TEST FUNCTIONS FOR SHIMURA VARIETIES: THE DRINFELD CASE

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1. Introduction

Let (G, X, K) be a Shimura datum with reflex field E. Choose a prime number p and a prime ideal $\mathfrak{p}$ of E lying over p. Suppose that $K_p \subset G(\mathbb{Q}_p)$ is a *parahoric* subgroup. Also assume that G splits over $\mathbb{Q}_p^{un}$, and hence that $E_\mathfrak{p}$ is an unramified extension of $\mathbb{Q}_p$. Let S_K denote a model over $\mathcal{O}_{E,\mathfrak{p}}$ of the corresponding Shimura variety. Then it is often the case that S_K has bad reduction. The singularities of the special fiber are very complicated in general, and somehow must be understood in order to study the local zeta function of the Shimura variety at $\mathfrak{p}$. Rapoport ([10]) has outlined a strategy to attack this problem. The first part is to find a convenient way to express

$$tr(Fr_q\,;\, R\Psi^{\mathcal{I}}_{x_0}(\overline{\mathbb{Q}}_l)),$$

where $R\Psi(\overline{\mathbb{Q}}_l)$ is the sheaf of *nearby cycles* on $(S_K)_{\overline{\mathbb{F}}_q}$, $q = p^j$ is such that $\mathbb{Q}_{p^j}$ contains $E_\mathfrak{p}$, $x_0 \in S_K(\mathbb{F}_q)$, Fr_q is the geometric Frobenius on $(S_K)_{\overline{\mathbb{F}}_q}$, and $\mathcal{I}$ is the inertia subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. (The study of this trace only directly relates to a *semi-simplified* version of the local zeta function, but it is nevertheless a first step towards the usual local zeta function.) The sheaf of nearby cycles can often be computed when one understands the global geometry of the special fiber, but in practice the global geometry of the special fiber is too complicated to be dealt with directly. To circumvent this problem Rapoport introduces a "local model" M^{loc} over $\mathcal{O}_{E,\mathfrak{p}}$ and a procedure that attaches to $x_0 \in S_K(\mathbb{F}_q)$ a point x in $M^{\mathrm{loc}}(\mathbb{F}_q)$. We consider now the special case where K_p is an Iwahori subgroup. The local model then has a stratification indexed by certain elements of the extended affine Weyl group of G (conjecturally the *μ-admissible* set; see definition in §2.1). Although the point x is not uniquely determined by x_0, it is contained in a well-defined stratum; we can thus also use the symbol x to denote the element of the extended affine Weyl group corresponding to this stratum. The local model at x is locally isomorphic to the special fiber at x_0, so by transport of structure we see that we need an expression for the trace of Frobenius on $R\Psi^{\mathcal{I}}_x(\overline{\mathbb{Q}}_l)$. This should have a purely group-theoretic interpretation, if we are eventually going to use the Arthur-Selberg trace formula to express the zeta function in terms of automorphic L-functions.

Such an interpretation has been conjectured by R. Kottwitz. To give his prescription we must fix an unramified extension F containing $E_\mathfrak{p}$ and assume that G is quasisplit over F. Associated to X is a minuscule cocharacter μ (defined up to conjugacy) of the group $G_{\overline{E}_\mathfrak{p}}$, and by definition of E the conjugacy class of μ is defined over $E_\mathfrak{p}$. Because G_F is quasisplit we can consider μ as a well-defined element of $X_*(A)/W_0$, where A is a maximal F-split torus of G_F and W_0 is the relative Weyl group of G_F (see Lemma 1.1.3 of [5]). Let $q = p^j$ denote the cardinality of the residue field of F, and let z_μ denote the Bernstein function corresponding to μ, which is an element of the center of the Iwahori-Hecke algebra of the p-adic group $G(F)$

(see definition in §2). Then z_μ is a linear combination (over $\mathbb{Z}[q^{1/2}, q^{-1/2}]$) of the generators T_w, where T_w denotes the characteristic function of the Iwahori double coset corresponding to the element w of the extended affine Weyl group of G_F. Denote the coefficient of T_w by $z_\mu(w)$. Then we have the following

Conjecture (*Kottwitz*): *If K_p is an Iwahori subgroup and x is an $\mathbb{F}_q$-rational point of the local model M^{loc} corresponding to the Shimura datum (G, X, K), then*

$$tr(Fr_q\,;\,R\Psi_x^{\mathcal{I}}(\overline{\mathbb{Q}}_l)) = q^{\dim(S_K)/2} z_\mu(x).$$

If $q^{\dim(S_K)/2} z_\mu$ has this property, then it follows that it is the correct "test function" for the Shimura variety, meaning that it gives the function at the prime p which is "plugged into" the twisted orbital integrals that come into the computation of the semi-simple local zeta function when one attempts to use the Arthur-Selberg trace formula.

There is also a version of this conjecture for the general parahoric case: the local model then has a stratification by certain "μ-admissible" elements of the double coset space $W_J \backslash \widetilde{W} / W_J$ (here W_J is a parabolic subgroup of the extended affine Weyl group $\widetilde{W}$). In the statement above the Bernstein function z_μ needs to replaced with its image in the parahoric Hecke algebra.

The main result of this paper (Theorem 4.3) is a formula for z_μ where μ is any *minuscule* coweight of any reduced root system. It applies to Hecke algebras with arbitrary parameters (see definition in §2). For simplicity we state the result in the special case corresponding to a *split* group (the parameter system is then given by $L(s) = 1$ for all $s \in S_a$; see notation in §2).

Theorem 1.1. *Let μ be a dominant minuscule coweight of a split connected reductive p-adic group G with root system $(X^*, X_*, R, \check{R}, \Pi)$. Let $\widetilde{W} = X_* \rtimes W$ denote the extended affine Weyl group of G. Let z_μ denote the Bernstein function corresponding to μ. Then*

$$q^{l(a^\mu)/2} z_\mu = \varepsilon_{a^\mu} \sum_{x\,:\,x \text{ is } \mu\text{-adm.}} \varepsilon_x R_{x, a^{t(x)}}(q) T_x,$$

where $\varepsilon_y = (-1)^{l(y)}$, x has decomposition $x = wa^{t(x)}$ ($w \in W$, $t(x) \in X_*$), and $R_{x,a^{t(x)}}(q)$ is the R-polynomial attached to $\widetilde{W}$ in Kazhdan-Lusztig theory (cf. §2).

In this paper we use Theorem 1.1 to prove Kottwitz' conjecture for a certain class of Shimura varieties with Iwahori type reduction. We examine Shimura varieties attached to the group of unitary similitudes $GU(1, d-1)$ defined by an imaginary quadratic extension E of $\mathbb{Q}$ in which the prime p splits (the "Drinfeld case"). In this case Rapoport gives formulae for the trace of the Frobenius Fr_q on the nearby cycles on the corresponding local model ([10] and [9]), the formulae being very explicit under some assumptions on the number $q = p^j$ (Proposition 5.1 of §5). Under the same assumptions on $q = p^j$ the group $G_{\mathbb{Q}_{p^j}}$ is split and we can use Theorem 1.1 to give an explicit formula for the coefficients $q^{l(a^\mu)/2} z_\mu(x)$ (comp. Proposition 5.2). Comparing the two explicit formulae, one sees that Kottwitz' conjecture holds in this case. (Theorem 1.1 is all that is needed for this particular application, because the group in question is split. However, in cases where the group is quasisplit but nonsplit we need to allow for arbitrary parameters and the more general Theorem 4.3 is necessary.)

More precisely, fix an integer $d > 2$ and let $(D, *)$ denote a central simple algebra D of dimension d^2 over an imaginary quadratic extension E of $\mathbb{Q}$, together with a positive involution $*$ which induces the nontrivial automorphism on E. Suppose the prime p splits in

E as the product $\mathfrak{p}_1\mathfrak{p}_2$, where $\mathfrak{p}_1$ is the prime ideal of E distinguished in §5. Let G denote the corresponding $\mathbb{Q}$-group (defined in §5), and let $K = K^pK_p$, where K_p is an Iwahori subgroup of $G(\mathbb{Q}_p)$. Let X be chosen as in §5, so that $G_{\mathbb{R}}$ is the group $GU(1, d-1)$. With this choice of X the corresponding $G(\overline{\mathbb{Q}}_p)$-conjugacy class $\{\mu\}$ of cocharacters of $G(\overline{\mathbb{Q}}_p)$ can be "identified" in a certain precise sense with the cocharacter $(1, 0^{d-1})$ of Gl_d, if we assume that $q = p^j$ is such that $j \cdot (\mathrm{inv}(D_{\mathfrak{p}_1}) \equiv 0 \,(\mathrm{mod}\; d)$. Indeed, let $F = \mathbb{Q}_{p^j}$, where j is chosen as above. Then via the isomorphism of F-groups $G_F = Gl_d \times \mathbb{G}_m$ described in §5, we may represent $\{\mu\}$ using the cocharacter $\mu = (1, 0^{d-1})t$ where t denotes the cocharacter $x \mapsto x$ in $X_*(\mathbb{G}_m)$ (see §5).

