by $W^*$ its dual. We shall consider the Newton diagram $\Gamma(f)$ as a subset of $W^*$. For any $w \in \sigma_+$ we can associate a number $\mu(w) = \min\limits_{m \in \Gamma_+(f)} \langle w, m \rangle$, where $\langle \cdot, \cdot \rangle$ stands for the pairing between $W$ and $W^*$, and a face

$$\gamma(w) = \{m \in \Gamma_+(f) | \langle w, m \rangle = \mu(w)\}$$

of the Newton polyhedron $\Gamma_+(f)$. Two vectors $w^1$ and $w^2 \in \sigma_+$ are called *equivalent with respect to* $\Gamma_+(f)$ if they cut the same face on $\Gamma_+(f)$. We shall write this equivalence relation as $w^1 \sim_f w^2$, so that

$$w^1 \sim_f w^2 \iff \gamma(w^1) = \gamma(w^2) .$$

It is not difficult to verify that closures of equivalence classes of $\sim_f$ are rational polyhedral cones and these cones posses all the properties necessary to form a fan. Denote this fan by $\Sigma'$. It will be called *the first Varchenko subdivision of* $\sigma_+$. To get $\Sigma$, subdivide $\Sigma'$ so that all cones of $\Sigma$ give non-singular affine toric varieties. This is equivalent to saying that every cone $\sigma \in \Sigma$ is simplicial and its skeleton (the set of primitive vectors of $\mathbb{Z}^{n+1} \subset W$ along the edges of $\sigma$) is a part of a basis for $\mathbb{Z}^{n+1}$. The toric variety $V(\Sigma)$ is smooth by the construction. The rest of needed properties of the birational morphism $\pi\colon V(\Sigma) \to \mathbb{C}^{n+1}$ follow from

**Lemma 5.1.** *(i) The strict transform $Y$ of $X$ by the morphism $\pi$ is smooth; (ii) all the toric strata $Z_\sigma$ of $V(\Sigma)$ corresponding to the cones $\sigma$ of $\Sigma$ are transversal to $Y$ in some neighborhood of $\pi^{-1}(0)$.*

*Proof.* (i) Let us take some of the affine pieces of $V(\Sigma)$, say $V(\sigma) \simeq \mathbb{C}^{n+1}$, corresponding to the $n+1$-dimensional cone $\sigma$ of $\Sigma$. Let $w^0, w^1, \dots, w^n$, be the skeleton of $\sigma$,

$$w^0 = (w_0^0, w_1^0, \dots, w_n^0), \dots, w^n = (w_0^n, w_1^n, \dots, w_n^n) .$$

Then the map $\pi$ restricted to $V(\sigma)$ is given by the formulae

$$\begin{aligned} x_0 &= y_0^{w_0^0} y_1^{w_0^1} \dots y_n^{w_0^n} , \\ x_1 &= y_0^{w_1^0} y_1^{w_1^1} \dots y_n^{w_1^n} , \\ &\dots\dots \\ x_n &= y_0^{w_n^0} y_1^{w_n^1} \dots y_n^{w_n^n} , \end{aligned} \tag{2}$$

where $y_i$ are the coordinates on $V(\sigma)$. The full transform of $X$ is given in $V(\sigma)$ by the equation

$$y_0^{\mu(w^0)} y_1^{\mu(w^1)} \dots y_n^{\mu(w^n)} f'(y_0, y_1, \dots, y_n) = 0 , \quad f'(0, 0, \dots, 0) \neq 0 ,$$

and the strict transform $Y$ is $\{f'(y_0, \dots, y_n) = 0\}$.

