Some remarks concerning holomorphically convex hulls and envelopes of holomorphy

Burglind Jöricke

Max-Planck-Gesellschaft
Arbeitsgruppe "Algebraische Geometrie und Zahlentheorie"
Mohrenstraße 39
O-1086 Berlin

Germany

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

Germany

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0. Introduction and formulation of the results.

In the last few years several papers appeared which concerned the connection between holomorphically convex hulls and envelopes of holomorphy in certain situations. We take here the following notion of holomorphically convex hulls. For a compact set K contained in the closure of a strictly pseudoconvex bounded domain in $\mathbb{C}^2$ we denote by $\hat{K}^{\bar{\Omega}}$ the hull with respect to the space $\mathcal{O}(\bar{\Omega})$ of functions holomorphic on $\bar{\Omega}$: $\hat{K}^{\bar{\Omega}} = \{z \in \bar{\Omega} : |f(z)| \leq \max |f| \text{ for all functions } f \text{ in } \mathcal{O}(\bar{\Omega})\}$. Hulls of this kind are interesting in approximation theory. The set K is called $\mathcal{O}(\bar{\Omega})$-convex if $K = \hat{K}^{\bar{\Omega}}$. The following theorem summarizes some of the known results concerning connections between holomorphically convex hulls and descriptions of certain envelopes of holomorphy.

We will say here that a continuous function u on a set A in $\mathbb{C}^2$ has analytic extension to a set D which is the union of Riemannian domains over $\mathbb{C}^2$ if there is a uniquely determined continuous function on $D \cup A$ which is analytic on D and coincides on A with u. Note that this definition implies that each connected component of D contains in its closure a sufficiently large part of A. The definition does not include non-schlicht analytic continuation to D.

Theorem A. *Let $\Omega \subset \mathbb{C}^2$ be a bounded strictly pseudoconvex domain with boundary of class C^2. Let $K \subset \partial\Omega$ be compact and let u be the restriction to $\partial\Omega \setminus K$ of a function which is analytic in a neighbourhood of $\partial\Omega \setminus K$. Then the following is true.*

1) *u has analytic extension to $\Omega \setminus \hat{K}^{\bar{\Omega}}$.*

2) *There is a one-one-correspondence between connected components of $\partial\Omega \setminus K$ and connected components of $\Omega \setminus \hat{K}^{\bar{\Omega}}$, namely, the boundary of each component of $\Omega \setminus \hat{K}^{\bar{\Omega}}$ contains exactly one connected component of $\partial\Omega \setminus K$ and does not intersect any other component of $\partial\Omega \setminus K$.*

3) *$\Omega \setminus \hat{K}^{\bar{\Omega}}$ is pseudoconvex (hence $\bar{\Omega} \setminus \hat{K}^{\bar{\Omega}}$ is the envelope of holomorphy of $\partial\Omega \setminus K$).*

Remark. Clearly each component of $\partial\Omega \setminus K$ is contained in the boundary of some component of $\Omega \setminus \hat{K}^{\bar{\Omega}}$. Indeed, consider for $\zeta \in \partial\Omega \setminus K$ a peak function for Ω, we see that for some neighbourhood U_ζ of ζ the set $U_\zeta \cap \Omega$ is contained in $\Omega \setminus \hat{K}^{\bar{\Omega}}$. In this way we get for each connected component of $\partial\Omega \setminus K$ a connected open set in $\Omega \setminus \hat{K}^{\bar{\Omega}}$ whose boundary contains the mentioned component of $\partial\Omega \setminus K$.

Part 2 is interesting since it gives a geometric relation between compact sets and their holomorphically convex hulls, the problem of geometric descriptions of holomorphically convex sets being up to now a difficult problem. Part 1 was proved by Stout [6] and Lupacciolu [11] under the restriction $\partial\Omega \setminus K$ being connected. They used integral formulas for the proof. The 3. part was proved by Słodkowski [M13], see also [12] for eliminating from Słodkowski's proof what is needed in this situation. The proof of part 2 is contained in the work of Alexander and Stout [2]. It uses a deep theorem of Stolzenberg [15]. For another proof in case Ω being the ball see [1]. After part 2 was proved, part 1 followed in full generality [17].

Part 1 of the theorem was generalized by Lupacciolu and Stout [6], [11]. They considered continuous CR-functions (instead of analytic functions) and replaced the condition of strict pseudoconvexity of Ω by the following one: Ω is a bounded pseudoconvex domain in $\mathbf{C}^2$ with $\bar{\Omega}$ having a Stein neighbourhood basis and $\partial\Omega \setminus K$ is connected (alternatively, Ω is compactly contained in a Stein manifold X, $K = \hat{K}^X \cap \partial\Omega$ and $\Omega \setminus \hat{K}^{\bar{\Omega}}$ has to be replaced by $\Omega \setminus \hat{K}^X$ with $\hat{K}^X$ being the hull with respect to holomorphic functions on X).

We will remove here the condition of the existence of a Stein neighbourhood basis of $\bar{\Omega}$ (for examples of smoothly bounded pseudoconvex domains without Stein neighbourhood basis see [4]). To do this we divide the problem into two independant problems, the problem of analytic extension of continuous CR-functions from hypersurfaces to their one-sided neighbourhoods and the problem of describing envelopes of holomorphy of one-sided neighbourhoods of $\partial\Omega \setminus K$ for Ω being a bounded pseudoconvex domain and K a compact subset of $\partial\Omega$. For the second problem we use exhaustion of Ω by relatively compact strictly pseudoconvex domains and apply theorem A to these domains. Instead of the hulls $\hat{K}^{\bar{\Omega}}$ we have to consider slightly smaller hulls.

Definition. Let Ω be a bounded domain in $\mathbf{C}^2$ with boundary of class C^2. Denote by $A(\Omega)$ the space of functions continuous in $\bar{\Omega}$ and analytic in Ω. The $A(\Omega)$-hull of K is defined in the following way: $A(\Omega)\text{-hull}\,(K) = \left\{ z \in \overline{\Omega} : |f(z)| \le \max_K |f| \text{ for all } f \in A(\Omega) \right\}$. K is called $A(\Omega)$-convex if $A(\Omega)\text{-hull}\,(K) = K$.

Note that the $A(\Omega)$-hull of a compact set is always $A(\Omega)$-convex. For a real C^2 hypersurface H in $\mathbf{C}^2$ and a compact subset K of H we will use also the following definition.

Definition. K is called $CR(H)$-convex if for each $z \in H \setminus K$ there exists a continuous CR-function f on H with $f(z) = 1$ and $\max_K |f| < 1$.

As usually, a CR-function on a hypersurface is a function which satisfies the tangential Cauchy-Riemann equations in the weak sense. Note, that for a compact set K in the boundary $\partial\Omega$ of a bounded domain in $\mathbf{C}^2$ the set $\partial\Omega \cap A(\Omega)$-hull (K) is $CR(H)$-convex ($\partial\Omega$ is assumed to be of class C^2).

We need also the notion of one-sided neighbourhoods. Let z be a point in a hypersurface H in $\mathbf{C}^n$. Take a small neighbourhood U in $\mathbf{C}^n$ of z such that $U \setminus H$ consists of two connected components. Each component is called a one-sided neighbourhood of z (with respect to H). An open set in $\mathbf{C}^2$ wich contains a one-sided neighbourhood of each point of H is called a one-sided neighbourhood of H.

We will prove here the following two theorems.

Theorem 1. *Suppose H is a connected hypersurface of class C^2 in $\mathbf{C}^2$ with compact Leviflat part (i.e. the set of points of H, where the Leviform vanishes, is compact). Let $K \neq H$ be a compact subset of H which is $CR(H)$-convex. Then each continuous CR-function on $H \setminus K$ has analytic extension to a one-sided neighbourhood of each point $z \in H \setminus K$ (the one-sided neighbourhood not depending on the CR-function).*

Note that the condition on H is satisfied in particular if H is a connected closed hypersurface of class C^2 in $\mathbf{C}^2$. If H bounds a pseudoconvex domain, then the one-sided neighbourhoods are contained in the domain. Some condition like $CR(H)$-convexity of K is essential as is seen immediately from the remark that for the conclusion of theorem 1 it is necessary that no connected component of $H \setminus K$ is Leviflat.

