MULTIPLIER IDEALS AND ADJOINT LINEAR SYSTEMS ON SURFACES

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ABSTRACT. In this note, we generalize the results of Ein and Lazarsfeld on the multiplier ideals to length 2 subschemes. We use them to study the separation of tangent directions for adjoint linear systems on surfaces. We also discuss some results for threefolds and higher dimensional varieties.

INTRODUCTION

For the last few years, the following conjecture of Fujita [6] has attracted many mathematicians.

Conjecture 0.1. *Let X be a smooth projective variety of dimension n and A be an ample divisor on X. Then $K_X + tA$ is base point free if $t \geq n+1$, and $K_X + tA$ is very ample if $t \geq n+2$.*

However in spite of all the success in base point freeness, it has been less successful in obtaining effective results in the very ampleness beyond surfaces.

On surfaces, above conjecture can be proved using several methods. First, one could apply the vector bundle technique or the Zariski decomposition method (cf. [12], [15], [16], [19], [21]). Here, it is essential to used the Hodge index theorem. Second, one could use the Kawamata-Viehweg vanishing theorem in the spirit of Ein and Lazarsfeld [3]. The first result in this direction was obtained by Maşek [18], where he also treated surfaces with boundary.

In the first part of this note, we develop another approach for the separation of tangent directions, and apply them to adjoint linear systems on surfaces in §3. It is based on the work [4] of Ein and Lazarsfeld. Two key results are Theorem 1.11 and Theorem 1.14. With these, the strategy for the separation of tangent directions become analogous to that of the base point freeness or the separations of distinct points. We explain them below.

Let X be a smooth projective variety and H be an ample divisor on X. Let W be a length 2 subscheme in X supported at a closed point p. In [4], Ein and Lazarsfeld introduced the notion of critical varieties (minimal centers in [10]) in the context of base point freeness, which plays such a crucial role in many papers on base point freeness. This notion is naturally extended

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to W (see Definition 1.7). With a tie-breaking condition satisfied, it means the following; a Q-Cartier divisor G on X is called critical at W if $Z(G)$, the multiplier ideal scheme for G, contains W but $Z((1-\epsilon)G)$ does not for any $0<\epsilon<<1$. We call the support of $\mathcal{I}_{Z((1-\epsilon)G)}/\mathcal{I}_{Z(G)}$ the critical variety of G at W.

Suppose now that we have a Q-Cartier divisor G on X critical at W, and $H-G$ is nef and big. Then Theorem 1.11 implies that if the critical variety is a point, then $|K_X+H|$ separate W. Of course, there is no reason that the critical variety should be a point in the first place. However one can make the critical variety smaller using the nice lifting argument of Ein and Lazarsfeld thanks to Theorem 1.14. It especially takes care of the case where the multiplier ideals is non reduced alone the critical variety, which is one of the new phenomenon in the separation of tangent directions. Details are explained in the proof of the following theorem. For the definition of *order*, see Definition 3.1.

Theorem 0.2 (cf. [4], [20]). *Let X be a smooth projective surface. Let W be a subscheme of X with $l := \text{length}\, W \le 2$*[1]*. Let A be a nef and big Q-divisor on X. Let $\Delta = \lceil A \rceil - A$ and $\mu = \text{ord}_W \Delta$. Suppose that there exist positive numbers β_1 and β_2 satisfying the following conditions:*

$$A^2 > l\cdot\beta_2^2 \ge l\cdot(2-\mu)^2.$$
$$A\cdot C \ge (\text{length}\,(W\cap C))\cdot\beta_1 \text{ for all curve } C \text{ in } X.$$
$$\beta_1 \ge \min\left\{2-\mu, \frac{\beta_2}{\beta_2-1+\mu}\right\}.$$

Then $|K_X+\Delta+A|$ separates W. Especially, if $\mu\ge 2$, then A need to be only nef and big.

At least for the base point freeness part, it was known to Ein and Lazarsfeld [4]. The very ample part of the previous theorem is essentially due to Maşek [18], [20].

As one may have noticed, Theorem 0.2 does not implies the numerical bound of Reider [21] for the very ampleness part, which can be though of as a strong version of Conjecture 0.1. In fact, alone the same line we can also prove the following proposition. A similar bound was obtained by Langer [16].

Proposition 0.3. *Let X be a smooth projective surface. Let W be a length two subscheme of X consisting of a closed point p in X and a tangent direction v of X at p. Let A be a nef and big Q-Cartier divisor on X. Let $\Delta = \lceil A\rceil - A$ and $\mu_p := \text{ord}_p\Delta$. Then $|K_X+\Delta+A|$ separates W in the following two cases.*

1. $\mu_p \ge 3$.

[1]In this note, we only work out up to lenght 2 subscheme. But the method developed in §2 is general enough to deal with any subscheme of finte length and this theorem is expected to be true without the restirction on the length of W.

2. $\mu_p < 3$, *and there exists positive numbers* σ_1 *and* σ_2 *satisfying the following three inequalities:*

$$A^2 > \sigma_2^2 \geq (3 - \mu_p)^2. \tag{0.1}$$

$$A \cdot C \geq \sigma_1 \text{ for all curve } C \text{ in } X \text{ through } p. \tag{0.2}$$

$$\sigma_1 \geq \frac{2\sigma_2}{\sigma_2 - 1 + \mu_p}. \tag{0.3}$$

Up to this point, the proofs essentially depend only on vanishing theorem. However it seems inevitable to rely on some other methods to obtain his bound. So far, all the known method uses Hodge index theorem. In §2, we adopt a simple technique from the graded linear series (cf. [5], [7], [8], [9], [10]), and use them to obtain the bound at the end of §3.

I should mention that one important ingredient in the proofs of the previous results is a multiplicity bound for the critical curves on surfaces. It is easy to obtain in this case because they are divisors. In general, Helmke gave a bound in [8] for reduced W. But there are no known bounds otherwise. Any bound on them with non reduced W will lead to an effective result on very ampleness for arbitrary dimension.

From his bound, one might raise the following Content-Length: 8883 X-Lines: 288 Status: RO

Question 0.4. *Let* X *be a smooth variety of dimension* n *and let* W *be a non reduced length* 2 *subscheme in* X. *Let* $p = Supp W$. *Let* G *be a Q-Cartier divisor on* X *critical at* W *with the critical variety* Z *of dimension* $d > 0$. *Is it true that*

$$mult_p Z \leq \binom{n}{d}?$$

In his proof, it was crucial that (X, G) is log canonical at p so that the coefficients of the exceptional divisors over p is bounded. But when W is non reduced it is not clear how to bound the exceptional divisors which is not over the tangent direction corresponding to W.

In §4, we consider some partial results in higher dimensional varieties. The next two result are probably well known among the experts. First, for smooth fano threefolds we can verify Conjecture 0.1.

Theorem 0.5. *Let* X *be a smooth projective threefolds with an ample anticanonical divisor* $-K_X$. *Then* $|-tK_X|$ *is very ample for all* $t \geq 4$.

One can have a similar result for smooth threefolds X with ample K_X instead of $-K_X$. But the bound is weaker than the conjectural one.

Proposition 0.6. *Let* X *be a smooth projective threefolds with an ample canonical divisor* K_X. *Then* $|tK_X|$ *is very ample for all* $t \geq 10$.

In fact, the proof of Theorem 0.5 can be easily extended to arbitrary *smooth* projective varieties with an ample (anti)canonical divisor to produce cubic bounds on very ampleness.

In general, we only have the following

Theorem 0.7. *Let X be a normal Q-Gorenstein projective variety of dimension n. Let p be a point in X. Let Δ be an effective Q-divisor on X such that (X,Δ) is Kawamata log terminal at p. Let H be an ample Q-divisor on X such that $K_X+\Delta+H$ is Cartier. We assume that H satisfies the following inequality.*

$$H^d\cdot Z > \Big\{\sum_{k=1}^{n}\sqrt[k]{2}\cdot k\Big\}^d$$

for all subvariety Z through p, where $d=\dim Z>0$. Then there exist a length 2 subscheme W of X supported at p, and a divisor D in $|K_X+\Delta+H\otimes\mathcal{I}_p|$ such that $W\not\subset D$, where $\mathcal{I}_p$ is the sheaf of maximal ideal at p.

The following is an immediate corollary of this.

Corollary 0.8. *Let X be a smooth projective variety of dimension n with an ample divisor H. Then for any closed point p in X, generic elements of $|K_X+tH\otimes\mathcal{I}_p|$ are smooth at p for $t\geq n^2+n+1$.*

As for the proof, we again use the multiplier ideals method. But to avoid the difficulties coming from the singularities of X and critical varieties, we use the specialization argument of Anghern and Siu [1]. The difficulty in obtaining a similar bound for the separations of tangent direction lies in that the tangent direction which we want to separate might be contained in the embedded components of the tangent cone of the multiplier ideal scheme at p.

