An Analytic Discriminant for Polarized Algebraic K3 Surfaces

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Introduction and Summary of Results.

In this paper we will define and study an analytic discriminant on the moduli space of polarized algebraic K3 surfaces. The tools that we employ to define our analytic discriminant in the case of $K3$ surfaces include: the Piatetski-Shaprio and Shafarevich proof [PSS 71] of a global Torelli theorem for marked, polarized, algebraic $K3$ surfaces; Kulikov's proof [Ku 77] of the surjectivity of the period mapping and study of degenerating polarized, algebraic $K3$ surfaces of a fixed degree; Yau's theorem [Ya 78] on the existence of a Kähler-Einstein metric lying in any Kähler class; the analytic part of the arithmetic Riemann-Roch theorem, in various forms as in [Fa 92], [FS 90] or [GS 92], for the universal family of marked, polarized, algebraic $K3$ surfaces of a fixed degree; our construction of a particular holomorphic family of holomorphic 2-forms on the moduli space of marked, polarized, algebraic $K3$, which is analogous to the family of forms $\{dz\}$ for elliptic curves; and our construction of a $K3$ modular parameter, which is analogous to the q parameter on the hyperbolic upper half plane.

We begin by showing that one can express the discriminant of an elliptic curve through the Quillen metric on the determinant line associated to the trivial sheaf when metrized with a flat metric. These calculations lead us to our definition of an analytic discriminant for a polarized, algebraic $K3$ surfaces, which we then show has many properties analogous to those for the elliptic curve discriminant. Let us now summarize the results in this paper.

In §1 we recall the discriminant for elliptic curves in a manner suitable for our purposes. Let E denote an elliptic curve. By the uniformization theorem, there is a flat metric on E, which is unique up to a multiplicative constant. Choose such a metric μ, and let $\mathbf{\Delta}_\mu(E)$ be the associated Laplacian which acts on the space of smooth functions on E. Let $\det^*\mathbf{\Delta}_\mu(E)$ be the non-zero part of the determinant of the Laplacian $\mathbf{\Delta}_\mu(E)$ obtained through the usual zeta function regularization. It is elementary to show that the quotient

$$\det{}^*\mathbf{\Delta}_\mu(E)/\mathrm{vol}_\mu(E)$$

is independent of the scale of the flat metric μ, hence is an invariant of the elliptic curve E. We define the **(logarithmic) unmarked discriminant** $\delta_{\mathrm{unm}}(E)$ to be

$$\delta_{\mathrm{unm}}(E) = \log\left[\det{}^*\mathbf{\Delta}_\mu(E)/\mathrm{vol}_\mu(E)\right].$$

Further, we show that the unmarked discriminant is a potential for the canonical Weil-Petersson metric on the moduli space of elliptic curves. Specifically, if μ_{WP}

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denotes the Weil-Petersson metric on the hyperbolic upper half plane $\mathbf{h}$, then

$$-dd^c\delta_{\mathrm{unm}} = \mu_{\mathrm{WP}}.$$

Now, let us mark the elliptic curve E by choosing a canonical basis of the first homology group $H_1(E, \mathbf{Z})$. With this, E can be viewed as the complex plane $\mathbf{C}$ modulo the $\mathbf{Z}$ lattice generated by 1 and τ where $\tau \in \mathbf{C}$ with $\mathrm{Im}(\tau) > 0$, meaning τ is an element of $\mathbf{h}$. We shall denote the marked elliptic curve by E_τ. Let z be the usual holomorphic parameter on $\mathbf{C}$, so one can take dz to be a holomorphic 1-form on E_τ, and we have

$$\mathrm{Im}(\tau) = \|dz\|^2_{L^2} = \frac{i}{2}\int\limits_{E_\tau} dz \wedge d\bar{z}.$$

The function on $\mathbf{h}$ defined by

$$\det{}^*\mathbf{\Delta}_\mu(E_\tau)/[\mathrm{vol}_\mu(E_\tau)\mathrm{Im}(\tau)]$$

is, in fact, the (inverse square of the) Quillen norm of a section of the determinant line $\det H(\mathcal{O})$ over E_τ. By applying the arithmetic Riemann-Roch theorem, and using the fact that the canonical sheaf on an elliptic curve is trivial, we have that the above defined function on $\mathbf{h}$ is the absolute value of a non-vanishing holomorphic function. Recall that $\mathrm{Im}(\tau)$ is modular of weight -2 with respect to the action by the discrete subgroup $PSL(2, \mathbf{Z})$, and also recall that the quotient space $PSL(2, \mathbf{Z})\backslash\mathbf{h}$, has precisely two elliptic points of order 2 and 3. Using these facts, we conclude that there exists a non-vanishing holomorphic function f on $\mathbf{h}$ which is a modular form of weight 2 with respect to the action by $PSL(2, \mathbf{Z})$ such that

$$\det{}^*\mathbf{\Delta}_\mu(E_\tau)/[\mathrm{vol}_\mu(E_\tau)\mathrm{Im}(\tau)] = |f(\tau)|^2.$$

It is known that the space of non-vanishing modular forms of weight 12 with respect to the action of $PSL(2, \mathbf{Z})$ on $\mathbf{h}$ is 1 dimensional over $\mathbf{C}$, and this is generated by the Dedekind delta function $\Delta(\tau)$, which is the 24-th power of the Dedekind eta function $\eta(\tau)$. By direct computation, one can show that $f = \eta^2$, but, for the purposes of this paper, it suffices to note that, at this point, we have shown $f = c\eta^2$, for some constant c. Recall that if one views the elliptic curve E_τ as the zero set of a cubic equation in $\mathbf{P}^2$, then the function f^{12} is a multiple of the discriminant of the cubic. In this way, we have used Quillen metrics to obtain a realization of the classical discriminant of an elliptic curve.

We can now take the same approach to define and then study an analytic discriminant associated to any polarized, algebraic $K3$ surface. We use the term "analytic discriminant" because in this article we only consider analytic properties of this discriminant.

In §2 we present the background material needed to define our analytic discriminant in the case of a polarized, algebraic $K3$ surface. Recall that a $K3$ surface is a compact, complex, non-singular surface X, not necessarily algebraic, with trivial canonical sheaf $\mathcal{K}$ and with $H^1(X, \mathcal{O}) = 0$. For this paper, all $K3$ surfaces under consideration are algebraic. A polarization of the (algebraic) $K3$ surface is the choice of an ample divisor class on X. There is an associated Kähler form μ_{FS} which is a rational multiple of the pullback of the Fubini-Study (1,1)-form on $\mathbf{P}^N$ via a projective embedding of X induced by a power of the given ample divisor. By

Yau's theorem, there exists a Kähler-Einstein (1,1)-form, unique up to multiplicative constant, which is cohomologous to μ_{FS}. Choose a scale of this form, say μ. Using the associated Kähler-Einstein metric, one can compute the analytic torsion $\mathrm{tor}_\mu(X)$ of the determinant line associated to the trivial sheaf $\mathcal{O}$. We then prove that the difference

$$\mathrm{tor}_\mu(X) - \log \mathrm{vol}_\mu(X)$$

is independent of the scale of the Kähler-Einstein form μ. We define the **(logarithmic) unmarked discriminant** on (X, L) to be

$$\delta_{\mathrm{unm}}(X, L) = \mathrm{tor}_\mu(X) - \log \mathrm{vol}_\mu(X).$$

Then δ_{unm} is a function on the moduli space of polarized $K3$ surfaces of a fixed degree d, denoted by $\mathcal{M}^d_{\mathrm{p}}$. As in the case of elliptic curves, our unmarked discriminant is a potential for (a scaling of) the Weil-Petersson metric on $\mathcal{M}^d_{\mathrm{p}}$. Specifically, if μ_{WP} is the Kähler form of this metric, then

$$-dd^c\delta_{\mathrm{unm}} = \mu_{\mathrm{WP}},$$

which is analogous to the situation for elliptic curves. We prove this result by using the arithmetic Riemann-Roch theorem and results due to Tian [Ti 88] and Todorov [To 89] which state that the period map of a marked, polarized $K3$ surface also gives a potential for the Weil-Petersson metric.

By a marking ϕ of a $K3$ surface one means a choice of a canonical basis of the second integral homology group $H_2(X, \mathbf{Z})$. It is known that when endowed with the inner product coming from cup product, the second integral homology group $H_2(X, \mathbf{Z})$ is an even, unimodular lattice of rank 22 and signature $(3, 19)$. All such lattices are isomorphic, and a marking amounts to a choice of an isomorphism of $H_2(X, \mathbf{Z})$ with a fixed lattice Λ satisfying these properties. The triple (X, L, ϕ) is called a marked, polarized $K3$ surface. One defines the degree to be the integer d such that $(L \cdot L) = 2d - 2$. It was shown by Piatetski-Shapiro and Shafarevich that there exists a universal family of such triples, namely marked, polarized $K3$ surfaces of a fixed degree d, which we shall denote by

$$\pi : \mathcal{X} \to \mathcal{M}^d_{\mathrm{mp}},$$

where $\mathcal{M}^d_{\mathrm{mp}}$ is called the moduli space. Additional work due to Kulikov shows that the moduli space $\mathcal{M}^d_{\mathrm{mp}}$ can be represented by a Zariski open subset of the hermitian symmetric space

$$\mathbf{h}_{\mathrm{K3}} = SO(2, 19)/(SO(2) \times SO(19)).$$

By results due to Kulikov, there is a discrete subgroup Γ_d of $SO(2, 19)$ such that $\mathcal{M}^d_{\mathrm{p}}$ is a quotient

$$\mathcal{M}^d_{\mathrm{p}} \cong \Gamma_d \backslash \mathcal{M}^d_{\mathrm{mp}},$$

with natural projection

$$\pi_{\mathrm{unm}} : \mathcal{M}^d_{\mathrm{mp}} \to \mathcal{M}^d_{\mathrm{p}},$$

which is the map that discards the marking. Hence, our unmarked discriminant δ_{unm} is a real-valued function on a Zariski open subset $\mathcal{M}^d_{\mathrm{p}}$ of the $\Gamma_d \backslash \mathbf{h}_{\mathrm{K3}}$.

In order to define a marked discriminant on $\mathcal{M}^d_{\mathrm{mp}}$, we need to choose a non-vanishing holomorphically varying family of holomorphic 2-forms on $\mathcal{M}^d_{\mathrm{mp}}$, which is the analogue of chosing the family of forms $\{dz\}$ on the moduli space of marked elliptic curves. We choose such a family of forms $\{\eta\}$ as follows.

Since $\mathcal{M}^d_{\mathrm{mp}}$ is a Zariski open subset of a bounded hermitian symmetric domain, the sheaf $\pi_*\mathcal{K}_{\mathcal{X}/\mathcal{M}^d_{\mathrm{mp}}}$ is trivial. Kulikov shows that the boundary of $\Gamma_d\backslash\mathbf{h}_{K3}$ relative to the Baily-Borel compactification consists of points which can be represented by two types of singular surfaces: $\mathbf{P}^2$ noded along two distinct imbeddings of $\mathbf{P}^1$ with a node; and $\mathbf{P}^2$ noded along distinct two imbeddings of a non-singular elliptic curve. We show that from any point in $p \in \mathcal{M}^d_{\mathrm{mp}}$, we can embed a disc $\mathcal{D}_p$ into $\bar{\mathcal{M}}^d_{\mathrm{p}}$ that $\mathcal{D}_p$ contains $\pi_{\mathrm{unm}}(p)$ and the point in $\bar{\mathcal{M}}^d_{\mathrm{p}}$ corresponding to $\mathbf{P}^2$ noded along two imbeddings of unimoded $\mathbf{P}^1$, and such that $\mathcal{D}_p$ contains no other boundary point of $\bar{\mathcal{M}}^d_{\mathrm{p}}$. There exists a unique, up to sign, holomorphically varying family of holomorphic 2-forms $\{\eta\} = \{\eta_{(X,L,\phi)}\}$ such that, when restricted to the 1 parameter family of forms on the family of $K3$ surfaces over $\mathcal{D}_p$, with a family of markings constructed from the Clemens map as in Griffiths's paper [Gr 70], $\eta_{(X,L,\phi)}$ limits to a non-zero form on $\mathbf{P}^2$ whose Poincaré residue along the node $\mathbf{P}^1$ is a meromorphic form on $\mathbf{P}^1$ with residue ± 1 at the nodes of $\mathbf{P}^1$. A construction of this family of forms is given in §5.

With the 2-form $\eta_{(X,L,\phi)}$, we define, for each (X, L, ϕ),the L^2 norm

$$\|\eta_{(X,L,\phi)}\|^2_{L^2} = -\int_X \eta_{(X,L,\phi)} \wedge \eta_{(X,L,\phi)}.$$

If (X, L, ϕ) is a marked, polarized $K3$ surface, we define the **(logarithmic) marked discriminant** to be

$$\delta_{\mathrm{mar}}((X, L, \phi), \eta) = \mathrm{tor}_\mu(X) - \log\left(\mathrm{vol}_\mu(X) \cdot \|\eta_{(X,L,\phi)}\|^2_{L^2}\right).$$

Using the arithmetic Riemann-Roch theorem, we show that there exists a non-vanishing holomorphic function f on $\mathcal{M}^d_{\mathrm{mp}}$ such that

$$\delta_{\mathrm{mar}} = \log |f|^2.$$

Kondō proved that the commutator group $[\Gamma_d, \Gamma_d]$ is a finite group of order 16. Therefore, we have that f^{32} is a non-vanishing, holomorphic modular form on $\mathcal{M}^d_{\mathrm{mp}}$ of weight 32 with respect to the action by the discrete group Γ_d. These results are presented in §2.

