# Bergmann tau-function on Hurwitz spaces and its applications

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**Abstract.** The main result of this work is a computation of the Bergmann tau-function on Hurwitz spaces in any genus. This allowes to get an explicit formula for the G-function of Frobenius manifolds associated to arbitrary Hurwitz spaces, get a new expression for the determinant of the Laplace operator in the Poincaré metric on Riemann surfaces of arbitrary genus, and compute the Jimbo-Miwa tau-function of an arbitrary Riemann-Hilbert problem with quasi-permutation monodromies.

## 1 Introduction

The Hurwitz space $H_{g,N}$ is the space of equivalence classes of pairs $(\mathcal{L},\,\pi)$, where $\mathcal{L}$ is a compact Riemann surface of genus $g$ and $\pi$ is a meromorphic function of degree $N$. The Hurwitz space is stratified according to multiplicities of poles and critical points of function $\pi$ (see [19, 9]); in this paper we shall mainly work within the generic stratum $H_{g,N}(1,\dots,1)$, for which all critical points and poles of function $\pi$ are simple. Denote the critical points of function $\pi$ by $P_1,\dots P_M$ ($M=2N+2g-2$ according to the Riemann-Hurwitz formula); the critical values $\lambda_m=\pi(P_m)$ can be used as (local) coordinates on $H_{g,N}(1,\dots,1)$. The function $\pi$ defines a realization of the Riemann surface $\mathcal{L}$ as an $N$-sheeted branched covering of $\mathbb{C}P^1$ with ramification points $P_1,\dots,P_M$ and branch points $\lambda_m=\pi(P_m)$; enumerate the points at infinity of the branched covering in some order and denote them by $\infty_1,\dots,\infty_N$. In a neighbourhood of the ramification point $P_m$ the local coordinate is chosen to be $x_m(P)=\sqrt{\pi(P)-\lambda_m}$, $m=1,\dots,M$; in a neighbourhood of any point $\infty_n$ the local parameter is $x_{M+n}(P)=1/\pi(P)$, $n=1,\dots,N$.

Fix a canonical basis of cycles $(a_\alpha,\,b_\alpha)$ on $\mathcal{L}$ and introduce the prime-form $E(P,Q)$ on $\mathcal{L}$ and Bergmann bidifferential $B(P,Q)=d_Pd_Q\ln E(P,Q)$ (a symmetric bimeromorphic differential with zero $a$-periods). The Bergmann bidifferential has the second order pole as $Q\to P$ with the following local behaviour: $B(P,Q)/\{dx(P)dx(Q)\}=(x(P)-x(Q))^{-2}+\frac{1}{6}S_B(x(P))+o(1)$, where $x(P)$ is a local coordinate; $S_B(x(P))$ is the Bergmann projective connection.

We define the Bergmann $\tau$-function $\tau(\lambda_1,\dots,\lambda_M)$ locally by the system of equations

$$\frac{\partial}{\partial\lambda_m}\ln\tau=-\frac{1}{12}S_B(x_m)|_{x_m=0}\,,\qquad m=1,\dots,M\,; \tag{1.1}$$

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compatibility of this system is a simple corollary of the Rauch variational formulas [14]. In global terms, $\tau$ is a horizontal holomorphic section of the flat holomorphic line bundle $\mathcal{T}_B$ (see [14]) over the space $\widehat{H_{g,N}}(1,\dots,1)$, which covers $H_{g,N}(1,\dots,1)$, and consists of pairs (weakly marked Riemann surface $\mathcal{L}$; meromorphic function $\pi$ with simple poles and critical points). This covering space appears due to dependence of the Bergmann bidifferential (and, therefore, the Bergmann projective connection) on the choice of homology basis on $\mathcal{L}$.

Definition (1.1) was motivated by appearance of the function $\tau$ as a part of the isomonodromic Jimbo-Miwa tau-function corresponding to Riemann-Hilbert problems with quasi-permutation monodromy matrices [16]. Later it turned out [15] that this object also coincides with another isomonodromic tau-function, the one which was intruduced by Dubrovin [2] in connection with semisimple Frobenius manifolds associated to Hurwitz spaces. Therefore, according to theorem by Dubrovin and Zhang [4], up to a simple additional term, $\tau$ coincides with $G$-function (solution of Getzler's equation, which has the field theoretic meaning of genus one free energy of Dijkgraaf and Witten) of this class of Frobenius manifolds. In [14] it was also revealed the role of the function $\tau$ in the problem of holomorphic factorization of the determinant of the Laplacian on Riemann surfaces: namely, up to a factor involving an appropriate regularized Dirichlet integral and matrix of $b$-periods of a Riemann surface, the determinant of Laplace operator (in the Poincaré metric) acting in the trivial line bundle over Riemann surface is given by $|\tau|^2$.

For $N = 2$ and arbitrary $g$ the Riemann surface $\mathcal{L}$ is hyperelliptic, and can be given by equation $w^2 = \prod_{m=1}^{2g+2}(\lambda-\lambda_m)$. In this case $\tau = \det\mathbf{A}\prod_{m\neq n}^{M}(\lambda_m-\lambda_n)^{1/4}$ [12], where $\mathbf{A}$ is the matrix of $a$-periods of non-normalized holomorphic differentials $\lambda^{\alpha-1}d\lambda/w$ , $\alpha = 1,\dots,g$. This expression coincides with Zamolodchikov's partition function of Ashkin-Teller model [25]; from heuristic conformal field theory point of view this espression has the interpretation of the determinant of Cauchy-Riemann operator acting in the trivial line bundle [13].

In other simple case, when $g = 0, 1$ and $N$ is arbitrary,the function $\tau$ was found in [14]; this result allowed to compute the $G$-function of Frobenius manifold related to the extended affine Weyl group $\tilde{W}(A_{N-1})$ (originally found in [21]) and the $G$-function of Frobenius manifold related to the Jacobi group $J(A_{N-1})$ (conjectured in [21]).

The goal of this paper is to compute the Bergmann tau-function $\tau$ on arbitrary Hurwitz space $H_{g,N}(1,\dots,1)$, and discuss the applications of this result in the contexts mentioned above.

Consider the divisor $\mathcal{D}$ of the differential $d\pi$: $\mathcal{D} = \sum_{k=1}^{M+N} d_k D_k$, where $D_m = P_m$ , $d_m = 1$ for $m = 1,\dots,M$ and $D_{M+n} = \infty_n$ , $d_{M+n} = -2$ for $n = 1,\dots,N$. Here and below, if the argument of a differential coincides with a point $D_k$ of divisor $\mathcal{D}$, we evaluate this differential at this point with respect to local parameter $x_k$. In particular, for the prime form we shall use the following conventions:

$$E(D_k, D_l) := E(P, Q)\sqrt{dx_k(P)}\sqrt{dx_l(Q)}|_{P=D_k,\,Q=D_l}\ , \tag{1.2}$$

for $k, l = 1,\dots,M+N$. The next notation correspond to prime-forms, evaluated at points of divisor $\mathcal{D}$ with respect to only one argument:

$$E(P, D_k) := E(P, Q)\sqrt{dx_k(Q)}|_{Q=D_k}\ , \tag{1.3}$$

$k = 1,\dots,M+N$; in contrast to $E(D_k, D_l)$, which are just scalars, $E(P, D_k)$ are $-1/2$-forms with respect to $P$.

