Dimensions of products of hyperbolic spaces

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Abstract

We give estimates on asymptotic dimensions of products of general hyperbolic spaces with following applications to the hyperbolic groups. We give examples of strict inequality in the product theorem for the asymptotic dimension in the class of the hyperbolic groups; and examples of strict inequality in the product theorem for the hyperbolic dimension. We prove that $\mathbb{R}$ is dimensionally full for the asymptotic dimension in the class of the hyperbolic groups.

1 Introduction

The purpose of this article is to answer some questions in asymptotic dimension theory (in particular posed in [Dr3]). For this we give different estimates on asymptotic dimensions of general hyperbolic spaces and products of hyperbolic spaces and apply them to hyperbolic groups.

To formulate our results we need some definitions. We say that metric spaces $\{X_\alpha : \alpha \in \mathcal{A}\}$ have the linearly controlled dimension $\ell\text{-dim} \le k$ uniformly if there is a constant $\delta \in (0,1)$ such that for every sufficiently small $\tau > 0$ and for every $\alpha \in \mathcal{A}$ there exists a $(k+1)$-colored open covering $\mathcal{U}_\alpha$ of X_α with $\operatorname{mesh}(\mathcal{U}_\alpha) \le \tau$ and with Lebesgue number $L(\mathcal{U}_\alpha) \ge \delta\tau$.

For a metric space X and $a > 0$, we denote by aX the metric space obtained from X by multiplying all distances by a.

In sect. 2.3, we recall the definitions of the asymptotic dimension asdim and the linearly controlled asymptotic dimension ℓ-asdim used in the following theorem, these are some quasi-isometry invariants of metric spaces.

Theorem 1.1. *Let X_1, X_2 be visual Gromov hyperbolic spaces and assume that the metric spaces of the family $\{a\partial_\infty X_1 \times b\partial_\infty X_2 : a \ge 1, b \ge 1\}$ have $\ell\text{-dim} \le k$ uniformly, where the boundaries at infinity $\partial_\infty X_1$, $\partial_\infty X_2$ are taken with some visual metrics. Then*

$$\operatorname{asdim}(X_1 \times X_2) \le k + 2.$$

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The property of a Gromov hyperbolic space X to be visual is a rough version of the property that every point $x \in X$ lies on a geodesic ray emanating from a fixed point $x_0 \in X$; for the precise definition see sect. 2.

Remark 1.2. A similar result holds true in the case of an arbitrary finite number ≥ 2 of factors, and furthermore the estimate remains true if we replace asdim by ℓ-asdim, that is:

Let $X_1, \dots, X_n$ be visual Gromov hyperbolic spaces and assume that the metric spaces of the family $\{a_1\partial_\infty X_1 \times \cdots \times a_n\partial_\infty X_n : a_i \geq 1, i \in \{1, \dots, n\}\}$ have ℓ-dim $\leq k$ uniformly. Then

$$\operatorname{asdim}(X_1 \times \cdots \times X_n) \leq k + n$$

and

$$\ell\text{-asdim}(X_1 \times \cdots \times X_n) \leq k + n.$$

For simplicity of notations, we shall consider the case of two spaces and the asymptotic dimension asdim.

Theorem 1.3. *Let $X_1, \dots, X_n$ be geodesic Gromov hyperbolic spaces. Then*

$$\operatorname{asdim}(X_1 \times \cdots \times X_n) \geq \ell\text{-dim}(\partial_\infty X_1 \times \cdots \times \partial_\infty X_n \times [0,1]^n).$$

Corollary 1.4. *Let Γ_1, Γ_2 be Gromov hyperbolic groups. Then*

$$\operatorname{asdim}(\Gamma_1 \times \Gamma_2) = \dim(\partial_\infty \Gamma_1 \times \partial_\infty \Gamma_2) + 2.$$

As an application of this corollary, we

1) give examples of strict inequality in the product theorem for the asymptotic dimension in the class of hyperbolic groups (cf. [Dr3, Problem 23]; a corresponding example in the class of general metric spaces is given in [BL]);

2) give examples of strict inequality in the product theorem for the hyperbolic dimension;

3) prove the equality $\operatorname{asdim}(\Gamma \times \mathbb{R}) = \operatorname{asdim}(\Gamma) + 1$ for any hyperbolic group Γ.

The last equality is not true for general metric spaces. An example of a metric space X of bounded geometry with finite asymptotic dimension for which $\operatorname{asdim}(X \times \mathbb{R}) = \operatorname{asdim} X$ is constructed in [Dr4], and the question is, if the equality above is true for groups ([Dr3, Problem 21]).

Theorem 1.5. *Let X be a visual Gromov hyperbolic space. Then*

$$\ell\text{-asdim}\, X \leq \ell\text{-dim}(\partial_\infty X \times [0,1]).$$

This estimate strengthens the estimate $\operatorname{asdim} X \leq \ell\text{-dim}\, \partial_\infty X + 1$ proved in [Bu1, Bu2]. Unfortunately, we do not know whether the space $[0,1]$ is dimensionally full for the linearly controlled dimension, i.e. whether $\ell\text{-dim}(Y \times [0,1]) = \ell\text{-dim}\, Y + 1$ for any (compact or proper) metric space Y.

Corollary 1.6. *Let X be a visual, geodesic Gromov hyperbolic space. Then*

$$\ell\text{-asdim}\, X = \operatorname{asdim} X = \ell\text{-dim}(\partial_\infty X \times [0,1]).$$

The first equality does not hold in general and the question is if it holds for all finitely presented groups ([Dr3, Problem 41]).

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2 Preliminaries

Here, we recall notions and facts necessary for the paper.

2.1 Coverings

Let Z be a metric space. For U, $U' \subset Z$ we denote by $\operatorname{dist}(U, U')$ the distance between U and U', $\operatorname{dist}(U, U') = \inf\{|uu'| : u \in U,\ u' \in U'\}$ where $|uu'|$ is the distance between u, u'. For $r > 0$ we denote by $B_r(U)$ the open r-neighborhood of U, $B_r(U) = \{z \in Z : \operatorname{dist}(z, U) < r\}$, and by $\overline{B}_r(U)$ the closed r-neighborhood of U, $\overline{B}_r(U) = \{z \in Z : \operatorname{dist}(z, U) \le r\}$. We extend these notations over all real r putting $B_r(U) = U$ for $r = 0$, and defining $B_r(U)$ for $r < 0$ as the complement of the closed $|r|$-neighborhood of $Z \setminus U$, $B_r(U) = Z \setminus \overline{B}_{|r|}(Z \setminus U)$.

Given a family $\mathcal{U}$ of subsets in a metric space Z we define $\operatorname{mesh}(\mathcal{U}) = \sup\{\operatorname{diam} U : U \in \mathcal{U}\}$. The *multiplicity* of $\mathcal{U}$, $m(\mathcal{U})$, is the maximal number of members of $\mathcal{U}$ with nonempty intersection. We say that a family $\mathcal{U}$ is *disjoint* if $m(\mathcal{U}) = 1$.

A family $\mathcal{U}$ is called a *covering* of Z if $\cup\{U : U \in \mathcal{U}\} = Z$. A covering $\mathcal{U}$ is said to be *colored* if it is the union of $m \ge 1$ disjoint families, $\mathcal{U} = \cup_{a \in A} \mathcal{U}^a$, $|A| = m$. In this case we also say that $\mathcal{U}$ is m-colored. Clearly, the multiplicity of a m-colored covering is at most m.

Let $\mathcal{U}$ be a family of open subsets in a metric space Z which cover $A \subset Z$. Given $z \in A$, we let

$$L(\mathcal{U}, z) = \sup\{\operatorname{dist}(z, Z \setminus U) : U \in \mathcal{U}\}$$

be the Lebesgue number of $\mathcal{U}$ at z, $L(\mathcal{U}) = \inf_{z \in A} L(\mathcal{U}, z)$ the Lebesgue number of the covering $\mathcal{U}$ of A. For every $z \in A$, the open ball $B_r(z) \subset Z$ of radius $r = L(\mathcal{U})$ centered at z is contained in some member of the covering $\mathcal{U}$.

We shall use the following obvious fact (see e.g. [Bu1]).

Lemma 2.1. *Let $\mathcal{U}$ be an open covering of $A \subset Z$ with $L(\mathcal{U}) > 0$. Then for every $s \in (0, L(\mathcal{U}))$ the family $\mathcal{U}_{-s} = B_{-s}(\mathcal{U})$ is still an open covering of A.* $\square$

2.2 Hyperbolic spaces

Here we recall necessary facts from the hyperbolic spaces theory. For more details one can see e.g. [BoS]. We also assume that the reader is familiar with notions like of a geodesic metric space, a geodesic ray etc.

Let X be a metric space. We use notation $|xx'|$ for the distance between x, $x' \in X$. For $o \in X$ and for x, $x' \in X$, put $(x|x')_o = \frac{1}{2}(|xo| + |x'o| - |xx'|)$. The number $(x|x')_o$ is nonnegative by the triangle inequality, and it is called the Gromov product of x, x' w.r.t. o.

A metric space X is called *δ-hyperbolic*, $\delta \geq 0$, if the following inequality

$$(x|y)_o \geq \min\{(x|z)_o, (z|y)_o\} - \delta$$

holds for every base point $o \in X$ and all x, y, $z \in X$.

Let X be a hyperbolic space, i.e. X is δ-hyperbolic for some $\delta \geq 0$. We denote by $\partial_\infty X$ the (Gromov) boundary of X at infinity. The Gromov product based at $o \in X$ naturally extends to $\partial_\infty X$.

If a hyperbolic space X is geodesic then every geodesic ray in X represents a point at infinity. Conversely, if a geodesic hyperbolic space X is proper (i.e. closed balls in X are compact), then every point at infinity is represented by a geodesic ray.

A metric d on the boundary at infinity $\partial_\infty X$ of X is said to be *visual*, if there are $o \in X$, $a > 1$ and positive constants c_1, c_2, such that

$$c_1 a^{-(\xi|\xi')_o} \leq d(\xi, \xi') \leq c_2 a^{-(\xi|\xi')_o}$$

for all ξ, $\xi' \in \partial_\infty X$. In this case, we say that d is visual with respect to the base point o and the parameter a. The boundary at infinity is bounded and complete w.r.t. any visual metric, moreover, if X is proper then $\partial_\infty X$ is compact. If $a > 1$ is sufficiently close to 1, then a visual metric with respect to a does exist.

A hyperbolic space X is called *visual*, if for some base point $x_0 \in X$ there is a positive constant D such that for every $x \in X$ there is $\xi \in \partial_\infty X$ with $|xx_0| \leq (x|\xi)_{x_0} + D$ (one easily sees that this property is independent of the choice of x_0).

2.3 Definitions of linearly controlled and asymptotic dimensions

We shall say that a covering $\mathcal{U}$ is an (L, M, n)-covering if it is n-colored with $\mathrm{mesh}(\mathcal{U}) \leq M$ and $L(\mathcal{U}) \geq L$.

The notion of the linearly controlled metric dimension of a metric space is introduced in [Bu1] where it is called the 'capacity dimension'. One of a number equivalent definitions of this notion is the following.

The linearly controlled dimension or ℓ-dimension of a metric space Z, $\ell\text{-}\dim(Z)$, is the minimal integer $m \geq 0$ with the following property: There is a constant $\delta \in (0,1)$ such that for every sufficiently small $\tau > 0$ there exists an open $(\delta\tau, \tau, m+1)$-covering of Z. Here (and in all similar definitions of a dimension below) we assume that $\ell\text{-}\dim Z = \infty$, if the number m with the indicated property does not exist.

