CUT-AND-STACK SIMPLE WEAKLY MIXING MAP WITH COUNTABLY MANY PRIME FACTORS

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Abstract. Via the cut-and-stack construction we produce a 2-fold simple weakly mixing transformation which has countably many proper factors and all of them are 2-to-1 and prime.

0. Introduction

Let T be an (invertible) transformation of a probability space $(X, \mathfrak{B}, \mu)$. A measure λ on $X \times X$ is called a 2-*fold self-joining* of T if it is $T \times T$-invariant and it projects onto μ on both coordinates. Denote by $J_2^e(T)$ the set of all ergodic 2-fold self-joinings of T. Let $C(T)$ stand for the centralizer of T, i.e. the group of all μ-preserving transformations commuting with T. Given $S \in C(T)$, we let $\mu_S(A \times B) := \mu(A \cap SB)$ for all $A, B \in \mathfrak{B}$. Of course, $\mu_S \in J_2^e(T)$. If $J_2^e(T) \subset \{\mu_S \mid S \in C(T)\} \cup \{\mu \times \mu\}$ then T is called 2-*fold simple* [Ve], [dJR]. By a *factor* of T we mean a T-invariant sub-σ-algebra of $\mathfrak{B}$. If T has no non-trivial proper factors then T is called *prime*. In [Ve] it was shown that if T is 2-fold simple then for each non-trivial factor $\mathfrak{F}$ of T there exists a compact (in the strong operator topology) subgroup $K_{\mathfrak{F}} \subset C(T)$ such that

$$\mathfrak{F} = \{A \in \mathfrak{B} \mid \mu(kA \triangle A) = 0 \text{ for all } k \in K_{\mathfrak{F}}\}.$$

Hence $\mathfrak{F}$ (or, more precisely, the restriction of T to $\mathfrak{F}$) is prime if and only if $K_{\mathfrak{F}}$ is a maximal compact subgroup of $C(T)$.

The purpose of our paper is to produce via cutting-and-stacking a 2-fold simple weakly mixing transformation which has countably many factors and all of them are prime. We also specify which of these factors are conjugate.

Note that the only known example of a 2-fold simple T with non-unique prime factors was constructed by Glasner and Weiss [GlW] as an inverse limit of certain horocycle flows, i.e. in a quite different way. The subtle results of M. Ratner on joinings of horocycle flows [Ra], as well as the existence of a lattice in $SL_2(\mathbb{R})$ with rather special properties play a crucial role in [GlW]. We notice also that T has many non-prime factors as well. Note that for some time it was not obvious at all whether it is possible to construct such an example by means of the more elementary cutting-and-stacking technique (see [Th]). To achieve this purpose we use the idea suggested first in [dJ] (see also [Ma], [Da3], [Da4]): we construct a rank-one action of an auxiliary non-Abelian group $G = \mathbb{Z} \times (\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z})$ such that the $\mathbb{Z}$-subaction

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is 2-fold simple and has centralizer coinciding with the full G-action. It remains to list all non-trivial finite subgroups of G:

$$\{\{(0,b,1),(0,0,0)\} \mid b\in\mathbb{Z}, b\neq 0\}$$

and note that all of them are maximal. While constructing this action we follow the (C,F)-formalism developed in [Da4, Section 6].

1.(C,F)-construction

We remind here the (C,F)-construction of funny rank-one actions (see [Da1]–[Da4], [DaS] and [dJ] for details). Let G be a countable group. Given a finite subset $F\subset G$, we denote by λ_F the probability equidistributed on F. Now let $(F_n)_{n\geq 0}$ and $(C_n)_{n\geq 1}$ be two sequences of finite subsets in G such that the following are satisfied:

$$F_0=\{0\},\ \#C_n>1, \tag{1-1}$$

$$F_nC_{n+1}\subset F_{n+1}, \tag{1-2}$$

$$F_nc\cap F_nc'=\emptyset \text{ for all } c\neq c'\in C_{n+1}, \tag{1-3}$$

$$\lim_{n\to\infty}\frac{\#F_n}{\#C_1\cdots\#C_n}<\infty. \tag{1-4}$$

We put $X_n:=F_n\times C_{n+1}\times C_{n+2}\times\cdots$ and define a map $i_n:X_n\to X_{n+1}$ by setting