Denote by $v_1,\dots,v_g$ the normalized ($\oint_{a_\alpha} v_\beta = \delta_{\alpha\beta}$ ) holomorphic differentials on $\mathcal{L}$; $\mathbf{B}_{\alpha\beta} = \oint_{b_\alpha} v_\beta$ is the corresponding matrix of $b$-periods; $\Theta(z|\mathbf{B})$ is the theta-function. Let us dissect the Riemann surface $\mathcal{L}$ along its basic cycles to get its fundamental polygon $\hat{\mathcal{L}}$; choose some initial point $P \in \hat{\mathcal{L}}$

and introduce the corresponding vector of Riemann constants $K^P$ and the Abel map $\mathcal{A}_\alpha(Q) = \int_P^Q v_\alpha$, computed along path which does not intersect $\partial\hat{\mathcal{L}}$. Since the vectors $\mathcal{A}(\mathcal{D})$ and $-2K^P$ define the same point on the Jacobian $J(\mathcal{L})$, we can introduce two integer vectors $\mathbf{r}$ and $\mathbf{s}$ such that

$$\mathcal{A}(\mathcal{D}) + 2K^P + \mathbf{B}\mathbf{r} + \mathbf{s} = 0 \tag{1.4}$$

The following theorem, together with its applications, is the main result of this paper

**Theorem 1** *The Bergmann tau-fuction (1.1) of Hurwitz space $H_{g,N}(1,\dots,1)$ is given by the following expression:*

$$\tau = \mathcal{F}^{2/3} \prod_{k,l=1\ \ k<l}^{M+N} [E(D_k, D_l)]^{\frac{d_k d_l}{6}} \tag{1.5}$$

*where the quantity $\mathcal{F}$ defined by*

$$\mathcal{F} = \frac{[d\pi(P)]^{\frac{g-1}{2}} e^{-\frac{\pi i}{2}\langle \mathbf{r}, K^P\rangle}}{W(P)} \left\{ \prod_{k=1}^{M+N} [E(P, D_k)]^{\frac{(1-g)d_k}{2}} \right\} \sum_{\alpha_1,\dots,\alpha_g=1}^{g} \frac{\partial^g \Theta(K^P)}{\partial z_{\alpha_1} \dots \partial z_{\alpha_g}} v_{\alpha_1}(P) \dots v_{\alpha_g}(P) \tag{1.6}$$

*is independent of the point $P \in \mathcal{L}$. Here $\Theta$ is the theta-function of $\mathcal{L}$; integer vector $\mathbf{r}$ is defined by (1.4); $\langle\ ,\ \rangle$ is the scalar product in $\mathbb{C}^g$, ($\langle x, y\rangle = \sum_{\alpha=1}^{g} x_\alpha y_a$);*

$$W(P) := \det_{1\le\alpha,\beta\le g} ||v_\beta^{(\alpha-1)}(P)|| \tag{1.7}$$

*denotes the Wronskian determinant of holomorphic differentials at the point $P$.*

The proof of this theorem is contained in Section 2 and is rather indirect; the first main ingredient of the proof is a variational formula for the part of expression $\mathcal{F}$ which depends only on the moduli of Riemann surface $\mathcal{L}$ ([8], p.58); we notice that $\mathcal{F}$ is a natural generalization of genus 1 expression $\Theta'(\frac{\mathbf{B}+1}{2})$ (which is equal to $\eta^3(\mathbf{B})e^{-\pi\mathbf{B}/4}$, where $\eta$ is the Dedekind eta-function) to higher genus.

The second main ingredient of the proof is the technique of variation and holomorphic factorization of a Dirichlet integral on $\mathcal{L}$.

Without any modification the same formula gives the Bergmann tau-function on an arbitrary stratum of Hurwitz space (when divisor $\mathcal{D}$ of differential $d\pi$ has an arbitrary allowed (i.e. such that $\sum d_k = 2g-2$) set of multiplicities $d_k$).

In section 3 we discuss applications of formula (1.5). First, we show how it allows to compute the $G$-function of Frobenius manifolds corresponding to Hurwitz spaces, proving that

$$G = -\frac{1}{2}\ln\tau - \frac{1}{48}\sum_{m=1}^{M} \ln \operatorname{Res}_{P_m} \frac{\varphi^2}{d\lambda}\ , \tag{1.8}$$

where $\varphi$ is a primary differential defing the Frobenius manifold, and $\tau$ is the Bergmann tau-function (1.5). Second, the formula (1.5) is used to get a new expression (valid up to a constant independent of moduli of the Riemann surface) for the determinant of the Laplacian on Riemann surface $\mathcal{L}$ in Poincaré metric.

Finally, we apply the formula (1.5) to computation of the isomonodromic tau-function of an arbitrary Riemann-Hilbert problem with quasi-permutation monodromy matrices [16].

## 2 Proof of the main theorem

We start from introducing some useful objects on the Riemann surface $\mathcal{L}$ [7, 8]. For any two points $P, Q \in \hat{\mathcal{L}}$ we define

$$\mathbf{s}(P,Q) := \exp\left\{ -\sum_{\alpha=1}^{g} \oint_{a_\alpha} v_\alpha(R) \ln \frac{E(R,P)}{E(R,Q)} \right\} \tag{2.1}$$

This object is a non-vanishing holomorphic $g/2$-differential on $\hat{\mathcal{L}}$ with respect to $P$ and non-vanishing holomorphic $-g/2$-differential with respect to $Q$. Under tracing along the cycles $a_\alpha$ and $b_\alpha$ it gains the multipliers 1 and $\exp[(g-1)\pi i \mathbf{B}_{\alpha\alpha} + 2\pi i K_\alpha^P]$ with respect to $P$ and the multipliers 1 and $\exp[(1-g)\pi i \mathbf{B}_{\alpha\alpha} - 2\pi i K_\alpha^Q]$ with respect to $Q$.

As in the case of prime form, if one of the arguments coincides with a point of $\mathcal{D}$, we evaluate $\mathbf{s}$ in the corresponding local parameter:

$$\mathbf{s}(D_k, Q) := \mathbf{s}(P,Q)(dx_k(P))^{-g/2}|_{P=D_k} \tag{2.2}$$

where $k = 1, \dots, M+N$. For arbitrary two points $P, Q \notin D$ we introduce the following notation:

$$\sigma(P,Q) := \mathbf{s}(P,Q)(d\pi(P))^{-g/2}(d\pi(Q))^{g/2} \tag{2.3}$$

Let us fix two points $P_0$, $Q_0 \in \mathcal{L}$ (for convenience in the sequel we assume that these points do not coincide with points of $\mathcal{D}$), and introduce another object which plays an important role below, the following holomorphic 1-differential on $\hat{\mathcal{L}}$:

$$\omega(P) = \mathbf{s}^2(P, Q_0) E(P, P_0)^{2g-2} (d\pi(Q_0))^g (d\pi(P_0))^{g-1} \; ; \tag{2.4}$$

in agreement with the previous notations, $\omega(D_k) := \frac{\omega(P)}{dx_k(P)}|_{P=D_k}$. The differential $\omega(P)$ has multipliers 1 and $\exp(4\pi i K_\alpha^{P_0})$ along the basic cycles $a_\alpha$ and $b_\alpha$ respectively. The only zero of the 1-form $\omega$ on $\hat{\mathcal{L}}$ is $P_0$; its multiplicity equals $2g-2$.

