QUANTUM COHOMOLOGY OF LAGRANGIAN AND ORTHOGONAL GRASSMANNIANS

HARRY TAMVAKIS

Abstract. This is the written version of my talk at the Mathematische Arbeitstagung 2001, which reported on joint work with Andrew Kresch (papers [KT2] and [KT3]).

The Classical Theory

For the purposes of this talk we will work over the complex numbers, and in the holomorphic (or algebraic) category. There are three kinds of Grassmannians that we will consider, depending on their Lie type:

- In type A, let $G(k,N) = SL_N/P_k$ be the Grassmannian which parametrizes k-dimensional linear subspaces of $\mathbb{C}^N$.
- In type C, let $LG = LG(n,2n) = Sp_{2n}/P_n$ be the Lagrangian Grassmannian.
- In types B and D, let $OG = OG(n,2n+1) = SO_{2n+1}/P_n$ denote the odd orthogonal Grassmannian, which is isomorphic to the even orthogonal Grassmannian $OG(n+1,2n+2) = SO_{2n+2}/P_{n+1}$.

We begin by defining the Lagrangian Grassmannian $LG(n,2n)$, and will discuss the Grassmannians for the other types later in the talk. Let V be the vector space $\mathbb{C}^{2n}$, equipped with a symplectic form. A subspace Σ of V is *isotropic* if the restriction of the form to Σ vanishes. The maximal possible dimension of an isotropic subspace is n, and in this case Σ is called a *Lagrangian* subspace. The Lagrangian Grassmannian $LG = LG(n,2n)$ is the projective complex manifold which parametrizes Lagrangian subspaces Σ in V.

There is a stratification of LG by *Schubert varieties* X_λ, one for each strict partition $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_\ell > 0)$ with $\lambda_1 \leqslant n$. The number $\ell = \ell(\lambda)$ is the *length* of λ, and we let $\mathcal{D}_n$ denote the set of strict partitions λ with $\lambda_1 \leqslant n$. To describe X_λ more precisely, fix an *isotropic flag* of subspaces F_i of V:

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_n \subset V$$

with $\dim(F_i) = i$ for each i, and F_n Lagrangian. Then

$$X_\lambda = \{\, \Sigma \in LG \mid \dim(\Sigma \cap F_{n+1-\lambda_i}) \geqslant i \quad \text{for} \;\; 1 \leqslant i \leqslant \ell \,\}.$$

The variety X_λ has codimension $|\lambda| := \sum \lambda_i$ in LG, and determines a *Schubert class* $\sigma_\lambda = [X_\lambda]$ in $H^{2|\lambda|}(LG,\mathbb{Z})$. The classes $\sigma_1, \ldots, \sigma_n$ are called *special*, and $H^2(LG)$ has rank 1, generated by σ_1. We have an isomorphism of abelian groups

$$H^*(LG,\mathbb{Z}) = \sum_{\lambda \in \mathcal{D}_n} \mathbb{Z} \cdot \sigma_\lambda$$

Date: June 16, 2001.

which describes the additive structure of the cohomology $H^*(LG)$. As all cohomology classes occur in even degrees, we adopt the convention that a class α in the cohomology of a complex manifold X has degree k when α lies in $H^{2k}(X)$.

Classically, there are three ingredients necessary for an understanding of the multiplicative structure of $H^*(LG)$.

a) A *presentation* of $H^*(LG)$ in terms of generators and relations. This is due to Borel [Bo], and can be derived as follows. Consider the universal exact sequence of vector bundles over LG

$$0 \longrightarrow E \longrightarrow V_{LG} \longrightarrow E^* \longrightarrow 0.$$

Here V_{LG} is the trivial rank $2n$ vector bundle over LG, E is the tautological subbundle of V_{LG}, and we have identified the quotient bundle V_{LG}/E with E^* by using the symplectic form on V. The special Schubert class σ_i is equal to the i-th Chern class $c_i(E^*)$, for $1 \leqslant i \leqslant n$. The cohomology ring of LG can then be presented as a quotient of the polynomial ring in the Chern classes of E by the relations coming from the Whitney sum formula

$$c(E)c(E^*) = (1 - \sigma_1 + \sigma_2 - \cdots)(1 + \sigma_1 + \sigma_2 + \cdots) = 1.$$

Expanding this equation, one obtains that $H^*(LG)$ is a quotient of $\mathbb{Z}[\sigma_1, \ldots, \sigma_n]$ by the relations

$$\sigma_i^2 + 2\sum_{k=1}^{n-i}(-1)^k \sigma_{i+k}\sigma_{i-k} = 0$$

for $1 \leqslant i \leqslant n$, where it is understood that $\sigma_j = 0$ if $j < 0$.