$$i_n(f_n,d_{n+1},d_{n+2},\dots):=(f_nd_{n+1},d_{n+2},\dots).$$

In view of (1-1), X_n endowed with the infinite product topology is a compact Cantor space. It follows from (1-2) and (1-3) that i_n is well defined and it is a topological embedding of X_n into X_{n+1}. Denote by X the topological inductive limit of the sequence $(X_n,i_n)_{n=1}^{\infty}$. In the sequel we will suppress the canonical embedding maps and just write $X=\bigcup_{n\geq 0}X_n$ with $X_0\subset X_1\subset\cdots$. Clearly, X is a locally compact Polish totally disconnected space without isolated points. We define a finite measure μ_n on X_n by setting

$$\mu_n:=\alpha_n(\lambda_{F_n}\times\lambda_{C_{n+1}}\times\lambda_{C_{n+2}}\times\cdots),$$

where α_n is a positive coefficient such that

$$\alpha_0:=1 \ \text{ and } \ \alpha_{n+1}:=\alpha_n\frac{\#F_{n+1}}{\#F_n\#C_{n+1}}.$$

The latter 'matching' condition yields that $\mu_{n+1}\restriction X_n=\mu_n$. Hence there exists a unique σ-finite measure μ on the standard Borel σ-algebra $\mathfrak{B}$ of X generated by the topology such that $\mu\restriction X_n=\mu_n$. In particular, $\mu(X_n)=\alpha_n$ for all $n\geq 0$. It is easy to check that $\mu(X)<\infty$ if and only if (1-4) holds. After a normalization (i.e. by an appropriate change of α_0) we may assume that $\mu(X)=1$. Suppose also that the following is satisfied:

(1-5) for any $g\in G$, there exists $m\geq 0$ with $gF_nC_{n+1}\subset F_{n+1}$ for all $n\geq m$.

For such n, take any $x\in X_n\subset X$ and write the expansion $x=(f_n,c_{n+1},c_{n+2},\dots)$ with $f_n\in F_n$ and $c_{n+i}\in C_{n+i}$, $i>0$. Then we let

$$T_gx:=(gf_nc_{n+1},c_{n+2},\dots)\in X_{n+1}\subset X.$$

It follows from (1-5) that T_g is a well defined homeomorphism of X. Moreover, $T_gT_{g'}=T_{gg'}$, i.e. $T:=(T_g)_{g\in G}$ is a topological action of G on X.

Definition 1.1. We call $(X, \mathfrak{B}, \mu, T)$ the (C, F)-action of G associated to the sequence $(F_n, C_{n+1})_{n=0}^{\infty}$ (cf. [dJ], [Da1], [Da4], [DaS]).

We list without proof several properties of T. They can be verified easily by the reader.

— T is a minimal uniquely ergodic (i.e. strictly ergodic) free action of G.
— Two points $x = (f_n, c_{n+1}, c_{n+2}, \dots)$ and $x' = (f'_n, c'_{n+1}, c'_{n+2}, \dots) \in X_n$ are T-orbit equivalent if and only if $c_i = c'_i$ eventually (i.e. for all large enough i). Moreover, $x' = T_g x$ if and only if

$$g = \lim_{i \to \infty} f'_n c'_{n+1} \cdots c'_{n+i} c_{n+i}^{-1} \cdots c_{n+1}^{-1} f_n^{-1}.$$

For each $A \subset F_n$, we let $[A]_n := \{x = (f_n, c_{n+1}, \dots) \in X_n \mid f_n \in A\}$ and call it an *n-cylinder*. The following holds:

$$\begin{aligned}
&[A]_n \cap [B]_n = [A \cap B]_n, \text{ and } [A]_n \cup [B]_n = [A \cup B]_n,\\
&[A]_n = \bigsqcup_{c \in C_{n+1}} [Ac]_{n+1},\\
&T_g[A]_n = [gA]_n \text{ if } gA \subset F_n,\\
&\mu([Ac]_{n+1}) = \frac{1}{\# C_{n+1}} \mu([A]_n) \text{ for any } c \in C_{n+1},\\
&\mu([A]_n) = \mu(X_n) \lambda_{F_n}(A).
\end{aligned}$$

Moreover, for each measurable subset $B \subset X$,

$$\lim_{n \to \infty} \min_{A \subset F_n} \mu\left(B \triangle \bigsqcup_{a \in A} T_a[0]_n\right) = 0. \tag{1-6}$$

This means that T has *funny rank one* (see [Fe] for the case of $\mathbb{Z}$-actions and [So] for the general case).

2. Main result

Let $G = \mathbb{Z} \times \mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ with multiplication as follows

$$(n, m, a)(n', m', a') = (n + n', m + (-1)^a m', a + a').$$

Then the center $C(G)$ of G is $\mathbb{Z} \times \{0\} \times \{0\}$. Each finite subgroup of G coincides with $G_b := \{(0, b, 1), (0, 0, 0)\}$ for some $b \in \mathbb{Z}$. Notice also that G_b is a maximal finite subgroup of G if $b \neq 0$.

