GAUSS MAP ON THE SPACE OF INTERVAL EXCHANGE TRANSFORMATIONS. FINITENESS OF THE INVARIANT MEASURE. LYAPUNOV EXPONENTS.

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ABSTRACT. We construct a map on the space of interval exchange transformations, which generalizes the classical Gauss map on the interval, related to continuous fraction expansion. This map is based on Rauzy induction, but unlike its relatives known up to now, the map is ergodic with respect to some *finite* absolutely continuous measure on the space of interval exchange transformations. We present the prescription for calculation of this measure based on technique developed by W.Veech for Rauzy induction.

We study Lyapunov exponents related to this map and show that when the number of intervals is m, and the genus of corresponding surface is g, there are $m-2g$ Lyapunov exponents, which are equal to zero, while the rest $2g$ ones are distributed into pairs $\theta_i = -\theta_{m-i+1}$. We present an explicit formula for the highest one, which proves in particular, that it is greater than zero.

1. INTRODUCTION

Consider an orientable measured foliation on a closed orientable surface M_g^2 of genus g with singularities of the saddle type. Throughout the paper we will assume, that the foliation has neither closed singular leaves, nor saddle connections. We will also assume, that the foliation is uniquely ergodic. A generic orientable measured foliation can be decomposed to ones which obey all the indicated properties (see [20]), as a consequence of unique ergodicity of a generic interval exchange transformation (see [7], [18]). Recall, that we can define an orientable measured foliation as a foliation of leaves of a closed 1-form ω. Any leaf of the orientable measured foliation as described above winds around the surface along one and the same cycle from the first homology group $H_1(M_g^2, \mathbb{R})$ of the surface, which is called asymptotic cycle. This

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cycle is just Poincaré dual to the cohomology class $[\omega]$ of corresponding 1-form. In a sense asymptotic cycle gives the first term of approximation of dynamics of leaves.

Study of further terms of approximation gives the following picture (see [19] for details). Computer experiments show, that taking the next term of approximation we get a two-dimensional subspace in $H_1(M_g^2, \mathbb{R})$, i.e., with a good precision leaves deviate from the asymptotic cycle not arbitrary, but inside one and the same two-dimensional subspace $\mathcal{H}^2$ in the first homology. Taking further steps $n = 3, ..., g$ of approximation we get subspaces $\mathcal{H}^k$ of dimension k for the k-th step; collection of the subspaces generates a flag $\mathcal{H}^1 \subset \mathcal{H}^2 \subset \cdots \subset \mathcal{H}^g$ of subspaces in the first homology group. The largest, g-dimensional subspace, gives a Lagrangian subspace in $2g$-dimensional symplectic space $H_1(M_g^2, \mathbb{R})$, with the intersection form considered as a symplectic form. We stop at level g since deviation from corresponding Lagrangian subspace is in a sense already negligible. The main conjecture of [19] claims existence of this asymptotic Lagrangian flag for almost all orientable measured foliations on surfaces as described above.

Having an orientable measured foliation on a surface, one can consider interval exchange transformation induced by the first return map on a piece of transversal. Taking shorter and shorter pieces of transversal we will get longer and longer pieces of leaf bounded by the point of first return. Joining the ends of the piece of leaf along transversal we get a closed cycle, representing an element of the first homology. The asymptotic behavior of this cycle is what we need to investigate. To trace modifications of our cycles we use special procedure for shortening our piece of transversal. Namely, we use iterates of Rauzy induction for corresponding interval exchange transformation (see [13] as well as later expositions in [18] and [6]). The transformation operator representing modification of our cycles after k steps of Rauzy induction is the product of k elementary matrices $A_{i_k} \cdots A_{i_1}$ related to each step of Rauzy induction. We now need to study properties of these products of matrices.

Though the mapping $\mathcal{T} : X \to X$ corresponding to Rauzy induction on the space X of interval exchange transformations is ergodic with respect to some absolutely continuous invariant measure on X ([18]), we can not immediately use multiplicative ergodic theorem to study products of matrices $A_{i_k} \cdots A_{i_1}$ since the invariant measure is not finite.

We construct another map $\mathcal{G} : X \to X$, which assigns to a point $x \in X$ some power $\mathcal{G}(x) = \mathcal{T}^{n(x)}(x)$ of the map $\mathcal{T}$ evaluated at x, where $n(x)$ depends on the point x. The numbers $n(x), n(\mathcal{G}(x)), \ldots$ here are analogous to the entries of continuous fraction expansion for a real number. In the simplest case of interval exchange transformation of two intervals the numbers $n(x), n(\mathcal{G}(x)), \ldots, n(\mathcal{G}^k(x)), \ldots$ are exactly the entries of corresponding continuous fraction, and the map $\mathcal{G}$ coincides with the classical Gauss map (up to duplication and conjugation). We prove that for any number of intervals the Gauss map $\mathcal{G}$ is ergodic with respect to some absolutely continuous invariant measure on X, and this measure is already *finite*.

Note that initial matrix-valued function $A(x)$ on X related to Rauzy induction induces a new cocycle, $B(x) = A(\mathcal{T}^{n(x)-1}(x)) \cdot A(\mathcal{T}^{n(x)-2}(x)) \cdots A(x)$. This time we are already able to apply Oseledec theorem to study products of matrices B. Consider the collection of corresponding Lyapunov exponents $\theta_1 \leq \cdots \leq \theta_m$.

We prove, that $\theta_{g+1} = \cdots = \theta_{m-g} = 0$, where g is the genus of the original surface. As for the rest Lyapunov exponents, we prove, that they are grouped into pairs $\theta_i = -\theta_{m-i+1}$. We calculate explicitly the largest Lyapunov exponent θ_1; in particular we show, that $\theta_1 > 0$.

We prove that Lyapunov exponents of the differential $D\mathcal{G}$ are represented by $\theta_1 + \theta_1, \theta_2 + \theta_1, \ldots, \theta_{m-1} + \theta_1$.

Presumably Lyapunov exponents $\theta_2, \ldots, \theta_g$ are also nonzero, and hence positive, and all of them have multiplicities one. This conjecture implies existence of asymptotic *Lagrangian* flag in the first homology of the surface, responsible for approximation of the leaves. Still to avoid overloading the paper we decided to discuss existence of asymptotic flag separately, in some other paper.

2. Interval exchange transformations and Rauzy induction

Recall the notion of interval exchange transformation. Consider an interval, and cut it into m subintervals of lengths $\lambda_1, \ldots, \lambda_m$. Now glue the subintervals together in another order, according to some permutation $\pi \in \mathfrak{S}_m$ and preserving the orientation. We again obtain an interval I of the same length, and hence we defined a mapping $T : I \to I$, which is called interval exchange transformation. Our mapping is piecewise linear, and it preserves the orientation and Lebesgue measure. It is singular at the points of cuts, unless two consecutive intervals separated by a point of cut are mapped to consecutive intervals in the image.

Remark 1. Note, that actually there are two ways to glue the subintervals "according to permutation π". We may send the interval number k to the place $\pi(k)$, or we may have the intervals in the image to appear in the order $\pi(1), \ldots, \pi(m)$. Following [18] we use the first way; under this choice the second way corresponds to permutation π^{-1}.

Given an interval exchange transformation T corresponding to a pair (λ, π), $\lambda \in \mathbb{R}_+^n$, $\pi \in \mathfrak{S}_n$, set $\beta_0 = 0$, $\beta_i = \sum_{j=1}^{i} \lambda_j$, and $X_i = [\beta_{i-1}, \beta_i[$. Define skew-symmetric $m \times m$-matrix $S(\pi)$ as follows:

$$\Omega_{ij}(\pi) = \begin{cases} 1 & \text{if } i < j \text{ and } \pi(i) > \pi(j) \\ -1 & \text{if } i > j \text{ and } \pi(i) < \pi(j) \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

Consider a translation vector

$$\delta = \Omega(\pi)\lambda \tag{2.2}$$

Our interval exchange transformation T is defined as follows:

$$T(x) = x + \delta_i, \qquad \text{for } x \in X_i,\ 1 \leq i \leq n$$

Note, that if for some $k < m$ we have $\pi\{1,\dots,k\} = \{1,\dots,k\}$, then our map T decomposes into two interval exchange transformations. We consider only the class $\mathfrak{S}^0_m$ of *irreducible* permutations — those which have no invariant subsets of the form $\{1,\dots,k\}$, where $1 \leq k < m$. We can also confine ourselves to the class of *nondegenerate* permutations — those which obey the property $\pi(k) + 1 \neq \pi(k+1)$, $1 \leq k < m$, since an interval exchange transformation defined by a degenerate permutation coincides with an interval exchange transformation of smaller number of intervals. Both subsets of permutations are invariant under operation of taking inverse.

Having an interval exchange transformation T corresponding to the pair (λ, π) one can construct a closed orientable surface M^2_g, a closed 1-form ω on M^2_g, and a nonselfintersecting curve γ in M^2_g, such that γ would be transversal to leaves of ω, and the induced Poincaré (first return) map $\gamma \to \gamma$ would coincide with the initial interval exchange transformation T (see corresponding constructions in [18] and in [7]). The genus g of the surface is defined by combinatorics of the permutation π as follows (see [18]).

Let $\pi \in \mathfrak{S}^0_m$. Define permutation $\sigma = \sigma(\pi)$ on $\{0, 1, \dots, m\}$ by

$$\sigma(j) = \begin{cases} \pi^{-1} - 1 & j = 0 \\ m & j = \pi^{-1} \\ \pi^{-1}(\pi(j) + 1) - 1 & \text{otherwise} \end{cases} \tag{2.3}$$

Let

$$S(j) = \{j, \sigma(j), \sigma^2(j), \dots\} \subset \{0, 1, 2, \dots, m\} \quad j = 0, 1, \dots, m \tag{2.4}$$

be the cyclic subset for the permutation σ. To each subset S of this form assign the vector $b_S \in \mathbb{R}^m$, which is presented in components as follows:

$$b^j_S = \chi^{j-1}_S - \chi^j_S \tag{2.5}$$

where

$$\chi^j_S = \begin{cases} 1 & \text{if } j \in S \\ 0 & \text{otherwise} \end{cases}$$

Let

$$\Sigma(\pi) := \{\text{set of cyclic subsets for } \sigma(\pi)\} \tag{2.6}$$

$$\Sigma_0(\pi) := \Sigma(\pi) \backslash S(0) \tag{2.7}$$

$$N(\pi) := \mathrm{Card}\Sigma(\pi) \tag{2.8}$$

According to [18] the genus g of the surface M_g^2 is

$$g = \frac{m - (N(\pi) - 1)}{2} \tag{2.9}$$

To each permutation $\pi \in \mathfrak{S}_m$ we assign $m \times m$ permutation matrix

$$P_{i,j}(\pi) = \begin{cases} 1 & \text{if } j = \pi(i), \\ 0 & \text{otherwise} \end{cases} \tag{2.10}$$

We denote by $\tau_k \in \mathfrak{S}_m$, $1 \le k < m$ the following permutation:

$$\begin{aligned} \tau_k &= \{1, 2, \ldots, k, k+2, \ldots, m, k+1\} \quad 1 \le k < m-1 \\ \tau_{m-1} &= \{1, 2, \ldots, m\} = id \end{aligned}$$

Permutation τ_k cyclically moves one step forward all the elements occurring after the element k.

Define the norm $|\lambda|$ of $\lambda \in \mathbb{R}_+^m$ to be $|\lambda| = \sum_{i=1}^m \lambda_i$. By Δ^{m-1} we denote the standard simplex $\Delta^{m-1} = \{\lambda \,|\, \lambda \in \mathbb{R}_+^m;\ |\lambda| = 1\}$. Having an interval exchange transformation, defined by a pair (λ, π), where vector $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}_+^m$, defines the lengths of subintervals, and π is a permutation, $\pi \in \mathfrak{S}_m$, we can renormalize vector λ to $\lambda/|\lambda| \in \Delta^{m-1}$. Interval exchange transformation corresponding to the pair $(\lambda/|\lambda|\,,\ \pi)$ is obviously conjugate to the initial one.