The asymptotic dimension is a quasi-isometry invariant of a metric space introduced in [Gr]. There are also several equivalent definitions, see [Gr], [BD], and we use the following one. The asymptotic dimension of a metric space X is the minimal integer $\operatorname{asdim} X = m$ such that for every positive d there is an open covering $\mathcal{U}$ of X with $m(\mathcal{U}) \le m+1$, $\operatorname{mesh}(\mathcal{U}) < \infty$ and $L(\mathcal{U}) \ge d$.

The linearly controlled asymptotic dimension is also a quasi-isometry invariant of a metric space. We use the following definition. The linearly controlled asymptotic dimension of a metric space X is the minimal integer $\ell\text{-}\operatorname{asdim} X = n$ with the following property: There is a constant $C > 1$ such that for every sufficiently large d there is an open $(d, Cd, n+1)$-covering $\mathcal{U}$ of X.

2.4 Coverings of the product

Throughout the paper, we assume that the product of metric spaces $X \times Y$ is endowed with l_∞-metric, that is $|(x,y)(x',y')| = \max\{|xx'|, |yy'|\}$ for each (x,y), $(x',y') \in X \times Y$.

Let $\mathcal{U}$ be an open covering of the product $X \times Y$ of metric spaces. We say that a pair (p,q), $p, q \ge 0$, dominates $\operatorname{mesh}^\times(\mathcal{U})$ and write $\operatorname{mesh}^\times(\mathcal{U}) \le (p,q)$ or $(p,q) = \operatorname{mesh}^\boxtimes(\mathcal{U})$, if for each member of $\mathcal{U}$ there exists a point $(x,y) \in X \times Y$ so that this member is contained in the product $\overline{B}_p(x) \times \overline{B}_q(y)$.

We also say that the function $f(z) \in [0,\infty) \times [0,\infty)$, $z \in X \times Y$, dominates the pointed $\operatorname{mesh}^\times(\mathcal{U}, z)$ and write $\operatorname{mesh}^\times(\mathcal{U}, z) \le f(z)$ or $f(z) = \operatorname{mesh}^\boxtimes(\mathcal{U}, z)$, if for each member of $\mathcal{U}$ that contains z there exists a point $(x,y) \in X \times Y$ so that this member is contained in $\overline{B}_p(x) \times \overline{B}_q(y)$, where $(p,q) = f(z)$. In this sense, the $\operatorname{mesh}^\times$ (the pointed $\operatorname{mesh}^\times$) is determined basically by the meshes (the meshes at the projected points) of the coverings projected to the factors.

Similarly, we say that the *Lebesgue number* $L^\times$ at a point $z = (x,y) \in X \times Y$ of the covering $\mathcal{U}$ *dominates* $f(z) = (p,q) \in [0,\infty) \times [0,\infty)$, $L^\times(\mathcal{U}, z) \ge f(z)$, if the product of the balls $\overline{B}_p(x) \times \overline{B}_q(y)$ is contained in some member of the covering. In this case, we also write $f(z) = L^\boxtimes(\mathcal{U}, z)$.

We write $L^\times(\mathcal{U}) \ge (p,q)$ or $(p,q) = L^\boxtimes(\mathcal{U})$, if $L^\times(\mathcal{U}, z) \ge (p,q)$ for every $z \in X \times Y$. There is no way in general to recover $L^\times$ from the Lebesgue numbers of the projected coverings.

For the product $X \times Y$, it is straightforward to check that the following two properties are equivalent.

(1) The spaces of the family $\{aX \times bY : a \ge 1, b \ge 1\}$ have $\ell\text{-}\dim \le k$ uniformly.

(2) There is a constant $\delta \in (0,1)$ such that for every sufficiently small $\tau > 0$ and for every α, $\beta \in (0,1]$ there exists a $(k+1)$-colored open covering $\mathcal{U}$ of $X \times Y$ with $\mathrm{mesh}^{\times}(\mathcal{U}) \leq (\alpha\tau, \beta\tau)$ and $L^{\times}(\mathcal{U}) \geq (\alpha\delta\tau, \beta\delta\tau)$.

Let $\mathcal{U}$, $\mathcal{V}$ be open coverings of $X \times Y$. We write $\mathrm{mesh}^{\times}(\mathcal{U}) \leq (p,q)L^{\times}(\mathcal{V})$ for some p, $q \geq 0$, if $\mathrm{mesh}^{\times}(\mathcal{U}) \leq (pr_1, qr_2)$ for some $(r_1, r_2) = L^{\boxtimes}(\mathcal{V})$. In the case $p = q$, we write $\mathrm{mesh}^{\times}(\mathcal{U}) \leq pL^{\times}(\mathcal{V})$. Note that if $\mathrm{mesh}^{\times}(\mathcal{U}) \leq L^{\times}(\mathcal{V})$ then the covering $\mathcal{U}$ is *inscribed* in the covering $\mathcal{V}$, that is, every member of $\mathcal{U}$ is contained in some member of $\mathcal{V}$.

We write $\mathrm{mesh}^{\times}(\mathcal{U}, z) \leq \mathrm{mesh}^{\times}(\mathcal{V}, z)$, if every pair $\mathrm{mesh}^{\boxtimes}(\mathcal{V}, z)$ is also a pair $\mathrm{mesh}^{\boxtimes}(\mathcal{U}, z)$. We write $L^{\times}(\mathcal{U}, z) \geq sL^{\times}(\mathcal{V}, z)$ for $s \geq 0$, if for any p, $q \geq 0$ with $L^{\times}(\mathcal{V}, z) \geq (p,q)$ we have $L^{\times}(\mathcal{U}, z) \geq (sp, sq)$.

For $A \subset Z = X \times Y$ and r_1, $r_2 > 0$, we denote by $B_{(r_1,r_2)}(A)$ the union of all products $B_{r_1}(x) \times B_{r_2}(y)$ with $(x,y) \in A$, $B_{(r_1,r_2)}(A) = \cup\{B_{r_1}(x) \times B_{r_2}(y) : (x,y) \in A\}$, and by $B_{(-r_1,-r_2)}(A) = Z \setminus \overline{B}_{(r_1,r_2)}(Z \setminus A)$.

We also write $(p,q) \geq (p', q')$ for reals p, q, p', q', if and only if $p \geq p'$ and $q \geq q'$.

3 Auxiliary facts

3.1 Separate and qualified families of coverings

In this section we give a generalization of the following lemma, see e.g. [BL].

Lemma 3.1. *Suppose, that Z is a metric space and A, $B \subset Z$. Let $\mathcal{U}$ be an open covering of A, $\mathcal{V}$ an open covering of B both with multiplicity at most m. If $\mathrm{mesh}(\mathcal{V}) \leq L(\mathcal{U})/2$ then there exists an open covering $\mathcal{W}$ of $A \cup B$ with multiplicity at most m and $\mathrm{mesh}(\mathcal{W}) \leq \max\{\mathrm{mesh}(\mathcal{V}), \mathrm{mesh}(\mathcal{U})\}$, $L(\mathcal{W}) \geq \min\{L(\mathcal{U})/2, L(\mathcal{V})\}$.* □

Let $\mathcal{U}_1$, $\mathcal{U}_2$ be families of subsets in a metric space Z. We say that $\mathcal{U}_1$ and $\mathcal{U}_2$ are separate, if every member of $\mathcal{U}_1$ intersects no member of $\mathcal{U}_2$.

Let $Z = X \times Y$ be the product of metric spaces and $Z_\alpha \subset Z$, $\alpha \in \mathcal{A}$. Let $\mathcal{U}_\alpha$ be a covering of Z_α. Let $S = \{0, \dots, N\}$ be the set of 'scales' and $i : \mathcal{A} \to S$ a *scale function*.

We say that the family $\mathcal{U}_\alpha$, $\alpha \in \mathcal{A}$ is *separate with the scale function* i if for every $U \in \mathcal{U}_\alpha$, $V \in \mathcal{U}_{\alpha'}$ with $\alpha \neq \alpha'$ and $i(\alpha) = i(\alpha')$, we have $U \cap V = \emptyset$.

We say that the family $\mathcal{U}_\alpha$ is *qualified by the scale function* i if the condition $U \cap V \neq \emptyset$ for some $U \in \mathcal{U}_\alpha$, $V \in \mathcal{U}_{\alpha'}$ with $i(\alpha) < i(\alpha')$ implies $L^{\boxtimes}(\mathcal{U}_{\alpha'}) \geq 4\,\mathrm{mesh}^{\boxtimes}(\mathcal{U}_\alpha)$ for appropriate dominating numbers fixed for each covering $\mathcal{U}_\alpha$.

Proposition 3.2. *Suppose that $Z = X \times Y$ is the product of metric spaces and $Z_\alpha \subset Z$, $\alpha \in \mathcal{A}$. Let $S = \{0, \dots, N\}$ and let $i : \mathcal{A} \to S$ be a scale function. Let $\mathcal{U}_\alpha$ be an open m-colored covering of Z_α for every $\alpha \in \mathcal{A}$ so that the family of the coverings $\mathcal{U}_\alpha$, $\alpha \in \mathcal{A}$ is separate with and qualified* *by the scale function i. Then there exists an open m-colored covering $\mathcal{W}$ of $\cup_\alpha Z_\alpha$ with* $\operatorname{mesh}^\times(\mathcal{W}, z) \le \operatorname{mesh}^\times(\cup_\alpha \mathcal{U}_\alpha, z)$*,* $L^\times(\mathcal{W}, z) \ge L^\times(\cup_\alpha \mathcal{U}_\alpha, z)/2$ *for every $z \in \cup_\alpha Z_\alpha$.*

Proof. Let A be the set of colors, $|A| = m$, that we may think to be common for all coverings $\mathcal{U}_\alpha$, so that $\mathcal{U}_\alpha$ is the union of disjoint families, $\mathcal{U}_\alpha = \bigcup_{a \in A} \mathcal{U}_\alpha^a$. For $s \in S$, we denote by $\mathcal{A}_s := \{\alpha \in \mathcal{A} : i(\alpha) = s\}$.

For every $\alpha \in \mathcal{A}$, $a \in A$, we consider the family $\mathcal{V}_\alpha^a = B_{-L^\boxtimes(\mathcal{U}_\alpha)/2}(\mathcal{U}_\alpha^a)$, where the pair $L^\boxtimes(\mathcal{U}_\alpha)$ is dominated by $L^\times(\mathcal{U}_\alpha)$ as in the definition of coverings qualified by a scale function.

Assume that $U \in \mathcal{U}_\alpha$, $V = B_{-L^\boxtimes(\mathcal{U}_\alpha)/2}(V')$ for $V' \in \mathcal{U}_{\alpha'}^a$. We have

(∗) if $i(\alpha) < i(\alpha')$ and $U \cap V \ne \emptyset$, then $U \cup V \subset V'$.

This easily follows from the fact that $L^\boxtimes(\mathcal{U}_{\alpha'}) \ge 4 \operatorname{mesh}^\boxtimes(\mathcal{U}_\alpha)$ in this case. In particular,

(∗∗) for every $U \in \mathcal{U}_\alpha$, every $\alpha' \in \mathcal{A}$ with $i(\alpha) < i(\alpha')$ and every $a \in A$ there exists at most one $V \in \mathcal{V}_{\alpha'}^a$ with $U \cap V \ne \emptyset$,

because the family $\mathcal{U}_{\alpha'}^a$ is disjoint.

We construct an m-colored covering $\mathcal{W}$ by induction over $j \in S$, independently for each color $a \in A$.