In §3 we consider the special case of Kummer surfaces. Recall that a Kummer surface is a particular example of a $K3$ surface obtained from an abelian surface A by taking the quotient of A by the involution $z \to -z$, then blowing up the 16 singular points. Any Kummer surface can be given a marking from the lattice of a marked abelian surface, and, in addition, a polarization of degree 2 coming from the theta divisor on the abelian surface. In the special case when the Kummer surface comes from a product of elliptic curves, we show that the marked discrminant of the Kummer surface is, up to a constant depending solely on the degree of the polarization, the product of the discriminants of the elliptic curves.

In §4 we consider the behavior of our marked and unmarked discriminants for degenerating families of $K3$ surfaces. As stated above, Kulikov determined the types of degenerating families of polarized, algebraic $K3$ surfaces that exist. For any point in $\mathcal{M}_{\mathrm{p}}^d$, we define a $K3$ modular parameter q which is analogous to the elliptic modular parameter $q_\tau = \exp(2\pi i\tau)$ on the upper half plane $\mathbf{h}$. Using the $K3$ modular parameter, we determine the asymptotic behavior of our marked and unmarked $K3$ discriminants for a degenerating one parameter family of algebraic $K3$ surfaces $\{(X_t, L_t, \phi_t)\}$ with $t \to 0$. For example, if the limit $K3$ surface is $\mathbf{P}^2$ noded along two embeddings of a (marked) elliptic curve E_τ, then the constant term in the asymptotic expansion of the $K3$ discriminant is essentially the discriminant of the elliptic curve E_τ. More precisely, there exist universal constants c_1 and c_2 such that

$$6 \log \delta_{\mathrm{mar}}((X_t, L_t, \phi_t), \eta_t) = \log |q([X_t, L_t])| + 6 \log \delta_{\mathrm{mar}}(E_\tau) + c_1 + o(1),$$

and

$$\begin{aligned} 6 \log \delta_{\mathrm{unm}}(X_t, L_t) &= \log |q([X_t, L_t])| + 6 \log \log |q([X_t, L_t])| \\ &\quad + 6 \log \delta_{\mathrm{unm}}(E_\tau) + c_2 + o(1). \end{aligned}$$

These formulas are analogous to known asymptotic expansions for the marked and unmarked discriminants for elliptic curves.

As stated above, this paper concentrates on the analytic aspects of our $K3$ discriminants. In forthcoming papers, we will extend the analytic aspects of our discriminants to define and study discriminants on Calabi-Yau manifolds and on hyper-Kählerian manifolds. In the case of the $K3$ discriminant, we will establish a type of Kronecker's limit formula which will relate our discriminant to the constant term in an expansion of an Eisenstein series on $\mathcal{M}_{\mathrm{mp}}^d$. This result will use work of Indik on non-holomorphic Eisenstein series on certain orthogonal groups. As stated above, in the case of Kummer surfaces associated to products of elliptic curves, our discriminant is essentially a product of the discriminants of the elliptic curves. So, for this case, our analytic discriminant is essentially algebraic. In a future article we will give an algebraic definition for our analytic discriminant.

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§1. The Discriminant for an Elliptic Curve.

We begin our study of discriminants by reviewing the existing theory of analytic discriminants for non-singular elliptic curves. The basic results of this section are well established, but, as far as we know, have not been formulated in a manner that readily extends to the setting of polarized $K3$ surfaces, which will be given in the next section.

Throughout this section, we let E denote a connected, non-singular elliptic curve that is defined over $\mathbf{C}$. By a **marking** on E we mean a choice of a canonical basis L of the first homology group $H_1(E, \mathbf{Z})$, which has an intersection pairing of signature $(1,1)$. By the uniformization theorem, the marked elliptic curve (E, L) can be realized as the complex plane $\mathbf{C}$ modulo the $\mathbf{Z}$ lattice that is generated by the complex numbers 1 and τ where $\tau = a + ib$ with $b > 0$, so τ is a point in the upper half plane $\mathbf{h}$. We shall denote the marked elliptic curve by E_τ. The marking L then corresponds to the choice of cycles in $H_1(E_\tau, \mathbf{Z})$ given by the boundary of the usual period parallelogram of E_τ in $\mathbf{C}$ spanned by 1 and τ. Throughout this section we shall denote an elliptic curve by E and a marked elliptic curve by E_τ.

Let $\mathcal{M}_{\text{mar}}$ denote the moduli space of isomorphism classes of marked elliptic curves with the requirement that an isomorphism preserves the complex structure and the marking. The space $\mathcal{M}_{\text{mar}}$ can be realized as either the upper half plane $\mathbf{h}$ or the open unit disc $\mathcal{D}$ in $\mathbf{C}$ under the analytic isomorphism

$$\mathbf{h} \to \mathcal{D}$$

given by

$$\tau \mapsto \frac{\tau - i}{\tau + i}.$$

The space

$$\mathcal{M}_{\text{unm}} = PSL(2, \mathbf{Z})\backslash\mathbf{h} \tag{1.1}$$

parameterizes the isomorphism classes of unmarked elliptic curves. The moduli space $\mathcal{M}_{\text{mar}}$ possesses a natural invariant metric, called the **Weil-Petersson metric**, which in the upper half plane model of $\mathcal{M}_{\text{mar}}$ can be expressed through the positive $(1,1)$-form

$$\mu_{\text{WP}} = -dd^c \log(\text{Im}(\tau)). \tag{1.2}$$

The form (1.2) is invariant under the action by $PSL(2, \mathbf{Z})$, hence descends to a metric on $\mathcal{M}_{\text{unm}}$. The Weil-Petersson metric is characterized by the fact that $\mathcal{M}_{\text{mar}}$ has constant Griffiths function (i.e., negative Gauss curvature) and the moduli space (1.1) has volume 1/12.

Let $\mathcal{K}$ denote the canonical sheaf on E. Any positive $(1,1)$-form μ on E induces a metric ρ on the canonical sheaf $\mathcal{K}$ and, by duality, induces a (trivial) metric on $\mathcal{O}$, meaning the metric ρ is equal to the transition functions of $\mathcal{O}$ (see page 94 of [La 87a]). If ω denotes any non-zero holomorphic 1-form on E, then we have the associated metric on $\mathcal{K}$ defined through the positive $(1,1)$-form

$$\mu = \frac{i}{2}\omega \wedge \bar{\omega} \tag{1.3}$$

and volume

$$\text{vol}_\mu(E) = \int_E \mu. \tag{1.4}$$

The metric (1.3) is called a **flat metric** on E since such a metric has Griffiths function that is identically zero (see page 100 of [La 87a]). Any two flat metrics are real, non-zero, scalar multiples of each other, which corresponds to the fact that any two non-zero holomorphic one-forms scalar multiples of each other.

From the metric (1.3), we have an L^2-norm of ω, which is

$$\|\omega\|_{L^2}^2 = \frac{i}{2}\int\limits_E \omega \wedge \bar{\omega} \tag{1.5}$$

(see page 5 of [La 88]). There is a canonical choice of holomorphic 1-form ω on any marked, elliptic curve E_τ, namely

$$\omega = dz, \tag{1.6}$$

where z is the standard local coordinate on the complex plane $\mathbf{C}$. For this choice of 1-form, we have

$$\|dz\|_{L^2}^2 = \frac{i}{2}\int\limits_{E_\tau} dz \wedge d\bar{z} = \mathrm{Im}(\tau). \tag{1.7}$$

The discussion given in [Fy 73], beginning on page 51, yields the following intrinsic characterization of the form (1.6). Consider the degenerating family of unmarked elliptic curves E_τ obtained by letting $\tau \to i\infty$ in (1.1) and $|\mathrm{Re}(\tau)| \le 1/2$. The limit algebraic curve is holomorphically equivalent to a uninoded $\mathbf{P}^1$, which we denote by $\mathbf{P}^1_{\mathrm{nod}}$. We can view $\mathbf{P}^1_{\mathrm{nod}}$ as $\mathbf{P}^1$ with two distinct points p and q identified. The form (1.6) on E_τ varies holomorphically over the moduli space $\mathcal{M}_{\mathrm{unm}}$ and limits to a section of the canonical sheaf on $\mathbf{P}^1_{\mathrm{nod}}$ (see chapter 3 of [Fy 73]), which lifts to a section of the line sheaf $\mathcal{K}(-p-q)$ on $\mathbf{P}^1$. The family of forms (1.6) is uniquely characterized (up to sign) by the fact that the family varies holomorphically over the moduli space (1.1) and limits to the meromorphic 1-form on $\mathbf{P}^1$ with residue 1 at p and -1 at q.

Alternatively, the family (1.6) of holomorphically varying 1-forms can be described as follows. Let E_τ be the degenerating family of marked elliptic curves described above. Let $\{\omega_\tau\}$ be any holomorphically varying family of 1-forms, so ω_τ has two periods on E_τ. If we divide ω_τ by the period of the vanishing cycle A in $H_1(E_\tau, \mathbf{Z})$, we obtain the family (1.6).

Since $H^0(E, \mathcal{K}) = 1$ for any elliptic curve E, two families of holomorphically varying 1-forms that vary over $\mathcal{M}_{\mathrm{mar}}$ differ by a multiplicative factor which is a non-vanishing holomorphic function on $\mathcal{M}_{\mathrm{mar}}$. Hence if ω is a family of holomorphic 1-forms that vary holomorphically over $\mathcal{M}_{\mathrm{mar}}$, the quantity

$$dd^c \log \|\omega\|_{L^2}^2$$

is well-defined. Combining this observation with (1.2) and (1.7) allows us to record the following result.

Proposition 1.1. *Let $\{\omega\}$ be a family of holomorphic 1-forms that vary holomorphically over $\mathcal{M}_{\mathrm{mar}}$. Then we have*

$$-dd^c \log \|\omega\|_{L^2}^2 = \mu_{\mathrm{WP}}.$$

In other words, $-\log \|\omega\|_{L^2}^2$ *is a potential for the Weil-Petersson metric on* $\mathcal{M}_{\mathrm{mar}}$.

Let us now recall briefly the definition of analytic torsion and Quillen norms associated to the trivial sheaf on E. This will lead natually to our definition of discriminants associated to E_τ

Fix a positive $(1,1)$-form μ on E and associated hermitian metric ρ on $\mathcal{O}$. With this data, there is a **Laplacian** $\boldsymbol{\Delta}_\mu$ that acts on the space of continuously twice-differentiable sections of $\mathcal{O}$. The Laplacian is positive and self-adjoint, and has a purely discrete spectrum. We denote the non-zero eigenvalues of $\boldsymbol{\Delta}_\mu$ by

$$0 < \lambda_1(\mu) \le \lambda_2(\mu) \dots .$$

By Weyl's Law, we can define the **spectral zeta function** $\zeta_\mu(s)$ for $\mathrm{Re}(s)$ sufficiently large by

$$\zeta_\mu(s) = \sum_{j=1}^{\infty} \lambda_j^{-s}(\mu) = \frac{1}{\Gamma(s)} \int_0^\infty \Big(\sum_{j=1}^{\infty} e^{-\lambda_j(\mu)t} \Big) t^s \frac{dt}{t}. \tag{1.8}$$

The exponential sum in (1.8) is the trace of the heat kernel associated to the Laplacian $\boldsymbol{\Delta}_\mu$ minus the integer $h^0(E,\mathcal{O}) = 1$, which is equal to the dimension of the zero eigenspace of $\boldsymbol{\Delta}_\mu$. By the small time asymptotics of the heat kernel (see [MP 49] or [See 67], for example), $\zeta_\mu(s)$ has a meromorphic continuation to $\mathbf{C}$ which is holomorphic at $s=0$. The exponential of **analytic torsion**, also called the **determinant of the Laplacian**, is defined by

$$\det{}^*\boldsymbol{\Delta}_\mu = \exp(-\zeta'_\mu(0)).$$

The asterisk reflects the fact that the zero eigenvalues have been omitted in (1.8).

Theorem 1.2. *Let* μ *denote any flat metric on the elliptic curve* E. *Then the quotient*

$$\det{}^*\boldsymbol{\Delta}_\mu / \mathrm{vol}_\mu(E)$$

is independent of the scale of the flat metric μ *on* E, *hence is an invariant of the unmarked elliptic curve* E.