Theorem 2. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^2$ with boundary $\partial\Omega$ of class C^2. Suppose $K \subset \partial\Omega$, $K \neq \partial\Omega$, is a compact $CR(\partial\Omega)$-convex set (i.e. $K = \partial\Omega \cap A(\Omega)$-hull$(K)$, but K is not necessarily $A(\Omega)$-convex). Then the following is true.*

1. *Let $\mathcal{O}$ be a one-sided neighbourhood of $\partial\Omega \setminus K$, $\mathcal{O} \subset \Omega \setminus A$-hull$(K)$. Suppose each connected component of $\mathcal{O}$ contains in its boundary exactly one component of $\partial\Omega \setminus K$ and no other point of $\partial\Omega \setminus K$. Let u be holomorphic in $\mathcal{O}$. Then u has (uniquely determined) analytic extension to $\Omega \setminus A(\Omega)$-hull$(K)$.*

2. *The same one-one- correspondence between connected components of $\partial\Omega \setminus K$ and connected components of $\Omega \setminus A(\Omega)$-hull$(K)$ is true as in part 2 of theorem A.*

By the assumption of the theorem $(\partial\Omega \setminus K) \cap A(\Omega)\text{-hull}(K) = \emptyset$, so each point of $\partial\Omega \setminus K$ has a neighbourhood in $\mathbb{C}^2$ not intersecting $A(\Omega)$-hull(K). This shows that a one-sided neighbourhood $\mathcal{O}$ of $\partial\Omega \setminus K$ with the properties described in the first part of the theorem always exists.

Note that the condition of $CR(\partial\Omega)$-convexity of K is essential for theorem 2. So, simple examples show that for K not $CR(H)$-convex the correspondence between connected components of $\partial\Omega \setminus K$ and those of $\partial\Omega \setminus A(\Omega)$-hull$(K)$ can be rather complicated.

Example 1. $\Omega = \mathbb{B}^2 \cap \{Re z_1 < 1 - \varepsilon\}$ ($\mathbb{B}^2$ being the unit ball in $\mathbb{C}^2, \varepsilon > 0$), $K = \partial\mathbb{B}^2 \cap \{Re z_1 = 1 - \varepsilon\}$. $\partial\Omega \setminus K$ consists of two connected components. The complement in $\partial\Omega$ of the set $\partial\Omega \cap A(\Omega)\text{-hull}(K) = \overline{\mathbb{B}}^2 \cap \{Re z_1 = 1 - \varepsilon\}$ is connected. The example can easily be modified to give a domain with smooth boundary. This is also a counterexample for part 1 of theorem 1 in case K is not $CR(\partial\Omega)$-convex.

Example 2. Ω is as in example 1, $K = \partial\mathbb{B}^2 \cap \{Im z_1 = 0\} \cap \{Re z_1 \leq 1 - \varepsilon\}$. $\partial\Omega \setminus K$ is connected. The complement in $\partial\Omega$ of the set $\partial\Omega \cap A(\Omega)\text{-hull}(K) = \partial\Omega \cap \{Im z_1 = o\}$ has two connected components.

In the second part of this article we prove theorem 1 and in part 3 we deduce theorem 2 from theorem A. In the first part we give a new proof of the first two assertions of theorem A. This proof seems to us very natural. It is based on the observation that Oka's characterization principle for holomorphically convex hulls (see, for example, [4] p. 263/264) is in complex dimension 2 essentially the same as what is needed for getting analytic continuation along one-parameter families of one-dimensional analytic manifolds via Kontinuitätssatz. Oka's underlying idea was to approximate functions which are holomorphic near a compact set by functions which are holomorphic in a fixed larger domain by continuously moving poles of meromorphic functions to the outside of the domain. So, as for the characterization of complements of holomorphically convex hulls, as for analytic continuation via Kontinuitätssatz we need "curves" (i.e. continuous one-parameter families) of one-dimensional analytic varieties (or, by Sard's theorem of analytic manifolds) which don't meet a compact set, the curve connecting a variety through a given point with another variety outside the domain. The only additional point for the proof of theorem A is some "monodromy consideration" for showing that the envelope of holomorphy is schlicht.

1. Analytic continuation along families of analytic manifolds.

Proof of theorem A. Let $z \in \Omega \setminus \hat{K}^{\Omega}$. By definition of $\hat{K}^{\Omega}$ there exists a function $f = f_z$ which is holomorphic in a neighbourhood of $\bar{\Omega}$ such that $f(z) = 1$ and $\max_K |f| < 1 - \delta$ for some $\delta > 0$. By Sard's theorem we can assume that 1 is a regular (non-critical) value of the function Ref in a neighbourhood of $\bar{\Omega}$. Indeed, almost all values of Ref are regular. Since f is a non-constant analytic function, for each neighbourhood U of z (in C^2) the set $f(U)$ is a neighbourhood of $f(z)$ in C. So there are points $z + \eta$ arbitrarily close to z such that $Imf(z+\eta) = 0$ and $(Ref)\ (z+\eta)$ is a regular value for Ref. Take instead of f the function f_1, $f_1(\zeta) = f(\zeta + \eta).\ (f(z+\eta))^{-1}$, which is analytic in a neighbourhood of $\bar{\Omega}$ if η is small enough. So, assume from the beginning that 1 is a regular value of Ref and denote by Ω_f a (connected) neighbourhood of $\bar{\Omega}$, such that f is analytic near $\bar{\Omega}_f$. We will use the notation $R_f = \{\zeta \in \Omega_f : Ref(\zeta) = 1\}$. It is clear that $\bar{R}_f \cap K = \emptyset$. Since the gradient of Ref does not vanish on $\bar{R}_f$ the same is true (by Cauchy-Riemann equations) for Imf. So R_f is foliated into a one-parameter family of analytic manifolds $V_{f,t} = \{\zeta \in \Omega_f : f(\zeta) = 1 + it\}$ of complex dimension one. If $V_{f,t}$ is not empty, then each of its connected components has non-empty boundary contained in $\partial\Omega_f$ (since there are no compact analytic manifolds contained in C^2). We assume (by shrinking Ω_f eventually) that the function u has analytic extension to a neighbourhood of $\bar{R}_f \setminus \Omega$. Put $r_f = R_f \cap \Omega$ and $v_{f,t} = V_{f,t} \cap \Omega$. It is convenient to use the following

Definition. Let X be a topological space and S a subset of X. We call a neighbourhood U of S nicely choosen if each connected component of U intersects S.

Consider a small nicely choosen neighbourhood of $\partial\Omega \setminus K$ (in C^2) which carries an analytic function with restriction to $\partial\Omega \setminus K$ being equal to u (the function which occured in the formulation of theorem A). Denote this analytic function also by u.

The following proposition realizes analytic continuation of the function u along the family of complex manifolds $v_{f,t}$.

Proposition 1. *There exists a nicely choosen neighbourhood $\mathcal{O}_f$ of $\bar{r}_f$ and a (uniquely determined, univalent) analytic function u_f in $\mathcal{O}_f$ such that u_f coincides with u at all points of $\mathcal{O}_f$ which are sufficiently close to $\partial\Omega$.*

Proof. The function f will be fixed during the proof of proposition 1, so for shortness we will write v_t $(V_t, resp.)$ instead of $v_{f,t}$ $(V_{f,t}, resp.)$. Set $t_{\max} = \max_{\bar{r}_f} Imf$, $t_{\min} = \min_{\bar{r}_f} Imf$.