Let S_K denote a model over $\mathcal{O}_{E_{\mathfrak{p}_1}}$ of the Shimura variety determined by the datum (G, X, K). A first step towards understanding the semi-simple local zeta function of S_K is provided by the following theorem.

Theorem 1.2. *Let (G, X, K) denote the Shimura datum above, and let M^{loc} denote the corresponding local model. Let j be such that $j \cdot (inv(D_{\mathfrak{p}_1})) \equiv 0 \;(mod\, d)$. Then Kottwitz' conjecture holds for M^{loc} and $q = p^j$.*

Corollary 1.3. *If $D_{\mathfrak{p}_1}$ is a matrix algebra, then Kottwitz' conjecture holds for M^{loc} and any q.*

One can also use Theorem 1.1 to predict the trace of Frobenius on nearby cycles for the local models attached to other Shimura varieties. When a calculation of this trace is possible, Kottwitz' conjecture can be verified using a stratum-by-stratum comparison of explicit formulae, as in Theorem 1.2. For example, consider the Shimura variety with Iwahori type reduction attached to the group $GU(2, 3)$ determined by an imaginary quadratic extension E of $\mathbb{Q}$ in which the prime p splits. This comes from a central simple algebra D over E as above. We again consider only j such that the condition in Theorem 1.2 holds. This means that we are essentially dealing with the local model M^{loc} attached to GL_5 and $\mu = (1, 1, 0, 0, 0)$ (see the proof of Theorem 1.2 in §5). Let $\tau \in \widetilde{W}$ denote element indexing the "most singular stratum" of the local model; it turns that $a^{t(\tau)} = a^{(0,0,0,1,1)} = s_3s_4s_2s_3s_1s_2\tau$ (here s_i is the transposition $(i\; i+1)$, for $1 \leq i \leq 4$). Then Theorem 1.1 and Kottwitz' conjecture predict that $tr(Fr_q; R\Psi^{\mathcal{I}}_{\tau}(\overline{\mathbb{Q}}_l))$ is

$$\varepsilon_\tau \varepsilon_{a^\mu} R_{\tau, s_3s_4s_2s_3s_1s_2\tau}(q) = (1-q)^4(1+q^2).$$

A calculation by U. Görtz of the trace of Frobenius on nearby cycles for this "most singular point" produced exactly this expression. Also, Görtz calculated the trace of Frobenius on the nearby cycles for all 33 strata for the local model attached to Gl_4 and $\mu = (1, 1, 0, 0)$. Comparing the results with the formula for z_μ in Theorem 1.1, he verifed that Kottwitz' conjecture holds for the local model of the Shimura variety with Iwahori type reduction attached to the group $GU(2, 2)$ determined by an imaginary quadratic extension of $\mathbb{Q}$ in which p splits.

We now outline the contents of the paper. In §2, we give further notation and prove some elementary lemmas. In §3 we present an efficient method of computing Bernstein functions for minuscule coweights as linear combinations of the basis elements $\tilde{T}_w$, $w \in \widetilde{W}$ (or equivalently, the elements T_w). In §4 we prove the main theorem (Theorem 4.3). Moreover we prove that when μ is minuscule, the support of z_μ is precisely the μ-admissible set (Proposition 4.6). In §5 we discuss the Shimura varieties in the Drinfeld case, and deduce the truth of Kottwitz' conjecture for the special case in Theorem 1.2 above, using Rapoport's formulae (Proposition 5.1) and the explicit formula for z_μ in this case (Proposition 5.2).

2. Notation

For the most part we will use the notation in [8], except that the affine Weyl groups and Hecke algebras we consider will be "dual" to Lusztig's.

2.1. **The Affine Weyl Group of a Root System.** Let $(X^*, X_*, R, \check{R}, \Pi)$ be a (based) root system, where Π denotes the simple positive roots. Let R^+ (resp. R^-) denote the set of positive (resp. negative) roots; we often use $\alpha > 0$ to denote $\alpha \in R^+$. We assume throughout this paper that the root system is *reduced*.

Corresponding to $\alpha \in \Pi$ we have the simple reflection s_α, acting on X_* (resp. X^*) by $s_\alpha(x) = x - \langle \alpha, x \rangle \check{\alpha}$ (resp. $s_\alpha(y) = y - \langle y, \check{\alpha} \rangle \alpha$). The Weyl group W_0 is the subgroup of $GL(X_*)$ (or $GL(X^*)$) generated by $S = \{s_\alpha \mid \alpha \in \Pi\}$. It is known that (W_0, S) is a finite Coxeter group.

Let $\preceq$ denote the partial order on X_* (resp. X^*) defined by $\lambda \preceq \mu \Leftrightarrow \mu - \lambda$ is a linear combination with ≥ 0 integer coefficients of elements of $\{\check{\alpha} \mid \alpha \in \Pi\}$ (resp.$\{\alpha \mid \alpha \in \Pi\}$) . Let Π_m denote the set of $\beta \in R$ such that β is a minimal element of $R \subset X^*$ with respect to $\preceq$. If the root system is irreducible, $\Pi_m = \{-\tilde{\alpha}\}$, where $\tilde{\alpha}$ is the unique highest root.

Let $\widetilde{W}$ be the semidirect product $W_0 \ltimes X_* = \{wa^x \mid w \in W_0,\ x \in X_*\}$ (a is a fixed symbol). The multiplication is given by $w'a^{x'}wa^x = w'wa^{w^{-1}(x')+x}$. Define a function $l : \widetilde{W} \to \mathbb{Z}$ by the formula

$$l(wa^x) = \sum_{\alpha \in R^+ : w(\alpha) \in R^-} |\langle \alpha, x \rangle + 1| + \sum_{\alpha \in R^+ : w(\alpha) \in R^+} |\langle \alpha, x \rangle|.$$

Define an action of $\widetilde{W}$ on $X^* \times \mathbb{Z}$ by $wa^x(y,k) = (w(y), k - \langle y, x \rangle)$. Let $\widetilde{R} = \widetilde{R}^+ \cup \widetilde{R}^- \subset X^* \times \mathbb{Z}$ be defined by

$$\widetilde{R}^+ = \{(\alpha, k) \mid \alpha \in R,\ k > 0\} \cup \{(\alpha, 0) \mid \alpha > 0\},$$
$$\widetilde{R}^- = \{(\alpha, k) \mid \alpha \in R,\ k < 0\} \cup \{(\alpha, 0) \mid \alpha < 0\}.$$

Let

$$\widetilde{\Pi} = \{(\alpha, 0) \mid \alpha \in \Pi\} \cup \{(\alpha, 1) \mid \alpha \in \Pi_m\} \subset \widetilde{R}^+,$$
$$S_a = \{s_\alpha \mid \alpha \in \Pi\} \cup \{s_\alpha a^{\check{\alpha}} \mid \alpha \in \Pi_m\} \subset \widetilde{W}.$$

There is a bijection $\widetilde{\Pi} \leftrightarrow S_a$ $(A \leftrightarrow s_A)$.

Let

$$X_{\text{dom}} = \{x \in X_* \mid \langle \alpha, x \rangle \geq 0,\ \forall \alpha \in \Pi\}$$
$$= \{x \in X_* \mid l(s_\alpha a^x) = l(a^x) + 1,\ \forall \alpha \in \Pi\}.$$

Note that $w \in W_0,\ x \in X_{\text{dom}} \Rightarrow l(wa^x) = l(w) + l(a^x)$.

Let $\check{Q}$ denote the subgroup of X_* generated by $\check{R}$. Then the subgroup $W_a = W_0\check{Q}$ of $\widetilde{W}$ is a Coxeter group with S_a the set of simple reflections, the length function being the restriction of l. This subgroup is normal in $\widetilde{W}$ and admits a complement $\Omega = \{w \in \widetilde{W} \mid w(\widetilde{\Pi}) = \widetilde{\Pi}\} = \{w \in \widetilde{W} \mid l(w) = 0\}$. It is known that Ω is an abelian group isomorphic to $X_*/\check{Q}$.