Proof. Let μ be any flat metric on E and $c \in \mathbf{R}^+$. Trivially we have

$$\mathrm{vol}_{c\mu}(E) = c \cdot \mathrm{vol}_\mu(E).$$

By the definition of the Laplacian, the sequence of eigenvalues associated to $\boldsymbol{\Delta}_{c\mu}$ is related to the sequence of eigenvalues of $\boldsymbol{\Delta}_\mu$ through multiplication by the scalar c^{-1}. Therefore, we have

$$\zeta_{c\mu}(s) = c^s \zeta_\mu(s),$$

from which we obtain the relation

$$\det{}^*\boldsymbol{\Delta}_{c\mu} = c^{-\zeta_\mu(0)} \det{}^*\boldsymbol{\Delta}_\mu.$$

Since μ is a flat metric, the small time asymptotics of the heat kernel are of the form

$$\sum_{j=1}^{\infty} e^{-\lambda_j(\mu)t} = \frac{\mathrm{vol}_\mu(E)}{4\pi t} - 1 + O(e^{-c/t}) \quad \text{as } t \to 0,$$

for some positive constant c (see page 149 of [Ch 84] or page 84 of [BGV 92]). Combining this expansion with the proof of the meromorphic continuation of the spectral zeta function (see, for example, section 1 of [JoLa 93]) yields

$$\zeta_\mu(0) = -1,$$

from which the theorem follows. □

Definition 1.3. *Let E be an unmarked elliptic curve defined over* **C**, *and let μ denote any flat metric on E. The* **unmarked (logarithmic) discriminant** *$\delta_{\mathrm{unm}}(E)$ of E is defined to be*

$$\delta_{\mathrm{unm}}(E) = \log[\det{}^*\boldsymbol{\Delta}_\mu/\mathrm{vol}_\mu(E)] = -\zeta_\mu'(0) - \log \mathrm{vol}_\mu(E).$$

The unmarked (logarithmic) discriminant is a function

$$\delta_{\mathrm{unm}} : \mathcal{M}_{\mathrm{unm}} \to \mathbf{R}.$$

To continue, let us relate the unmarked discriminant to the Weil-Petersson metric via Quillen norms, which we now describe. The **determinant line** $\det H(\mathcal{O})$ associated to $\mathcal{O}$ is defined to be the 1-complex dimensional vector space

$$\det H(\mathcal{O}) = H^0(E,\mathcal{O}) \otimes [H^1(E,\mathcal{O})]^{-1} \cong H^0(E,\mathcal{O}) \otimes H^0(E,\mathcal{K}).$$

The above isomorphism is via Serre duality, which is an isometry of metrized line sheaves (see page 97 of [De 88]). Let η denote a non-zero element of $H^0(E,\mathcal{O})$, which we can view as a constant function on E, and ω denote a non-zero element of $H^0(E,\mathcal{K})$, which we can view as a non-zero holomorphic 1-form on E. A **metric** or norm on the line $\det H(\mathcal{O})$ is equivalent to the assignment of a length to the element

$$\Upsilon_E = \Upsilon = \eta \wedge \omega$$

in $\det H(\mathcal{O})$. The square of the $\mathbf{L}^2$**-norm** on $\det H(\mathcal{O})$ is defined by

$$\|\Upsilon\|_{L^2}^2 = \langle \eta,\eta\rangle\langle\omega,\omega\rangle = |\eta|^2 \mathrm{vol}_\mu(E)\cdot \|\omega\|_{L^2}^2, \tag{1.9}$$

and the square of the **Quillen norm** on $\det H(\mathcal{O})$ is defined by

$$\|\Upsilon\|_Q^2 = \|\Upsilon\|_{L^2}^2 \cdot (\det{}^*\boldsymbol{\Delta}_\mu)^{-1}.$$

If we consider the marked elliptic curve E_τ and let $\omega = dz$, as in (1.6), and $\eta = 1$, then

$$\log\|\Upsilon_{E_\tau}\|_Q^2 = -\delta_{\mathrm{unm}}(E_\tau) + \log\|dz\|_{L^2}^2 = -\delta_{\mathrm{unm}}(E_\tau) + \log(\mathrm{Im}(\tau)). \tag{1.10}$$

The element Υ_{E_τ} in (1.9) is defined for a fixed marked elliptic curve E_τ. We then view (1.10) as the function

$$\mathcal{M}_{\mathrm{mar}} \to \mathbf{R}_{>0}$$

given by

$$E_\tau \mapsto \|\Upsilon_{E_\tau}\|_Q^2$$

The following result shows that the unmarked discriminant can be used to obtain a second potential for the Weil-Petersson metric on $\mathcal{M}_{\mathrm{mar}}$.

Theorem 1.4. *Let $\{\omega_{E_\tau}\}$ be a family of holomorphically varying 1-forms on the moduli space $\mathcal{M}_{\mathrm{mar}}$. Let $\Upsilon_{E_\tau} = 1 \wedge \omega_{E_\tau}$ where 1 corresponds to the constant function 1 on E_τ. Then*

$$dd^c \log \|\Upsilon\|_Q^2 = 0.$$

Equivalently, we have

$$-dd^c \delta_{\mathrm{unm}} = \mu_{\mathrm{WP}}.$$

In other words, $-\delta_{\mathrm{unm}}$ is a potential for the Weil-Petersson metric on $\mathcal{M}_{\mathrm{unm}}$.

Proof. The first assertion follows from the Quillen-Grothendieck-Riemann-Roch theorem, using the fact that the canonical sheaf of an elliptic curve is trivial. The second assertion follows from the first assertion, Proposition 1.1 and (1.7). There are a number of references with general theorems that contain the statement of Theorem 1.4, for example [BK 86] (see remark 12 on page 228), [BGS 87], [Fa 92], [FS 90], and [Qn 86]. The reader is referred to Theorem 3.10 of [Fy 92]. □

Corollary 1.5. *There exists a holomorphic function f on* **h** *such that if E_τ is a marked elliptic and μ is any flat metric on E_τ, then*

$$\det{}^* \mathbf{\Delta}_\mu / [\mathrm{vol}_\mu(E_\tau)\mathrm{Im}(\tau)] = |f(\tau)|^2 .$$

Further, the function f^{12} is a non-vanishing weight 12 modular form on the moduli space $PSL(2, \mathbf{Z})\backslash\mathbf{h}$.

Proof. The equation

$$dd^c \log\left(\det{}^* \mathbf{\Delta}_\mu / [\mathrm{vol}_\mu(E_\tau)\mathrm{Im}(\tau)]\right) = 0$$

follows directly from Theorem 1.4 and (1.9). Hence, locally on **h** the function

$$\log\left(\det{}^* \mathbf{\Delta}_\mu / [\mathrm{vol}_\mu(E_\tau)\mathrm{Im}(\tau)]\right)$$

is the real part of a holomorphic function, which we shall write as $\log f(\tau)^2$ (see page 82 of [Kr 82]). Since **h** is simply connected, $\log f(\tau)^2$ is globally defined, thus establishing the first assertion. It is immediate that f^2 is non-vanishing on **h**. Finally, the invariance assertion follows from the observations that δ_{unm} is $PSL(2, \mathbf{Z})$ invariant and that the map

$$\tau \mapsto \mathrm{Im}(\tau)$$

is of weight -2 with respect to action by the group $PSL(2, \mathbf{Z})$. □

In Corollary 1.5, it is natural to consider the function f^{12} since the moduli space $\mathcal{M}_{\mathrm{unm}}$ has precisely two elliptic points, with orders 2 and 3 (see page 6 of [La 76] or page 86 of [Ser 73]). This point will be discussed further in §3.

Definition 1.6. *Let E_τ be a marked elliptic curve defined over* $\mathbf{C}$, *and let μ denote any flat metric on E_τ. The* **marked (logarithmic) discriminant** *$\delta_{\mathrm{mar}}(E_\tau)$ is defined to be*

$$\begin{aligned}\delta_{\mathrm{mar}}(E_\tau) &= \log\left[\det{}^*\boldsymbol{\Delta}_\mu/[\mathrm{vol}_\mu(E_\tau)\mathrm{Im}(\tau)]\right] \\ &= \log|f(\tau)|^2.\end{aligned}$$

We also define

$$\Delta_{\mathrm{mar}} = f^{12}.$$

Let us now examine the asymptotic behavior of $\delta_{\mathrm{unm}}(E_\tau)$ for a degenerating family of elliptic curves obtained by letting $\tau \to i\infty$ and $|\mathrm{Re}(\tau)| \leq 1/2$. By considering a contour integral for a fundamental domain for (1.1) in $\mathbf{h}$, one shows, in the notation of Corollary 1.5, that the function $f(\tau)^{12}$ vanishes to first order as $\tau \to i\infty$ (see page 6 of [La 76] or page 85 of [Ser 73]). The asymptotic behavior of the period $\|dz\|^2_{L^2}$ is given on page 53 of [Fy 73]. Combining these results with Definition 1.6, we obtain the following theorem.

Theorem 1.7. *Let $\{E_\tau\}$ denote the degenerating family of marked elliptic curves obtained by letting $\tau \to i\infty$ with $|\mathrm{Re}(\tau)| \leq 1/2$. Let $q_\tau = \exp(2\pi i\tau)$. Then there exist constants c_1 and c_2 such that*

$$6\delta_{\mathrm{unm}}(E_\tau) = \log|q_\tau| + 6\log\log|q_\tau| + c_1 + o(1)$$

and

$$6\delta_{\mathrm{mar}}(E_\tau) = \log|q_\tau| + c_2 + o(1).$$

It is important to note that the asymptotics of δ_{unm} are independent of the marking of E_τ.

By combining (1.6), Corollary 1.5 and Definition 1.6, we obtain the following realization of the analytic discriminant $\delta_{\mathrm{unm}}(E_\tau)$.

Theorem 1.8. *There is a unique family of holomorphic 1-forms $\{\omega_{\mathrm{Ar},E_\tau}\}$, varying holomorphically over $\mathcal{M}_{\mathrm{mar}}$, such that*

$$\delta_{\mathrm{unm}}(E_\tau) - \frac{1}{6}c_2 = \log\|\omega_{\mathrm{Ar},E_\tau}\|^2_{L^2}.$$

Proof. Set $C = e^{c_2}$. Then, with notation as above, one sets

$$\omega_{\mathrm{Ar},E_\tau}(z) = C^{1/12}\cdot f(\tau)dz,$$

which is valid since f is non-vanishing on the simply connected space $\mathbf{h}$. We then have

$$\log\|\omega_{\mathrm{Ar},E_\tau}\|^2_{L^2} = \frac{i}{2}\int\limits_E \omega_{\mathrm{Ar},E_\tau} \wedge \bar{\omega}_{\mathrm{Ar},E_\tau} = C^{1/6}|f(\tau)|^2\mathrm{Im}(\tau).$$

The rest follows from Corollary 1.5 and Definition 1.6. □

We shall call the form $\omega_{\mathrm{Ar},E_\tau}$ the **Arakelov form** on E_τ, and the corresponding scale of the flat metric the **Arakelov metric** on the marked elliptic curve E.

By Theorem 1.2 and Corollary 1.5, notice that if we scale the flat metric on the marked elliptic curve E_τ so that

$$\mathrm{vol}_\mu(E_\tau)\mathrm{Im}(\tau)|f(\tau)| = 1,$$

then the determinant of the Laplacian is necessarily equal to 1. Such a scale of the flat metric will be called the **Ray-Singer metric** (see page 174 of [RS 73]).

To conclude this section, let us express many of the above functions through special functions. For any $\tau \in \mathbf{h}$, consider the function

$$\Delta(\tau) = (2\pi)^{12} e^{2\pi i\tau} \prod_{n=1}^{\infty} (1 - e^{2\pi i n\tau})^{24}. \tag{1.11}$$

The function $\Delta(\tau)$ is, up to multiplicative constant, the unique cusp form of weight 12 with respect to the action on the symmetric space

$$\mathbf{h} = SL(2, \mathbf{R})/SO(2)$$

by the arithmetic subgroup

$$\Gamma = PSL(2, \mathbf{Z})$$

of the full group of isometries $PSL(2, \mathbf{R})$ of $\mathbf{h}$. We have chosen the scale in (1.11) for algebraic significance. We use the notation

$$||\Delta||(\tau) = (\mathrm{Im}(\tau))^6 |\Delta(\tau)|.$$

The function $\Delta(\tau)$ can be realized as a special value of the Riemann theta function

$$\theta(z) = \theta(z, \tau) = \sum_{n=-\infty}^{\infty} \exp(\pi i n^2 \tau + 2\pi i n z)$$

through the expression

$$\left[\exp\left(\frac{\pi i \tau}{4}\right) \frac{\partial \theta}{\partial z}\left(\frac{\tau + 1}{2}\right)\right]^8 = (2\pi)^{-4} \Delta(\tau).$$

The Dedekind eta function $\eta(\tau)$ is a particular 24th root of $\Delta(\tau)$.

By carefully studying the asymptotic behavior of analytic torsion, one can show that in Theorem 1.7 and Theorem 1.8 we have

$$c_2 = -12 \log(2\pi)$$

(see [Jo 90] or [Wen 91]). The connection between the analytic function $\Delta(\tau)$ and the discriminant of the marked elliptic curve E_τ is as follows. The Arakelov 1-form is given by

$$\omega_{\mathrm{Ar}, E_\tau}(z) = \Delta(\tau)^{1/12} dz,$$

so

$$f(\tau) = (2\pi)^{-1} \Delta(\tau)^{1/12} = (2\pi)^{-1} \eta(\tau)^2 \tag{1.12}$$

(see [Fa 84], [Jo 90] or [Wen 91]). This yields the formula

$$6\delta_{\mathrm{unm}}(E_\tau) + 12\log(2\pi) = \log\|\Delta\|(\tau)$$

and

$$6\delta_{\mathrm{mar}}(E_\tau) + 12\log(2\pi) = \log|\Delta_{\mathrm{mar}}(E_\tau)| + 12\log(2\pi) = \log|\Delta(\tau)|$$

(see [Fy 92], [Jo 91], or [RS 73]). From this, one can directly verify Theorem 1.4 and Corollary 1.5, specifically we have

$$dd^c\log\|\Delta\| = -6\mu_{\mathrm{WP}}.$$

Finally, let us note an important number theoretic realization of the unmarked discriminant. If we view E_τ as the zero set of a cubic equation in $\mathbf{P}^2$, given in Weierstrass form, then $\Delta(\tau)$ is equal to the discriminant of the cubic (see page 214 of [Hu 87], pages 43-45 of [La 87b], or pages 343-349 of [Sil 86]). In particular, if our marked elliptic curve is defined over a number field K, then, up to a factor of complex modulus one, the invariant $\|\omega_{\mathrm{Ar},E_\tau}\|^6$ is algebraic, and, in particular, is expressible in terms of the primes of bad reduction of E over K.

§2. The Discriminant for a Polarized $K3$ Surface.