Notations. We will work over the field $\mathbb{C}$ of complex numbers. The following is the list of the notations that might be used differently by other people.

- $=$ equivalence as Weil divisor.
- $\underset{lin}{\overset{Q}{\approx}}$ Q-linear equivalence.
- $\lfloor\cdot\rfloor$ round-down(integral part).
- $\lceil\cdot\rceil$ round-up.
- $\{\cdot\}$ fractional part.
- T_pX the tangent space of X at p.
- $\mathbb{P}(T_pX)$ the projectivised tangent space of X at p.
- C_pX the tangent cone of X at p.
- $\mathbb{P}(C_pX)$ the projectivised tangent cone of X at p.

1. Multiplier Ideals

Here we collect some properties of multiplier ideals which are needed later in the proofs. The main reference for this section is [2], [4]. We also extend some of the results there to subschemes of finite length. All the proofs can be found there except Proposition 1.10 and Theorem 1.11, 1.14. But we give the proofs for the convenience of readers.

The following vanishing theorem of Kawamata and Viehweg is our basic tool.

Theorem 1.1 (Theorem 1-2-3 in [11]). *Let X be a smooth variety and let $f : X \longrightarrow S$ be a proper morphism onto a variety S. Let D be a f-nef and f-big Q-divisor whose support is in normal crossing. Then*

$$R^i f_* \mathcal{O}_X(K_X + \lceil D \rceil) = 0$$

for all $i > 0$.

Lemma 1.2 (Lemma 1-3-2 in [11]). *Let $f : Y \longrightarrow X$ be a proper birational morphism from a nonsingular variety Y onto a variety X. Let P be an effective f-exceptional Cartier divisor and let D be a $(-f)$-nef Q-Cartier divisor on Y. Then*

$$f_* \mathcal{O}_P(P + \lfloor D \rfloor) = 0.$$

1.1. **Critical Varieties.** The notations introduced here, we keep them throughout §1.

Let X be a normal Q-Gorenstein variety, i.e. K_X is Q-Cartier. Let W be a 0-dimensional subscheme in X. Let G be an effective Q-Cartier divisor on X. Let $f : Y \longrightarrow X$ be an embedded resolution of G. Define b_i and g_i by

$$K_Y - f^* K_X = \sum b_i F_i \qquad \text{and} \qquad f^* G = \sum g_i F_i,$$

where F_i's are the distinct irreducible smooth divisors in simple normal crossing. Let

$$\sum \lceil (b_i - g_i) \rceil F_i = P - N,$$

where P and N are effective divisors with no common components. We observe that P is f-exceptional. Thus by Lemma 1.2

$$f_* \mathcal{O}_P(P - N) = f_* \mathcal{O}_P(P) = 0.$$

Then

$$f_* \mathcal{O}_Y(P - N) = f_* \mathcal{O}_Y(-N) \subset \mathcal{O}_X.$$

Definition 1.3. $f_* \mathcal{O}_Y(P - N)$ is called the multiplier ideal of G. Let $Z(G)$ be the scheme defined by this ideal and we will note the multiplier ideal by $\mathcal{I}_{Z(G)}$. One can check that $\mathcal{I}_{Z(G)}$ is independent of the choice of the embedded resolutions.

Proposition 1.4.
1. *$R^1 f_* \mathcal{O}_Y(-N) = 0$ and $R^i f_* \mathcal{O}_Y(P - N) = 0$ for all $i > 0$.*
2. *Assume that X is projective and let H be an Cartier divisor on X such that $H - K_X - G$ is nef and big. Then $H^i(X, \mathcal{O}_X(H)) = 0$ for all $i > 0$.*

Proof. (1) follows from Theorem 1.1 and Lemma 1.2. For (2), we note that

$$H^i(Y, \mathcal{O}_Y(P - N) \otimes f^* \mathcal{O}_X(H)) = 0$$

for all $i > 0$. Thus it follows from the Serre's spectral sequence and (1). □

Definition 1.5. We say that (G, f), or simply G, satisfies tiebreaking if $b_i - g_i \neq b_j - g_j$ for all $i \neq j$.

Remark 1.6. If G is ample, for arbitrary small $0 < \epsilon << 1$, one can always find a Q-Cartier divisor $G' \underset{lin}{\overset{Q}{\sim}} (1+\epsilon)G$ together with an embedded resolution of G' so that G' satisfies tiebreaking.

Definition 1.7.
1. We say that G is critical at W if it satisfies tiebreaking, $W \subset Z(G)$, but $W \not\subset Z((1-\epsilon)G)$ for any $\epsilon > 0$.
2. Suppose G is critical at W. Then there is a unique F_i such that
$$\lceil K_Y - f^*(K_X + G)\rceil = \lceil K_Y - f^*(K_X + (1-\epsilon)G)\rceil - F_i$$
for a sufficiently small $0 < \epsilon << 1$. We call F_i the critical component of G and $f(F_i)$ the critical variety of G at W.
3. Let $l \in \mathbb{N}$ and let p be a closed point in X. We say that G is l-critical at p if
   (a) G is critical at a length l subscheme of X whose support is p.
   (b) For all $\lambda < 1$, $Z(\lambda G)$ does not contain any length l subschemes in X supported at p.
4. Suppose G is l-critical at p. Then G is critical at some W with the length l and $Supp\, W = p$. We call the critical component(resp. variety) of G at W as the l-critical component(resp. variety) of G at p. By 1.(a) they are well defined.

Proposition 1.8. *We assume that W is a closed point in X. If G is critical at p with the critical variety Z, then Z is normal at p.*

Proof. Let F be the critical component. Since it is local, we may assume that $N = F$.

From Proposition 1.4,
$$0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{O}_X \longrightarrow f_*\mathcal{O}_F \longrightarrow 0.$$
Thus Z is normal at p. □

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Proposition 1.9. *Let H be an irreducible normal Cartier divisor on X intersecting G properly and let $G_H = G|_H$. Then,*
$$\mathcal{I}_{Z(G_H)} \subset Img\,(\mathcal{I}_{Z(G)} \longrightarrow \mathcal{O}_H.$$

Proof. We may assume that $f : Y \longrightarrow X$ is also an embedded resolution of H. Let V be the proper transform of H in Y.

From Proposition 1.4, we have
$$f_*\mathcal{O}_Y(K_Y + V - \lfloor f^*(K_X + G + H)\rfloor) \to$$
$$f_*\mathcal{O}_V(K_V - \lfloor f^*(K_H + G_H)\rfloor) \to 0.$$
The first term is a subsheaf of $\mathcal{I}_{Z(G)}$. The second term is $\mathcal{I}_{Z(G_H)}$. Thus we have the inclusion. □

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The following proposition is implicit in [4], [14].

Proposition 1.10. *Assume that W is a length 2 subscheme of X supported at a closed point p. Let G be critical at W with the critical variety Z of dimension $d > 0$ and the critical component F. Suppose that the following two conditions hold.*

1. *Z is smooth at p, and $f|_F$ is smooth over a neighborhood of p.*
2. *$F \cap f^{-1}(p)$ is not contained in P.*

Let $\overline{\phi} \in \mathcal{I}_{W/Z}^{kd+1}$ for some integer $k > 0$. Then for any lifting ϕ of $\overline{\phi}$,

$$W \subset Z(D + (1-\epsilon)G),$$

where $0 < \epsilon << 1$ and $D = \frac{1}{k} div(\phi)$.

Proof. Since it is a local problem, we may assume that $P - N = -F$ by assumption 2.

From the surjection $\mathcal{I}_{W/X}^{kd+1} \longrightarrow \mathcal{I}_{W/Z}^{kd+1} \longrightarrow 0$, we can pick a local lifting $\phi \in \mathcal{I}_{W/X}^{kd+1}$ of $\overline{\phi}$. Let $D = \frac{1}{k} div(\phi)$. From Theorem 1.14, it is enough to show that $W \subset Z(D + (1-\epsilon)G)$. Content-Length: 4558 X-Lines: 152 Status: RO

Let C_1 be an irreducible fiber of f over p. Let $g_1 : Y_1 \longrightarrow Y$ be the blowing up at C_1 with exceptional divisor E_1. Then there is a unique subvariety C_2 of $\dim C_2 = n - 1 - d$ in E_1 corresponding to W. Let $g_2 : Y_2 \longrightarrow Y_1$ be the blowing up at C_2 with the exceptional divisor E_2. We set $g = g_1 \circ g_2$. Note that

1. $K_{Y_2} - g^*K_Y = dg_2^*E_1 + dE_2$.
2. $g^*f^*D \geq g_*^{-1}f^*D + dg_2^*E_1 + dE_2$.
3. $g^*F = g_*^{-1}F + g_2^*E_1 + E_2$ and $(f \circ g)_*\mathcal{O}_{Y_2}(-g_2^*E_1 - E_2) \subset \mathcal{I}_W$.

