TORIC RESIDUE AND GLOBAL DUALITY

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ABSTRACT. The notion of toric residue was introduced by D. Cox. Let X be a compact n-dimensional toric variety. Then the toric residue is associated with $n+1$ divisors $E_0, E_1, \ldots, E_n$ on X having empty intersection and defined by a sequence of polynomials $F_0, F_1, \ldots, F_n$ in homogeneous coordinate ring of X. In the first (algebraic) part of this paper, we prove a *Global Duality Theorem* for toric residues which says that a homogeneous polynomial H belongs to the ideal $\langle F_0, F_1, \ldots, F_n\rangle$ if and only if $\mathrm{Res}_F(H \cdot G) = 0$ for all homogeneous polynomials G. In the second (analytic) part, using the Weil integral formula, we show that it is sufficient to verify the vanishing of the residue only for finite number of polynomials G and give an algorithm of how to choose them.

1. Introduction

We begin this section with a review of some basic properties of multidimensional residues (see [GH, Ts]). Let $I(f) = \langle f_1, \ldots, f_n\rangle$ be a polynomial ideal in the ring $\mathbb{C}[z] = \mathbb{C}[z_1, \ldots, z_n]$ defined by a system of polynomials $f = (f_1, \ldots, f_n)$ with discrete set of zeros $Z_f \subset \mathbb{C}^n$. Given another polynomial $h \in \mathbb{C}[z]$, consider the meromorphic n-form on $\mathbb{C}^n$

$$\omega_h = \frac{h}{f_1 \cdots f_n}\, dz_1 \wedge \ldots \wedge dz_n.$$

We denote by

$$\mathrm{Res}_f(h) = \sum_{a \in Z_f} \mathrm{res}_a(\omega_h)$$

the sum of Grothendieck point residues, the so-called *global residue*. If $h \in I(f)$, then it is well-known that $\mathrm{Res}_f(h) = 0$. Since Z_f is finite, the quotient ring $R(f) = \mathbb{C}[z]/I(f)$ is a finite dimensional $\mathbb{C}$-space. Therefore, the residue defines a $\mathbb{C}$-linear map

$$\mathrm{Res}_f : R(f) \to \mathbb{C}.$$

The classical *Global Duality Theorem* for residues says that the polynomial $h \in \mathbb{C}[z]$ belongs to the ideal $I(f)$ if and only if global residue vanishes on all possible products of h with polynomials $g \in \mathbb{C}[z]$:

$$h \in I(f) \iff \mathrm{Res}_f(h \cdot g) = 0, \quad \forall g \in \mathbb{C}[z].$$

The global duality has the following algebraic interpretation. Denote by $R(f)^* = \mathrm{Hom}_{\mathbb{C}}(R(f), \mathbb{C})$ the dual to $R(f)$ vector space. Then there is pairing

$$R(f) \times R(f)^* \to R(f)^*, \quad (h, \varphi) \mapsto (h.\varphi) \tag{1}$$

given by means of multiplication $(h.\varphi)(g) = \varphi(h \cdot g)$. Then the global duality is equivalent to saying that the global residue Res_f defines a generator of $R(f)^*$ as a $R(f)$-module and the pairing (1) is nondegenerate.

The aim of the first part of this paper is to show that the similar duality holds for toric residue. The notion of toric residue was introduced by D. Cox [Cox2]. It is defined for every collection of $n+1$ effective divisors $E_0, E_1, \ldots, E_n$ having empty intersection on a complete toric variety X of dimension n. Let Σ be a fan defining toric variety with r one-dimensional generators $e_1, \ldots, e_r$. Then there are homogeneous coordinates $x_1, \ldots, x_r$ of the ring $S = \mathbb{C}[x_1, \ldots, x_r]$ graded by $A_{n-1}(X)$ which are associated with $e_1, \ldots, e_r$ [Cox1]. If $F = (F_0, F_1, \ldots, F_n)$ is a collection of homogeneous polynomials in S of degrees $\alpha_0, \alpha_1 \ldots, \alpha_n \in A_{n-1}(X)$ and defining $E_0, E_1, \ldots, E_n$, then there is a toric residue map

$$\mathrm{Res}_F : S_\rho / \langle F_0, F_1, \ldots, F_n \rangle_\rho \to \mathbb{C} \tag{2}$$

on the graded part of the quotient $S/\langle F_0, F_1, \ldots, F_n\rangle$ of the so-called *critical degree*

$$\rho = \sum_{i=0}^{n} \alpha_i - \sum_{j=1}^{r} \deg(x_j).$$

It was recently shown in [CD] that if $F_0, F_1, \ldots, F_n$ satisfy the properties: F_i don't vanish simultaneously on X and α_i is in $\mathrm{Pic}(X)$ such that $\mathcal{O}_X(\alpha_i)$ is generated by global sections and each polyhedron Δ_i associated with α_i is of dimension n, then

$$\dim_{\mathbb{C}} S_\rho / \langle F_0, F_1, \ldots, F_n \rangle_\rho = 1,$$

from which follows that the mapping (2) is an isomorphism. It is a nice open question to find the explicit description of this mapping. For example, if all degrees of $F_0, F_1, \ldots, F_n$ are equal, i.e. $\alpha = \alpha_0 = \alpha_1 = \cdots = \alpha_n$, the isomorphism (2) is given by the toric Jacobian J_F [Cox2]:

$$\mathrm{Res}_F(J_F) = \mathrm{Vol}(\Delta),$$

where Δ is the polyhedron associated with α. Another realizations of this mapping which gives residue one can be found in [CCD] and [AK].