Now we remind construction of Rauzy induction [13]. Whenever it is possible we try to use notations in [18]. We also use some notations from [6].

Consider two maps $a, b : \mathfrak{S}_m^0 \to \mathfrak{S}_m^0$ on the set of irreducible permutations:

$$\begin{aligned} a(\pi) &= \pi \cdot \tau_{\pi^{-1}(m)}^{-1} \\ b(\pi) &= \tau_{\pi(m)} \cdot \pi \end{aligned} \tag{2.11}$$

where one should consider product of permutations as composition of operators — from right to left. Considering permutation as a map from one ordering of $1, 2, \ldots, m$ to another, operator b corresponds to the modification of the image ordering by cyclically moving one step forward those letters occurring after the image of the last letter in the domain, i.e., after the letter m. Operation a corresponds to the modification of the ordering of the domain by cyclically moving one step forward those letters occurring after one going to the last place, i.e., after $\pi^{-1}(m)$.

Note, that

$$(a(\pi))^{-1} = b(\pi^{-1}) \tag{2.12}$$

In components the maps a, b are as follows:

$$
(2.13)\qquad
\begin{aligned}
a(\pi)(j) &= \begin{cases} \pi(j) & j \le \pi^{-1}(m) \\ \pi(m) & j = \pi^{-1}(m)+1 \\ \pi(j-1) & \text{other } j \end{cases} \\
b(\pi)(j) &= \begin{cases} \pi(j) & \pi(j) \le \pi(m) \\ \pi(j)+1 & \pi(m) < \pi(j) < m \\ \pi(m)+1 & \pi(j) = m \end{cases}
\end{aligned}
$$

The Rauzy class $\mathfrak{R}(\pi_0)$ of an irreducible permutation π_0 is the subset of those permutations $\pi \in \mathfrak{S}_m^0$ which can be obtained from π_0 by some composition of mappings a and b. We will also denote by the same symbol $\mathfrak{R}(\pi_0)$ the oriented graph, which vertices are indexed by elements $\pi \in \mathfrak{R}(\pi_0)$, and which directed edges are either of the type $\pi \mapsto a(\pi)$, or of the type $\pi \mapsto b(\pi)$.

Denote by E identity $m\times m$-matrix, and by $I_{i,j}$ square $m\times m$-matrix, which has only one nonzero entry, which equals one, at the (i,j) place. For any $\pi \in \mathfrak{S}_m^0$ define matrices $A(\pi, a)$, $A(\pi, b)$ as follows:

$$
(2.14)\qquad
\begin{aligned}
A(\pi,a) &= (E + I_{\pi^{-1}(m),m}) \cdot P(\tau_{\pi^{-1}(m)}) = P(\tau_{\pi^{-1}(m)}) + I_{\pi^{-1}(m),m} \\
A(\pi,b) &= E + I_{m,\pi^{-1}(m)}
\end{aligned}
$$

Consider an interval exchange transformation T corresponding to a pair (λ, π), where $\lambda = (\lambda_1, \dots, \lambda_m) \in \Delta^{m-1}$, $\pi \in \mathfrak{S}_m^0$. Compare the lengths λ_m and $\lambda_{\pi^{-1}(m)}$ of the last subinterval in the domain and in the image of T. Suppose they are not equal. Let $\nu = \min(\lambda_m, \lambda_{\pi^{-1}(m)})$. Cut off an interval of the length ν from the right hand side of the initial interval and consider induction of the map T to the subinterval $[0, 1-\nu[$. According to [13] the new map would be again an interval exchange transformation of m subintervals corresponding to a pair (λ', π'), where

$$
(\lambda', \pi') = \begin{cases} (A^{-1}(\pi,a)\lambda\,,\ a(\pi)) & \lambda_m < \lambda_{\pi^{-1}(m)} \\ (A^{-1}(\pi,b)\lambda\,,\ b(\pi)) & \lambda_m > \lambda_{\pi^{-1}(m)} \end{cases}
$$

Rescaling the vector λ' we get the transformation

$$
(2.15)\qquad
\begin{aligned}
\mathcal{T} : \Delta^{m-1} \times \mathfrak{S}_m^0 &\to \Delta^{m-1} \times \mathfrak{S}_m^0 \\
(\lambda, \pi) &\mapsto \left(\tfrac{\lambda'}{|\lambda'|}, \pi'\right)
\end{aligned}
$$

Consider restriction of this map to invariant subsets of the form $\Delta^{m-1} \times \mathfrak{R}(\pi)$. In [18] Veech proves, that Rauzy induction $\mathcal{T}$ is conservative and ergodic on each $\Delta^{m-1} \times \mathfrak{R}(\pi)$. It admits unique up to a scalar multiple absolutely continuous invariant measure, but this measure is infinite.

To complete this section we make one more note on the requirements on measured foliations on closed orientable surfaces, which we will consider in this paper. We will consider only orientable measured foliations. We always assume, that the foliation does not have minima or maxima, i.e., all singularities are of saddle type only. We assume that there are no saddle connections and no closed singular leaves. In this case the foliation is minimal — every leaf is dense in the surface. A generic orientable measured foliation on a closed orientable surface can be decomposed to ones as described above in the following sense. Suppose a foliation has minima, maxima, and hence closed loops. Assume, that the corresponding closed 1-form has maximal rank, i.e., all periods of the form are rationally independent. Note, that this is the generic situation. One can decompose the surface into several components filled with closed leaves, and several components, where each nonsingular leaf would be closed. Component filled with closed leaves is homeomorphic either to a disk (a *trap*), or to a cylinder (a *neck*). Consider a minimal component, where each nonsingular leaf is dense. It is a compact orientable surface with several holes. Each hole, i.e., each component of the boundary is represented by a closed singular leaf. Shrinking each hole to a point we will get a closed surface with smooth foliation on it; critical points corresponding to the holes would be eliminated. By construction the foliation would be minimal. For more details see, e.g., [20].

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3. Gauss map $\mathcal{G}$

Fix some $\pi_0 \in \mathfrak{S}^0_m$ and confine ourselves to the class $\mathfrak{R}(\pi_0) = \mathfrak{R}$. We denote

$$\begin{aligned} (\lambda^{(k)}, \pi^{(k)}) &:= \mathcal{T}^k(\lambda, \pi) \\ (\lambda^{(0)}, \pi^{(0)}) &:= (\lambda, \pi) \end{aligned}$$

We subdivide each simplex $\Delta^{m-1} \times \pi$, $\pi \in \mathfrak{R}$ into two subsimplexes

$$\Delta^{m-1} \times \pi = \left(\Delta^+(\pi) \cup \Delta^-(\pi)\right) \times \pi$$

where

$$\begin{aligned} \Delta^+(\pi) &= \{\lambda \in \Delta^{m-1} \mid \lambda_m > \lambda_{\pi^{-1}(m)}\} \\ \Delta^-(\pi) &= \{\lambda \in \Delta^{m-1} \mid \lambda_m < \lambda_{\pi^{-1}(m)}\} \end{aligned} \tag{3.1}$$

Similarly define positive cones $\Lambda^+(\pi) \cup \Lambda^-(\pi) = \mathbb{R}^m_+$.

For almost all points on $\Delta^{m-1} \times \mathfrak{R}$ we can define the function

$$n(\lambda, \pi) = \min_{k=1,2,\ldots} k \ \text{ such that } \begin{cases} \lambda^{(k)} \in \Delta^-(\pi^{(k)}) & \text{when } \lambda \in \Delta^+(\pi) \\ \lambda^{(k)} \in \Delta^+(\pi^{(k)}) & \text{when } \lambda \in \Delta^-(\pi) \end{cases} \tag{3.2}$$

In other words we iterate Rauzy induction and count how many consecutive transformations of the same type (a or b, see (2.11), (2.13)) we can make.

Definition 1. We define the *Gauss map* $\mathcal{G}$ related to Rauzy induction $\mathcal{T}$ to be

$$\begin{gathered}\mathcal{G} : \sqcup_{\pi\in\mathfrak{R}}(\Delta^+(\pi)\sqcup\Delta^-(\pi)) \to \sqcup_{\pi\in\mathfrak{R}}(\Delta^+(\pi)\sqcup\Delta^-(\pi)) \\ \mathcal{G}(\lambda,\pi) := \mathcal{T}^{n(\lambda,\pi)}(\lambda,\pi)\end{gathered} \tag{3.3}$$

One should consider domain of $\mathcal{G}$ as $\sqcup_{\pi\in\mathfrak{R}}(\Delta^+(\pi)\sqcup\Delta^-(\pi))$ forgetting that simplexes $\Delta^+(\pi)$ and $\Delta^-(\pi)$ were once glued into one. Note that the map $\mathcal{G}$ maps simplexes Δ^+ to Δ^- and vice versa.

Define the following matrix-valued function $B(\lambda,\pi)$ on a subset of complete measure in $\sqcup_{\pi\in\mathfrak{R}}(\Delta^+(\pi)\sqcup\Delta_-(\pi))$ as

$$B(\lambda,\pi) := \prod_{k=1}^{n(\lambda,\pi)-1} A(\lambda^{(k)},\pi^{(k)}) \tag{3.4}$$

where matrix-valued function $A(\lambda,\pi)$ is defined by (2.14). By definition if $(\lambda',\pi') = \mathcal{G}(\lambda,\pi)$, then $\lambda' = B(\lambda,\pi)\cdot\lambda/\|B(\lambda,\pi)\cdot\lambda\|$ (see also explicit formulae (3.6) and (3.7) below). Note, that $\det B(\lambda,\pi) = \pm 1$.

We can give also a direct definition of $\mathcal{G}$ as follows. Let

$$\nu(\lambda,\pi) := \begin{cases} \max\limits_{n\ge 1} s_n^+ \mid s_n^+ \le \lambda_{\pi^{-1}(m)} & \text{when } \lambda_m < \lambda_{\pi^{-1}(m)} \\ \quad\text{where } s_n^+ = \underbrace{\lambda_m + \lambda_{m-1} + \dots + \lambda_{\pi^{-1}(m)+1} + \lambda_m + \lambda_{m-1} + \dots}_{n \text{ terms}} & \\ & \\ \max\limits_{n\ge 1} s_n^- \mid s_n^- \le \lambda_m & \text{when } \lambda_m > \lambda_{\pi^{-1}(m)} \\ \quad s_n^- = \underbrace{\lambda_{\pi^{-1}(m)} + \lambda_{\pi^{-1}(m-1)} + \dots + \lambda_{\pi^{-1}(\pi(m)+1)} + \lambda_{\pi^{-1}(m)} + \dots}_{n \text{ terms}} & \end{cases} \tag{3.5}$$

Note that the maximal possible number n involved in definition above coincides with $n(\lambda,\pi)$ in (3.2). Consider an interval exchange transformation T corresponding to a pair (λ,π), $\lambda\in\Delta^{m-1}$, $\pi\in\mathfrak{S}_m^0$. Cut off an interval of the length $\nu(\lambda,\pi)$ from the right hand side of the initial interval and consider induction of the map T to the subinterval $[0, 1-\nu(\lambda,\pi)]$. The new map would be again an interval exchange transformation of m subintervals corresponding to the pair (λ',π'). There would be two cases.

Case a $\lambda_m < \lambda_{\pi^{-1}(m)}$. In this case

$$\pi' = \pi\cdot\tau_{\pi^{-1}(m)}^{-q}$$

and

$$\lambda'_j = \begin{cases} \lambda_j & j < \pi^{-1}(m) \\ \lambda_{\pi^{-1}(m)} - \nu(\lambda,\pi) & j = \pi^{-1}(m) \\ \lambda_{m+\pi^{-1}(m)+q-j} & \pi^{-1}(m) < j \le \pi^{-1}(m) + q \\ \lambda_{\pi(j-q)} & \text{other } j \end{cases} \tag{3.6}$$

Case b $\lambda_m > \lambda_{\pi^{-1}(m)}$. In this case

$$\pi' = \tau^q_{\pi(m)} \cdot \pi$$

and

$$\lambda'_j = \begin{cases} \lambda_j & j < m \\ \lambda_m - \nu(\lambda,\pi) & j = m \end{cases} \tag{3.7}$$

One can see, that the matrix $B(\lambda,\pi)$ defined by (3.4) is the matrix of transformation (3.6) when $\lambda_m < \lambda_{\pi^{-1}(m)}$, and of transformation (3.7), when $\lambda_m > \lambda_{\pi^{-1}(m)}$.