Since for $\zeta \in \Omega$ the set $f(\Omega)$ contains a neighbourhood of $f(\zeta)$ in C, the set $S_{t_{\max}} = \{\zeta \in \bar{\Omega} : f(\zeta) = 1 + it_{\max}\}$ is contained in $\partial\Omega$. We will say that for $t \in [t_{\min}, t_{\max}]$ the continuation property (CP) holds, if there is an analytic function u_t in some nicely choosen neighbourhood U_t of $\bar{r}_f \cap \{Imf \geq t\}$ which coincides with u near $\partial r_f \cap \{Imf \geq t\}$. For proving proposition 1 we have to show that CP holds for $t = t_{\min}$. For t close to $t_{\max}$ CP holds since for those t the set $\bar{r}_f \cap \{Imf \geq t\} \subset \{z \in \bar{\Omega} : f(z) = 1 + i\tau : t \leq \tau \leq t_{\max}\}$ is contained in a small neighbourhood of $S_{t_{\max}} \subset \partial\Omega$ so that u is defined near this set. Set $t_o = inf\{t : CP \text{ holds for } t\}$ and suppose $t_o > t_{\min}$. The main step in the proof of proposition 1 is the following

Lemma 1. *There exists an analytic function v_{t_o} defined in a nicely choosen neighbourhood w_{t_o} of V_{t_o}, which coincides with u near $V_{t_o} \setminus v_{t_o} \subset R_f \setminus r_f$.*

Proof. By the definition of t_o for $t > t_o$ there exists a nicely choosen neighbourhood U_t of $\bar{r}_f \cap \{Imf \geq t\}$ and an analytic function u_t in U_t which coincides with u near $\partial r_f \cap \{Imf \geq t\}$. Since the function u is defined near the set $\bar{R}_f \setminus r_f$, it is easy to see that there exists a nicely choosen neighbourhood $\tilde{U}_t$ of $\bar{R}_f \cap \{Imf \geq t\}$ and an analytic function $\tilde{u}_t$ in $\tilde{U}_t$ which coincides with u near $(\bar{R}_f \setminus r_f) \cap \{Imf \geq t\}$.

Now we can apply the Cartan-Thullen argument (see [20], III. 16.1.). Suppose u is analytic in an ε-neighbourhood of ∂R_f for some $\varepsilon > 0$. Let $z \in V_{t_o}$ and $z_t \in V_t$ $(t > t_o)$ be close to z, say $|z - z_t|$ much smaller than ε. By the Cartan-Thullen argument applied to V_t and $\partial V_t \subset \partial R_f$ the Taylor series of $\tilde{u}_t$ around the point z_t converges near z and defines an analytic function near z. It is easy to see that this function does not depend on the choice of the point $z_t \in V_t$ for $t > t_o$ and z_t being sufficiently close to z. (Here we used, that R_f and V_t are smooth manifolds and the Taylor expansions around points z_t, $t > t_o$, come from a unique analytic function near $\bar{R}_f \cap \{Imf > t_o\}$.) So, there is a nicely choosen neighbourhood w_{t_o} of V_{t_o} and a uniquely determined analytic function v_{t_o} in w_{t_o} which (by construction) coincides with u near $V_{t_o} \setminus \bar{\Omega}$, since all points in $V_{t_o} \setminus \bar{\Omega}$ are limit points of $V_t \setminus v_t$ $(t > t_o)$ and $V_t \setminus v_t \subset (R_f \setminus r_f) \cap \{Imf \geq t\}$. It remains to show that v_{t_o} coincides with u near $V_{t_o} \cap \partial\Omega$. This follows from the next lemma and the construction of the function v_{t_o}.

Lemma 2. *Let C be a connected component of $V_{t_o} \cap \partial\Omega$. There exists a point $z \in C$ which is the limit point of a sequence $\{z_{t_n}\}_{t_n > t_o}$, $z_{t_n} \in V_{t_n} \setminus v_{t_n}$.*

Proof. Suppose, in contrary, there is a small connected neighbourhood N (in $\mathbf{C}^2$) of C, say $N \subset \Omega_f$, which does not intersect $V_t \setminus v_t$ for $t > t_o$. Then $N \cap V_t \subset v_t$ for $t > t_o$ close to t_o. But by Kontinuitätssatz this contradicts the fact that Ω is pseudoconvex.

End of the proof of the proposition 1. There is some $\sigma > 0$ such that w_{t_o} covers $\bar{r}_f \cap \{t_o - \sigma \leq Imf \leq t_o + \sigma\}$ and v_{t_o} coincides with u near the set $\partial r_f \cap \{t_o - \sigma \leq Imf \leq t_o + \sigma\}$. Indeed, w_{t_o} is a nicely choosen neighbourhood of V_{t_o}. The compact set $S_{t_o} = \{\zeta \in \bar{\Omega} : f(\zeta) = 1 + it_o\}$ is contained in V_{t_o}, and $\bar{r}_f \cap \{Imf = t_o\} \subset S_{t_o}$, so w_{t_o} covers $\bar{r}_f \cap \{t_o - \sigma \leq Imf \leq t_o + \sigma\}$ for small $\sigma > 0$ (since on the compact set $\bar{r}_f \setminus w_{t_o}$ the continuous function $Imf - t_o$ does not vanish, it is greater than some $\sigma > 0$ there). Further, v_{t_o} coincides with u in some neighbourhood $\mathfrak{N}_{t_o}$ (in $\mathbf{C}^2$) of $V_{t_o} \setminus v_{t_o} = V_{t_o} \setminus \Omega \supset \partial r_f \cap \{Imf = t_o\}$. Since $\partial r_f \setminus \mathfrak{N}_{t_o}$ is compact and $Imf \neq t_o$ on $\partial r_f \setminus \mathfrak{N}_{t_o}$ we have $\partial r_f \cap \{t_o - \sigma \leq Imf \leq t_o + \sigma\} \subset \mathfrak{N}_{t_o}$ for small $\sigma > 0$.

Now the set $w_{t_o} \cup U_{t_o + \frac{\sigma}{2}}$ covers $\bar{r}_f \cap \{Imf \geq t_o - \sigma\}$ ($U_{t_o+\frac{\sigma}{2}}$ comes from the fact that CP holds for $t = t_o + \frac{\sigma}{2} > t_o$). For proving CP for $t_o - \sigma$ we set $\tilde{U}_{t_o - \sigma} = (w_{t_o} \cap \{Imf < t_o + \frac{\sigma}{2}\}) \cup \left(U_{t_o+\frac{\sigma}{2}} \cap \{Imf > t_o + \frac{\sigma}{2}\}\right) \cup \Gamma_{t_o}$, where Γ_{t_o} is the union of all connected components of $U_{t_o+\frac{\sigma}{2}} \cap w_{t_o} \cap \{Imf = t_o + \frac{\sigma}{2}\}$ which intersect $\bar{r}_f$. Γ_{t_o} is an open subset of $R_f \cap \{Imf = t_o + \frac{\sigma}{2}\}$, which contains $\bar{r}_f \cap \{Imf = t_o + \frac{\sigma}{2}\}$, so it is not hard to see that $\tilde{U}_{t_o-\sigma}$ is open (in $\mathbf{C}^2$) and contains $\bar{r}_f \cap \{Imf \geq t_o - \sigma\}$. After removing from $\tilde{U}_{t_o-\sigma}$ superfluous connected components (i.e. those which don't intersect $\bar{r}_f \cap \{Imf \geq t_o - \sigma\}$) we get a nicely choosen neighbourhood $U_{t_o-\sigma}$ of $\bar{r}_f \cap \{Imf \geq t_o - \sigma\}$. Now the function $u_{t_o-\sigma}$ defined to be equal to $u_{t_o+\frac{\sigma}{2}}$ in $U_{t_o-\sigma} \cap \{Imf \geq t_o + \frac{\sigma}{2}\}$ and equal to v_{t_o} in $U_{t_o-\sigma} \cap \{Imf \leq t_o + \frac{\sigma}{2}\}$ is a correctly defined analytic function in $U_{t_o-\sigma}$ which coincides with u near $\partial r_f \cap \{Imf \geq t_o - \sigma\}$. This contradicts the definition of t_0. Proposition 1 is proved.