Suppose that $Y$ is singular in some point $Q = (y_0^0, \dots, y_n^0)$. We assume that $y_0^0, \dots, y_k^0 \neq 0$, $y_{k+1}^0 = \dots = y_n^0 = 0$, $0 \leq k \leq n$. We can write

$$f'(y_0, \dots, y_k) = g(y_0, \dots, y_k) + y_{k+1}(\dots) + \dots + y_n(\dots) .$$

Here $g$ comes from those monomials $x^m$ of $f$ whose degrees $\langle w^j, m\rangle$ with respect to weight-vectors $w^{k+1}, \dots, w^n$ are exactly $\mu(w^{k+1}), \dots, \mu(w^n)$. In other words, $g = f'_\gamma$ (strict transform of $f_\gamma$) for the face

$$\gamma = \gamma(w^{k+1}) \cap \gamma(w^{k+2}) \cap \dots \cap \gamma(w^n) .$$

At the same time the point $Q$ must lie on the exceptional divisor of $\pi$. Thus some of $\mu(w^{k+1}), \dots, \mu(w^n)$ are strictly positive. It follows that the face $\gamma$ is compact and hence $f'_\gamma$ is a polynomial.

The hypersurface $\{f'_\gamma(y_0, \dots, y_k) = 0\}$ is singular at the point $Q$ and, moreover, in every point $(y_0^0, \dots, y_k^0, y_{k+1}, \dots, y_n)$ for arbitrary numbers $y_{k+1}, \dots, y_n \in \mathbb{C}$. Take, for instance, a point

$$Q' = (y_0^0, \dots, y_k^0, y'_{k+1}, \dots, y'_n)$$

for some $y'_{k+1}, \dots, y'_n \neq 0$. But the morphism $\pi$ is a local isomorphism at $Q'$, thus the hypersurface $\{f_\gamma = 0\}$ is singular at the point $P = \pi(Q) \in (\mathbb{C}^*)^{n+1}$, but this contradicts the non-degeneracy.

(ii) In the notation of the case (i), let us show that $Y$ is transversal to the toric stratum $L = \{y_{k+1} = \dots = y_n = 0\}$. Take some point $Q \in L \cap Y$, $\pi(Q) = 0$. It is sufficient to prove that one of the partial derivatives

$$\frac{\partial f'}{\partial y_i}(Q) \neq 0 , \quad i = 1, \dots, k .$$

Let $Q = (y_0^0, \dots, y_l^0, 0 \dots, 0)$, $0 \le l \le k$. Now the same argument as in the case (i) shows that non-transversality of $L$ and $Y$ at $Q$ would contradict non-degeneracy. □

*Remark* 5.2. The assertions of Lemma 5.1 essentially hold and can be proved in the same manner also in the case when $\sigma$ is an $(n+1)$-dimensional simplicial cone contained in one of the cones of the fan $\Sigma'$, and we take $\pi$ to be the birational morphism $V(\sigma) \to \mathbb{C}^{n+1}$. Now we do not assume the skeleton of $\sigma$ to be a part of a basis, so that $V(\sigma)$ is a quotient of $\mathbb{C}^{n+1}$ by some abelian group $G$. We have to replace "smooth" of (i) by "quasismooth" indicating that the cover $\widetilde{Y}$ of $Y$ is smooth in $\mathbb{C}^{n+1}$. The transversality of (ii) also must be understood as transversality of $\widetilde{Y}$ and toric strata in the covering $\mathbb{C}^{n+1}$.

5.2. **Dual complex for non-degenerate singularities.** Naturally, the dual complex $\Delta(Z)$ associated to a resolution of a non-degenerate isolated hypersurface singularity

$$(X, o) = (\{f = 0\}, 0) \subset (\mathbb{C}^{n+1}, 0)$$

is connected to its Newton diagram $\Gamma(f)$. However, one should be careful applying the embedded toric resolution to the calculation of $\Delta(Z)$ because Varchenko-Hovanskiĭ resolution is not a resolution in rigorous sense. When one says that $\pi\colon (Y, Z) \to (X, o)$ is a resolution of singularity $(X, o)$, it is usually meant that $Y \setminus Z \simeq X \setminus \{o\}$. But Varchenko-Hovanskiĭ resolution can involve blow-ups with centers different from the singular point $o$. Indeed, look at the example

$$(V, 0) = (\{x^4 + y^4 + xz + yz = 0\}, 0) \subset (\mathbb{C}^3, 0)\,.$$

This is a non-degenerate isolated surface singularity. Its Newton polyhedron is shown in Fig. 3.

