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QUATERNIONIC TRANSFORMATIONS OF A NON-POSITIVE QUATERNIONIC-KÄHLER MANIFOLD*

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ABSTRACT. Let (M, g, Q) be a simply connected, complete, quaternionic Kähler manifold without flat de Rham factor. Then any 1-parameter group of transformations of M which preserve the quaternionic structure Q preserves also the metric g. Moreover, if (M, g) is irreducible then the quaternionic Kähler metric g on (M, Q) is unique up to a homothety.

1. Introduction.

Let Q be an *almost quaternionic structure* on a $4n$-dimensional manifold M, that is a 3-dimensional subbundle of the bundle of endomorphisms locally generated by three anticommuting almost complex structures $J_\alpha, \alpha = 1, 2, 3$, with $J_3 = J_1 J_2$. We will say that $H = (J_\alpha)$ is a (local) *admissible basis* for Q. Q is called a *quaternionic structure* if there exists a torsionless linear connection ∇ which preserves Q. Such connection ∇ (called a *quaternionic connection*) is not unique. Any other quaternionic connection ∇' can be written as

$$\nabla' = \nabla + S^\xi \tag{1}$$

where $\xi \in \Lambda^1 M$ is a 1-form and S^ξ is a $(1, 2)$-tensor given by

$$S^\xi_X = \xi(X) Id + X \otimes \xi - \sum_{\alpha=1}^{3} [\xi(J_\alpha X) J_\alpha + J_\alpha X \otimes (\xi \circ J_\alpha)] \qquad (X \in TM)$$

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where $H = (J_\alpha)$ is an admissible basis for Q (See [1]).

Definition 1. 1) A Riemannian metric g on a manifold M with a quaternionic structure Q is called *Q-Hermitian* if all endomorphisms $J \in Q$ are skew-symmetric with respect to g.

2) a Q-Hermitian metric g is called *Q-Kähler* if the Levi-Civita connection ∇^g is a quaternionic connection.

A triple (M, g, Q) is called a *quaternionic Hermitian* (resp., *quaternionic Kähler*) *manifold* if g is a Q-Hermitian (resp., Q-Kähler) metric.

We will assume that $\dim M = 4n > 4$. Then it is well known that any Q-Kähler metric g is Einstein and the curvature tensor R of g can be written as

$$R = \nu R_1 + W \tag{2}$$

where $\nu = \frac{K}{4n(n+2)}$ is the *reduced scalar curvature*, K is the scalar curvature, R_1 is the curvature tensor of the standard quaternionic Kähler metric of the quaternionic projective space $\mathbb{H}P^n$,

$$R_1(X, Y) = \frac{1}{4}[S_X^{g \circ Y} - S_Y^{g \circ X}] \qquad (X, Y \in TM)$$

and W is the *quaternionic Weyl tensor* which satisfies the conditions

$$\mathrm{Ric}(W) = 0 \quad , \quad [W(X, Y), J_\alpha] = 0 \quad (\alpha = 1, 2, 3) \qquad X, Y \in TM$$

for any admissible basis $H = (J_\alpha)$ of Q.

Definition 2. Let (M, g, Q) be a quaternionic Kähler manifold. A transformation of M is called a *quaternionic transformation* (resp., *quaternionic isometry*) if it preserves Q (resp., Q and g).

A vector field on M is called to be *quaternionic* (resp., *quaternionic Killing*) if it generates a local 1-parameter group of quaternionic transformations (resp., quaternionic isometries).

We denote by $\mathrm{Aut}(M, Q)$, $\mathrm{Aut}(M, Q, g)$ or, shortly, by $\mathrm{Aut}(Q)$, $\mathrm{Aut}(Q, g)$ the group of all quaternionic transformations and quaternionic isometries respectively, and by $\mathrm{aut}(Q)$, $\mathrm{aut}(Q, g)$ the Lie algebra of quaternionic and quaternionic Killing vector fields on M. We will use subscript 0 to denote the connected component of unity G_0 of a group G.

Remark that $\mathrm{aut}(Q, g)$ is the Lie algebra of the Lie group $\mathrm{Aut}(Q, g)$ if the metric g is complete since any Killing vector field on a complete Riemannian manifold is complete [8]. We will denote by $\mathrm{aut}_c(Q)$ the Lie algebra of the Lie group $\mathrm{Aut}(Q)$. It is a subalgebra of $\mathrm{aut}(Q)$ consisting of all complete quaternionic vector fields.

Recall ([3]) that the Lie derivative of the Levi-Civita connection ∇^g with respect to a quaternionic vector field $Z \in \mathrm{aut}(Q)$ is given by

$$Z \cdot \nabla^g = S^\xi \tag{3}$$

where ξ is a 1-form,

$$\xi = df_Z \qquad , \qquad f_Z = \frac{1}{4(n+1)} \operatorname{Trace} \nabla^g Z \tag{4}$$

The form ξ is called the *1-form associated to* Z.

Note that if $\nu \neq 0$ then the quaternionic structure Q is canonically defined by the metric g and, hence, $\operatorname{Aut}(Q,g) = \operatorname{Aut}(g)$ (the group of isometries), $\operatorname{aut}(Q,g) = \operatorname{aut}(g)$ (the Lie algebra of Killing vector fields).