Thus

$$\begin{aligned} &K_{Y_2} - (f \circ g)^*(K_X + (1-\epsilon)G + D) \\ &\qquad = K_{Y_2/Y} + g^*(K_{Y/X} - f^*((1-\epsilon)G + D)) \leq -g_2^*E_1 - E_2, \end{aligned}$$

for $0 < \epsilon << 1$ and the assertion follows. □

1.2. **Two key theorems.** In this section we give the proofs of the theorems mentioned in the introduction. The next theorem appears in several papers including [1], [2], [3], [8], [9], [10], [13] in the context of freeness. The following version is due to Ein.

Theorem 1.11. *We assume that X is projective. Let H be a Cartier divisor on X such that $H - K_X - G$ is nef and big. We assume that $\mathcal{I}_{Z(G)} \subset \mathcal{I}_W$. Let $N = N_0 + N_+$ be a decomposition of N such that for all $F_i \subset Supp\, N_0$, $F_i \subset f^{-1}W$ or $F_i \cap f^{-1}W = \emptyset$. Let Z_+ be the scheme defined by the ideal sheaf $f_*\mathcal{O}_Y(P - N_+)$, i.e. $\mathcal{I}_{Z_+} = f_*\mathcal{O}_Y(P - N_+)$. Then for any Σ with $W \cap Z_+ \subset \Sigma \subset W$, there is $D \in |H|$ such that $\Sigma = D \cap W$, i.e. we have the following surjection:*

$$H^0(X, \mathcal{I}_{Z_+} \otimes \mathcal{O}_X(H)) \longrightarrow \mathcal{I}_{W \cap Z_+/W} \longrightarrow 0. \tag{1.1}$$

Proof. Let $\mathcal{J} = \mathcal{I}_{Z_+}/\mathcal{I}_{Z(G)} \subset \mathcal{O}_{Z(G)}$. By Proposition 1.4,

$$H^0(X, \mathcal{I}_{Z_+} \otimes \mathcal{O}_X(H)) \to H^0(Z(G), \mathcal{J} \otimes \mathcal{O}_{Z(G)}(H)) \to 0. \tag{1.2}$$

By the construction, $\mathcal{J}$ is trivial in a punctured neighborhood of W. So we can write $\mathcal{J} = \mathcal{J}_1 \oplus \mathcal{J}_2$ so that $Supp\, \mathcal{J}_1 \subset W$ and $Supp\, \mathcal{J}_2 \cap W = \emptyset$. Thus

$$H^0(\mathcal{J} \otimes \mathcal{O}_{Z(G)}(H)) \cong \mathcal{J}_1 \oplus H^0(Z(G), \mathcal{J}_2 \otimes \mathcal{O}_{Z(G)}(H)). \tag{1.3}$$

Since $\mathcal{I}_{Z(G)} \subset \mathcal{I}_W$, we have

$$\mathcal{J}_1 \longrightarrow \mathcal{I}_{W \cap Z_+/W} \longrightarrow 0. \tag{1.4}$$

Now (1.1) follows from (1.2), (1.3), and (1.4). □

Corollary 1.12. *Let X, H, and G be as in Theorem 1.11 and we assume further that G is critical at W.*

1. *Then there is $D \in |H|$ such that $length\, W = length\,(D \cap W) + 1$.*
2. *We assume that $W = \{p, v\}$, where $v \in \mathbb{P}T_pX$.*
   (a) *If the subscheme of $Z(G)$ which consists of positive dimensional components does not contains W, then $|H|$ separates v at p.*
   (b) *If the critical variety of G is a point, then $|H|$ separates v at p.*

Remark 1.13. Corollary 1.12.1 can be used to separates any subschemes of finite length inductively.

The next Theorem is essentially due to Ein and Lazarsfeld.

Theorem 1.14. *Suppose G is critical at W with the critical variety Z. Let B be a non-zero effective Q-Cartier divisor on Z. Let D_1 and D_2 be two liftings of B, i.e. $B = D_i|_Z$ for $i = 1, 2$. If*

$$W \subset Z((1-s)D_1 + (1-t)G)$$

for all sufficiently small s and t, then

$$W \subset Z((1-s)D_2 + (1-t)G)$$

for all sufficiently small s and t.

Proof. We may assume that $f : Y \longrightarrow X$ is also embedded resolution of D_1 and D_2. Let F be the critical component. Then taking t sufficiently small, we may assume that

$$K_Y - \lfloor f^*(K_X + G) \rfloor = K_Y - \lfloor f^*(K_X + (1-t)G) \rfloor - F.$$

Let $P_i - N_i := K_Y - \lfloor f^*(K_X + (1-t)G + (1-s)D_i) \rfloor$ for $i = 1, 2$. Then taking s small and generic, we may assume further that

$$K_Y - \lfloor f^*(K_X + G + (1-s)D_i) \rfloor = P_i - N_i - F.$$

Since D_1 and D_2 are the liftings of the same non-zero effective divisor B on Z, $F \not\subset Supp\, f^*D_i$ for $i = 1, 2$, and $f^*D_1|_F = f^*D_2|_F$. We note that $ord_F N_1 = ord_F N_2$. Then since f is an embedded resolution for G, D_1, and D_2,

$$P_1 - N_1 - (ord_F N_1)F|_F = P_2 - N_2 - (ord_F N_2)F|_F.$$

Thus

$$\mathcal{O}_F(P_1 - N_1) = \mathcal{O}_F(P_2 - N_2). \tag{1.5}$$

From Proposition 1.4 we have the following commutative diagram.

$$\begin{array}{ccccccccc}
0\longrightarrow & \mathcal{I}_{Z(G)} & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_{Z(G)} & \longrightarrow 0 \\
& \uparrow & & \uparrow & & \uparrow & \\
0\longrightarrow & \mathcal{I}_{Z(G+(1-s)D_i)} & \longrightarrow & \mathcal{I}_{Z((1-t)G+(1-s)D_i)} & \longrightarrow & \mathcal{J}_i & \longrightarrow 0
\end{array}$$

where $\mathcal{J}_i = f_*\mathcal{O}_F(P_i - N_i)$. From (1.5) $\mathcal{J}_1 = \mathcal{J}_2$. So the theorem follows from $W \subset Z(G)$. □

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Definition 1.15. Suppose G is critical at W with the critical variety Z. Let B be an effective Q-Cartier divisor on Z. An effective Q-Cartier divisor D is said to be a nice lifting of B with respect to G if

$$D|_Z = B \qquad \text{and} \qquad Supp\, \mathcal{J} \subset Z$$

where $\mathcal{J}$ is defined by

$$0 \to \mathcal{I}_{Z((1-t)G+D)} \to \mathcal{I}_{Z((1-t)G)} \to \mathcal{J} \to 0.$$

for $0 < t << 1$

Proposition 1.16. *Suppose G is critical at W with the critical variety Z. Let B be an effective Q-Cartier divisor on Z. Let A be a Q-ample Cartier divisor on X such that $B \underset{lin}{\overset{Q}{\sim}} A|_Z$. Then there is a nice lifting D of B with respect to G such that $D \underset{lin}{\overset{Q}{\sim}} A$.*

Proof. We fix $m >> 0$ so that $\mathcal{O}_X(mA) \otimes \mathcal{I}_Z$ is globally generated and

$$H^1(X, \mathcal{O}_X(mA) \otimes \mathcal{I}_Z) = 0.$$

Let

$$\phi : H^0(X, \mathcal{O}_X(mA)) \to H^0(Z, \mathcal{O}_Z(mA)) \to 0.$$

Let $\Lambda \subset |mA|$ be the sublinear system consisting of those divisors D' such that $Z \subset D'$ or $D'|_Z = mB$. Since $\mathcal{O}_X(mA) \otimes \mathcal{I}_Z$ is globally generated, $\mathbb{P}(\phi^{-1}(0))$ is base point free on $X - Z$. Thus Λ is also base point free on $X - Z$. Now the generic elements of $\frac{1}{m}\Lambda$ give nice liftings. □

2. Graded Linear Series

Here we extend a property of graded linear series in [4] to length two subscheme. The idea comes from [7]. For more details on the graded linear series we refer to [4], [5].