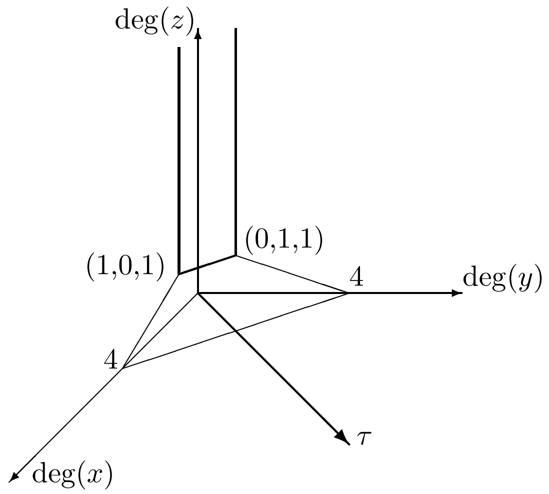


FIGURE 3. Newton diagram for $x^4 + y^4 + xz + yz$.

The corresponding first Varchenko subdivision $\Sigma'$ contains the ray $\tau = \langle(1,1,0)\rangle$ "orthogonal" to the face bordered by bold lines in Fig. 3. The cone $\tau$ gives rise to some exceptional divisor $Z_\tau$ in Varchenko-Hovanskiĭ resolution of $(V,0)$; the center of $Z_\tau$ in $\mathbb{C}^3$ is the $z$-axis which is also contained in $V$.

Let us recall from [20] how $\Delta(Z)$ looks when $Z$ is the exceptional divisor of some toric birational morphism $\pi\colon V(\Sigma) \to V(\sigma_+) = \mathbb{C}^{n+1}$ corresponding to a subdivision $\Sigma$ of $\sigma_+$ such that all cones of $\Sigma$ are simplicial. Prime exceptional divisors of $\pi$ are in 1-to-1 correspondence with 1-dimensional cones (rays) of $\Sigma$. Two such divisors intersect iff corresponding rays are faces of some 2-dimensional cone of $\Sigma$. Add to $Z$ the divisor $T = \sum_{i=0}^{n} T_i$, where $T_i$ correspond to the ray $\langle e_i = (0,\dots,\underset{i}{1},\dots,0)\rangle$. If we take a hyperplane $H$ in $\mathbb{R}^{n+1}$ such that it intersects all the rays $\langle e_i\rangle$ in a point different from 0, then we get a compact polyhedron $K = H \cap \sigma_+$. The fan $\Sigma$ determines some triangulation of $K$. It is clear that $K$ with this triangulation is exactly $\Delta(Z+T)$. To obtain $\Delta(Z)$, we have only to throw away the vertices $T_i$ and all incidental to them cones.

One may object that earlier the complex $\Delta(Z)$ was introduced only for divisors with simple normal crossings, but here even the ambient space $V(\Sigma)$ can be singular. But since all the cones of $\Sigma$ are simplicial, we can understand simple normal crossings in the "orbifold sense" as in Remark 5.2, or consider $\Delta(Z)$ as the underlying complex of the polyhedral complex associated to the toroidal (here simply toric) embedding $V(\Sigma) \setminus Z \subset V(\Sigma)$.

Now let us come back to the singularity $(X,o)$. Assume that

> **(R):** the first Varchenko subdivision $\Sigma'$ for $(X,o)$ does not contain any rays on the border of cone $\sigma_+$ with the exception of $\langle e_0\rangle, \dots, \langle e_n\rangle$.

We shall refer to this as to the property (R). This guarantees that the second subdivision $\Sigma$ can also be chosen with the property (R), so that the obtained Varchenko-Hovanskiĭ resolution is a resolution in rigorous sense.

Denote by $\sigma_1,\dots,\sigma_N$ all the $(n+1)$-dimensional cones of $\Sigma'$ and let us consider one more subdivision $\Sigma''$ of $\Sigma'$ satisfying property (R) and such that all its cones are simplicial. We do not demand $V(\Sigma'')$ to be smooth, but Lemma 5.1 applies to $\Sigma''$ (see Remark 5.2).