We use $\leq$ to denote the Bruhat order on $\widetilde{W}$. This is defined on the Coxeter group (W_a, S_a) as usual, and it is then extended to $\widetilde{W}$ by declaring $x\tau \leq x'\tau'$ $(x, x' \in W_a,\ \tau, \tau' \in \Omega)$ if $x \leq x'$ and $\tau = \tau'$.

Definition 2.1. For $\mu \in X_{dom}$, we say $x \in \widetilde{W}$ is *μ-admissible* if $x \leq a^{w(\mu)}$ for some $w \in W_0$. The set of all such x's is called the *μ-admissible set.*

2.2. **The Hecke Algebra.** Let B be the group with generators T_w, $(w \in \widetilde{W})$ and relations

$$T_w T_{w'} = T_{ww'} \text{ whenever } l(ww') = l(w) + l(w').$$

We call B the *braid group* of $\widetilde{W}$. For any $x \in X_*$ we define an element $\overline{T}_x = T_{a^{x_1}} T_{a^{x_2}}^{-1}$, where $x = x_1 - x_2$ and $x_1, x_2 \in X_{\mathrm{dom}}$. This is independent of the choice of the x_i.

Fix a parameter set $L : S_a \to \mathbb{N}$ for the root system. This means that $L(s) = L(s')$ whenever s , $s' \in S_a$ are conjugate in $\widetilde{W}$. Equivalently, L is the restriction to S_a of a function $L' : \widetilde{W} \to \mathbb{N}$ with $L'(ww') = L'(w) + L'(w')$ whenever w , $w' \in \widetilde{W}$ satisfy $l(ww') = l(w) + l(w')$. We denote both functions simply by L. We also use to L to denote the unique homomorphism $L : B \to \mathbb{Z}$ such that $L(T_w) = L(w)$.

Let v be an indeterminate (thought of as $q^{1/2}$), and let $\mathbb{Z}' = \mathbb{Z}[v, v^{-1}]$. The Hecke algebra $\mathcal{H}$ is by definition the quotient of the group algebra of B (over $\mathbb{Z}'$) by the two sided ideal generated by the elements

$$(T_s + 1)(T_s - v^{2L(s)}), \quad s \in S_a.$$

The image of T_w (resp. $\overline{T}_x$) in $\mathcal{H}$ is still denoted by T_w (resp. $\overline{T}_x$).

Remark 2.2. Let G be a split connected reductive group over a p-adic field F, with root system $(X^*, X_*, R, \check{R}, \Pi)$. Suppose q is the size of the residue field of F, and let $v = q^{1/2}$. Let L be the parameter set given by $L(s) = 1$ for every $s \in S_a$. Choose an Iwahori subgroup $I \subset G(F)$ whose "reduction mod p" is the Borel corresponding to the choice of simple positive roots Π. Define convolution in the algebra $C_c(I \backslash G(F)/I)$ using the Haar measure on $G(F)$ which gives I measure 1. Then there is a canonical isomorphism of $\mathbb{Z}[q^{1/2}, q^{-1/2}]$-algebras

$$\mathcal{H} \cong C_c(I \backslash G(F)/I).$$

Define for any $w \in \widetilde{W}$ a renormalization $\tilde{T}_w = v^{-L(w)} T_w$. The elements T_w (resp. $\tilde{T}_w$) $(w \in \widetilde{W})$ form a $\mathbb{Z}'$-basis for $\mathcal{H}$. For $x \in X_*$, define

$$\Theta_x = v^{-L(\overline{T}_x)} \overline{T}_x = v^{-L(x_1) + L(x_2)} T_{a^{x_1}} T_{a^{x_2}}^{-1} = \tilde{T}_{a^{x_1}} \tilde{T}_{a^{x_2}}^{-1},$$

where $x = x_1 - x_2,\ x_i \in X_{\mathrm{dom}}$. It is known that the elements $\Theta_x T_w$ $(x \in X_*,\ w \in W_0)$ form a $\mathbb{Z}'$-basis for $\mathcal{H}$ ([8], Prop. 3.7).

Definition 2.3. For each W_0-orbit M in X_* define the *Bernstein function* attached to M by $z_M = \sum_{\lambda \in M} \Theta_\lambda$.

When the W_0-orbit M contains the dominant element μ, this function will usually be denoted by z_μ.

The following theorem is due to Bernstein in the special case where $L(s)$ is independent of s and the roots generate a direct summand of X^*, and to Lusztig in general (see [8], Prop. 3.11):

Theorem 2.4. *(Bernstein, Lusztig) Let $Z(\mathcal{H})$ denote the center of $\mathcal{H}$. Then $Z(\mathcal{H})$ is the free $\mathbb{Z}'$-module with basis z_M, where M runs over the W_0-orbits in X_*.*

Define for each $s \in S_a$ an indeterminate $Q_s = v^{-L(s)} - v^{L(s)}$. With the normalizations above, the usual relations in $\mathcal{H}$ can be written simply as

$$\tilde{T}_s \tilde{T}_w = \begin{cases} \tilde{T}_{sw}, & \text{if } l(sw) = l(w) + 1, \\ -Q_s \tilde{T}_w + \tilde{T}_{sw}, & \text{if } l(sw) = l(w) - 1, \end{cases}$$

if $w \in \widetilde{W}$ and $s \in S_a$ (and similarly for $\tilde{T}_w \tilde{T}_s$). In particular, $\tilde{T}_s^{-1} = \tilde{T}_s + Q_s$.

For any $y \in \widetilde{W}$, choose a reduced expression $y = s_1 \cdots s_r \tau$, $(s_i \in S_a,\ \tau \in \Omega)$. Then for any $x \in \widetilde{W}$ we can define a polynomial expression $\tilde{R}_{x,y}(Q_S)$ in variables $Q_S = \{Q_s,\ s \in S_a\}$, by the formula

$$\tilde{T}_{y^{-1}}^{-1} = \sum_{x \in \widetilde{W}} \tilde{R}_{x,y}(Q_S)\tilde{T}_x. \tag{1}$$

The coefficient of $\tilde{T}_x$ appearing in the above expression is *a priori* just an element of the ring $\mathbb{Z}' = \mathbb{Z}[v, v^{-1}]$. However it is clear that we can also think of it as a polynomial expression in the indeterminates Q_s because of the identity

$$\tilde{T}_{y^{-1}}^{-1} = (\tilde{T}_{s_1} + Q_{s_1}) \cdots (\tilde{T}_{s_r} + Q_{s_r})\tilde{T}_\tau. \tag{2}$$

The expression $\tilde{R}_{x,y}(Q_S)$, viewed as an element of $\mathbb{Z}'$, does not depend on the choice of reduced expression for y, although viewed formally as a polynomial in $|S_a|$ variables, it does depend on the choice of reduced expression. The fact that the indeterminates Q_s are not independent (indeed not even distinct) variables will not affect any of the arguments we make using them. It will only be necessary sometimes to verify that some polynomial expressions in Q_S are not zero in the ring $\mathbb{Z}'$.

These $\widetilde{R}$-functions are analogous to the R-polynomials introduced in [4]. These are defined in the context of an affine Hecke algebra with trivial parameters, meaning that we take the parameter set L given by $L(w) = l(w)$, $(w \in \widetilde{W})$. In this case set $v = q^{1/2}$ and $Q = q^{-1/2} - q^{1/2} = Q_s$, $\ (s \in S_a)$. Following [4] define $R_{x,y}(q)$ by the equation

$$T_{y^{-1}}^{-1} = \sum_x \varepsilon_x \varepsilon_y q^{-l(y)} R_{x,y}(q) T_x,$$

where $\varepsilon_x = (-1)^{l(x)}$. Then we have

$$\varepsilon_x \varepsilon_y R_{x,y}(q) = q^{(l(y)-l(x))/2} \widetilde{R}_{x,y}(Q).$$

It is easy to prove the following facts.

Lemma 2.5. *For x, $y \in \widetilde{W}$ and $s \in S_a$ we have*

1. $sx < x,\ sy < y \Rightarrow \widetilde{R}_{x,y}(Q_S) = \widetilde{R}_{sx,sy}(Q_S)$,
2. $x < sx,\ sy < y \Rightarrow \widetilde{R}_{x,y}(Q_S) = Q_s\widetilde{R}_{x,sy}(Q_S) + \widetilde{R}_{sx,sy}(Q_S)$,
3. $\widetilde{R}_{x,y}(Q_S) \in \mathbb{Z}_+[Q_S]$,
4. $deg_{Q_S}\widetilde{R}_{x,y}(Q_S) = l(y) - l(x)$ *if* $x \leq y$,
5. $\widetilde{R}_{x,y}(Q_S) \neq 0$ *in* $\mathbb{Z}' \iff x \leq y$.