Consider the following Schwarzian derivative (which depends on the chosen point $P_0$, but is obviously independent of the point $Q_0$ from (2.4))

$$S_{Fay}^{P_0}(x(P)) := \left\{ \int^P \omega(P),\; x(P) \right\} \;, \tag{2.5}$$

where $x(P)$ is a local coordinate on $\mathcal{L}$; $S_{Fay}^{P_0}$ is a projective connection (see e.g. [22]) on $\mathcal{L}$; this object was introduced and exploited by Fay [8] in a different form (and without mentioning that it is a projective connection).

Now we turn to a local analytic consideration; with a slight abuse of terminology we denote the branched covering of the Riemann sphere defined by the function $\pi$ on the Riemann surface $\mathcal{L}$ by the same letter $\mathcal{L}$; the coordinate on the covered Riemann sphere will be denoted by $\lambda$. In this local treatment the zeros and poles of the differential $d\pi(P)$ are handled differently (although they enter almost uniformly in the final result (1.5). The zeros of $d\pi$ are the ramification point of the branched covering $\mathcal{L}$; the local parameter in a neighbourhood of $P_m$ is $x_m = \sqrt{\lambda - \lambda_m}$, according to the notations from introduction. For the local parameter near the point at infinity $\infty_n$ in this section we shall use notation $\zeta_n := 1/\lambda$, which is the same as the parameter $x_{M+n}$ from the introduction.

## 2.1 Dirichlet integral: variational formulas and holomorphic factorization

### 2.1.1 Definition of regularized Dirichlet integral

Let us cut the branched covering $\mathcal{L}$ into $N$ sheets by a family of contours connecting ramification points $P_m$; in addition, we dissect it along all $a$-cycles. On each sheet of the covering $\mathcal{L}$ dissected in this way we can define a real-valued function

$$\varphi(P) = \ln\left|\frac{\omega(P)}{d\pi(P)}\right|^2 \tag{2.6}$$

The difference of values of function $\varphi$ on different sides of cycle $a_\alpha$ equals $4\pi i(K_\alpha^P - \overline{K_\alpha^P})$. Function $\varphi$ is singular at all points of divisor $\mathcal{D}$ i.e. ramification points $P_1, \dots, P_M$ and points at infinity $\infty_1, \dots, \infty_N$ of the branch covering $\mathcal{L}$. The derivative $\partial_\lambda \varphi$ (where $\lambda = \pi(P)$) is holomorphic outside of the singularities of the function $\varphi$ and does not change under tracing along the $a$-cycles.

**Lemma 1** *Projective connection (2.5) is related to function $\varphi$ (2.6) everywhere outside of the divisor $\mathcal{D}$ as follows:*

$$S_{Fay}^{P_0}(\lambda) = \varphi_{\lambda\lambda} - \frac{1}{2}\varphi_\lambda^2 \tag{2.7}$$

The proof of this lemma is a simple standard computation.

In terms of the function $\varphi$ we define $M$ functions $\varphi^{int}(x_m)$, which are analytic in corresponding neighborhoods of the ramification points $P_m$, as follows:

$$e^{\varphi^{int}(x_m)}|dx_m|^2 = e^{\varphi(P)}|d\lambda|^2 \; ; \tag{2.8}$$

in analogy to (2.7) we get $S_{Fay}^{P_0}(x_m) = \varphi^{int}_{x_m x_m} - \frac{1}{2}(\varphi^{int}_{x_m})^2$.

Similarly, in a neighborhood of any point at infinity $\infty_n$ we define the function $\varphi^\infty(\zeta_n)$ of the local parameter $\zeta_n$ by the equality $e^{\varphi^\infty}|d\zeta_n|^2 = e^{\varphi}|d\lambda|^2$. The projective connection $S_{Fay}^{P_0}$ in the parameter $\zeta_n$ coincides with $\varphi^\infty_{\zeta_n\zeta_n} - \frac{1}{2}(\varphi^\infty_{\zeta_n})^2$.

Using the interplay between the functions $\varphi$, $\varphi^{int}$ and $\varphi^\infty$, we find the following asymptotics near the ramification points $P_m$ and the poles $\infty_n$:

$$|\varphi_\lambda(P)|^2 = \frac{1}{4}|\lambda - \lambda_m|^{-2} + O(|\lambda - \lambda_m|^{-3/2}) \quad \text{as } P \to P_m \tag{2.9}$$

and

$$|\varphi_\lambda(P)|^2 = 4|\lambda|^{-2} + O(|\lambda|^{-3}) \quad \text{as} \quad P \to \infty_n \; . \tag{2.10}$$

At the zero $P_0$ of the differential $\omega$ one gets

$$|\varphi_\lambda(P)|^2 = 4(g-1)^2|\lambda - \lambda_0|^{-2} + O(|\lambda - \lambda_0|^{-1}) \quad \text{as} \quad P \to P_0. \tag{2.11}$$

where $\lambda_0 := \pi(P_0)$.

These asymptotics enable us to introduce the following regularized Dirichlet integral

$$\mathbb{D} = \frac{1}{\pi}\text{reg}\int_{\mathcal{L}} |\varphi_\lambda|^2 \widehat{d\lambda} = \frac{1}{\pi}\lim_{\rho\to 0}\left\{I_\rho + \pi(M + 8N + 8(g-1)^2)\ln\rho\right\}, \tag{2.12}$$

where $\widehat{d\lambda} = |d\lambda \wedge d\bar{\lambda}|/2$ and

$$I_\rho = \sum_{n=1}^{N} \int_{\mathcal{L}_\rho^{(n)}} |\varphi_\lambda|^2 \widehat{d\lambda}\,. \tag{2.13}$$

Here $\mathcal{L}_\rho^{(n)}$ is the subdomain of the $n$-th sheet of covering $\mathcal{L}$ obtained by cutting off the (small) discs of radius $\rho$ centered at the ramification points and (if applicable) $P_0$ from the (large) disc $\{\lambda \in \mathcal{L}^{(n)} : |\lambda| < 1/\rho\}$.

### 2.1.2 Holomorphic factorization of Dirichlet integral

The following theorem shows how to compute the Dirichlet integral (2.12) in terms of the local data at the points of divisor $\mathcal{D}$ and points $P_0$ and $Q_0$.