Rescaling the vector λ' we get the transformation

$$\begin{aligned} \mathcal{G} : \bigsqcup_{\pi\in\mathfrak{R}} (\Delta^+(\pi) \sqcup \Delta^-(\pi)) &\to \bigsqcup_{\pi\in\mathfrak{R}} (\Delta^+(\pi) \sqcup \Delta^-(\pi)) \\ (\lambda,\pi) &\mapsto \left(\tfrac{\lambda'}{|\lambda'|}, \pi'\right) \end{aligned} \tag{3.8}$$

In other words at one step of the new induction we are shortening one and the same interval λ_m or $\lambda_{\pi^{-1}(m)}$, whichever is larger, as much as possible, cutting cyclically from its right-hand side intervals of lengths $\lambda_{\pi^{-1}(m)}, \lambda_{\pi^{-1}(m-1)}, \dots, \lambda_{\pi^{-1}(\pi(m)+1)}$ in the first case, and intervals of lengths $\lambda_m, \lambda_{m-1}, \dots, \lambda_{\pi^{-1}(m)+1}$ in the second case. The lengths of the rest intervals stay unchanged (modulo reenumeration in the first case).æ

4. Formulation of results

Theorem 1. *Let $m > 1$, and let $\mathfrak{R} \in \mathfrak{S}^0_m$ be a fixed Rauzy class. Then the Gauss map $\mathcal{G}$ is ergodic on*

$$\bigsqcup_{\pi\in\mathfrak{R}} \Delta^+(\pi) \sqcup \Delta^-(\pi)$$

and $\mathcal{G}$ admits an absolutely continuous invariant probability measure μ. The density of μ is the restriction to $\bigsqcup_{\pi\in\mathfrak{R}} \Delta^+(\pi) \sqcup \Delta^-(\pi)$, of a function on $\bigsqcup_{\pi\in\mathfrak{R}} \Lambda^+(\pi) \sqcup \Lambda^-(\pi)$ which is positive, rational, and homogeneous of degree $-m$.

Theorem 1 is proved in section 9

Consider some norm in the space $\mathbb{R}^m_+$. Define norm of a matrix $B \in GL(m)$ to be

$$\|B\| := \max_{\|v\|=1} \|B \cdot v\|.$$

Proposition 1. *Function* $\left|\log\|B(\lambda,\pi)\|\right|$ *is integrable over* $\bigsqcup_{\pi\in\mathfrak{R}}\Delta^+(\pi)\sqcup\Delta^-(\pi)$ *with respect to the measure* μ.

$$\int\limits_{\bigsqcup_{\pi\in\mathfrak{R}}(\Delta^+(\pi)\sqcup\Delta^-(\pi))} \left|\log\|B(\lambda,\pi)\|\right|\mu(dx) < \infty$$

Apply multiplicative ergodic theorem to products

$$\left(B^{(k)}\right)^{-1} = B^{-1}(\mathcal{G}^{k-1}(\lambda,\pi))\cdot\ldots\cdot B^{-1}(\mathcal{G}(\lambda,\pi))\cdot B^{-1}(\lambda,\pi) \tag{4.1}$$

of matrices $B^{-1}(\lambda,\pi)$ taken at trajectories of the Gauss map $\mathcal{G}$. Let $\theta_1\geq\theta_2\geq\cdots\geq\theta_m$ be corresponding Lyapunov exponents.

Theorem 2. *The middle* $m-2g$ *Lyapunov exponents are equal to zero*

$$\theta_{g+1}=\theta_{g+2}=\cdots=\theta_{m-g}=0$$

The rest $2g$ *Lyapunov exponents are distributed in pairs*

$$\theta_k = -\theta_{m-k+1} \qquad \textit{for } \ k=1,\ldots,g$$

One can consider differential $D_{(\lambda,\pi)}\mathcal{G}$ as another "matrix-valued function" on the space $\bigsqcup_{\pi\in\mathfrak{R}}\Delta^+(\pi)\sqcup\Delta^-(\pi)$, and compute collection of corresponding Lyapunov exponents. The dimension of the space is $m-1$, so there would be $m-1$ Lyapunov exponents for the differential.

Proposition 2. *Collection of Lyapunov exponents for the differential* $D\mathcal{G}$ *of the Gauss map coincides with the collection*

$$\theta_1+\theta_1,\theta_2+\theta_1,\ldots,\theta_{m-1}+\theta_1$$

Theorem 3. *The largest Lyapunov exponent* θ_1 *equals*

$$\theta_1 = -\sum_{\pi\in\mathfrak{R}}\int\limits_{\Delta^\pm(\pi)}\left(\log\|B^{-1}(\lambda,\pi)\cdot\lambda\| - \log\|\lambda\|\right)d\mu =$$

$$= \frac{1}{m}\sum_{\pi\in\mathfrak{R}}\int\limits_{\Delta^\pm(\pi)}\det D\mathcal{G}\,d\mu =$$

$$= -\sum_{\pi\in\mathfrak{R}}\int\limits_{\Delta^\pm(\pi)}\log(1-\nu(\lambda,\pi))d\mu = \tag{4.2}$$

$$= \sum_{\pi\in\mathfrak{R}}\int\limits_{\Delta^\pm(\pi)}\left|\log(1-\lambda_m)-\log(1-\lambda_{\pi^{-1}(m)})\right|d\mu \tag{4.3}$$

Corollary 1. *The highest Lyapunov exponent is strictly positive,* $\theta_1>0$.
Proof. □

Conjecture 1. *The top g Lyapunov exponents are distinct and strictly positive*

$$\theta_1 > \theta_2 > \cdots > \theta_g > 0$$

5. Rauzy induction in dimension two, Euclidean algorithm, and classical Gauss map

Before treating the general case we want to illustrate relation between the Gauss map $\mathcal{G}$ and Rauzy induction $\mathcal{T}$ in the simplest case, when we have interval exchange transformation of just two intervals, $m = 2$. In this case Rauzy class $\mathfrak{R}(\pi_0)$ of the permutation $\pi_0 = \{2, 1\}$ contains the only element $\mathfrak{R}(\pi_0) = \{\pi_0\}$; the simplex $\Delta^{m-1} = \Delta^1$ is just the interval, and hence the space $\Delta^{m-1} \times \mathfrak{R}(\pi_0)$ is just an interval.

We remind the definition of the Gauss map $G : I \to I$ on the interval $I = [0, 1]$

$$G(x) = \left\{\frac{1}{x}\right\}$$

where $\{\}$ denotes the fractional part of the number. Gauss map is ergodic; the density function of corresponding invariant probability measure is as follows

$$g(x) = \frac{1}{\log 2} \cdot \frac{1}{x+1}$$

We remind, that Gauss map G is related to one step of Euclidean algorithm in the following sense. Suppose we have two intervals. Rescale them proportionally so that the longer one would have a unit length. Let the shorter one have length x, $0 < x \leq 1$ after rescaling. We cut the shorter interval from the larger one as many times as possible, that is

$$n_1 = \left[\frac{1}{x}\right]$$

times. Consider the remainder, which has the length

$$\delta = 1 - \left[\frac{1}{x}\right] \cdot x = x\left(\frac{1}{x} - \left[\frac{1}{x}\right]\right) = x\left\{\frac{1}{x}\right\}$$

We got two intervals of the lengths x and δ, $\delta < x$. Rescale them proportionally to make the larger interval have the unit length. Then the smaller one would have the length

$$G(x) = \frac{\delta}{x} = \left\{\frac{1}{x}\right\}$$

Note, that if we will go on, then the consecutive integer numbers $n_1, n_2, \ldots$ would give the continuous fraction expansion of the real number x

$$x = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cdots}}}$$

Now let us consider Rauzy induction. In coordinates (λ_1, λ_2), where $\lambda_1 + \lambda_2 = 1$, and $\lambda_1, \lambda_2 \geq 0$, Rauzy induction $\mathcal{T} : \Delta^1 \to \Delta^1$ is as follows

$$\mathcal{T}(\lambda_1, \lambda_2) = \begin{cases} \left(\frac{\lambda_1}{\lambda_2}, \frac{\lambda_2 - \lambda_1}{\lambda_2}\right) & \text{if } \lambda_1 < \lambda_2 \\ \\ \left(\frac{\lambda_1 - \lambda_2}{\lambda_1}, \frac{\lambda_2}{\lambda_1}\right) & \text{if } \lambda_1 > \lambda_2 \end{cases}$$

Density $f_{\pi_0}(\lambda)$ of the invariant measure, computed according prescription in [18] equals

$$f_{\pi_0}(\lambda) = \frac{1}{\lambda_1 \lambda_2}$$

The map $\mathcal{T}$ corresponds to "slow", or "step by step" Euclidian algorithm. This time we use slightly different normalization in comparison with one used in the definition of the Gauss map G above. Having a pair of intervals we proportionally rescale them to make the sum of their lengths λ_1 and λ_2 have the unit length. Then we cut the smaller interval from the larger one, but this time only once. Then we again proportionally rescale the intervals, to make the sum of their lengths have the unit length. We called such map "slow" Euclidian algorithm, because several iterations of this process lead to one step of a usual Euclidean algorithm, when we cut from the larger interval the largest possible number of the smaller ones at once.

In coordinate $x \in I = [0,1]$, $\lambda_1 = x$, $\lambda_2 = 1 - x$ Rauzy induction $\mathcal{T} : I \to I$ is represented as follows

$$\mathcal{T}(x) = \begin{cases} \frac{1}{1-x} - 1 & \text{if } 0 \leq x < \frac{1}{2}, \\ \\ 2 - \frac{1}{x} & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

Density of invariant measure in this coordinates is represented by the function

$$\rho(x) = \frac{1}{x(1-x)} \tag{5.1}$$

and integral of $\rho(x)$ over the interval obviously diverges though the map $\mathcal{T}$ is ergodic.

It would be easy to predict that invariant measure would not be finite. Consider an orbit of some point $x_0 \in I$ under action of $\mathcal{T}$. For almost all initial points after certain number of steps we will get close to the endpoint of the interval, say, to the point 0. Then the orbit will stay near this endpoint for a long time, since

$$\mathcal{T}(x) - x = o(x) \qquad \text{as } x \sim 0$$

Let us improve the situation as follows. We speed up Rauzy induction to make it correspond to the usual Euclidean algorithm. That is for any $(\lambda_1, \lambda_2) \in \Delta^1$ define

$$n_1(\lambda) = \begin{cases} \left[\frac{\lambda_2}{\lambda_1}\right] & \text{if } \lambda_1 < \lambda_2 \\ \\ \left[\frac{\lambda_1}{\lambda_2}\right] & \text{if } \lambda_1 \geq \lambda_2 \end{cases}$$

Now define

$$\mathcal{G}(\lambda) = \mathcal{T}^{n(\lambda)}(\lambda)$$

In other words, starting with the point λ we iterate the map $\mathcal{T}$ while the iterates obey the same type of inequality $\lambda_1 < \lambda_2$ or $\lambda_1 > \lambda_2$ as the initial point. Note, that by definition image of $\mathcal{G}$ obeys the opposite type of inequality.