Let X be a projective variety of dimension n and H be a nef and big Q-Cartier divisor on X. Let W be a length 2 subscheme of X. We also fix positive dimensional subvariety Z through W. Let β be a nonnegative real number. We consider the following graded linear series.

$$\Omega^\beta = \Omega^\beta(W,H) = \bigoplus_{k=0}^{\infty} \Omega_k^{\beta k}$$

where

$$\Omega_k^{k\beta} = \{s \in H^0(X, \mathcal{O}_X(kH)) | s \in \mathcal{I}_W^{\lceil \beta k \rceil} \text{ for } 1 \le i \le h\}$$

if kH is Cartier and $\Omega_k^{k\beta} = 0$ otherwise. We define the degree of Ω^β as

$$\deg(\Omega^\beta) = \limsup_k \left\{ \frac{n!}{k^n} \dim \Omega_k^{\beta k} \right\}.$$

Let s_k be a general section of $\Omega_k^{\beta k}$. We define

$$M(\beta) = M(\beta, W, Z) = \liminf_k \left\{ \frac{1}{k} mult_Z(s_k) \right\}.$$

Proposition 2.1. *$M(\beta)$'s are lower convex on $\mathbb{R}_{\ge 0}$ for $1 \le i \le h$.*

Proof. First we show that $M(x)$ is lower convex on $\mathbb{Q}_{\ge 0}$. Let β_1, β_2 and $\lambda \in \mathbb{Q}_{>0}$ with $0 < \lambda < 1$. Let k_1 and k_2 be two sufficiently large and divisible integers such that $\lambda = \frac{k_1}{k_1+k_2}$. Let s_j, $j = 1, 2$, be generic section in

$$s_j \in H^0(X, \mathcal{O}_X(k_j H) \otimes \bigotimes_{i=1}^{h} \mathcal{I}_i^{\lceil \beta_j k_j \rceil})$$

such that

$$M(\beta_j) = \frac{mult_Z s_j}{k_j} + \delta_j$$

for some δ_1, $\delta_2 << 1$. Since

$$\lceil \beta_1 k_1 \rceil + \lceil \beta_2 k_2 \rceil \ge \lceil (\lambda\beta_1 + (1-\lambda)\beta_2)(k_1 + k_2) \rceil,$$

we have

$$M(\lambda\beta_1 + (1-\lambda)\beta_2) \le \frac{mult_Z(s_1 s_2)}{k_1 + k_2} = \lambda \frac{mult_Z s_1}{k_1} + (1-\lambda) \frac{mult_Z s_2}{k_2}.$$

Thus $M(x)$ is lower convex on $\mathbb{Q}_{\ge 0}$. Now this implies that $M(x)$ is continuous on $\mathbb{Q}_{\ge 0}$ and, since $M(x)$ is increasing function, $M(x)$ is continuous on $\mathbb{R}_{\ge 0}$. Thus $M(x)$ is lower convex on $\mathbb{R}_{\ge 0}$. ☐

The idea of the following proposition is due to Fujita [7]. Kawamata [10] and Helmke [8] also have similar result. Here we state it for only surfaces, which suit our purpose. But it can be easily extended to arbitrary dimensions.

Proposition 2.2. *We assume that X is a surface and that Z is a curve. Let β, γ in $\mathbb{R}_{\geq 0}$ such that $\deg(\Omega^x) > 0$ for all $\beta \leq x < \gamma$.*

1. *Assume that $W = \{p, q\}$, two distinct points. Let $m_p = ord_p Z$ and $m_q = ord_q Z$. Then*
$$\deg(\Omega^\gamma) \geq \deg(\Omega^\beta) - \int_\beta^\gamma 2(t - m_p M(t))dt - \int_\beta^\gamma 2(t - m_q M(t))dt.$$
2. *Assume that $W = \{p, v\}$ where $v \in \mathbb{P}(T_p X)$. Let $m = 2ord_W Z - ord_p Z$. Then*
$$\deg(\Omega^\gamma) \geq \deg(\Omega^\beta) - 2\int_\beta^\gamma 2(t - mM(t))dt.$$

Proof. The proofs are similar, so we only prove (2). Let $f : X_p \to X$ be the blowing up at p with the exceptional divisor E and let $g : X_v \to X_p$ be the blowing up at v with exceptional divisor F_v. Let Z_v be the proper transform of Z in X_v. Let F_p be the proper transform of E, $F := F_p + 2F_v$, and let $h = f \circ g$.

Let k be sufficiently large and divisible positive integer. For $\beta k \leq j \leq \gamma k$, we have the following exact sequences.

$$\begin{array}{rl} 0 \longrightarrow & H^0(X_v, \mathcal{O}_{X_v}(h^*kL - (j+1)F)) \\ \longrightarrow & H^0(X_v, \mathcal{O}_{X_v}(h^*kL - jF)) \xrightarrow{g_{k_j}} H^0(F, \mathcal{O}_F(-jF)) \end{array}$$

and

$$0 \longrightarrow \mathcal{O}_{F_p}(-jF - 2F_v) \longrightarrow \mathcal{O}_F(-jF) \longrightarrow \mathcal{O}_{2F_v}(-jF) \longrightarrow 0.$$

Since $H^0(F_p, \mathcal{O}_{F_p}(-jF - 2F_v)) = 0$,

$$H^0(F, \mathcal{O}_F(-jF)) \subset H^0(2F_v, \mathcal{O}_{2F_v}(-jF)).$$

Z appears with multiplicity $kM(\frac{j}{k})$ in the base locus of $|\Omega_k^j|$. Thus from Proposition 2.1 and $m = F_v \cdot Z_v$,

$$\begin{aligned} Img\,(g_{k,j}) &\subset H^0(2F_v, \mathcal{O}_{2F_v}(-jF - \lceil mkM(\frac{j}{k})\rceil)) \\ &\subset H^0(2F_v, \mathcal{O}_{2F_v}(-jF - \lceil j\frac{m}{\beta}M(\beta)\rceil)). \end{aligned}$$

We set

$$\alpha := \lceil j\frac{m}{\beta}M(\beta)\rceil.$$

Now from the the following exact sequence,

$$0 \longrightarrow \mathcal{O}_{F_v}(-jF - F_v - \alpha) \longrightarrow \mathcal{O}_{2F_v}(-jF - \alpha) \longrightarrow \mathcal{O}_{F_v}(-jF - \alpha) \longrightarrow 0.$$

we obtain

$$\dim Img\,(g_{k,j}) \leq 2 \cdot \left(j - \lceil j \frac{m}{\beta} M(\beta) \rceil \right) + 1.$$

Thus

$$\deg(\Omega^{\gamma}) \geq \deg(\Omega^{\beta}) - 2\left(1 - \frac{m}{\beta} M(\beta)\right)(\gamma^2 - \beta^2).$$

So by taking Riemann sum, we obtain the inequality. □

3. On Surfaces

In this section, first we will state and give proofs of three propositions, Proposition 3.2, 3.3, and 3.5, which together imply Theorem 0.2. The arguments are essentially the same and we could give one proof that works for all three of them. But, it is our intention to emphasize the handling of the separation of tangent directions and to show new phenomenon. So with the risk of boring the reader, we separate them into three propositions. One can see the similarities among the proofs as well as the differences. As for the separation of tangent directions (Proposition 3.5), they were very much inspired by Maşek [18], [20] and we follows his argument very closely.

In the second part of this section, we will give the proof of Proposition 0.3 and the numerical bound of Reider using graded linear series.

First we clarify what we mean by *order* of Q-divisors.

Definition 3.1. Let X be a variety. Let G be a Q-Cartier divisor on X.

1. Let p be a closed point of X. We define the order of G at p, ord_pG, as the coefficient of the exceptional divisor of the blowing up of X at p in the total transform of G.
2. Let W be a length two subscheme of X consists of two distinct closed points p and q in X. We define the order of G at W, ord_WG, as the minimum between ord_pG and ord_qG.
3. Let W be a non reduced length two subscheme of X supported at p, i.e. $W = \{p, v\}$ for some $v \in \mathbb{P}(T_pX)$. Let $h : U \longrightarrow X$ be the blowing up of X at p with the exceptional divisor E and let $g : V \longrightarrow U$ be the blowing up of U at v with the exceptional divisor F. We define the order of G at W, ord_WG, as the coefficient of $g^*E + F$.

3.1. **Proof of Theorem 0.2.**

First we deal with the case of freeness (cf. [4], [20]).

Proposition 3.2. *Let X be a smooth projective surface. Let A be a nef and big Q-divisor on X. Let $\Delta = \lceil A \rceil - A$ and let $\mu := ord_p\Delta$. Then $|K_X + \Delta + A|$ is free at p in the following two cases.*

1. $\mu \geq 2$.