**Theorem 2** *The regularized Dirichlet integral admits the following representation:*

$$\mathbb{D} = \ln \left| \frac{\sigma^{4-4g}(P_0, Q_0) \prod_{m=1}^{M} \omega(P_m)}{\prod_{n=1}^{N} \omega^2(\infty_n)} \exp\left\{4\pi i \langle \mathbf{r}, K^{P_0} \rangle\right\} \right|^2 + 2M \ln 2, \tag{2.14}$$

*where vector* $\mathbf{r}$ *has integer components given by*

$$2\pi r_\alpha := Var|_{a_\alpha} \left\{ Arg \frac{\omega(P)}{d\pi(P)} \right\}\,. \tag{2.15}$$

**Proof.** Applying the Stokes theorem, we get

$$I_\rho = \frac{1}{2i} \left\{ \sum_{m=1}^{M} \oint_{P_m} + \sum_{n=1}^{N} \oint_{\infty_n} + \oint_{P_0} + \sum_{\alpha=1}^{g} \int_{a_\alpha^+ \cup a_\alpha^-} \right\} \varphi_\lambda \varphi \, d\lambda, \tag{2.16}$$

Here $\oint_{P_m}$ and $\oint_{P_0}$ are integrals over clock-wise oriented circles of radius $\rho$ around the points $P_m$ and $P_0$ (it should be noted that each of the points $P_m$ belongs to two sheets simultaneously and, therefore, the integration in $\int_{P_m}$ goes over two circles). The $\oint_{\infty_n}$ denotes the integral over the counterclock-wise oriented circle of radius $1/\rho$ on the $n$-th sheet; $a_\alpha^+$ and $a_\alpha^-$ are different shores of the cycle $a_\alpha$ with the opposite orientation. One has the equality

$$\frac{1}{2i} \int_{a_\alpha^+ \cup a_\alpha^-} \varphi_\lambda \varphi \, d\lambda = \pi r_\alpha \ln |\exp 4\pi i K_\alpha^{P_0}|^2\,. \tag{2.17}$$

We note that $\varphi_\lambda = \partial_\lambda \ln(\omega(P)/d\pi(P))$, where the function $\omega(P)/d\pi(P)$ is single-valued on the cycle $a_\alpha$; since the $a$-cycles are assumed not to contain the point $P_0$, function $\omega(P)/d\pi(P)$ does not vanish on $a_\alpha$.

We have also

$$\begin{aligned}\frac{1}{2i} \oint_{P_m} &= \frac{1}{2i} \oint_{|x_m| = \sqrt{\rho}} \left\{ \varphi_{x_m}^{int} \frac{1}{2x_m} - \frac{1}{2x_m^2} \right\} \{\varphi^{int} - 2\ln|x_m| - 2\ln 2\} 2x_m \, dx_m = \\ &= \pi \varphi^{int}(x_m)|_{x_m=0} + 2\pi \ln 2 - \pi \ln \rho + o(1),\end{aligned} \tag{2.18}$$

$$\frac{1}{2i} \oint_{\infty_n} = \frac{1}{2i} \oint_{|\lambda| = 1/\rho} \left\{ -\varphi_{\zeta_n}^{\infty} \lambda^{-2} - \frac{2}{\lambda} \right\} \{\varphi^{\infty} - 4\ln|\lambda|\} \, d\lambda =$$

$$= -2\pi\varphi^\infty(\zeta_n)|_{\zeta_n=0} - 8\pi\ln\rho + o(1), \tag{2.19}$$

and

$$\frac{1}{2i}\oint_{P_0} = \frac{1}{2i}\oint_{|\lambda-\lambda_0|=\rho} \ln|\sigma^2(P_0,Q_0)(\lambda-\lambda_0)^{2g-2}\{1+O(\lambda-\lambda_0)\}|^2\times$$
$$\times\left(\frac{2g-2}{\lambda-\lambda_0}+O(1)\right)d\lambda = -\pi\ln|\sigma^{4g-4}(P_0,Q_0)|^2 - 8\pi(g-1)^2\ln\rho + o(1), \tag{2.20}$$

as $\rho\to 0$. These asymptotics together with (2.12) and (2.17) imply (2.14). $\square$

### 2.1.3 Variational formulas for Dirichlet integral

¿From now on we assume that the projections $\pi(P_0)$ and $\pi(Q_0)$ of the points $P_0$ and $Q_0$ from (2.4) are independent of $\{\lambda_m\}$.

**Theorem 3** *The variation of the regularized Dirichlet integral $\mathbb{D}$ (2.12), (2.14) with respect to branch points $\lambda_m$ is given by values of projective connection (2.5) at the ramification points $P_m$ i.e.*

$$\frac{\partial\mathbb{D}}{\partial\lambda_m} = S_{Fay}^{P_0}(x_m(P))|_{P=P_m}, \quad m=1,\dots,M\ . \tag{2.21}$$

We start from the following

**Lemma 2** *On every sheet of the covering $\mathcal{L}$, dissected in addition along all $a$ and $b$-cycles, the derivatives of function $\varphi(P)$ with respect to $\lambda$ and $\lambda_m$ are related as follows:*

$$\varphi_{\lambda_m} + F_m\varphi_\lambda + (F_m)_\lambda = 0\ , \quad m=1,\dots,M, \tag{2.22}$$

*where functions $F_m(P)$ are defined on the dissected covering $\mathcal{L}$ as follows:*

$$F_m(P) = -\frac{\mathcal{U}(P)_{\lambda_m}}{\mathcal{U}(P)_\lambda}\ ,$$

*and $\mathcal{U}(P)=\int_{P_0}^P\omega$. Near ramification points and points at infinity the functions $F_m$ have the following asymptotics:*

$$F_m(P) = O(|\lambda|^2)\ , \qquad \text{as} \qquad P\to\infty_n\ ,$$
$$F_m(P) = \delta_{lm} + o(1)\ , \qquad \text{as} \qquad P\to P_l\ ,$$

*where $\delta_{lm}$ is the Kronecker symbol.*

The proof of relation (2.22) can be obtained by direct differentiation (one needs to use the fact that the map $\mathcal{U}$ depends on $\{\lambda_m\}$ holomorphically). The proof of the asymptotical behaviour of the functions $F_m$ essentially repeats Lemma 5 from [14].

Considering the exact differentials $d((F_m)_\lambda\varphi)$, $d(F_m\varphi_\lambda\varphi)$, $d(F_m\varphi_\lambda)$ and making use of (2.22), we get the following

**Corollary 1** *The following two 1-forms are exact:*

$$\{(\varphi_\lambda\varphi)_{\lambda_m}d\lambda\} - \{F_m|\varphi_\lambda|^2d\bar\lambda + (F_m)_\lambda\varphi_{\bar\lambda}d\bar\lambda\ \}\ , \tag{2.23}$$

*and*

$$\{F_m|\varphi_\lambda|^2d\bar\lambda - (F_m)_\lambda\varphi_\lambda d\lambda + (F_m)_\lambda\varphi_{\bar\lambda}d\bar\lambda\} - \{F_m(2\varphi_{\lambda\lambda}-\varphi_\lambda^2)d\lambda + \varphi\varphi_{\lambda\lambda_m}d\lambda\}\ . \tag{2.24}$$

*Proof of theorem 3* (the idea of such a proof, including lemma 2 initially goes to [23]).
By (2.16) we get

$$\frac{\partial I_\rho}{\partial \lambda_m} = \frac{1}{2i}\left\{\oint_{P_m}(\varphi_\lambda\varphi)_{\lambda_m} + (\varphi_\lambda\varphi)_\lambda\, d\lambda + \left(\sum_{l\neq m}\oint_{P_l} + \sum_{n=1}^{N}\oint_{\infty_n} + \oint_{P_0}\right)(\varphi_\lambda\varphi)_{\lambda_m}\, d\lambda\right\} +$$

$$+\frac{1}{2i}\sum_{\alpha=1}^{g}\int_{a_\alpha^+\cup a_\alpha^-}(\varphi_\lambda\varphi)_{\lambda_m}\, d\lambda\,. \tag{2.25}$$

(One may assume that the projections of basic cycles $\pi(a_\alpha)$ are independent of $\{\lambda_m\}$.)