Proof. The first two statements are consequences of the definition, and these immediately imply (3) and (4) by induction on $l(y)$. Finally (5) ($\Rightarrow$) is easy using the equation (2) above. It remains to prove (5) ($\Leftarrow$). This is a consequence of (4) and the fact that no nontrivial polynomial expression in the indeterminates Q_s with nonnegative integer coefficients can be 0 in $\mathbb{Z}'$ (multiply by a sufficiently high power of v to get a polynomial in v, and note that the leading coefficient is nonzero). $\square$

We write $u < (v, w) < z$ for $u, v, w, z \in \widetilde{W}$ when $u < v < z$ and $u < w < z$ (in the Bruhat order). The first two statements of the lemma easily yield the following:

Corollary 2.6. *For x, $z \in \widetilde{W}$ and $s \in S_a$ we have*

1. $sxs < (sx, xs) < x,\ \ zs < (z, szs) < sz \Longrightarrow \widetilde{R}_{x,z}(Q_S) = \widetilde{R}_{sxs,szs}(Q_S) - Q_s\widetilde{R}_{xs,szs}(Q_S)$,

2. $xs < (x, sxs) < sx, \quad zs < (z, szs) < sz \Longrightarrow \widetilde{R}_{x,z}(Q_S) = \widetilde{R}_{sxs,szs}(Q_S).$

3. Computing Bernstein Functions for Minuscule Coweights

In this section we present an efficient method to compute the Bernstein function z_μ as a linear combination of the normalized basis elements $\tilde{T}_w$ in the case where μ is *minuscule*, i.e., $\langle \alpha, \mu \rangle \in \{-1, 0, 1\}, \ \forall \alpha \in R$.

Throughout this section fix a dominant and minuscule coweight $\mu \in X_*$. Let λ denote an element in the W_0-orbit of μ. Let μ^- denote the unique antidominant coweight in $W_0(\mu)$. Because our root system is reduced we have $\lambda \preceq \mu$ and so we can write $\mu - \lambda = \sum_{i=1}^{p} \check{\alpha}_i$, where α_i ranges over a subset of the simple roots (possibly with repeats). In the case $\lambda = \mu^-$ we see that $p = l(a^\mu)$ (let 2ρ be the sum of the positive roots, apply $\langle 2\rho, \cdot \rangle$ to both sides, and use $\langle 2\rho, \check{\alpha}_i \rangle = 2$). Thus for general λ we have $p \le l(a^\mu)$.

In what follows we will write s_i in place of s_{α_i}.

Lemma 3.1. *For μ and λ as above, there exists a sequence of simple roots $\alpha_1, \alpha_2, \ldots, \alpha_p$ such that*

$$\begin{aligned} s_1(\mu) &= \mu - \check{\alpha}_1, \\ s_2 s_1(\mu) &= \mu - \check{\alpha}_1 - \check{\alpha}_2, \\ &\cdot \\ &\cdot \\ \lambda = s_p \cdots s_1(\mu) &= \mu - \check{\alpha}_1 - \cdots - \check{\alpha}_p. \end{aligned}$$

Proof. Use induction on p. If $p = 0$, then $\lambda = \mu$ and there is nothing to prove. If $p = 1$, write $\lambda = \mu - \check{\alpha}_1$. Then λ is not dominant, so there exists $\alpha \in \Pi$ such that $\langle \alpha, \lambda \rangle = -1$ (since λ is minuscule). Thus $s_\alpha(\lambda) = \lambda + \check{\alpha}$ and

$$\check{\alpha}_1 - \check{\alpha} = \mu - s_\alpha(\lambda).$$

Because the right hand side is either 0 or a sum of positive coroots and α_1 is a simple root we must have $\alpha_1 = \alpha$ and $\mu = s_{\alpha_1}(\lambda)$. It follows that $\lambda = s_1(\mu) = \mu - \check{\alpha}_1$ as desired.

Now suppose that $p > 1$ and the result holds for $p - 1$. Since $\mu - \lambda$ is a sum of p simple coroots, λ is not dominant, so there exists $\alpha \in \Pi$ such that $\langle \alpha, \lambda \rangle = -1$. Thus $s_\alpha(\lambda) = \lambda + \check{\alpha}$ and

$$\mu - \lambda = (\mu - s_\alpha(\lambda)) + \check{\alpha}.$$

It follows that $\mu - s_\alpha(\lambda)$ is a sum of $p-1$ simple coroots, so the induction hypothesis applied to $s_\alpha(\lambda)$ yields a sequence $\alpha_1, \ldots, \alpha_{p-1}$ such that

$$\begin{aligned} s_1(\mu) &= \mu - \check{\alpha}_1, \\ &\cdot \\ &\cdot \\ s_\alpha(\lambda) = s_{p-1} \cdots s_1(\mu) &= \mu - \check{\alpha}_1 - \cdots - \check{\alpha}_{p-1}. \end{aligned}$$

Now taking $\alpha_p = \alpha$ easily yields the desired result. $\square$

Lemma 3.2. *Let $x \in X_*$ be a (nonzero) coweight for which there exists a sequence of simple roots $\alpha_1, \dots, \alpha_p$ $(0 \leq p \leq r = l(a^x))$ such that*

$$s_1(x) = x - \check{\alpha}_1,$$
$$\cdot$$
$$\cdot$$
$$s_p \cdots s_1(x) = x - \check{\alpha}_1 - \cdots - \check{\alpha}_p.$$

Then there exists a reduced expression for a^x of the form

$$a^x = t_1 \cdots t_{r-p}({}^\tau s_p) \cdots ({}^\tau s_1)\tau,$$

where $\tau \in \Omega$ is such that $a^x \in W_a\tau$, and $t_j \in S_a$, for $1 \leq j \leq r-p$.

Proof. We use induction on p. If $p = 1$, then $\langle \alpha_1, x\rangle = 1$, so that $l(a^x s_1) < l(a^x)$. Then the Exchange property of the Coxeter group (W_a, S_a) shows that $a^x\tau^{-1}$ has a reduced expression ending with ${}^\tau s_1$, as desired.

Now suppose $p > 1$ and the result holds for $p-1$. By the induction hypothesis $s_1 a^x s_1 = a^{s_1(x)}$ has a reduced expression of the form

$$s_1 a^x s_1 = t_1 \cdots t_{r-p+1}({}^\tau s_p) \cdots ({}^\tau s_2)\tau,$$

and thus

$$a^x = s_1 t_1 \cdots t_{r-p+1}({}^\tau s_p) \cdots ({}^\tau s_2)({}^\tau s_1)\tau.$$

This last expression becomes reduced upon omitting exactly two letters. We must omit the first letter s_1, for otherwise $s_1 a^x < a^x$, contrary to $\langle \alpha_1, x\rangle = 1 > 0$. If the other omitted letter is one of the t_j's, we are done. We need to show therefore that the other omitted letter cannot be one of the ${}^\tau s_i$'s. Suppose it were. Then

$$a^x = t_1 \cdots t_{r-p+1}({}^\tau s_p) \cdots (\widehat{{}^\tau s_i}) \cdots ({}^\tau s_1)\tau.$$

Comparing with the expression for $a^{s_1(x)}$ above it is easy to see

$$a^{s_1(x)} = a^x . s_1 \cdots s_i s_{i-1} \cdots s_2 \in X_* W_0.$$

Since the s_l terms are in W_0, this implies $s_1(x) = x$, a contradiction. □

We need the following lemma, which is due to Bernstein and Lusztig:

Lemma 3.3. *(Bernstein, Lusztig) Let $x \in X_*$, let $\alpha \in \Pi$, and write $s = s_\alpha$. Suppose that $\langle \alpha, x\rangle = 1$. Then*

$$\tilde{T}_s^{-1}\Theta_x \tilde{T}_s^{-1} = \Theta_{s(x)}.$$

Proof. This is a consequence of (a dual version of) Proposition 3.6 of [8] when the parameter set $L : S_a \to \mathbb{N}$ is arbitrary (noting that $\langle \alpha, x\rangle = 1 \Rightarrow \alpha \notin 2X^*$). It is due to Bernstein (see Lemma 4.4, [7]) in the case where $L(s) = 1$, $\forall s \in S_a$. □

Proposition 3.4. *Let μ be a dominant and minuscule coweight, and let $\tau \in \Omega$ be the unique element such that $a^\mu \in W_a\tau$. Let $\lambda \in W_0(\mu)$. Suppose $\mu - \lambda$ is a sum of p simple coroots $(0 \leq p \leq l(a^\mu) = r)$. Then there exists a sequence of simple roots $\alpha_1, \dots, \alpha_p$ such that the following hold (setting $s_i = s_{\alpha_i}$):*

1. $\langle\, \alpha_i \;,\; s_{i-1} \cdots s_1(\mu)\,\rangle = 1, \;\; \forall 1 \leq i \leq p,$
2. *There is a reduced expression for a^μ of the form $a^\mu = t_1 \cdots t_{r-p}({}^\tau s_p) \cdots ({}^\tau s_1)\tau$,*
3. *There is a reduced expression for a^λ of the form $a^\lambda = s_p \cdots s_1 t_1 \cdots t_{r-p}\tau$,*

4. $\Theta_\lambda = \tilde{T}_{s_p}^{-1} \cdots \tilde{T}_{s_1}^{-1} \tilde{T}_{t_1} \cdots \tilde{T}_{t_{r-p}} \tilde{T}_\tau$,

where $t_j \in S_a, \ \forall j \in \{1, 2, \ldots, r-p\}$.