Let $H := \mathbb{Z}^2$. Given $a > 0$, we denote by $I[a]$ the symmetric interval $\{m \in \mathbb{Z} \mid |m| < a\}$. We also set $I_+[a] := I[a] \cup \{a\}$. The Cartesian square of $I[a]$ and $I_+[a]$ are denoted by $I^2[a]$ and $I_+^2[a]$ respectively. Let $(r_n)_{n=0}^{\infty}$ be an increasing sequence of positive integers such that

$$\lim_{n \to \infty} n^4 / r_n = 0. \tag{2-1}$$

Below—just after Lemma 2.1—one more restriction on the growth of $(r_n)_{n=0}^\infty$ will be imposed. We define recurrently two other sequences $(a_n)_{n=0}^\infty$ and $(\widetilde{a}_n)_{n=0}^\infty$ by setting

$$a_0 = \widetilde{a}_0 = 1,\ a_{n+1} := \widetilde{a}_n(2r_n - 1),\ \widetilde{a}_{n+1} := a_{n+1} + (2n+1)\widetilde{a}_n.$$

For each $n \in \mathbb{N}$, we let

$$H_n := I^2[r_n],\ F_n := I_+^2[a_n] \times \mathbb{Z}/2\mathbb{Z},\ \widetilde{F}_n := I_+^2[\widetilde{a}_n] \times \mathbb{Z}/2\mathbb{Z}\ \text{ and }$$
$$S_n := I_+^2[(2n-1)\widetilde{a}_{n-1}] \times \mathbb{Z}/2\mathbb{Z}.$$

We also consider a homomorphism $\phi_n : H \to G$ given by

$$\phi_n(i,j) := (2i\widetilde{a}_n, 2j\widetilde{a}_n, 0).$$

We then have

$$S_n \subset F_n, \qquad F_nS_n = S_nF_n \subset \widetilde{F}_n \subset G, \tag{2-2}$$

$$F_{n+1} = \bigsqcup_{h\in H_n} \widetilde{F}_n\phi_n(h) = \bigsqcup_{h\in H_n} \phi_n(h)\widetilde{F}_n \text{ and} \tag{2-3}$$

$$S_n = \bigsqcup_{h\in I^2[n]} \widetilde{F}_{n-1}\phi_{n-1}(h). \tag{2-4}$$

Now fix a sequence $\epsilon_n \to 0$ as $n \to \infty$. For any two finite sets A, B and a map $\phi : A \to B$, the probability $\frac{1}{\#A}\sum_{a\in A}\delta_{\phi(a)}$ on B will be denoted by $\mathrm{dist}_{a\in A}\phi(a)$. Given two measures κ, ρ on a finite set B, we let $\|\kappa - \rho\|_1 := \sum_{b\in B}|\kappa(b) - \rho(b)|$.

Lemma 2.1[dJ]**.** *If r_n is sufficiently large then there exists a map $s_n : H_n \to S_n$ such that for any $\delta \geq n^{-2}r_n$,*

$$\|\mathrm{dist}_{t\in I[\delta]\times\{0\}}(s_n(h+t), s_n(h'+t)) - \lambda_{S_n} \times \lambda_{S_n}\|_1 < \epsilon_n$$

whenever $h \neq h' \in H_n$ with $\{h,h'\} + (I[\delta]\times\{0\}) \subset H_n$.

From now on we will assume that r_n is large so that the conclusion of Lemma 2.1 is satisfied. For every $n \in \mathbb{N}$, we fix a map s_n whose existence is asserted in the lemma. Without loss of generality we may assume that the following *boundary condition* holds

$$s_n(i, r_n - 1) = s_n(i, 1 - r_n) = s_n(r_n - 1, i) = s_n(1 - r_n, i) = 0 \text{ for all } i \in I[r_n]. \tag{2-5}$$

Now we can define a map $c_{n+1} : H_n \to F_{n+1}$ by setting $c_{n+1}(h) := s_n(h)\phi_n(h)$. We also put $C_{n+1} := c_{n+1}(H_n)$. It is easy to derive from (2-2) and (2-3) that (1-1)–(1-3) are satisfied for the sequence $(F_n, C_{n+1})_{n=0}^\infty$. Moreover,

$$\frac{\#F_{n+1}}{\#F_n\#C_{n+1}} = \frac{a_{n+1}^2}{a_n^2(2r_n-1)^2} = \frac{\widetilde{a}_n^2}{a_n^2} = \left(1 + \frac{(2n-1)\widetilde{a}_{n-1}}{a_n}\right)^2 = \left(1 + \frac{2n-1}{2r_{n-1}-1}\right)^2.$$