Using the holomorphy of $(F_m)_\lambda\varphi_\lambda$ and the relation $(\varphi_\lambda\varphi)_\lambda\, d\lambda = d(\varphi_\lambda\varphi) - \varphi_\lambda\varphi_{\bar\lambda}d\bar\lambda$, we rewrite the r. h. s. of (2.25) as

$$\frac{1}{2i}\Big[-\oint_{P_m}|\varphi_\lambda|^2 d\bar\lambda + \left\{\sum_{l=1}^{M}\oint_{P_l} + \sum_{n=1}^{N}\oint_{\infty_n} + \oint_{P_0}\right\}\{F_m|\varphi_\lambda|^2 - (F_m)_\lambda\varphi_\lambda\, d\lambda + (F_m)_\lambda\varphi_{\bar\lambda}\, d\bar\lambda\} -$$

$$-\sum_{\alpha=1}^{g}\left[\int_{a_\alpha^+\cup a_\alpha^-} + \int_{b_\alpha^+\cup b_\alpha^-}\right](F_m)_\lambda\varphi_\lambda\, d\lambda + \sum_{\alpha=1}^{g}\int_{a_\alpha^+\cup a_\alpha^-}(\varphi_\lambda\varphi)_{\lambda_m}\, d\lambda\Big]. \tag{2.26}$$

By means of the asymptotical expansions of the integrands at the ramification points and points $\infty_n$ (the asymptotics from Lemma 2 play here a central role; cf. the proof of Theorem 4 in [14]) one gets the relation

$$\frac{1}{2i}\left\{\sum_{l=1}^{M}\oint_{P_l} + \sum_{n=1}^{N}\oint_{\infty_n}\right\}\{F_m|\varphi_\lambda|^2 d\bar\lambda - (F_m)_\lambda\varphi_\lambda d\lambda + (F_m)_\lambda\varphi_{\bar\lambda}\, d\bar\lambda\} =$$

$$= \frac{1}{2i}\oint_{P_m}|\varphi_\lambda|^2\, d\bar\lambda - \frac{3\pi}{4}\sum_{l=1}^{M}\frac{d^2F_m}{(dx_l)^2}\Big|_{x_l=0} + o(1) \tag{2.27}$$

as $\rho\to 0$.

By (2.24) we have

$$\frac{1}{2}\oint_{P_0}F_m|\varphi_\lambda|^2 d\bar\lambda - (F_m)_\lambda\varphi_\lambda d\lambda + (F_m)_\lambda\varphi_{\bar\lambda}\, d\bar\lambda = \frac{1}{2i}\oint_{P_0}F_m(2\varphi_{\lambda\lambda} - (\varphi_\lambda)^2)d\lambda + o(1). \tag{2.28}$$

The Cauchy theorem, the asymptotical expansions at $P_l$ and $\infty_n$ and relation (2.7) imply that

$$0 = \frac{1}{2i}\left\{\sum_{l=1}^{M}\oint_{P_l} + \sum_{n=1}^{N}\oint_{\infty_n} + \oint_{P_0} + \sum_{\alpha=1}^{g}\left[\int_{a_\alpha^+\cup a_\alpha^-} + \int_{b_\alpha^+\cup b_\alpha^-}\right]\right\}F_m(2\varphi_{\lambda\lambda} - (\varphi_\lambda)^2)\, d\lambda =$$

$$= \frac{3\pi}{4}\sum_{l=1}^{M}\frac{d^2F_m}{(dx_l)^2}\Big|_{x_l=0} + \pi S_{Fay}^{P_0}(x_m)\Big|_{x_m=0} + \left\{\oint_{P_0} + \sum_{\alpha=1}^{g}\left[\int_{a_\alpha^+\cup a_\alpha^-} + \int_{b_\alpha^+\cup b_\alpha^-}\right]\right\}F_m(2\varphi_{\lambda\lambda} - (\varphi_\lambda)^2)\, d\lambda + o(1) \tag{2.29}$$

(cf. [14], Lemma 6). Observe that

$$\int_{a_\alpha^+\cup a_\alpha^-}(\varphi_\lambda\varphi)_{\lambda_m}\, d\lambda = \int_{a_\alpha^+\cup a_\alpha^-}[(2\varphi_{\lambda\lambda} - (\varphi_\lambda)^2)F_m + (F_m)_\lambda\varphi_\lambda]\, d\lambda$$

due to (2.24) and an obvious equality

$$\int_{a_\alpha^+\cup a_\alpha^-}\varphi\varphi_{\lambda\lambda_m}\,d\lambda=0.$$

Similarly,

$$\int_{b_\alpha^+\cup b_\alpha^-}[(2\varphi_{\lambda\lambda}-(\varphi_\lambda)^2)F_m+(F_m)_\lambda\varphi_\lambda]\,d\lambda=0,$$

due to the equality

$$\frac{\partial}{\partial\lambda_m}\int_{b_\alpha^+\cup b_\alpha^-}\varphi_\lambda\varphi\,d\lambda=0.$$

To finish the proof it remains to collect together equations (2.25)-(2.29), and make use of the fact that all $o(1)$ in the above equalities are uniform with respect to $(\lambda_1,\dots,\lambda_M)$ belonging to a compact neighborhood of the initial point $(\lambda_1^0,\dots,\lambda_M^0)$. $\square$

## 2.2 Bergmann tau-function and Fay's variational formula

Let us introduce the following $g(1-g)/2$-differential on $\hat{\mathcal{L}}$ which has multipliers 1 and $\exp\{-\pi i(g-1)^2\mathbf{B}_{\alpha\alpha}-2\pi i(g-1)K_\alpha^P\}$ along basic cycles $a_\alpha$ and $b_\alpha$, respectively:

$$\mathcal{C}(P)=\frac{1}{W[v_1,\dots,v_g](P)}\sum_{\alpha_1,\dots,\alpha_g=1}^{g}\frac{\partial^g\Theta(K^P)}{\partial z_{\alpha_1}\dots\partial z_{\alpha_g}}v_{\alpha_1}\dots v_{\alpha_g}(P)\,, \tag{2.30}$$

where $W(P)$ is the Wronskian (1.7) from the introduction. The differential $\mathcal{C}$ is an essential ingredient of Mumford measure on the moduli space of Riemann surfaces of given genus [8]. Multiplicative differential $\mathbf{s}$ (2.1) is expressed in terms of $\mathcal{C}$ as follows [8]:

$$\mathbf{s}(P,Q)=\left(\frac{\mathcal{C}(P)}{\mathcal{C}(Q)}\right)^{1/(1-g)} \tag{2.31}$$