Having established various analytic properties for marked and unmarked discriminants associated to elliptic curves, we now proceed to develop an analogous theory for polarized algebraic $K3$ surfaces. In this section we will define the marked and unmarked discriminants associated to polarized $K3$ surfaces, following Definition 1.3 and Definition 1.6. Let us begin by recalling the properties of $K3$ surfaces which will be necessary in our work. We refer to [Ast 85] for a more complete and detailed discussion. For additional background material, we refer to the following sources: [BPV 84], [Bea 83], [Be 87], [GH 78], [LP 80], or [Sh 67]. We will attempt to address carefully, although quite briefly, all of the main points that we need.

A **K3 surface** X is a compact, connected complex analytic surface that is regular, meaning $h^1(X,\mathcal{O})=0$, and its canonical sheaf $\mathcal{K}$ is trivial. Let

$$H^2(X,\mathbf{C}) = H^{2,0}(X,\mathbf{C}) \oplus H^{1,1}(X,\mathbf{C}) \oplus H^{0,2}(X,\mathbf{C})$$

be the Hodge decomposition. On any $K3$ surface X there is a unique (up to multiplicative constant) holomorphic and non-vanishing 2-form; choose such a form ω. The cohomology class

$$[\omega] \in H^2(X,\mathbf{C})$$

of ω spans the subspace $H^{2,0}(X,\mathbf{C})$ and satisfies the **Riemann bilinear relations**

$$[\omega]\cdot[\omega]=0 \quad \text{and} \quad [\omega]\cdot[\bar{\omega}]>0. \tag{2.1}$$

Equivalently, one can express (2.1) by

$$\int_X \omega\wedge\omega = 0 \quad \text{and} \quad \int_X \omega\wedge\bar{\omega}>0.$$

Set

$$H^{1,1}(X,\mathbf{R}) = H^{1,1}(X,\mathbf{C}) \cap H^2(X,\mathbf{R});$$

or equivalently

$$H^{1,1}(X,\mathbf{R}) = \{c \in H^2(X,\mathbf{R}) : c\cdot[\omega]=0\}.$$

We shall assume throughout that X is algebraic.

A **polarization** of X is the choice of an ample line sheaf (or bundle) up to isomorphism, or equivalently a divisor class for linear equivalence (see page 548 of [PSS 71] or page 146 of [LP 80]). A pair (X,L) consisting of a $K3$ surface and a polarization is called a **polarized $K3$ surface.** There is an integer $d\geq 2$ such that

$$(L\cdot L) = 2d-2,$$

and this integer is called the **degree** of the polarization.

Associated to a polarization, one has a Kähler form which is a rational multiple of the pullback of the Fubini-Study form on $\mathbf{P}^N$ via a projective embedding induced by a power of the ample divisor class. Let $[L]$ be the cohomology class in $H^{1,1}(X,\mathbf{R})$ given by the imaginary part of this form. Yau ([Ya 78]) proves that if an element $[L]$ of $H^{1,1}(X,\mathbf{R})$ can be represented by a Käher form, then it can be represented by a unique (up to multiplicative constant) Kähler form with zero Ricci curvature, called **compatible with the polarization.** Such forms are called **Kähler-Einstein forms**. Hence, one has a unique (up to multiplicative constant) Kähler-Einstein form for every polarized $K3$ surface (X, L), which yields a picture which is analogous to the existence of a flat metric on every elliptic curve. Throughout this section, the only metrics that we consider are the Kähler-Einstein metrics, and these metrics will be represented by a Kähler-Einstein $(1,1)$ form μ.

A relation between a Kähler-Einstein form μ and the holomorphic 2-form ω on X comes from Bochner's principle, which implies that any holomorphic tensor on a Kähler manifold X with a Kähler-Einstein form is parallel with respect to the Levi-Civita connection of the Kähler-Einstein form, meaning

$$\nabla_\mu \omega = 0$$

(see Theorem 6.1 on page 119 of [KM 71] or page 194 of [GHL 90]). As a result, the volume element vol_μ associated to the Kähler-Einstein form μ can be realized as

$$\mathrm{vol}_\mu = -\omega \wedge \bar{\omega}, \tag{2.2}$$

for some $\omega \in H^0(X, \mathcal{K})$, where $\mathcal{K}$ is the canonical sheaf. The volume of X is given by

$$\mathrm{vol}_\mu(X) = -\int_X \omega \wedge \bar{\omega} = \|\omega\|^2_{L^2}.$$

By (2.2), the choice of a scale of the Kähler-Einstein form μ on X compatible with a given polarization L determines the scale of a holomorphic 2-form $\omega \in H^0(X, \mathcal{K})$.

From Noether's formula (see page 9 of [Bea 83] or page 438 of [GH 78]), it can be shown that the second integral homology group $H_2(X, \mathbf{Z})$ has dimension $h_2(X, \mathbf{Z}) = 22$. The group $H_2(X, \mathbf{Z})$ is torsion-free (see page 212 of [Sh 67]) and, when endowed with the symmetric bilinear form given by cup product, is an even unimodular lattice of signature (3,19). From the structure theorem of even unimodular lattices (see page 54 of [Ser 73]), there exists a basis

$$\phi = \{\gamma_1, \ldots, \gamma_{22}\}$$

of $H_2(X, \mathbf{Z})$ such that the intersection matrix $Q = (\gamma_i \cdot \gamma_j)$ is block diagonal of the form

$$Q = H^3 \oplus E_8^2, \tag{2.3}$$

where H is the 2×2 hyperbolic matrix and E_8 is the 8×8 matrix corresponding to the root system of type E_8 (see page 52 of [Se 73]). Any basis ϕ of $H_2(X, \mathbf{Z})$ satisfying (2.3) will be called a **canonical basis**. We shall call the pair (X, ϕ) a **marked K3 surface** if X is a $K3$ surface and ϕ is a canonical basis of $H_2(X, \mathbf{Z})$. Two canonical bases of $H_2(X, \mathbf{Z})$ differ by the action of an element of $SO_0(3, 19; \mathbf{Z})$. If X is polarization in addition, we call the triple (X, L, ϕ) a **marked, polarized, $K3$ surface.**

Associated to every marked $K3$ surface and $\omega \in H^0(X, \mathcal{K})$, one has a **period mapping**

$$\gamma \mapsto \int_\gamma \omega \quad \text{for } \gamma \in H_2(X, \mathbf{Z}). \tag{2.4}$$

If γ is an algebraic homology cycle, then the integral in (2.4) is zero. If X is marked with a canonical basis of homology ϕ, then the point

$$\psi(X;\phi) = \left(\int_{\gamma_1} \omega, \ldots, \int_{\gamma_{22}} \omega\right) \in \mathbf{P}^{21} \tag{2.5}$$

is the **period** associated to the marked $K3$ surface (X, ϕ). By the Riemann bilinear relations (2.1), the period (2.5) can be viewed as a point in the space

$$\Omega = \{z \in \mathbf{P}(\Lambda \otimes \mathbf{C}) | \langle z, \bar{z}\rangle = 0, \langle z, z\rangle > 0\}. \tag{2.6}$$

In fact, the Riemann bilinear relations (2.3) can be reformulated as follows. Let

$$Q = (\gamma_i \cdot \gamma_j)$$

be the symmetric matrix defined by the intersection of the cycles γ_j which form a canonical basis of $H_2(X, \mathbf{Z})$. Then the bilinear relations are equivalent to the statements

$$\psi(X;\phi) \cdot Q \cdot {}^t\psi(X;\phi) = 0, \quad \text{and} \quad \psi(X;\phi) \cdot Q \cdot {}^t\bar{\psi}(X;\phi) > 0. \tag{2.7}$$

From (2.7) one can show that (2.6) can be realized as

$$\Omega \cong SO_0(3,19)/SO(2) \times SO(1,19).$$

At this point, one is led to consider the image of the period mapping into the space Ω, when considering marked $K3$ surfaces and when considering marked, polarized $K3$ surfaces.

Let $\mathcal{Y} \to \mathcal{D}$ be a family of non-singular $K3$ surfaces over a polycylinder $\mathcal{D}$. As a C^∞ manifold, $\mathcal{Y}$ is diffeomorphic to $\mathcal{D} \times Y$, where Y is a fixed $K3$ surface (see page 257 of [BPV 84]). It follows that if we choose a marking of one fibre then we have marked all Y_t in the family $\mathcal{Y} \to \mathcal{D}$, where $t \in \mathcal{D}$. It is a theorem, due to Tiurina, Kodaira, Andreotti, and A. Weil, that if we choose $\omega = \omega_t$ to vary holomorphically for $t \in \mathcal{D}$, then the period map per : $\mathcal{D} \to \Omega$ is holomorphic on $\mathcal{D}$. The **local Torelli theorem** for $K3$ surfaces asserts that the periods give local moduli for small deformations of a given $K3$ surface (see, for example, [Ko 64] and [PSS 71] as well as [Lo 80] and page 254 of [BPV]).

A global Torelli theorem for marked, polarized $K3$ surfaces was first given by Piatetski-Shapiro and Shafarevich (see [PSS 71] or [Lo 80], [LP 80], and [Sh 67]). In [PSS 71] it is proved that a moduli space $\mathcal{M}^d_{\text{mp}}$ of marked, polarized algebraic $K3$ surfaces of degree d exists and is a 19-dimensional complex manifold. Moreover, as discussed by Piatetski-Shapiro and Shafarevich, there is a universal family

$$\pi : \mathcal{X} \longrightarrow \mathcal{M}^d_{\text{mp}}$$

of marked, polarized, algebraic $K3$ surfaces of degree d, from which one can define the period map

$$\text{per} : \mathcal{M}^d_{\text{mp}} \to \Omega,$$

as follows. From the definition of the polarized $K3$ surface (X, L), $[L]$ is a class of type (1,1) with respect to the complex structure, so

$$[\omega] \cdot [L] = 0 \quad \text{for} \quad \omega \in H^0(X, \mathcal{K}).$$

Let Λ be the lattice $H^2(X, \mathbf{Z})$ in $H^2(X, \mathbf{R})$, and let H be the hyperplane

$$H = \{z \in \mathbf{P}(\Lambda \otimes \mathbf{C}) | \langle z, [L] \rangle = 0\}.$$

Define

$$\Omega^d_{\mathrm{mp}} = H \cap \Omega.$$

Then

$$\Omega^d_{\mathrm{mp}} = H \cap \Omega \cong SO(2,19)/(SO(2) \times SO(19)).$$

The **global Torelli theorem** for marked, polarized, algebraic $K3$ surfaces of degree d asserts that the image of the period map lies in Ω^d_{mp}, and that the period map is a holomorphic embedding. In particular, a marked, polarized, algebraic $K3$ surface of degree d is uniquely determined by its periods, and the image of $\mathcal{M}_{\mathrm{mp}}$ in Ω is a countable union of analytic submanifolds, indexed by the degree d of the polarization.

The global Torelli theorem, together with results due to Kulikov [Ku 77], show that there is a discrete subgroup Γ_d of $SO(2,19)$ such that $\Gamma_d \backslash \Omega^d_{\mathrm{mp}}$ contains $\Gamma_d \backslash \mathcal{M}^d_{\mathrm{mp}}$ as a Zariski open subset, and this subset, which we denote by $\mathcal{M}^d_{\mathrm{p}}$ is a coarse moduli space for polarized $K3$ surfaces of degree d without regard to marking. Thus

$$\mathcal{M}^d_{\mathrm{p}} = \Gamma_d \backslash \mathcal{M}^d_{\mathrm{mp}}.$$

For further discussion, see also [Lo 80], [LP 80], [PeP 81], [Siu 83], and [To 80].

Remark 1. By using results due to Kobayashi-Todorov [KT 89], one can analyze the points in Ω^d_{mp} which are not in $\mathcal{M}^d_{\mathrm{mp}}$. These points correspond to morphisms $\psi : X \to X^\#$ where $X^\#$ is a singular surface with only isolated double points which come from blowing down (-2)-curves on the $K3$ surface. A further analysis of this topic, and its connection with the results in this paper, will be considered elsewhere.

As in the case of elliptic curves, there exists a canonical metric called the **Weil-Petersson metric** on the moduli space $\mathcal{M}^d_{\mathrm{mp}}$. We give the definition in a manner similar to that of §1.

Since $H^0(X, \mathcal{K}) = 1$ for any marked, polarized, algebraic $K3$ surface X of degree d, two families of holomorphically varying 2-forms on $K3$ surfaces that vary over $\mathcal{M}^d_{\mathrm{mp}}$ differ by a multiplicative factor which is a non-vanishing holomorphic function on $\mathcal{M}^d_{\mathrm{mp}}$. Hence if ω is a family of holomorphic 2-forms that vary holomorphically over $\mathcal{M}^d_{\mathrm{mp}}$, the quantity

$$dd^c \log \|\omega\|^2_{L^2}$$

is a well-defined closed $(1,1)$ form. We define the **Weil-Petersson metric** on $\mathcal{M}^d_{\mathrm{mp}}$ to be the Kähler metric corresponding to the Kähler form

$$\mu_{\mathrm{WP}} = -dd^c \log \|\omega\|^2_{L^2},$$

which we call the **Weil-Petersson form.** Thus, by definition, $-\log\|\omega\|_{L^2}^2$ is a potential for the Weil-Petersson metric on $\mathcal{M}_{\mathrm{mp}}^d$. In an appendix to this section we shall recall the definition of the Weil-Petersson metric given more classically in terms of harmonic forms and deformation theory. However, we emphasize here that only the definition we have given will be used in the present paper.