Proof. Lemma 3.1 and 3.2 applied to μ and λ immediately imply the existence of a sequence $\alpha_1, \ldots, \alpha_p$ of simple roots such that (1) and (2) hold. But (3) follows from (2) and the fact that $s_p \cdots s_1(\mu) = \lambda$. Note that (1) implies that the hypotheses of Lemma 3.3 are satisfied for $x = s_{i-1} \cdots s_1(\mu)$ and $\alpha = \alpha_i, \ \forall i$. Therefore starting with the expression

$$\Theta_\mu = \tilde{T}_{t_1} \cdots \tilde{T}_{t_{r-p}} \tilde{T}_{^\tau s_p} \cdots \tilde{T}_{^\tau s_1} \tilde{T}_\tau$$

and applying Lemma 3.3 repeatedly yields (4). □

Corollary 3.5. *With the hypotheses above,* $a^{\mu^-} \in W_0 \tau$.

Proof. Take $\lambda = \mu^-$, so that $p = r$. Then the result is obvious from the reduced expression for a^{μ^-} given in the Proposition. □

In practice the sequence $\alpha_1, \ldots, \alpha_p$ for any λ is easily computed. The expansion of the resulting expression for Θ_λ in terms of the basis $\{\tilde{T}_w, \ w \in \widetilde{W}\}$ is also a straightforward matter, using the relation $\tilde{T}_s^{-1} = \tilde{T}_s + Q_s$. Summing over λ then yields the Bernstein function z_μ as a linear combination of the elements $\tilde{T}_w$. One can then use this expression to write z_μ in terms of the usual basis elements T_w.

Example: *The Bernstein function for* $G = GSp_6, \ \mu = (1, 1, 1, 0, 0, 0)$.

Identifying $X_* \otimes \mathbb{R} = \mathbb{R}^6$ with $X^* \otimes \mathbb{R}$ using the standard inner product, we can write the simple roots and coroots as follows: $\alpha_1 = (1/2, -1/2, 0, 0, 1/2, -1/2)$, $\check{\alpha}_1 = (1, -1, 0, 0, 1, -1)$ $\alpha_2 = (0, 1/2, -1/2, 1/2, -1/2, 0)$, $\check{\alpha}_2 = (0, 1, -1, 1, -1, 0)$, $\alpha_3 = \check{\alpha}_3 = (0, 0, 1, -1, 0, 0)$. Write s_i instead of s_{α_i}, and let $s_0 = (1, 0, 0, 0, 0, -1)(1\ 6)$ denote the simple affine reflection.

Note that $a^\mu = s_0 s_1 s_0 s_2 s_1 s_0 \tau$ is a reduced expression, where $\tau \in \Omega$ permutes the simple affine roots by interchanging the subscripts $0 \leftrightarrow 3$ and $1 \leftrightarrow 2$.

Consider the sequence of simple roots $\alpha_3, \alpha_2, \alpha_1, \alpha_3, \alpha_2, \alpha_3$. This satisfies (1) in Proposition 3.4, and we see that

$$\begin{aligned}
\Theta_{(1,1,1,0,0,0)} &= \tilde{T}_0 \tilde{T}_1 \tilde{T}_0 \tilde{T}_2 \tilde{T}_1 \tilde{T}_0 \tilde{T}_\tau, \\
\Theta_{(1,1,0,1,0,0)} &= \tilde{T}_3^{-1} \tilde{T}_0 \tilde{T}_1 \tilde{T}_0 \tilde{T}_2 \tilde{T}_1 \tilde{T}_\tau, \\
\Theta_{(1,0,1,0,1,0)} &= \tilde{T}_2^{-1} \tilde{T}_3^{-1} \tilde{T}_0 \tilde{T}_1 \tilde{T}_0 \tilde{T}_2 \tilde{T}_\tau, \\
\Theta_{(0,1,1,0,0,1)} &= \tilde{T}_1^{-1} \tilde{T}_2^{-1} \tilde{T}_3^{-1} \tilde{T}_0 \tilde{T}_1 \tilde{T}_0 \tilde{T}_\tau, \\
\Theta_{(0,1,0,1,0,1)} &= \tilde{T}_3^{-1} \tilde{T}_1^{-1} \tilde{T}_2^{-1} \tilde{T}_3^{-1} \tilde{T}_0 \tilde{T}_1 \tilde{T}_\tau, \\
\Theta_{(0,0,1,0,1,1)} &= \tilde{T}_2^{-1} \tilde{T}_3^{-1} \tilde{T}_1^{-1} \tilde{T}_2^{-1} \tilde{T}_3^{-1} \tilde{T}_0 \tilde{T}_\tau, \\
\Theta_{(0,0,0,1,1,1)} &= \tilde{T}_3^{-1} \tilde{T}_2^{-1} \tilde{T}_3^{-1} \tilde{T}_1^{-1} \tilde{T}_2^{-1} \tilde{T}_3^{-1} \tilde{T}_\tau,
\end{aligned}$$

where we write $\widetilde{T}_i$ instead of $\widetilde{T}_{s_i}$ for every $s_i \in S_a$. The missing term $\Theta_{(1,0,0,1,1,0)}$ can be computed using a slightly different sequence. Let $z_\mu(x)$ denote the coefficient of T_x in the expression for z_μ. We find that $z_\mu(x)$ is 0 unless x is in the μ-admissible set (a set with 79 elements) and that for μ-admissible x the coefficient is as follows:

$$q^{l(\mu)/2}z_\mu(x) = \begin{cases} (1-q)^2(1-q+q^2-q^3+q^4), & \text{if } l(x)=0,\\ (1-q)^3(1+q^2), & \text{if } l(x)=1,\\ (1-q)^2(1-q+q^2), & \text{if } l(x)=2,\\ (1-q)(1-q+q^2), & \text{if } x\in S_1,\\ (1-q)^3, & \text{if } x\in S_2,\\ (1-q)^{l(\mu)-l(x)}, & \text{if } l(x)>3. \end{cases}$$

Here S_1 and S_2 are the following sets of μ-admissible elements of length three:

$$S_1 = \{s_{321}\tau, s_{232}\tau, s_{123}\tau, s_{210}\tau, s_{101}\tau, s_{012}\tau\},$$
$$S_2 = \{s_{323}\tau, s_{312}\tau, s_{212}\tau, s_{213}\tau, s_{230}\tau, s_{310}\tau, s_{320}\tau, s_{120}\tau, s_{301}\tau, s_{201}\tau, s_{101}\tau\},$$

where the symbol s_{ijk} stands for the product $s_is_js_k$.

4. A Formula for Bernstein Functions in the Minuscule Case

In this section we present a formula for z_μ in the case where μ is minuscule. First we need some preparation.

For any $x \in \widetilde{W}$, there is a unique expression $x = w.a^{t(x)}$, where $w \in W_0$ and $t(x) \in X_*$. It is obvious that if $\alpha \in \Pi$ and $s = s_\alpha$, then $t(sx) = t(x)$ and $t(xs) = t(sxs) = s(t(x))$. We need the following further properties of $t(x)$:

Lemma 4.1. *Let $x = w.a^{t(x)} \in \widetilde{W}$ as above and let $s = s_\alpha$ for $\alpha \in \Pi$. Then*

1. $xs < x \Leftrightarrow \langle\alpha, t(x)\rangle > 0$ *or* $(\langle\alpha, t(x)\rangle = 0 \;\;\&\;\; ws < w)$,
2. $x < xs \Leftrightarrow \langle\alpha, t(x)\rangle < 0$ *or* $(\langle\alpha, t(x)\rangle = 0 \;\;\&\;\; w < ws)$,
3. $sx < sxs \Leftrightarrow \langle\alpha, t(x)\rangle < 0$ *or* $(\langle\alpha, t(x)\rangle = 0 \;\;\&\;\; sw < sws)$,
4. $sxs < sx \Leftrightarrow \langle\alpha, t(x)\rangle > 0$ *or* $(\langle\alpha, t(x)\rangle = 0 \;\;\&\;\; sws < sw)$.