Fix a color $a \in A$ and set $\mathcal{W}_0^a := \cup_{\alpha \in \mathcal{A}_0} U_\alpha^a$. Then the family $\mathcal{W}_0^a$ is disjoint and moreover, it possesses properties (2), (3) below. Assume that for some $j \in S$ we have already constructed a family $\mathcal{W}_j^a$ with the following properties:

(1) $\mathcal{W}_j^a$ is disjoint;

(2) for every $U \in \mathcal{V}_\alpha^a$ with $i(\alpha) \le j$ there exists $W \in \mathcal{W}_j^a$ so that $U \subset W$;

(3) for every $W \in \mathcal{W}_j^a$ there exists $U \in \mathcal{U}_\alpha^a$ with $i(\alpha) \le j$ so that $W \subset U$.

In view of (∗∗), property (3) implies that for every $W \in \mathcal{W}_j^a$ there exists at most one member $V \in \bigcup_{\mathcal{A}_{j+1}} \mathcal{V}_\alpha^a$ so that $V \cap W \ne \emptyset$. We set $I(W) = V$ if such V exists and $I(W) = \emptyset$ otherwise; so we have the function $I : \mathcal{W}_j^a \to \bigcup_{\mathcal{A}_{j+1}} \mathcal{V}_\alpha^a \cup \{\emptyset\}$.

We put

$$\mathcal{W}_{j+1}^a = \{W \in \mathcal{W}_j^a : I(W) = \emptyset\} \cup \{(V \cup I^{-1}(V) : V \in \bigcup_{\mathcal{A}_{j+1}} \mathcal{V}_\alpha^a\}.$$

The family $\bigcup_{\mathcal{A}_{j+1}} \mathcal{V}_\alpha^a$ is disjoint. Thus in view of (∗) and (1), the family $\mathcal{W}_{j+1}^a$ is also disjoint. Furthermore, $\mathcal{W}_{j+1}^a$ possesses property (2) (with j replaced by $j+1$) by construction. Property (3) also holds for $\mathcal{W}_{j+1}^a$ by the inductive assumption and in view of (∗).

Then proceeding by induction we obtain the family $\mathcal{W}_N^a$ for each $a \in A$. We set $\mathcal{W}^a = \mathcal{W}_N^a$ and $\mathcal{W} = \cup_a \mathcal{W}^a$. Property (2) for $\mathcal{W}_N^a$ implies that $\mathcal{W}$ is a covering of $\cup_\alpha Z_\alpha$. By property (1) for $\mathcal{W}_N^a$, $\mathcal{W}$ is m-colored. Property (3) implies $\operatorname{mesh}^\times(\mathcal{W}, z) \leq \operatorname{mesh}^\times(\cup_\alpha \mathcal{U}_\alpha, z)$ and property (2) implies $L^\times(\mathcal{W}, z) \geq L^\times(\cup_\alpha \mathcal{U}_\alpha, z)/2$ for every $z \in \cup_\alpha Z_\alpha$. That is, $\mathcal{W}$ possesses all required properties. $\square$

3.2 Barycentric triangulation of products

Recall some standard constructions related to simplicial polyhedra. Given an index set J, we let R^J be the Euclidean space of functions $J \to \mathbb{R}$ with finite support. For x, $x' \in \mathbb{R}^J$, the distance $|xx'|$ is well defined by

$$|xx'|^2 = \sum_{j \in J} (x_j - x'_j)^2.$$

Let $\Delta^J \subset R^J$ be the standard simplex, i.e., $x \in \Delta^J$ iff $x_j \geq 0$ for all $j \in J$ and $\sum_{\in J} x_j = 1$. The metric of R^J induces a metric on Δ^J and on every subcomplex $K \subset \Delta^J$. If J is finite then $|J| - 1 = \dim \Delta^J$ is the (combinatorial) dimension of Δ^J.

For every simplicial polyhedron K, there is the canonical embedding $u : K \to \Delta^J$, where J is the vertex set of K, which is affine on every simplex. Its image $K' = u(K)$ is called the *uniformization* of K. The (combinatorial) dimension of K is the maximal dimension of its simplices.

Given a vertex $v \in K$, its *star* $\overline{\mathrm{st}}_v \subset K$ consists of all simplices of K containing v. The *open star* st_v of v is the star without faces which miss v. If a simplicial polyhedron K is uniform, then the open star st_v of any vertex $v \in K$ is an open neighborhood of v.

There are several ways to triangulate the product of simplicial complexes. One needs for that to choose some ordering of simplices. Since the barycentric triangulation is canonically ordered, we prefer to use the following construction which can be found e.g. in the forthcoming book [BS3]. Given an index set J, we denote by $\overline{\mathrm{ba}}\Delta^J$ the barycentric subdivision of Δ^J, that is a simplicial complex isometric to Δ^J. The vertices of this complex are barycenters of all simplices (including 0-dimensional). The simplices of this complex are convex hulls of all sets S of vertices with the property: if s, $s' \in S$ then they are barycenters of two simplices, one of which is contained in the other. If J is finite then the covering of Δ^J by the open stars of the vertices of its barycentric subdivision is $|J|$-colored: as the color of a star st_v we can take the dimension of the face for which v is the barycenter.

Now, we describe the barycentric triangulation of the product of two simplices Δ^{J_1}, Δ^{J_2}. Regard this product $\Delta^{J_1} \times \Delta^{J_2}$ as a complex with faces that are products of standard simplices. Let b_1, b_2 be barycenters of simplices $S_1 \subset \Delta^{J_1}$, $S_2 \subset \Delta^{J_2}$. We call the point $(b_1, b_2) \in \Delta^{J_1} \times \Delta^{J_2}$ the barycenter of the face $S_1 \times S_2$. We define the *barycentric subdivision* of

$\Delta^{J_1} \times \Delta^{J_2}$ as a simplicial complex $\Delta^{J_1} \overline{\times}_s \Delta^{J_2}$, isometric to $\Delta^{J_1} \times \Delta^{J_2}$, in the following way. The vertices of this complex are the barycenters of all faces (including 0-dimensional). The simplices of this complex are convex hulls of all sets S of vertices with the property: if s, $s' \in S$ then they are barycenters of two faces, one of which is contained in the other.

We denote the uniformization of $\Delta^{J_1} \overline{\times}_s \Delta^{J_2}$ by $\Delta^{J_1} \times_s \Delta^{J_2}$ and call it the *barycentric triangulation* of the product $\Delta^{J_1} \times \Delta^{J_2}$.

The canonical bijection $\varphi : \Delta^{J_1} \times \Delta^{J_2} \to \Delta^{J_1} \times_s \Delta^{J_2}$ is called the *barycentric triangulation map.*

Now, let K_1, K_2 be uniform simplicial polyhedra, which we can identify with subpolyhedra of the standard simplices Δ^{J_1}, Δ^{J_2}; we define the *barycentric triangulation* of $K_1 \times K_2$ as $\varphi(K_1 \times K_2)$.

Note that the covering of $K_1 \times_s K_2$ by open stars of its vertices is $(n_1 + n_2 + 1)$-colored, where $n_1 = \dim K_1$, $n_2 = \dim K_2$: one can take as the color of a star st_v the dimension of the minimal simplex of $\Delta^{J_1 \cup J_2}$ containing v.

We give without proof the following two technical lemmas, (for the proof see [BS3]).

Lemma 3.3. *Let K_1, K_2 be uniform simplicial polyhedra having both finite dimensions. The barycentric triangulation map*

$$\varphi : K_1 \times K_2 \to K_1 \times_s K_2$$

is bilipschitz with bilipschitz constant depending only on dimensions of K_1, K_2. $\square$

Lemma 3.4. *Let K_1, K_2 be uniform simplicial polyhedra, $K = K_1 \times_s K_2$. For every vertex $v \in K$, there are vertices $v_1 \in K_1$, $v_2 \in K_2$ such that $\varphi^{-1}(st_v) \subset st_{v_1} \times st_{v_2}$.* $\square$

Recall that the *nerve* $\mathcal{N}(\mathcal{U})$ of a covering $\mathcal{U} = \{U_j\}_{j \in J}$ is the simplicial complex whose vertices are the members of the covering and a collection of vertices spans a simplex if and only if the corresponding members of $\mathcal{U}$ have nonempty intersection. We assume that the nerves we consider are uniform unless the opposite is explicitly stated; moreover, we can regard $\mathcal{N}(\mathcal{U})$ as subcomplex of Δ^J.

Let $\mathcal{U}$ be an open locally finite covering of a metric space Z, so that no member of it coincides with Z. The barycentric map $p : Z \to \mathcal{N}(\mathcal{U})$ is defined as follows. For every $j \in J$ we put $q_j(z) = \mathrm{dist}(z, Z \setminus U_j)$. Since the covering is open, $\sum_{j \in J} q_j(z) > 0$. Since no member of $\mathcal{U}$ coincides with Z and the covering is locally finite, $\sum_{j \in J} q_j(z) < \infty$ for every point $z \in Z$. Then we define the coordinate functions of the map $p : Z \to \mathbb{R}^J$ by $p_j(z) = q_j(z) / \sum_{j \in J} q_j(z)$. Clearly $p(Z) \subset \mathcal{N}(\mathcal{U})$. For every vertex $v \in \mathcal{N}(\mathcal{U})$ the preimage $p^{-1}(\mathrm{st}_v)$ coincides with corresponding element of the covering.

Suppose in addition that the multiplicity $m(\mathcal{U}) \leq m+1$ is finite and $L(\mathcal{U}) \geq d > 0$. Then, it is not difficult to prove (see e.g. [BS1]) that the map p is Lipschitz with Lipschitz constant

$$\mathrm{Lip}(p) \leq (m+2)^2/d.$$

Lemma 3.5. *Let X_1, X_2 be metric spaces. Let $\mathcal{U}_1$ be an open covering of X_1, $\mathcal{U}_2$ an open covering of X_2, both with multiplicities at most n_1+1 and n_2+1 respectively. Then there exists an (n_1+n_2+1)-colored open covering $\mathcal{W}$ of $X_1 \times X_2$ with*

$$L^\times(\mathcal{W}) \geq q(L(\mathcal{U}_1), L(\mathcal{U}_2)) \quad \textit{and} \quad \mathrm{mesh}^\times(\mathcal{W}) \leq (\mathrm{mesh}(\mathcal{U}_1), \mathrm{mesh}(\mathcal{U}_2)),$$

where the constant q depends only on n_1, n_2.

Proof. According to the definitions of $L^\times$ and $\mathrm{mesh}^\times$, rescaling the metric of $X_1 \times X_2$ in one of the factors does not change the problem, i.e. if we construct a required covering for say $aX_1 \times X_2$, then the rescaling the metric back gives a required covering for $X_1 \times X_2$.

So, by rescaling of the metric, we can assume that $L(\mathcal{U}_2) = \frac{(n_2+2)^2}{(n_1+2)^2} L(\mathcal{U}_1)$. Denote

$$\lambda := \frac{(n_1+2)^2}{L(\mathcal{U}_1)} = \frac{(n_2+2)^2}{L(U_2)}.$$

Let K_i be the nerve of U_i and $p_i : X_i \to K_i$ the corresponding barycentric map, $i = 1, 2$. So we have

$$\mathrm{Lip}(p_i) \leq \lambda.$$

Then for the map $p : X_1 \times X_2 \to K_1 \times K_2$, defined by $p(x_1, x_2) = (p_1(x_1), p_2(x_2))$, we have $\mathrm{Lip}(p) \leq \lambda$.