We show in Section 1, that under the same conditions on $F_0, F_1, \ldots, F_n$ as above, there is a duality similar to (1):

$$R(F)_\alpha \times R(F)_{\rho-\alpha} \to R(F)_\rho, \quad \alpha \in \mathrm{Pic}(X)$$

if the polyhedron Δ associated with α is of dimension n, which is a perfect pairing given by multiplication. The proof consists in considering a differential of some spectral sequence. This duality implies the Global Duality for toric residues: the polynomial H belongs to the ideal $\langle F_0, F_1, \ldots, F_n\rangle$ if and only if $\mathrm{Res}_F(H \cdot G) = 0$ for all $G \in S$.

In Section 2, we show that it is sufficient to check the equality $\mathrm{Res}_F(H \cdot G) = 0$ only for finite numbers of polynomials $G \in S$ and propose an algorithm of how to choose them by using the Weil integral formula. Also, this helps to find the generators of the graded piece of the quotient ring $S/\langle F_0, F_1, \ldots, F_n\rangle$ of degree α.

Again, this repeats in some sense the classical situation. Let $\{\mathcal{U}_a\}_{a\in Z_f}$ be a family of disjoint neighborhoods of the point $a \in Z_f$, where $f = (f_1, \ldots, f_n)$ is a polynomial mapping $f : \mathbb{C}^n \to \mathbb{C}^n$ with discrete set of zeros $Z_f \subset \mathbb{C}^n$ as before. Choose $\varepsilon \in \mathbb{R}^n_+$ so small that the polyhedron

$$\Pi^\varepsilon_a = \{z \in \mathcal{U}_a : |f_1(z)| < \varepsilon_1, \ldots, |f_n(z)| < \varepsilon_n\}$$

is relatively compact in $\mathcal{U}_a$. Set $\Pi^\varepsilon = \cup_{a\in Z_f} \Pi^\varepsilon_a$ and denote by Γ^ε its distinguished boundary. The Weil integral representation generalize the Cauchy integral formula. It recovers the value of the function $h \in \mathbb{C}[z]$ in the polyhedron Π^ε from its values on the distinguished boundary Γ^ε:

$$h(z) = \left(\frac{1}{2\pi i}\right)^n \int_{\Gamma^\varepsilon} \frac{h(\xi)\, H(\xi, z)}{f(\xi) - f(z)}\, d\xi_1 \wedge \ldots \wedge d\xi_n, \quad z \in \Pi^\varepsilon.$$

Here $H(\xi, z)$ is the determinant of the matrix $||P_{jk}(\xi, z)||_{1\le j,k\le n}$ in the *Hefer expansion*

$$f_j(\xi) - f_j(z) = \sum_{k=1}^{n} (\xi_k - z_k) P_{jk}(\xi, z), \quad j = 1, \ldots, n.$$

Since $|f_j(z)| < |f_j(\xi)|$ for $z \in \Pi^\varepsilon$, $\xi \in \Gamma^\varepsilon$, $j = 1, \ldots, n$, the integrand can be expanded into a multiple geometric progression

$$h(\xi) \cdot H(\xi, z) \cdot [f(\xi) - f(z)]^{-1} = h(\xi) \cdot H(\xi, z) \cdot \sum_{|\alpha|\ge 0} \frac{f^\alpha(z)}{f^{\alpha+I}(\xi)}$$

which converges uniformly on the compact subsets of $\Gamma^\varepsilon \times \Pi^\varepsilon$. After term-by-term integration, we obtain

$$h(z) = \mathrm{Res}_f(h(\xi)H(\xi, z)) + \sum_{|\alpha|\ge 1} \left(\frac{1}{2\pi i}\right)^n \left[\int_{\Gamma^\varepsilon} \frac{h(\xi)H(\xi, z)d\xi}{f^{\alpha+I}(\xi)}\right] f^\alpha(z).$$

Note that the second summand is in the ideal $\langle f_1, \ldots, f_n\rangle$ in $\mathcal{O}(\Pi^\varepsilon)$. Using the fact that $Z_f \subset \Pi^\varepsilon$ and $\mathrm{Res}_f(h(\xi)H(\xi, z))$ is a polynomial in z, we can write modulo the ideal $I(f)$:

$$h(z) = \mathrm{Res}_f(h(\xi)H(\xi, z)),$$

since analytic and algebraic memberships are equivalent. Now expanding the Heferian into the sum of products of functions in disjoint variables

$$H(\xi, z) = \sum_{j=1}^{p} \varphi_j(z) \cdot \psi_j(\xi),$$

we obtain modulo the ideal $I(f)$ the equality

$$h(z) = \sum_{j=1}^{p} \mathrm{Res}_f(h(\xi) \cdot \psi_j(\xi)) \cdot \varphi_j(z)$$

which implies that $h(z) \in I(f)$ if and only if $\mathrm{Res}_f(h(\xi) \cdot \psi_j(\xi)) = 0$ for all $j = 1, \ldots, p$.

We apply the same idea for finding the polynomials $G \in S$ for which it is sufficient to check the vanishing $\mathrm{Res}_F(H \cdot G) = 0$. To do this, we use an analytic expression of the toric residue as a sum of residues in the torus from [CCD] and Weil integral formulas for the sum of residues in the torus from [BY1, BY2].