Proof. This can be deduced from Proposition 1.28 of [3]. Alternatively, one notes that in the notation of §2.1, if $A = (\beta, k) \in \widetilde{R}^+$ and $y \in \widetilde{W}$, we have $y^{-1}(\beta, k) \in \widetilde{R}^- \Leftrightarrow s_Ay \le y$. Apply this with $A = (\alpha, 0)$ and $y = x^{-1}$ and use the definition of the action of $\widetilde{W}$ on $\widetilde{R}$ given in §2.1. $\square$

Lemma 4.2. *Let μ be dominant and minuscule. Then any μ-admissible element x has the property that $t(x) \in W_0(\mu)$.*

Proof. Recall that the Bruhat ordering on $\widetilde{W}$ descends to give an order on $W_0\backslash\widetilde{W}/W_0$ and that for λ and μ dominant we have

$$W_0a^\lambda W_0 \le W_0a^\mu W_0 \Longleftrightarrow \lambda \preceq \mu.$$

Now write $t(x) = w(\lambda)$, where λ is dominant and $w \in W_0$. Then using the hypothesis on x we have $W_0a^\lambda W_0 = W_0a^{t(x)}W_0 = W_0xW_0 \le W_0a^\mu W_0$, so that $\lambda \preceq \mu$. But μ (being minuscule) is minimal with respect to $\preceq$, and so $\lambda = \mu$. Therefore $t(x) = w(\mu) \in W_0(\mu)$. $\square$

Theorem 4.3. *Let μ be a dominant minuscule coweight. Then*

$$z_\mu = \sum_{x\,:\,x \text{ is } \mu-adm.} \widetilde{R}_{x,a^{t(x)}}(Q_S)\tilde{T}_x.$$

This will follow immediately if we establish that for $\lambda \in W_0(\mu)$, we have

$$\Theta_\lambda = \sum_x \widetilde{R}_{x,a^\lambda}(Q_S)\tilde{T}_x,$$

where x runs over elements in the μ-admissible set such that $t(x) = \lambda$ (using Lemma 4.2, which asserts that each $t(x)$ must be equal to some $\lambda \in W_0(\mu)$). But the only elements x giving a nonzero contribution to the above sum are those for which $x \leq a^\lambda$ (by Lemma 2.5 (5)), and so the condition that x be μ-admissible is redundant. Therefore we need to prove the following proposition:

Proposition 4.4. *Let μ be minuscule and dominant and suppose $\lambda \in W_0(\mu)$. Then*

$$\Theta_\lambda = \sum_{x\,:\,t(x)=\lambda} \widetilde{R}_{x,a^\lambda}(Q_S)\tilde{T}_x.$$

Proof. Suppose that $\mu - \lambda$ is a sum of p simple coroots ($0 \leq p \leq r = l(a^\mu)$). We proceed by downwards induction on p. First assume $p = r$. Then $\lambda = \mu^-$ and in view of $\Theta_{\mu^-} = \tilde{T}^{-1}_{a^{-\mu^-}}$ (since $-\mu^-$ is dominant) and equation (1), the statement to be proved is

$$\sum_x \widetilde{R}_{x,a^{\mu^-}}(Q_S)\tilde{T}_x = \sum_{x\,:\,t(x)=\mu^-} \widetilde{R}_{x,a^{\mu^-}}(Q_S)\tilde{T}_x.$$

Taking into account Lemma 2.5 (5) we need only show that $x \leq a^{\mu^-} \Rightarrow t(x) = \mu^-$. But this is clear because $a^{\mu^-} \in W_0\tau$ (Corollary 3.5) implies that $x \in W_0\tau = W_0a^{\mu^-}$, and so $t(x) = \mu^-$.

Now assume $p < r$ and the result holds for $p+1$. Since λ is not antidominant, there exists $s = s_\alpha$ ($\alpha \in \Pi$) such that $\langle \alpha, \lambda \rangle = 1$. Thus $s(\lambda) = \lambda - \breve{\alpha} \prec \lambda$ and hence the induction hypothesis applies to $s(\lambda)$; we multiply the resulting equality for $s(\lambda)$ on both sides by $\tilde{T}_s$ to get

$$\tilde{T}_s\Theta_{s(\lambda)}\tilde{T}_s = \sum_{y\,:\,t(y)=s(\lambda)} \widetilde{R}_{y,a^{s(\lambda)}}(Q_S)\tilde{T}_s\tilde{T}_y\tilde{T}_s.$$

Since $\langle \alpha, \lambda \rangle = 1$, Lemma 3.3 implies that the left hand side is Θ_λ. Therefore we must show that the right hand side is

$$\sum_{x\,:\,t(x)=\lambda} \widetilde{R}_{x,a^\lambda}(Q_S)\tilde{T}_x.$$

Now for every x such that $t(x) = \lambda$ we have $\langle \alpha, t(x) \rangle = \langle \alpha, \lambda \rangle = 1$, hence by Lemma 4.1 we have $xs < x$ and $sxs < sx$, so either $xs < (x, sxs) < sx$, or $sxs < (xs, sx) < x$. Similarly, for y such that $t(y) = s(\lambda)$ we have $\langle \alpha, t(y) \rangle = \langle \alpha, s(\lambda) \rangle = -1$, hence by Lemma 4.1 we have $y < ys$ and $sy < sys$, so either $y < (ys, sy) < sys$, or $sy < (y, sys) < ys$. Furthermore note that for such y we have

$$\tilde{T}_s\tilde{T}_y\tilde{T}_s = \begin{cases} \tilde{T}_{sys}, & \text{if } y < (ys, sy) < sys, \\ -Q_s\tilde{T}_{ys} + \tilde{T}_{sys}, & \text{if } sy < (y, sys) < ys. \end{cases}$$

Therefore we need to prove that the expression

(3)
$$\sum_{\substack{y\,:\,t(y)=s(\lambda) \\ y<(ys,sy)<sys}} \widetilde{R}_{y,a^{s(\lambda)}}(Q_S)\tilde{T}_{sys} + \sum_{\substack{y\,:\,t(y)=s(\lambda) \\ sy<(y,sys)<ys}} \widetilde{R}_{y,a^{s(\lambda)}}(Q_S)(-Q_s\tilde{T}_{ys}) + \sum_{\substack{y\,:\,t(y)=s(\lambda) \\ sy<(y,sys)<ys}} \widetilde{R}_{y,a^{s(\lambda)}}(Q_S)\tilde{T}_{sys}$$

is equal to

$$\sum_{\substack{x\,:\,t(x)=\lambda \\ sxs<(xs,sx)<x}} \widetilde{R}_{x,a^\lambda}(Q_S)\tilde{T}_x + \sum_{\substack{x\,:\,t(x)=\lambda \\ xs<(x,sxs)<sx}} \widetilde{R}_{x,a^\lambda}(Q_S)\tilde{T}_x. \tag{4}$$

In the first and third sums in (3), replace y with sxs. In the second sum replace y with xs. Then (3) becomes

(5)
$$\sum_{\substack{x\,:\,t(x)=\lambda \\ sxs<(sx,xs)<x}} \widetilde{R}_{sxs,a^{s(\lambda)}}(Q_S)\tilde{T}_x + \sum_{\substack{x\,:\,t(x)=\lambda \\ sxs<(xs,sx)<x}} \widetilde{R}_{xs,a^{s(\lambda)}}(Q_S)(-Q_s\tilde{T}_x) + \sum_{\substack{x\,:\,t(x)=\lambda \\ xs<(sxs,x)<sx}} \widetilde{R}_{sxs,a^{s(\lambda)}}(Q_S)\tilde{T}_x$$

Note that $\langle\alpha,\lambda\rangle = 1$ implies a^λ satisfies $a^\lambda s < (a^\lambda, sa^\lambda s) < sa^\lambda$, by Lemma 4.1. Furthermore $a^{s(\lambda)} = sa^\lambda s$. Thus Corollary 2.6 with $z = a^\lambda$ shows that (5) is indeed (4). The proposition follows, and thus Theorem 4.3 is proved. □