By Lemma 3.3, the barycentric triangulation map

$$\varphi : K_1 \times K_2 \to K_1 \times_s K_2$$

is Lipschitz with Lipschitz constant c, depending only on n_1, n_2. Consider the covering $\mathcal{V}$ of $K_1 \times_s K_2$ by open stars of its vertices that is (n_1+n_2+1)-colored. Since the polyhedron $K_1 \times_s K_2$ is uniform, the Lebesgue number of this covering is bounded below by a positive constant l depending only on $\dim(K_1 \times_s K_2)$.

Now let $\mathcal{W}$ be the open covering of $X_1 \times X_2$ by preimages of elements of $\mathcal{V}$ under the map $\varphi \circ p$. Then $\mathcal{W}$ is (n_1+n_2+1)-colored and

$$L(\mathcal{W}) \geq \lambda^{-1} c^{-1} l = \frac{l}{c(n_1+2)^2} L(\mathcal{U}_1) = \frac{l}{c(n_2+2)^2} L(\mathcal{U}_2),$$

in particular, $L^\times(\mathcal{W}) \geq q(L(\mathcal{U}_1), L(\mathcal{U}_2))$ with $q = \frac{l}{c \max\{(n_1+2)^2, (n_2+2)^2\}}$.

By Lemma 3.4 for every $W \in \mathcal{W}$ there are $U_1 \in \mathcal{U}_1$, $U_2 \in \mathcal{U}_2$ so that $W \subset U_1 \times U_2$, thus $\mathrm{mesh}^\times(\mathcal{W}) \leq (\mathrm{mesh}\,\mathcal{U}_1, \mathrm{mesh}\,\mathcal{U}_2)$. The lemma follows. $\square$

We denote the covering from this lemma by $\mathcal{W} = \mathcal{U} * \mathcal{V}$. In the following corollary, we use the natural extension of our notations $\operatorname{mesh}^{\times}$ and $L^{\times}$ to the case of more than two factors.

Corollary 3.6. *Let $\mathcal{U}_i$ be an open covering with multiplicity at most $n_i + 1$ of a metric space X_i, $i \in \{1, \dots, k\}$. We let $\mathcal{W}_i = \mathcal{W}_{i-1} * \mathcal{U}_i$ be the covering of $X_1 \times \cdots \times X_i$ for $i = 1, \dots, k$, where $\mathcal{W}_1 = \mathcal{U}_1$. Then $\mathcal{W} = \mathcal{W}_k$ is an open m-colored covering of $X_1 \times \cdots \times X_k$ with $m = n_1 + \cdots + n_k + 1$,*

$$L^{\times}(\mathcal{W}) \geq q(L(\mathcal{U}_1), \dots, L(\mathcal{U}_k))$$

and

$$\operatorname{mesh}^{\times}(\mathcal{W}) \leq (\operatorname{mesh}(\mathcal{U}_1), \dots, \operatorname{mesh}(\mathcal{U}_k)),$$

where the constant q depends only on $n_1, \dots, n_k$.

Proof. Arguing as at the beginning of the proof above and rescaling of the metrics of the factors, we can assume that $L(\mathcal{U}_1) = \cdots = L(\mathcal{U}_k) = r$.

Note that the property $L^{\times}(\mathcal{U}) \geq (r, \dots, r)$ for a covering $\mathcal{U}$ of a product and the property $L(\mathcal{U}) \geq r$ are equivalent. Using this and proceeding by induction, we obtain $L^{\times}(\mathcal{W}) \geq q(r, \dots, r)$ for some constant q depending only on $n_1, \dots, n_k$.

By Lemma 3.4 for every $W \in \mathcal{W}_i$ there are $W' \in \mathcal{W}_{i-1}$, $U \in \mathcal{U}_i$ so that $W \subset W' \times U$ for every $i = 2, \dots, k$. Hence

$$\operatorname{mesh}^{\times}(\mathcal{W}) \leq (\operatorname{mesh} \mathcal{U}_1, \dots, \operatorname{mesh} \mathcal{U}_k).$$

□

We shall use the following

Lemma 3.7. *Let X, X_1, X_2 be metric spaces.*

1. *(1) Assume that $\ell\text{-}\dim(X \times [0,1]) \leq k$. Then there exist $\tau_0 > 0$, $\sigma > 1$ so that for every $0 < \tau < \tau_0$ and $\tau' > 0$ there exists a $(k+1)$-colored open covering $\mathcal{U}$ of $X \times \mathbb{R}$ with $\operatorname{mesh}^{\times}(\mathcal{U}) \leq (\sigma\tau, \sigma\tau')$ and $L^{\times}(\mathcal{U}) \geq (\tau, \tau')$;*
2. *(2) Assume that the spaces of the family $aX_1 \times bX_2$ have ℓ-dim $\leq k$ uniformly in a, $b \geq 1$. Then there exist $\tau_0 > 0$, $\sigma > 1$ so that for every $0 < \tau_1, \tau_2 < \tau_0$ and τ_1', $\tau_2' > 0$ there exists a $(k+3)$-colored open covering $\mathcal{W}$ of $Z = X_1 \times X_2 \times \mathbb{R} \times \mathbb{R}$ with $\operatorname{mesh}^{\times}(\mathcal{W}) \leq \sigma\tau$, $L^{\times}(\mathcal{W}) \geq \tau$, where $\tau = (\tau_1, \tau_2, \tau_1', \tau_2')$.*

Proof. (1) First, we show that $\ell\text{-}\dim(X \times \mathbb{R}) \leq k$. To this end, we represent $X \times \mathbb{R} = Z_1 \cup Z_2$, where $Z_1 = X \times \cup_{k \in \mathbb{Z}}(k, k + 3/4)$, $Z_2 = X \times \cup_{k \in \mathbb{Z}}(k - 1/2, k + 1/4)$. Taking two appropriate qualified coverings of these sets and applying Proposition 3.2, we see that $\ell\text{-}\dim(X \times \mathbb{R}) \leq k$. Then taking an appropriate covering of $X \times \mathbb{R}$ and applying a homothety in the $\mathbb{R}$ factor, we find a desired covering.

(2) Consider the 2-colored covering $\mathcal{U} = \mathcal{U}^1 \cup \mathcal{U}^2$ of $\mathbb{R}$, where $\mathcal{U}^1 = \{(2k, 2k+2) : k \in \mathbb{Z}\}$, $\mathcal{U}^2 = \{(2k+1, 2k+3) : k \in \mathbb{Z}\}$. Clearly, $L(\mathcal{U}) = 1/2$, $\operatorname{mesh}(\mathcal{U}) = 2$.

There is a constant $\delta \in (0,1)$ such that for every sufficiently small $\tau_1,\ \tau_2 > 0$ there exists a $(k+1)$-colored open covering $\mathcal{V}$ of $X \times Y$ with $\operatorname{mesh}^\times(\mathcal{V}) \leq (\tau_1, \tau_2)$ and $L^\times(\mathcal{V}) \geq \delta(\tau_1, \tau_2)$.

For the product covering $\mathcal{W} = (\mathcal{V} * \mathcal{U}) * \mathcal{U}$ of Z we have by Corollary 3.6 $L^\times(\mathcal{W}) \geq q(\tau_1, \tau_2, 1/2, 1/2)$ and $\operatorname{mesh}^\times(\mathcal{W}) \leq (\tau_1, \tau_2, 2, 2)$ for some constant $q > 0$ depending only on k, δ. Then applying homotheties independently in two $\mathbb{R}$-factors and changing the constants, we obtain a desired covering. $\square$

3.3 Locally self-similar spaces

Let $\lambda \geq 1$ and $R > 0$ be given. A map $f : Z \to Z'$ between metric spaces is *λ-quasi-homothetic with coefficient R* if for all z, $z' \in Z$, we have

$$R|zz'|/\lambda \leq |f(z)f(z')| \leq \lambda R|zz'|.$$

A metric space Z is *locally self-similar*, if there is $\lambda \geq 1$ such that for every sufficiently large $R > 1$ and every $A \subset Z$ with $\operatorname{diam} A \leq 1/R$, there is a λ-quasi-homothetic map $f : A \to Z$ with coefficient R.

It is proved in [BL] that the linearly controlled dimension $\ell\text{-}\dim Z$ of every compact locally self-similar metric space Z is finite and coincides with $\dim Z$, $\ell\text{-}\dim Z = \dim Z$.

We shall use the following facts obviously implied by the definition of a quasi-homothetic map.

Lemma 3.8. *Let $h : Z \to Z'$ be a λ-quasi-homothetic map with coefficient R. Let $V \subset Z$ and let $\widetilde{\mathcal{U}}$ be an open covering of $h(V)$, $\mathcal{U} = h^{-1}(\widetilde{\mathcal{U}})$. Then*

(1) $R \operatorname{mesh}(\mathcal{U})/\lambda \leq \operatorname{mesh}(\widetilde{\mathcal{U}}) \leq \lambda R \operatorname{mesh}(\mathcal{U})$;

(2) $\lambda R \cdot L(\mathcal{U}) \geq L(\widetilde{\mathcal{U}}) \geq R \cdot L(\mathcal{U})/\lambda$, *where $L(\mathcal{U})$ is the Lebesgue number of $\mathcal{U}$ as a covering of V.* $\square$

Proposition 3.9. *If compact metric spaces X, Y are locally self-similar then the spaces of the family $aX \times bY$ have $\ell\text{-}\dim \leq n$, $n = \dim(X \times Y)$, uniformly in a, $b \geq 1$.*

Proof. By the remark above, $\ell\text{-}\dim X = N$ is finite. Furthermore, the space $Z = X \times Y$ is also compact and locally self-similar. Thus $\ell\text{-}\dim Z = n$ is finite.

It suffices to prove that there is a constant $\delta \in (0,1)$ such that for every sufficiently small $\tau > 0$ and for every α, $\beta \in (0,1]$ there exist $(n+1)$-colored open coverings $\mathcal{U}$, $\mathcal{U}'$ of Z with $\operatorname{mesh}^\times(\mathcal{U}) \le (\alpha\tau,\tau)$, $L^\times(\mathcal{U}) \ge \delta(\alpha\tau,\tau)$; $\operatorname{mesh}^\times(\mathcal{U}') \le (\tau,\beta\tau)$ and $L^\times(\mathcal{U}') \ge \delta(\tau,\beta\tau)$.

Let δ_1, δ_2, δ_3 be the constants from the definition of ℓ-dim for X, Y, Z respectively, and let $\delta' = \min\{\delta_1,\delta_2,\delta_3\}$. We can assume that the constant λ from the definition of self-similarity is common for both X and Y.

We explain how to construct the covering $\mathcal{U}$, the covering $\mathcal{U}'$ is constructed similarly. Fix a positive $\tau < \min\{\delta'/\lambda, \operatorname{diam} Y\}$. For every $\alpha \in (0,1]$, we construct a covering with $\operatorname{mesh}^\times \le (\alpha\tau,\tau)$ and $L^\times \ge \delta(\alpha\tau,\tau)$, where $\delta = (\delta'/4\lambda^2)^{N+1}/2$.

The argument is similar to that in [BL, Theorem 1.1]. We fix an $(N+1)$-colored open covering $\mathcal{V}''$ of X with $\operatorname{mesh}(\mathcal{V}'') \le \alpha/\lambda$, $L(\mathcal{V}'') \ge \delta'\alpha/\lambda$ and put $\mathcal{V} = \{V \times Y : V \in \mathcal{V}''\}$. Then $\mathcal{V}$ is an $(N+1)$-colored open covering of Z with $L(\mathcal{V}) \ge \delta'\alpha/\lambda$, and we assume that $\mathcal{V}$ is colored by the set $S = \{0,\dots,N\}$, $\mathcal{V} = \cup_{a\in S}\mathcal{V}^a$.