We denote also by

$$\mathcal{P} = \mathcal{P}(Q,g) = \{Z = \operatorname{grad} f = g^{-1} \circ df \in \operatorname{aut}(Q)\}$$

the space of all *gradient quaternionic vector fields*.

Now we state the main results.

Theorem 1. *Let* (M,g,Q) *be a simply connected complete quaternionic Kähler 4n-manifold,* $n > 1$. *Assume that*

$$\operatorname{Aut}_0(Q) \neq \operatorname{Aut}_0(Q,g).$$

If M *is compact, it is isometric to the quaternionic projective space with the standard quaternionic Kähler structure.*

If M *is not compact, it has zero scalar curvature and its de Rham decomposition has an Euclidean factor* $(\mathbb{H}^k, g_0, Q_0,), k > 0$. *The converse is also true.*

In the compact case the conclusion holds under the weaker condition $\operatorname{Aut}(Q) \neq \operatorname{Aut}(Q,g)$, see ([3],[10]).

In the case of zero scalar curvature we have the following more general result. To state it we note that the quaternionic structure Q of a complete simply connected quaternionic Kähler manifold (M,g,Q) with zero scalar curvature is generated by a parallel hypercomplex structure $H = (J_1, J_2, J_3)$, where J_α, $\alpha = 1,2,3$, are parallel anticommuting complex structures. We fix such H, which is defined up to a rotation from $SO(3)$, and we write $Q =< H >$ to indicate that Q is generated by H. It is well known that the de Rham decomposition of the manifold (M,g,Q) may be written as follows:

$$M = \mathbb{H}^k \times M_1 \times \ldots \times M_l$$

$$g = g_0 \oplus g_1 \oplus \ldots \oplus g_l \tag{5}$$

$$H = H_0 \oplus H_1 \oplus \ldots \oplus H_l,$$

where $(\mathbb{H}^k, g_0, < H_0 >)$ is the 4k-dimensional flat quaternionic Kähler manifold and $(M_i, g_i, < H_i >), i = 1, ...l$, is an irreducible quaternionic Kähler manifold with the holonomy group $Sp(n_i)$, $\dim M_i = 4n_i$.

Theorem 2. *Let (M, g, Q) be a complete simply connected quaternionic Kähler manifold with zero scalar curvature and let (5) be its de Rham decomposition.*

(1) *Assume that the metric $g = g_0$ is flat, that is M is identified with the quaternionic vector space $\mathbb{H}^n$ with the standard quaternionic structure Q and the standard metric g_0. Then any Ricci-flat Q-Kähler metric g' on $\mathbb{H}^n$ is flat and has the form $g' = g \circ A$, where A is a positively defined symmetric endomorphism of $\mathbb{H}^n = \mathbb{R}^{4n}$ which commutes with Q. Moreover, any Q-Kähler metric g' with the reduced scalar curvature $\nu \neq 0$ has constant positive quaternionic curvature and can be written as*

$$g'(x) = \frac{4}{q\nu}[h_0 - \frac{1}{q}(h_0 \circ x \otimes h_0 \circ x + \sum_\alpha h_0 \circ J_\alpha x \otimes h_0 \circ J_\alpha x)] \qquad x \in \mathbb{H}^n$$

where $h_0 = g \circ A$ is a flat Q-Kähler metric and

$$q = h_0(x, x) + c \quad , \quad c = const > 0.$$

(2) *If the metric g is not flat, any Q-Kähler metric g' of (M, Q) is Ricci flat and may be written as*

$$g' = g'_0 \oplus \lambda_1 g_1 \oplus ... \oplus \lambda_l g_l,$$

where $\lambda_i = \mathrm{const} > 0$ and g'_0 is a flat quaternionic-Kähler metric on $\mathbb{H}^k$.

Corollary 1. *Under the assumptions of the theorem*

(1) *any quaternionic transformation of (M, g, Q) is affine :*

$$\mathrm{Aut}(Q) \subset \mathrm{Aut}(\nabla^g).$$

(2)

$$\mathrm{Aut}_0(Q) \neq \mathrm{Aut}_0(Q, g)$$

iff there is the flat factor in (5), i.e. $k > 0$.

(3)

$$\mathrm{Aut}(Q) \neq \mathrm{Aut}(Q, g) \quad , \quad \mathrm{Aut}_0(Q) = \mathrm{Aut}_0(Q, g)$$

iff $k = 0$ and for some i, j the manifolds $(M_i, g_i), (M_j, g_j)$ are homothetic but not isometric.

2. Quaternionic transformations of the spaces of constant quaternionic curvature.

We describe the groups $\mathrm{Aut}(M, Q)$ and $\mathrm{Aut}(M, g, Q)$ for the standard quaternionic Kähler manifolds $M = \mathbb{H}P^n, \mathbb{H}^n, \mathbb{H}\Lambda^n$ of constant quaternionic curvature $1, 0, -1$ respectively.

Proposition 1.