The following proposition shows that the analytic continuation to $\Omega \setminus \hat{K}^{\bar{\Omega}}$ is univalent.

Proposition 2. *Let $z \in \Omega \setminus \hat{K}^{\bar{\Omega}}$. Suppose $f, g \in \mathcal{O}(\bar{\Omega})$ satisfy the conditions $f(z) = g(z) = 1$, $\max_K |f| < 1$, $\max_K |g| < 1$ and 1 is a regular value for both functions, Ref and Reg, on $\bar{\Omega}_f \cap \bar{\Omega}_g$. Let u_f and u_g be the functions defined by proposition 1. Then u_f and u_g are analytic in a (connected) neighbourhood of z and coincide there.*

Proof. Denote by Ω_z the connected component of $\Omega_f \cap \Omega_g$ which contains $\bar{\Omega}$. Consider the analytic manifolds $\{\zeta \in \Omega_z : f(z) = 1\}$ $(\{\zeta \in \Omega_z : g(z) = 1\}$, resp.) and let V_f (V_g, resp.) be the connected components, containing z. We may assume, that V_f and V_g intersect transversally at z and therefore the functions $f - 1$ and $g - 1$ define a coordinate system in a neighbourhood of z. Indeed, otherwise we take instead of f the function $f + l_z$, where $l_z(\zeta) = l(\zeta) - l(z)$ $(\zeta \in \Omega_f)$ for a complex linear function l with sufficiently small coefficients. Clearly, one can assume that for some domain $\Omega_{f+l_z} \supset \overline{\Omega}$, $\overline{\Omega}_{f+l_z} \subset \Omega_f$, the set $\{\zeta \in \bar{\Omega}_{f+l_z} : Re(f + l_z)(\zeta) = 1\}$ is contained in a small neighbourhood of R_f and the gradient of $Re(f + l_z)$ does not vanish on the intersection of this neighbourhood with $\bar{\Omega}_{f+l_z}$. So $u_f = u_{f+l_z}$ near z.

Take a curve γ_f on V_f which joines the point z with a point of $V_f \setminus \bar{\Omega}$ (such a curve always exists, otherwise V_f must be contained in $\bar{\Omega}$ and, therefore, being relatively closed in Ω_z, it must be compact, which is not possible). $V_f \cap V_g$ consists of isolated points, so we can assume that z is the only point on the curve which is contained in $V_f \cap V_g$. Take a similar curve γ_g on V_g. Let U_f be a connected neighbourhood of γ_f on which the analytic function $\tilde{u}_f$ is defined ($\tilde{u}_f$ is equal to u_f on $U_f \cap \Omega$ and equal to u on $U_f \setminus \Omega$, see the proof of lemma 1). Let U_g be a similar neighbourhood of γ_g and assume that $U_f \cap U_g$ is a ball B_ρ around z of small radius ρ in coordinates $(f - 1, g - 1)$, $B_\rho \subset \Omega_z$. Define in Ω_z the function $F = 1 + (f - 1)(g - 1)$. Then $F = 1$ on $\gamma_f \cup \gamma_g$ and $\nabla F = (f - 1) \nabla g + \nabla f(g - 1) \neq 0$ on $\gamma_f \cup \gamma_g \setminus \{z\}$. For small $\varepsilon > 0$ with $1 + \varepsilon$ being a regular value of ReF put $F_\varepsilon = (1 + \varepsilon)^{-1} F$, $V_{F_\varepsilon} = \{\zeta \in \Omega_z : F_\varepsilon(z) = 1\}$ and let u_{F_ε} be the analytic function near $\bar{r}_{F_\varepsilon}$ constructed in proposition 1.

Lemma 3. *If $\varepsilon > 0$ is small enough and $1+\varepsilon$ is a regular value of F then there exists a curve γ^ε on $V_{F_\varepsilon} \cap (U_f \cup U_g)$ of the following kind: γ^ε is the union of three curves, $\gamma^\varepsilon = \gamma^\varepsilon_f \cup \gamma^\varepsilon_i \cup \gamma^\varepsilon_g$ such that $\gamma^\varepsilon_f \subset V_{F_\varepsilon} \cap U_f$ connects a point in $U_f \setminus \Omega$ with a point $z_f \in U_f \cap U_g$, similar conditions hold for γ^ε_g, and $\gamma^\varepsilon_i \subset U_f \cap U_g$ joines z_f with z_g.*

From lemma 3 proposition 2 follows. Indeed, the analytic function $\tilde{u}_{F_\varepsilon}$ is defined on $\gamma^\varepsilon \subset U_f \cup U_g$. Since $\gamma^\varepsilon_f \cup \gamma^\varepsilon_i$ is connected and is contained in U_f, and, moreover, near $\gamma^\varepsilon_f \setminus \Omega$ we have $\tilde{u}_{F_\varepsilon} = u$ and $\tilde{u}_f = u$, the equality $\tilde{u}_{F_\varepsilon} = \tilde{u}_f$ holds near the whole curve $\gamma^\varepsilon_f \cup \gamma^\varepsilon_i$. Similar arguments are true for $\gamma^\varepsilon_g \cup \gamma^\varepsilon_i$, therefore near γ^ε_i $\tilde{u}_f = \tilde{u}_{F_\varepsilon} = \tilde{u}_g$. So, on the connected set $U_f \cap U_g \subset \Omega$, which contains γ^ε_i, the desired equality $u_f = u_g$ holds.

Proof of lemma 3. Consider a connected part of γ_f, which does not contain z and joines a point in $U_f \setminus \Omega$ with a point in $U_f \cap U_g$. On this part we have $F = 1$ and $\nabla F \neq 0$. From this fact the existence of γ^ε_f for small $\varepsilon > 0$ is clear. The existence of γ^ε_g follows in the same way. For joining the endpoint z_f of γ^ε_f with the endpoint z_g of γ^ε_g by a curve γ^ε_i on $V_{F_\varepsilon} \cap$

$U_f \cap U_g$, we will show that the latter set is connected. In coordinates $z_1 = f-1$, $z_2 = g-1$ the set has the form $\{(z_1, z_2) \in B_\rho,\ z_1 z_2 = \varepsilon\}$, where $B_\rho = \{|z_1|^2 + |z_2|^2 < \rho^2\}$. This set is topologically the product of a piece of a hyperbola $\{r_1 r_2 = \varepsilon : r_1 > 0, r_2 > 0, r_1^2 + r_2^2 < \rho^2\}$ and the circle $\{(\zeta, \zeta^{-1}) : \zeta \text{ in the unit circle of the complex plane}\}$. Therefore $V_{F_\varepsilon} \cap U_f \cap U_g$ is connected and lemma 3 is proved.

<u>**End of the proof of theorem A.**</u> From proposition 1 and 2 follows the existence of a uniquely determined analytic function u_Q in $Q = \Omega \setminus \hat{K}^{\Omega}$ which coincides with u near some points of $\partial\Omega \setminus K$, namely, near points contained in $\partial r_f (\subset \partial\Omega \setminus K)$ for some $f \in \mathcal{O}(\bar{\Omega})$ with $\max_K |f| < 1$ and 1 being a regular value for $Re f$ in $\bar{\Omega}$. Clearly, if a connected component of $\partial\Omega \setminus K$ contains such points, then u_Q coincides with u near the whole component. But it is easy to see that each component ∂ of $\partial\Omega \setminus K$ contains such points. Indeed, take a point p of ∂ and consider a peak function f_p, $f_p(p) = 1$, $|f_p| < 1$ on $\bar{\Omega} \setminus \{p\}$. Let $1 - \varepsilon < 1$ be a regular value of the function $Re f_p$ with $1 - \varepsilon$ sufficiently close to 1. Then for the function $f = (1-\varepsilon)^{-1} f_p$ we have $|f| < 1$ on K and the set $\partial r_f \subset \partial\Omega$ is close to p and therefore it is contained in ∂. So, part 1 of theorem A is proved.