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2. Global Duality Theorem

We begin with some notations and and review basic facts on toric varieties and toric residues. Let us fix a lattice $N \cong \mathbb{Z}^n$ and its dual lattice $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. Their real extensions will be denoted by $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$ respectively. A fan Σ is a collection of convex rational polyhedral cones in $N_{\mathbb{R}}$ such that each face of a cone in Σ belongs to Σ and the intersection of two cones in Σ is a face of each. Let $X = X_\Sigma$ be a compact toric variety over the field $\mathbb{C}$ of dimension n defined by the complete fan Σ. The variety X contains the torus $\mathbb{T} = N \otimes \mathbb{C}^* \cong (\mathbb{C}^*)^n$ as a dense open subset.

The set of minimal edge generators of the fan Σ will be denoted by $G(\Sigma) = \{e_1, \ldots, e_r\}$. Each vector e_j corresponds to the torus-invariant divisor D_i of X and to the variable x_j of the *homogeneous coordinate ring* $S = \mathbb{C}[x_1, \ldots, x_r]$ introduced by Cox [Cox1]. The ring S is graded by the Chow group $A_{n-1}(X)$ as follows: a monomial $x^a = \prod_{i=1}^r x_i^{a_i}$ has degree $\alpha = \deg(x^a) = [\sum_{i=1}^r a_i D_i] \in A_{n-1}(X)$. Denote by S_α the α-graded part of S, i.e. the linear combinations of all monomials in S of degree α. The subring S_α corresponding to the class $\alpha = [D]$ can be identified with the space of global sections $H^0(X, \mathcal{O}_X(D))$ corresponding to the invertible sheaf $\mathcal{O}_X(D)$. We define the convex polyhedron associated with the torus-invariant divisor $D = \sum_{i=1}^r a_i D_i$ by the formula

$$\Delta_D = \{m \in M_{\mathbb{R}} : \langle m, e_i \rangle \geq -a_i, \quad i = 1, \ldots, r\} \subset M_{\mathbb{R}}.$$

In fact, since the polyhedrons corresponding to divisors in the same class in $A_{n-1}(X)$ are isomorphic, the polyhedron Δ_D depends only on the class of D, and we may assign polyhedron Δ_α to the class $\alpha = [D] \in A_{n-1}(X)$ by choosing any representative in α.

Given homogeneous polynomials $F_0, F_1, \ldots, F_n$ in S of degrees $\alpha_0, \alpha_1, \ldots, \alpha_n$, we denote by

$$\rho = \sum_{j=0}^{n} \alpha_j - \sum_{i=1}^{r} \deg(x_i) \in A_{n-1}(X)$$

their *critical, or Macaulay degree*. Note, that it is follows from the definition of ρ that the polyhedron Δ_ρ is the relative interior of the Minkowski sum of the polyhedrons $\Delta_0, \Delta_1, \ldots, \Delta_n$ corresponding to $\alpha_0, \alpha_1, \ldots, \alpha_n$:

$$\Delta_\rho = (\Delta_0 + \Delta_1 + \cdots + \Delta_n)^\circ.$$

Suppose that the divisors $E_0, E_1, \ldots, E_n \subset X$ of polynomials $F_0, F_1, \ldots, F_n$ are such that

$$E_0 \cap E_1 \cap \ldots \cap E_n = \emptyset.$$

Then, the charts $U_i = \{x \in X : F_i(x) \neq 0\}$ define the open covering $\mathcal{U} = \{U_i, i = 0, \ldots, n\}$ of X. Denote by Ω the *Euler differential form* on X (see [BC]), which is a generator of $H^0(X, \widehat{\Omega}^n_X(\sum_{i=1}^r D_i))$, where $\widehat{\Omega}^n_X$ is a sheaf of n-forms of Zariski on X. Then for any polynomial $H \in S_\rho$, the differential form

$$\omega_F(H) = \frac{H\,\Omega}{F_0F_1\cdots F_n} \in \widehat{\Omega}^n_X(U_0 \cap U_1 \cap \ldots \cap U_n)$$

is a well-defined Čech cocycle with respect to the covering $\mathcal{U}$. Since the sheaf $\widehat{\Omega}^n_X$ is the dualizing sheaf for X [D], there exists a *trace morphism* $\mathrm{Tr_X} : H^n(X, \widehat{\Omega}^n_X) \to \mathbb{C}$.

Definition 2.1. (D. Cox, [Cox2]) The *toric residue* of the form $\omega_F(H)$ (or, of the polynomial H) is defined by the trace morphism

$$\mathrm{Res}_F(H) := \mathrm{Tr}_X([\omega_F(H)]).$$

If X is simplicial toric variety, then trace map can be realized explicitly via the integration over X:

$$\mathrm{Res}_F(H) = \left(\frac{-1}{2\pi i}\right)^n \int_X \eta_\omega,$$

where η_ω is (n,n)-form corresponding to $\omega_F(H)$ under the Dolbeaut isomorphism

$$H^n(X, \widehat{\Omega}^n_X) \cong H^{n,n}_{\overline{\partial}}(X) \cong H^{2n}(X).$$