b) A *Giambelli formula* which identifies the polynomials which represent the Schubert classes σ_λ in the above presentation. For the type A Grassmannians $G(k,N)$, the answer to this question is quite classical (and due to Giambelli). In contrast, an analogous formula for the Lagrangian Grassmannian was proved much more recently. According to Pragacz [P], the Giambelli formula for LG is obtained in two stages: for $i > j > 0$, we have

$$\sigma_{i,j} = \sigma_i\sigma_j + 2\sum_{k=1}^{n-i}(-1)^k \sigma_{i+k}\sigma_{j-k}$$

while for any $\lambda \in \mathcal{D}_n$ of length at least 3,

$$\sigma_\lambda = \text{Pfaffian}[\sigma_{\lambda_i,\lambda_j}]_{1 \leqslant i < j \leqslant r}.$$

Here r is the least even integer which is $\geqslant \ell(\lambda)$.

c) *Schubert calculus:* By this we mean algorithms for evaluating the structure constants $e_{\lambda\mu}^{\nu}$ which occur in the cup product expansion

$$\sigma_\lambda \sigma_\mu = \sum_{|\nu| = |\lambda| + |\mu|} e_{\lambda\mu}^{\nu}\, \sigma_\nu \tag{1}$$

in $H^*(LG)$. Recall that for type A Grassmannians, this problem is solved by showing that the multiplication of Schubert classes agrees with the corresponding product of Schur S-polynomials (defined by Schur in his celebrated 1901 thesis). In [P], Pragacz showed that the product (1) agrees with the corresponding product of two Schur Q-polynomials. The latter polynomials were used by Schur [S] to study the projective representations of the symmetric and alternating groups. Moreover, combinatorial algorithms for computing in the ring of Schur Q-polynomials were obtained by Boe and Hiller [BH] and Stembridge [St].

Quantum Cohomology of $LG(n, 2n)$

We will work throughout with the *small* quantum cohomology ring $QH^*(LG)$. This is a deformation of $H^*(LG)$ which first occured in the work of string theorists. The multiplicative structure of the quantum cohomology ring of LG encodes the enumerative geometry of rational curves in LG, in the form of *Gromov–Witten invariants.*

The ring $QH^*(LG)$ is an algebra over $\mathbb{Z}[q]$, where q is a formal variable of degree $n+1$; one recovers the classical cohomology ring by setting $q = 0$. To describe the product structure, recall that for each Schubert class σ_λ one has a (Poincaré) dual class $\sigma_{\lambda'}$, defined so that

$$\int_{LG} \sigma_\lambda \, \sigma_\mu = \delta_{\lambda\mu'}$$

for all partitions λ, μ in $\mathcal{D}_n$. The parts of the partition λ' complement the parts of λ in the set $\{1, \ldots, n\}$. Using dual classes, we can write the classical structure constant $e^\nu_{\lambda\mu}$ in (1) as a triple intersection number:

$$e^\nu_{\lambda\mu} = \langle \sigma_\lambda, \sigma_\mu, \sigma_{\nu'} \rangle_0 = \int_{LG} \sigma_\lambda \, \sigma_\mu \, \sigma_{\nu'} \, .$$

In $QH^*(LG)$ there is a formula

$$\sigma_\lambda \cdot \sigma_\mu = \sum \langle \sigma_\lambda, \sigma_\mu, \sigma_{\nu'} \rangle_d \, \sigma_\nu \, q^d, \tag{2}$$

the sum over $d \geqslant 0$ and strict partitions ν with $|\nu| = |\lambda| + |\mu| - d(n+1)$. The Gromov–Witten invariant $\langle \sigma_\lambda, \sigma_\mu, \sigma_{\nu'} \rangle_d$ is a non-negative integer, which counts the number of degree-d rational maps $f : \mathbb{P}^1 \to LG$ such that $f(0) \in X_\lambda$, $f(1) \in X_\mu$ and $f(\infty) \in X_{\nu'}$, for general translates of the three Schubert varieties X_λ, X_μ and $X_{\nu'}$. Here *degree* d means that $f_*[\mathbb{P}^1] = d \, \sigma_{1'}$. The 'miracle' is that with this definition, (2) gives an *associative* product.