Corollary 4.5. *Let μ be dominant and minuscule, let $s \in S_a$, $x \in \widetilde{W}$, and $\tau \in \Omega$. Then*

1. *If $l(sxs) = l(x)$, and x is μ-admissible, then $\widetilde{R}_{x,a^{t(x)}}(Q_S) = \widetilde{R}_{sxs,a^{t(sxs)}}(Q_S)$,*
2. *If $l(sxs) = l(x) - 2$ and x is μ-admissible, then $\widetilde{R}_{x,a^{t(x)}}(Q_S) = \widetilde{R}_{sxs,a^{t(sxs)}}(Q_S) - Q_s\widetilde{R}_{xs,a^{t(xs)}}(Q_S)$,*
3. *If x is μ-admissible, then $\widetilde{R}_{x,a^{t(x)}}(Q_S) = \widetilde{R}_{\tau x\tau^{-1},a^{t(\tau x\tau^{-1})}}(Q_S)$.*

Proof. These follow directly from the conditions on central elements of $\mathcal{H}$ proved in §3 of [2]. □

For $\phi = \sum_x a_x(Q_S)\tilde{T}_x \in \mathcal{H}$ define $supp(\phi) = \{x \mid a_x(Q_S) \neq 0\}$. It is obvious from Theorem 4.3 that $supp(z_\mu)$ is a subset of the μ-admissible set. We can now prove that these sets are in fact equal in this case.

Proposition 4.6. *Let μ be minuscule and dominant. Then*

$$supp(z_\mu) = \{x \in \widetilde{W} \mid x \text{ is } \mu\text{-admissible}\}.$$

Proof. The left hand side is clearly contained in the right hand side, by Theorem 4.3. To prove the other inclusion it is enough to prove (by Lemma 2.5 (5)) that if x is μ-admissible, then $x \leq a^{t(x)}$. By Lemma 4.2 it is enough to prove, for every $\lambda \in W_0(\mu)$, the statement

$Hyp(\lambda)$: $\quad x$ is μ-admissible and $t(x) = \lambda \Rightarrow x \leq a^\lambda$.

Suppose $\mu - \lambda$ is a sum of p simple coroots. We prove the statement $Hyp(\lambda)$ by induction on p.

Suppose first that $p = 0$. Then $\lambda = \mu$ and it is enough to show that if x is μ-admissible and $t(x) = \mu$, then $x = a^\mu$. Write $x = wa^\mu$ for $w \in W_0$. Then $l(x) = l(w) + l(a^\mu)$ (see §2.1) and so x can be μ-admissible only if $w = 1$, i.e., $x = a^\mu$.

Now suppose that $p > 0$ and that $Hyp(\lambda')$ is true for $p - 1$. Since λ is not dominant there exists $s = s_\alpha$ ($\alpha \in \Pi$) such that $\langle\alpha,\lambda\rangle = -1$. Now suppose x is μ-admissible and $t(x) = \lambda$. Then $\langle\alpha, t(x)\rangle < 0$ so Lemma 4.1 implies $x < xs$ and $sx < sxs$, so that either (I) $x < (xs, sx) < sxs$, or (II) $sx < (x, sxs) < xs$.

Consider case (I). It follows from Corollary 4.6 of [2] that sx and xs are both μ-admissible. But also $t(xs) = s(\lambda) = \lambda + \check{\alpha} \succ \lambda$, so the induction hypothesis applied to $s(\lambda)$ and xs shows that $xs \leq a^{t(xs)}$ and so $\widetilde{R}_{xs,a^{t(xs)}}(Q_S) \neq 0$ (Lemma 2.5 (5)). On the other hand, Corollary 4.5 (1) above (with xs for x) then implies $\widetilde{R}_{sx,a^{t(sx)}}(Q_S) \neq 0$, so again using Lemma 2.5 (5) we see $x < sx \leq a^{t(sx)} = a^{t(x)} = a^{\lambda}$, as desired.

Finally consider (II). Since $l(sxs) = l(x)$, sxs is μ-admissible (see Lemma 4.5 of [2]). Since $t(sxs) = s(\lambda)$ the induction hypothesis applied to $s(\lambda)$ and sxs yields $sxs \leq a^{s(\lambda)} = a^{t(sxs)}$. The same argument as in Case (I) applies (using Corollary 4.5 (1) and Lemma 2.5 (5)) to give first $\widetilde{R}_{x,a^{t(x)}}(Q_S) \neq 0$ and then $x \leq a^{t(x)} = a^{\lambda}$ as desired. □

The following answers a question of Rapoport affirmatively.

Proposition 4.7. *If λ' and λ are distinct elements of $W_0(\mu)$, then $supp(\Theta_{\lambda'}) \cap supp(\Theta_{\lambda}) = \emptyset$.*

Proof. It follows from Proposition 4.4 that $supp(\Theta_{\lambda}) \subset \{x \mid t(x) = \lambda\}$ for any $\lambda \in W_0(\mu)$. This immediately implies the result. □

Remark 4.8. One can show using the example of Gl_3, $\mu = (2,1,0)$ that this "Disjointness Property" is not true if one removes the hypothesis that μ be minuscule.

5. The Bernstein Function in the Drinfeld Case

In this section we explain how to use Theorem 4.3 to prove Theorem 1.2. Roughly speaking the method is to compare the formula in Theorem 4.3 for GL_d and $\mu = (1, 0^{d-1})$ with Rapoport's formula for the trace of Frobenius on nearby cycles in the Drinfeld case (see Proposition 5.1).

First we recall the setup. Fix a positive integer $d > 2$ and a prime number p. Let E be an imaginary quadratic extension of $\mathbb{Q}$ such that p decomposes as $\mathfrak{p}_1\mathfrak{p}_2$ in $\mathcal{O}_E$. We fix embeddings $\sigma : E \hookrightarrow \overline{\mathbb{Q}} \subset \mathbb{C}$ and $\phi : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ such that $\phi \circ \sigma$ determines the place $\mathfrak{p}_1$ of E. Let $(D, *)$ be a central simple algebra over E of dimension d^2 together with a positive involution * which induces the nontrivial automorphism on E. (Thus $*$ is necessarily of the second type since E is imaginary). Fix an isomporphism $D \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\sim} M_d(\mathbb{C})$ such that * is carried over to the standard involution $A \mapsto \overline{A}^t$ on $M_d(\mathbb{C})$. Let G be the $\mathbb{Q}$-group whose points in any commutative $\mathbb{Q}$-algebra R are given by

$$G(R) = \{x \in D \otimes_{\mathbb{Q}} R \mid xx^* \in R^{\times}\}.$$

Choose $i = \sqrt{-1} \in \mathbb{C}$ and let $h_0 : \mathbb{C} \to D \otimes_{\mathbb{Q}} \mathbb{R} = M_d(\mathbb{C})$ be given by

$$h_0(a + ib) = \mathrm{diag}(a + ib, a - ib, \dots, a - ib).$$

The restriction of h_0 to $\mathbb{C}^{\times}$ gives a homomorphism

$$h : R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m) \to G_{\mathbb{R}}.$$

Let X denote the $G(\mathbb{R})$-conjugacy class of h. Let $\mathcal{O}_D$ be a $*$-stable order in D which is a maximal order at p. Let $K = K^p K_p$ be a compact open subgroup which leaves $\mathcal{O}_D \otimes \hat{\mathbb{Z}}$ invariant (acting by multiplication on the right), where K^p is a sufficiently small subgroup of $G(\mathbb{A}_f^p)$ and where K_p is an Iwahori subgroup of $G(\mathbb{Q}_p)$. Note that if $D_{\mathfrak{p}_i}$ is a division algebra ($i = 1, 2$), then K_p is the unique maximal compact subgroup. The triple (G, X, K) is a Shimura datum with reflex field E (see [6]) giving rise to a quasi-projective scheme S_K over $\mathcal{O}_{E,\mathfrak{p}_1}$ (see [1]). The homomorphism h gives rise to a $G(\overline{\mathbb{Q}})$-conjugacy class $\{\mu\}$ of cocharacters $\mu : (\mathbb{G}_m)_{\overline{\mathbb{Q}}} \to G_{\overline{\mathbb{Q}}}$, and $\{\mu\}$ is defined over the field E.