For every $V \in \mathcal{V}$, consider the slightly smaller subset $V' = B_{-\delta'\alpha/2\lambda}(V)$. Then, the sets $Z_a = \cup_{V\in\mathcal{V}^a} V' \subset Z$, $a \in S$, cover Z, $Z = \cup_{a\in S} Z_a$, because $L(\mathcal{V}) \ge \delta'\alpha/\lambda$. The idea is to construct a family of open $(n+1)$-colored coverings $\mathcal{U}_a$ of Z_a, $a \in S$, which is separated with and qualified by the scale function $i : S \to \{0,\dots,N\}$, $i(a) = N - a$, and then to construct the desired $\mathcal{U}$ using Proposition 3.2. The construction of $\mathcal{U}_a$ is based on self-similarity of X.

Using that $\ell\text{-}\dim(X \times Y) = n$ and assuming that τ is sufficiently small, we find for every $a \in S$ an $(n+1)$-colored covering $\widetilde{\mathcal{U}}_a$ of $X \times Y$ with $\operatorname{mesh}(\widetilde{\mathcal{U}}_a) \le (\delta'/4\lambda^2)^a\tau/4$ and $L(\widetilde{\mathcal{U}}_a) \ge \delta'(\delta'/4\lambda^2)^a\tau/4$.

Given $V \in \mathcal{V}$, we fix a map $h_V = (h_V^1, \mathrm{id}) : V \to X \times Y$, where h_V^1 is λ-quasi-homothetic with coefficient $R = \lambda/\alpha$, and put $\widetilde{V} = h_V(V')$. Now, for every $a \in S$, $V \in \mathcal{V}^a$ consider the family $\widetilde{\mathcal{U}}_{a,V} = \{\widetilde{U} \in \widetilde{\mathcal{U}}_a : \widetilde{V} \cap \widetilde{U} \neq \emptyset\}$ which is obviously an $(n+1)$-colored covering of $\widetilde{V}$. Then,

$$\mathcal{U}_{a,V} = \{h_V^{-1}(\widetilde{U}) : \widetilde{U} \in \widetilde{\mathcal{U}}_{a,V}\}$$

is an open $(n+1)$-colored covering of V'.

Note that $U = h_V^{-1}(\widetilde{U})$ is contained in V for every $\widetilde{U} \in \widetilde{\mathcal{U}}_{a,V}$ because $\operatorname{mesh}^\times \widetilde{U} \le (\tau/4,\tau/4)$, therefore $\operatorname{mesh}^\times U \le (\alpha\tau/4,\tau/4)$ by Lemma 3.8, and hence $U \subset B_{(\alpha\tau/2,\operatorname{diam} Y)}(V') \subset V$ by the choice of τ. Thus the family $\mathcal{U}_{a,V}$ is contained in V. Now, the family $\mathcal{U}_a = \cup_{V\in\mathcal{V}^a}\mathcal{U}_{a,V}$ covers the set Z_a of the color a, and it has the following properties

(1) for every $a \in S$, the covering $\mathcal{U}_a$ is $(n+1)$-colored;

(2) $L(\mathcal{U}_a)^{\boxtimes} \ge 4\operatorname{mesh}^{\boxtimes}\mathcal{U}_{a+1}$ for every $a \in S$, $a \le N-1$;

(3) $\operatorname{mesh}^\times(\cup_{a\in S}\mathcal{U}_a) \le (\alpha\tau,\tau)$, $L^\times(\cup_{a\in S}\mathcal{U}_a) \ge (\delta'/4\lambda^2)^{N+1}(\alpha\tau,\tau)$.

Indeed, distinct V_1, $V_2 \in \mathcal{V}^a$ are disjoint and thus any $U_1 \in \mathcal{U}_{a,V_1}$, $U_2 \in \mathcal{U}_{a,V_2}$ are disjoint because $U_1 \subset V_1$, $U_2 \subset V_2$. This proves (1).

Applying Lemma 3.8, we see that $\mathrm{mesh}^\times(\mathcal{U}_a) \le \mathrm{mesh}(\widetilde{\mathcal{U}}_a)(\alpha,1)$ and $L^\times(\mathcal{U}_a) \ge L(\widetilde{\mathcal{U}}_a)(\alpha/\lambda^2, 1/\lambda^2)$ for every $a \in S$. These estimates together with the estimates on $\mathrm{mesh}(\widetilde{\mathcal{U}}_a)$, $L(\widetilde{\mathcal{U}}_a)$ yield (3), and together with the inequalities

$$4\lambda^2 \,\mathrm{mesh}(\widetilde{\mathcal{U}}_{a+1}) \le \delta'(\delta'/4\lambda^2)^a\tau/4 \le L(\widetilde{\mathcal{U}}_a)$$

for every $a \in S$, $a \le N-1$, yield (2).

In view (1) and (2), the family of coverings $\mathcal{U}_a$, $a \in S$, satisfies the condition of Proposition 3.2. Applying this proposition and using (3), we obtain an open $(n+1)$-colored covering $\mathcal{U}$ of $X \times Y$ with $\mathrm{mesh}^\times(\mathcal{U}) \le (\alpha\tau, \tau)$ and $L^\times(\mathcal{U}) \ge \delta(\alpha\tau,\tau)$, where $\delta = (\delta'/4\lambda^2)^{N+1}/2$. □

3.4 'Hyperbolic cone' map

Let Z be a bounded metric space. Assuming that $\mathrm{diam}\, Z > 0$, we put $\mu = \pi/\mathrm{diam}\, Z$ and note that $\mu|zz'| \in [0,\pi]$ for every z, $z' \in Z$. Recall that the hyperbolic cone $\mathrm{Co}(Z)$ over Z is the space $Z \times [0,\infty)/Z \times \{0\}$ with metric defined as follows. Given $x = (z,t)$, $x' = (z',t') \in \mathrm{Co}(Z)$ we consider a triangle $\overline{o}\,\overline{x}\,\overline{x}' \subset \mathrm{H}^2$ with $|\overline{o}\,\overline{x}| = t$, $|\overline{o}\,\overline{x}'| = t'$ and the angle $\angle_{\overline{o}}(\overline{x},\overline{x}') = \mu|zz'|$. Now, we put $|xx'| := |\overline{x}\,\overline{x}'|$. In the degenerate case $Z = \{\mathrm{pt}\}$, we define $\mathrm{Co}(Z) = \{\mathrm{pt}\} \times [0,\infty)$ as the metric product. The point $o = Z \times \{0\} \in \mathrm{Co}(Z)$ is called the *vertex* of $\mathrm{Co}(Z)$.

We let $h : Z \times [0,\infty) \to \mathrm{Co}(Z)$ be the canonical projection, and $\delta = \delta_{\mathrm{H}^2}$ the hyperbolicity constant of H^2.

It is well known that the hyperbolic cone $\mathrm{Co}(Z)$ is a hyperbolic space which satisfies the δ-inequality w.r.t. the vertex o. Furthermore, there is a canonical inclusion $Z \subset \partial_\infty \mathrm{Co}(Z)$, and the metric of Z is visual w.r.t. the base point o and the parameter e,

$$e^{-(\xi|\xi')_o - c_0} \le d(\xi,\xi') \le e^{-(\xi|\xi')_o + c_0}$$

for some $c_0 \ge 0$ and all ξ, $\xi' \in Z$. In general, $\mathrm{Co}(Z)$ is not geodesic, however, for every point $z \in \mathrm{Co}(Z)$ there is a geodesic segment $oz \subset \mathrm{Co}(Z)$ and every $\xi \in Z$ is represented by a geodesic ray $o\xi \subset \mathrm{Co}(Z)$.

The following lemma is similar to [BoS, Lemma 5.1].

Lemma 3.10. *Let X be a δ-hyperbolic space. Given o, z_1, $z_2 \in X$ and $x_1 \in oz_1$, $x_2 \in oz_2$, we have*

$$\min\{|ox_1|, |ox_2|, (z_1|z_2)_o\} - 2\delta \le (x_1|x_2)_o \le \min\{|ox_1|, |ox_2|, (z_1|z_2)_o\} + 2\delta.$$

Proof. Applying the δ-inequality twice, we obtain

$$\begin{aligned}(z_1|z_2)_o &\ge \min\{(z_1|x_1)_o, (x_1|x_2)_o, (x_2|z_2)_o\} - 2\delta\\ &= \min\{|ox_1|, (x_1|x_2)_o, |ox_2|\} - 2\delta = (x_1|x_2)_o - 2\delta.\end{aligned}$$

Similarly we have $(x_1|x_2)_o \ge \min |ox_1|, (z_1|z_2)_o, |ox_2| - 2\delta$. The lemma follows. □

Corollary 3.11. *Let X be a δ-hyperbolic space. Suppose that points $x_1 \in o\xi_1$, $x_2 \in o\xi_2$ satisfy $|x_1 o| = |x_2 o|$ and $|x_1 x_2| > 4\delta$, where ξ_1, $\xi_2 \in \partial_\infty X$. Then*

$$(x_1|x_2)_o - 2\delta \le (\xi_1|\xi_2)_o \le (x_1|x_2)_o + 2\delta.$$

Proof. In this case $(x_1|x_2)_o < |ox_1| - 2\delta = |ox_2| - 2\delta$. Hence by the previous lemma, $\min\{|ox_1|, |ox_2|, (z_1|z_2)_o\} = (z_1|z_2)_o$ for each $z_1 \in (x_1, \xi_1)$, $z_2 \in (x_2, \xi_2)$. Now, the statement follows from that lemma. $\square$

We put $c = c_0 + 2\delta$.

Claim 1. *For every $R > 0$, the map $h|_{Z\times\{R\}} : Z \times \{R\} \to \mathrm{Co}(Z)$ possesses the following properties*

(1) $|h(z, R)h(z', R)| \ge 2D$ for every $D > 0$ and each z, $z' \in Z$ with $|zz'| \ge e^{-R+D+c}$;

(2) $|h(z, R)h(z'R)| \le 2D$ for every $D > 2\delta$ and each z, $z' \in Z$ with $|zz'| \le e^{-R+D-c}$.

Proof. (1) We identify $z = \xi$, $z' = \xi' \in \partial_\infty \mathrm{Co}(Z)$ and denote $h(z, R) = x$, $h(z', R) = x'$. Thus

$$e^{-R+D+c} \le |\xi\xi'| \le e^{-(\xi|\xi')_o+c_0},$$

and we obtain $(\xi|\xi')_o \le R - D - 2\delta$. By Lemma 3.10,

$$(x|x')_o \le \min\{R, (\xi|\xi)_o\} + 2\delta \le R - D.$$

Hence, $|xx'| = 2(R - (x|x')_o) \ge 2D$.

(2) Using the notations as above, we obtain $e^{-(\xi|\xi')_o-c_0} \le |\xi\xi'| \le e^{-R+D-c}$. Hence, $(\xi|\xi')_o \ge R - D + 2\delta$. In the case $|xx'| \le 4\delta$, we see that $|xx'| < 2D$ by the assumption $D > 2\delta$. In the case $|xx'| > 4\delta$, we apply Corollary 3.11 and obtain $(x|x')_o \ge (\xi|\xi')_o - 2\delta \ge R - D$. Thus $|xx'| = 2(R - (x|x')_o) \le 2D$. $\square$

The proof of the following claim is similar, and we leave it as an exercise to the reader. We only mention that at some point one should use monotonicity of the Gromov product, $(x|x')_o \le (\xi|\xi')_o$ for every $x \in o\xi$, $x' \in o\xi'$.