One of the main properties of the toric residue is: if $H \in \langle F_0, F_1, \ldots, F_n\rangle$, then $\mathrm{Res}_F(H) = 0$ [Cox2]. Indeed, if $H \in \langle F_0, F_1, \ldots, F_n\rangle$, then $\omega_F(H)$ is a Čech coboundary and its class $[\omega_F(H)]$ in $H^n(\mathcal{U}, \widehat{\Omega}^n_X)$ is zero. Therefore, toric residue defines the mapping

$$\text{Res}_F : S_\rho/\langle F_0, F_1, \dots, F_n\rangle_\rho \to \mathbb{C}, \tag{3}$$

where the subscript ρ in $\langle F_0, F_1, \dots, F_n\rangle_\rho$ means that we take ρ-homogeneous part of the ideal $\langle F_0, F_1, \dots, F_n\rangle$. Denote by $R(F)$ the quotient ring $R(F) = S/\langle F_0, F_1, \dots, F_n\rangle$ which is graded by the Chow group $A_{n-1}(X)$. It was proved in [CD, Corollary 2.5], that under certain assumptions on $F_0, F_1, \dots, F_n$ the quotient ring $R(F)_\rho$ is one-dimensional, which implies that the mapping (3) is an isomorphism. We refer to the original work [CD] for the comprehensive study of the estimates of the dimension of the quotient $R(F)_\rho$.

Theorem 2.2. (D. Cox, A. Dickenstein, [CD]) *Suppose that X is a compact toric variety of dimension n. Let $F = (F_0, F_1, \dots, F_n)$ be a sequence of polynomials regular in S, where their degrees $\alpha_0, \alpha_1, \dots, \alpha_n \in \text{Pic}(X)$ are globally generated by sections and the corresponding polyhedra $\Delta_0, \Delta_1, \dots, \Delta_n$ are n-dimensional. Then*

$$\dim_{\mathbb{C}} R(F)_\rho = 1$$

and the toric residue mapping (3) is an isomorphism.

The main result of this section is the following duality theorem.

Theorem 2.3. *Under assumptions of the theorem 2.2 on $F = F_0, F_1, \dots, F_n$, for any class $\alpha \in \text{Pic}(X)$ of a Cartier divisor on X generated by sections, such that the polyhedron Δ corresponding to α is of dimension n, the pairing*

$$R(f)_\alpha \times R(f)_{\rho-\alpha} \to R(f)_\rho$$

given by multiplication is perfect.

This implies immediately the *Global Duality Theorem for toric residue* which answers to the question when the polynomial H belongs to the ideal $\langle F_0, F_1, \dots, F_n\rangle$ in terms of toric residues.

Theorem 2.4. *The bilinear form*

$$B(\cdot,\cdot) : R(f)_\alpha \times R(f)_{\rho-\alpha} \to \mathbb{C}$$

defined by the formula $B(\overline{H}, \overline{G}) = \text{Res}_F(H \cdot G)$ is nondegenerate, that is, if one has $\text{Res}_F(H \cdot G) = 0$ for all $G \in S_{\rho-\alpha}$, then $H \in \langle F_0, F_1, \dots, F_n\rangle_\alpha$.

Also, as a corollary of theorem 2.3, we get the following.

Proposition 2.5. *The α-graded part $R(F)_\alpha$ of $R(F)$ is nonzero if and only if the polyhedron Δ_α lies in the relative interior of the Minkowski sum of the polyhedrons $\Delta_0, \Delta_1, \dots, \Delta_n$:*

$$R(F)_\alpha \neq 0 \iff \Delta_\alpha \subset (\Delta_0 + \Delta_1 + \dots + \Delta_n)^\circ.$$

Proof of the theorem 2.3. To prove the theorem, we explore a spectral sequence, similar to used in [CD]. Denote by $\mathcal{E}$ the sheaf $\mathcal{E} = \oplus_{i=0}^{n}\mathcal{O}_X(-\alpha_i)$, and consider the exact sequence

$$0 \to \mathcal{E}^0 \to \mathcal{E}^1 \to \cdots \to \mathcal{E}^{n+1} \to 0,$$

where

$$\mathcal{E}^p = \bigwedge^{n+1-p} \mathcal{E}(\alpha) \cong \bigoplus_{|J|=n+1-p} \mathcal{O}_X(\alpha - \sum_{j\in J}\alpha_j),$$

i.e.

$$0 \to \bigwedge^{n+1}\mathcal{E}(\alpha) \to \bigwedge^{n}\mathcal{E}(\alpha) \to \cdots \to \mathcal{E}(\alpha) \to \mathcal{O}_X(\alpha) \to 0,$$

obtained by taking a tensor multiple of the Koszul complex

$$0 \to \mathcal{O}_X(-\sum_i \alpha_i) \to \cdots \to \bigoplus_i \mathcal{O}_X(-\alpha_i) \to \mathcal{O}_X \to 0$$

by $\mathcal{O}_X(\alpha)$ over $\mathcal{O}_X$. Then the spectral sequence associated with the naive filtration

$$F^p : 0 \to \mathcal{E}^p \to \cdots \to \mathcal{E}^{n+1} \to 0.$$

has the terms

$$E_1^{p,q} = H^q(X, \mathcal{E}^p) \cong \bigoplus_{|J|=n+1-p} H^q(X, \mathcal{O}_X(\alpha - \sum_{j\in J}\alpha_j))$$

and convergent to 0. Since $E_1^{n,0} \to E_1^{n+1,0} \to E_1^{n+2,0}$ is

$$\bigoplus_{k=0}^{n} S_{\alpha-\alpha_k} \to S_\alpha \to 0,$$

one has $E_2^{n+1,0} = R(F)_\alpha$. Similarly, the sequence $0 \to E_1^{0,n} \to E_1^{1,n}$ gives

$$0 \to H^n(X, \bigwedge^{n+1}\mathcal{E}(\alpha)) \to H^n(X, \bigwedge^{n}\mathcal{E}(\alpha)),$$

from which follows that

$$E_2^{0,n} = \ker\left(H^n(X, \bigwedge^{n+1}\mathcal{E}(\alpha)) \to H^n(X, \bigwedge^{n}\mathcal{E}(\alpha)) \right).$$