**Proposition 5.3.** *Let $\tau$ be a cone of $\Sigma''$ such that the corresponding toric stratum $Z_\tau$ of $V(\Sigma'')$ is exceptional. Denote by $Y''$ the strict transform of $X$ in $V(\Sigma'')$. Then $Y'' \cap Z_\tau \neq \varnothing$ iff $\tau$ does not contain any point from interior of some $\sigma_i$, $i = 1,\dots,N$. If $\dim Z_\tau \geq 2$, then the intersection $Y'' \cap Z_\tau$ is irreducible.*

*Proof.* The fact that if $\tau$ contains a point in the interior of some $\sigma_i$, then $Y'' \cap Z_\tau = \varnothing$ follows from the construction of $\Sigma'$. Thus suppose that $\tau$ is contained in an $n$-dimensional cone $\sigma'$ of $\Sigma'$. Also let $\tau \subset \sigma$, where $\sigma$ is one of the $(n+1)$-dimensional cones of $\Sigma''$. In the affine piece $V(\sigma)$ the morphism $\pi''\colon V(\Sigma'') \to V(\sigma_+) \simeq \mathbb{C}^{n+1}$ is given by formulae (2), where $y_i$ are now the coordinates on $\mathbb{C}^{n+1}$ which covers $V(\sigma)$ (see Remark 5.2). Now $Z_\tau$ is defined, say, by $y_0 = \dots = y_k = 0$, and the intersection $Z_\tau \cap Y''$ by the system

$$\begin{cases} y_i = 0,\ 0 \le i \le k\,, \\ g(y_{k+1}, \dots, y_n) = f'(0, \dots, 0, y_{k+1}, \dots, y_n) = 0\,, \end{cases}$$

where $f'$ is the equation (of the cover) of the strict transform of $X$.

Let $w^0, \dots, w^k$ be the skeleton of $\tau$. The polynomial $g$ comes from those monomials of $f$ which belong to the face $\gamma = \gamma(w^0) \cap \dots \cap \gamma(w^k)$. But all $w_i$ belong to an $n$-dimensional cone of $\Sigma'$, thus $\gamma$ contains at least 2 vertices of $\Gamma(f)$. Therefore $Z_\tau \cap Y''$ is indeed non-empty. At the same time, $Z_\tau$ is transversal to $Y''$ in the sense of Remark 5.2, and, moreover, $Y''$ is transversal to all the exceptional strata. This shows that $Z_\tau \cap Y''$ is irreducible. □

These considerations allow us to introduce the dual complex $\Delta(Z'')$ for the exceptional divisor $Z''$ of the morphism $\alpha\colon Y'' \to X$. Since all the strata $Z_\tau$, $\dim Z_\tau \ge 2$, have irreducible intersections with $X$, we can formulate the following receipt for finding $\Gamma(Z'')$. First construct the dual complex $\Delta(\bar{Z}'')$, where $\bar{Z}''_i$ are the exceptional divisors of the morphism

$$\bar{\alpha}\colon V(\Sigma'') \to \mathbb{C}^{n+1}\,,$$

$Z''_i = \bar{Z}''_i \cap Y''$. It can be built using a hyperplane section $H \cap \Sigma''$ as described above. Then remove from $\Delta(\bar{Z}'')$ all the interiors of $n$-dimensional cells of $\Sigma'' \cap H$ and glue additional $(n-1)$-simplexes for those 1-dimensional strata $Z_\tau$ which intersect $Y''$ in more than one point. To make this construction clearer let us illustrate it by an example (taken from [8]).

*Example* 5.4. Consider

$$(X, 0) = (\{x^8 + y^8 + z^8 + x^2y^2z^2 = 0\}, 0) \subset (\mathbb{C}^3, 0)\,.$$

This is a non-degenerate isolated surface singularity. Its Newton diagram is shown in Fig. 4, *a*.