Next we recall the definition of analytic torsion associated to the trivial sheaf $\mathcal{O}$ on X, from which we will obtain our definition of the unmarked discriminant.

Let (X, L) be a polarized, algebraic $K3$ surface of degree d, with compatible Kähler-Einstein form μ. Let $\bar{\partial}$ denote the Cauchy-Riemann operator associated to the Dolbeault complex

$$\cdots \longrightarrow \Omega^{0,q-1} \xrightarrow{\bar{\partial}_{q-1}} \Omega^{0,q} \xrightarrow{\bar{\partial}_q} \Omega^{0,q+1} \longrightarrow \cdots$$

where, as above, $\Omega^{0,q}$ is the sheaf of smooth forms of type $(0,q)$ with coefficients in the trivial sheaf $\mathcal{O}$, so

$$\Omega^{0,q} = \bar{\mathcal{K}}^q.$$

Relative to the inner product of sections on $\bar{\mathcal{K}}^q$, which is induced by the chosen form μ, the operator $\bar{\partial}$ admits a formal adjoint, which we shall denote by $\bar{\partial}^*$. The **Laplacians** we study are the operators of the form

$$\mathbf{\Delta}_q = \bar{\partial}_{q-1}\bar{\partial}_{q-1}^* + \bar{\partial}_q^*\bar{\partial}_q.$$

It is classical that $\mathbf{\Delta}_q$ has a discrete spectrum with associated eigensections that form an orthonormal basis of L^2 sections of $\bar{\mathcal{K}}^q$. Let us denote the non-zero eigenvalues of $\mathbf{\Delta}_q$ by the sequence

$$0 < \lambda_1^{(q)}(\mu) \leq \lambda_2^{(q)}(\mu) \leq \ldots.$$

Define the **spectral zeta function** associated to $\mathbf{\Delta}_q$ to be

$$\zeta_\mu^{(q)}(s) = \sum_{k=1}^{\infty} \lambda_k^{(q)}(\mu)^{-s}, \tag{2.8}$$

and the **full spectral zeta function** associated to $\mathcal{O}$ to be

$$\begin{aligned}\zeta_\mu(s) &= \sum_{q=0}^{2}(-1)^q q \zeta_\mu^{(q)}(s) \\ &= \sum_{q=0}^{2}(-1)^q q \int_0^\infty \left[\mathrm{Tr}K_\mu(X, \bar{\mathcal{K}}^q)(t) - h^q(X, \mathcal{O})\right] t^s \frac{dt}{t},\end{aligned} \tag{2.9}$$

where $\mathrm{Tr}K_\mu$ denote the trace of the appropriate heat kernel. By Weyl's law, (2.8) converges for $\mathrm{Re}(s)$ sufficiently large, and by Seeley's theorem [See 67], the series (2.9) has a meromorphic continuation to all s which is holomorphic in a neighborhood of $s = 0$. Following Ray and Singer [RS 73], the **analytic torsion** associated to $\mathcal{O}$ is defined by

$$\mathrm{tor}_\mu(X) = -\zeta_\mu'(0) = -\sum_{q=0}^{2}(-1)^q\, q \cdot \frac{\partial}{\partial s}\zeta_\mu^{(q)}(s)\Big|_{s=0}.$$

The analogue of Theorem 1.2 can now be stated.

Theorem 2.1. *Let (X, L) be a polarized, algebraic $K3$ surface of degree d, and let μ be any compatible Kähler-Einstein form. Then for any $c \in \mathbf{R}^+$, we have*

$$\operatorname{tor}_\mu(X) - \log \operatorname{vol}_\mu(X) = \operatorname{tor}_{c\cdot\mu}(X) - \log \operatorname{vol}_{c\cdot\mu}(X).$$

In other words, the quantity $\operatorname{tor}_\mu(X) - \log \operatorname{vol}_\mu(X)$ is an invariant of the polarized, algebraic $K3$ surface (X, L).

Proof. The proof is almost identical to that of Theorem 1.2, but will be repeated here for clarity. Let μ be any Kähler-Einstein form on (X, L) compatible with the polarization, and $c \in \mathbf{R}^+$. Trivially we have

$$\operatorname{vol}_{c\cdot\mu}(X) = c^2 \operatorname{vol}_\mu(X).$$

By definition of the Laplacian, the sequence of eigenvalues associated to $\mathbf{\Delta}_{c\cdot\mu}$ are related to the sequence of eigenvalues of $\mathbf{\Delta}_\mu$ through multiplication by the scalar c^{-1}. Therefore, we have

$$\zeta_{c\cdot\mu}(s) = c^s \zeta_\mu(s),$$

from which we obtain the relation

$$-\zeta'_{c\cdot\mu}(0) = -\zeta_\mu(0)\log c - \zeta'_\mu(0).$$

Since the form μ is Ricci flat, Seeley's theorem ([Se 67]) states

$$\operatorname{Tr} K(X, \bar{\mathcal{K}}^q)(t) = \frac{\operatorname{vol}_\mu(X)}{(4\pi t)^2} + O(t^N) \quad \text{as } t \to 0,$$

for any $N > 0$ (see also page 150 of [Ch 84] or page 84 of [BGV 92]). From the proof of the meromorphic continuation of the zeta function (see [JoLa 93]), this implies that

$$\zeta_\mu(0) = -\sum_{q=0}^{2} (-1)^q q h^q(X, \mathcal{O}) = -2,$$

from which the theorem follows. □

Definition 2.2. *Let (X, L) be a polarized, algebraic $K3$ surface of degree d, and let μ denote any Kähler-Einstein form compatible with the polarization. The* **unmarked (logarithmic) discriminant** *$\delta_{\mathrm{unm}}((X, L))$ of (X, L) is defined to be*

$$\delta_{\mathrm{unm}}(X, L) = \operatorname{tor}_\mu(X) - \log \operatorname{vol}_\mu(X).$$

As in the case of elliptic curves, the unmarked (logarithmic) discriminant is an invariant associated to any polarized, algebraic $K3$ surface, hence is a function on $\mathcal{M}^d_{\mathrm{p}}$ and should be written as

$$\delta_{\mathrm{unm}}(X, L) = \delta_{\mathrm{unm}}([X, L]).$$

To continue, we shall describe another potential for the Weil-Petersson metric via Quillen norms, analogous to Theorem 1.4, which will lead to our definition of the marked (logarithmic) discriminant.

For any marked, polarized, algebraic $K3$ surface (X, L, ϕ) of degree d, the **determinant line** $\det H(\mathcal{O})$ associated to the trivial sheaf $\mathcal{O}$ is defined to be the 1-complex dimensional vector space

$$\det H(\mathcal{O}) = \det H^0(X, \mathcal{O}) \otimes \det H^1(X, \mathcal{O})^{-1} \otimes \det H^2(X, \mathcal{O}).$$

Since X is a $K3$ surface, the cohomology group $H^1(X, \mathcal{O})$ is trivial, hence the vector space $\det H^1(X, \mathcal{O})$ is canonically isomorphic to $\mathbf{C}$, which yields the isomorphism

$$\begin{aligned} \det H(\mathcal{O}) &\cong \det H^0(X, \mathcal{O}) \otimes \det H^2(X, \mathcal{O}) \\ &\cong \det H^0(X, \mathcal{O}) \otimes [\det H^0(X, \mathcal{K})]^{-1}; \end{aligned} \tag{2.10}$$

where the isomorphism in (2.10) comes from Serre duality. Let η denote a non-zero element of $H^0(X, \mathcal{O})$, which we can view as a constant function on X, and let ω denote a non-zero element of $H^0(X, \mathcal{K})$, which we can view as a non-zero holomorphic 2-form on X. A **metric** or norm on the line $\det H(\mathcal{O})$ is equivalent to the assignment of a length to the element

$$\Upsilon = \eta \wedge \omega^{-1} \tag{2.11}$$

in $\det H(\mathcal{O})$. The square of the $\mathbf{L^2}$**-norm** on $\det H(\mathcal{O})$ is defined by

$$\|\Upsilon\|_{L^2}^2 = \langle \eta, \eta \rangle \langle \omega, \omega \rangle^{-1} = |\eta|^2 \mathrm{vol}_\mu(X) \cdot \|\omega\|_{L^2}^{-2},$$

and the square of the **Quillen norm** on $\det H(\mathcal{O})$ is defined by

$$\|\Upsilon\|_Q^2 = \|\Upsilon\|_{L^2}^2 \cdot \exp(\mathrm{tor}_\mu(X)).$$

If $\eta = 1$, we have

$$\log \|\Upsilon\|_Q^2 = -\delta_{\mathrm{unm}}(X, L) - \log \|\omega\|_{L^2}^2. \tag{2.12}$$

The element Υ in (2.11) is defined for a fixed marked, polarized, algebraic $K3$ surface (X, L, ϕ) of degree d. When necessary, we shall write this element as

$$\Upsilon = \Upsilon_{(X,L,\phi)}.$$

We then view (2.12) as the function

$$\mathcal{M}_{\mathrm{mp}}^d \to \mathbf{R}_{>0}$$

given by

$$(X, L, \phi) \mapsto \|\Upsilon_{(X,L,\phi)}\|_Q^2.$$

The following result, which is analogous to Theorem 1.4, shows that the unmarked discriminant can be used to obtain a potential for the Weil-Petersson metric on $\mathcal{M}_{\mathrm{mp}}^d$.

Theorem 2.3. *Let $\{\omega_{(X,L,\phi)}\}$ be a family of holomorphically varying 2-forms on the moduli space $\mathcal{M}^d_{\mathrm{mp}}$, and let $\Upsilon_{(X,L,\phi)} = 1 \wedge \omega^{-1}_{(X,L,\phi)}$, where 1 corresponds to the constant function 1. Then*

$$dd^c \log \|\Upsilon\|^2_Q = 2\mu_{\mathrm{WP}}.$$

Equivalently, we have

$$-dd^c\delta_{\mathrm{unm}} = \mu_{\mathrm{WP}}.$$

In other words, $-\delta_{\mathrm{unm}}$ is a potential for the Weil-Petersson metric on $\mathcal{M}^d_{\mathrm{mp}}$.

Proof. As in Theorem 1.4, this result follows from the Quillen-Grothendieck-Riemann-Roch theorem, using the fact that the canonical sheaf of a $K3$ surface is trivial, the second Chern class $\mathrm{ch}^{(2)}(\mathcal{T}_X)$ of the tangent sheaf $\mathcal{T}_X$ integrates to 24 (see page 46 of [Ast 85] or page 590 of [GH 78]), and that the degree two component of the Todd class of $\mathcal{T}_X$ is

$$\mathrm{td}^{(2)}(\mathcal{T}_X) = \frac{1}{12}\left((\mathrm{ch}^{(1)}(\mathcal{T}_X))^2 + \mathrm{ch}^{(2)}(\mathcal{T}_X)\right) = \frac{1}{12}\mathrm{ch}^{(2)}(\mathcal{T}_X)$$

(see page 20 of [FL 85]). The second assertion follows from the first assertion, definition of the Weil-Petersson form, and (2.12). For further details, see page 330 of [To 88] or pages 164-165 of [FS 90], which references [BGS 87] (see also the general arithmetic Riemann-Roch theorem for the full Chern character, as stated in [Fa 92]). □

Corollary 2.4. *Let $\{\omega_{X,L,\phi}\}$ be a holomorphic family of holomorphic 2-forms over $\mathcal{M}^d_{\mathrm{mp}}$. Then there exists a non-vanishing holomorphic function $f_\omega = f$ on $\mathcal{M}^d_{\mathrm{mp}}$ such that if (X, L, ϕ) is a marked, polarized, algebraic $K3$ surface of degree d and μ is any compatible Kähler-Einstein form, then*

$$\mathrm{tor}_\mu(X) - \log(\mathrm{vol}_\mu(X)\|\omega_{X,L,\phi}\|^2_{L^2}) = \log|f_\omega(X, L, \phi)|^2.$$

The function $|f_\omega|$ varies like a modular form of weight 2 on $\mathcal{M}^d_{\mathrm{mp}}$ with respect to the discrete arithmetic subgroup Γ_d.

Proof. The proof is immediate from the definition of the Weil-Petersson form and Theorem 2.3, following the pattern of the proof of Corollary 1.5. The computation of the weight of f comes from the invariance of

$$|f_\omega(X, L, \phi)|^2 \, \|\omega_{X,L,\phi}\|^2_{L^2}$$

on $\mathcal{M}^d_{\mathrm{mp}}$ with respect to the marking of (X, L). □

Remark 2. In [Ko 91] it is shown that the group $\Gamma_d/[\Gamma_d, \Gamma_d]$ is a finite group of order 16. Therefore, the function f^{32} is a non-vanishing holomorphic modular form on $\mathcal{M}^d_{\mathrm{mp}}$ of weight 32 with respect to Γ_d.

Remark 3. In a subsequent paper, we will show that f extends to a holomorphic function on the symmetric space Ω^d_{mp} which vanishes on the complement of $\mathcal{M}^d_{\mathrm{mp}}$.

This will allow us to give an algebraic realization of our discriminant in terms of automorphic forms on $SO(2,19)$.

From the definition of the Weil-Peterson form we see that the function f in Corollary 2.4 does not depend on the scale of the Kähler-Einstein forms μ, hence is a function of the marked, polarized, algebraic $K3$ surface (X, L, ϕ) and the holomorphic family of holomorphic 2-forms ω. For any marked elliptic curve, there is a canonical choice of holomorphic 1-form, namely dz.