Next we want to study the group G at the prime p. We will henceforth denote the group $G_{\mathbb{Q}_p}$ simply by G. Moreover from now on we will view $\{\mu\}$ as a $G(\overline{\mathbb{Q}}_p)$-conjugacy class of cocharacters of $G_{\overline{\mathbb{Q}}_p}$, via the choice of embedding $\phi : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ made above. Note that $D \otimes \mathbb{Q}_p = D_{\mathfrak{p}_1} \times D_{\mathfrak{p}_2}$ and that $D_{\mathfrak{p}_1} \xrightarrow{\sim} D_{\mathfrak{p}_2}^{\mathrm{op}}$ via $*$. For any algebra R over the field $\mathbb{Q}_p = E_{\mathfrak{p}_1}$ we can therefore identify $G(R)$ with the group

$$\{(x_1, x_2) \in (D_{\mathfrak{p}_1} \otimes R)^\times \times (D_{\mathfrak{p}_2} \otimes R)^\times \mid x_1 = cx_2^{-1}, \text{ for some } c \in R^\times\}.$$

Therefore there is an isomorphism of $\mathbb{Q}_p$-groups $G \cong D_{\mathfrak{p}_1}^\times \times \mathbb{G}_m$ given by $(x_1, x_2) \mapsto (x_1, c)$. Let H denote the $\mathbb{Q}_p$-group $D_{\mathfrak{p}_1}^\times$.

Now fix an unramified extension $F = \mathbb{Q}_{p^j}$ of $E_{\mathfrak{p}_1} = \mathbb{Q}_p$ such that $j \cdot (\mathrm{inv}(D_{\mathfrak{p}_1})) \equiv 0 \pmod d$. Then $H_F = Gl_d$ and $G_F = Gl_d \times \mathbb{G}_m$. Under these identifications we get isomorphisms on the level of affine Weyl groups: $\widetilde{W}(G_F) = \widetilde{W}(Gl_d) \times \widetilde{W}(\mathbb{G}_m)$, where $\widetilde{W}(Gl_d) = \mathbb{Z}^d \rtimes S_d$, $W_0(G_F) = S_d$, and $\widetilde{W}(\mathbb{G}_m) = X_*(\mathbb{G}_m)$. Furthermore we can identify the conjugacy class of μ with the S_d-orbit of $(((1, 0^{d-1}), 1), t) \in (\mathbb{Z}^d \rtimes S_d) \rtimes t^{\mathbb{Z}}$. Here t denotes the element $x \mapsto x$ in $X_*(\mathbb{G}_m)$. We will abuse notation and denote the element $(1, 0^{d-1})t$ by μ.

Let M^{loc} denote the local model attached to the datum (G, X, K) as in [10]. The following theorem follows from an explicit calculation of nearby cycles on M^{loc}:

Proposition 5.1. *(Rapoport) Let $q = p^j$ be such that $j \cdot (inv(D_{\mathfrak{p}_1})) \equiv 0$ (mod d). Let $y \in M^{loc}(\mathbb{F}_q)$. Then*

$$tr(Fr_q; R\Psi_y^{\mathcal{I}}(\overline{\mathbb{Q}}_l)) = (1-q)^{|S_y|-1},$$

where S_y denotes the set of strata of $M^{loc}_{\overline{\mathbb{F}}_p}$ that contain the point y.

Proof. For the case where D is a central *division* algebra over E, this is proved in Theorem 3.12 of [10]. The proof there works in the case we consider as well. □

As a consequence of Theorem 4.3 (or Theorem 1.1) we also have the following explicit formula for the Bernstein function for Gl_d and the coweight $(1, 0^{d-1})$.

Proposition 5.2. *Let $H = Gl_d(F)$ and $\nu = (1, 0^{d-1})$. Let z_ν denote the corresponding Bernstein function. For $x \in \widetilde{W}(H)$ let $z_\nu(x)$ denote the coefficient of T_x in the expression for element z_ν. Then*

$$q^{l(a^\nu)/2} z_\nu(x) = \begin{cases} 0, & \text{if } x \text{ is not } \nu\text{-admissible}, \\ (1-q)^{l(a^\nu)-l(x)}, & \text{if } x \text{ is } \nu\text{-admissible}. \end{cases}$$

Proof. Because $H = Gl_d$ is split, the parameters of the corresponding affine Hecke algebra are trivial: $L(s) = 1$, for every $s \in S_a$ (cf. §2). Therefore for each $s \in S_a$ we have $Q_s = Q$, where $Q = q^{-1/2} - q^{1/2}$. Now fix $x \in \widetilde{W}(H)$, which we assume is ν-admissible (the other case being trivial). Note that $l(a^{t(x)}) = l(a^\nu)$ by Lemma 4.2. Using the identity $q^{1/2}Q = 1 - q$ and recalling that $\widetilde{T}_x = q^{-l(x)/2}T_x$, we see by Theorem 4.3 that it is enough to show

$$Q^{l(a^{t(x)})-l(x)} = \widetilde{R}_{x,a^{t(x)}}(Q).$$

Now if one numbers the simple affine reflections for $W_a(Gl_d)$ in the standard way ($s_0 = (1, 0, \ldots, 0, -1)(1d)$, $s_1 = (12)$, $\ldots$, $s_{d-1} = (d-1, d)$), then it is easy to show that $a^\nu = s_0 s_{d-1} \cdots s_2 \tau$, where $\tau = (1, 0^{d-1})c \in \widetilde{W}(H)$ and $c = (12 \ldots d) \in S_d = W_0$. In particular the simple reflections in any reduced expression for a^ν are pairwise distinct. Since $\mathrm{Int}(\tau)$ acts transitively on $W_0(\nu)$, the same is true of $a^{t(x)} = t_1 \cdots t_{d-1}\tau$. We have

$$\begin{aligned} \tilde{T}^{-1}_{(a^{t(x)})^{-1}} &= \sum_{y \in \widetilde{W}} \tilde{R}_{y,a^{t(x)}}(Q)\tilde{T}_y \\ &= (\tilde{T}_{t_1} + Q) \cdots (\tilde{T}_{t_{d-1}} + Q)\tilde{T}_\tau. \end{aligned}$$

Because the t_j's are pairwise distinct, any two different subsets $\{j_1, \dots j_r\}$ and $\{j'_1, \dots j'_{r'}\}$ yield *distinct* elements $t_{j_1} \cdots t_{j_r}\tau$ and $t_{j'_1} \cdots t_{j'_{r'}}\tau$, and moreover these expressions are reduced. Taking into account these remarks, the proposition follows. □

We conclude with the proof of Kottwitz' conjecture in the Drinfeld case, for those $q = p^j$ where j is such that $j \cdot (\mathrm{inv}(D_{\mathfrak{p}_1})) \equiv 0 \,(\mathrm{mod}\ d)$.

Proof of Theorem 1.2: We first make the following remarks:

1. Let $F = \mathbb{Q}_{p^j}$. Then $G_F = Gl_d \times \mathbb{G}_m$ and the cocharacter $\mu = (1, 0^{d-1})t = \nu t$ of G_F gives rise via Bernstein's construction (§2) to the function $z_\mu = z_\nu \otimes T_t \in Z(\mathcal{H}(Gl_d)) \otimes \mathbb{Z}[T_t^{\pm}]$. Note that z_μ (resp. z_ν) can be viewed as an element of the Iwahori-Hecke algebra of $G(F)$ (resp. $Gl_d(F)$), by Remark 2.2. By definition of the Bruhat order (§2), an element $y \in \widetilde{W}(G_F)$ is in the μ-admissible set if and only if it is of the form $y = xt$, where x is an ν-admissible element of $\widetilde{W}(Gl_d)$. Furthermore we have $l(y) = l(x)$ and $z_\mu(y) = z_\nu(x)$ if $y = xt$.

2. The strata of M^{loc} in the Drinfeld case are indexed by elements of the form $y = xt \in \widetilde{W}(G_F) = \widetilde{W}(Gl_d) \rtimes t^{\mathbb{Z}}$, where y (resp. x) is μ-admissible (resp. ν-admissible).

3. If $y = xt$ is a μ-admissible, then

$$|S_y| - 1 = l(a^\mu) - l(y) = l(a^\nu) - l(x),$$

in the notation of Proposition 5.1.

The equality in Theorem 1.2 now follows easily by combining these three remarks with Propositions 5.1 and 5.2.

6. Acknowledgements

This paper was written during a visit to the Max-Planck-Institut fuer Mathematik in Bonn. I thank the Institut for their support and for providing an excellent working environment. I am especially grateful to Robert Kottwitz and Michael Rapoport for many interesting conversations concerning this material, and for asking thought-provoking questions. I am indebted to Michael Rapoport for explaining to me the generalization here (Proposition 5.1) of his calculations in [10]. The final version of this paper was written during a visit to the Mathematisches Institut at the University of Köln, which I thank for the hospitality and support.

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