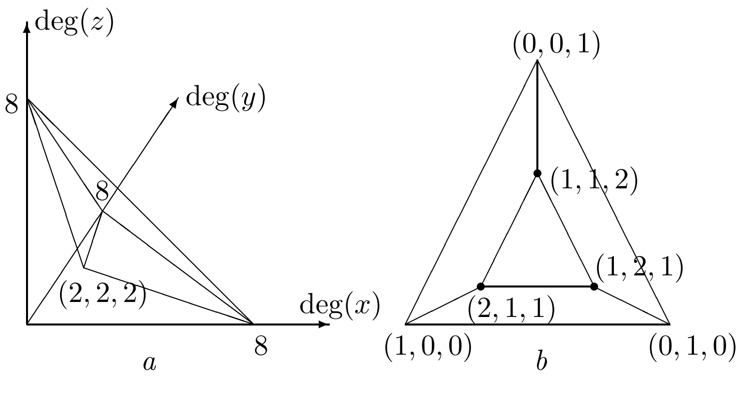


FIGURE 4. Newton diagram for $x^8 + y^8 + z^8 + x^2y^2z^2$.

The section $\Sigma' \cap H$ of the first Varchenko subdivision with an appropriate hyperplane is shown in Fig. 4, *b*. $(1,0,0)$ etc. are primitive vectors along the corresponding rays; bold points indicate the exceptional divisors. Three cells of $\Sigma' \cap H$ are not triangles. We can triangulate them by introducing some vectors into their interiors (i. e., by additional toric blow-ups); but by Proposition 5.3 the corresponding exceptional divisors does not intersect the strict transform of $X$, thus they will be deleted in the next step. The complex $\Delta(\bar{Z}'')$ is shown in Fig. 5, *a*. It is just the inner triangle of $\Sigma' \cap H$.

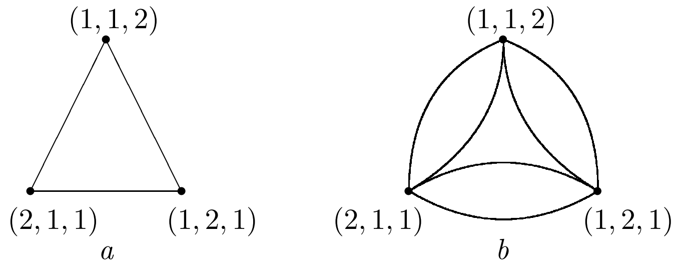


FIGURE 5. Dual complex of $x^8 + y^8 + z^8 + x^2y^2z^2$.

To get the dual complex $\Delta(Z'')$ we must delete the interior of the triangle in Fig. 5, and the only remaining question is whether its edges are multiple. In order to find this out let us consider the birational morphism $V(\Sigma'') \to \mathbb{C}^3$ restricted to the affine piece $V(\sigma) = \mathbb{C}^3/G$, $\sigma = \langle(2,1,1),(1,2,1),(1,1,2)\rangle$, and $G$ is a group of order

$$\begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 4\,.$$

Formulae (2) take the form

$$x = x_1^2 y_1 z_1,\ y = x_1 y_1^2 z_1,\ z = x_1 y_1 z_1^2\,.$$

The strict transform $Y''$ of $X$ is

$$x_1^8 + y_1^8 + z_1^8 + 1 = 0\,.$$

The intersection of $\widetilde{Y}''$ with, say, the stratum $Z_\tau$: $y_1 = z_1 = 0$ consists of 8 points; but we must account also the group action. Verification shows that $G = \mathbb{Z}_4$ acting via

$$x_1 \to \varepsilon x_1\,,\ y_1 \to \varepsilon y_1\,,\ z_1 \to \varepsilon z_1\,,$$

$\varepsilon^4 = 1$, thus $\sharp(Y'' \cap Z_\tau) = 2$. Therefore, the edges of triangle are double. The complex $\Delta(Z'')$ is shown in Fig. 5, *b*. It is homotopy to a bouquet of 4 circles; in particular, rank of $H_1(\Delta(Z''))$ is 4.