Definition 2.5. *Let (X, L, ϕ) be any marked, polarized, algebraic $K3$ surface of degree d. Let μ denote any compatible Kähler-Einstein form, and let $\omega_{X,L,\phi}$ be a holomorphic 2-form on (X, L, ϕ). The* **marked (logarithmic) discriminant** *$\delta_{\mathrm{mar}}([X, L, \phi], \omega)$ of (X, L, ϕ) associated to the choice of $\omega_{X,L,\phi}$ is defined to be*

$$\begin{aligned}\delta_{\mathrm{mar}}([X, L, \phi], \omega) &= \mathrm{tor}_\mu(X) - \log(\mathrm{vol}_\mu(X)\|\omega_{X,L,\phi}\|^2_{L^2}) \\ &= \log |f_\omega(X, L, \phi)|^2 .\end{aligned}$$

We also define

$$\Delta_{\mathrm{mar}} = f^{12}.$$

In the next section we will study our discriminant for marked Kummer surfaces of degree 2 as well as for certain Kummer surfaces of arbitrary degree associated to abelian surfaces that are products of elliptic curves.

Appendix: To conclude this section, let us give the definition of the Weil-Petersson metric via local deformation theory.

Let

$$\pi : (\mathcal{X}, \mathcal{L}, \Phi) \longrightarrow \mathcal{M}^d_{\mathrm{mp}}$$

be the universal family of marked, polarized, algebraic $K3$ surfaces of degree d. For any $t \in \mathcal{M}^d_{\mathrm{mp}}$, let $\{\mu_t\}$ be a C^∞ family of Kähler-Einstein forms such that for every $t \in \mathcal{M}^d_{\mathrm{mp}}$, μ_t is a Kähler-Einstein form on the polarized $K3$ surface (X_t, L_t), where $(X_t, L_t) = \pi^{-1}(t)$ and $[\mathrm{Im}(\mu_t)] = L_t$. Let $\mathcal{T}_t$ be the sheaf of holomorphic vector fields on X_t. We define a hermitian metric on $C^\infty(X_t, \mathcal{T}_t \otimes \Omega^{0,1}_t)$ for all $t \in \mathcal{M}^d_{\mathrm{mp}}$ as follows. For each $t \in \mathcal{M}^d_{\mathrm{mp}}$, μ_t defines an isomorphism of sheaves

$$\mu_t : \mathcal{T}_t \otimes \Omega^{0,1}_t \longrightarrow (\bar{\mathcal{T}}_t)^\vee \otimes \left(\bar{\Omega}^{0,1}_t\right)^\vee ,$$

so we have a map

$$\sigma_t : C^\infty \left(X_t, \mathcal{T}_t \otimes \Omega^{0,1}_t\right) \longrightarrow C^\infty \left(X_t, (\bar{\mathcal{T}}_t)^\vee \otimes \left(\bar{\Omega}^{0,1}_t\right)^\vee\right).$$

Let $\phi_1, \phi_2 \in C^\infty \left(X_t, \mathcal{T} \otimes \Omega^{0,1}\right)$ and define the hermitian inner product

$$\langle \phi_1, \phi_2 \rangle_{\mathrm{WP}} = \int_{X_t} \phi_1 \cdot \sigma_t(\phi_2) \cdot \mathrm{vol}_{\mu_t}.$$

For each $t \in \mathcal{M}^d_{\mathrm{mp}}$ let

$$\mathbf{H}(X_t, \mathcal{T}_t \otimes \Omega_t) \subset C^\infty(X_t, \mathcal{T}_t \otimes \Omega^{0,1}_t)$$

be the harmonic subspace, and identify

$$T_{t,\mathcal{M}^d_{\mathrm{mp}}} = \mathbf{H}(X_t, \mathcal{T}_t \otimes \Omega_t^{0,1}).$$

We now have defined a hermitian metric on the tangent space of $\mathcal{M}^d_{\mathrm{mp}}$. Any positive scalar multiple of this metric will be called **a Weil-Petersson metric**.

The following result is from [To 89] and [Ti 88].

Theorem 2.6. *There is a scalar multiple of the above metric such that the associated* $(1,1)$*-form is equal to* μ_{WP}.

For readers who are interested, the proof of Theorem 2.6 comes directly from the results on page 641 of [Ti 88] and Theorem 2.6 of [To 89] (see also [Na 86] and [Sch 85]).

§3. The Discriminant for a Polarized Kummer Surface.

Let us now examine the discriminant defined in §2 considered as a function on certain spaces of polarized Kummer surfaces of degree d. Recall that a Kummer surface X is constructed as follows. Let A be a projective abelian surface, and let $\tilde{A}$ be the non-singular surface obtained by blowing up the 2-torsion points on A, so $\tilde{A}$ has 16 exceptional curves. The map $z \mapsto -z$ of A to itself extends to an involution of $\tilde{A}$ (see, for example, page 99 of [Bea 83] or pages 171 and 246 of [BPV 84]), and the quotient of $\tilde{A}$ by this involution is the Kummer surface X associated to the abelian surface A. It is a reasonably straightforward exercise to show that X is a $K3$ surface (see page 99 of [Bea 83]).

The principal Siegel upper half space of dimension two, which we shall denote by $\mathcal{C}_2$, consists of all 2×2 symmetric matrices Ω such that $\mathrm{Im}(\Omega)$ is positive definite and all elementary divisors are 1. Let I_2 be the 2×2 identity matrix. For any $\Omega \in \mathcal{C}_2$, let $L(\Omega)$ denote the $\mathbf{Z}$ lattice generated by the columns of I_2 and Ω. Given any $\Omega \in \mathcal{C}_2$, we have an associated projective abelian surface $A(\Omega)$ given by

$$A(\Omega) = \mathbf{C}^2/L(\Omega). \tag{3.1}$$

The abelian surface (3.1) has a natural principal polarization corresponding to the hermitian form $\mathbf{H}$ whose associated matrix is $(\mathrm{Im}(\Omega))^{-1}$. Throughout we shall assume $A(\Omega)$ is given this polarization, which induces a polarization L of degree 2 on the associated Kummer surface X.

The abelian surface (3.1) has a canonical basis of the 4 dimensional vector space $H_1(A(\Omega), \mathbf{Z})$ given by the boundary of the period parallelogram in $\mathbf{C}^2$ spanned by the columns of I_2 and Ω. In an appendix to §5 of [PSS 71] by D. B. Fuchs, it is shown that one can construct a canonical basis of $H_2(X, \mathbf{Z})$ consisting of the 16 exceptional curves obtained by blowing up the two torsion points of the associated abelian surface, together with a particular basis of the 6 dimensional space $\bigwedge^2 H_1(A(\Omega), \mathbf{Z})$. In order to obtain a marking of the Kummer surface, one must take the 16 exceptional curves with exponent 1/2. Hence, the abelian surface $A(\Omega)$ induces a marking ϕ of the Kummer surface X.

Let

$$\pi : \tilde{A}(\Omega) \to (X, \phi)$$

be the projection map from the surface $\tilde{A}(\Omega)$ to the marked Kummer surface (X, ϕ). From the realization (3.1) of the abelian surface $A(\Omega)$, we can express a generator of the one complex dimensional vector space $H^0(A(\Omega), \mathcal{K})$ via the standard holomorphic coordinates z_1 and z_2 on $\mathbf{C}^2$. With these coordinates, we take our choice of holomorphic 2-form on $A(\Omega)$ to be

$$\omega_{A(\Omega)} = dz_1 \wedge dz_2. \tag{3.2}$$

In the case Ω is diagonal, so $A(\Omega)$ is a product of elliptic curves E_{τ_1} and E_{τ_2}, the form (3.2) can be characterized as follows. Let A_1, B_1 be a canonical basis of $H_1(E_{\tau_1}, \mathbf{Z})$, and let A_2, B_2 be a canonical basis of $H_1(E_{\tau_2}, \mathbf{Z})$. Then the form (3.2) is determined by the condition that its period relative to the cycle $A_1 \times A_2$ is 1.

From (3.2) we obtain a generator of $H^0(\tilde{A}(\Omega), \mathcal{K})$, which we write as $\omega_{\tilde{A}(\Omega)}$. Let $\omega_{X,\phi}$ be the holomorphic 2-form on (X, ϕ) such that

$$\omega_{\tilde{A}(\Omega)} = \pi^* \omega_{X,\phi}.$$

Lemma 3.1. *Let $A(\Omega)$ be a abelian surface with associated marked Kummer surface (X,ϕ). Then*

$$\|\omega_{X,\phi}\|_{L^2}^2 = 2\det(\mathrm{Im}(\Omega)).$$

Proof. It is easy to show

$$-\int_{A(\Omega)} \omega_{A(\Omega)} \wedge \bar{\omega}_{A(\Omega)} = 4\det(\mathrm{Im}(\Omega)).$$

From the formula,

$$-\|\omega_{X,\phi}\|_{L^2}^2 = \int_X \omega_{X,\phi} \wedge \bar{\omega}_{X,\phi} = \frac{1}{2}\int_{A(\Omega)} \omega_{A(\Omega)} \wedge \bar{\omega}_{A(\Omega)}$$

the lemma follows. □

Let us now study our discriminants for polarized, algebraic $K3$ surfaces when restricted to the certain spaces of polarized Kummer surfaces. With the above discussion and Definition 2.6, let us set the notation

$$\delta_{\mathrm{unm}}(\Omega) = \delta_{\mathrm{unm}}([X,L,\phi],\omega_{X,\phi}) \tag{3.3}$$

and

$$\Delta_{\mathrm{mar}}(\Omega) = \Delta_{\mathrm{mar}}([X,L,\phi],\omega_{X,\phi}), \tag{3.4}$$

where $\omega_{X,\phi}$ is the holomorphic 2-form from Lemma 3.1. We shall view (3.3) and (3.4) as functions on $\mathcal{C}_2$. From Definition 2.6 and Lemma 3.1, we have

$$\begin{aligned}\exp\left(6\delta_{\mathrm{unm}}(\Omega)\right) &= |\Delta_{\mathrm{mar}}(\Omega)| \cdot \|\omega_{X,\phi}\|_{L^2}^{12} \\ &= |\Delta_{\mathrm{mar}}(\Omega)| \cdot (2\det(\mathrm{Im}(\Omega)))^6. \end{aligned} \tag{3.5}$$

A **special Kummer surface** is a Kummer surface corresponding to an abelian surface which is a product of elliptic curves. The following theorem evaluates the discriminant for special Kummer surfaces with arbitrary polarizations.

Theorem 3.2. *For any positive integers d_1 and d_2, there is a constant c such that the following holds. Let $A(\Omega)$ be a projective abelian surface which is a product of elliptic curves, so*

$$A(\Omega) = E_{\tau_1} \times E_{\tau_2}.$$

Let $D = D_{d_1,d_2}$ be the divisor

$$D = d_1(E_{\tau_1} \times \{0\}) + d_2(\{0\} \times E_{\tau_2})$$

on $A(\Omega)$. Let $X = K(\Omega)$ be polarized by the image of D minus the union of the (-2)-curves on X which lie on the image of D, which indeed is a polarization $L = L_{d_1,d_2}$ of degree $d = 2(d_1 + d_2)$. Let X be given the marking induced *from the abelian surface $A(\Omega)$. Then the marked discriminant associated to the marked, polarized Kummer surface (X, L, ϕ) of degree d is given by*

$$\Delta_{\rm mar}((X, L, \phi), \omega_{X,\phi}) = c\Delta(\tau_1)\Delta(\tau_2).$$

Proof. Restrict the function $\Delta_{\rm mar}$ of (3.4) to the space of principally polarized abelian surfaces that are products of marked elliptic curves. Then the function f is viewed as a non-vanishing holomorphic function on $\mathbf{h} \times \mathbf{h}$ which is whose absolute value changes like a modular form of weight 2 with respect to action by the group $PSL(2, \mathbf{Z}) \times PSL(2, \mathbf{Z})$. One can show that any character χ of the fundamental group of the space

$$\mathcal{M}_{\rm unm} = PSL(2, \mathbf{Z})\backslash\mathbf{h}$$

is such that χ^6 is trivial (see page 4 of [La 76] or page 78 of [Ser 73]). Hence, f^6 is a modular form of weight 12. The space of such forms is one dimensional and is generated by the Dedekind delta function Δ (see page 11 of [La 76] or page 89 of [Ser 73]). □

Remark 1. Consider the case when a general Kummer surface $K(\Omega)$ is given the polarization which is the image of the principal theta polarization of the associated abelian surface $A(\Omega)$, minus the exceptional curves which lie on the theta divisor. If Ω is not equivalent to a diagonal matrix under the action of $\mathrm{Sp}_4(\mathbf{Z})$, then we indeed have a polarization, and in the other cases one does not. One can show that the marked $K3$ discriminant extends to zero across this subset of $\mathcal{C}_2$. Further, by arguing as in the proof of Theorem 3.2, using results due to Mumford [Mu 67] and Powell [Po 78], we can relate the $K3$ discriminant to the weight 10 cusp form χ_{10}, defined in [Ig 62].

Immediately from Corollary 2.5 we have the following analogue of Theorem 1.8, which is valid for all marked, polarized, algebraic $K3$ surfaces of degree d. We state the result here rather than in §2 because we needed Theorem 3.2 in order to determine the constant c of interest.

Theorem 3.3. *There is a unique family of holomorphic 2-forms $\{\omega_{\mathrm{Ar},(X,L,\phi)}\}$, varying holomorphically over $\mathcal{M}^d_{\rm mpa}$, such that, if $c = c(1, d-1)$ is the constant defined in Theorem 3.2, we have*

$$\delta_{\rm unm}(X, L) - \frac{1}{6}c = \log \|\omega_{\mathrm{Ar},(X,L,\phi)}\|^2_{L^2}.$$

We shall call $\omega_{\mathrm{Ar},(X,L,\phi)}$ the **Arakelov 2-form** on the marked, polarized, algebraic $K3$ surface (X, L, ϕ) of degree 2. By choosing a particular holomorphic 2-form, we have determined a scale of the Kähler-Einstein $(1,1)$-form compatible with the given polarization. We shall call this $(1,1)$-form the **Arakelov-Kähler-Einstein** form on (X, L, ϕ).