Claim 2. *For every $R > 0$, the map $h|_{Z\times\{R\}} : Z \times \{R\} \to \mathrm{Co}(Z)$ possesses the following properties*

(1) $|zz'| \ge e^{-R+D-c}$ for every $D > 2\delta$ and each $x = h(z, R)$, $x' = h(z', R)$ with $|xx'| \ge 2D$;

(2) $|zz'| \le e^{-R+D+c}$ *for every* $D > 0$ *and each* $x = h(z, R)$, $x' = h(z', R)$ *with* $|xx'| \le 2D$.

$\square$

Claim 3. *(1) Given* $l > 0$, $R \ge R_1 \ge l$, *we let* $r = e^{-R_1+3l/2+c}$. *Then*

$$h(B_r(x) \times [R-l, R+l]) \supset B_l(h(x, R));$$

(2) given $l > 2\delta$, $t > 0$, $R_2 \ge R \ge t$, *we let* $r = e^{-R_2+l-c}$. *Then*

$$h(B_r(x) \times [R-t, R+t]) \subset B_{3t+2l}(h(x, R)).$$

Proof. (1) Given $y' \in \mathrm{Co}(Z)$ with $|y'h(x, R)| < l$, we represent $y' = h(y, R_0)$. Using the definition of distances in $\mathrm{Co}(Z)$, we obtain $R_0 \in [R-l, R+l]$ and $|h(y, R')h(x, R')| < l$, where $R' = \min\{R_0, R\}$. Then by Claim 2(2), we have

$$|yx| \le e^{-R'+l/2+c} \le e^{-R+l+l/2+c} \le e^{-R_1+3l/2+c}.$$

Claim 3(1) follows.

(2) Assume that $(y, R_0) \in B_r(x) \times [R-t, R+t]$. Then $R_0 \in [R-t, R+t]$ and $|yx| < e^{-R_2+l-c} \le e^{-R_0+t+l-c}$. By Claim 1(2), we have $|h(y, R_0)h(x, R_0)| < 2t + 2l$. Then by the triangle inequality,

$$|h(x, R)h(y, R_0)| < 3t + 2l.$$

Claim 3(2) follows. $\square$

Claim 4. *(1) Given* $R_1 \ge R > 0$, $l > 8\delta$, *we let* $r_1 = e^{-R_1-l/4-c}$. *Then* $h^{-1}(B_l(h(x, R))) \supset B_{r_1}(x) \times [R-l/2, R+l/2]$*;*

(2) given $l > 0$, $R \ge R_2 \ge l$, *we let* $r_2 = e^{-R_2+3l/2+c}$. *Then* $h^{-1}(B_l(h(x, R))) \subset B_{r_2}(x) \times [R-l, R+l]$.

Proof. (1) Assume that $(y, R_0) \in B_{r_1}(x) \times [R-l/2, R+l/2]$. Then $R_0 \in [R-l/2, R+l/2]$ and $|xy| < e^{-R_1-l/4-c} \le e^{-R_0+l/4-c}$. Using Claim 1(2), we obtain $|h(y, R_0)h(x, R_0)| < l/2$. Then by the triangle inequality, $|h(y, R_0)h(x, R)| < l$. Claim 4(1) follows.

(2) Claim 4(2) follows from Claim 3(1). $\square$

4 Proofs

Proposition 4.1. *Let* Z_1, Z_2 *be bounded metric spaces such that the spaces of the family* $\{aZ_1 \times bZ_2 : a \ge 1, b \ge 1\}$ *have* ℓ-dim $\le k$ *uniformly. Then*

$$\mathrm{asdim}(\mathrm{Co}(Z_1) \times \mathrm{Co}(Z_2)) \le k + 2.$$

We need some preparation for the proof. We use the following notations

$$
\begin{aligned}
\overline{P_0}(T) &= Z_1 \times [0,T] \times Z_2 \times [0,T],\\
\overline{P_1}(T) &= Z_1 \times [T,\infty] \times Z_2 \times [0,T],\\
\overline{P_2}(T) &= Z_1 \times [0,T] \times Z_2 \times [T,\infty],\\
\overline{P_3}(T) &= Z_1 \times [T,\infty] \times Z_2 \times [T,\infty],
\end{aligned}
$$

and $P_i = (h_1,h_2)(\overline{P_i})$, where $h_i : Z_i \times [0,\infty) \to \operatorname{Co} Z_i$ is the canonical projection, $i = 1,2$.

Lemma 4.2. *For every $L > 0$ there exist T_3, $M > 0$ so that for every $T > T_3$ there exists a $(L,M,k+3)$-covering of $P_3(T)$.*

Proof. Let $\sigma > 1$ be the constant from Lemma 3.7(2). We fix a constant $L > 0$, put $H = 512\sigma^4 L$ and for every integer m, $n \geq 0$ consider the product

$$A_{m,n} = Z_1 \times Z_2 \times [Hm, Hm+H] \times [Hn, Hn+H].$$

For some $T > 0$, we first construct a $(k+3)$-colored covering $\mathcal{U}$ of $\overline{P_3}(T)$, so that for all sufficiently large integers m, n the following holds

$$L^\times(\mathcal{U})|_{A_{m,n}} \geq (e^{-Hm+3L/2+c}, e^{-Hn+3L/2+c}, L, L);$$

$$\operatorname{mesh}^\times(\mathcal{U})|_{A_{m,n}} \leq (e^D e^{-Hm}, e^D e^{-Hn}, H/4, H/4)$$

for some $D > 0$, where the constant $c > 0$ is defined just before Claim 1.

It is convenient to use the following notations $c_1 = 2e^{3L/2+c}$, $b = e^{-H}$. Using Lemma 3.7(2), we fix a sufficiently large N so that for each integers m, $n \geq N$ and for every $i = 0,1,2,3$ there exists a $(k+3)$-colored covering $\widetilde{\mathcal{U}}^i_{m,n}$ of $Z_1 \times Z_2 \times \mathbb{R} \times \mathbb{R}$ with $L^\times(\widetilde{\mathcal{U}}^i_{m,n}) \geq (4\sigma)^i(c_1 b^{m-i}, c_1 b^{n-i}, 2L, 2L)$ and $\operatorname{mesh}^\times(\widetilde{\mathcal{U}}^i_{m,n}) \leq \sigma(4\sigma)^i(c_1 b^{m-i}, c_1 b^{n-i}, 2L, 2L)$. For m, $n \geq N$, we consider the subfamily $\mathcal{U}^i_{m,n}$ of $\widetilde{\mathcal{U}}^i_{m,n}$ consisting of all members that intersect $A_{m,n}$.

Next, we define the function $i : \mathbb{N} \times \mathbb{N} \to \{0,1,2,3\}$ by $i(m,n) := m \bmod 2 + 2(n \bmod 2)$ and put $\mathcal{U}_{m,n} := \mathcal{U}^{i(m,n)}_{m,n}$. Now, we show that the family of coverings $\mathcal{U}_{m,n}$ of $A_{m,n}$, m, $n \geq N$, is separated with and qualified by the scale function i.

We have $\sigma(4\sigma)^3 2L = H/4 < H/2$ by the choice of the constants. It follows from the estimate on $\operatorname{mesh}^\times(\widetilde{\mathcal{U}}^i_{m,n})$ that any member $U \in \mathcal{U}_{m,n}$ is disjoint with any $U' \in \mathcal{U}_{m',n'}$, if $i(m,n) = i(m',n')$ and $(m,n) \neq (m',n')$. That is, the family $\mathcal{U}_{m,n}$, m, $n \geq N$, is separated with the scale function i. In particular, if $U \cap U' \neq \emptyset$ for some $U \in \mathcal{U}_{m,n}$, $U' \in \mathcal{U}_{m',n'}$ then the pairs (m,n) and (m',n') are adjacent, i.e., $|m'-m| \leq 1$ and $|n'-n| \leq 1$.

Assume that pairs (m',n'), (m,n) are adjacent and $i(m',n') > i(m,n)$. We have $(4\sigma)^{i+1}2L = 4\sigma(4\sigma)^i 2L$ and $(4\sigma)^{i+1}c_1 b^{m-(i+1)} > 4\sigma(4\sigma)^i c_1 b^{m-i}$, since $b < 1$. These inequalities together with the estimates on $L^{\times}(\widetilde{\mathcal{U}}^i_{m,n})$, $\mathrm{mesh}^{\times}(\widetilde{\mathcal{U}}^i_{m,n})$, show that

$$L^{\boxtimes}(\mathcal{U}_{m',n'}) \geq 4\,\mathrm{mesh}^{\boxtimes}(\mathcal{U}_{m,n}).$$

It follows that these coverings are qualified by the scale function i. So applying Proposition 3.2 to this family of coverings, we obtain a $(k+3)$-colored covering $\mathcal{U}$ of $\overline{P}_3(T)$, $T = HN$, for which the following holds

$$\begin{aligned}
L^{\times}(\mathcal{U})|_{A_{m,n}} &\geq \frac{1}{2}(c_1e^{-Hm}, c_1e^{-Hn}, 2L, 2L) \\
&= (e^{-Hm+3L/2+c}, e^{-Hn+3L/2+c}, L, L); \\
\mathrm{mesh}^{\times}(\mathcal{U})|_{A_{m,n}} &\leq \sigma(4\sigma)^3(c_1e^{-H(m-i(m,n))}, c_1e^{-H(n-i(m,n))}, 2L, 2L) \\
&\leq (e^De^{-Hm}, e^De^{-Hn}, H/4, H/4)
\end{aligned}$$

for some $D > 0$.

Now, we estimate the Lebesgue number and the mesh of the covering $\mathcal{U}' = (h_1, h_2)(\mathcal{U})$. Applying Claim 3(1) with $l = L$, $R_1 = H_m$, $R \in [Hm, Hm+H]$ (resp. $R_1 = Hn$, $R \in [Hn, Hn+H]$) to the map h_1 (resp. h_2), we obtain $L(\mathcal{U}') \geq L$.

Next, we represent the estimate above for $\mathrm{mesh}^{\times}(\mathcal{U})|_{A_{m,n}}$ as follows

$$\mathrm{mesh}^{\times}(\mathcal{U})|_{A_{m,n}} \leq (e^{-(Hm+H)+H+D+c-c}, e^{-(Hn+H)+H+D+c-c}, H/4, H/4).$$

Now, applying Claim 3(2) with $l = H + D + c$ (note that $l > 2\delta = 2\delta_{\mathrm{H}^2}$), $t = H/4$, $R_2 = Hm + H$, $R \in [Hm, Hm+H]$ (resp. $R_2 = Hn + H$, $R \in [Hn, Hn+H]$) to the map h_1 (resp. h_2), we obtain $\mathrm{mesh}(\mathcal{U}') \leq M = 2(3H/4 + 2(H+D+c))$.

It follows that $\mathcal{U}'$ is a $(L, M, k+3)$-covering of $P_3(T)$. □

Lemma 4.3. *Given $L > 0$, there is T_2 so that for every $T > T_2$ there exist $(L, M, k+2)$-coverings of $P_1(T)$, $P_2(T)$ with $M > 0$ depending on T, L.*

Proof. The proof is similar to that of Lemma 4.2. For simplicity, we construct a desired covering of $P_1(T)$. For $P_2(T)$ the construction is the same with obvious modifications.