Part 2 follows immediately. Considering an analytic function near $\partial\Omega \setminus K$ which is equal to different constants on different components of $\partial\Omega \setminus K$ shows that the boundary of each connected component of Q intersects no more than one connected component of $\partial\Omega \setminus K$. By the maximum principle there is no component of Q with boundary not intersecting $\partial\Omega \setminus K$ (and, therefore, contained in $\hat{K}^{\Omega}$).

Theorem A is proved completely.

<u>2. Analytic extension of CR-functions to one-sided neighbourhoods of hypersurfaces.</u>

<u>**Proof of theorem 1.**</u> Let $z \in H \setminus K$. If through z there is no germ of a one-dimensional analytic manifold contained in H (in other words, z is a minimal point of H) by the theorem of Trepreau [19], [21] each continuous CR-function on $H \setminus K$ has analytic extension to a one-sided neighbourhood of z not depending on the CR-function). Suppose $z \in H \setminus K$ is not minimal and denote by X_z the maximal connected one-dimensional analytic manifold through z contained in H. By a manifold M contained in H we always mean the image of an abstract manifold under a smooth injective inclusion into H with injective differential. Note that M must not be a relatively closed submanifold of H, moreover, the manifold topology on M must not coincide with the topology on M induced by the topology of H. The fact that X_z is the maximal connected one-dimensional analytic manifold through z contained in H means that if Y_z is a connected one-dimensional analytic manifold, $z \in Y_z \subset H$, then $Y_z \subset X_z$.

Since K is $CR(H)$-convex and $z \notin K$, there exists a continuous CR-function f on H such that $f(z) = 1$, $\max_K |f| = 1 - \delta < 1$. Consider the part $\{\zeta \in X_z : |f(\zeta)| > 1 - \frac{\delta}{2}\}$ of X_z which is "far" from K and denote by D_z the connected component of this set which contains z. We will prove two lemmas.

<u>**Lemma 4.**</u> *There exists a point $p \in D_z$ such that each continuous CR-function on $H \setminus K$ has analytic extension to a one-sided neighbourhood of p (not depending on the CR-function).*

The second lemma is known. It is a result on propagation of one-sided analytic extendability of CR-functions along analytic submanifolds (for the propagation of analyticity see [8], for the one-sided variant see [20]). For convenience of the reader who is not familiar with microlocal technics we give the sketch of a simple proof based only on the theorem of Trepreau [19], [21].

__Lemma 5.__ *Let M be a connected hypersurface of class C^2 in $\mathbf{C}^n, n \geq 2$. Let $X \subset M$ be a connected analytic manifold (in the same sense as before) of complex dimension $n-1$. Suppose for some point $p \in X$ all CR-functions on M have analytic extension to a one-sided neighbourhood of p (not depending on the CR-function). Then the same is true for all points $q \in X$.*

The two lemmas together imply the theorem.

__Proof of lemma 4.__ Consider the closure (in H) $\bar{D}_z$ of D_z. Since D_z is contained in the Leviflat part of H the set $\bar{D}_z$ is compact. Take a point $\eta \in \bar{D}_z$ such that $|f(\eta)| = \max\limits_{\bar{D}_z} |f| (\geq |f(z)| = 1)$. If η is a minimal point of H we are done. Indeed, by Trepreau's theorem continuous CR-functions on $H \setminus K$ have analytic extension to a one-sided neighbourhood of η and one can take for p a point of D_z sufficiently close to η.

__Remark.__ Since continuous CR-functions on H are analytic on analytic manifolds contained in H one can assume by the maximum principle (applied to $f|D_z$) that $\eta \in \bar{D}_z \setminus D_z$. If, for example, D_z is an analytic disc with smooth boundary ∂_z, smoothly inbedded into H, then the proof is easy. Indeed, in this case $\bar{D}_z \setminus D_z = \partial_z$. If $\zeta \in \partial_z$ is not a minimal point then obviously $\zeta \in X_z$ and $|f(\zeta)| = 1 - \frac{\delta}{2}$. Since $|f(\eta)| \geq 1$ and $\eta \in \partial_z$, η is minimal. The general case needs a more detailed consideration of the set $\bar{D}_z$.

So, suppose now, η is not minimal. Denote by X_η the maximal connected one-dimensional analytic manifold through η contained in H. Let B_η be a small ball around η intersected with H such that $|f| > 1 - \frac{\delta}{4}$ on B_η. If B_η is small enough, the real and the imaginary part of the complex tangent vector to H at points of B_η define two linearly independant real vector fields of class C^1. Denote by $\mathfrak{V}_\eta$ all real vector fields on B_η of unit length which are linear combinations (with coefficients being real C^1-functions) of these two vector fields.

Let Δ_η be a small analytic disc on X_η around η with smooth boundary and compact closure $\bar{\Delta}_\eta$ in X_η, such that $\bar{\Delta}_\eta \subset B_\eta$. Each point p of $\bar{\Delta}_\eta$ can be joined with η by an integral curve of some vector field $v \in \mathfrak{V}_\eta$, that means by a curve $\gamma = \gamma_{v,p} : [0, T_\gamma] \to \bar{\Delta}_\eta$, such that $\gamma(0) = \eta$, $\gamma(T_\gamma) = p$, $\gamma'(t) = v(\gamma(t))$ for $t \in [0, T_\gamma]$. By the compactness of $\bar{\Delta}_\eta$ one can take these curves from a set Γ of curves with uniformly bounded length, so $T_\gamma \leq T < \infty$ for all $\gamma \in \Gamma$.

Let ζ_n be a sequence of points of D_z tending to η, ζ_n close enough to η. Then for each $\gamma \in \Gamma$ and each n one can define the integral curve $\gamma_n : [0, T_\gamma] \to H$ of the vector field v by the conditions $\gamma_n(0) = \zeta_n$, $\gamma_n'(t) = v(\gamma_n(t))$ for $t \in [0, T_\gamma]$ and $\gamma_n([0, T_\gamma]) \subset B_\eta$. (This fact is well known, see, for example [9] Corollary V.4.1.). If for some $\gamma \in \Gamma$, some n and certain $t \in [0, T_\gamma]$ the point $\gamma_n(t)$ is minimal, we are done. Indeed, set $T_\gamma^{(n)} = \sup \{t \in [0, T_\gamma] : \gamma_n(\tau) \text{ are not minimal points of } H \text{ for } 0 \leq \tau \leq t\}$. Obviously, $0 < T_\gamma^{(n)} \leq T_\gamma$ and $\gamma_n\left(T_\gamma^{(n)}\right)$ is a minimal point of H. For $0 \leq t < T_\gamma^{(n)}$ the $\gamma_n(t)$ are not minimal points of H. Since $|f| > 1 - \frac{\delta}{4}$ on $\gamma_n([0, T_\gamma])$ and $\gamma_n(0) = \zeta_n \in D_z$ the set $\gamma_n\left(\left[0, T_\gamma^{(n)}\right]\right)$ is contained in D_z. So, $\gamma_n\left(T_\gamma^{(n)}\right)$ is a minimal point contained in the closure $\bar{D}_z$ and we conclude as before.

In the other case the whole disc $\bar{\Delta}_\eta$ is contained in $\bar{D}_z$ (see [9] Corollary V.4.1.). By the maximum principle applied to $f|\Delta_\eta$ we get $f|\Delta_\eta \equiv \text{const} = f(\eta)$ with $|f(\eta)| \geq 1$ and therefore $f|X_\eta \equiv f(\eta)$ and $X_\eta \subset H \setminus K$. It is now enough to prove lemma 4 for X_η (instead of X_z). Indeed, suppose this is done, then by lemma 5 one-sided analytic extension holds for all points of X_η hence also for η and so also for points $\zeta_n \in D_z$, ζ_n close to η.