2. $\mu < 2$, *and there are positive constants* β_1, β_2 *satisfying the following three inequalities.*

$$A^2 > \beta_2^2 \geq (2-\mu)^2. \tag{3.1}$$

$$A \cdot C \geq \beta_1 \text{ for all curve } C \text{ through } p. \tag{3.2}$$

$$\beta_1 \geq \min\left\{2-\mu, \frac{\beta_2}{\beta_2 - 1 + \mu}\right\}. \tag{3.3}$$

Proof. (1) From the assumption, we already have $\mathcal{I}_{Z(\Delta)} \subset \mathcal{I}_p$. Furthermore the connected component of $Z(\Delta)$ at p has dimension 0. Thus it follows from Theorem 1.11.

(2) From (3.1) and Riemann-Roch theorem, we can create a Q-Cartier divisor D on X such that

$$\eta := ord_p D > \beta_2 \geq 2 - \mu. \tag{3.4}$$

Then by tie-breaking technique, we can create a Q-divisor $G \underset{lin}{\overset{Q}{\sim}} \lambda A$ such that $\Delta + G$ is critical at p with the critical variety Z, and $\lambda < 1$. We note that essentially, $G = \lambda D$.

If the critical variety is a point, then Theorem 1.11 immediately implies that $|K_X + \Delta + A|$ is free at p. So we assume that Z is a curve. We note that Z is smooth at p by Proposition 1.8.

Again from Riemann-Roch theorem and (3.2), we can create a Q-divisor $B \underset{lin}{\overset{Q}{\sim}} H|_Z$ on Z with $ord_p B > \beta_1 - \epsilon$ for some $0 < \epsilon << 1$. Since we have surjections

$$\mathcal{I}^k_{p/X} \longrightarrow \mathcal{I}^k_{p/Z} \longrightarrow 0$$

for all $k \in \mathbb{N}$, we can find a local lifting D^{loc} of B with $ord_p D^{loc} > \beta_1 - \epsilon$. Let D' be a nice lifting of B with respect to $\Delta + G$.

From Theorem 1.14, there exist $0 < \lambda'$ such that

$$p \subset Z(\Delta + (1-\delta)G + \lambda' D')$$

and

$$\lambda'(\beta_1 - \epsilon) \leq 2 - \mu - ord_p(1-\delta)G$$

for some $0 < \delta << 1$. Content-Length: 2698 X-Lines: 90 Status: RO

We note that from (3.4), $\frac{2-\mu-\lambda\eta}{1-\lambda}$ is a decreasing function of λ. Also note that since the critical variety is a divisor, $\mu + \lambda\eta \geq 1$. Thus

$$\frac{2-\mu-\lambda\eta}{1-\lambda} < \min\left\{2-\mu, \frac{\beta_2}{\beta_2 - 1 + \mu}\right\}.$$

Taking ϵ and δ sufficiently small, $\lambda' + \lambda < 1$ by (3.3). Since the critical variety of $\Delta + (1-\delta)G + \lambda' D'$ is a point, by Theorem 1.11 $|K_X + \Delta + A|$ is free at p. □

Next we consider the separation of distinct points (cf [18], [20]).

Proposition 3.3. *Let X be a smooth projective surface. Let W be a length two subscheme consisting of two distinct closed points p and q of X. Let A be a nef and big Q-divisor on X. Let $\Delta = \lceil A \rceil - A$ and $\mu := ord_W \Delta$. Then $|K_X + \Delta + A|$ separates p and q in the following two cases.*

1. *$\mu \geq 2$.*
2. *$\mu < 2$, and there are positive constants β_1, β_2 satisfying the following three inequalities.*

$$A^2 > 2\beta_2^2 \geq 2(2-\mu)^2. \tag{3.5}$$

$$A \cdot C \geq (length\,(W \cap C)) \cdot \beta_1 \text{ for all curve } C \text{ in } X. \tag{3.6}$$

$$\beta_1 \geq \min\left\{2-\mu, \frac{\beta_2}{\beta_2 - 1 + \mu}\right\}. \tag{3.7}$$

Proof. (1) From the assumption, we already have $\mathcal{I}_{Z(\Delta)} \subset \mathcal{I}_W$. Furthermore the connected component of $Z(\Delta)$ at p has dimension 0. Thus it follows from Theorem 1.11.

(2) Since the length of W is two, from (3.5) and Riemann-Roch theorem, we can create a Q-Cartier divisor D on X such that

$$\eta := ord_W D > \beta_2 \geq 2 - \mu. \tag{3.8}$$

Then by tie-breaking technique, we can create a Q-divisor $G \underset{lin}{\overset{Q}{\approx}} \lambda A$ such that $\Delta + G$ is critical at W with the critical variety Z, and $\lambda < 1$. We note that essentially, $G = \lambda D$.

If the critical variety is a point, then Theorem 1.11 immediately implies that $|K_X + \Delta + A|$ separates W. So we assume that Z is a curve. We note that Z is smooth at $W \cap Z$ by Proposition 1.8.

Again from Riemann-Roch theorem and (3.6), we can create a Q-divisor $B \underset{lin}{\overset{Q}{\approx}} H|_Z$ on Z with $ord_{W\cap Z} B > \beta_1 - \epsilon$ for some $0 < \epsilon << 1$. Since we have surjections

$$\mathcal{I}^k_{W\cap Z/X} \longrightarrow \mathcal{I}^k_{W\cap Z/Z} \longrightarrow 0 \tag{3.9}$$

for all $k \in \mathbb{N}$, we can find a local lifting D^{loc} of B with

$$ord_{W\cap Z} D^{loc} > \beta_1 - \epsilon. \tag{3.10}$$

Let D' be a nice lifting of B with respect to $\Delta + G$.

From Theorem 1.14, there exist $0 < \lambda'$ such that

$$W \subset Z(\Delta + (1-\delta)G + \lambda' D')$$

and

$$\lambda'(\beta_1 - \epsilon) \leq 2 - \mu - ord_W (1-\delta)G, \tag{3.11}$$

for some $0 < \delta << 1$. Also since the critical variety is a divisor, by examining the coefficient of the exceptional divisor of the blowing up at $Z \cap W$ we obtain

$$\lambda'(\beta_1 - \epsilon) \leq 1. \tag{3.12}$$

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We note that from (3.8), $\frac{2-\mu-\lambda\eta}{1-\lambda}$ is a decreasing function of λ. Thus

$$\frac{2-\mu-\lambda\eta}{1-\lambda} < 2-\mu.$$

Also note that

$$\min\left\{\frac{2-\mu-\lambda\eta}{1-\lambda}, \frac{1}{1-\lambda}\right\} \leq \frac{\eta}{\eta-1+\mu}.$$

Thus

$$\min\left\{\frac{2-\mu-\lambda\eta}{1-\lambda}, \frac{1}{1-\lambda}\right\} < \min\left\{2-\mu, \frac{\beta_2}{\beta_2-1+\mu}\right\}. \tag{3.13}$$

Taking ϵ and δ sufficiently small, we have $\lambda'+\lambda<1$ by (3.7), (3.11), (3.12), and (3.13). Since the critical variety of $\Delta+(1-\delta)G+\lambda' D'$ is a point, by Theorem 1.11 $|K_X+\Delta+A|$ separates W. $\square$

Remark 3.4. Proposition 4.2 in [20] can also be proved in a similar manner.

Now we consider the separation of tangent directions (cf. [18], [20]).

Proposition 3.5. *Let X be a smooth projective surface. Let W be a length two subscheme of X consisting of a closed point p in X and a tangent direction v of X at p. Let A be a nef and big Q-Cartier divisor on X. Let $\Delta=\lceil A\rceil - A$ and $\mu = ord_W\Delta$. Then $|K_X+\Delta+A|$ separates v at p in the following two cases.*

1. $\mu \geq 2$.
2. $\mu < 2$, *and there are positive constants β_1, β_2 satisfying the following three inequalities.*

$$A^2 > 2\beta_2^2 \geq 2(2-\mu)^2. \tag{3.14}$$

$$A\cdot C \geq (length\,(W\cap C))\cdot\beta_1 \textit{ for all curve } C \textit{ in } X. \tag{3.15}$$

$$\beta_1 \geq \min\left\{2-\mu, \frac{\beta_2}{\beta_2-1+\mu}\right\}. \tag{3.16}$$

Proof. (1) From the assumption, we already have $\mathcal{I}_{Z(\Delta)} \subset \mathcal{I}_W$. Furthermore the connected component of $Z(\Delta)$ at p has dimension 0. Thus it follows from Theorem 1.11.

(2) Since the length of W is two, from Riemann-Roch theorem and (3.14), we can create a Q-Cartier divisor D on X such that

$$\eta := ord_W D > 2-\mu. \tag{3.17}$$

Then by tie-breaking technique, we can create a Q-divisor $G \underset{lin}{\overset{Q}{\sim}} \lambda A$ such that $\Delta+G$ is critical at W with the critical variety Z, and $\lambda<1$. We note that essentially, $G=\lambda D$.