Dependence of the differential $\mathcal{C}$ on moduli of the Riemann surface $\mathcal{L}$ is given by the following lemma, which is a slight reformulation of the variational formula given by Fay ([8], p.58):

**Lemma 3** *The infinitesimal variation of the differential $\mathcal{C}$ under the variation of the conformal structure of the Riemann surface defined by a Beltrami differential $\mu$ is given by the following expression:*

$$\delta_\mu\ln\mathcal{C}(P_0)=-\frac{1}{8\pi i}\int_{\mathcal{L}}\mu\left\{S_B-S_{Fay}^{P_0}\right\}\,, \tag{2.32}$$

*where $S_B$ is the Bergmann projective connection; $S_{Fay}^{P_0}$ is the projective connection (2.5); $S_B-S_{Fay}^{P_0}$ is a quadratic differential on $\mathcal{L}$.*

Substituting in (2.32) the Beltrami differential

$$\mu_m=-\frac{1}{2\epsilon^2}\mathbf{1}_{\{|x_m|\le\epsilon\}}\frac{d\bar{x}_m}{dx_m}$$

(which coincides with the Schiffer variation, describing the variation of the conformal structure under the infinitesimal shift of $\lambda_m$; see, e. g., [14]), we get the equality

$$\partial_{\lambda_m}\ln\mathcal{C}(P_0)=\left(S_B(x_m(P))-S_{Fay}^{P_0}(x_m(P))\right)\Big|_{P=P_m}. \tag{2.33}$$

Now we are in position to formulate the following

**Theorem 4** *The Bergmann tau-function is given by the following expression, which is independent of the choice of the points $P_0$ and $Q_0$:*

$$\tau^{-6} = \frac{\{\mathbf{s}(P_0, Q_0)\}^{2-2g} e^{2\pi i \langle \mathbf{r}, K^{P_0}\rangle}}{\mathcal{C}^4(P_0)(d\pi(P_0))^{g-1}} \prod_{k=1}^{M+N} \{\mathbf{s}(D_k, Q_0)\}^{d_k} \{E(D_k, P_0)\}^{(g-1)d_k} , \tag{2.34}$$

*where the integer vector $\mathbf{r}$ is defined as follows:*

$$\mathcal{A}(\mathcal{D}) + 2K^{P_0} + \mathbf{B}\mathbf{r} + \mathbf{s} = 0 ; \tag{2.35}$$

*the initial point of the Abel map coincides with $P_0$ and all the paths are chosen inside the same fundamental polygon $\hat{\mathcal{L}}$.*

**Proof.** Expression (2.34) is an immediate corollary of theorems 2, 3 and formula (2.33). The only thing one needs to check is the coincidence of the vector $\mathbf{r}$ defined by formula (2.15) with the vector $\mathbf{r}$ defined by (2.35). This coincidence is easy to prove for $g \geq 2$. Namely, we know that (2.34), where the components of the vector $\mathbf{r}$ are given by (2.15), gives the Bergmann tau-function (1.1). Therefore, as $P_0$ encircles the basic $b$-cycle, the tau-function can only gain a $\{\lambda_m\}$-independent factor. Computing the monodromy of expression (2.34) along cycle $b_\alpha$, we see that this implies (2.35) unless $g - 1 \neq 0$. For $g = 1$ relation (2.34) follows from the formula for the Bergmann tau-function which was found in [14].

Now, since (2.35) is proved, we can show that expression (2.34) is independent of $P_0$ and $Q_0$. Simple counting of tensor weight shows that expression (2.34) is a 0-differential with respect to each argument $P_0$ and $Q_0$, which is, moreover, free of singularities. Due to relation (2.35) and multiplicative properties of the differentials $\mathcal{C}$ and $\mathbf{s}$ we can also check that it is single-valued on $\mathcal{L}$ with respect to each of these arguments, and, therefore, is independent of both of them.

□

To transform expression (2.34) further to the form (1.5) we shall use the following

**Lemma 4** *For any two points $P, Q \in \mathcal{L}$ the function $\mathbf{s}(P,Q)$ can be written as follows in terms of prime-forms:*

$$\mathbf{s}^2(P,Q) = \frac{d\pi(P)}{d\pi(Q)} \prod_{k=1}^{M+N} \left(\frac{E(D_k,Q)}{E(D_k,P)}\right)^{d_k} \tag{2.36}$$

*Proof.* Consider the expression

$$(d\pi(P))^{1-g} \mathcal{C}^{-2}(P) \prod_{k=1}^{M+N} E^{d_k(g-1)}(D_k, P) . \tag{2.37}$$

Summing up the tensor weights of all ingredients of this expression, we see that this is a 0-differential with respect to $P$. Moreover, it is single-valued under tracing along all the basic cycles (this can be easily checked using multiplicative properties of the prime-forms and $\mathcal{C}(P)$), and does not have either zeros or poles on $\mathcal{L}$ (the poles and zeros induced by the prime-forms are canceled by the poles and zeros of $d\pi(P)$). Therefore, expression (2.37) is independent of the point $P$. Taking its ratio at arbitrary two points $P$ and $Q$ and using expression (2.31) of $\mathbf{s}(P,Q)$ in terms of ratio of the differential $\mathcal{C}(P)$ at these two points, we get (2.36). □

Now, choosing in formula (2.34) $P_0 = Q_0$ and expressing $\mathbf{s}(D_k, P_0)$ in terms of the prime-forms using (2.36), we get expression (1.5) for the Bergmann tau-function stated in the introduction.

### 2.2.1 Genus 1 case

The differential $\mathcal{C}(P)$ in elliptic case does not depend on $P$ ([8], p.21):

$$\mathcal{C} = \eta^3(\mathbf{B})e^{-\pi i\mathbf{B}/4} ,$$

where $\eta(\mathbf{B})$ is the Dedekind eta-function. The differential $\mathbf{s}(P, Q)$ is given by

$$\mathbf{s}(P, Q) = \exp\left\{\pi i \int_Q^P v\right\} \frac{\sqrt{v(P)}}{\sqrt{v(Q)}} \tag{2.38}$$

Substituting these expressions to (2.34) and taking into account (2.35), we get the following expression:

$$\tau = \eta^2(\mathbf{B}) \prod_{k=1}^{M+N} \{v(D_k)\}^{-d_k/12} , \tag{2.39}$$

where according to our usual conventions $v(D_k) := v(P)/dx_k(P)|_{P=D_k}$, $k = 1, \dots, M + N$. This formula was independently proved in [14]; this confirmes correctness of the choice of integer $\mathbf{r}$ in (2.34) for $g = 1$.