1) $\mathrm{Aut}(\mathbb{H}P^n, Q) = PGL_n(\mathbb{H}) = GL_{n+1}(\mathbb{H})/\mathbb{R}^* \supsetneqq \mathrm{Aut}(\mathbb{H}P^n, g, Q) = Sp_{n+1}/\mathbb{Z}_2$

2) $\mathrm{Aut}(\mathbb{H}^n, Q) = GL_n(\mathbb{H}) \rtimes \mathbb{H}^n \supsetneqq \mathrm{Aut}(\mathbb{H}^n, g, Q) = Sp_n \rtimes \mathbb{H}^n$

3) $\mathrm{Aut}(\mathbb{H}\Lambda_n, Q) = \mathrm{Aut}(\mathbb{H}\Lambda_n, g, Q) = Sp_{1,n}/\mathbb{Z}_2$

where $\rtimes$ indicates the semidirect product.

Proof. 1) and 2) are well known (see [11], [9]). To prove 3) we realize the quaternionic Lobachevsky space $\mathbb{H}\Lambda^n$ as the open orbit $B = Sp_{1,n}[(1,0,...,0)] \subset \mathbb{H}P^n$ of the subgroup $Sp_{1,n}$ of the projective group $PGL_n(\mathbb{H})$ which preserves the quaternionic quadric Q :

$$x^0\bar{x}^0 - \sum_{\alpha=1}^{n} x^\alpha\bar{x}^\alpha = 0.$$

The quaternionic structure of $\mathbb{H}\Lambda^n$ is induced by the canonical locally flat quaternionic structure of $\mathbb{H}P^n$. Any quaternionic transformation of $B = \mathbb{H}\Lambda^n$ can be extended to a unique quaternionic transformation φ of $\mathbb{H}P^n$; see ([11], [9]). Since Q is the boundary of B, the transformation φ preserves Q, that is it belongs to $Sp_{1,n}$.

Now we pass to the general case.

3. Quaternionic transformations and gradient quaternionic vector fields.

Let (M, g, Q) be a quaternionic Kähler manifold. For any vector field Z on M we denote by L_Z the field of endomorphisms $X \mapsto \nabla_X Z$, $X \in TM$, where $\nabla = \nabla^g$ is the Levi-Civita connection.

Lemma 1 ([3]). *A vector field Z (resp. a gradient vector field $Z = \mathrm{grad}\, f, f \in C^\infty(M)$) is quaternionic iff $[L_Z, Q] \subset Q$ (resp. $[L_Z, Q] = 0$).*

Note that if M is simply connected a vector field Z is gradient iff the operator L_Z is symmetric (with respect to g). Hence, we have

Corollary 2. *Let M be simply connected. Then a vector field Z is gradient quaternionic field iff $g \circ L_Z = \nabla(g \circ Z)$ is a symmetric Q-hermitian form.*

Now we prove the following

Proposition 2. *Let g' be a quaternionic Q-Kähler metric on a simply connected quaternionic Kähler manifold (M, g, Q). If $\nabla^{g'} \neq \nabla^g$, then there exists a non zero gradient quaternionic vector field $Z = \mathrm{grad}\, f = g^{-1} \circ df$ on M , where $f = \mathrm{div}\, Z = tr\nabla^g Z$ is an eigenfunction of the Laplacian with the eigenvalue $\nu_1 = 2\nu(n+1)$.*

Proof. By (1) we have

$$\nabla^{g'} - \nabla^g = S^\xi$$

for some $0 \neq \xi \in \Lambda^1 M$. Then ([2]) the Ricci tensors of the connections $\nabla^{g'}$, ∇^g are related by

$$\mathrm{Ric}' = \mathrm{Ric} - 4\rho^s + 4(n+1)\rho + 8\Pi\rho^s \tag{6}$$

where

$$\rho = \xi \otimes \xi - \sum_{\alpha=1}^{3} (\xi \circ J_\alpha) \otimes (\xi \circ J_\alpha) - \nabla\xi$$

ρ^s is the symmetric part of the bilinear form ρ and Π is the projection of the space of bilinear forms onto the space of Q-Hermitian forms given by

$$\Pi : \omega \mapsto \Pi\omega = \frac{1}{4}[\omega + \sum_\alpha \omega(J_\alpha \cdot, J_\alpha \cdot)].$$

Using (2), we can rewrite (6) as

$$\frac{\nu'}{4} g' = \frac{\nu}{4} g + \xi \otimes \xi - \sum_{\alpha=1}^{3} (\xi \circ J_\alpha) \otimes (\xi \circ J_\alpha) - \nabla\xi$$

where ν' is the reduced scalar curvature of the metric g'. It implies that the bilinear form $\nabla\xi - 2\xi \otimes \xi$ is symmetric and Q-Hermitian; in particular $d\xi = \mathrm{Alt}(\nabla\xi) = \mathrm{Alt}(\nabla\xi - 2\xi \otimes \xi) = 0$ and hence $\xi = dh$ for some function h. Now we put $\eta := e^{-2h}\xi$. Then $\nabla\eta = e^{-2h}[\nabla\xi - 2\xi \otimes \xi]$ is a symmetric Q-hermitian form and $\eta = df$ for $f = -\frac{1}{2}e^{-2h}$. Hence, by Corollary 2, $Z := g^{-1} \circ \eta = \mathrm{grad}\, f$ is a non zero gradient quaternionic vector field. The last statement was proved in [3].