§4. Asymptotic Behavior of the $K3$ Discriminant Under Degeneration.

In this section we will study the asymptotic behavior of the marked and unmarked discriminants for a one dimensional family of marked, polarized $K3$ surfaces of degree d. Analytically, any family of $K3$ surfaces is a flat proper map

$$\pi : X \to \mathcal{D} \tag{4.1}$$

from a threefold X to the unit disc $\mathcal{D}$ in $\mathbf{C}$ such that for all $t \neq 0$ the fibres $X_t = \pi^{-1}(t)$ are non-singular algebraic $K3$ surfaces. The family is said to be a **polarized family** if there is a flat proper map

$$\tilde{\pi} : S \to \mathcal{D}$$

from a surface S to the unit disc $\mathcal{D}$ such that for all $t \neq 0$ the fibres $C_t = \tilde{\pi}^{-1}(t)$ are non-singular algebraic curves that can be imbedded into the $K3$ surface X_t, and such that the imbedded curve, also denoted by C_t, is an ample divisor on X_t.

The family is said to be a **marked family** if we can choose representatives of a canonical basis ϕ_t of $H_2(X_t, \mathbf{Z})$ that vary continuously for all $t \neq 0$. This naturally brings up the question of monodromy, which will be discussed later. Also, it is convenient to assume that the family (4.1) is semi-stable, meaning the special fibre X_0 is reduced and has only normal crossing singularities. Locally, such singularities are of the form $z_1 z_2 = t$, in the case of a developing double point, or $z_1 z_2 z_3 = t$, in the case of a developing triple point. By Mumford's semi-stable reduction theorem (see [Mu 73]), any family of $K3$ surfaces can be reduced to a semi-stable family. So, throughout this paper, we shall assume all families of $K3$ surfaces are semi-stable.

The main result of this section is an analogue of Theorem 1.7 which determines the asymptotic behavior of the marked and unmarked discriminants for a family of $K3$ surfaces. In fact, the asymptotic results obtained in this section are derived from Theorem 1.7 by appealing to the fact that special Kummer surfaces are dense in the moduli space of all marked, polarized $K3$ surfaces of degree d (see page 256 of [BPV 84]) and the explicit evaluations the marked and unmarked discriminants obtained in Lemma 3.1 and Theorem 3.2.

An outline of the discussion of this section is as follows. After discussing background information concerning the topology of degenerating polarized $K3$ surfaces, we shall determine a particular family of holomorphically varying holomorphic 2-forms, and we will choose a particular marking of any given family of Kulikov family of $K3$ surfaces. With these choices, we then determine the asymptotics of the associated marked discriminant as well as asymptotics of the associated L^2 norm of the chosen family of 2-forms. By combining these asymptotic formulas, we then obtain the asymptotics of the unmarked discriminants. In the end, one should note that the asymptotics of the unmarked discriminant is independent of the given choices, hence is determined solely by the Kulikov family being considered.

When considering a family of algebraic curves that degenerate to the boundary of the Deligne-Mumford (stable) compactification of the moduli space of algebraic curves of a fixed genus, one has a complete description of the asymptotic behavior of a canonical basis of the first homology group as well as a dual basis of holomorphic 1-forms (see chapter 3 of [Fa 73]). One could present a similar theory for families of $K3$ surfaces (see, for example, various results that appear in the articles [Cl 77], [Gr 70], [Pe 77], [St 77] and [To 76]). We will leave the complete presentation of this picture for a future article.

We begin with the following theorem which is due to Kulikov [Ku 77] and Persson-Pinkham [PeP 81]. We quote directly from [PeP 81].

Theorem 4.1. *Let $\pi : X \to \mathcal{D}$ be a semi-stable degeneration of surfaces such that:*

a) *The generic fiber X_t for $t \neq 0$ has trivial canonical sheaf;*
b) *All components of the special fiber X_0 are algebraic.*

Then there exists a semi-stable modification $\pi' : X' \to \mathcal{D}$ of $\pi : X \to \mathcal{D}$ such that the canonical sheaf of the total space X' is trivial.

Further details and background information concerning Theorem 4.1 are given in [PeP 81] and [Ku 77]. A family of algebraic $K3$ surfaces that fulfills the above theorem will be called a **Kulikov family** of $K3$ surfaces. Throughout this section we will assume that our family of $K3$ surfaces is semi-stable and is such that the canonical sheaf of the total space is trivial.

Since $\pi_1(\mathcal{D}^*) \cong \mathbf{Z}$, it is not necessarily possible to attach a consistent set of markings to the polarized surfaces in a Kulikov family. Attached to each Kulikov family, there is a **monodromy operator** T which can be described explicitly by selecting $t \in \mathcal{D}^*$ and viewing

$$T \in \mathrm{Aut}(H_2(X_t, \mathbf{Z}), L_t)$$

as the Picard-Lefschetz transformation obtained by transporting cycles around the origin $t = 0$ in $\mathcal{D}$ while preserving the polarization class L_t. The operator T is quasi-unipotent, meaning $(T^n - I)^3 = 0$ for some positive integer n. If we assume that the family of surfaces is semi-stable, then $(T - I)^3 = 0$, which is equivalent to saying that its logarithm

$$N = \log T = (T - I) - \frac{1}{2}(T - I)^2 \tag{4.2}$$

is a nilpotent endomorphism of $H_2(X, \mathbf{Z})$ satisfying $N^3 = 0$ (see [To 76]).

There are three types of Kulikov families of $K3$ surfaces which are distinguished by the structure of the logarithm of the monodromy operator (4.2) of the family. It was proved in [To 76] that in the case of a semi-stable family of $K3$ surfaces, we have the following possiblities for the Jordan decomposition of the monodromy operator (see also of Theorem II on page 957 of [Ku 77]).

I. $T = I$, or $N = 0$;
II. T has two Jordan cells of dimension 2, or $N^2 = 0$ and $N \neq 0$;
III. T has one Jordan cell of dimension 3, or $N^3 = 0$ and $N^2 \neq 0$.

In [Ku 77] Kulikov proves a classification theorem for Kulikov families of $K3$ surfaces. Let $\pi : X \to \mathcal{D}$ be a semi-stable family of polarized $K3$ surfaces such that the canonical sheaf of the total space X is trivial. Then we have the following topological classification of Kulilov families.

I. X_0 is smooth;
II. X_0 is a chain of elliptic ruled surfaces with rational surfaces on either end, X_0 contains only double curves, and all double curves are (isomorphic) elliptic curves, say E;
III. X_0 has components that are all are rational surfaces whose double curves on each component form a cycle of rational curves, X_0 contains triple points, and the dual graph of X_0 is a triangulation of $\mathbf{P}^1$.

For type II and type III families, the special fibre X_0 is a singular algebraic variety, and the family is said to be a **degeneration**. The non-triviality of the monodromy operator for degenerating Kulikov families is analogous to the similar phenomenon in the case of elliptic curves, and of degenerating algebraic curves whose limits are irreducible uninoded stable curves. In these situations, one can not choose a consistent basis of homology for all $t \in \mathcal{D}$; rather, in order to consider degenerating *marked* varieties, one must restrict t to a sector $\mathcal{D}_\alpha$ of the form

$$\mathcal{D}_\alpha = \{t \in \mathcal{D} : 0 < \arg(t) < \alpha < 2\pi\}$$

(see page 51 of [Fa 73]).

From the Kulikov classification theorem and the discussion in the beginning of §3, any limit point of a semi-stable family of $K3$ surfaces can be obtained by degenerating special Kummer surfaces. Let us assume that the family of Kummer surfaces are marked with an admissible basis of homology, meaning a basis of homology induced from the associated abelian surface

$$A(\Omega) = E_{\tau_1} \times E_{\tau_2}.$$

The types of degenerating Kulikov families occur in the following situations:

II. τ_2 is fixed and $\tau_1 \to i\infty$, with $|\mathrm{Re}(\tau_1)| \leq 1/2$;
III τ_1 and τ_2 approach $i\infty$, with $|\mathrm{Re}(\tau_1)| \leq 1/2$ and $|\mathrm{Re}(\tau_2)| \leq 1/2$.

In the case of type II degeneration, there is a marking induced on any elliptic curve that lies along the node of the limit, singular $K3$ surface.

The following result, due to Borel [Bo 72], Griffiths [Gr 70], Piatetski-Shapiro and Shafarevich [PSS 71], relates degeneration and compactification.

Theorem 4.2. *Let $\bar{\mathcal{M}}^d_{\mathrm{p}}$ be the Baily-Borel compactification of $\Gamma_d\backslash\mathbf{h}_{K3}$, and assume $\pi : X \to \mathcal{D}$ is a Kulikov family of polarized, algebraic $K3$ surfaces of degree d. Then the induced period map*

$$\mathrm{per} : \mathcal{D}^* \mapsto \mathcal{M}^d_{\mathrm{p}}$$

extends to a holomorphic map of $\mathcal{D}$ into $\bar{\mathcal{M}}^d_{\mathrm{p}}$. If the family is of type II or III, meaning $\pi : X \to \mathcal{D}$ is a degeneration, then

$$\mathrm{per}(0) \in \bar{\mathcal{M}}^d_{\mathrm{p}} \setminus \mathcal{M}^d_{\mathrm{p}}.$$

As in the case of elliptic curves, one must choose a family of holomorphically varying forms for all $\mathcal{M}^d_{\mathrm{mp}}$ in order to study the asymptotics of the marked discriminant. In the case of Kummer surfaces, such a family $\{\omega_{(X,\phi)}\}$ was determined in Lemma 3.1. The following result states that this family of forms has an extension to a holomorphic family of holomorphic 2-forms over the entire moduli space $\mathcal{M}^d_{\mathrm{mp}}$.

Theorem 4.3. *There exists a holomorphically varying family of holomorphic 2-forms $\{\eta\}$ on $\mathcal{M}^d_{\mathrm{mp}}$ such that if $[X, L, \phi] \in \mathcal{M}^d_{\mathrm{mp}}$ is the marked, polarized Kummer surface corresponding to the abelian surface $A(\Omega)$ with arbitrary polarization, then*

$$\eta_{(X,\phi)} = \omega_{(X,\phi)}$$

Outline of Proof. Let $\pi : \mathcal{X} \to \mathcal{M}^d_{\mathrm{mp}}$ be the universal family of marked, polarized $K3$ surfaces of degree d. By the existence of the universal family, the sheaf

$$\pi_* \mathcal{K}_{\mathcal{X}/\mathcal{M}^d_{\mathrm{mp}}}$$

is trivial. Therefore, there exists a globally defined non-zero section, which corresponds to the existence of a globally defined holomorphically varying family of holomorphic 2-forms, say $\{\tilde{\eta}\}$. The problem remains to scale this family appropriately, which can be done as follows. Consider a degenerating Kulikov family of type III, and let γ be the vanishing cycle of homology (see [To 76] and pages 280-287 of [Gr 70]) . For any polarized $K3$ surface X_t in this family, let $\phi(t)$ be the period of $\tilde{\eta}$ with respect to the vanishing cycle; that is,

$$\phi(t) = \int_\gamma \tilde{\eta}_t.$$

From the vanishing cycle γ in this particular Kulikov family, one can use the deformation theory of $K3$ surfaces to define a cycle for any marked, polarized $K3$ surface of degree d, hence ϕ extends to a well-defined holomorphic function on all of $\mathcal{M}^d_{\mathrm{mp}}$. With this, we define the new family of holomorphic 2-forms by $\eta = \tilde{\eta}/\phi$. This new family of holomorphic 2-forms is well-defined provided ϕ is never zero, which can be established by showing that any point in $\mathcal{M}^d_{\mathrm{mp}}$ lies on a degenerating Kulikov family of type III. Details will be given in the next section. □

As a corollary of the proof of Theorem 4.3, we have the following result.

Corollary 4.4. *Let $\{\eta\}$ be the holomorphically varying family of holomorphic 2-forms defined in Theorem 4.3.*

a) *Let $\pi : X \to \mathcal{D}$ be a degenerating Kulikov family of type II. Then η limits to a non-zero meromorphic 2-form η_0 on X_0 with singularities at any elliptic curves E that lie along the nodes. Further, the Poincaré residue of η_0 is a non-vanishing holomorphic 1-form on E.*
b) *Let $\pi : X \to \mathcal{D}$ be a degenerating Kulikov family of type III. Then η limits to a non-zero meromorphic 2-form η_0 on X_0 with singularities at the uninoded rational curves $\mathbf{P}^1_{\mathrm{nod}}$ that lie along the nodes. Further, the Poincaré residue of η_0 is a meromorphic 1-form on $\mathbf{P}^1_{\mathrm{nod}}$ that has residues equal to 1 and -1.*

In this way, we see that the family of forms given by Theorem 4.3 is quite analogous to the family of forms $\{dz\}$ we considered in the setting of elliptic curves.