If the critical variety is a point, then again Theorem 1.11 immediately implies that $|K_X+\Delta+A|$ separates W. So we assume that Z is a curve. Here Z may be singular at p. But from tie-breaking, we may assume that

$$mult_p\lfloor\Delta+G\rfloor \leq 2,$$

thus $mult_p Z \leq 2$. Below, we separate into two cases.

(2.1) First, assume that $mult_p Z = 1$. Near p, one of the following cases should occur.

$$\lfloor \Delta + G \rfloor = Z \quad \text{and} \quad v \notin \mathbb{P}(T_p Z) \tag{3.18}$$

$$\lfloor \Delta + G \rfloor = Z \quad \text{and} \quad v \in \mathbb{P}(T_p Z) \tag{3.19}$$

$$\lfloor \Delta + G \rfloor = Z + Z', \quad Z \neq Z', \quad \text{and} \quad v \in \mathbb{P}(T_p Z) \tag{3.20}$$

$$\lfloor \Delta + G \rfloor = Z + Z', \quad Z \neq Z', \quad \text{and} \quad v \notin \mathbb{P}(T_p Z) \tag{3.21}$$

$$\lfloor \Delta + G \rfloor = 2Z \quad \text{and} \quad v \notin \mathbb{P}(T_p Z) \tag{3.22}$$

where Z' is another curve through p. As for (3.18), since $\mathcal{I}_Z(\Delta + G) \subset \mathcal{I}_W$ and $\mathcal{O}_X(-\lfloor \Delta + G \rfloor) \not\subset \mathcal{I}_W$, again Theorem 1.11 implies that $|K_X + \Delta + A|$ separates W. For the remaining cases, we can argue all together as follows. The argument here is similar to the one in the previous proposition.

As before, from (3.15) and the Riemann-Roch theorem, we can create a Q-divisor $B \underset{lin}{\overset{Q}{\sim}} A|_Z$ on Z so that $ord_{W \cap Z} B > \beta_1 - \epsilon$ for some $0 < \epsilon << 1$. Since we have surjections

$$\mathcal{I}^k_{W/X} \longrightarrow \mathcal{I}^k_{W \cap Z/Z} \longrightarrow 0 \tag{3.23}$$

for all $k \in \mathbb{N}$, we can find a local lifting D^{loc} of B with

$$ord_W D^{loc} > \beta_1 - \epsilon. \tag{3.24}$$

We note the difference between (3.9),(3.10) and (3.23), (3.24). Let D' be a nice lifting of B with respect to $\Delta + G$.

From Theorem 1.14, there exist $0 < \lambda'$ such that Content-Length: 5646 X-Lines: 184 Status: RO

$$\lambda'(\beta_1 - \epsilon) \leq 2 - \mu - (1 - \delta) ord_W G$$

and

$$W \subset Z(\Delta + (1 - \delta)G + \lambda' D')$$

for some $0 < \delta << 1$.

We note that from (3.17), $\frac{2-\mu-\lambda\eta}{1-\lambda}$ is a decreasing function of λ. Also note that $\mu + \lambda\eta \geq 1$ since Z contains W for (3.19) and (3.20), and $mult_p \lfloor \Delta + G \rfloor = 2$ for the rest of the cases. Thus

$$\frac{2 - \mu - \lambda\eta}{1 - \lambda} < \min\left\{2 - \mu, \frac{\beta_2}{\beta_2 - 1 + \mu}\right\}.$$

Taking ϵ and δ sufficiently small, $\lambda' + \lambda < 1$ by (3.16). Since the critical variety of $\Delta + (1 - \delta)G + \lambda' D'$ is a point, by Theorem 1.11 $|K_X + \Delta + A|$ separates W.

So far, we have shown the handling of all the possible cases in general except when Z is singular at p. For surfaces, it is bounded by 2 as we have already mentioned. Unfortunately, even for surfaces it is already too big to argue as (2.1). So we take an alternative and restrict the linear system to the proper transform of Z in the blowing up of X at p using Lemma 3.6.(1), and apply Lemma 3.6.(2). The reason to restrict to the proper transform of Z in the blowing of X at p rather than in an embedded resolution of G is as follows.

Let $f_1 : V_1 \longrightarrow X$ and $f_2 : V_2 \longrightarrow X$ be two birational morphisms from smooth surfaces V_1 and V_2 such that $f_2 = f_1 \circ g$ for some $g : V_2 \longrightarrow V_1$. Then

$$f_{2*}\mathcal{O}_{V_2}(K_{V_2} - \lfloor f_2^*(K_X + G)\rfloor) \subset f_{1*}\mathcal{O}_{V_1}(K_{V_1} - \lfloor f_1^*(K_X + G)\rfloor).$$

So, even though we do need to take advantage of the fact that (X, G) is not log canonical at p, to use Lemma 3.6.(2) most effectively we need to take the *smallest* resolution whose corresponding ideal sheaf on X for $(1-\epsilon)G$, $0 < \epsilon << 1$, is non trivial at p. In the following situation, the blowing up of X at p turned out to be the right one for us (cf. [18])

(2.2) Assume that $mult_p Z = 2$. Here we need the following well-known

Lemma 3.6. 1. *Let S be a smooth projective surface and let M be a nef and big Q-Cartier divisor on M. Then $H^i(S, \mathcal{O}_S(K_S + \lceil M\rceil)) = 0$ for all $i > 0$.*
2. *Let C be an irreducible Gorenstein curve and B be a Cartier divisor on C. Then $\mathcal{O}_C(K_C + B)$ is globally generated if $degB > 1$.*

Proof. (1) is Lemma 1.1 in [3]. For (2), we note that from the Serre's duality and the assumption $H^1(C, \mathcal{I}_p \otimes \mathcal{O}_C(K_C + B)) = Hom(\mathcal{I}_p, \mathcal{O}_C(-B)) = 0$ for all closed point p in C. □

Now let $f : Y \longrightarrow X$ be the blowing up of X at p with the exceptional divisor E. Let F be the proper transform of Z in Y. Put $H = K_X + \Delta + A$. We have the following two exact sequences

$$0 \longrightarrow \mathcal{O}_Y(-E - F + f^*H) \longrightarrow \mathcal{O}_Y(-E + f^*H) \longrightarrow \mathcal{O}_F(-E + f^*H) \longrightarrow 0$$

and

$$0 \longrightarrow \quad \mathcal{O}_X(-Z + H) \quad \longrightarrow \quad \mathcal{I}_p \otimes \mathcal{O}_X(H) \quad \longrightarrow \quad \mathcal{I}_{p/Z} \otimes \mathcal{O}_Z(H) \quad \longrightarrow 0,$$

where the 0-th cohomology groups of the first two columns are isomorphic. Note that $\lceil K_Y - f^*(K_X + G)\rceil = -F - E$ and $\lceil -G\rceil = Z$, Then since

$$H^1(Y, \mathcal{O}_Y(-E - F + f^*H)) = H^1(X, \mathcal{O}_X(-Z + H)) = 0$$

by Lemma 3.6.(1), the third column also induces an isomorphism between 0-the cohomology groups. Thus to separates v at p, it is enough to show that $\mathcal{O}_F(-E - F + f^*H)$ is globally generated. Since $\lceil -f^*(G - Z)\rceil = 0$, we have

$$\begin{aligned}\mathcal{O}_F(-E-F+f^*H) &= \mathcal{O}_F(K_F+\lceil f^*(\Delta+A-G)\rceil)\\ &= \mathcal{O}_F(K_F+f^*(\Delta+A-Z))\\ &= \mathcal{O}_F(K_F+f^*\lceil\Delta+A-G\rceil).\end{aligned}$$

Thus, from Lemma 3.6.(2) it is enough to show

$$Z\cdot\lceil(\Delta+A-G)\rceil>1. \tag{3.25}$$

But

$$\begin{aligned}&Z\cdot\lceil(\Delta+A-\lambda D)\rceil\\ =&Z\cdot((1-\lambda)A)+Z\cdot(\Delta+\{\lambda D\})\\ \geq&(1-\lambda)2\beta_1+2\cdot ord_p(\Delta+\{\lambda D\})\\ \geq&(1-\lambda)2\beta_1+2\cdot ord_W(\Delta+\{\lambda D\})\\ \geq&(1-\lambda)2\beta_1+2\cdot\lambda\eta+2\cdot\mu-3 \text{ if } v\in\mathbb{P}(C_pZ), \text{ or}\\ \geq&(1-\lambda)2\beta_1+2\cdot\lambda\eta+2\cdot\mu-2 \text{ if } v\notin\mathbb{P}(C_pZ).\end{aligned}$$

So (3.25) in turn follows from

$$\beta_1>\frac{2-\mu-\lambda\eta}{1-\lambda}$$

by taking ϵ and δ small enough. But $\lambda\eta+\mu\geq 1$ since $mult_pZ=2$. Thus as before,

$$\frac{2-\mu-\lambda\eta}{1-\lambda}<\min\left\{2-\mu,\frac{\beta_2}{(\beta_2-1+\mu)}\right\}.$$

□

3.2. **Reider's Bound.** The theorem of Reider [21] implies the following

Proposition 3.7. *Let X be a smooth projective surface. Let A be a Cartier divisor on X. Then K_X+A is very ample if $A^2\geq 10$ and $A\cdot C\geq 3$ for all curves C in X.*

Proposition 0.3 implies the part of the separation of tangent directions, which we prove below.