From this and (2-1) we deduce that (1-4) holds. Moreover, (2-5) implies (1-5). Thus the conditions (1-1)—(1-5) are all satisfied for $(F_n, C_{n+1})_{n=0}^\infty$. Hence the associated (C,F)-action $T = (T_g)_{g\in G}$ of G is well defined on a standard probability space $(X, \mathfrak{B}, \mu)$.

We now state the main result of the paper.

Theorem 2.2. *The transformation $T_{(1,0,0)}$ is weakly mixing and 2-fold simple. All non-trivial proper factors of $T_{(1,0,0)}$ are 2-to-1 and prime. They are as follows: $\mathfrak{F}_{G_b}$, $b \in \mathbb{Z} \setminus \{0\}$. Two factors $\mathfrak{F}_{G_b}$ and $\mathfrak{F}_{G_{b'}}$ are isomorphic if and only if b and b' are of the same evenness.*

To prove Theorem 2.2 we will need an auxiliary lemma.

Lemma 2.3. *Let $f = f'\phi_{n-1}(h)$ with $f' \in \widetilde{F}_{n-1}$ and $h \in H$.*

(i) *Then we have*

$$\widetilde{F}_{n-1}\phi_{n-1}(h + I^2[n-1]) \subset fS_n \subset \widetilde{F}_{n-1}\phi_{n-1}(h + I^2[n+1]) \text{ and hence}$$
$$\frac{\#(fS_n \setminus \widetilde{F}_{n-1}\phi_{n-1}(h + I^2[n-1]))}{\#S_n} \le \frac{4}{n-1}.$$

(ii) *If, in addition, $fS_n \subset F_n$ then*

$$\frac{\#(AC_n \cap fS_n)}{\#S_n} = \lambda_{F_{n-1}}(A) \pm \frac{10}{n}$$

for any subset $A \subset F_{n-1}$.

Proof. (i) We have

$$fS_n = f'\phi_{n-1}(h)\widetilde{F}_{n-1}\phi_{n-1}(I^2[n]) = f'\widetilde{F}_{n-1}\phi_{n-1}(h + I^2[n]).$$

For each $u = (u_1, u_2) \in \mathbb{Z}^2$, we let $|u| := \max(|u_1|, |u_2|)$. Since $\widetilde{F}_{n-1}\widetilde{F}_{n-1} \subset \bigsqcup_{|u|\le 1} \widetilde{F}_{n-1}\phi_{n-1}(u)$, there exists a partition of $\widetilde{F}_{n-1}$ into subsets A_u, $|u| \le 1$, such that $f'A_u \subset \widetilde{F}_{n-1}\phi_{n-1}(u)$ for any u. Therefore

$$fS_n = \bigsqcup_{|u|\le 1} f'A_u\phi_{n-1}(u)^{-1}\phi_{n-1}(u + h + I^2[n]).$$

It remains to notice that $\bigsqcup_{|u|\le 1} f'A_u\phi_{n-1}(u)^{-1} = \widetilde{F}_{n-1}$.

(ii) If $fS_n \subset F_n$ then the subset $K := h + I^2[n-1]$ is contained in H_{n-1} by (i). Therefore

$$\begin{aligned}
\frac{\#(AC_n \cap fS_n)}{\#S_n} &= \frac{\sum_{h\in H_{n-1}} \#(As_{n-1}(h)\phi_{n-1}(h) \cap \widetilde{F}_{n-1}\phi_{n-1}(K)) \pm \frac{4\#S_n}{2n-1}}{\#S_n} \\
&= \sum_{k\in K} \frac{\#(As_{n-1}(h))}{\#S_n} \pm \frac{8}{n} \\
&= \frac{\#A}{\#F_{n-1}} \cdot \frac{\#F_{n-1}}{\#\widetilde{F}_{n-1}} \cdot \frac{\#K\#\widetilde{F}_{n-1}}{\#S_n} \pm \frac{8}{n} \\
&= \lambda_{F_{n-1}}(A)\left(1 \pm \frac{1}{n}\right)\frac{\#I^2[n-1]}{\#I^2[n]} \pm \frac{8}{n}. \quad \square
\end{aligned}$$

Now we are ready to prove the first half of the first claim of Theorem 2.2.