## 2.3 Tau-function for an arbitrary stratum of Hurwitz space

Here we briefly consider the general case, when the critical points and poles of function $\pi(P)$ have arbitrary multiplicities. Such Bergmann tau-function arises in the problem of computation of isomonodromic tau-function corresponding to Riemann-Hilbert problem with arbitrary permutation monodromies [16] (in this case multiplicities of the critical points are arbitrary) and in the theory of semisimple Frobenius manifolds related to Hurwitz spaces ([2, 15]) (in this case the multiplicities of poles of $\pi(P)$ can be arbitrary). As before, denote the branch points of the branched covering $\mathcal{L}$ by $P_1, \dots, P_M$ and assume that they have multiplicities $d_1, \dots, d_M$; the orders of the poles $\infty_1, \dots, \infty_L$ of $\pi$ we denote by $d_{M+1} - 1, \dots, d_{M+L} - 1$, respectively. One has $N = \sum_{s=1}^L d_{M+s}$. Then divisor $\mathcal{D} := (d\pi)$ can be formally written in the same form as before:

$$\mathcal{D} = \sum_{k=1}^{M+L} d_k D_k \tag{2.40}$$

where $D_m := P_m$, $m = 1, \dots, M$ and $D_{M+s} = \infty_s$, $s = 1, \dots, L$. The genus of the Riemann surface $\mathcal{L}$ is given in this case by the formula:

$$g = \frac{1}{2}\sum_{m=1}^M d_m - \sum_{s=1}^L d_{M+s} + 1 \tag{2.41}$$

The definition (1.1) generalizes as follows:

$$\frac{\partial}{\partial\lambda_m}\ln\tau = -\frac{1}{6(d_m-1)!(d_m+1)}\left(\frac{d}{dx_m(P)}\right)^{d_m-1} S_B(x_m(P))\Big|_{P_m=P} , \qquad m = 1, \dots, M . \tag{2.42}$$

Compatibility of the system (2.42) follows from Schlesinger equations [16]; it was checked directly in [14] using Rauch variational formulas. By using the same technique as in the case of simple branch points and infinities one can verify that the formula (1.5) stated in the introduction remains valid in the general case after substitution of corresponding multiplicities $d_k$ and genus $g$. We don't present the proof here since it does not contain any new essential ideas in comparison with the case of simple poles and zeros.

# 3 Applications of the Bergmann tau-function

## 3.1 G-function of Frobenius manifolds

Fix a stratum $H_{g,N}(k_1, \dots, k_L)$ of Hurwitz space, for which all the critical points of function $\pi(P)$ are simple, but infinities have arbitrary multiplicities $k_1, \dots, k_L$ (for simple infinities all $k_s = 1$). Then the divisor $\mathcal{D}$ (2.40) of the differential $d\pi$ looks as follows:

$$\mathcal{D}_{Frob} = \sum_{m=1}^{M} P_m - \sum_{s=1}^{L} (k_s + 1)\infty_s \tag{3.1}$$

i.e. $d_m = 1$ for $m = 1, \dots, M$ and $d_{M+n} = k_n + 1$ for $n = 1, \dots, L$. The structures of Frobenius manifold on any Hurwitz space of this type were introduced by Dubrovin [2]. We refer to [2] or [17] for definition of all ingredients (Frobenius algebra, prepotential, canonical and flat coordinates, Darboux-Egoroff metrics, $G$-function) of this construction. Here we shall only discuss the $G$-function (genus one free energy of Dijkgraaf and Witten), which gives a solution of Getzler equation (for classes of Frobenius manifolds related to quantum cohomologies the $G$-function is the generating function of elliptic Gromov-Witten invariants).

Recall that each Frobenius structure on the Hurwitz space corresponds to a so-called primary differential $\varphi$ on the covering $\mathcal{L}$. The arising Frobenius manifold will be denoted by $M_\varphi$. In [4, 5, 10] it was found the following expression for the $G$-function of an semisimple arbitrary Frobenius manifold:

$$G = \ln \left( \frac{\tau_I}{J^{\frac{1}{24}}} \right). \tag{3.2}$$

where $\tau_I$ is the isomonodromic tau-function associated to an arbitrary semisimple Frobenius manifold; $J$ is the Jacobian of the transformation from the flat to the canonical coordinates. In [15] it was further proved that for Frobenius structures on Hurwitz spaces one has $\tau_I = \tau^{-1/2}$, where $\tau$ is the Bergmann tau-function (1.1). Using the known expression of Jacobian $J$ in terms of diagonal coefficients of Darboux-Egoroff metric (see, e. g., [10]), we get the following

**Theorem 5** *The $G$-function of the Frobenius manifold $M_\varphi$ can be expressed as follows*

$$G = -\frac{1}{2} \ln \tau - \frac{1}{48} \sum_{m=1}^{M} \ln \operatorname{Res}_{P_m} \frac{\varphi^2}{d\lambda}, \tag{3.3}$$

*where $\tau$ is the Bergmann tau-function on the Hurwitz space $H_{g,N}(k_1, \dots, k_L)$ given by (1.5), with the divisor $\mathcal{D}$ given by (3.1).*

## 3.2 Determinant of Laplacian in Poincaré metric

Here we consider an application of the Bergmann tau-function to computation of the determinants of Laplacians on Riemann surface in the Poincaré metric (in the trivial line bundle); such determinants are defined in terms of the corresponding $\zeta$-functions as follows: $\det\Delta := \exp\{-\zeta'_\Delta(0)\}$. For elliptic case and the flat metric $|v|^2$ (where $v$ is holomorphic normalized differential) this determinant is given by the classical Kronecker formula (see [20]). Such explicit formula is absent for $g > 1$ for Poincaré metric, although variational formulas for $\det\Delta$ with respect to the moduli of the Riemann surface are well-known (see, e. g., [24], or [8] (formulae (5.4) and (4.58))). Namely, the variation of the determinant under the variation of conformal structure given by a Beltrami differential $\mu$ looks as follows:

$$\delta_\mu\left\{\frac{\det\Delta}{\det\Im\mathbf{B}}\right\} = -\frac{1}{12\pi i}\int_{\mathcal{L}}(S_B - S_{Fuchs})\mu \tag{3.4}$$

where $S_B$ is the Bergmann projective connection, and $S_{Fuchs}$ is the Fuchsian projective connection on $\mathcal{L}$.

Using these variational formulas, in [14] it was obtained a formula which expresses $\det\Delta$ in terms of the Bergmann tau-function. To formulate the theorem which combines this result with formula (1.5) we need to introduce a few new objects.

Let all the critical points and poles of the function $\pi$ be simple (this is sufficient for computation of $\det\Delta$ since on any Riemann surface we can find a meromorphic function with these properties). For $g > 1$ the Riemann surface $\mathcal{L}$ is biholomorphically equivalent to the quotient space $\mathbb{H}/\Gamma$, where $\mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$; $\Gamma$ is a strictly hyperbolic Fuchsian group. Denote by $\pi_F : \mathbb{H} \to \mathcal{L}$ the natural projection. Let $x$ be a local parameter on $\mathcal{L}$. Introduce the standard metric of the constant curvature $-1$ on $\mathcal{L}$:

$$e^{\chi}|dx|^2 = \frac{|dz|^2}{|\Im z|^2}\ , \tag{3.5}$$

where $z \in \mathbb{H}$, $\pi_F(z) = P$, $x = x(P)$.

In complete analogy to constructions of Sec. 2.1.1, introduce the real-valued functions $\chi(\lambda)$, $\chi^{int}(x_m)$, $m = 1, \dots, M$ and $\chi_k^\infty(\zeta_n)$, $n = 1, \dots, N$ by specifying the local parameter $x = \lambda$, $x = x_m$ and $x = \zeta_n$ (in a neighborhood of the point at infinity of the $n$-th sheet) in (3.5) respectively.