We fix $L > 0$, put $H = 40\sigma^2L$, σ is the constant from Lemma 3.7(1), and for every integer $m \geq 0$ consider the product

$$A_m = Z_1 \times Z_2 \times [Hm, Hm+H] \times [0, T].$$

For some $T > 0$, we first construct a $(k+2)$-colored covering $\mathcal{U}$ of $\overline{P_1}(T)$, so that for all $m \geq N$ the following holds

$$L(\mathcal{U})|_{A_m} \geq (e^{-Hm+3L/2+c}, \mathrm{diam}\, Z_2, L, L),$$

$$\mathrm{mesh}(\mathcal{U})|_{A_m} \le (e^D e^{-Hm}, \mathrm{diam}\, Z_2, H/4, T+L)$$

for some $D > 0$, where the constant $c = c_0 + 2\delta$ is introduced before Claim 1.

For convenience we use notations $c_1 = 2e^{3L/2+c}$, $b = e^{-H}$. Since Z_1 is isometrically embedded into $Z_1 \times Z_2$, we have $\ell\text{-}\dim Z_1 \le \ell\text{-}\dim(Z_1 \times Z_2) \le k$ and therefore $\ell\text{-}\dim(Z_1 \times [0,1]) \le k+1$. Then, using Lemma 3.7(1), we find a sufficiently large N so that for every integer $m \ge N$ and every $i = 0,1$ there exists a $(k+2)$-colored covering $\widetilde{\mathcal{U}}_m^i$ of $Z_1 \times \mathbb{R}$ with

$$\begin{aligned} L^\times(\widetilde{\mathcal{U}}_m^i) &\ge (4\sigma)^i (c_1 b^{m-i}, 2L) \\ \mathrm{mesh}^\times(\widetilde{\mathcal{U}}_m^i) &\le \sigma(4\sigma)^i (c_1 b^{m-i}, 2L). \end{aligned}$$

We take its subfamily $\mathcal{U}_m^i$ consisting of all members that intersect $\widehat{A}_m = Z_1 \times [Hm, Hm+h]$.

We define the function $i : \mathbb{N} \to \{0,1\}$ by $i(m) := m \mod 2$ and put $\mathcal{U}_m := \mathcal{U}_m^{i(m)}$. Now, we show that the family of the coverings $\mathcal{U}_m$ of $\widehat{A}_m$, $m \ge N$, is separated with and qualified by the scale function i.

We have $\sigma(4\sigma)2L < H/4$ by the choice of the constants. It follows from the estimate on $\mathrm{mesh}^\times(\widetilde{\mathcal{U}}_m^i)$ that any member $U \in \mathcal{U}_m$ is disjoint with any $U' \in \mathcal{U}_{m'}$, if $i(m) = i(m')$ and $m \ne m'$. That is, the family $\mathcal{U}_m$, $m \ge N$, is separated with the scale function i. In particular, if $U \cap U' \ne \emptyset$ for some $U \in \mathcal{U}_m$, $U' \in \mathcal{U}_{m'}$ then $|m' - m| \le 1$.

Assume that $|m' - m| = 1$ and $i(m') > i(m)$, i.e., $i(m') = 1, i(m) = 0$. Using the estimates on $L^\times(\widetilde{\mathcal{U}}_m^i)$, $\mathrm{mesh}^\times(\widetilde{\mathcal{U}}_m^i)$ and the fact that $b < 1$, we obtain

$$L^\boxtimes(\mathcal{U}_{m'}) \ge 4\,\mathrm{mesh}^\boxtimes(\mathcal{U}_m).$$

Thus these coverings are qualified by the scale function i. So, we apply Proposition 3.2 to the family of the coverings $\mathcal{U}_m$, $m \ge N$, and obtain a $(k+2)$-colored covering $\mathcal{U}'$ of $Z_1 \times [T, \infty)$, $T = HN$, with

$$\begin{aligned} L^\times(\mathcal{U}')|_{A_m} &\ge \frac{1}{2}(c_1 e^{-Hm}, 2L) = (e^{-Hm+3L/2+c}, L); \\ \mathrm{mesh}^\times(\mathcal{U}')|_{A_m} &\le 4\sigma^2 (c_1 e^{-H(m-i(m))}, 2L) \le (e^D e^{-Hm}, H/4) \end{aligned}$$

for some $D > 0$.

Now, the family $\mathcal{U} = \{U \times Z_2 \times [0, T+L] : U \in \mathcal{U}'\}$ is a $(k+2)$-colored covering of $\overline{P_1}(T)$ with required estimates on $L^\times$- and $\mathrm{mesh}^\times$-numbers.

Let us estimate the Lebesgue number and the mesh of the covering $\mathcal{U}'' = (h_1, h_2)(\mathcal{U})$. Applying Claim 3(1) with $l = L$, $R_1 = H_m$, $R \in [Hm, Hm+H]$ to the map h_1, we obtain $L(\mathcal{U}'') \ge L$.

Next, we represent the estimate above for $\mathrm{mesh}^\times(\mathcal{U})|_{A_m}$ as follows

$$\mathrm{mesh}^\times(\mathcal{U})|_{A_m} \le (e^{-(Hm+H)+H+D+c-c}, \mathrm{diam}\, Z_2, H/4, T+L).$$

Now, applying Claim 3(2) with $l = H + D + c$ (note that $l > 2\delta = 2\delta_{\mathrm{H}^2}$), $t = H/4$, $R_2 = Hm + 2H$, $R \in [Hm, Hm+H]$ to the map h_1 and using that the diameter of $h_2(Z_2 \times [0, T+L]) \subset \mathrm{Co}(Z_2)$ is at most $2(T+L)$, we obtain $\mathrm{mesh}(\mathcal{U}'') \le M = 2\max\{3H/4 + 2(2H+D+c), T+L\}$.

It follows that $\mathcal{U}''$ is a $(L, M, k+2)$-covering of $P_1(T)$. $\square$

Lemma 4.4. *For every T, $L > 0$ there exists a $(L, M, 1)$-covering of $P_0(T)$ with $M > 0$ depending only on T and L.*

Proof. The covering of $P_0(T)$ consisting of the unique element $U = h_1(Z_1 \times [0, T+2L)) \times h_2(Z_2 \times [0, T+2L))$ has the required properties. $\square$

Proof of Proposition 4.1. Given $L_0 > 0$, we construct a uniformly bounded covering $\mathcal{U}$ of $\mathrm{Co}(Z_1) \times \mathrm{Co}(Z_2)$ with $L(\mathcal{U}) \ge L_0$.

We apply Lemma 4.2 for $L = L_0$, find corresponding constants T_3, M_3 and let $\mathcal{U}_3$ be a $(L_0, M_3, k+3)$-covering of $P_3(T_3)$. We can assume that $M_3 \ge L_0$.

Next, we apply Lemma 4.3 first for $L = 2M_3$ and denote by T_2 the corresponding constant. Thus for $T_0 = \max\{T_2, T_3\}$ there is a $(2M_3, M_2, k+2)$-covering $\mathcal{U}_2$ of $P_2(T_0)$, where the constant $M_2 \ge 2M_3$ depends on T_0, M_3.

Then we again apply Lemma 4.3 for $L = 2M_2$ and denote by T_1 the corresponding constant. Thus for $T^* = \max\{T_0, T_1\}$ there is a $(2M_2, M_1, k+2)$-covering $\mathcal{U}_1$ of $P_1(T^*)$, where the constant $M_1 \ge 2M_2$ depends on T^*, M_2.

Finally, by Lemma 4.4, there is a $(2M_1, M, 1)$-covering $\mathcal{U}_0$ of $P_0(T^*)$, where $M > 0$ depend on M_1 and T^*.

We have

$$\mathrm{Co}(Z_1) \times \mathrm{Co}(Z_2) = P_0(T^*) \cup P_1(T^*) \cup P_2(T_0) \cup P_3(T_3).$$

Now we apply Lemma 3.1 consequently to the sequence of the coverings $\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3$ and obtain a bounded $(k+3)$-colored covering $\mathcal{U}$ of $\mathrm{Co}(Z_1) \times \mathrm{Co}(Z_2)$ with $L(\mathcal{U}) \ge L_0$. This shows that $\mathrm{asdim}(\mathrm{Co}(Z_1) \times \mathrm{Co}(Z_2)) \le k+2$. $\square$

Proposition 4.5. *Let Z be a bounded metric space. Then*

$$\ell\text{-asdim}\,\mathrm{Co}(Z) \le \ell\text{-dim}(Z \times [0,1]).$$

Proof. We can assume that $\ell\text{-dim}(Z \times [0,1]) = k$ is finite. We fix some $L > 4\delta$, $\delta = \delta_{\mathrm{H}^2}$. Let $\tau_0 > 0$, $\sigma > 1$ be the constants from Lemma 3.7(1). We put $H = 40\sigma^2 L$ and denote

$$A_m = Z \times [Hm, Hm+H].$$

There exists $N > 0$ such that for $m \ge N$ we have $e^{-H(m-1)+3L/2+c} < \tau_0$ uniformly in $L > 4\delta$, where as above the constant $c = c_0 + 2\delta$ is introduced before Claim 1.

Exactly as in the proof of Lemma 4.3, we construct a $(k+1)$-colored covering $\mathcal{U}$ of $Z \times [HN, \infty)$ such that for all $m \geq N$ we have

$$\begin{aligned} L^{\times}(\mathcal{U})|_{A_m} &\geq (e^{-Hm+3L/2+c}, L), \\ \operatorname{mesh}^{\times}(\mathcal{U})|_{A_m} &\leq (e^{D}e^{-Hm}, H/4) \end{aligned}$$

for some constant $D > 0$ linearly depending on L.

We have

$$\operatorname{mesh}^{\times}(\mathcal{U})|_{A_m} \leq (e^{-(Hm+H)+(H+3L/2+D+c)-c}, H/4).$$

Applying Claim 3 to the canonical map $h : Z \times [0, \infty) \to \operatorname{Co}(Z)$, we see that the covering $h(\mathcal{U}) = \mathcal{U}'$ has $L(\mathcal{U}') \geq L$ and $\operatorname{mesh}(\mathcal{U}') \leq (3H/4 + 2(H + D + 3L/2 + c))$.

Now for every $L > 4\delta$ we have the $(k+1)$-colored covering $\mathcal{U}'$ of $h(Z \times [CL, \infty))$, $L(\mathcal{U}') \geq L$, $\operatorname{mesh}(\mathcal{U}') \leq CL$ for some constant $C \geq 1$ independent of L. Then we cover the ball $B_{CL}(o) \subset \operatorname{Co} Z$ by the ball $\mathcal{U}'' = B_{3CL}(o)$ and apply Lemma 3.1 to $\mathcal{U}'$, $\mathcal{U}''$ to obtain a covering $\mathcal{W}$ of $\operatorname{Co} Z$ with multiplicity $m(\mathcal{W}) \leq k+1$, $L(\mathcal{W}) \geq L$ and $\operatorname{mesh}(\mathcal{W}) \leq 6CL$. This shows that $\operatorname{asdim} \operatorname{Co}(Z) \leq \ell\text{-}\dim(Z \times [0,1])$. □

Proposition 4.6. *Let $Z_1, \dots, Z_n$ be bounded metric spaces. Then*

$$\ell\text{-}\dim(Z_1 \times \dots \times Z_n \times [0,1]^n) \leq \operatorname{asdim}(\operatorname{Co} Z_1 \times \dots \times \operatorname{Co} Z_n).$$

We reduce the proof to the following two lemmas.