There is a long list of papers studying questions (a), (b) and (c) above for the quantum cohomology rings of homogeneous spaces. We now have answers to all three questions when $X = SL_N/P$ is a partial flag variety of $SL_N(\mathbb{C})$; see [W], [ST], [Be] and [BCF] for the case of the Grassmannian $G(k, N)$, and [KT2] for more references. For an arbitrary complex semisimple Lie group G, with Borel subgroup B, Kim [K] found a presentation of $QH^*(G/B)$, thus answering (a). However, one still lacked a presentation of the quantum ring for the other parabolic subgroups and a quantum Giambelli formula to compute the Gromov–Witten invariants, if the Lie group G is not of type A. This was due to serious difficulties in extending the methods used in the proofs in the SL_N case to the other Lie types (for a discussion of one such problem, see [KT1, §4]).

The remainder of this talk reports on joint work with Andrew Kresch [KT1] [KT2] [KT3]. We answer all three basic questions for the quantum cohomology rings of the type B, C and D Grassmannians mentioned previously. For the Lagrangian Grassmannian, questions (a) and (b) are addressed by the following Theorem.

Theorem 1. *The ring $QH^*(LG)$ is presented as a quotient of the polynomial ring $\mathbb{Z}[\sigma_1, \ldots, \sigma_n, q]$ by the relations*

$$\sigma_i^2 + 2\sum_{k=1}^{n-i}(-1)^k \sigma_{i+k}\sigma_{i-k} = (-1)^{n-i}\sigma_{2i-n-1}\, q.$$

The Schubert class σ_λ in this presentation is given by the Giambelli formulas

$$\sigma_{i,j} = \sigma_i\sigma_j + 2\sum_{k=1}^{n-i}(-1)^k \sigma_{i+k}\sigma_{j-k} + (-1)^{n+1-i}\sigma_{i+j-n-1}\, q$$

for $i > j > 0$, and

$$\sigma_\lambda = \mathrm{Pfaffian}[\sigma_{\lambda_i,\lambda_j}]_{1 \leqslant i < j \leqslant r}, \tag{3}$$

where quantum multiplication is employed throughout. In other words, quantum Giambelli for $LG(n,2n)$ coincides with classical Giambelli for $LG(n+1,2n+2)$, when the class $2\,\sigma_{n+1}$ is identified with q.

Note that Theorem 1 is not saying that the cohomology ring of $LG(n+1, 2n+2)$ is isomorphic to $QH^*(LG(n,2n))$, for the relation $\sigma_{n+1}^2 = 0$ in the former ring fails to hold in the latter (as $q^2 \neq 0$). It turns out that the algebra of Schur Q-polynomials is not sufficient to describe the multiplication in $QH^*(LG)$. Instead, we use *$\widetilde{Q}$-polynomials*; this is a family of symmetric polynomials modelled on Schur's Q-polynomials, and defined by Pragacz and Ratajski [PR] in their work on degeneracy loci. The $\widetilde{Q}$-polynomials seem to be the correct *geometric* analogue of Schur's S-polynomials in type C. Their multiplication also governs the *Arakelov theory* of LG, regarded as a scheme over the ring of rational integers [T].

One consequence of Theorem 1 is a *quantum Pieri rule* for multiplying a class σ_λ in $H^*(LG)$ by a special Schubert class, which extends the classical one of [BH]. For another, define an involution $*$ on $\mathcal{D}_n$ by

$$*\,(\lambda_1, ..., \lambda_\ell) = (n+1-\lambda_\ell, \ldots, n+1-\lambda_1),$$

and let ρ_n be the partition $(n, n-1, \ldots, 1)$ (which corresponds to the class of a point in LG). We then have the following rule for multiplying by the point class:

Theorem 2. $\sigma_\lambda \cdot \sigma_{\rho_n} = \sigma_{\lambda'^*}\, q^{\ell(\lambda)}$ *in $QH^*(LG)$. In particular, $\sigma_{\rho_n} \cdot \sigma_{\rho_n} = q^n$, which implies that there is a unique rational curve of degree n passing through* 3 *general points on $LG(n,2n)$.*

The above results are part of the *quantum Schubert calculus* for LG.

The Orthogonal Grassmannian $OG(n, 2n+1)$

Consider now $V' = \mathbb{C}^{2n+1}$ equipped with a non-degenerate symmetric form. The maximal isotropic subspaces in V' are n-dimensional and parametrized by the odd orthogonal Grassmannian $OG = OG(n, 2n+1)$. The cohomology ring $H^*(OG, \mathbb{Z})$ has an additive basis of Schubert classes τ_λ, for $\lambda \in \mathcal{D}_n$. As a $\mathbb{Z}$-module, $QH^*(OG)$ is identified with $H^*(OG) \otimes \mathbb{Z}[q]$, but this time the degree of q is $2n$.