Proposition 2.4. *The transformation $T_{(1,0,0)}$ is weakly mixing.*

Proof. Let $h_0 := (1,0) \in H$ and $g_n := \phi_n(h_0)$. Since $g_n = (1,0,0)^{2\widetilde{a}_n}$, it suffices to show that the sequence $(g_n)_{n=1}^{\infty}$ is *mixing* for T, i.e.

$$\lim_{n\to\infty} \mu(T_{g_n} D \cap D') = \mu(D)\mu(D')$$

for every pair of measurable subsets $D, D' \subset X$. Take any $A, B \subset F_n$. Since $g_n \in C(G)$ for all $n \in \mathbb{N}$, we have

$$g_n A c_{n+1}(h) = A s_n(h) \phi_n(h_0 + h) = A s_n(h) s_n(h_0 + h)^{-1} c_{n+1}(h_0 + h)$$

whenever $h, h_0 h \in H_n$. We set $F_n' := F_n \cap F_n S_n S_n^{-1}$, $A' := A \cap F_n'$, $B' := B \cap F_n'$, $H_n' := H_n \cap (h_0^{-1} H_n)$. Then

$$\begin{aligned}
\mu(T_{g_n}[A]_n \cap [B]_n) &= \mu(T_{g_n}[A']_n \cap [B']_n) \pm 2\mu([F_n \setminus F_n']_n) \\
&= \sum_{h\in H_n} \mu(T_{g_n}[A' c_{n+1}(h)]_{n+1} \cap [B']_n) + \overline{o}(1) \\
&= \sum_{h\in H_n'} \mu(T_{g_n}[A' c_{n+1}(h)]_{n+1} \cap [B']_n) + \overline{o}(1) \\
&= \sum_{h\in H_n'} \mu([A' s_n(h) s_n(h_0 + h)^{-1} c_{n+1}(h_0+h)]_{n+1} \cap [B']_n) + \overline{o}(1) \\
&= \sum_{h\in H_n'} \mu([(A' s_n(h) s_n(h_0 + h)^{-1} \cap B') c_{n+1}(h_0+h)]_{n+1}) + \overline{o}(1) \\
&= \frac{1}{\# H_n} \sum_{h\in H_n'} \mu([A' s_n(h) s_n(h_0 + h)^{-1} \cap B']_n) + \overline{o}(1) \\
&= \frac{1}{\# H_n} \sum_{h\in H_n'} \lambda_{F_n}(A' s_n(h) s_n(h_0 + h)^{-1} \cap B')\mu(X_n) + \overline{o}(1) \\
&= \frac{1}{\# H_n'} \sum_{h\in H_n'} \lambda_{F_n}(A' s_n(h) \cap B' s_n(h_0 + h)) + \overline{o}(1), \\
&= \frac{1}{\# H_n'} \sum_{h\in H_n'} \lambda_{F_n}(A s_n(h) \cap B s_n(h_0 + h)) + \overline{o}(1)
\end{aligned}$$

where $\overline{o}(1)$ means (here and below) a sequence that goes to 0 and that does not depend on the choice of $A, B \subset F_n$. Let $\xi_n := \mathrm{dist}_{h\in H_n'}(s_n(h), s_n(h_0+h))$. Notice that

$$\xi_n = \frac{1}{2r_n - 1} \sum_{i\in I[r_n]} \mathrm{dist}_{-r_n<t<r_n-1}(s_n(t,i), s_n(t+1,i)).$$

It follows from Lemma 2.1 that $\|\xi_n - \lambda_{S_n} \times \lambda_{S_n}\|_1 < \epsilon_n$. We define a function $f : S_n \times S_n \to \mathbb{R}$ by setting $f(v,w) := \lambda_{F_n}(Av \cap Bw)$. Then

$$\frac{1}{\# H_n'} \sum_{h\in H_n'} \lambda_{F_n}(A s_n(h) \cap B s_n(h_0 + h)) = \int f\, d\xi_n = \int f\, d(\lambda_{S_n} \times \lambda_{S_n}) \pm \epsilon_n.$$

Thus we obtain

$$\text{(2-6)} \qquad \mu(T_{g_n}[A]_n \cap [B]_n) = \int_{S_n \times S_n} \lambda_{F_n}(Av \cap Bw)\, d\lambda_{S_n}(v)\, d\lambda_{S_n}(w) + \overline{o}(1).$$