Corollary 3. *Let (M, g, Q) be a simply connected quaternionic Kähler manifold and $\varphi \in \mathrm{Aut}(M, Q)$ be a quaternionic transformation which is not affine (i.e. doesn't preserves ∇^g). (If (M, g) is irreducible it is sufficient to assume that φ is not an isometry.) Then there exists a non zero gradient quaternionic vector field $Z = \mathrm{grad}\, f$, where $f = \mathrm{div}\, Z$ is an eigenfunction of the Laplacian with eigenvalue $\nu_1 = 2\nu(n+1)$.*

Proof. It is sufficient to apply the Proposition 2 to $g' = \varphi^* g$.

4. Fundamental equation for gradient quaternionic vector fields.

We define the parallel (1, 3) tensor P on M by

$$\begin{aligned} P(X,Y)Z =& 2g(X,Z)Y + g(Z,Y)X + g(X,Y)Z \\ & - \sum_{\alpha=1}^{3} g(Z, J_\alpha Y) J_\alpha X - \sum_{\alpha=1}^{3} g(X, J_\alpha Y) J_\alpha Z \\ =& S_X^{g\circ Z} Y + S_Z^{g\circ X} Y \end{aligned}$$

Remark 1. *For any $X \in TM$ one has*

$$P(X,X)X = 4\|X\|^2 X$$

Proposition 3. *Let Z be a quaternionic vector field on (M,g,Q) and ξ the associated 1-form. Then*

1) *Z and ξ satisfy the following equation:*

$$\nabla_X L_Z + R(Z,X) = S^\xi_X \qquad \forall\, X \in \chi(M) \tag{7}$$

2) *if Z is a gradient field then*

$$\xi = -\frac{\nu}{2} g \circ Z \tag{8}$$

and Z satisfies the following fundamental equation

$$\nabla_X L_Z = -\frac{\nu}{4} P(X,\cdot\,)Z \qquad \forall\, X \in \chi(M) \tag{9}$$

Moreover

$$W(Z,\cdot\,) = 0 \tag{10}$$

and

$$[W(X,Y), L_Z] = 0 \tag{11}$$

for any $X, Y \in TM$, where W is the quaternionic Weyl tensor.

Remark 2. *If M is compact and ν is positive the inverse statement for 2) holds: any solution of the fundamental equation is a gradient quaternionic vector field (see [3]).*

Corollary 4. *If $\nu = 0$ then any gradient quaternionic field Z is affine ($Z \cdot \nabla = S^\xi = 0$). In particular, Z is complete if the manifold (M,g) is complete.*

Proof. 1) For any vector field Z on the Riemannian manifold (M,g) the following identity holds:

$$(Z\cdot\nabla)_X Y = (\nabla^2 Z)_{X,Y} + R(Z,X)Y \qquad (\forall X, Y \in \chi(M))$$

Taking into account the formula (3) we get (7). If $Z \in \mathcal{P}$ then L_Z is a symmetric endomorphism and consequently

$$2g(R(Z,X)Y,T) = g(S^\xi_X Y, T) - g(S^\xi_X T, Y) \qquad (\forall X, Y, T \in \chi(M))$$

By taking the trace, we obtain (8). Hence

$$R(Z,X) = \frac{\nu}{4}[S_Z^{g\circ X} - S_X^{g\circ Z}] \equiv \nu R_1(Z,X) \tag{12}$$

that is (10) holds. Then (9) follows from (7), (8) and (12). Now we prove (11). Taking the covariant derivative of the fundamental equation we get the identity

$$(\nabla^2 L_Z)_{Y,X} = -\frac{\nu}{4} P(X,\cdot\,) L_Z Y$$

since $\nabla P = 0$. By antisymmetrizing with respect to X, Y the Ricci identity gives

$$\begin{aligned}[R(X,Y), L_Z] &= \frac{\nu}{4}[P(X,\cdot\,)L_Z Y - P(Y,\cdot\,)L_Z X] \\ &= \frac{\nu}{4}[S_X^{g\circ L_Z Y} + S_{L_Z Y}^{g\circ X} - S_Y^{g\circ L_Z X} - S_{L_Z X}^{g\circ Y}]\end{aligned}$$

Recall now that

$$W(X,Y) = R(X,Y) - \nu R_1(X,Y) = R(X,Y) - \frac{\nu}{4}[S_X^{g\circ Y} - S_Y^{g\circ X}]$$

To prove the formula (11) it is sufficient to check that if $\nu \neq 0$ then

$$\begin{aligned}S_X^{g\circ L_Z Y} + S_{L_Z Y}^{g\circ X} - S_Y^{g\circ L_Z X} - S_{L_Z X}^{g\circ Y} &= 4[R_1(X,Y), L_Z] \\ &= [S_X^{g\circ Y} - S_Y^{g\circ X}, L_Z]\end{aligned}$$

This is established by the following Lemma 2.

Lemma 2. *Let A be a symmetric endomorphism which commutes with Q. Then for any $X, Y \in TM$ the following identities hold:*

$$1)\quad [S_X^{g\circ Y}, A] = S_X^{g\circ AY} - S_{AX}^{g\circ Y}$$

$$2)\quad [S_X^{g\circ Y}, A] - [S_Y^{g\circ X} A] = S_X^{g\circ AY} + S_{AY}^{g\circ X} - S_Y^{g\circ AX} - S_{AX}^{g\circ Y}$$

Proof. 1) is straightforward and then 2) follows from 1) immediately.