Then, we note that in the conditions of the statement, $E_1^{p,q} = 0$ if q is not 0 or n, and $E_\infty^{n+1,0} = 0$ that implies that the differential in

$$E_\infty^{n+1,0} = \operatorname{coker}\left(d_{n+1} : E_2^{0,n} \to E_2^{n+1,0}\right)$$

is an isomorphism. In other words, one has an isomorphism

$$d_{n+1} : \ker\left(H^n(X, \bigwedge^{n+1}\mathcal{E}(\alpha)) \to H^n(X, \bigwedge^{n}\mathcal{E}(\alpha)) \right) \to R(F)_\alpha.$$

Applying the Serre duality and the isomorphisms

$$\bigwedge^{n+1}\mathcal{E}(\alpha)=\mathcal{O}_X(\alpha-\sum_i\alpha_i),\quad \bigwedge^{n}\mathcal{E}(\alpha)=\mathcal{E}^*(\alpha-\sum_i\alpha_i),$$

we see that the map

$$H^n(X,\bigwedge^{n+1}\mathcal{E}(\alpha))\to H^n(X,\bigwedge^{n}\mathcal{E}(\alpha))$$

is canonically dual to the map

$$H^0\left(X,\mathcal{E}\left(\sum_i\alpha_i-\alpha-\beta_0\right)\right)\to H^0\left(X,\mathcal{O}_X\left(\sum_i\alpha_i-\alpha-\beta_0\right)\right),$$

where we write for simplicity $\beta_0=\sum_{i=1}^r\deg(x_i)$. Since the image of the last map is

$$\langle F_0,F_1,\dots,F_n\rangle_{\sum_i\alpha_i-\alpha-\beta_0}=\langle F_0,F_1,\dots,F_n\rangle_{\rho-\alpha},$$

the differential d_{n+1} provides an isomorphism

$$d_{n+1}:(R(F)_{\rho-\alpha})^*\cong R(F)_\alpha.$$

It is not hard to check that the isomorphism constructed above is given by the pairing

$$R(F)_\alpha\times R(F)_{\rho-\alpha}\to R(F)_\alpha$$

defined by the multiplication. Indeed, the bilinear pairing

$$(\cdot,\cdot):R(F)_\alpha\times R(F)_{\rho-\alpha}\to R(F)_\rho$$

given by multiplication

$$(G,H)\mapsto G\cdot H,\quad G\in R(F)_\alpha,\ H\in R(F)_{\rho-\alpha}$$

satisfies

$$(P\cdot G,H)=(G,P\cdot H),\quad \forall P\in R(F)_\beta,\ G\in R(F)_\alpha,\ H\in R(F)_{\rho-\alpha-\beta}.$$

In other words, the diagram

$$\begin{array}{ccc} R(F)_{\rho-\alpha} & \xrightarrow{\cong} & R(F)_\alpha \\ {\scriptstyle P^*}\downarrow & & \downarrow{\scriptstyle P} \\ R(F)_{\rho-\alpha-\beta} & \xrightarrow{\cong} & R(F)_{\alpha+\beta} \end{array}$$

commutes. Hence, the isomorphism provided by d_{n+1} is compatible with the multiplication and provides the bilinear pairing as required. $\square$

3. Weil Formula for Toric Residues

Throughout this section we assume that the conditions of the theorem 2.3 are fulfilled. We show that in order to check that the polynomial H is in the ideal $\langle F_0, F_1, \ldots, F_n\rangle$, it is sufficient to show that $\mathrm{Res}_F(H \cdot G) = 0$ in the Global Duality theorem 2.3 only for certain finite number of polynomials $G \in S$.

The choice of the basis $m_1, \ldots, m_n$ of the lattice $M \cong \mathbb{Z}^n$ establishes the choice of the local coordinates $t_1, \ldots, t_n$ on the torus $\mathbb{T} \cong (\mathbb{C}^*)^n$:

$$t_j = x^{\langle m_j, e\rangle} = \prod_{i=1}^{r} x_i^{\langle m_j, e_i\rangle}.$$

Given any Laurent polynomial

$$f(t) = f(t_1, \ldots, t_n) = \sum_{a \in \Delta \cap \mathbb{Z}^n} c_a t^a = \sum_{a \in \Delta \cap \mathbb{Z}^n} c_{a_1 \ldots a_n} t_1^{a_1} \cdots t_n^{a_n} \in \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$$

with Newton polyhedron $\Delta = \Delta(f)$, we define its Σ-homogenization with respect to the fan Σ generalizing the classical homogenization of polynomial in the case $X_\Sigma = \mathbb{P}^n$:

$${}^h f = F(x_1, \ldots, x_r) = \sum_{a \in \Delta \cap \mathbb{Z}^n} c_a \prod_{i=1}^{r} x_i^{\langle a, e_i\rangle + d_i} = \prod_{i=1}^{r} x_i^{d_i} f(x^{\langle m_1, e\rangle}, \ldots, x^{\langle m_n, e\rangle}),$$

where $d_i = -\min_{l \in \Delta}\langle l, e_i\rangle$. The obtained polynomial $F(x_1, \ldots, x_r)$ has degree $\alpha = [\sum_{i=1}^{r} d_i D_i]$.