Note that constructing $\Sigma''$ can be much easier than finding $\Sigma$ that gives the embedded resolution. However, knowing $\Sigma''$ is enough to determine the homotopy type of $\Delta(Z)$.

**Proposition 5.5.** *Let the singularity $(X, o)$ be such that the property (R) is satisfied, and let $\Sigma''$ be as described above. Then $\Delta(Z'')$ is homotopy equivalent to the dual complex $\Delta(Z)$ associated to any good resolution $(Y, Z) \to (X, o)$.*

*Proof.* The proposition is an easy consequence of the theory of toroidal embeddings. Indeed, let $Y''$ be the strict transform of $X$ in $V(\Sigma'')$. Since all cones of $\Sigma'$ are simplicial, $Y''$ has only toroidal quotient singularities. Then $Y'' \setminus Z'' \subset Y''$ is a toroidal embedding. In a similar way to the proof of Theorem 4.1 we can construct a good toroidal resolution $Y$ of $Y''$ such that its dual complex $\Delta(Z)$ is a subdivision of $\Delta Z''$. Hence $\Delta(Z)$ is homeomorphic to $\Delta(Z'')$. By Theorem 2.1 dual complex of any good resolution of $X$ is homotopy to $\Delta(Z)$ and thus to $\Delta(Z'')$. □

Applying Proposition 5.5 to Example 5.4, we deduce that $\Delta(Z)$ is homotopy equivalent to the bouquet of 4 circles for any good resolution $Y \supset Z$. This agrees with [8] where $(X, 0)$ is resolved by a sequence of appropriate blow-ups. One more application is the following

**Corollary 5.6.** *Let $(X, 0)$ be the Brieskorn singularity*

$$x_0^{a_0} + x_1^{a_1} + \cdots + x_n^{a_n} = 0\,.$$

*Let $(Y, Z)$ be its good resolution. Then $\Delta(Z)$ has trivial homotopy type.*

*Proof.* In view of Proposition 5.5 the task becomes trivial. The first Varchenko subdivision in this case is just the subdivision of the positive cone $\sigma_+$ by the ray $\langle w \rangle$,

$$w = \left( \frac{m}{a_0}, \frac{m}{a_1}, \ldots, \frac{m}{a_n} \right)\,,$$

where $m$ is the least common multiple of the integers $a_0,\ a_1, \ldots, a_n$. Thus $\Sigma'$ consists of simplicial cones and possesses the property (R). We can put $\Sigma'' = \Sigma'$. There is only one prime exceptional divisor, hence $\Delta(Z'')$ is a point. Therefore $\Delta(Z)$ is homotopy equivalent to a point for any good resolution. □

We have seen that for a partial resolution $Y'' \subset V(\Sigma'')$ of a non-degenerate singularity the complex $\Delta(Z'')$ is just the $(n-1)$-skeleton of $\Delta(\bar{Z}'')$ to which, maybe, several additional $(n-1)$-simplexes have been glued. But $\Delta(\bar{Z}'')$ is homotopy trivial. Indeed, we can make all its cones simplicial by introducing new rays only in its $n$-skeleton. Also, since $\Sigma'$ satisfies the property (R), we can put all these rays to the interior of $\sigma_+$. Then $\Delta(\bar{Z}'')$ is obtained from $\Delta(T + \bar{Z}'')$ by deleting the vertices corresponding to the divisor $T$. This does not change the homotopy type of the dual complex and, on the other hand, $\Delta(T + \bar{Z}'')$ is obviously homotopy trivial. We come to the following result about which we have a feeling that it must hold in a much wider situation.

**Corollary 5.7.** *Let $(X, 0)$ be a non-degenerate isolated $n$-dimensional hypersurface singularity satisfying the property (R). Let $(Y, Z) \to (X, 0)$ be a good resolution. Then all the intermediate homologies of $\Delta(Z)$ vanish:*

$$H_k(\Delta(Z), \mathbb{Z}) = 0 \text{ for } 0 < k < n - 1\,.$$

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