Proof of Proposition 0.3. If $\mu_p\geq 3$, then it follows from Theorem 1.11. Actually, it generates 1-jet. So we assume that $\mu_p<3$. We set $\mu=ord_W\Delta$.

From Riemann-Roch theorem and (0.1), we can create a Q-Cartier divisor D on X such that

$$\eta_p:=ord_pD>3-\mu_p. \tag{3.26}$$

We set $\eta:=ord_WD$. Then by tie-breaking technique, we can create a Q-divisor $G\underset{lin}{\overset{Q}{\sim}}\lambda A$ such that $\Delta+G$ is critical at W with the critical variety

Z, and $\lambda < 1$. We note that essentially, $G = \lambda D$. Content-Length: 156 X-Lines: 4 Status: RO

If the critical variety is a point, then again Theorem 1.11 immediately implies that $|K_X + \Delta + A|$ separates W. So we assume that Z is a curve.

From tie-breaking, we may assume that

$$mult_p\lfloor \Delta + G \rfloor \leq 2,$$

thus $mult_p Z \leq 2$. Below, we separate into three cases. Content-Length: 55 X-Lines: 1 Status: RO

(2.1) First, assume that $mult_p Z = 1$ and $v \in T_p Z$.

From (0.2) and the Riemann-Roch theorem, we can create a Q-divisor $B \underset{lin}{\overset{Q}{\approx}} A|_Z$ on Z so that $2 \cdot ord_W B > \sigma_1 - \epsilon$ for some $0 < \epsilon << 1$. Since we have surjections

$$\mathcal{I}^k_{W/X} \longrightarrow \mathcal{I}^k_{W/Z} \longrightarrow 0$$

for all $k \in \mathbb{N}$, we can find a local lifting D^{loc} of B with

$$2 \cdot ord_W D^{loc} > \sigma_1 - \epsilon. \tag{3.27}$$

Let D' be a nice lifting of B with respect to $\Delta + G$. From Theorem 1.14, there exist $0 < \lambda'$ such that

$$\lambda'(\sigma_1 - \epsilon) \leq 4 - 2 \cdot \mu - 2 \cdot (1 - \delta) ord_W G$$

and

$$W \subset Z(\Delta + (1 - \delta)G + \lambda' D')$$

for some $0 < \delta << 1$. From the assumption, $\mu_p + \lambda\eta_p \geq 1$ and $2\mu + 2\lambda\eta - (\mu_p + \lambda\eta_p) \geq 1$. Thus

$$\frac{4 - 2 \cdot \mu - 2 \cdot \lambda\eta}{1 - \lambda} \leq \frac{3 - \mu_p - \lambda\eta_p}{1 - \lambda} < \frac{2\sigma_2}{\sigma_2 - 1 + \mu_p}.$$

Thus $\lambda + \lambda' < 1$ and $|K_X + \Delta + A|$ separates W. Content-Length: 819 X-Lines: 27 Status: RO

(2.2) Second, assume that $mult_p Z = 1$ but $v \notin T_p Z$. In this case, we may assume that

$$ord_p\lfloor \Delta + G \rfloor = 2. \tag{3.28}$$

Otherwise, for the same reason as the case (3.18), Theorem 1.11 would imply that $|K_X + \Delta + A|$ separates W.

Now from (0.2) and the Riemann-Roch theorem, we can create a Q-divisor $B \underset{lin}{\overset{Q}{\approx}} A|_Z$ on Z so that $ord_p B > \sigma_1 - \epsilon$ for some $0 < \epsilon << 1$. Since we have surjections

$$\mathcal{I}^k_{W/X} \longrightarrow \mathcal{I}^k_{W \cap Z/Z} \longrightarrow 0$$

for all $k \in \mathbb{N}$, we can find a local lifting D^{loc} of B with

$$ord_W D^{loc} > (\sigma_1 - \epsilon). \tag{3.29}$$

Note the difference between (3.27) and (3.29). Let D' be a nice lifting of B with respect to $\Delta + G$.

From Theorem 1.14, there exist $0 < \lambda'$ such that

$$2 \cdot \lambda'(\sigma_1 - \epsilon) \leq 4 - 2 \cdot \mu - 2 \cdot (1-\delta) ord_W G$$

and

$$W \subset Z(\Delta + (1-\delta)G + \lambda' D')$$

for some $0 < \delta << 1$. Since $\mu_p + \lambda\eta_p \geq 2$,

$$\frac{4 - 2\cdot\mu - 2\cdot\lambda\eta}{1-\lambda} \leq \frac{4 - \mu_p - \lambda\eta_p}{1-\lambda} < \frac{2\sigma_2}{\sigma_2 - 2 + \mu_p}.$$

But from (0.1),

$$\frac{2\sigma_2}{\sigma_2 - 2 + \mu_p} \leq \frac{4\sigma_2}{\sigma_2 - 1 + \mu_p}.$$

Thus $\lambda + \lambda' < 1$ by (0.3). Content-Length: 6181 X-Lines: 202 Status: RO

(2.3) Assume that $mult_p Z = 2$. The proof is identical to that of case (2.2) in Proposition 3.5, except the parts following inequality (3.25). We replace them with the followings.

$$\begin{aligned}
&Z \cdot \lceil(\Delta + A - \lambda D)\rceil \\
=&Z \cdot ((1-\lambda)A) + Z \cdot (\Delta + \{\lambda D\}) \\
\geq&(1-\lambda)\sigma_1 + ord_p(\Delta + \{\lambda D\}) \\
\geq&(1-\lambda)\sigma_1 + \lambda\eta_p + \mu_p - 2.
\end{aligned}$$

So (3.25) follows from

$$\sigma_1 > \frac{3 - \mu_p - \lambda\eta_p}{1-\lambda}$$

taking ϵ and δ small enough. But $\lambda\eta_p + \mu_p \geq 1$, (in fact ≥ 2). Thus as before,

$$\frac{3 - \mu_p - \lambda\eta_p}{1-\lambda} < \frac{2\sigma_2}{\sigma_2 - 1 + \mu_p}.$$

□

Next we use Proposition 2.2 to separates distinct points in Proposition 3.7. Most of the proof is overlapping with that of Proposition 3.3. So we only sketch them.

Sketch of Proof. Here we keep the same notation introduced in Proposition 3.3. The only case that need to be considered is when the critical variety Z is a curve and G is critical at p and also at q. Here we need to show that the following inequality holds

$$2 - \lambda\eta < (1-\lambda)2. \tag{3.30}$$

One can easily check that if $ord_W D > 3$, then (3.30) holds. So from now on, we will show that in fact we can create such D. We may assume that

$$\eta - 2 << 1.$$

Let $\Omega^x = \Omega^x(W, A)$ and $M(x) = M(x, W, Z)$. Since $M(x)$ is a convex function of x, $\frac{M(\eta)}{\eta} \geq \frac{M(\lambda\eta)}{\lambda\eta}$. Thus

$$M(\eta) \geq \frac{1}{\lambda} \geq \frac{\eta}{2}.$$

Let s_γ be the left derivative of $M(x)$ at $\gamma > 2$. Let $s = \lim_{\gamma \to 2} s_\gamma$ and $T(x) = s(x-2) + 1$. Again, since $M(x)$ is convex, we may assume that $T(x) \leq M(x)$.

Suppose $T(x) = M(x)$, for all $x \geq 2$ with $\deg \Omega^x > 0$. We may assume that $\eta - 2 << 1$ so that $T(\eta) = s(\eta - 2) + 1 < \eta - 1$. Then, since $\frac{1}{\lambda} = M(\eta) = T(\eta)$,

$$\frac{2 - \lambda\eta}{1 - \lambda} = \frac{2M(\eta) - \eta}{M(\eta) - 1} = \frac{2T(\eta) - \eta}{T(\eta) - 1} < 1.$$

Thus (3.30) holds and $|K_X + A|$ separates p and q.