We explore the analytic properties of the toric residue. It was proved in [CCD] that toric residue with respect to the system of divisors $E_0, E_1, \ldots, E_n$ defined by polynomials $F_0, F_1, \ldots, F_n$ can be written as a sum of local residues. This theorem is analogous to the situation in $\mathbb{P}^n$ considered in [Ts, Corollary 1, p. 35].

Theorem 3.1. (E. Cattani, D. Cox, A. Dickenstein, [CCD, Theorem 0.4]) *Let X be a complete toric variety and $F_0, F_1, \ldots, F_n \in S$ are homogeneous polynomials of degrees $\alpha_0, \alpha_1, \ldots, \alpha_n$ respectively whose corresponding divisors $E_0, E_1, \ldots, E_n$ defined by $E_i = \{x \in X : F(x) = 0\}$ are such that*

$$E_0 \cap E_1 \cap \ldots \cap E_n = \emptyset$$

and for some index $k \in \{0, 1, \ldots, n\}$

$$V_k = \{E_0 \cap \ldots \cap \widehat{E_k} \cap \ldots \cap E_n\} \subset \mathbb{T}$$

is a finite set of points. Then for any polynomial $H \in S_\rho$ the toric residue $\mathrm{Res}_F(H)$ can be written as a sum of local residues

$$\mathrm{Res}_F(H) = (-1)^k \sum_{t \in V_k} \mathrm{res}_t \left(\left. \frac{(H/F_k)\Omega}{F_0 \cdots \widehat{F_k} \cdots F_n} \right|_{\mathbb{T}} \right), \tag{4}$$

where $|_{\mathbb{T}}$ *denotes restriction of the form to the torus* $\mathbb{T}$*.*

Remark that the formula in the last theorem can be considered as an analytic definition of the toric residue and, in fact, the sum of the local residues does not depend on the compactification of the torus $\mathbb{T}$. This formula was first proved for simplicial toric varieties and then extended in [CDS, Theorem 3.2] to arbitrary toric variety by the following arguments: if $\varphi : X' \to X$ is a birational morphism preserving the torus $\mathbb{T}$ and $F_i' = \varphi^*(F_i) \in H^0(X', \varphi^*(\mathcal{O}_X(D)))$, $H' = \varphi^*(H)$, then

$$\mathrm{Res}_F(H) = \mathrm{Tr}_X([\omega_F(H)]) = \mathrm{Tr}_{X'}([\omega_{F'}(H')])$$

by the property of the trace map.

Definition 3.2. For any Laurent polynomials $h(t), f_1(t), \dots, f_n(t) \in \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$, where $V = \{t \in \mathbb{T} : f_1(t) = \cdots = f_n(t) = 0\} \subset \mathbb{T}$ is a discrete set of isolated points, we denote the sum of the local residues in the torus by

$$\mathrm{Res}\left[\begin{array}{c} h\, dt \\ f_1, \dots, f_n \end{array}\right]_{\mathbb{T}} := \sum_{t \in V} \mathrm{res}_t \left(\frac{h}{f_1 \cdots f_n} \frac{dt_1}{t_1} \wedge \ldots \wedge \frac{dt_n}{t_n} \right).$$

If Laurent polynomials $h(t), f_0(t), \dots, f_n(t) \in \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ are such that their homogenizations with respect to the fan Σ of X are $H, F_0, \dots, F_n$ respectively, then the formula (4) can be rewritten as follows:

$$\mathrm{Res}_F(H) = (-1)^k \mathrm{Res}\left[\begin{array}{c} h/f_k\, dt \\ f_0, \dots, \widehat{f_k}, \dots, f_n \end{array}\right]_{\mathbb{T}}. \tag{5}$$

To find an analogue of the Weil integral formula for toric residue, we associate with the sequence of polynomials $f_0, f_1, \dots, f_n$ their *Hefer expansions.* Let us write any Laurent monomial $t^\gamma \in \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ as a quotient of monomials of nonnegative degrees: $t^\gamma = (t')^\alpha / (t'')^\beta$, where t' and t'' are disjoint sets of coordinates. Then, extending the equalities

$$\xi^\gamma - t^\gamma = \frac{(\xi')^\alpha (t'')^\beta - (t')^\alpha (\xi'')^\beta}{(\xi'')^\beta (t'')^\beta} = \frac{\sum_{k=1}^n q_k(\xi, t)(\xi_k - t_k)}{(\xi'')^\beta (t'')^\beta}$$

by linearity, we obtain for any point $(\xi, t) \in \mathbb{T}_\xi \times \mathbb{T}_t$ the Hefer expansions:

$$f_j(\xi) - f_j(t) = \sum_{k=1}^n p_{kj}(\xi, t)\, (\xi_k - t_k), \quad j = 0, \dots, n \tag{6}$$

of Laurent polynomials $f_0(t), f_1(t), \dots, f_n(t) \in \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. This expansion is of course not unique. For example:

$$\begin{aligned} \xi_1 \xi_2 - t_1 t_2 &= t_1(\xi_2 - t_2) + (\xi_1 - t_1) t_2 + (\xi_1 - t_1)(\xi_2 - t_2) \\ &= \xi_2(\xi_1 - t_1) + t_1(\xi_2 - t_2) \\ &= t_2(\xi_1 - t_1) + \xi_1(\xi_2 - t_2). \end{aligned}$$

The sufficient geometric conditions on $\Delta_j = \Delta(f_j)$ so that the Laurent polynomials $p_{kj}(\xi,t)$ can be chosen with supports in Δ_j are the following.