Theorem 3. *The ring $QH^*(OG)$ is presented as a quotient of the polynomial ring $\mathbb{Z}[\tau_1, \dots, \tau_n, q]$ modulo the relations*

$$\tau_i^2 + 2\sum_{k=1}^{i-1}(-1)^k \tau_{i+k}\tau_{i-k} + (-1)^i \tau_{2i} = 0$$

for all $i < n$, together with the quantum relation

$$\tau_n^2 = q.$$

The Schubert class τ_λ in this presentation is given by the Giambelli formulas

$$\tau_{i,j} = \tau_i\tau_j + 2\sum_{k=1}^{j-1}(-1)^k \tau_{i+k}\tau_{j-k} + (-1)^j \tau_{i+j}$$

for $i > j > 0$, and

$$\tau_\lambda = \mathrm{Pfaffian}[\tau_{\lambda_i,\lambda_j}]_{1\leqslant i<j\leqslant r}, \tag{4}$$

where quantum multiplication is employed throughout. In other words, classical Giambelli and quantum Giambelli coincide for $OG(n, 2n+1)$.

We thus see that the answers to questions (a) and (b) for OG are directly analogous to those for type A Grassmannians. The quantum Schubert calculus is governed this time by the multiplication of $\widetilde{P}$*-polynomials.* The quantum Pieri rule for OG implies that for each $\lambda \in \mathcal{D}_{n-1}$, we have $\tau_n\,\tau_\lambda = \tau_{(n,\lambda)}$, while $\tau_n\,\tau_{(n,\lambda)} = \tau_\lambda\,q$ in $QH^*(OG)$. It follows that to compute all the Gromov–Witten invariants for OG, it suffices to calculate $\langle \tau_\lambda, \tau_\mu, \tau_\nu\rangle_d$ for all $\lambda, \mu, \nu \in \mathcal{D}_{n-1}$.

Partitions in $\mathcal{D}_{n-1}$ also parametrize the Schubert classes σ_λ in the (quantum) cohomology ring of the Lagrangian Grassmannian $LG(n-1, 2n-2)$. Let us compare the two spaces at hand in a table:

X	$\dim_{\mathbb{C}}(X)$	Schubert classes	$\deg(q)$
$OG(n, 2n+1)$	$n(n+1)/2$	$\tau_\lambda\ ,\ \lambda \in \mathcal{D}_n$	$2n$
$LG(n-1, 2n-2)$	$n(n-1)/2$	$\sigma_\lambda\ ,\ \lambda \in \mathcal{D}_{n-1}$	n

In the next result, $*$ and $'$ denote the involutions that were defined previously, but on the set $\mathcal{D}_{n-1}$.

Theorem 4. *Suppose that $\lambda \in \mathcal{D}_{n-1}$ is a partition with $\ell(\lambda) = 2d+e+1$ for some nonnegative integers d and e. For any $\mu, \nu \in \mathcal{D}_{n-1}$, we have an equality*

$$\langle \tau_\lambda, \tau_\mu, \tau_\nu\rangle_d = \langle \sigma_{\lambda^*}, \sigma_{\mu'}, \sigma_{\nu'}\rangle_e \tag{5}$$

of Gromov–Witten invariants for $OG(n, 2n+1)$ and $LG(n-1, 2n-2)$, respectively. If λ is zero or $\ell(\lambda) < 2d+1$, then $\langle \tau_\lambda, \tau_\mu, \tau_\nu\rangle_d = 0$.

Remarks. 1) The left hand side of (5) is obviously symmetric in λ, μ and ν, unlike the right hand side. This reflects a $(\mathbb{Z}/2\mathbb{Z})^3$-symmetry enjoyed by the Gromov–Witten invariants for both LG and OG.

2) We lack a purely geometric argument that would 'explain' Theorem 4. Instead, we first prove the main Theorems 1 and 3 and develop the quantum Schubert calculus for both spaces, then deduce Theorem 4 using algebra.

Proofs of the Main Results

The most difficult part of our approach to Theorems 1 and 3 is to establish the quantum Giambelli formulas (3) and (4). The arguments needed come from two sources: algebraic geometry and combinatorics. In conclusion, we mention three key ingredients that occur in the proofs; for precise definitions of the terms that follow, see [KT1], [KT2] and [KT3].

- Intersection theory on the *Lagrangian (resp. orthogonal) Quot scheme* which compactifies the moduli space of degree d maps from $\mathbb{P}^1$ to LG (resp. OG). These Quot schemes are singular in general.
- Degeneracy loci for *isotropic morphisms* of vector bundles.
- Combinatorics of $\widetilde{P}$, $\widetilde{Q}$ and *Schubert polynomials* for the Lie types B, C and D.

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University of Pennsylvania and Max-Planck-Institut
E-mail address: harryt@math.upenn.edu