Lemma 4.7. *Assume that* $\operatorname{asdim}(\operatorname{Co} Z_1 \times \dots \times \operatorname{Co} Z_n) \leq k$. *Then for some $\overline{\varepsilon} > 0$ there exists a function $\delta : [0, \overline{\varepsilon}] \to [0,1]$ such that for every $\varepsilon \in (0, \overline{\varepsilon})$, $\tau \in (0,1)$ there exists a $(\delta(\varepsilon)\tau, \varepsilon\tau, k+1)$-covering of $Z_1 \times \dots \times Z_n \times [0,\tau]^n$ by open subsets of $Z_1 \times \dots \times Z_n \times \mathbb{R}^n$.*

Proof. We let $Z = Z_1 \times \cdots Z_n$, $X = \operatorname{Co} Z_1 \times \dots \times \operatorname{Co} Z_n$, and $h : Z \times \mathbb{R}^n \to X$ the product of the canonical projections. By the assumption, for every $L > 0$ there exists a $(L, M, k+1)$-covering $\mathcal{U}$ of X with some $M < \infty$. We fix such a covering for $L > 8\delta_{\mathrm{H}^2}$ together with $\varepsilon \in (0, \overline{\varepsilon})$, $\tau \in (0,1)$ and let $r = -\ln(\varepsilon\tau) + 5M/2 + c$, where the constant $c = c_0 + 2\delta_{\mathrm{H}^2}$ is defined before Claim 1.

The annulus $\operatorname{An}_{[r, r+M/\varepsilon]} \subset X$ which consists of all $(x_1, \dots, x_n) \in X$ with $r \leq |x_i o| \leq r + M/\varepsilon$, $i = 1, \dots, n$, is covered by $\mathcal{U}(\varepsilon, \tau) = \{U \in \mathcal{U} : U \cap \operatorname{An}_{[r, r+M/\varepsilon]} \neq \emptyset\}$. We lift this covering to the covering $\mathcal{U}'(\varepsilon, \tau) = h^{-1}(\mathcal{U}(\varepsilon, \tau))$ of $Z \times [r, r+M/\varepsilon]^n$ (the later we identify with $Z \times [0, M/\varepsilon]^n$).

We have $\operatorname{mesh}(\mathcal{U}(\varepsilon, \tau)) \leq M$. Applying Claim 4(2) with $l = M$, $R_2 = r - M$ (note that $R_2 \geq l$), $R \geq R_2$, we obtain $\operatorname{mesh}^{\times} \mathcal{U}'(\varepsilon, \tau) \leq (\varepsilon\tau, M)$ due to our choices. Applying Claim 4(1) with $R_1 = r + M/\varepsilon$, $l = L$, $R \leq R_1$, we obtain

$$L^{\times}\mathcal{U}'(\varepsilon, \tau) \geq (e^{-(r+M/\varepsilon)-L/4-c}, L/2) = (e^{-M/\varepsilon - L/4 - 5M/2 - 2c}\varepsilon\tau, L/2),$$

the later equality holds by the choice of r.

Consider the homothety 'along the second factor', $Z\times[0,M/\varepsilon]^n\to Z\times[0,\tau]^n$, with coefficient $\varepsilon\tau/M$ and the image $\mathcal{V}$ of $\mathcal{U}'$ under this homothety. Then $\operatorname{mesh}^\times(\mathcal{V})\le(\varepsilon\tau,\varepsilon\tau)$, $L^\times(\mathcal{V})\ge(\delta_0(\varepsilon)\tau,\varepsilon\tau L/(2M))$ for $\delta_0(\varepsilon)=c_1\varepsilon e^{-M/\varepsilon}$, $c_1=e^{-L/4-5M/2-2c}$, and we obtain the covering with the required properties taking $\delta(\varepsilon)=\min\{\delta_0(\varepsilon),L\varepsilon/(2M)\}$. $\square$

Lemma 4.8. *Let Z be a metric space. Suppose that for some $\overline{\varepsilon}>0$ there exists a function $\delta:[0,\overline{\varepsilon}]\to[0,1]$ such that for every $\varepsilon\in(0,\overline{\varepsilon})$, $\tau\in(0,1)$ there exists a $(\delta(\varepsilon)\tau,\varepsilon\tau,k+1)$-covering of $Z\times[0,\tau]^n$ by open subsets of $Z\times[0,\infty)^n$. Then $\ell\text{-}\dim(Z\times[0,1]^n)\le k$.*

Proof. It suffices to find for sufficiently large $m\in\mathbb{N}$ a $(c/m,C/m,k+1)$-covering of $Z\times[0,1]^n$ with c, C depending only on Z.

We represent

$$Z\times[0,1]^n=\bigcup_{(l_1,\dots,l_n)\in A} Z\times\left[\frac{l_1}{m},\frac{l_1+1}{m}\right]\times\cdots\times\left[\frac{l_n}{m},\frac{l_n+1}{m}\right],$$

where $A=\{0,\dots,m-1\}^n$. We define $i:A\to S=\{0,\dots,2^n-1\}$ by

$$i(l_1,\dots,l_n)=(l_1\mod 2)2^0+\cdots+(l_n\mod 2)2^{n-1}.$$

Let $\varepsilon_0=\min\{1/4,\overline{\varepsilon}\}$, $\varepsilon_{i+1}=\delta(\varepsilon_i)/2$, $i\in S$. We fix a $(\delta(\varepsilon_i)/m,\varepsilon_i/m,k+1)$-covering $\mathcal{U}_{(l_1,\dots,l_n)}$ of $Z\times\left[\frac{l_1}{m},\frac{l_1+1}{m}\right]\times\cdots\times\left[\frac{l_n}{m},\frac{l_n+1}{m}\right]$, where $i=i(l_1,\dots,l_n)$. We put for $s\in S$

$$\mathcal{U}_s=\bigcup_{(l_1,\dots,l_n)\in i^{-1}(s)}\mathcal{U}_{(l_1,\dots,l_n)}.$$

In view of the condition $\varepsilon_{i+1}=\delta(\varepsilon_i)/2$ we have $\operatorname{mesh}(\mathcal{U}_{s+1})\le L(\mathcal{U}_s)/2$. Proceeding by induction over $s\in S$ and applying Lemma 3.1, we obtain a $(c/m,C/m,k+1)$-covering of $Z\times[0,1]^n$ with $c=\delta(\varepsilon_{2^n-1})$, $C=\varepsilon_0$. $\square$

Proof of Proposition 4.6. Proposition 4.6 follows immediately from previous two lemmas. $\square$

4.1 Proof of main results

It is known that any visual hyperbolic space X can be quasi-isometrically embedded in the hyperbolic cone over its boundary at infinity $\partial_\infty X$ and that for a geodesic hyperbolic space Y the hyperbolic cone over its boundary at infinity can be quasi-isometrically embedded in the space itself, see e.g. [BS3].

It follows that in these cases

$$\operatorname{asdim}X\le\operatorname{asdim}\operatorname{Co}(\partial_\infty X)$$

$$\operatorname{asdim} Y \geq \operatorname{asdim} \operatorname{Co}(\partial_\infty Y)$$

and similar estimates hold for products of such spaces.

Then Theorem 1.1 follows immediately from Proposition 4.1, Theorem 1.3 from Proposition 4.6, Theorem 1.5 from Proposition 4.5.

For the proof of Corollary 1.4, we note that any Gromov hyperbolic group is a visual hyperbolic space with the compact, locally self-similar boundary at infinity, see [BL]. Thus by Proposition 3.9, the spaces of the family $\{a\partial_\infty\Gamma_1 \times b\partial_\infty\Gamma_2 : a, b \geq 1\}$ have ℓ-dim $\leq n$ uniformly, $n = \dim(\partial_\infty\Gamma_1 \times \partial_\infty\Gamma_2)$, for any hyperbolic groups Γ_1, Γ_2. By Theorem 1.1,

$$\begin{aligned} \operatorname{asdim}(\Gamma_1 \times \Gamma_2) &\leq \ell\text{-}\dim(\partial_\infty\Gamma_1 \times \partial_\infty\Gamma_2) + 2 \\ &= \dim(\partial_\infty\Gamma_1 \times \partial_\infty\Gamma_2) + 2, \end{aligned}$$

where the last equality follows from the fact that $\partial_\infty\Gamma_1 \times \partial_\infty\Gamma_2$ is a compact, locally self-similar space, and [BL]. The opposite inequality

$$\operatorname{asdim}(\Gamma_1 \times \Gamma_2) \geq \dim(\partial_\infty\Gamma_1 \times \partial_\infty\Gamma_2) + 2$$

follows from Theorem 1.3 and the inequalities

$$\ell\text{-}\dim(Z \times [0,1]) \geq \dim(Z \times [0,1]) = \dim Z + 1,$$

which hold for any metric space Z.

Corollary 1.6 follows from Theorem 1.3 and Theorem 1.5.

5 Applications

As applications of Corollary 1.4, we have

1) examples of the strict inequality in the product theorem for the asymptotic dimension in the class of hyperbolic groups. Namely, it is proved in [Dr1, Dr2] that for every prime p, there is a hyperbolic Coxeter group Γ_p with a Pontryagin surface Π_p as the boundary at infinity. Then by Corollary 1.4, we have

$$\begin{aligned} \operatorname{asdim}(\Gamma_p \times \Gamma_q) &= \dim(\Pi_p \times \Pi_q) + 2 \\ &< \dim \Pi_p + \dim \Pi_q + 2 \\ &= \operatorname{asdim} \Gamma_p + \operatorname{asdim} \Gamma_q \end{aligned}$$

for prime $p \neq q$ (the last equality follows from the main result of [BL]);

2) examples of strict inequality in the product theorem for the hyperbolic dimension (the hyperbolic dimension is quasi-isometry invariant of metric space introduced in [BS2]). Indeed, let Γ_p, Γ_q for $p \neq q$ be as in 1). Then

$$\begin{aligned} \mathrm{hypdim}(\Gamma_p \times \Gamma_q) &\leq \mathrm{asdim}(\Gamma_p \times \Gamma_q) \\ &< \dim \Pi_p + \dim \Pi_q + 2 \\ &= \mathrm{hypdim}(\Gamma_p) + \mathrm{hypdim}(\Gamma_q) \end{aligned}$$

(the last equality also follows from the main result of [BL] and [BS2]);

3) the equality $\mathrm{asdim}(\Gamma \times \mathbb{R}) = \mathrm{asdim}(\Gamma) + 1$ for any hyperbolic group Γ.

It is known that for a visual, proper, geodesic hyperbolic space X we have

$$\dim(\partial_\infty X) + 1 \leq \mathrm{asdim}\, X \leq \ell\text{-}\dim(\partial_\infty X) + 1. \qquad (*)$$

The estimate from below is simple and the ideas of the proof are contained for example in [Gr, 1.E$'_1$]. The estimate from above is proved in [Bu1, Bu2]. Corollary 1.6 gives a better estimate from below and it allows to show that the first inequality in $(*)$ might be strict. More exactly, we give an example of a hyperbolic space with the asymptotic dimension arbitrarily larger than the topological dimension of its boundary at infinity. Let $Z = \{0\} \cup \{1/m : m \in \mathbb{N}\}$ and $X = \mathrm{Co}\, Z^k$. It is known that $\ell\text{-}\dim(Z^k \times [0,1]) = k+1$, see [BL]. Then $\mathrm{asdim}\, X = k+1$ while $\dim(\partial_\infty X) = 0$. The natural conjecture is that for every compact metric space Y we have $\ell\text{-}\dim(Y \times [0,1]) = \ell\text{-}\dim(Y) + 1$. If this is true then both inequalities in $(*)$ become the equalities.

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