The new induction in coordinates λ before normalization is as follows:

$$(\lambda_1, \lambda_2) \mapsto \begin{cases} \left(\lambda_1, \lambda_2 - \left[\frac{\lambda_2}{\lambda_1}\right]\right) & \text{if } \lambda_1 < \lambda_2 \\ \\ \left(\lambda_1 - \left[\frac{\lambda_1}{\lambda_2}\right], \lambda_2\right) & \text{if } \lambda_1 \geq \lambda_2 \end{cases}$$

The direct definition of $\mathcal{G}$ in coordinate $x \in I = [0,1]$, $\lambda_1 = x$, $\lambda_2 = 1 - x$: is as follows:

$$\mathcal{G}(x) = \begin{cases} \frac{1}{1+\frac{1}{x}} & \text{if } \lambda_1 < \lambda_2 \\ \\ 1 - \frac{1}{1+\frac{1}{1-x}} & \text{if } \lambda_1 > \lambda_2 \end{cases}$$

Density of the new invariant measure is

$$\rho(\lambda) = \begin{cases} \frac{1}{\lambda_1+\lambda_2} \cdot \frac{1}{\lambda_2} & \text{if } \lambda_1 < \lambda_2 \\ \\ \frac{1}{\lambda_1+\lambda_2} \cdot \frac{1}{\lambda_1} & \text{if } \lambda_1 > \lambda_2 \end{cases}$$

and now it is already finite.

In coordinate x corresponding normalized probability measure $\mu = \rho(x)\,dx$ is as follows

$$\rho(x) = \begin{cases} \frac{1}{2\ln 2} \cdot \frac{1}{1-x} & \text{if } 0 \leq x < \frac{1}{2}, \\ \\ \frac{1}{2\ln 2} \cdot \frac{1}{x} & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

Note, that up to duplication and conjugation the maps G and $\mathcal{G}$ are the same.

$$\mathcal{G}(x) = \begin{cases} \frac{1}{2}\left(1 + G\left(\frac{x}{1-x}\right)\right) & \text{if } 0 \le x < \frac{1}{2}, \\ \frac{1}{2}G\left(\frac{1-x}{x}\right) & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

6. Construction of the invariant measure

In this section we remind construction of the *space of zippered rectangles* from [18]. Then we define some particular subspace in it and an automorphism of the subspace, which projects to the Gauss map $\mathcal{G}$. Finally we define a measure on the space of interval exchange transformations invariant under $\mathcal{G}$. Since we are extensively using the technique in [18] we need to remind briefly some definitions and results from there.

For $\pi \in \mathfrak{S}_m^0$ define $H(\pi) \subset \mathbb{R}^m$ as the annulator of the system of vectors b_S, $S \in \Sigma(\pi)$ (see (2.5))

$$H(\pi) = \{h \in \mathbb{R}^m \mid h \cdot b_S = 0 \text{ for all } S \in \Sigma(\pi)\} \tag{6.1}$$

Remark 2. There is a natural local identification of the space $H(\pi)$ with $H_1(M_g^2; \mathbb{R})$ (see Proposition 4 and Remark 3 below).

Define the parallelepiped $Z(h, \pi)$ to be the set of solutions $a \in \mathbb{R}^m$ to the following system of equations and inequalities (which are equations (2.3) and inequalities (3.1) in [18]):

$$\begin{array}{rcll} h_i - a_i & = & h_{\sigma(i)+1} - a_{\sigma(i)} & (0 \le i \le m) \\ h_i & \ge & 0 & (1 \le i \le m) \\ a_i & \ge & 0 & (1 \le i < m) \\ -h_{\pi^{-1}m} & \le a_m & \le h_m & \\ a_{\pi^{-1}m} & \le & h_{\pi^{-1}m+1} & \\ a_i & \le & \min(h_i, h_{i+1}) & (0 < i < m,\ i \ne \pi^{-1}m) \end{array} \tag{6.2}$$

and define the cone

$$H^+(\pi) = \{h \in H(\pi) \mid Z(h,\pi) \text{ is nonempty}\} \tag{6.3}$$

The *zippered rectangles space of type* π is the set of triples (λ, h, a), $\lambda \in \mathbb{R}_+^m, h \in H^+(\pi), a \in Z(h,\pi)$. Parameters h and a are responsible for the structure of the Riemann surface corresponding to the interval exchange transformation (λ, π) (see [18] for details).

Define $\Omega(\mathfrak{R})$ to be the set of zippered rectangles (λ, h, a, π) such that $\pi \in \mathfrak{R}$ is in a given Rauzy class $\mathfrak{R}$, and $\lambda \cdot h = 1$. Define also a codimension-one subspace $\Upsilon(\mathfrak{R}) \subset \Omega(\mathfrak{R})$ by additional constraint $|\lambda| = 1$.

In [18] Veech defines a flow $P^t, t \in \mathbb{R}$ on $\Omega(\mathfrak{R})$, $P^t(\lambda, h, a, \pi) = (e^t\lambda, e^{-t}h, e^{-t}a, \pi)$, and a mapping $\mathcal{U} : \Omega(\mathfrak{R}) \to \Omega(\mathfrak{R})$

$$\mathcal{U}(\lambda, h, a, \pi) = \begin{cases} (A^{-1}(\pi, a)\lambda, A^T(\pi, a)h, a', a(\pi)) & \text{when } \lambda_m < \lambda_{\pi^{-1}m} \\ (A^{-1}(\pi, b)\lambda, A^T(\pi, b)h, a'', b(\pi)) & \text{when } \lambda_m > \lambda_{\pi^{-1}m} \end{cases}$$

where matrices $A(\pi, a)$ and $A(\pi, b)$ are defined by equations (2.14); transformations $a(\pi), b(\pi)$ are defined by eqrefeq:ab (and (2.13)); and vectors a', a'' are defined as follows:

$$\text{(6.4)} \qquad \begin{aligned} a'_j &= \begin{cases} a_j & j < \pi^{-1}m \\ h_{\pi^{-1}m} + a_{m-1} & j = \pi^{-1}m \\ a_{j-1} & \pi^{-1}m < j \leq m \end{cases} \\ a''_j &= \begin{cases} a_j & 0 \leq j < m \\ -(h_{\pi^{-1}m} - a_{\pi^{-1}m-1}) & j = m \end{cases} \end{aligned}$$

Define $t(x), x \in \Upsilon(\mathfrak{R})$, by $t(x) = -\log\big(1 - \min(\lambda_m, \lambda_{\pi^{-1}m})\big)$. Consider a mapping $\mathcal{S} : \Upsilon(\mathfrak{R}) \to \Upsilon(\mathfrak{R})$, $\mathcal{S}x = \mathcal{U}P^{t(x)}x$.

The following measure

$$\text{(6.5)} \qquad \eta = \sum_{\pi \in \mathfrak{R}} c(\pi)\eta_\pi$$

on $\Upsilon(\mathfrak{R})$ is constructed in [18] as a measure invariant under transformation $\mathcal{S}$. Here $c(\pi)$ are the constants,

$$\text{(6.6)} \qquad c(\pi) = \Big(\text{Volume of fundamental domain in } \mathbb{Z}^m \cap H(\pi)\Big)^{-1}$$

and

$$\eta_\pi = \int_{\Delta^{m-1}} \int_{H^+_\lambda} \int_{Z(h,\pi)} \xi_h(da)\, \frac{\mu_\lambda(dh)}{\|Q\lambda\|}\, \omega_{m-1}(d\lambda)$$

Here $H^+_\lambda = \{h \in H^+(\pi) \,|\, h \cdot \lambda = 1\}$; $\xi_h(da)$ is Euclidean measure on $Z(h, \pi)$ in dimension $N(\pi) - 1$; $\mu_\lambda(dh)$ is the measure on H^+_λ induced by Euclidean metric; $\|Q\lambda\|$ is the Euclidean norm of the orthogonal projection of the vector λ on $H(\pi)$; and $\omega_{m-1}(d\lambda)$ is Euclidean measure on Δ^{m-1}.

Finally we remind that the following diagram

$$\begin{array}{ccc} \Upsilon(\mathfrak{R}) & \xrightarrow{\mathcal{S}} & \Upsilon(\mathfrak{R}) \\ {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \rho} \\ \Delta^{m-1} \times \mathfrak{R} & \xrightarrow{\mathcal{T}} & \Delta^{m-1} \times \mathfrak{R} \end{array}$$

is commutative (see [18]), where $\rho : (\lambda, h, a, \pi) \mapsto (\lambda, \pi)$ is the natural projection. Hence the measure $\rho\eta$ is invariant under Rauzy induction $\mathcal{T}$ on the space $\Delta^{m-1} \times \mathfrak{R}$ of interval exchange transformations.

* * *

Having reminded constructions in [18] we now modify them to get a measure μ on the space of interval exchange transformations invariant under the Gauss map $\mathcal{G}$.

Define the parallelepipeds

$$Z^+(h,\pi) = \{a \in Z(h,\pi) \,|\, a_m \geq 0\}$$
$$Z^-(h,\pi) = \{a \in Z(h,\pi) \,|\, a_m \leq 0\} \tag{6.7}$$

Define the subcones

$$H^{++}(\pi) = \{h \in H^+(\pi) \,|\, Z^+(h,\pi) \text{ is nonempty}\}$$
$$H^{+-}(\pi) = \{h \in H^+(\pi) \,|\, Z^-(h,\pi) \text{ is nonempty}\} \tag{6.8}$$

For a given Rauzy class $\mathfrak{R}$ define

$$\Omega^+(\mathfrak{R}) = \{(\lambda, h, a, \pi) \in \Omega(\mathfrak{R}) \,|\, \lambda \in \Delta^+(\pi); h \in H^{++}(\pi); a \in Z^+(h,\pi)\}$$
$$\Omega^-(\mathfrak{R}) = \{(\lambda, h, a, \pi) \in \Omega(\mathfrak{R}) \,|\, \lambda \in \Delta^-(\pi); h \in H^{+-}(\pi); a \in Z^-(h,\pi)\}$$

Define also

$$\Upsilon^+(\mathfrak{R}) = \Upsilon(\mathfrak{R}) \cap \Omega^+(\mathfrak{R})$$
$$\Upsilon^-(\mathfrak{R}) = \Upsilon(\mathfrak{R}) \cap \Omega^-(\mathfrak{R})$$
$$\Upsilon^\pm(\mathfrak{R}) = \Upsilon^+(\mathfrak{R}) \cup \Upsilon^-(\mathfrak{R})$$

Consider the map $\mathcal{F} : \Upsilon^\pm(\mathfrak{R}) \to \Upsilon(\mathfrak{R})$ as follows:

$$\mathcal{F}(\lambda, h, a, \pi) = \mathcal{S}^{n(\lambda,\pi)}(\lambda, h, a, \pi)$$

where $n(\lambda, \pi)$ is defined by (3.2).

Lemma 6.1. *The map $\mathcal{F}$ is the induction of the map $\mathcal{S}$ to the subspace $\Upsilon^\pm(\mathfrak{R}) \subset \Upsilon(\mathfrak{R})$.*

Proof. We need to prove, that the image of $\mathcal{F}$ belongs to $\Upsilon^\pm(\mathfrak{R})$, and, that $n(\lambda, \pi)$ is the first return time, i.e., the "time", when trajectory of a point $x = (\lambda, h, a, \pi) \in \Upsilon^\pm(\mathfrak{R})$ returns to $\Upsilon^\pm(\mathfrak{R})$ under iterations of the mapping $\mathcal{S}$. Suppose $\lambda \in \Delta^+(\pi)$. Then $x^{(1)} = (\lambda^{(1)}, h^{(1)}, a^{(1)}, \pi^{(1)}) = \mathcal{S}x$ is obtained by transformation "of the type b". Recall the remark in [18], saying that the image a' in (6.4) of the transformation of the "type a", $\lambda_m < \lambda_{\pi^{-1}m}$, satisfies $a'_m \geq 0$, and the image a'' in (6.4) of the transformation of the "type b", $\lambda_m > \lambda_{\pi^{-1}m}$, satisfies $a''_m \leq 0$. Hence, if $\lambda^{(1)} \in \Delta^+(\pi^{(1)})$ and $a^{(1)} \neq 0$, then the point $x^{(1)} = \mathcal{S}$ does not belong to $\Upsilon^\pm(\mathfrak{R})$ since $a^{(1)} < 0$. The first time the iterate would get back to the space $\Upsilon^\pm(\mathfrak{R})$ is the first time vector $\lambda^{(k)} = \mathcal{T}^k\lambda$ would get to the simplex of the type Δ^- (we neglect the set of measure zero of the points $\{x = (\lambda, h, a, \pi) \in \Upsilon(\mathfrak{R}) \mid a = 0\}$). But this is exactly the definition (3.2) of the function $n(\lambda, \pi)$.