So, we will prove lemma 4 for X_η. Cover H by small relatively open sets B_{η_k}, $\eta_k \in B_{\eta_k} \subset H$, such that for B_{η_k} the set of vector fields $\mathfrak{V}_{\eta_k}$ can be defined as above. We will consider piecewise integral curves of vector fields from $\mathfrak{V}_{\eta_k}$; i.e. curves $s : [0,T] \to H$ such that the intervall $[0,T]$ can be devided into subintervalls $[T_j, T_{j+1}]$, $j = 0, \cdots, T_N$, with $0 = T_0 < T_1 < \cdots < T_N = T$, and to each j corresponds an integral curve of some $\mathfrak{V}_{\eta_k}$, say $\mathfrak{V}_{\eta_k(j)}$: $s([T_j, T_{j+1}])$ is contained in $B_{\eta_k(j)}$ and $s'(t) = v_j(s(t))$ for $t \in [T_j, T_{j+1}]$ for some $v_j \in \mathfrak{V}_{\eta_k(j)}$. Following [8] for a point $p \in H$ the set $\{q \in H : q \text{ can be joined with } p \text{ by a piecewise integral curve}\}$ is called the orbit through p. By [8] an orbit of H is a manifold contained in H the tangent space of which at each point contains the complex tangent space of H at the same point. So, as is easily seen directly, an orbit is either an open subset of H either an analytic manifold.

It is clear now that X_η consists of all points q which can be joined with η by a piecewise integral curve s of vector fields from $\mathfrak{V}_{\eta_k}$ **with the whole curve contained in the set of non-minimal points of** H . For proving lemma 4 for X_η it is enough to show that there is a piecewise integral curve $s^* : [0, T^*] \to H$ with $s^*(0) = \eta$, $s^*(t)$ being non-minimal for $t < T^*$ but $s^*(T^*)$ being a minimal point of H. In other words, we have to show that X_η is not an orbit. Suppose, in contrary, it is: Using again Corollary V.4.1. of [9] we see that the set $\bar{X}_\eta$ (closure in H) is the union of orbits. No orbit, contained in $\bar{X}_\eta$ can be an open subset of H (the points of such an orbit would have a neighbourhood not intersecting X_η). So, all orbits in X_η are analytic manifolds contained in H. But X_η is contained in the Leviflat part of H and therefore, by assumption on H, $\bar{X}_\eta$ is compact. This is impossible, see [6] page 309, and lemma 4 is proved for X_η.

<u>**Sketch of the proof of lemma 5.**</u> Suppose the lemma is not true. Let $Y \subsetneq X$ be the non-empty (relatively open) set of points in X for which each continuous CR-function on M has analytic extension to a one-sided neighbourhood (not depending on the function). Consider a smooth curve $\gamma : [0,1] \to X$ connecting a point in Y with a point in $X \setminus Y$. For a continuous CR-function u on M denote by A_u the maximum of the following two numbers: the maximum of u over a compact set in M containing $\gamma([0,1])$ in its interior and the supremum of the analytic extension of u to a fixed one-sided neighbourhood $\mathcal{O}_0$ of $\gamma(0)$, $\overline{\mathcal{O}}_0$ being contained in the union of K and the one-sided neighbourhood of $\gamma(0)$ to which all continuous CR-functions on M have analytic extension. Let a be the supremum of all $t \in [0,1]$ for which each continuous CR-function u on M has analytic extension to a one-sided neighbourhood (not depending on u) of $\gamma(t)$, the extension being bounded by A_u. Reparametrizing we will assume that $a = 1$ and will show that there is analytic extension (the extension bounded by A_u) to a one-sided neighbourhood of $\gamma(1)$. Take a sequence $t_n \in [0,1), t_n \uparrow 1$ and let V_n be small disjoint neighbourhoods (in $\mathbf{C}^n$) of $\gamma(t_n)$,

such that $\gamma(1) \notin V_n$ for all n. Make a small (but non-trivial) deformation of $H \cap V_n$ for all n in such a way that we obtain a new hypersurface H_1 of class C^2 with H_1 close to H and $(H_1 \setminus H) \cap V_n$ contained in the one-sided (with respect to H) neighbourhood $\mathcal{O}_n$ of $\gamma(t_n)$ to which all continuous CR-functions on H have analytic extension. For a continuous CR-function u on H define a function u_1 on H_1 by $u_1 = u$ on $H_1 \cap H$ and u_1 being equal to the analytic extension of u on $V_n \cap (H_1 \setminus H)$ for all n. It is standard to verify that for H_1 close to H the function u_1 is a continuous CR-function on H_1. This is clear near points of $H \setminus \left(\bigcup_n V_n \cup \{\gamma(1)\} \right)$ and follows easily from the definition of one-sided analytic extension near points of $H_1 \cap V_n$ $(n \geq 1)$. It remains to see that u_1 is continuous at $\gamma(1)$. For this we use that the one-sided analytic extension of u near points of $\gamma([0,1))$ is bounded by A_u. Further, if the deformations of $H \cap V_n$ are non-zero only on a sufficiently small compact part of V_n, then the harmonic measure of $V_n \cap H$ with respect to $\mathcal{O}_n$ at points of $V_n \cap (H_1 \setminus H)$ is bounded away from zero uniformly for n. These facts together imply that $\max\limits_{\zeta \in V_n \cap (H_1 \setminus H)} |u_1(\zeta) - u(\gamma(1))| \to 0$ for $n \to \infty$.

Now $\gamma(1) \in H_1$ is a minimal point of H_1. Indeed, H_1 contains a large piece of the analytic manifold X, namely $X \setminus \bigcup_n V_n$, with $\gamma(1)$ being in the closure of this set. An analytic manifold through $\gamma(1)$ of dimension $n-1$ contained in H_1 must contain all points of this set in a small neighbourhood of $\gamma(1)$ and so, by uniqueness theorems for analytic manifolds, it must coincide with X in a small neighbourhood of $\gamma(1)$. But by the construction of H_1 no neighbourhood of $\gamma(1)$ on X is contained in H_1. So, $\gamma(1)$ is a minimal point of H_1. Trépreau's theorem gives an analytic extension of u_1 to a one-sided neighbourhood (with respect to H_1) of $\gamma(1)$ (not depending on u), the extension being bounded by A_u, and it is clear now that u has an analytic extension to a one-sided neighbourhood of $\gamma(1)$ (with respect to H), which does not depend on u. So, in contrast to the assumption, $\gamma(1) \in Y$. The contradiction proves that $Y = X$.

3. Reduction to the case of strictly pseudoconvex domains.

Proof of theorem 2. Let $\mathcal{O}$ be the one-sided neighbourhood of $\partial\Omega \setminus K$ described in the formulation of theorem 2.

Take two other one-sided neighbourhoods $\mathcal{O}_1$ and $\mathcal{O}_2$ of $\partial\Omega \setminus K$ with $\overline{\mathcal{O}_1 \cup \partial\Omega} \subset \mathcal{O}_2 \cup \partial\Omega$ and $\overline{\mathcal{O}_2 \cup \partial\Omega} \subset \mathcal{O} \cup \partial\Omega$. Assume that each connected component of $\mathcal{O}_1$ ($\mathcal{O}_2$ resp.) contains in its boundary exactly one connected component of $\partial\Omega \setminus K$ and no other point of $\partial\Omega \setminus K$.