Proposition 4. *Let (M, g, Q) be a complete quaternionic Kähler manifold with non-zero scalar curvature. Then the Lie algebra $\mathrm{aut}_c(Q)$ admits a reductive decomposition*

$$\mathrm{aut}_c(Q) = \mathrm{aut}(Q,g) + \mathcal{P}_c,$$

$$[\mathrm{aut}(Q,g), \mathcal{P}_c] \subset \mathcal{P}_c \qquad , \qquad \mathrm{aut}(Q,g) \cap \mathcal{P}_c = 0$$

where $\mathcal{P}_c$ is the space of complete gradient quaternionic vector fields.

If

$$\mathrm{Aut}_0(Q) \neq \mathrm{Aut}_0(Q,g)$$

then $\mathcal{P}_c \neq 0$.

Proof. For any $X \in \mathrm{aut}_c(Q)$ we construct a gradient quaternionic vector field Z as follows. Let $\xi = df_X$ be the 1-form associated to X, see sect.1. By using formula (3) we find

$$X \cdot \mathrm{Ric} = -4(n+1)\nabla\xi + 4[\nabla\xi]^s - 8\Pi[\nabla\xi]^s$$

where "$\cdot$" indicates the Lie derivative. Since $X \cdot Ric$ is symmetric and Q-Hermitian we deduce that *the bilinear form* $\nabla\xi$ *is symmetric, Q-Hermitian and*

$$X \cdot \mathrm{Ric} = -4(n+2)\nabla\xi$$

Hence

$$\nu X \cdot g = -4\nabla\xi$$

On the other hand, from the formula for Lie derivative we get

$$(g^{-1} \circ \xi) \cdot g = 2\nabla\xi$$

Hence

$$Y = X + \frac{2}{\nu} g^{-1} \circ \xi$$

is a Killing vector field and

$$Z = -\frac{2}{\nu} g^{-1} \circ \xi$$

is a gradient quaternionic vector field. Moreover, $Z = X - Y$ is complete, since $X \in \mathrm{aut}_c(Q)$ and $Y \in \mathrm{aut}(Q, g) \subset \mathrm{aut}_c(Q)$. For any $Y \in \mathrm{aut}(Q, g)$, $Z = \mathrm{grad}\, f \in \mathcal{P}_c$ we have

$$[Y, Z] = \mathrm{grad}(Y \cdot f) \in \mathcal{P}_c,$$

since Y preserves g. Suppose now that $Z \in \mathrm{aut}(Q, g) \cap \mathcal{P}_c$. Then the endomorphism $L_Z = \nabla Z$ is both symmetric and skew-symmetric, hence, zero. The assumptions of the proposition imply that the metric g is irreducible. This implies that $Z = 0$.

5. Quaternionic distribution associated with a gradient quaternionic vector field.

Let Z be a gradient quaternionic vector field and $L_Z = \nabla Z$. Denote by $\mathcal{L}(Z)$ the space of vector fields spanned by vector fields $Z, L_Z Z, ...L_Z^k Z, ...$

Proposition 5. *$\mathcal{L}(Z)$ is a Lie subalgebra of the Lie algebra $\chi(M)$ of vector fields and its orbits (leaves of the corresponding singular integrable distribution, see* [15]*), are totally geodesic totally real submanifolds.*

The proof follows from the Lemma below.

Lemma 3.

$$1)\quad < L^k Z, JL^h Z >= 0, \qquad \forall\, J \in Q; h, k \in \mathbb{Z}^+$$

$$2)\quad \nabla_{L^i Z} L^h Z = -\frac{\nu}{4}\{2h < L^i Z, Z > L^{h-1} Z$$

$$+ \sum_{r=1}^{h} [< Z, L^{h-r} Z > L^{i+r-1} Z + < L^i Z, L^{h-r} Z > L^{r-1} Z]\} + L^{i+h+1} Z$$

where $L^i \equiv L^i_Z$ *and the sum in right member of 2) has to be considered only for* $h > 0$.

Proof of Lemma. 1) Since L_Z is a symmetric operator which commutes with J we need only to prove that $< L^k Z, JZ >= 0$ for any positive integer k. It can be done as follows: for k odd the operator JL_Z is skew-symmetric and hence $< Z, JL^k_Z Z >= 0$; for $k = 2l$ we have $< L^k_Z Z, JZ >=< L^l_Z Z, JL^l_Z Z >= 0$.

2) By definition, we have

$$\nabla_{L^i Z} Z = L^{i+1} Z$$

which gives 2) for $h = 0$. By using (9), we have

$$\begin{aligned}\nabla_{L^i Z} L^1 Z &= (\nabla_{L^i Z} L_Z) Z + L_Z(\nabla_{L^i Z} Z)\\ &= -\frac{\nu}{4} P(L^i Z, Z) Z + L^{i+2} Z\end{aligned}$$

By using 1) we get

$$\nabla_{L^i Z} L^1 Z = -\frac{\nu}{4}\{2 < L^i Z, Z > Z + < Z, Z > L^i Z + < L^i Z, Z > Z\} + L^{i+2} Z$$

which establishes 2) for $h = 1$. Moreover, for $h > 1$,

$$\nabla_{L^i Z} L^h Z = (\nabla_{L^i Z} L_Z) L^{h-1} Z + L_Z(\nabla_{L^i Z} L^{h-1} Z)$$

Then 2) follows by induction on h.