The case, when we have $\lambda \in \Delta^-(\pi)$ for the initial point is analogous to one discussed above. □

Corollary 2. *The map $\mathcal{F}$ is almost everywhere one-to-one map on $\Upsilon^{\pm}(\mathfrak{R})$. The measure η from* (6.5) *confined to $\Upsilon^{\pm}(\mathfrak{R})$ is invariant under $\mathcal{F}$.*
Proof. □

Lemma 6.2. *The following diagram is commutative*

$$\begin{array}{ccc} \Upsilon^{\pm}(\mathfrak{R}) & \xrightarrow{\mathcal{F}} & \Upsilon^{\pm}(\mathfrak{R}) \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\rho} \\ \bigsqcup\limits_{\pi\in\mathfrak{R}} (\Delta^+(\pi) \sqcup \Delta^-(\pi)) & \xrightarrow{\mathcal{G}} & \bigsqcup\limits_{\pi\in\mathfrak{R}} (\Delta^+(\pi) \sqcup \Delta^-(\pi)) \end{array}$$

Proof. This is just a straightforward corollary of definitions $\mathcal{F} = \mathcal{S}^{n(\lambda,\pi)}$; $\mathcal{G} = \mathcal{T}^{n(\lambda,\pi)}$; and of commutativity of the initial diagram above. □

Define measure μ on $\bigsqcup_{\pi\in\mathfrak{R}} (\Delta^+(\pi) \sqcup \Delta^-(\pi))$ as $\mu = \rho\eta$.

Theorem 4. *Let $m > 1$, and let $\mathfrak{R}$ be a Rauzy class. The Gauss map $\mathcal{G}$ on the space of interval exchange transformations $\bigsqcup_{\pi\in\mathfrak{R}} (\Delta^+(\pi) \sqcup \Delta^-(\pi))$ admits the invariant measure*

$$\mu = \sum_{\pi\in\mathfrak{R}} c(\pi)\Big(f_\pi^+\omega^+(\pi) + f_\pi^-\omega^-(\pi)\Big) \otimes \delta_\pi$$

where $\delta_\pi, \pi \in \mathfrak{R}$, is the unit mass at π; $c(\pi)$ are constants specified above; and $\omega^+(\pi)$ ($\omega^-(\pi)$) is the Euclidean measure on $\Delta^+(\pi)$ ($\Delta^-(\pi)$). For each $\pi \in \mathfrak{R}$ the density f_π^+ (correspondingly f_π^-) is the restriction to $\Delta^+(\pi)$ (correspondingly $\Delta^-(\pi)$) of a function which is rational, positive, and homogeneous of degree $-m$ on $\mathbb{R}_+^m$.

The measure μ is finite.

The invariance of the measure follows from its definition. The statement about the concrete form of the measure is just the original theorem 11.6 in [18] for the initial measure invariant under Rauzy induction $\mathcal{T}$. What is new (and rather essential for us) is that the measure μ is now *finite*, which would be proved in the next two sections.

Morally, we claim, that the fiber $\rho^{-1}(\lambda, \pi)$ is "iceberg-like", i.e., there is a huge "underwater part" specified by inequality $a_m < 0$ for $\lambda \in \Delta^+$ (and $a_m > 0$ for $\lambda \in \Delta^-$) which gives an impact to the measure leading to it infiniteness; while the rest part of the "iceberg", which is "above the water", and which volume gives us our density function, leads to the finite measure.

7. The cones $H^{++}(\pi, W)$ and $H^{+-}(\pi, W)$

This section is parallel to §12 in [18], but dealing with the spaces $H^{++}(\pi)$ and $H^{+-}(\pi, W)$ we are able to improve the estimate of Proposition 12.8 in [18]. Here we would not exclude the subsets containing m and $\pi^{-1}m$ anymore.

Let $m > 1$, and fix $\pi \in \mathfrak{S}^0_m$. Consider $W \subset \{1, 2, \ldots, m\}$ such that $W \neq \emptyset$; $W \neq \{1, 2, \ldots, m\}$. Define $\Sigma_0(W)$ (cf. (2.7)) to be the set of $S \in \Sigma_0(\pi)$ such that

$$T(S) = S \cup \{S+1\} \subset W$$

Here $\{S+1\} = \{j+1 | j \in S\ , j \neq m\}$.

Next define $H^{++}(\pi, W)$ ($H^{+-}(\pi, W)$) to be the subset of those $h \in H^{++}(\pi)$ (correspondingly , $H^{+-}(\pi)$) which are supported on W i.e., $h_j = 0, j \notin W, h \in H^{++}(\pi)$ (correspondingly $h_j = 0, j \notin W, h \in H^{+-}(\pi)$). We use the same definition for $H^+(\pi, W)$ as in [18], except that we do not assume $\pi^{-1}m$, $m \notin W$ anymore, unless it is specially indicated.

Lemma 7.1. *In both of the following cases*

(1) $m \in W$, $\pi^{-1}m \notin W$ *and* $h \in H^{+-}(\pi, W)$, $a \in Z^-(h, \pi)$;
(2) $m \notin W$, $\pi^{-1}m \in W$; *and* $h \in H^{++}(\pi, W)$, $a \in Z^+(h, \pi)$;

the following equality is valid:

$$0 \le a_j \le h_j, h_{j+1} \qquad (0 \le j \le m)$$

Proof. Case (1): $h \in H^{+-}(\pi, W)$, $a \in Z^-(h, \pi)$, and $m \in W$, $\pi^{-1}m \notin W$.
In this case $a_m \le 0$. Since $\pi^{-1}m \notin W$, we get $h_{\pi^{-1}m} = 0$. Since by definition $h_{m+1} = 0$ we may combine equation

$$h_{\pi^{-1}m} - a_{\pi^{-1}m} = h_{m+1} - a_m$$

from (6.2) with inequality $a_{\pi^{-1}(m)} \ge 0$ to obtain

$$0 \le a_{\pi^{-1}m} = a_m \le 0,$$

Combining this with inequalities from (6.2) we prove the lemma for this case.

Case (2): $h \in H^{++}(\pi, W)$, $a \in Z^+(h, \pi)$, and $m \notin W$, $\pi^{-1}m \in W$.
In this case $a_m \ge 0$. Since $m \notin W$, we get $h_m = 0$. Since from (6.2) $a_m \le h_m$, we obtain $0 \le a_m \le h_m = 0$, and hence

$$a_m = 0. \tag{7.1}$$

Using the following equation from (6.2)

$$h_{\pi^{-1}m} - a_{\pi^{-1}m} = h_{m+1} - a_m$$

we get

$$a_{\pi^{-1}m} = h_{\pi^{-1}m}$$

Combining this equation and equation 7.1 with inequalities (6.2) we complete the proof of Lemma 7.1 □

Lemma 7.2. *In both of the following cases*

(1) $m \in W$, $\pi^{-1}m \notin W$ *and* $h \in H^{+-}(\pi, W)$, $a \in Z^{-}(h, \pi)$;

(2) $m \notin W$, $\pi^{-1}m \in W$; *and* $h \in H^{++}(\pi, W)$, $a \in Z^{+}(h, \pi)$;

the following strict inequality is valid:

$$\dim H^{+\pm}(\pi, W) + \operatorname{Card} \Sigma_0(W) < \operatorname{Card} W$$

Proof. The proof is the same as the proof of Proposition 12.8 in [18]. □

Now we consider one more case. We stress that the statement below is formulated for the subcone $H^{+}(\pi, W) \subset H^{+}(\pi)$ in the "old" cone from [18].

Lemma 7.3. *Let* $\pi^{-1}m$, $m \in W$, *where* W *is as above. Then*

$$\dim H^{+}(\pi, W) + \operatorname{Card} \Sigma_0(W) < \operatorname{Card} W$$

Proof. We have to consider three cases separately.

Case (i). $(S(m) \cup \{S(m) + 1\}) \setminus \{0\} \subset W$.

(Here we do not assume, that necessarily $0 \in S(m)$.) In this case we apply the arguments similar to those, which follow Lemma 12.3 in [18]. Slight modifications are as follows. Choose $i \in W$, $i \neq m$, so that $i + 1 \notin W$, and define $l \geq 1$ to be the first integer such that one of $\sigma^{l}i$, $\sigma^{l}i+1$ fails to belong to W. Since $W \neq \{1, 2, \ldots, m\}$ and $W \neq \emptyset$ such i exists. Note, that from assumptions for this case it follows that $i \notin S(m) = S(\pi^{-1}m)$. It means that $i \neq \pi^{-1}m$, m, and $\sigma^{l}i \neq \pi^{-1}m$, m. Hence $a_i = 0$ because $h_{i+1} = 0$, and $a_{\sigma^{l}i} = 0$ because one of $h_{\sigma^{l}i}, h_{\sigma^{l}i+1}$ is equal to zero. From this point we can apply the same arguments as in [18] — we get additional equation

$$h_i - h_{\sigma i+1} + h_{\sigma i} - \cdots + h_{\sigma^{l-1}i} - h_{\sigma^{l}i+1} = 0 \qquad (h \in H^{+}(\pi, W)), \tag{7.2}$$

(cf. equation (12.6) in [18]) and we rewrite it as $h \cdot b = 0$, $(h \in H^{+}(\pi, W))$. Then we prove, that vector b and vectors b_S, $S \in \Sigma_0(W)$ are linearly independent. We skip special consideration of the case $H^{+}(\pi, W) = \{0\}$. Since by construction the sets $\{i, \sigma i + 1, \ldots, \sigma^{l-1}i, \sigma^{l}i + 1\}$ and $\cup_{S \in \Sigma_0(W)}(S \cup \{S + 1\})$ do not intersect, linear dependence holds in only case when equation 7.2 is tautologically trivial on $H^{+}(\pi, W)$, which contradicts Lemma 12.7 in [18]. As b_S, $S \in \Sigma_0(W)$, and b (restricted to W) are orthogonal to $H^{+}(\pi, W)$ and linearly independent we get desired inequality, and prove case (i).

Case (ii). $S(m) \notin \Sigma_0(W)$ and $0 \notin S(m)$.

In this case we get additional vector b, $b \perp H^{+}(\pi, W))$, independent from b_S, $S \in \Sigma_0(W)$ as a restriction of $b_{S(m)}$ to W (i.e., by letting $b_j = 0$, for all $j \notin W$). Indeed, this restriction is nontrivial, because $b_{\pi^{-1}m} = 1$ since $\pi^{-1}m \in W$, and $\pi^{-1}m \in S(m)$, while $\pi^{-1}m - 1 = \sigma 0 \notin S(m)$. And this vector is independent from b_S, $S \in \Sigma_0(W)$

since $\pi^{-1}m \notin S$ for all $S \in \Sigma_0(W)$, and hence component $\pi^{-1}m$ vanishes for all these vectors.

Case (iii). $(S(m)\backslash\{0\}) \cup \{S(m)+1\} \not\subset W$ and $0 \in S(m)$.
Note that in this case $S(m) = S(\pi^{-1}m) = S(0)$. Choose $i \in W$, $i \neq m$, so that $i+1 \notin W$. If $i \notin S(m)$, further proof is analogous to case (i). Suppose $i \in S(m)$. We represent orbit $S(m)$ as follows

$$\{m \mapsto \sigma m \mapsto \cdots \mapsto \sigma^p m = 0 \mapsto \sigma 0 = \pi^{-1}m - 1 \mapsto \cdots \mapsto \sigma^q 0 = \pi^{-1}m \mapsto m\}$$

.