Definition 3.3. (C. Berenstein, A. Yger, [BY1, §3.2], [BY2, Definition 4.1]) The convex polyhedron $\Delta_j \subset \mathbb{R}^n$ is called *good polyhedron*, if

$$\forall x \in \Delta_j,\ \{y \in \mathbb{R}^n : |y_k| \le |x_k|,\, x_k y_k \ge 0,\, k = 1,\dots,n\} \subset \Delta_j.$$

Definition 3.4. Given a Hefer expansion of the Laurent polynomials $f_0, f_1, \dots, f_n$, we denote by $W(\xi,t)$ their *Weil polynomial*

$$W(\xi,t) = \xi_1 \cdots \xi_n \cdot \det \begin{pmatrix} f_0(\xi) & f_1(\xi) & \cdots & f_n(\xi) \\ p_{10}(\xi,t) & p_{11}(\xi,t) & \cdots & p_{1n}(\xi,t) \\ \vdots & \vdots & \ddots & \vdots \\ p_{n0}(\xi,t) & p_{n1}(\xi,t) & \cdots & p_{nn}(\xi,t) \end{pmatrix},$$

where $p_{jk}(\xi,t)$ are from the Hefer expansion (6).

It is easy to see that if the polynomials $f_0(t), f_1(t), \dots, f_n(t)$ have good supports $\Delta_0, \Delta_1, \dots, \Delta_n$, then the Weil polynomial $W(\xi,t)$ has support in the sum of the polyhedrons $\Delta_0 + \Delta_1 + \cdots + \Delta_n + \mathbf{1}$, where $\mathbf{1} = (1,\dots,1)$. We denote by $W(y,x)$ the polynomial in homogeneous coordinates $(y_1,\dots,y_r,x_1,\dots,x_r)$ of $X \times X$ obtained by the $\Sigma \times \Sigma$-homogenization of $W(\xi,t)$.

Now we are ready to formulate the main result of this section - the *toric Weil formula.*

Theorem 3.5. *Let X be a toric variety of dimension n defined by a fan Σ. Suppose that $F_0(x), F_1(x), \dots, F_n(x)$ are homogeneous polynomials which do not have common zeros in X and obtained as Σ-homogenizations of the Laurent polynomials $f_0(t), f_1(t), \dots, f_n(t)$ with good supports $\Delta_0, \Delta_1, \dots, \Delta_n$ of dimensions n. Let $W(y,x)$ be a $\Sigma \times \Sigma$-homogenization of the Weil polynomial. Then,*

$$H(x) = \mathrm{Res}_F(H(y)\, W(y,x)) \mod \langle F_0(x), F_1(x), \dots, F_n(x)\rangle,$$

where the toric residue is taken with respect to the variables y.

Proof. Denote by $H_k(\xi,t)$ the determinant of the minor of $W(\xi,t)$ obtained by deleting the first line and the k-th row of $W(\xi,t)$. Then, by the torc analogue of the Weil formula (see, e.g. [BY2, Proposition 4.2]), for any index $k \in \{0,\dots,n\}$ one has the equality

$$\begin{aligned} h(t) \quad &= \quad \mathrm{Res}\left[\begin{array}{c} h(\xi)\, H_k(\xi,t)\, d\xi \\ f_0,\dots,\widehat{f_k},\dots,f_n \end{array}\right]_{\mathbb{T}} \\ &+ \sum_{\nu \in \mathbb{N}^n \setminus \{0\}} \mathrm{Res}\left[\begin{array}{c} h(\xi)\, H_k(\xi,t)\, d\xi \\ f_0^{\nu_0+1},\dots,\widehat{f_k},\dots,f_n^{\nu_n+1} \end{array}\right]_{\mathbb{T}} f_0(t)^{\nu_0} \cdots \widehat{f_k(t)} \cdots f_n(t)^{\nu_n}. \end{aligned}$$

Note that the second sum over $(\nu_0,\dots,\widehat{\nu}_k,\dots,\nu_n)\in\mathbb{N}^n\backslash\{0\}$ is in fact finite under assumption that $\Delta_0,\dots,\Delta_n$ are good by the Jacobi-Khovanskii formula [Kh]. Indeed, let Δ be a support polyhedron for $h(t)$. Writing the Hefer expansions for any k as

$$H_k(\xi,t)=\sum_{\substack{\alpha,\beta\in\mathbb{Z}^n\\ \alpha+\beta+\mathbf{1}\in\Delta[k]}} c^k_{\alpha,\beta}\,\xi^\alpha t^\beta,$$

where $\Delta[k]=\Delta_0+\cdots+\widehat{\Delta}_k+\cdots+\Delta_n$, the second sum in the right hand side of integral representation of $h(t)$ can be rewritten as