Denote by $\mathcal{D}(Z)$ the (eventually singular) quaternionic (i.e. Q-invariant) distribution defined by

$$M \ni x \mapsto \mathcal{D}_x(Z) = \mathcal{L}_x(Z) + Q_x \mathcal{L}_x(Z)$$

and define the kernel of the Weyl tensor W as follows:

$$Ker W = \{X \in TM \mid W(X, \cdot) = 0\}$$

Proposition 6.

1) $\mathcal{D}(Z) \subset KerW$

2) $\mathcal{D}(Z)$ *is integrable*

3) *a regular orbit N of $\mathcal{D}(Z)$ is a totally geodesic quaternionic submanifold with constant quaternionic curvature, that is $W_{|N} \equiv 0$.*

Proof. 2) Let be $X = L^k Z, Y = L^l Z$ and J a local section of Q. Then $[X, Y] = \nabla_X Y - \nabla_Y X$ belongs to $\mathcal{D}(Z)$ by 2) of Lemma 3. $\nabla_X(JY) = (\nabla_X J)Y + J\nabla_X Y$ belongs to $\mathcal{D}(Z)$ since $\nabla_X J \in Q$ and $\nabla_X Y \in \mathcal{D}(Z)$. Now it is sufficient to prove that $\nabla_{JX} Y \in \mathcal{D}(Z)$. It can be done by using induction on l:

$$\begin{aligned}\nabla_{JX} Y &= \nabla_{JX}(L^l Z) = (\nabla_{JX} L_Z)(L^{l-1} Z) + L_Z \nabla_{JX}(L^{l-1} Z) \\ &= -\frac{\nu}{4} P(JX, Z) L^{l-1} Z + L_Z \nabla_{JX}(L^{l-1} Z).\end{aligned}$$

The first term belongs to $\mathcal{D}(Z)$ by inductive hypothesis. This proves 2). Now we prove 1). By using identities (10) and (11), for any $X, Y \in TM$ and $J \in Q$ we have for any natural k:

$$W(X, Y) L^k Z = L^k W(X, Y) Z = 0$$

and

$$W(X, Y) J L^k Z = J W(X, Y) L^k Z = 0.$$

Hence the conclusion follows. 3) follows immediately from 1) and 2).

6. Completeness of a totally geodesic submanifold of an analytic Riemannian manifold.

Recall that a submanifold N of a Riemannian manifold (M, g) is called to be totally geodesic if any geodesic of the submanifold $(N, g|N)$ is a geodesic of the manifold (M, g). A submanifold N of a Riemannian manifold (M, g) is totally geodesic iff the Lie algebra $\mathcal{X}(N)$ of vector fields tangent to N is invariant under covariant derivatives in the directions of vector fields from $\mathcal{X}(N)$:

$$\nabla_{\mathcal{X}(N)} \mathcal{X}(N) \subset \mathcal{X}(N)$$

In general, a totally geodesic submanifold of a complete Riemannian manifold can not be extended to a complete totally geodesic submanifold. However, we prove that this is true if the manifold (M, g) is analytic.

Proposition 7. *Any (embedded) totally geodesic submanifold N of a complete analytic Riemannian manifold (M, g) admits a unique extension to a complete totally geodesic (immersed) submanifold.*

Proof. The proof is based on the following lemma.

Lemma 4. *Let (M, g) be an analytic Riemannian manifold and r the radius of injectivity in a point $p \in M$. Denote by B the open ball of radius $r/2$ in the tangent space T_pM and set $U = \exp B$. Then any (embedded) totally geodesic submanifold $N \in p$ of $(U, g|U)$ admits a unique extension to a maximal totally geodesic submanifold $\tilde{N} = \exp(T_pN \cap B) \subset U$.*

Proof of Lemma. Let $\mathbf{e}_1, \cdots, \mathbf{e}_n$ be an orthonormal basis of T_pM such that the vectors $\mathbf{e}_1, \cdots, \mathbf{e}_k$ form a basis of T_pN. Denote by x_i the corresponding geodesic coordinates in U and set $\partial_i = \partial/\partial x_i$. The (analytic) submanifold $\tilde{N} = \exp(T_pN \cap B)$ of U is totally geodesic iff the (analytic) functions

$$\Gamma^a_{ij} = g(\nabla_{\partial_i}\partial_j, \partial_a), \; i, j \leq k, \qquad a > k$$

vanish identically on $\tilde{N}$. This is true, since they vanish in the open submanifold N of $\tilde{N}$. This proves Lemma.