Suppose $i = \sigma^k m$, $1 \le k < p$. Define $l \ge k+1$ to be the first integer such that one of $\sigma^l m$, $\sigma^l m + 1$ fails to belong to W. We note, that $l \le p$. If $l < p$, then since $\sigma^l m \neq \pi^{-1}m$, m we have $0 \le a_{\sigma^l m} \le h_{\sigma^l m}, h_{\sigma^l m+1}$ and hence $a_{\sigma^l m} = 0$. If $l = p$, then $a_{\sigma^l m} = a_0 = 0$ by definition. We again obtain additional equation

$$h_{\sigma^k m} - h_{\sigma^{k+1}m+1} + h_{\sigma^{k+1}m} - \cdots + h_{\sigma^{l-1}m} - h_{\sigma^l m+1} = 0 \qquad (h \in H^+(\pi, W)),$$

and proceed further same as before.

Suppose $i = \sigma^k 0$, $2 \le k \le q$. We exclude case $k=1$ since $\sigma 0 = \pi^{-1}m - 1$, and we assume $\pi^{-1}m \in W$. By the same arguments as above $a_{\sigma^k 0} = 0$. Now we start with 0 to get

$$h_0 - a_0 = h_{\sigma 0+1} - a_{\sigma 0}$$

or

$$a_{\pi^{-1}m-1} = h_{\pi^{-1}m}.$$

Going on and using (2.4) in [18] we come up to equation

$$-h_{\pi^{-1}m} + h_{\pi^{-1}m-1} - \cdots + h_{\sigma^{k-1}0+1} = 0$$

We excluded term $h_{\sigma^k 0+1}$ since by assumption $\sigma^k 0 + 1 \notin W$, and hence $h_{\sigma^k 0+1} = 0$ for $h \in H^+(\pi, W)$. To prove that the new condition is linearly independent with others we only need to prove, that the equation is not tautologically trivial, which is so, say, because the component $\pi^{-1}m$ of corresponding vector b does not vanish. Lemma 7.3 is proved. □

8. Finiteness of the measure

In this section we will prove that the integrals of the density functions

$$\begin{aligned} f_\pi^+(\lambda) &= \int_{H_\lambda^{++}} Volume\,(Z^+(h,\pi))\,dh \\ f_\pi^-(\lambda) &= \int_{H_\lambda^{+-}} Volume\,(Z^-(h,\pi))\,dh \end{aligned} \tag{8.1}$$

of the measure μ in Theorem 4 over corresponding simplexes $\Delta^\pm(\pi)$ are finite. We use the scheme similar to one in §13 in [18]. In particular we use the following bound

(4.12) from there:

$$Volume\ Z(h,\pi) \leq \prod_{S\in\Sigma_0(\pi)} B(\lambda,S)^{-1} \quad \text{for } h\in H_\lambda^+(\pi) \tag{8.2}$$

where

$$B(\lambda,S) = \sum_{j\in S\cup\{S+1\}} \lambda_j \qquad (m\notin S\in\Sigma_0(\pi))$$

and

$$B(\lambda,S(m)) = \mathrm{Min}(\lambda_{\pi^{-1}m},\lambda_m) + \sum_{\substack{j\in S(m)\cup\{S(m)+1\}\\ j\neq\pi^{-1}m,\,m}} \lambda_j \qquad (S(m)\in\Sigma_0(\pi))$$

(see (4.7), (4.8) in [18]).

From now on we fix the permutation π and one of the subsimplexes $\Delta^+(\pi) = \{\lambda\in\Delta(\pi)\mid \lambda_m\geq\lambda_{\pi^{-1}m}\}$ or $\Delta^-(\pi)=\{\lambda\in\Delta(\pi)\mid\lambda_m\leq\lambda_{\pi^{-1}m}\}$. Corresponding cone $H^{++}(\pi)$ in the first case and $H^{+-}(\pi)$ in the second case can be subdivided to a finite union of cones with simplex base. Note, that this subdivision is not canonical unless $H^{++}(\pi)$ (correspondingly $H^{+-}(\pi)$) is itself a cone with a simplex base. Fix some subdivision. Each cone $\mathcal{C}$ intersects with the hyperplane $(h\cdot\lambda)=1$ by simplex Δ_λ. Integral 8.1 decomposes to the sum of integrals like

$$\int_{\Delta_\lambda} Volume\left(Z^\pm(h,\pi)\right)dh \tag{8.3}$$

According to bound 8.2, and obvious inequality

$$Volume\left(Z^\pm(h,\pi)\right) \leq Volume\left(Z(h,\pi)\right)$$

each of integrals 8.3 is bounded by

$$Volume(\Delta_\lambda)\cdot\prod_{S\in\Sigma_0(\pi)} B(\lambda,S)^{-1}$$

Let $v_1,\ldots,v_d$, where $d=\dim H^{+\pm}(\pi)=\dim\mathcal{C}=m-N(\pi)+1$ be extremals which span $\mathcal{C}$. We can choose vectors v_j to be positive. They are defined up to multiplication by positive scalars, and do not depend on λ. Fix collection of v_j. The vertices of the simplex Δ_λ are given by points $(v_j\cdot\lambda)v_j$, where v_j does not depend on λ. Hence

$$Volume(\Delta_\lambda) = const\cdot\prod_{j=1}^{d}\frac{1}{v_j\cdot\lambda}$$

(cf. 13.5 in [18]) where *const* is a constant, which does not depend on λ.

Proposition 3. *For each subsimplex $\Delta^+(\pi)$, $\Delta^-(\pi)$, $\pi \in \mathfrak{R} \subset \mathfrak{S}_m^0$, and for each cone C in the corresponding space $H^{+\pm}(\pi)$ function*

$$f(\lambda) = \left(\prod_{j=1}^{d} \frac{1}{v_j \cdot \lambda}\right) \prod_{S \in \Sigma_0(\pi)} B(\lambda, S)^{-1} \tag{8.4}$$

is integrable over corresponding subsimplex $\Delta^{\pm}(\pi)$.

Proof. Consider one of the simplexes $\Delta^+(\pi)$, $\Delta^-(\pi)$ in the standard simplex $\Delta^{m-1} = \{\lambda \in \mathbb{R}_+^m \mid \sum_{1\le i\le m} \lambda_i = 1\}$. We use the following change of coordinates to replace domain of f by standard simplex Δ^{m-1}. For $\Delta^+(\pi)$, that is for subsimplex $\lambda_m \geq \lambda_{\pi^{-1}m}$ we define

$$\begin{cases} \lambda_{\pi^{-1}m} &= \frac{1}{2}\lambda'_{\pi^{-1}m} \\ \lambda_m &= \lambda'_m + \frac{1}{2}\lambda'_{\pi^{-1}m} \\ \lambda_j &= \lambda'_j \quad \text{for } j \neq \pi^{-1}m, m \end{cases} \tag{8.5}$$

For $\Delta^-(\pi)$ we define

$$\begin{cases} \lambda_{\pi^{-1}m} &= \lambda'_{\pi^{-1}m} + \frac{1}{2}\lambda'_m \\ \lambda_m &= \frac{1}{2}\lambda'_m \\ \lambda_j &= \lambda'_j \quad \text{for } j \neq \pi^{-1}m, m \end{cases} \tag{8.6}$$

Consider induced function

$$f(\lambda') = \left(\prod_{j=1}^{d} \frac{1}{v'_j \cdot \lambda'}\right) \prod_{S \in \Sigma_0(\pi)} B'(\lambda', S)^{-1} \tag{8.7}$$

on Δ^{m-1}.

Lemma 8.1. *Consider a subset $W \subset \{1, 2, \dots, m\}$, $0 < \operatorname{Card} W < m$. Then number of factors $N(W)$ in 8.7 which depend only on the variables with subscripts in W is strictly less than* Card W.

$$N'(W) < \operatorname{Card} W$$

Proof. Note that $N'(W) \le N(W)$, where $N(W)$ is the number $N(W)$ of those factors in 8.4 which depend only on the variables with subscripts in W. Hence for those W, such that $\pi^{-1}m$, $m \notin W$ the statement of the Lemma follows from linear independence of vectors $v_j \in H^{+\pm}(\pi) \subset H^+(\pi)$ and Proposition 12.8 in [18].

Similarly, for those W, which contain both $\pi^{-1}m$, $m \in W$, Lemma 7.3 proves that $N(W) < \operatorname{Card} W$, and hence the statement is valid for this case as well.

Now we have to consider cases of subsimplex $\Delta^+(\pi)$ and $\Delta^-(\pi)$ separately. Suppose we started with the subsimplex $\Delta^+(\pi)$. Then the case $\pi^{-1}m \in W$, $m \notin W$ follows from Lemma 7.2. Consider the rest case, when $\pi^{-1}m \notin W$, $m \in W$. Due to our change of coordinates 8.5, each factor in 8.7 containing variable λ'_m would necessarily contain $\lambda'_{\pi^{-1}m}$. Hence none of them would be counted towards $N'(W)$ for the W like ours. Hence

$$N'(W) = N'(W \setminus m) \le N(W \setminus m) \le \operatorname{Card} W - 1,$$

and we obtain desired strict inequality.

We use similar arguments for the subsimplex $\Delta^-(\pi)$ to complete the proof of Lemma 8.1. □

To complete the proof of Proposition 3 we apply Proposition 13.2 in [18] to function $f(\lambda')$. For every subset $W \subset \{1, 2, \dots, m\}$, $0 < \operatorname{Card} W < m$ we define $f_W(\lambda')$ to be the product of all factors in 8.7 which have subscripts in W, and we define $D(W, \lambda')$ to be the product of the rest factors. Functions $f_W(\lambda')$, and $D(W, \lambda')$ obey all conditions of Proposition 13.2 in [18], except that $f(\lambda')$ is homogeneous of degree $-m$ on $\mathbb{R}^m_+$, which does not affect the proof of this proposition. Proposition 3, and hence Theorem 4 are proved. □

9. Ergodicity of the map $\mathcal{G}$

Now we can prove ergodicity of the map $\mathcal{G}$. The proof is similar to the proof of ergodicity of Rauzy induction $\mathcal{T}$ (c.f. Theorem 13.8 in [18], and Theorem 1.11 in [6]).

Proof. Let A be a matrix such that $\det A = 1$, with some of the entries possibly negative. Consider projective linear map $T_A : \lambda \mapsto \frac{A\lambda}{|A\lambda|}$ and suppose T_A maps some compact subset $K \subset \Delta^{m-1}$ into Δ^{m-1}, $\operatorname{Im}(K) \subseteq \Delta^{m-1}$. Let J_A be Jacobian of T_A. Then according to 7.1 and 7.2 in [17]

$$\sup_{\lambda, \lambda' \in K} \frac{J_A(\lambda)}{J_A(\lambda')} \le \sup_{\substack{\lambda, \lambda' \in K \\ 1 \le i \le m}} \left(\frac{\lambda_i}{\lambda'_i} \right)^m$$

Consider a subset $\Delta_\epsilon = \{\lambda \,|\, \lambda_i \ge \epsilon, \;\; i = 1, \dots, m; \;\; \sum \lambda_i = 1\}$ Then for any $K \subseteq \Delta_\epsilon$ and any matrix $A \in \mathrm{SL}(m)$ such that $A(K) \subseteq \Delta^{m-1}$ we get from the estimate above, that

$$\sup_{\lambda, \lambda' \in K} \frac{J_A(\lambda)}{J_A(\lambda')} \le \left(\frac{1}{\epsilon} \right)^m$$

Note, that this estimate does not depend neither on A nor on the subset K anymore. We remind that

$$\mathcal{G}^k(\lambda, \pi_0) = \left(\frac{A\lambda}{|A\lambda|}, \pi \right), \qquad \det A = 1$$

Consider the set $\Delta_{\mathcal{G}}(\lambda, \pi_0, k) \in \Delta^{m-1}$ of (λ', π_0) for which $\mathcal{G}^k$ uses the same matrix A. Then $\mathcal{G}^k(\lambda, \pi_0)$ maps $\Delta_G(\lambda, \pi_0, k)$ onto one of the $(\Delta^+(\pi), \pi)$, $(\Delta^-(\pi), \pi)$.