$$\sum_{\substack{\alpha,\beta\in\mathbb{Z}^n\\ \alpha+\beta+\mathbf{1}\in\Delta[k]}}\ \sum_{\substack{\nu\in\mathbb{N}^n\backslash\{0\}\\ \Delta+\alpha\not\subset(\langle\nu,\Delta\rangle)^\circ}} c^k_{\alpha,\beta}\,\mathrm{Res}\left[\begin{array}{c} h(\xi)\,\xi^\alpha\,d\xi\\ f_0^{\nu_0+1},\dots,\widehat{f}_k,\dots,f_n^{\nu_n+1}\end{array}\right]_{\mathbb{T}} t^\beta f_0(t)^{\nu_0}\cdots\widehat{f_k(t)}\cdots f_n(t)^{\nu_n},$$

where $\langle\nu,\Delta\rangle=(\nu_0+1)\Delta_0+\cdots+\widehat{\Delta}_k+\cdots+(\nu_n+1)\Delta_n$. So, we have

$$h(t)=\mathrm{Res}\left[\begin{array}{c} h(\xi)\,H_k(\xi,t)\,d\xi\\ f_0,\dots,\widehat{f}_k,\dots,f_n\end{array}\right]_{\mathbb{T}}\mod\langle f_0(t),f_1(t),\dots,f_n(t)\rangle. \tag{7}$$

On the other hand, expand the Weil polynomial by the first line

$$W(\xi,t)=\sum_{l=0}^n(-1)^l\,f_l(\xi)\,H_k(\xi,t)$$

and consider the residue

$$\mathrm{Res}\left[\begin{array}{c} h(\xi)\,W(\xi,t)\,d\xi\\ f_0,\dots,\widehat{f}_k,\dots,f_n\end{array}\right]_{\mathbb{T}}=\sum_{l=0}^n(-1)^l\mathrm{Res}\left[\begin{array}{c} h(\xi)\,f_l(\xi)\,H_l(\xi,t)\,d\xi\\ f_0,\dots,\widehat{f}_k,\dots,f_n\end{array}\right]_{\mathbb{T}}.$$

Taking into account the formula for toric residue (5) and the property of residue that

$$\mathrm{Res}\left[\begin{array}{c} g(\xi)\,d\xi\\ f_0,\dots,\widehat{f}_k,\dots,f_n\end{array}\right]_{\mathbb{T}}=0,\quad\text{if }\ g(\xi)\in\langle f_0(\xi),f_1(\xi),\dots,f_n(\xi)\rangle,$$

we see that

$$\mathrm{Res}\left[\begin{array}{c} h(\xi)\,W(\xi,t)\,d\xi\\ f_0,\dots,\widehat{f}_k,\dots,f_n\end{array}\right]_{\mathbb{T}}=(-1)^k\mathrm{Res}\left[\begin{array}{c} h(\xi)\,f_k(\xi)\,H_k(\xi,t)\,d\xi\\ f_0,\dots,\widehat{f}_k,\dots,f_n\end{array}\right]_{\mathbb{T}}.$$

Then, after the Σ-homogenization, the last equality together with (7) and analytic property of the toric residue (5) yields

$$H(x)=\mathrm{Res}_F(H(y)\,W(y,x))\mod\langle F_0(x),F_1(x),\dots,F_n(x)\rangle,$$

as required. $\square$

We can work over arbitrary algebraically closed subfield k of $\mathbb{C}$. By theorem 3.5, after taking a specialization $W(a)(x,y)$ of the polynomial $W(x,y)$ at the point $a \in X$, such that $F_0(a), F_1(a), \dots, F_n(a)$ do not have zeros at a, we get

$$H(a)(x) = \mathrm{Res}_F(H(y)W(a)(x,y))$$

in the quotient ring $S/\langle F_0(a)(x), F_1(a)(x), \dots, F_n(a)(x)\rangle$. If we expand $W(x,y)$ in the sum

$$W(x,y) = \sum_\mu \varphi_\mu(x) \cdot \psi_\mu(y), \tag{8}$$

then, for any polynomial $H(x) \in S_\alpha$

$$H(a)(x) = \sum_\mu \mathrm{Res}_F(H(y)\psi_\mu(y)) \cdot \varphi_\mu(a)(x) \mod \langle F_0(a)(x), F_1(a)(x), \dots, F_n(a)(x)\rangle,$$

where $\varphi_\mu(a)(x)$ is the specialization of $\varphi_\mu(x)$ at a, and the sum of residues is taken over the terms of degree $\alpha = \rho - \deg(\psi_\mu(y))$. Therefore, we have the following corollary of the theorem 3.5.

Corollary 3.6. *The polynomials $\{\varphi_\mu(a)(x)\}$ in the expansion (8) whose degrees are $\alpha = \rho - \deg(\psi_\mu(y))$, generates the specialization of the graded piece of the quotient ring $R(F) = S/\langle F_0, F_1, \dots, F_n\rangle$ of degree α at the point $a \in X$ such that $\{F_0(a) = F_1(a) = \dots = F_n(a) = 0\} = \emptyset$.*

The polynomial $H(x) \in S$ belongs to the ideal $\langle F_0(x), F_1(x), \dots, F_n(x)\rangle$ if and only if $\mathrm{Res}_F(H(y)\psi_\mu(y)) = 0$ for all polynomials $\{\psi_\mu(y)\}$ in the expansion (8).

The last statement in the corollary above claims that it is sufficient to check the vanishing of the residue in the theorem 2.4 only for finite set of polynomials $\{\psi_\mu(y)\}$ which can be done, for example, using Gröbner basis algorithm.

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