Proof of Proposition 7. To prove Proposition 7) it is sufficient to show that an embedded totally geodesic submanifold N can be extended along any geodesic $\gamma(t)$ which is tangent to N starting from a point $\gamma(0) \in N$. Let $q = \gamma(t_0)$ be a point of the geodesic γ such that $\gamma([0, t_0)) \subset N$ but $\gamma(t_0) \notin N$. Let r be the injectivity radius of a compact neighbourhood of q and $p = \gamma(t_0 - r/3)$. Denote by B the open ball of radius $\frac{r}{2}$ in T_pM. By Lemma 4, $V = \exp(T_pN \cap B)$ is a totally geodesic submanifold of M which extends $N \cap \exp B$. So $\tilde{N} = N \cup V$ gives an extension of N to an (immersed) totally geodesic submanifold which contains $\gamma(0, t_0 + \epsilon)$. More precisely, $\tilde{N}$ is defined as follows. If $(\varphi, N), \varphi : N \to M$ is the immersed totally geodesic submanifold, then $\varphi(N) \cap V$ is a disjoint union of totally geodesic connected submanifolds V_i and we define the extension $(\tilde{\varphi}, \tilde{N})$ by glueing to N in a natural way the components V_i which are open in V. This proves the Proposition.

7. Proof of the main theorems.

We prove Theorem 1 under the assumption that the reduced scalar curvature is negative, $\nu < 0$. For $\nu > 0$ the theorem was proved in [3], [10] and for $\nu = 0$ it follows from Theorem 2. Assume that

$$\mathrm{Aut}_0(Q) \neq \mathrm{Aut}_0(Q, g).$$

By Proposition 4 there exists a complete non zero gradient quaternionic vector field Z on M. It generates the 1-parameter group A of quaternionic transformations which preserves the (integrable) distribution $\mathcal{D}(Z)$ associated with Z, see sect. 5. A leaf N of this distribution is a totally geodesic quaternionic submanifold of M of constant quaternionic curvature. Since the quaternionic Kähler manifold is analytic, we can extend N to a complete totally geodesic quaternionic Kähler manifold $\tilde{N}$ of constant negative quaternionic curvature. The group A preserves $\tilde{N}$ and induces on $\tilde{N}$ a one-parameter group of non isometric quaternionic transformations. This is impossible by Proposition 1, since the universal cover of the $\tilde{N}$ is isometric to the quaternionic Lobachevsky space. This contradiction proves the Theorem.

Proof of Theorem 2. 1) Let $M = \mathbb{H}^n$ be the quaternionic vector space with the standard quaternionic structure Q and the standard flat metric g_0. The Levi-Civita connection ∇' of any Q-Kähler metric g' is related with the Levi-Civita connection ∇^0 of g_0 by

$$\nabla' = \nabla^0 + S^\xi$$

where ξ is an exact 1-form, say

$$\xi = -\frac{1}{2}df \ ,$$

and

$$\frac{\nu'}{4}g' = \xi \otimes \xi - \sum_\alpha (\xi \circ J_\alpha) \otimes (\xi \circ J_\alpha) - \nabla^0 \xi \tag{13}$$

(See the proof of Prop.2). This formula may be written as

$$\nu' g' = 2e^{-f}[\nabla^0 \eta - 2e^{-f}\Pi(\eta \otimes \eta)] \tag{14}$$

where

$$\eta := de^f = -2e^f \xi \tag{15}$$

and $Z = \operatorname{grad} e^f = g^{-1} \circ \eta$ is a gradient quaternionic vector field.

Since g_0 is Ricci flat, Z is affine (see corollary 4) and hence it can be written as

$$Z(x) = Ax + b \qquad , \quad A \in gl_n^+(\mathbb{H}), \quad b \in \mathbb{H}^n \qquad\qquad x \ \in \ \mathbb{H}^n$$

where $gl_n^+(\mathbb{H})$ is the space of symmetric quaternionic linear endomorphisms of $\mathbb{H}^n$. Indeed $\nabla^0 Z = A$ is an endomorphism of $\mathbb{H}^n$ which commutes with Q, by Lemma 1, and it is symmetric with respect to g_0. Hence the potential function of Z may be written as

$$e^f = \frac{1}{2}g_0(Ax, x) + g_0(b, x) + c_1 \qquad\qquad x \ \in \ \mathbb{H}^n \tag{16}$$

and

$$\eta_{|x} = de^f{}_{|x} = g_0(Ax, \cdot) + g_0(b, \cdot).$$

Remark that A is not negatively defined : $A \geq 0$. In fact, the following more strong statement is true.

Lemma 5. *Either A is positively defined or $A = 0$.*

Proof. Assume that $Ax = -\lambda x$, $x \neq 0$, $\lambda > 0$. Then restriction of (16) to the line tx gives

$$e^{f(tx)} = -\frac{1}{2}t^2\lambda g_0(x,x) + tg_0(b,x) + c_1 \qquad \forall\, t \in \mathbb{R}$$

and this is a contradiction.

Now we prove that $b \in ImA$. Indeed, let write $b = b_1 + b_2$ where $b_1 \in ImA$ and $b_2 \in (ImA)^\perp$: then

$$e^{f(tb_2)} = tg_0(b_2, b_2) + c_1 \qquad \forall t \in \mathbb{R}$$

and hence $b_2 = 0$.