Consider analogous subsimplexes $\Delta_{\mathcal{T}}(\lambda, \pi_0, k)$ corresponding to Rauzy induction $\mathcal{T}$. It is known that diameters of subsimplexes $\Delta_{\mathcal{T}}(\lambda, \pi_0, k)$ tend to zero as $k \to \infty$ for almost all λ (see [18]and [6]). Since $\Delta_{\mathcal{G}}(\lambda, \pi_0, k) = \Delta_{\mathcal{T}}(\lambda, \pi_0, l(k))$ for some $l(k)$ we conclude, that diameters of the subsimplexes $\Delta_G(\lambda, \pi_0, k)$ tend to zero for almost all λ as well. Hence up to a set of measure zero we can subdivide Δ_ϵ to subsimplexes $\Delta_{\mathcal{G}}(\lambda_j, \pi_0, k)$, $\lambda_j \in \Delta_\epsilon$ Suppose now E is an invariant subset under the mapping $\mathcal{G}$. If for some $\epsilon > 0$ we have $\mu(E \cap \Delta_\epsilon) < \mu(\Delta_\epsilon)$, then, probably refining our subdivision, for any $\delta > 0$ we will find a subsimplex $\Delta_0 = \Delta_G(\lambda_0, \pi_0, k_0)$ from our subdivision such that $\mu(E \cap \Delta_0)/\mu(\Delta_0) < \delta$. Let $(\Delta^{\pm}(\pi), \pi) = \mathcal{G}^{k_0}(\Delta_G(\lambda_0, \pi_0, k_0)$. Then $\mu(E \cap (\Delta^{\pm}, \pi)) \leq \delta/\epsilon^m$. Since δ is arbitrary small, we can find some π, such that $\mu(E \cap (\Delta^{\pm}, \pi)) = 0$. Combining this with the following Lemma we complete the proof of ergodicity of $\mathcal{G}$. □

Lemma 9.1. *The only invariant subcollections of simplexes of the form* $(\Delta^{\pm}, \pi)$, $\pi \in \mathfrak{R}(\pi_0)$ *are* $\emptyset$ *and* $(\Delta^+ \cup \Delta^-) \times \mathfrak{R}(\pi_0)$.

Proof. Consider the oriented graph representing Rauzy class $\mathfrak{R}(\pi_0)$. Any ordered pair of vertices of this graph can be joined by an oriented path (see [18]).

Consider the following oriented graph, responsible for the Gauss map $\mathcal{G}$. We enumerate the set of vertices of the new graph by duplicated set $\mathfrak{R}(\pi_0)$, providing each $\pi \in \mathfrak{R}(\pi_0)$ with additional superscript "+" or "−". We join π_1^+ with π_2^-, $\pi_1, \pi_2 \in \mathfrak{R}(\pi_0)$, by an arrow, if there is some $(\lambda, \pi_1) \in (\Delta^+(\pi_1), \pi_1)$ which is mapped by $\mathcal{G}$ to $(\Delta^-(\pi_2), \pi_2)$. Similarly we join π_1^- with π_2^+, $\pi_1, \pi_2 \in \mathfrak{R}(\pi_0)$, by an arrow, if there is some $(\lambda, \pi_1) \in (\Delta^-(\pi_1), \pi_1)$ which is mapped by $\mathcal{G}$ to $(\Delta^+(pi_2), \pi_2)$. (Note that points of $\Delta^{\pm}$ are always mapped to points of $\Delta^{\mp}$.) To prove the Lemma we need to prove that any ordered pair of vertices of the graph just constructed can be connected by an oriented path.

First note that for each $\pi \in \mathfrak{R}(\pi_0)$ there is a pair of arrows going in opposite directions joining π^+ and π^-. Next note that for each edge of the graph, corresponding to Rauzy induction, which goes from the vertex π_1 to vertex π_2, there is corresponding edge of the new graph, which joins either edges π_1^+ and π_2^- or edges π_1^- and π_2^+, depending on whether the initial edge of the Rauzy graph was of type "a" or "b" correspondingly (see (2.11)). Note also that there is a natural orientation preserving projection of the new graph to Rauzy graph, which sends each pair of vertices π^+ and π^- to vertex π, and each edge of the new graph to the oriented chain of the edges of Rauzy graph.

Now having an arbitrary pair of vertices π_1^+ and π_2^- we construct an oriented path in the Rauzy graph joining π_1 and π_2. Taking into consideration remarks above it is easy to "lift this path up" to the new graph. Lemma is proved, and hence the Theorem 1 is proved as well. □

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10. Lyapunov exponents

First we will prove Proposition 1.

Proof. Recalling the definitions 3.4–3.7, of the function $B(\lambda,\pi)$ we see, that the following inequalities for the entries of matrix B are valid:

$$|B_{i,j}(\lambda,\pi)| \le \frac{\lambda_{\pi^{-1}(m)}}{\lambda_m} + 1 \le \frac{1}{\lambda_m} + 1 \quad \text{when } \lambda \in \Delta^+(\pi)$$

and

$$|B_{i,j}(\lambda,\pi)| \le \frac{\lambda_m}{\lambda_{\pi^{-1}(m)}} + 1 \le \frac{1}{\lambda_{\pi^{-1}(m)}} + 1 \quad \text{when } \lambda \in \Delta^-(\pi)$$

Hence to prove integrability of the function $\Big|\log\|B(\lambda,\pi)\|\Big|$ over $\bigsqcup_{\pi\in\mathfrak{R}} \Delta^+(\pi) \sqcup \Delta^-(\pi)$ with respect to measure μ, it is sufficient to prove integrability of the function

$$h(\lambda,\pi) = \begin{cases} \log\lambda_m & \text{if } \lambda \in \Delta^+(\pi) \\ \log\lambda_{\pi^{-1}(m)} & \text{if } \lambda \in \Delta^-(\pi) \end{cases}$$

To prove integrability of h, it is sufficient to prove integrability of the product $h(\lambda')f(\lambda')$ of h with the function f in 8.7, bounding the density of μ, over each simplex $\Delta^\pm(\pi)$, now already with respect to Lebesgue measure. We do it the same way, as we proved integrability of f in section 8. We use Lemma 8.1 and then trivially modify proof of Proposition 13.2 in [18] to fit our case. Proposition 1 is proved. □

Since function $\Big|\log\|B(\lambda,\pi)\|\Big|$ is integrable, we can use multiplicative ergodic theorem to study products $B^{-1}(\mathcal{G}^k(\lambda,\pi)) \cdot \ldots \cdot B^{-1}(\lambda,\pi)$. To prove Theorem 2 let us first prove the following

Lemma 10.1. *At least $m-2g$ Lyapunov exponents are equal to zero*

$$\theta_j = \theta_{j+1} = \cdots = \theta_{j+m-2g-1} = 0$$

Proof. Recall, that there is natural local identification between the space $\mathbb{R}^m_+$ of interval exchange transformations with fixed permutation $\pi \in \mathfrak{S}^0_m$ and the first relative cohomology $H^1(M^2_g, \{\text{saddles}\}; \mathbb{R})$ of corresponding surface M^2_g with respect to the set of saddles of corresponding foliation (see [5]). Recall, that the saddles are enumerated by the classes $S \in \Sigma(\pi)$ (see section 6 in [18]). Consider the following terms of the exact sequence of the pair $\{\text{set of saddles}\} \subset M^2_g$

$$\begin{array}{llll} 0 = & H^0(M^2_g, \{\text{saddles}\}; \mathbb{R}) \to & H^0(M^2_g; \mathbb{R}) = \mathbb{Z} \to & H^0(\text{saddles}; \mathbb{R}) \to \\ \to & H^1(M^2_g, \{\text{saddles}\}; \mathbb{R}) \to & H^1(M^2_g; \mathbb{R}) \to & H^1(\text{saddles}; \mathbb{R}) = 0 \end{array}$$

From results in [19] the following Lemmas easily follow

Lemma 10.2. *Under identification with cohomology, vector b_S represents the image of an element in $H^0(\text{saddles}; \mathbb{R})$ dual to the saddle corresponding to the class S. In particular the $(m-2g)$-dimensional image of $H^0(\text{saddles}; \mathbb{R})$ is spanned by vectors b_S, $S \in \Sigma(\pi)$.*

Proof. □

Lemma 10.3. *For those powers $\mathcal{G}^k$, of the Gauss map, which send (λ, π) to (λ', π), i.e., for those powers, which preserve permutation π, the corresponding composition of operators*

$$\left(B^{(k)}\right)^{-1}(\lambda,\pi) = \left(B^{-1}\right)^{(k)}(\lambda,\pi) = B^{-1}(\mathcal{G}^{k-1}(\lambda,\pi)) \cdot \ldots \cdot B^{-1}(\mathcal{G}(\lambda,\pi)) \cdot B^{-1}(\lambda,\pi)$$

preserves the collection of vectors b_s, $S \in \Sigma(\pi)$, and hence it preserves the subspace

$$K = \operatorname{Im}\left(H^0(saddles; \mathbb{R}) \to H^1(M_g^2, \{saddles\}; \mathbb{R})\right) \tag{10.1}$$

Proof. □

Note now, that since the Gauss map is ergodic with respect to *finite* invariant measure, it is minimal, i.e., trajectory of almost every point is dense in the compact space $\bigsqcup_{\pi\in\mathfrak{R}}(\Delta^+(\pi) \sqcup \Delta^-(\pi))$ In particular trajectory of almost any point (λ_0, π_0) would visit initial simplex $\Delta^\pm(\pi_0) \ni \lambda_0$ infinite number of times. Hence for almost every point (λ_0, π_0) we have $m - 2g$ linearly independent vectors b_S, $S \in \Sigma_0(\pi_0)$ such that for some infinite sequence of growing integer numbers (depending on the point (λ_0, π_0)) the following inequalities are valid

$$C_1 \le \|\left(B^{(k_i)}\right)^{-1}(\lambda_0, \pi_0) \cdot b_S\| \le C_2 \quad i = 1, 2, \ldots$$

where C_1 and C_2 depend only on the point (λ_0, π_0). Hence at least $m - 2g$ Lyapunov exponents are equal to zero. Lemma 10.1 is proved. □

Let us now prove the relation

$$\theta_k = -\theta_{m-k+1} \qquad \text{for } k = 1, \ldots, g \tag{10.2}$$

Recall, that $B^{(k)}$ preserves "degenerate symplectic form" $\Omega(\pi)$ defined by (2.1) in the following sense (see [9]):

$$\Omega(\pi) = \left(B^{(k)}\right)^T \cdot \Omega(\pi^{(k)}) \cdot B^{(k)} \tag{10.3}$$

Due to Proposition 2 in [19] the kernel of $\Omega(\pi)$ coincides with the subspace K in (10.1) spanned by vectors b_S. Hence $\Omega(\pi)$ is already nondegenerate on the quotient over subspace K, and hence it induces the symplectic structure there. Regarding exact sequence above and Lemma 10.2 we get the following

Proposition 4. *Under local identification of the space of interval exchange transformations with relative cohomology $H^1(M_g^2, saddles; \mathbb{R})$ the quotient space over the subspace spanned by vectors b_S, $S \in \Sigma(\pi)$, coincides with the absolute cohomology $H^1(M_g^2; \mathbb{R})$. The symplectic structure induced by $\Omega(\pi)$ on the quotient space coincides with the intersection form on cohomology.*

Proof. □ (See [19] for details.)

Consider once more a power k of the Gauss map $\mathcal{G}^k$, which sends (λ, π) to (λ', π), i.e., which preserves the permutation π. As we already mentioned the corresponding operator $B^{(k)}(\lambda, \pi)$ preserves subspace K, and hence we can define induced operator on the quotient over K, i.e., on cohomology $H^1(M_g^2; \mathbb{R})$. According to (10.3) induced operator would be symplectic — it preserves the intersection form.