Now we take $A := A^* C_n$ and $B := B^* C_n$ for some subsets $A^*, B^* \subset F_{n-1}$. Then the integral in the righthand side of (2-6) equals to the sum

$$\text{(2-7)} \qquad \sum_{a \in A^*} \sum_{b \in B^*} \sum_{h, h' \in H_{n-1}} \frac{\#(ac_n(h)S_n \cap bc_n(h')S_n \cap F_n)}{(\#S_n)^2 \# F_n}.$$

It follows from the definition of c_n and Lemma 2.3(i) that

$$ac_n(h)S_n \cap bc_n(h')S_n \subset \widetilde{F}_{n-1}\phi_{n-1}(h + I^2[n+1]) \cap \widetilde{F}_{n-1}\phi_{n-1}(h' + I^2[n+1]).$$

Hence $ac_n(h)S_n \cap bc_n(h')S_n \neq \emptyset$ only if $h' - h \in I^2[2n+1]$. If the latter is satisfied we say that h and h' are *partners*. Denote by $P(h)$ the set of partners of h that belong to H_{n-1}. Clearly, $\#P(h) \leq (4n+1)^2$. Therefore we deduce from (2-6), (2-7) and Lemma 2.3(i) that

$$\begin{aligned}
&\mu(T_{g_n}[A^*]_{n-1} \cap [B^*]_{n-1}) \\
&\quad = \sum_{a \in A^*} \sum_{b \in B^*} \sum_{h \in H_{n-1}} \sum_{h' \in P(h)} \frac{\#(c_n(h)S_n \cap c_n(h')S_n \cap F_n) \pm \frac{4\#S_n}{n-1}}{(\#S_n)^2 \# F_n} + \overline{o}(1) \\
&\quad = \frac{\#A^* \#B^*}{(\#F_{n-1})^2} \cdot \theta_n \pm \frac{(\#F_{n-1})^2 \# H_{n-1} \cdot (4n+1)^2 \cdot 4\#S_n}{(\#S_n)^2 \#F_n \cdot (n-1)} + \overline{o}(1) \\
&\quad = \lambda_{F_{n-1}}(A^*)\lambda_{F_{n-1}}(B^*)\theta_n \pm 67n \frac{\#F_{n-1}}{\#S_n} + \overline{o}(1),
\end{aligned}$$

where θ_n is a positive number. Substituting $A^* = B^* = F_{n-1}$ and passing to the limit, we obtain that $\theta_n \to 1$ as $n \to \infty$. Hence

$$\text{(2-8)} \qquad \mu(T_{g_n}[A^*]_{n-1} \cap [B^*]_{n-1}) = \mu([A^*]_{n-1})\mu([B^*]_{n-1}) + \overline{o}(1).$$

Since $\overline{o}(1)$ does not depend on the choice of A^* and B^* inside F_{n-1}, it follows from (1-6) and (2-8) that $(g_n)_{n=1}^\infty$ is mixing for T. $\square$

Notice that slyghtly modifying the techniques from Ornstein's work [Or] one can show that $T_{(1,0,0)}$ is mixing indeed (cf. [Ma]). However we will not need this.

Our next task is to describe all ergodic 2-fold self-joinings of $T_{(1,0,0)}$.

Theorem 2.5. *The transformation $T_{(1,0,0)}$ is 2-fold simple and*

$$C(T_{(1,0,0)}) = \{T_g \mid g \in G\}.$$

Proof. Take any joining $\nu \in J_2^e(T_{(1,0,0)})$. Let $I_n := I[n^{-2}a_n]$, $J_n := I[n^{-2}r_n]$ and $\Phi_n := I_n + 2\widetilde{a}_n J_n$. We first notice that $(\Phi_n)_{n=1}^\infty$ is a Følner sequence in $\mathbb{Z}$. Since

$$\frac{a_n}{n^2} + \frac{2\widetilde{a}_n r_n}{n^2} < \frac{\widetilde{a}_n(2r_n+1)}{n^2} < \frac{2a_{n+1}}{(n+1)^2},$$

it follows that $\Phi_n \subset I_{n+1} + I_{n+1}$ and hence $\bigcup_{m=1}^n \Phi_m \subset I_{n+1} + I_{n+1}$. This implies that

$$\#\left(\Phi_{n+1} - \bigcup_{m=1}^{n} \Phi_m\right) \le 3\#\Phi_{n+1} \quad \text{for every } n \in \mathbb{N},$$

i.e. Shulman's condition [Li] is satisfied for $(\Phi_n)_{n=1}^\infty$. By [Li], the pointwise ergodic theorem holds along $(\Phi_n)_{n=1}^\infty$ for any ergodic transformation. Hence