Let us put now $y = x - x_0$ where $Z(x_0) \equiv Ax_0 + b = 0$. Then in new coordinates y the vector field Z is given by

$$Z(y) = Ay$$

and (14) can be written as

$$\frac{\nu'}{2}e^{f(y)}g' = g_0 \circ A - 2e^{-f(y)}\Pi(g_0 \circ Ay \otimes g_0 \circ Ay) \qquad (17)$$

$$e^{f(y)} = \frac{1}{2}g_0(Ay, y) + c_1.$$

In the origin $y = 0$ we have

$$\frac{\nu'}{2}c_1 g' = g_0 \circ A$$

If $\nu' = 0$ then $A = 0$. If $\nu' \neq 0$ then A is positively defined and $\nu' > 0$. This proves Lemma.

Continuation of proof of Theorem 2. Now we finish the proof of the first part of the Theorem.

If $A = 0$ then $\xi = 0$ and $\nabla' = \nabla^0$ is a flat connection: hence g' is flat.

If $A > 0$ then (17) gives

$$g'_{|y} = \frac{4}{\nu' q}[h_0 - \frac{4}{q}\Pi(h_0 \circ y \otimes h_0 \circ y)]$$

where $h_0 = g_0 \circ A$ is a flat quaternionic Kähler metric on $\mathbb{H}^n$ and $q = h_0(y,y) + c_1$. This is exactly the canonical expression for a standard quaternionic Kähler metric of $\mathbb{H}P^n$ (See for example [6]).

To prove the second part we need the following lemma.

Lemma 6. *Let (M, g) be a simply connected complete Riemannian manifold with the de Rham decomposition*

$$M = \mathbb{R}^k \times M_1 \times \cdots \times M_l$$

$$g = g_0 + g_1 + \cdots + g_l,$$

where g_0 is the flat metric and g_i, $i > 0$ is an irreducible metric on M_i. Then any Riemannian metric on M with the same Levi-Civita connection as g is given by

$$\bar{g} = g_0 \circ A_0 + \lambda_1 g_1 + \cdots + \lambda_l g_l$$

where $\lambda_i = const > 0$ and A_0 is a positively defined endomorphism of $\mathbb{R}^k$.

Proof of the Lemma. The field of endomorphisms $A = g^{-1} \circ \bar{g}$ is parallel with respect to the Levi-Civita connection of the metric g and, hence, it commutes with the holonomy group. By Schur lemma, it can be written as

$$A = \mathrm{diag}(A_0, \lambda_1 \,\mathrm{Id}, \ldots, \lambda_l \,\mathrm{Id}),$$

where $A_0 > 0$ is a constant endomorphism. This proves the lemma.

Now we prove the statement 2).

Proof of the statement 2. Let (M, g, Q) be a non flat quaternionic Kähler manifold with $\nu = 0$ and g' a Q-Kähler metric on M. Denote by Z the gradient quaternionic vector field on M, associated with g, g' by Proposition 2. The proposition 3 and Corollary 4 show that Z is an affine (complete) vector field and the field L_Z is parallel. Applying the lemma to the metrics $g, \bar{g} = (\exp tZ)^* g$, one can easily check that the field Z can be written as $Z = Z_0 + Z_1 + \cdots + Z_l$, where Z_i is an affine gradient vector field on (M_i, g_i). Moreover, $L_{Z_i} = \lambda_i \,\mathrm{Id}$ for $i > 0$, that is Z_i is an infinitesimal homothety. Since on an irreducible manifold (M_i, g_i) there is no non trivial homothetic transformation and parallel vector field , we conclude that $\lambda_1 = \cdots = \lambda_l = 0$ and, hence, $Z_i = 0$ for $i > 0$. This implies that the metric g' can be decomposed into the direct sum of some metric $\bar{g}$ on $\tilde{M} = M_1 \times \cdots \times M_l$ which has the same Levi-Civita connection as $g_1 + \cdots + g_l$ and a Ricci-flat Q-Kähler metric g_0' on H^k. The statement 2) follows now from statement 1) and the Lemma.

Proof of the corollary. 1) Let φ be a quaternionic transformation of (M, g, Q). Applying Theorem 2 to the metric $g' = \varphi^* g$, we get $\varphi^* \nabla^g = \nabla^{\varphi^* g} = \nabla^g$. (In the flat case we take into account that the metric $\varphi^* g$ is flat and hence $\varphi^* g = g \circ A$ for some constant endomorphism A.) 2) Now we will assume that there is no flat factor in the de Rham decomposition (5) and we denote by D_i the tangent distribution of the factor M_i, $i = 1, \ldots, l$. Since the distributions D_i depend only on the connection ∇^g and any quaternionic transformation of M is affine, any one-parametric group φ_t of quaternionic transformations preserves the distributions D_i and, hence, induces on (M_i, g_i) an one-parametric group H_i of affine transformations. Since (M_i, g_i) is an irreducible manifold, the group H_i preserves the metric. This shows that $\mathrm{Aut}_0(Q) \subset \mathrm{Aut}_0(Q, g)$ and proves the direct statement of 2). The inverse statement is immediate. 3) We may assume as before that there is no flat factor in (5). Let φ be a quaternionic transformation. If it preserves all distributions D_i we conclude as before that it is an isometry. In the opposite case it induces some non trivial permutation of the set of the distributions. Let choose the index i such that $\varphi^* D_i = D_j$, $i \neq j$. The lemma shows that φ induces an homothetic diffeomorphism of M_i onto M_j. This proves the corollary.

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