Now we assume that $T(x) < M(x)$ for some $x \geq 2$ with $\deg \Omega^x > 0$. We have $\deg \Omega^x \geq 10 - 2x^2$. Let us fix $r > 2$ and let $\beta_0 = 2 - \frac{1}{s}$. Let

$$f(s) = 10 - 2 \cdot \beta_0^2 - 2 \cdot \int_{\beta_0}^{r} 2(x - T(x))dx.$$

Then simple calculation shows that $f(s)$ achieves the minimum $g(r) = -6 + 8r - 2r^2$. $g(r) > 0$ for $2 \leq r < 3$. Since $M(x) > T(x)$ for some $x \geq 2$ with $\deg \Omega^x > 0$, by Proposition 2.2, $g(3) \geq 0$ already implies that $\deg \Omega^{\{3+\epsilon\}} > 0$ for a sufficiently small $\epsilon > 0$. Thus $\deg \Omega^3 > 0$. ☐

4. On Higher Dimensions

Proof of Theorem 0.5. First, $|-nK_X|$ induces morphism for $n \geq 2$ and $|-nK_X|$ separates two distinct points for $n \geq 4$ (cf. Corollary 2.6, 2.10 in [17]).

Now let p be a closed point in X and $v \in \mathbb{P}(T_pX)$. We choose a generic D from $|-2K_X \otimes \mathcal{I}_p|$. We separates into two cases according to the singularities of D at p.

(1) Suppose that D is singular at p. Here we argue with the multiplier ideals. Let D_1, D_2 be two generic elements of $|-2K_X|$ through p. Then by assumption, $mult_p D_i \geq 2$ for $i = 1, 2$. Furthermore, since $|-2K_X|$ induces a finite morphism, we may assume that D_1 and D_2 are smooth and intersects properly in a punctured neighborhood of p. We pick a Q-divisor $\Delta \underset{lin}{\overset{Q}{\sim}} \epsilon \cdot (-K_X)$ with $0 < \epsilon << 1$. Then

$$\mathcal{I}_{Z((1-\delta)(\Delta + D_1 + D_2))} \subset \mathcal{I}_p^2$$

and $Z((1-d)(\Delta+D_1+D_2))$ has no positive dimensional components through p for sufficiently small $0<\delta<<1$. Thus by Theorem 1.11, $|-tK_X|$ is very ample.

(2) Now assume that D is smooth at p. If $v \notin \mathbb{P}(T_pD)$, then $|-tK_X|$ separates $\{p,v\}$ by the remark given in the beginning of the proof and the assumption $t \geq 4$. So we assume that $v \in \mathbb{P}(T_pX)$.

We note that D is connected since $H^1(X, \mathcal{O}_X(-2K_X)) = 0$, and $K_D = -K_X|_D$ and $-tK_X|_D = tK_D$. We may assume that D is smooth. In fact, D is normal by the assumption. Thus (if necessary) we can work on the minimal resolution of D instead of D. Then the pull back of $-K_X|_D$ is no longer ample but it is still nef and big and the bound of the intersection numbers with the curves through p is unchanged.

Now we apply Proposition 3.7 to (D, tK_D). Since $K_D^2 \geq 2$ and $t \geq 4$, $|tK_D|$ separate v at p. Then since $H^1(X, \mathcal{O}_X(-(2+t)K_X)) = 0$, $|-tK_X|$ also separates v at p. $\square$

Remark 4.1. The argument in (1) works with D being smooth at p. Only then the bound is worse. This way, one can obtain effective bounds on smooth varieties with (anti)canonical ample divisors.

The proof of Proposition 0.6 is similar, so we leave it to the interesting readers.

Proof of Theorem 0.7. We begin with the following simple lemma.

Lemma 4.2. *With the same notations as above, we assume further that there is a Q-divisor G in X satisfying the following conditions:*

1. *$H-G$ is nef and big.*
2. *$\Delta+G$ is 2-critical at p, and the 2-critical variety is a point.*

Then for generic D in $|K_X+\Delta+H\otimes\mathcal{I}_p|$, $W \not\subset D$ where W is the connected component of $Z(\Delta+G)$ at p.

Proof. From the assumptions, we have the following exact sequence.

$$0 \longrightarrow \mathcal{O}_X(K_X+\Delta+H)\otimes\mathcal{I}_{Z(\Delta+G)} \longrightarrow \mathcal{O}_X(K_X+\Delta+H)\otimes\mathcal{I}_{Z(\Delta+(1-\epsilon)G)} \longrightarrow \mathcal{J} \longrightarrow 0,$$

where the connected component of $Z(\Delta+(1-\epsilon)G)$ at p is p itself, and $\mathcal{J}$ is supported at p for $0<\epsilon<<1$. Thus $H^0(X,\mathcal{J}) \neq 0$. Content-Length: 0 X-Lines: 0 Status: RO

From Proposition 1.4, we also have the exact sequence in 0-th cohomology groups derived from the above exact sequence. Now we assume the contrary, i.e. every section of $H^0(X, \mathcal{O}_X(K_X+\Delta+H)\otimes\mathcal{I}_p)$ is singular at p. Then since W has length 2 by the assumption 2,

$$H^0(X, \mathcal{O}_X(K_X+\Delta+H)\otimes\mathcal{I}_{Z(\Delta+G)}) = H^0(X, \mathcal{O}_X(K_X+\Delta+H)\otimes\mathcal{I}_{Z(\Delta+(1-\epsilon)G)}),$$

and $H^0(X,\mathcal{J}) = 0$. So it is a contradiction. $\square$

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Below we will create a Q-divisor G satisfying the two conditions in the previous lemma.

Let $q(k)$, $k = 1, \cdots n$, be a sequence of rational numbers such that

$$H^d \cdot Z > \left\{ \sum_{k=1}^{n} q(k) \right\}^d$$

for all subvariety Z through p where $d = \dim Z$, and $q(k) > \sqrt[k]{2} \cdot k$ for all $k = 1, \cdots, n$. We set

$$A := \frac{1}{\sum_{k=1}^{n} q(k)} H.$$

Let W_0 be a length 2 subscheme of X supported at p and we put $d_0 := n$.

(1-st step)

From the Riemann-Roch theorem we can create a Q-divisor $D_1 \underset{lin}{\overset{Q}{\sim}} q(d_0)A$ such that $ord_{W_0} D_1 > n$. Then by tie-breaking technique, we can create a Q-Cartier divisor

$$G_1 \underset{lin}{\overset{Q}{\sim}} \lambda_1 q(d_0) A$$

for some $\lambda_1 < 1$ so that $\Delta + G_1$ is 2-critical at p with the 2-critical variety Z_1. We note that essentially $G_1 = \lambda_1 D_1$. Content-Length: 6673 X-Lines: 253 Status: RO

If Z_1 is a point, then Lemma 4.2 finishes the proof. So we assume that $d_1 := \dim Z_1 > 0$.

((i+1)-th step) Suppose that we have created a Q-Cartier divisor

$$G_i \underset{lin}{\overset{Q}{\sim}} \lambda_i (\sum_{k=1}^{i} q(d_{k-1})) A$$

with $\lambda_i < 1$, such that $\Delta + G_i$ is 2-critical at p with the 2-critical variety Z_i at p with $d_i := \dim Z_i > 0$.

Let T be a curve on Z_i obtained by generic hyperplanes through p. Now from the Riemann-Roch theorem, we can create a family of Q-divisors $\{B_c\}_{c \in T-p}$ on Z_i such that

$$B_c \underset{lin}{\overset{Q}{\sim}} q(d_i) A|_{Z_i}$$

and

$$ord_{W_c} B_c > d_i$$

where W_c is the length two subscheme of T supported at c.

Let D_c^{loc} be a local lifting of B_c such that $ord_{W_c} D_c^{loc} > d_i$. Then from Proposition 1.10,

$$W_c \subset Z(\Delta + (1-\delta)G_i + D_c^{loc}) \tag{4.1}$$

for sufficiently small $0 < \delta << 1$. Now we take the limit (cf. Theorem 6.7.1 [14])

$$B_0 := limit_{c \to 0} B_c$$

and let D_{i+1} be a nice lifting of B_0 with respect to $\Delta + G_i$. Then from Proposition 1.9, Theorem 1.14 and (4.1), $Z(\Delta + (1-\delta)G_i + D_{i+1})$ contains a length two subscheme. Thus we can find a Q-divisor

$$G_{i+1} \underset{lin}{\overset{Q}{\sim}} \lambda_{i+1} (\sum_{k=1}^{i+1} q(d_{k-1}))A$$

for some $\lambda_{i+1} < 1$ so that $\Delta + G_{i+1}$ is 2-critical at p. □

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