$$\text{(2-9)} \qquad \frac{1}{\#\Phi_n} \sum_{i \in \Phi_n} \chi_D(T_{(i,0,0)}x)\chi_{D'}(T_{(i,0,0)}x') \to \nu(D \times D')$$

as $n \to \infty$ at ν-a.a. $(x, x') \in X \times X$ for all cylinders $D, D' \subset X$. We call such (x, x') a *generic point* for $(T_{(1,0,0)} \times T_{(1,0,0)}, \nu)$. Fix one of them. Then $x, x' \in X_n$ for all sufficiently large n and we have the following expansions

$$x = (f_n, c_{n+1}(h_n), c_{n+2}(h_{n+1}), \dots,),$$
$$x' = (f'_n, c_{n+1}(h'_n), c_{n+2}(h'_{n+1}), \dots,)$$

with $f_n, f'_n \in F_n$ and $h_i, h'_i \in H_i$, $i > n$. We let $H_n^- = I^2[(1 - n^{-2})r_n] \subset H_n$. Then $\#H_n^-/\#H_n \ge 1 - 0.5n^{-2}$. Since the marginals of ν are both equal to μ, by Borel-Cantelli lemma we may assume without loss of generality that $h_n, h'_n \in H_n^-$ for all sufficiently large n. This implies, in turn, that

$$f_{n+1} = f_n c_{n+1}(h_n) \in \widetilde{F}_n \phi_n(H_n^-) \subset I_+^2[(2r_n(1 - n^{-2}) - 1)\widetilde{a}_n] \times \mathbb{Z}/2\mathbb{Z},$$

and, similarly, $f'_{n+1} \in I_+^2[(2r_n(1 - n^{-2}) - 1)\widetilde{a}_n] \times \mathbb{Z}/2\mathbb{Z}$. Notice that given $g \in \Phi_n$, we have $(g, 0, 0) = (b, 0, 0)\phi_n(t, 0)$ for some uniquely determined $b \in I_n$ and $t \in J_n$. Moreover, $(t, 0, 0)h_n \in H_n$. We also claim that

$$\text{(2-10)} \qquad (b, 0, 0)f_n S_n S_n \subset F_n \quad \text{and} \quad (b, 0, 0)f_n S_n S_n^{-1} \subset F_n$$

if n is large enough. To verify this, it suffices to show that

$$\frac{a_n}{n^2} + 2r_{n-1}\left(1 - \frac{1}{(n-1)^2}\right)\widetilde{a}_{n-1} + 4n\widetilde{a}_{n-1} < a_n,$$

which follows from (2-1) in a routine way. Hence

$$(g, 0, 0)f_n s_n(h_n)\phi_n(h_n) = dc_{n+1}((t, 0) + h_n) \quad \text{and}$$
$$(g, 0, 0)f'_n s_n(h'_n)\phi_n(h'_n) = d'c_{n+1}((t, 0) + h'_n),$$

where $d := (b, 0, 0)f_n s_n(h_n)s_n((t, 0) + h_n)^{-1}$ and $d' := (b, 0, 0)f'_n s_n(h'_n)s_n((t, 0) + h'_n)^{-1}$ belong to F_n by (2-10). Now take any $B, B' \subset F_{n-1}$ and set $A := BC_n \subset F_n$ and $A' := B'C_n \subset F_n$. We have

$$\begin{aligned}
&\frac{\#\{g \in \Phi_n \mid (T_{(g,0,0)}x, T_{(g,0,0)}x') \in [A]_n \times [A']_n\}}{\#\Phi_n} \\
&= \frac{1}{\#I_n} \sum_{b \in I_n} \frac{\#\{t \in J_n \mid d \in A, d' \in A'\}}{\#J_n} \\
&= \frac{1}{\#I_n} \sum_{b \in I_n} \xi_n(A^{-1}(b, 0, 0)f_n s_n(h_n) \times A'^{-1}(b, 0, 0)f'_n s_n(h'_n)),
\end{aligned}$$

where $\xi_n := \mathrm{dist}_{t\in J_n}(s_n((t,0)h_n), s_n((t,0)h'_n))$.