Our aim is now to construct strictly pseudoconvex domains Ω_n relatively compact in Ω $(\Omega_n \Subset \Omega)$ and compact sets $K_n \subset \partial\Omega_n$ such that $\Omega_n \setminus \hat{K}^{\Omega_n} \subset \Omega_{n+1} \setminus \hat{K}^{\Omega_{n+1}}$ and $\bigcup_n \left(\Omega_n \setminus \hat{K}^{\Omega_n} \right) \supset \Omega \setminus A(\Omega)\text{-hull}(K)$. After that we will apply theorem A to each Ω_n and K_n.

Take an arbitrary small number $\delta_1 > 0$. Denote by $U(\delta_1)$ the δ_1-neighbourhood in $\mathbb{C}^2$ of $A(\Omega)\text{-hull}(K), U(\delta_1) = \{\zeta \in \mathbb{C}^2 : \text{dist}(\zeta, A(\Omega)\text{-hull}(K)) < \delta_1\}$. Since $(\partial\Omega \setminus K) \cup U(\delta_1)$ covers $\partial\Omega$, the set $\mathcal{O}_1 \cup U(\delta_1)$ covers a one-sided neighbourhood of $\partial\Omega$.

By theorem 2.6.11. in [10] and Sard's theorem we can choose a strictly pseudoconvex domain Ω_1 with C^2 boundary, $\Omega_1 \Subset \Omega$ and $\partial\Omega_1 \subset U(\delta_1) \cup \mathcal{O}_1$. Consider the compact set $\overline{\partial\Omega_1 \cap \mathcal{O}_1}$

and cover it with a finite number of connected components of $\partial\Omega_1 \cap \mathcal{O}_2$ (each component intersecting $\mathcal{O}_1$). Call the components $C_j^{(1)}$, $j \in J_1$ with J_1, containing a finite number of elements. Put $K_1 = \partial\Omega_1 \setminus \bigcup_{j\in J_1} C_j^{(1)}$. Since $\overline{\bigcup_{j\in J_1} C_j^{(1)}}$ is contained in $\mathcal{O}\cap\partial\Omega_1$ and $\mathcal{O}\cap A(\Omega)$-hull$(K) = \emptyset$, the interior $\operatorname{int} K_1$ of K_1 with respect to $\partial\Omega_1$ is a neighbourhood (in $\partial\Omega_1$) of $A(\Omega)$-hull$(K) \cap \partial\Omega_1$. Take for each $j \in J_1$ a curve $\gamma_j^{(1)}$ contained in $\mathcal{O}_1 \cup (\partial\Omega \setminus K)$ which connects a point in Ω_1 which is very close to some point in $C_j^{(1)} \cap \mathcal{O}_1$ with a point in $\partial\Omega \setminus K$. Clearly $\gamma_j^{(1)}$ does not intersect $A(\Omega)$-hull(K).

Suppose now, that for $l \le n$ the strictly pseudoconvex domains $\Omega_l \Subset \Omega$, the compact sets $K_l \subset \partial\Omega_l$, the finite sets J_l and the curves $\gamma_j^{(l)}$, $j \in J_l$, are constructed and construct Ω_{n+1} and K_{n+1}. The compact set $\overline{\partial\Omega_n \setminus K_n} \cup \bigcup_{j\in J_n} \gamma_j^{(n)}$ does not intersect $A(\Omega)$-hull(K), therefore for sufficiently small $\delta_{n+1} > 0$, $\delta_{n+1} < \delta_n$, each compact subset S of $U(\delta_{n+1}) \cap \bar{\Omega}$ has the property $A(\Omega)\text{-hull}(S) \cap \left\{ \overline{\partial\Omega_n \setminus K_n} \cup \bigcup_{j\in J_n} \gamma_i \right\} = \emptyset$. Indeed, for each $z \in \overline{\Omega_n \setminus K_n} \cup \bigcup_{j\in J_n} \gamma_j$ there exists a function $f_z \in A(\Omega)$ with $f_z(z) = 1$ and $\max_{A(\Omega)-\text{hull}(K)} |f_z| \le 1 - \delta_z$, $\delta_z > 0$. Cover $\overline{\partial\Omega_n \setminus K_n} \cup \bigcup_{j\in J_n} \gamma_j$ with finitely many balls B_{z_l} around z_l such that $|f_{z_l} - 1| < \frac{\delta_{z_l}}{3}$ on B_{z_l} and take $\delta_{n+1} > 0$ so small that $|f_{z_l}(\zeta)| \le 1 - \frac{2}{3}\delta_{z_l}$ for each l and each $\zeta \in U(\delta_{n+1})$. Let now Ω_{n+1} be a strictly pseudoconvex domain with C^2 boundary, $\Omega_n \Subset \Omega_{n+1} \Subset \Omega$, such that $\partial\Omega_{n+1} \subset \mathcal{O}_1 \cup U(\delta_{n+1})$. Cover the compact set $\overline{\partial\Omega_{n+1} \cap \mathcal{O}_1}$ with a finite number $C_j^{(n+1)}$ $(j \in J_{n+1})$ of connected components of $\partial\Omega_{n+1} \cap \mathcal{O}_2$, each connected component intersecting $\mathcal{O}_1$. Define $K_{n+1} = \partial\Omega_{n+1} \setminus \bigcup_{j\in J_{n+1}} C_j^{(n+1)}$. It is clear that $K_{n+1} \subset U(\delta_{n+1}) \cap \partial\Omega_{n+1}$, $A(\Omega)$-hull$(K) \cap \partial\Omega_{n+1} \subset \operatorname{int} K_{n+1}$. The curves $\gamma_j^{(n+1)}(j \in J_{n+1})$ are constructed as before: $\gamma_j^{(n+1)} \subset \mathcal{O}_1 \cup (\partial\Omega \setminus K)$ and connects a point in Ω_{n+1} close to some point in $C_j^{(n+1)} \cap \mathcal{O}_1$ with a point in $\partial\Omega \setminus K$.

Now by the choice of δ_{n+1} we have $A(\Omega)\text{-hull}(K_{n+1}) \cap \left(\overline{\partial\Omega_n \setminus K_n} \cup \bigcup_{j\in J_n} \gamma_j \right) = \emptyset$, therefore, since $\hat{K}_{n+1}^{\bar{\Omega}_{n+1}} \subset A(\Omega)$-hull (K_{n+1}), we have $\hat{K}_{n+1}^{\bar{\Omega}_{n+1}} \cap \partial\Omega_n \subset \operatorname{int} K_n$ (the interior of K_n with respect to $\partial\Omega_n$) and $\bar{\Omega}_{n+1} \setminus \hat{K}_{n+1}^{\bar{\Omega}_{n+1}} \supset \bigcup_{j\in J_n} \gamma_j \cap \bar{\Omega}_{n+1}$.

By the Runge approximation property ([10], theorem 4.3.2) for each Ω_n and each compact $K \subset \bar{\Omega}_n$ the equality $\hat{K}^{\bar{\Omega}_n} = \hat{K}^{\Omega}$ holds, where $\hat{K}^{\Omega}$ denotes the hull of K with respect to the space $\mathcal{O}(\Omega)$ of all functions holomorphic in Ω, $\hat{K}^{\Omega} = \left\{ z \in \Omega : |f(z)| \le \max_K |f| \text{ for all } f \in \mathcal{O}(\Omega) \right\}$. So $\hat{K}_{n+1}^{\Omega} \cap \partial\Omega_n \subset \operatorname{int} K_n$ and by the local maximum principle ([10], theorem 7.2.10 and [5] theorem III.8.2) the inclusion $\hat{K}_{n+1}^{\Omega} \cap \bar{\Omega}_n \subset \hat{K}_n^{\Omega}$ holds. The inclusion $\Omega_{n+1} \setminus \hat{K}_{n+1}^{\bar{\Omega}_{n+1}} \supset \Omega_n \setminus \hat{K}_n^{\bar{\Omega}_n}$ is proved.