Recall now, that eigenvalues of a symplectic matrix are distributed into pairs — together with eigenvalue κ it has eigenvalue $1/\kappa$. Taking the logarithms, and using the fact, that generically the iterates of $\mathcal{G}$ will come back to initial permutation infinite number of times, once more, we prove relation (10.2), and complete the proof of Theorem 2. □

Remark 3. Recalling the definition 6.1 of the space $H(\pi)$ as the annulator of the subspace spanned by vectors b_S, $S \in \Sigma(\pi)$, we see, that $H(\pi)$ is locally identified in our setting with the absolute homology $H_1(M_g^2; \mathbb{R})$.

Remark 4. Due to Perron—Frobenius theorem the highest eigenvalue of $B^{(k)}(\lambda, \pi)$ is real and positive for almost all (λ, π), and k large enough. Normalize corresponding eigenvector λ_0 so that $\lambda \in \Delta^{m-1}$. Vector λ_0 would be very close to initial vector λ for large k. In particular $B^{(k)}(\lambda, \pi) = B^{(k)}(\lambda_0, \pi)$. Suppose $\mathcal{G}^k$ preserves the permutation π as above. Consider measured foliation corresponding to interval exchange transformation defined by (λ_0, π). Veech constructs in [18] the pseudo Anosov diffeomorphism which preserves the foliation thus constructed. We note, that the automorphism in cohomology $H^1(M_g^2; \mathbb{R})$ defined by $B^{(k)}(\lambda, \pi)$ above, coincides with the automorphism in cohomology, induced by corresponding pseudo Anosov transformation.

Let us prove now Theorem 3.

Proof. To prove, that some number θ belongs to the collection of Lyapunov exponents it is sufficient to present for a set of points (λ, π) of nonzero measure a vector $v(\lambda, \pi) \in \mathbb{R}^m$ such that

$$\lim_{k \to +\infty} \frac{1}{k} \log \frac{\left\| \left(B^{(k)}\right)^{-1} (\lambda, \pi) \cdot v(\lambda, \pi) \right\|}{\|v(\lambda, \pi)\|} = \theta$$

Let

$$v(\lambda, \pi) := \lambda$$

Denote

$$(\lambda^{(k)}, \pi^{(k)}) := \mathcal{G}^k(\lambda, \pi)$$

We see, that vectors $\left(B^{(k)}\right)^{-1}(\lambda, \pi) \cdot \lambda$ and $\lambda^{(k)}$ are proportional. Let

$$r(\lambda, \pi) := \frac{\|B^{-1}(\lambda, \pi) \cdot \lambda\|}{\|\lambda\|}$$

Due to the comment above under our choice of vector $v(\lambda, \pi)$ we will have

$$\frac{1}{k} \log \frac{\left\| \left(B^{(k)}\right)^{-1}(\lambda, \pi) \cdot v(\lambda, \pi) \right\|}{\|v(\lambda, \pi)\|} =$$

$$= \frac{1}{k} \log\Bigl(r(\mathcal{G}^{k-1}(\lambda, \pi)) \cdot \ldots r(\mathcal{G}(\lambda, \pi)) \cdot r(\lambda, \pi)\Bigr) =$$

$$= \frac{1}{k}\Bigl(\log r(\lambda, \pi) + \log r(\mathcal{G}(\lambda, \pi)) + \cdots + \log r(\mathcal{G}^{k-1}(\lambda, \pi))\Bigr)$$

Applying Ergodic theorem to the sum above we prove, that the following number θ is present in the collection of Lyapunov exponents

$$\theta = \sum_{\pi \in \mathfrak{R}} \int_{\Delta^{\pm}(\pi)} (\log \|B^{-1}(\lambda, \pi) \cdot \lambda\| - \log \|\lambda\|)\, d\mu \tag{10.4}$$

Note that in fact we have absolute freedom in choosing the norm $\|\ \|$. Choosing the norm $\|v\| := |v_1| + \cdots + |v_m|$ we will get for $\lambda \in \Delta^{\pm}$

$$\log \|B^{-1}(\lambda, \pi) \cdot \lambda\| - \log \|\lambda\| = \log(1 - \nu(\lambda, \pi)) - \log 1$$

where $\nu(\lambda, \pi)$ is defined by (3.5). Choosing for $v \in \mathbb{R}^m \times \pi$ another norm

$$\|v\| = |v_1| + \cdots + |v_{\pi^{-1}(m)-1}| + |v_{\pi^{-1}(m)+1}| + \cdots + |v_{m-1}| + \max(|v_m|, |v_{\pi^{-1}(m)}|)$$

we get

$$\log \|B(\lambda, \pi) \cdot \lambda\| - \log \|\lambda\| = \begin{cases} \log(1 - \lambda_{\pi^{-1}(m)}) - \log(1 - \lambda_m) & \text{for } \lambda \in \Delta^-(\pi) \\ \log(1 - \lambda_m) - \log(1 - \lambda_{\pi^{-1}(m)}) & \text{for } \lambda \in \Delta^+(\pi) \end{cases}$$

Remark 5. Note, that the second norm is different for the spaces $\mathbb{R}^m$ corresponding to different $\Delta^{\pm}(\pi)$. In fact, we should consider $\mathbb{R}^m$ as a fiber of a trivialized vector bundle over $\bigsqcup_{\pi \in \mathfrak{R}} \Delta^+(\pi) \sqcup \Delta^-(\pi)$, and we can even choose the norm, which would differ (continuously) from fiber to fiber. It is easy to see, that the integral (10.4) would be the same anyway.

Note, that expressions (4.2) and (4.3) for θ_1 in the statement of Theorem 3 differ from the corresponding expressions for θ above only by a sign. Since we already proved, that $\theta_1 = -\theta_m$, to complete the proof of Theorem 3 we just need to prove, that Lyapunov exponent θ computed above is the smallest one, i.e., that $\theta = \theta_m$. This is true since for almost every point (λ, π)

$$\lim_{k \to \infty} \frac{B^{(k)}(\lambda, \pi) v}{\|B^{(k)}(\lambda, \pi) v\|} = \lambda \quad \text{for any } v \in \Delta^{m-1},$$

and hence for almost all $v \in \mathbb{R}^m$, $|v| = 1$. Theorem 3 is proved. □

We complete this section by proving Proposition 2.

Proof. Consider the trivialized vector bundle with the base $\bigsqcup_{\pi\in\mathfrak{R}} \Delta^+(\pi) \sqcup \Delta^-(\pi)$ and a fiber $\mathbb{R}^m$. The Gauss map $\mathcal{G}$ extends to the map on the total space of this bundle $(x, v) \mapsto (\mathcal{G}(x), B^{-1}(x)v)$, where $x = (\lambda, \pi)$ is a point in the base, and v is a vector in the fiber. There is a tautological one-dimensional vector subbundle with a fiber spanned by vector λ over a point (λ, π). This subbundle is obviously invariant under the action above. Now note, that the quotient of the trivialized bundle over tautological bundle is isomorphic to the tangent bundle over our base. Moreover, it is easy to see, that the composition of the induced action in the total space of the quotient bundle with fiberwise homotety with coefficient $|B^{-1}\lambda|^{-1}$ coincides with the action of the differential $D\mathcal{G}$ under suggested identification. In fact, we just use canonical isomorphism $TG_1(m) \cong \mathrm{Hom}(\gamma, \gamma^\perp)$, where γ is the tautological, and $\gamma^\perp$ is the normal bundle to the Grassmann manifold $G_1(m) = \mathbb{R}P^{m-1}$.

Multiplicative ergodic theorem guaranties existence of the flag of subspaces $\mathbb{R}^m = E_1 \supset E_2 \supset \cdots \supset E_q \supset E_{q+1} = \emptyset$ (depending on the point of the base) in almost all fibers of the trivialized bundle, such that for any $v \in E_j, v \notin E_{j+1}$ the relation

$$\lim_{k\to+\infty} \frac{1}{k} \log \frac{\|\left(B^{(k)}\right)^{-1} v\|}{\|v\|} = \theta^{(j)} \tag{10.5}$$

holds. Here by $\theta^{(1)}, \ldots, \theta^{(q)}$ we denote Lyapunov exponents repeated without multiplicities.

From the proof of Theorem 3 it follows, that the one-dimensional fiber of the tautological bundle belongs (and presumably coincides) to the subspace $E_q(\lambda, \pi)$. Hence, the flag of subspaces E_j induces the flag of the subspaces in the fiber of the quotient bundle, with the same property 10.5. The only difference is that the multiplicity of the smallest Lyapunov exponent would be reduced by one. The impact of the homotety can be easily computed since homotety commutes with our induced cocycle in the quotient bundle (and, actually with any fiberwise linear mapping). This impact is just a shift of all Lyapunov exponents by θ_1. Theorem 2 is proved. □

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References

1. V.I.Arnold, *A-graded algebras and continued fractions*, Comm. on Pure and Appl. Math., **42** (1989), no.7, 993-1000.
2. I.P.Cornfeld, S.V.Fomin, Ya.G.Sinai. *Ergodic Theory*. Springer-Verlag, 1982.
3. A.Fathi, F.Laudenbach, V.Poénaru. *Travaux de Thurston sur les surfaces*, Astérisque **66–67** (1979).
4. D.Fried, *Anosov foliations and cohomology*, Ergod. Th & Dynam. Sys. **6** (1986), 9–16; **8** (1988), 491–492.
5. A.B.Katok, *Invariant measures of flows on oriented surfaces*, Soviet Math. Dokl. **14** (1973), 1104–1108.
6. S.P.Kerckhoff, *Simplicial systems for interval exchange maps and measured foliations*, Ergod. Th. & Dynam. Sys. **5** (1985), 257–271
7. H. Masur, *Interval exchange transformations and measured foliations*, Ann. of Math., **115:1** (1982), 169–200.
8. H. Masur, J.Smillie, *Hausdorf dimension of sets of nonergodic measured foliations*, Ann. of Math., **134** (1991), 455–543.
9. A.Nogueira, D.Rudolph, *Topological weakly mixing of interval exchange maps*
10. A.Nogueira, *The 3-dimensional Poincaré continued fraction algorithm*, Preprint ENSL 93 (1993), 1–25.
11. S.P.Novikov, *The Hamiltonian formalism and a many-valued analogue of Morse theory*, Russian Math. Surveys, **37:5**, (1982), 1–56.
12. V.I.Oseledec, *A Multiplicative Ergodic Theorem. Ljapunov characteristic numbers for dynamical systems*, Trans. Moscow Math. Soc., **19**, (1968), 197–231.
13. G.Rauzy, *Echanges d'intervalles et transformations induites.* Acta Arith. **34** (1979), 315–328.
14. D.Ruelle, *Chaotic evolution and strange attractors*, Cambridge University Press, 1989, 91pp.
15. D.Ruelle, D.Sullivan, *Currents, flows and diffeomorphisms.* Topology **14** (1975), 319–328.
16. S.Schwartzman, *Asymptotic cycles*, Annals of Mathematics 66 (1957), 270–284
17. W.A.Veech, *Projective swiss cheeses and uniquely ergodic interval exchange transformations*, Ergodic Theory and Dynamical Systems, Vol. I, in Progress in Mathematics, Birkhauser, Boston, 1981, 113–193.
18. W.A.Veech, *Gauss measures for transformations on the space of interval exchange maps.* Annals of Mathematics **115** (1982), 201–242
19. A.Zorich, *Asymptotic Flag of an Orientable Measured Foliation*, Séminaire de Théorie Spectrale et Géométrie, **11** (1993) Institute Fourier, Grenoble 1993
20. A.Zorich, *Hamiltonian Flows of Multivalued Hamiltonians on Closed Orientable Surfaces*, Preprint of Max-Plank-Institut für Matematik, MPI/94-29 (1994), 1–19.

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