We consider separately two cases. Suppose first that $h_n = h'_n$ for all n greater than some N. Then it is easy to deduce from Lemma 2.1 that $\|\xi_n - \Delta\|_1 < \epsilon_n$, where Δ is the probability equidistributed on the diagonal of $S_n \times S_n$. Moreover, $f_n {f'_n}^{-1} = f_N {f'_N}^{-1} =: k$ for all $n > N$. This yields

$$\begin{aligned}\frac{1}{\#I_n}\sum_{b\in I_n}&\xi_n(A^{-1}(b,0,0)f_n s_n(h_n)\times {A'}^{-1}(b,0,0)f'_n s_n(h'_n))\\ &= \frac{1}{\#I_n}\sum_{b\in I_n}\lambda_{S_n}(A^{-1}(b,0,0)f_n s_n(h_n)\cap {A'}^{-1}(b,0,0)f'_n s_n(h_n)) \pm \epsilon_n\\ &= \frac{1}{\#I_n}\sum_{b\in I_n}\frac{\#(A\cap kA'\cap(b,0,0)f_n s_n(h_n)S_n^{-1})}{\#S_n} \pm \epsilon_n.\end{aligned}$$

Notice that $(b,0,0)f_n s_n(h_n)S_n \subset F_n$ by (2-10). We now set $\widetilde{B} := B'\cap k^{-1}F_{n-1}$. Since $k \in F_N F_N^{-1}$, it follows that $\#(B'\setminus\widetilde{B})/\#F_{n-1} = \overline{o}(1)$. Then Lemma 2.3(ii) yields

$$\begin{aligned}&\frac{\#(BC_n\cap kB'C_n\cap(b,0,0)f_n s_n(h_n)S_n^{-1})}{\#S_n}\\ &\quad= \frac{\#((B\cap k\widetilde{B})C_n\cap(b,0,0)f_n s_n(h_n)S_n)}{\#S_n} + \overline{o}(1)\\ &\quad= \lambda_{F_{n-1}}(B\cap k\widetilde{B}) + \overline{o}(1)\\ &\quad= \frac{\mu([B\cap k\widetilde{B}]_{n-1})}{\mu(X_{n-1})} + \overline{o}(1)\\ &\quad= \mu([B]_{n-1}\cap T_k[B']_{n-1}) + \overline{o}(1).\end{aligned}$$

Therefore it follows from (2-9) that

$$\nu([B]_{n-1}\times[B']_{n-1}) = \mu_{T_k}([B]_{n-1}\times[B']_{n-1}) + \overline{o}(1).$$

Then we deduce from (6-6) that $\nu = \mu_{T_k}$.

Now consider the second case, where $h_n \neq h'_n$ for infinitely many, say *bad* n. It follows from Lemma 2.1 that $\|\xi_n - \lambda_{S_n}\times\lambda_{S_n}\| < \epsilon_n$ for all such n. Hence

$$\begin{aligned}\frac{1}{\#I_n}\sum_{b\in I_n}&\xi_n(A^{-1}(b,0,0)f_n s_n(h_n)\times {A'}^{-1}(b,0,0)f'_n s_n(h'_n))\\ &= \frac{1}{\#I_n}\sum_{b\in I_n}\lambda_{S_n}(A^{-1}(b,0,0)f_n s_n(h_n))\lambda_{S_n}({A'}^{-1}(b,0,0)f'_n s_n(h'_n)) \pm \epsilon_n.\end{aligned}$$

Now we derive from Lemma 2.3(ii) and (2-10) that

$$\lambda_{S_n}(A^{-1}(b,0,0)f_n s_n(h_n)) = \frac{\#(A\cap(b,0,0)f_n s_n(h_n)S_n^{-1})}{\#S_n} = \lambda_{F_{n-1}}(B) + \overline{o}(1)$$

and, in a similar way, $\lambda_{S_n}({A'}^{-1}(b,0,0)f'_n s_n(h'_n)) = \lambda_{F_{n-1}}(B') + \overline{o}(1)$. Hence

$$\nu([B]_{n-1}\times[B']_{n-1}) = \mu([B]_{n-1})\mu([B']_{n-1}) + \overline{o}(1)$$

provided that n is bad. It remains to note that (1-6) holds along any subsequence, in particular along the subsequence of bad n. Hence $\nu = \mu \times \mu$. $\square$

Proof of Theorem 2.2. follows now from Proposition 2.4, Theorem 2.5 and the fact that $\mathfrak{F}_{G_b}$ and $\mathfrak{F}_{G_{b'}}$ are isomorphic if and only if the subgroups G_b and $G_{b'}$ are conjugate in $C(T_{(1,0,0)})$ [dJR]. $\square$

Notice that with some additional conditions on s_n (cf. [Da4, Section 6]) one can show that $T_{(1,0,0)}$ is actually simple of all orders. For the definitions of higher order simplicity we refer to [dJR]. (In fact, 3-fold simplicity implies the simplicity of any order [GHR].) This would imply in turn that $T_{(1,0,0)}$ is mixing of any order whenever it is mixing.

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