Consider domains $\mathcal{L}_\rho^{(n)}$ of $\mathcal{L}$ as in the integrals (2.13). (Recall that the domain $\mathcal{L}_\rho^{(n)}$ is obtained from the $n$-th sheet of $\mathcal{L}$ by deleting small discs around ramification points belonging to this sheet, and the disc around infinity.)

Define the regularized Dirichlet integral analogous to (2.12):

$$\mathbb{D}_F := \frac{1}{\pi}\lim_{\rho\to 0}\left(\sum_{n=1}^{N}\int_{\mathcal{L}_\rho^{(n)}}|\partial_\lambda\chi|^2\widehat{d\lambda} + (8N + M)\pi\ln\rho\right). \tag{3.6}$$

Define the function $\mathbb{S}_F$ by

$$\mathbb{S}_F(\lambda_1, \dots, \lambda_M) = \frac{1}{12}\mathbb{D}_F + \frac{2}{3}\sum_{m=1}^{M}\chi^{int}(x_m)\Big|_{x_m=0} - \frac{2}{3}\sum_{n=1}^{N}\chi_n^\infty(\zeta_n)\Big|_{\zeta_n=0}\ ; \tag{3.7}$$

Now we are in a position to formulate the following

**Theorem 6** *Consider the Hurwitz space $H_{g,N}(1, \dots, 1)$. Let the pair $(\mathcal{L}, \pi)$ belong to $H_{g,N}(1, \dots, 1)$. Then the determinant of the Laplace operator on $\mathcal{L}$ (acting in the trivial line bundle) in Poincaré metric is given by the following expression:*

$$\det\Delta = c_{g,N}\det\Im\mathbf{B}\, e^{\mathbb{S}_F}\,|\tau|^2. \tag{3.8}$$

*where $c_{g,N}$ is a constant independent of the point $(\mathcal{L}, \pi) \in H_{g,N}(1, \dots, 1)$; $\mathbf{B}$ is the matrix of $b$-periods on $\mathcal{L}$; $\tau$ is the Bergmann tau-function (1.5).*

### 3.3 Riemann-Hilbert problems with quasi-permutation monodromies and isomonodromic tau-function

The Riemann-Hilbert problem of construction of $GL(N)$-valued function on the universal covering of puctured Riemann sphere $\mathbb{C}P^1 \setminus \{\lambda_1, \dots, \lambda_M\}$ with prescribed monodromy representation in general case (for an arbitrary representation) can not be solved in terms of known special functions. For an arbitrary quasi-permutation monodromy group (i.e. such that each monodromy matrix has exactly one non-vanishing entry in each of its columns and each of its raws) the RH problem was solved in [16] outside of a divisor in the space of monodromy data (the so-called Malgrange divisor, or the divisor of zeros of the Jimbo-Miwa tau-function) following previous works [12, 1], where the $2 \times 2$ case was solved. One usually requires that the solution $\Psi$ of the Riemann-Hilbert problem is normalized to $I$ at some point $\lambda_0 \in \mathbb{C}P^1$, which does not coincide with singularities $\{\lambda_m\}$; we shall denote such normalized solution by $\Psi(\lambda, \lambda_0)$.

**Theorem 7** *[16] Let the set of the monodromy data lies outside of the Malgrange divisor. Then the solution $\Psi(\lambda, \lambda_0)$ of an arbitrary Riemann-Hilbert problem with quasi-permutation monodromy representation is given by the analytical continuation on universal covering of the punctured sphere of the following expression defined in a neighbourhood of the normalization point (all objects in this formula correspond to the $N$-sheeted branched covering $\mathcal{L}$, associated with the quasi-permutation monodromy representation):*

$$\Psi_{kj}(\lambda_0, \lambda) = \frac{\lambda - \lambda_0}{\sqrt{d\lambda d\lambda_0}} \frac{\Theta\left[{}^{\mathbf{p}}_{\mathbf{q}}\right]\left(\mathcal{A}(\lambda^{(j)}) - \mathcal{A}(\lambda_0^{(k)}) + \Omega\right)}{\Theta\left[{}^{\mathbf{p}}_{\mathbf{q}}\right](\Omega) E(\lambda^{(j)}, \lambda_0^{(k)})} \prod_{m=1}^{M} \prod_{l=1}^{N} \left[\frac{E(\lambda^{(j)}, \lambda_m^{(l)})}{E(\lambda_0^{(k)}, \lambda_m^{(l)})}\right]^{r_m^{(l)}} \tag{3.9}$$

*where*

$$\Omega := \sum_{m=1}^{M} \sum_{j=1}^{N} r_m^{(j)} \mathcal{A}(\lambda_m^{(j)}) ; \tag{3.10}$$

*$\lambda^{(k)}$ denotes the point of $\mathcal{L}$ which belongs to the $k$th sheet and has projection $\lambda$ on $\mathbb{C}P^1$; $\mathbf{p}, \mathbf{q} \in \mathbb{C}^g$ are constant vectors; $r_m^{(k)}$ are constants assigned to all points from $\pi^{-1}(\lambda_m)$ (if two points from $\pi^{-1}(\lambda_m)$ coincide, the constants $r_m^{(k)}$ are assumed to coincide, too). The logarithms of the matrix elements of monodromy matrices are linear functions of the constants $\mathbf{p}, \mathbf{q}$ and $r_m^{(k)}$. The Malgrange divisor is defined by the equation $\Theta\left[{}^{\mathbf{p}}_{\mathbf{q}}\right](\Omega) = 0$.*

If the elements of monodromy matrices (or, equivalently, the constants $\mathbf{p}, \mathbf{q}$ and $r_m^{(k)}$) are independent of positions of singularities $\{\lambda_m\}$, function $\Psi$ defines a solution of the Schlesinger system, together with isomonodromic tau-function of Jimbo-Miwa [11], defined as follows:

$$\frac{\partial}{\partial \lambda_m} \ln \tau_{JM} = \frac{1}{2} \text{res}|_{\lambda=\lambda_m} \text{tr}\left(\Psi_\lambda \Psi^{-1}\right)^2 \tag{3.11}$$

In [16] it was proved the following

**Theorem 8** *The Jimbo-Miwa tau-function corresponding to solution (3.9) of the Riemann-Hilbert problem, is given by the following formula:*

$$\tau_{JM} = \tau^{-1/2} \prod_{m,l=1}^{M} (\lambda_m - \lambda_l)^{r_{ml}} \Theta\left[{}^{\mathbf{p}}_{\mathbf{q}}\right](\Omega|\mathbf{B}) \tag{3.12}$$

*where $\tau$ is the Bergmann tau-function defined by (1.1);*

$$r_{mn} = \sum_{k=1}^{N} r_m^{(k)} r_n^{(k)} \; .$$

This theorem, together with expression (1.5) proved in this paper, gives the explicit formula for Jimbo-Miwa tau-function corresponding to general Riemann-Hilber problem with quasi-permutation monodromies. We notice that for monodromy groups corresponding to hyperlliptic curves this tau-function was found in [12]; for $Z_N$ curves with $N > 2$ it was computed in [6].

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