Now for each n theorem A gives an analytic function u_n in $\Omega_n \setminus \hat{K}_n^{\bar{\Omega}_n}$ which coincides with u near $\partial\Omega_n \setminus K_n = \bigcup_{j\in J_n} C_j^{(n)}$. Prove, that $u_{n+1}|\Omega_n \setminus \hat{K}_n^{\bar{\Omega}_n} = u_n$. By theorem A the set $\Omega_n \setminus \hat{K}_n^{\bar{\Omega}_n}$ has $\operatorname{card} J_n$ components $A_j^{(n)}$, and $C_j^{(n)} \subset \partial A_j^{(n)}$ $(j \in J_n)$. For each

$j \in J_n$ the set $A_j^{(n)} \cup \tilde{\gamma}_j^{(n)}$ is connected, where $\tilde{\gamma}_j^{(n)}$ is a connected part of $\gamma_j^{(n)} \cap \overline{\Omega}_{n+1}$ which joines a point of Ω_n with a point of $\partial\Omega_{n+1} \cap \mathcal{O}_1$. Therefore $A_j^{(n)} \cup \left(\tilde{\gamma}_j^{(n)}\right)$ is contained in $A_l^{(n+1)} \cup C_l^{(n+1)}$ for some connected component $A_l^{(n+1)}$ of $\Omega_{n+1} \setminus \hat{K}_{n+1}^{\bar{\Omega}_{n+1}}$ with $C_l^{(n+1)} \subset \partial A_l^{(n+1)}$. So u_{n+1} coincides with u near $\tilde{\gamma}_j^{(n)}$, and therefore $u_{n+1} = u_n$ in $A_j^{(n)}$.

So, we get a well defined analytic function on the set $Q \overset{\text{def}}{=} \bigcup_n \Omega_n \setminus \hat{K}_n^{\bar{\Omega}_n}$. Call it u_Q. Suppose the domains Ω_n are choosen such that $\cup\Omega_n = \Omega$ and the numbers $\delta_n > 0$ tend to zero. It is easy to see that then $\bigcup_{n\geq 1} \Omega_n \setminus \hat{K}_n^{\bar{\Omega}_n} \supset \Omega \setminus A(\Omega)\text{-hull}\,(K)$. Indeed, if $z \in \Omega \setminus A(\Omega)\text{-hull}\,(K)$, then $z \in \Omega_n$ for $n \geq n_0$ and there exists $f \in A(\Omega)$ with $f(z) = 1$ and $\max_K |f| < 1 - \delta$ for some $\delta > 0$. For some neighbourhood U of $A(\Omega)\text{-hull}\,(K)$ we have by continuity $|f| < 1 - \delta$ on $U \cap \bar{\Omega}$. By construction $K_n \subset U_{\delta_n} \cap \bar{\Omega}_n$, so for $n \geq n_0$ $K_n \subset U \cap \bar{\Omega}_n$ and therefore $z \notin \hat{K}_n^{\bar{\Omega}_n}$. So $Q \supset \Omega \setminus A(\Omega)\text{-hull}\,(K) \supset \mathcal{O}_1$ and from the construction of u_Q it is immediately clear that $u_Q = u$ in $\mathcal{O}_1$. (Indeed, $u_Q = u_n$ in $\Omega_n \setminus \hat{K}_n^{\bar{\Omega}_n}$, u_n is equal to u at least near points of $\partial\Omega_n \cap \mathcal{O}_1$ and for each component of $\mathcal{O}_1$ the intersection with $\partial\Omega_n$ is not empty for n large enough). Part 1 of theorem 2 is proved. Part 2 follows immediately as in the proof of theorem A.

Note, that part 2 of theorem 2 holds also with respect to $Q = \bigcup_n \left(\Omega_n \setminus \hat{K}_n^{\bar{\Omega}_n}\right)$ (instead of $\Omega \setminus A(\Omega)\text{-hull}\,(K)$). The only thing which, maybe, is not obvious, is that the boundary of each connected components of Q meets $\partial\Omega \setminus K$. But $Q \supset \mathcal{O}$, so if a connected component of Q intersects a component of $\mathcal{O}$ then it contains the whole component of $\mathcal{O}$. Since each component of Q intersects $\mathcal{O}$ (for large n each component of Q contains some component of $\Omega_n \setminus \hat{K}^{\overline{\Omega}_n}$) the assertion is clear.

__Remark 1.__ In case Ω has C^∞ boundary it follows from [7] and [3] that $Q = \Omega \setminus A(\Omega)$-hull$\,(K)$. Indeed, by [7] $\overline{\Omega}$ is the spectrum of the algebra $A(\Omega)$, so $A(\Omega)\text{-hull}\,(K)$ is the spectrum of the uniform closure of the algebra of restrictions of elements of $A(\Omega)$ to K. By the local maximum principle ([15], theorem III.8.2) and the fact that $A(\Omega)$-hull$\,(K) \cap \partial\Omega_n \subset K_n$ we have $A(\Omega)\text{-hull}\,(K) \cap \overline{\Omega}_n \subset A(\Omega)\text{-hull}\,(K_n)$. By [3] $A(\Omega)$-hull$\,(K_n) = \mathcal{O}(\Omega)\text{-hull}\,(K_n)$. So, (see the remark below) for bounded pseudoconvex domains $\Omega \subset \mathbf{C}^2$ with C^∞ boundary the envelope of holomorphy of one-sided neighbourhoods $\mathcal{O}$ of $\partial\Omega \setminus K$ ($\mathcal{O}$ and K as in the in theorem 2) is equal to $\Omega \setminus A(\Omega)\text{-hull}\,(K)$.

__Remark 2.__ The set $\Omega_n \setminus \hat{K}_n^{\bar{\Omega}_n}$ is pseudoconvex by the local maximum principle and Słodkowski's theorem [13] (see also [12]). So $\bigcup_n \left(\Omega_n \setminus \hat{K}_n^{\bar{\Omega}_n}\right)$ is pseudoconvex by the Behnke – Stein theorem (see [22], III.16.10). The proof of theorem 2 shows that the envelope of holomorphy of the one-sided neighbourhood $\mathcal{O}$ of $\partial\Omega \setminus K$ ($\mathcal{O}, \Omega$ and K as in theorem 2) is schlicht and coincides with $\Omega \setminus IRA(\Omega)\text{-hull}\,(K)$ where $IRA(\Omega)\text{-hull}\,(K)$ denotes the "inner regularization" of the $A(\Omega)\text{-hull}\,(K)$ defined in the following way. A smoothly bounded strictly pseudoconvex domain D with compact closure in Ω is called admissible if it has the form $D = \{u < 0\}$ for a smooth plurisubharmonic function u in Ω with $\{u < c\}$ relatively compact in Ω for each $c \in \mathbf{R}$. Note that for admissible domains $\mathcal{O}(\Omega)$ is dense in $\mathcal{O}(\bar{D})$. For an admissible domain D put $K_D = \partial D \cap A(\Omega)\text{-hull}\,(K)$. The inner regularization of $A(\Omega)\text{-hull}\,(K)$ is now defined as follows: $z \in \Omega \cap (IR\text{-}A(\Omega)$-hull$\,(K))$ iff $z \in \mathcal{O}(\Omega)\text{-hull}\,(K_D)$ for all admissible domains D containing z. It is clear, that $IRA(\Omega)$-hull $(K) \subset A(\Omega)$-hull (K) (since $K_D \subset A(\Omega)$-hull (K) for each D, we have $\mathcal{O}(\Omega)$-hull $(K_D) \subset A(\Omega)$-hull $(K_D) \subset A(\Omega)$-hull (K).) We don't know if in general the inner regularization coincides with the $A(\Omega)$-hull (K).

Note also, that the smoothness assumption for $\partial\Omega$ in theorem 2 is not essential (in that case we will not speak on $CR(\partial\Omega)$-convex sets, but on compact sets K with $K = \partial\Omega \cap A(\Omega)$-hull (K).)

The present methods can be applied to other two-dimensional Stein manifolds instead of $\mathbf{C}^2$. We will not formulate here corresponding results.

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