Given a degenerating Kulikov family of polarized $K3$ surfaces defined over a sector $\mathcal{D}_\alpha$, one needs to give a marking to this family. As discussed in the above proof of Theorem 4.3, this point is discussed in detail in [To 76] and in [Gr 70], pages 280-284. We refer the reader to these references for futher details. Roughly, one can argue as follows. First, note that in the special case of families of special Kummer surfaces, a marking is induced from the associated abelian surface (see discussion in the beginning of §3). By deforming this family, we can extend this marking to any degenerating Kulikov family, defined over $\mathcal{D}_\alpha$. Similarly, one can argue that the limit of any marked degenerating Kulikov family of type II defined over $\mathcal{D}_\alpha$ induces a marking on any elliptic curve lying at the node.

Before we can prove the analogue of Theorem 1.7, we need to define a parameter measuring the degeneration for any degenerating Kulikov family. In the case of elliptic curves, one has the elliptic modular parameter

$$q_\tau = \exp(2\pi i\tau).$$

The following definition establishes a q-parameter for any type III degenerating Kulikov family of polarized, algebraic $K3$ surfaces of degree d.

As discussed above, if $\pi : X \to \mathcal{D}$ is a degenerating Kulikov family of polarized $K3$ surfaces of type III, then the monodromy operator T has a single Jordan cell of dimension 3. Hence, there is a free, three-dimensional submodule $W(X_t, L_t) \subset H_2(X_t, \mathbf{Z})$ for which the action of the monodromy operator is unipotent. That is, with respect to a continously varying basis $\{A_t, B_t, C_t\}$ of $W(X_t, L_t)$, the action of the monodromy is by the matrix

$$\begin{pmatrix} 1 & 1 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus, there is a unique invariant 1-dimensional submodule, generated by $\pm A_t$ for $t \in \mathcal{D}$. Let η_t be as in Theorem 4.3. In §5 we will prove

$$\int_{A_t} \eta_t = \pm 1.$$

The **vanishing cycle** is the cycle A_t such that the above integral is equal to 1. An element B_t in $W(X_t, L_t)$ for which $T(B_t) = B_t + A_t$ will be called a **transverse cycle**. Two transverse cycles differ by an additive factor of the form nA_t where n is an integer.

Definition 4.5. *Let $\pi : X \to \mathcal{D}$ be a degenerating Kulikov family of polarized $K3$ surfaces of degree d. Let A_t be the vanishing cycle of homology, and let B_t be a transverse cycle. The $K3$* **modular parameter** *associated to this family is defined by*

$$q_\pi(t) = \exp\left(2\pi i \int_{B_t} \eta_t\right).$$

Equivalently, if $\tilde{\eta}$ is any non-zero holomorphic 2-form on X_t, then

$$q_\pi(t) = \exp\left(2\pi i \int_{B_t} \tilde{\eta} \Big/ \int_{A_t} \tilde{\eta}\right).$$

We call $q_\pi : \mathcal{D} \to \mathbf{C}$ the $K3$ **modular parameter** associated to the Kulikov family $\pi : X \to \mathcal{D}$. The reason why we call q_π a parameter is that it is indeed a holomorphic parameter at 0 in $\mathcal{D}$, that is

$$q_\pi(0) = 0 \quad \text{and} \quad q'_\pi(0) \neq 0.$$

This follows directly from the asymptotics of the periods as given in [Gr 70]. Also, notice that q_π is independent of the choice of transverse cycle.

Various properties of the $K3$ modular parameter will be presented in a separate article. For this paper, we only need the following result, whose proof is based on results that will be obtained in §5.

Theorem 4.6. *Let $\pi_1 : X_1 \to \mathcal{D}_1$ and $\pi_2 : X_2 \to \mathcal{D}_2$ be two degenerating Kulikov families of type III. Assume there exists $t_1 \in \mathcal{D}_1$ and $t_2 \in \mathcal{D}_2$ such that $\pi_1^{-1}(t_1)$ and $\pi_2^{-1}(t_2)$ are isomorphic polarized $K3$ surfaces, which we shall call (X, L). Then*

$$q_{\pi_1}(t_1) = q_{\pi_2}(t_2).$$

Proof. In Lemma 5.7 we will show that the families of algebraic, polarized $K3$ surfaces

$$\pi_1^* : X_1^* \to \mathcal{D}_1^* \quad \text{and} \quad \pi_2^* : X_2^* \to \mathcal{D}_2^*$$

are diffeomorphic. From the construction of the vanishing and transverse cycles via the Clemens map (see §5 or [Gr 70]), any such diffeomorphism necessarily maps the vanishing cycle to vanishing cycle. Hence, the choice of an A cycle is determined. After the vanishing cycle has been determined, the choice of a transverse cycle is then determined up to additive factors of the form nA. Finally, since $H_0(X_t, \mathcal{K})$ is 1-dimensional over $\mathbf{C}$, it follows that the modular parameter is independent of the choice of the holomorphic 2-form from which one computes A and B periods. Therefore, the q-parameter is well-defined and is independent of π. □

By Theorem 4.6, we can write

$$q([X_t, L_t]) = q_\pi(t)$$

where $\pi^{-1}(t)$ is isomorphic to the polarized $K3$ surface (X_t, L_t). It is shown in §5 that any point in $\mathcal{M}_{\mathrm{p}}^d$ can viewed as a point in some degenerating Kulikov family of type III. Since the A and B periods are locally holomorphic, q is also holomorphic. As stated above, we shall investigate further properties of the $K3$ modular parameter q in a forthcoming article.

In order to consider the asymptotic behavior of the marked discriminant on a degenerating Kulikov family, we need to marked the family over each sector $\mathcal{D}_\alpha$. A construction of such a marking is given on pages 280-284 of [Gr 70]. We will assume this construction, and refer the reader to [Gr 70] for further details. However, there is one important point which we need to emphasize. When considering a degenerating Kulikov family of type II, the marking of the degenerating family over any sector is such that any elliptic curve E which lies along a node of the limit $K3$ surface has a natural induced marking. Hence, for type II degenerations, the nodes of the limit surface can be viewed as marked elliptic curves, which, by Kulikov's classification theorem and Griffiths construction of the family of markings, are isomorphic marked elliptic curves, which we denote by E_τ. We refer to page 282 of [Gr 70] for a more detailed discussion.

With all this, we have the following theorem which determines the asymptotic behavior of the marked discriminant for Kulikov families of $K3$ surfaces.

Theorem 4.7. *There exist constants c_1 and c_2 which depend only on the degree d such that the following asymptotic formulas hold:*

I. *If $\pi : X \to \mathcal{D}$ is a type I Kulikov family of marked, polarized $K3$ surfaces of degree d, then*

$$\log \delta_{\mathrm{mar}}((X_t, L_t, \phi_t), \eta_t) = \log \delta_{\mathrm{mar}}((X_0, L_0, \phi_0), \eta_0) + o(1);$$

II. *If $\pi : X \to \mathcal{D}_\alpha$ is a type II Kulikov family of marked, polarized $K3$ surfaces of degree d, then*

$$6 \log \delta_{\mathrm{mar}}((X_t, L_t, \phi_t), \eta_t) = \log |q([X_t, L_t])| + 6 \log \delta_{\mathrm{mar}}(E_\tau) + c_1 + o(1);$$

III. *If $\pi : X \to \mathcal{D}_\alpha$ is a type III Kulikov family of marked, polarized $K3$ surfaces of degree d, then*

$$6 \log \delta_{\mathrm{mar}}((X_t, L_t, \phi_t), \eta) = 2 \log |q([X_t, L_t])| + c_2 + o(1).$$

Proof. Part I follows directly from the holomorphicity, hence continuity, of the marked discriminant on $\mathcal{M}^d_{\mathrm{mp}}$ As for part II and part III, it suffices to consider Kulikov families of special Kummer surfaces. This follows since the marked discriminant is holomorphic, and $\mathcal{M}^d_{\mathrm{mp}}$ is a Zariski open subset of a domain of holomorphy, hence the asymptotic behavior of the unmarked discriminant out to $o(1)$ depends solely on the limit point of the degenerating family. Also, the marking of a degenerating Kulikov family as given in [Gr 70] coincides with the marking of the Kummer surfaces as described by Fuchs in [PSS 71], which is what we used to calculate the marked discriminants in §3. Finally, as noted above, any limit point can be obtained by considering degenerating families of special Kummer surfaces. By the evaluation of the marked discriminant for such families, as given in Theorem 3.2, the result follows from Theorem 1.7. □

By the Riemann bilinear relations, Theorem 4.7 and Definition 2.5, it suffices to understand the asymptotics of the periods of the holomorphic family of 2-forms $\{\eta\}$ for Kulikov families with admissible bases of homology in order to determine the asymptotics of the unmarked $K3$ discriminant, which, in the end, will not depend on the choice of homology. For the asymptotics of the periods of $\{\eta\}$, one can cite the results from page 286 of [Gr 70]. Alternatively, as in Theorem 4.7, one can reduce the problem in hand to understanding asymptotics of the periods for special Kummer surfaces, since the period map is holomorphic. However, the family of forms $\omega_{(X,\phi)}$ on families of Kummer surfaces is such that the periods are associated to the 16 exceptional curves are all equal to zero since these cycles are algebraic. Therefore, from our definition of an admissible basis of homology, the problem actually reduces to understanding the asymptotics of the periods of the abelian surface, which are reasonably well-known (see page 53 of [Fa 73]).

With all this, we obtain the following result.

Theorem 4.8. *There exist constants c_3 and c_4 which depend only on the degree d such that the following asymptotic formulas hold:*

I. *If $\pi : X \to \mathcal{D}$ is a type I Kulikov family of marked, polarized $K3$ surfaces of degree d, then*

$$\log \|\eta_t\|^2_{L^2} = \log \|\eta_0\|^2_{L^2} + o(1);$$

II. *If $\pi : X \to \mathcal{D}_\alpha$ is a type II Kulikov family of marked, polarized $K3$ surfaces of degree d, then*

$$\log \|\eta_t\|^2_{L^2} = \log\log |q([X_t, L_t])| + \log \mathrm{Im}(\tau) + c_3 + o(1);$$

III. *If $\pi : X \to \mathcal{D}_\alpha$ is a type III Kulikov family of marked, polarized $K3$ surfaces of degree d, then*

$$\log \|\eta_t\|^2_{L^2} = 2\log\log |q([X_t, L_t])| + c_4 + o(1).$$

Proof. One uses the asymptotic formula for periods of degenerating algebraic surfaces as given on page 287 of [Gr 70]. □

Finally, by combining Theorem 4.7 and Theorem 4.8, one has the following result for the asymptotic behavior of the unmarked discriminant for Kulikov families of polarized, algebraic $K3$ surfaces of degree d.

Theorem 4.9. *There exist constants c_5 and c_6 which depend only on the degree d such that the following asymptotic formulas hold:*

I. *If $\pi : X \to \mathcal{D}$ is a type I Kulikov family of polarized $K3$ surfaces of degree d, then*
$$\log \delta_{\text{unm}}(X_t, L_t) = \log \delta_{\text{unm}}(X_0, L_0) + o(1);$$

II. *If $\pi : X \to \mathcal{D}_\alpha$ is a type II Kulikov family of polarized $K3$ surfaces of degree d, then*

$$\begin{aligned} 6 \log \delta_{\text{unm}}(X_t, L_t) =& \log |q([X_t, L_t])| + 6 \log \log |q([X_t, L_t])| \\ &+ 6 \log \delta_{\text{unm}}(E_\tau) + c_5 + o(1); \end{aligned}$$

III. *If $\pi : X \to \mathcal{D}_\alpha$ is a type III Kulikov family of polarized $K3$ surfaces of degree d, then*

$$6 \log \delta_{\text{unm}}(X_t, L_t) = 2 \log |q([X_t, L_t])| + 12 \log \log |q([X_t, L_t])| + c_6 + o(1).$$

Even though we used Theorem 4.7 and Theorem 4.8 to prove Theorem 4.9, it is important to note that the result in Theorem 4.9 is independent of the family of markings on the Kulikov family of $K3$ surfaces.

Remark 1. Theorem 3.3 defines an Arakelov-Kähler-Einstein form on any marked, polarized, algebraic $K3$ surface. One can use Theorem 4.9 to show that, in an appropriate sense, the limiting behavior of the Arakelov-Kähler-Einstein form for a degenerating Kulikov family of type II has a residue that is the Arakelov form on elliptic curve that lies along the node.

§5. Proof of Theorem 4.3

In this section, we will give details of the proof of Theorem 4.3. The proof is given in three steps.

Step 1. The construction of a Kulikov family of type III consisting of special Kummer surfaces.

Step 2. Let γ be the vanishing invariant cycle of the Kulikov family constructed in Step 1, which, by a deformation argument, extends to give a well-defined choice of cycle for any marked, polarized $K3$ surface of degree d. Let $\mathcal{X} \to \mathcal{M}^d_{\mathrm{mp}}$ be the universal family of marked, polarized $K3$ surfaces of degree d, and choose a non-zero section $\tilde{\eta}$ of the trivial sheaf $\pi_*\mathcal{K}_{\mathcal{X}/\mathcal{M}^d_{\mathrm{mp}}}$. Then the function

$$\phi = \int_\gamma \tilde{\eta}$$

is a holomorphic and non-vanishing on $\mathcal{M}^d_{\mathrm{mp}}$.

Step 3. From Step 2, let $\eta = \tilde{\eta}/\phi$. Then if (X, L, ϕ) is a Kummer surface with arbitrary polarization, we have, in the notation of §3, $\eta_{(X,\phi)} = \omega_{(X,\phi)}$.

To begin, we need to review the Clemens map associated to a family of algebraic manifolds. For now, let $\pi : Y \to \mathcal{D}$ be a family of algebraic manifolds of complex dimension n defined over the unit disc $\mathcal{D}$ such that:

a) π is proper;
b) The singular fiber $Y_0 = \pi^{-1}(0)$ is a divisor with normal crossing;
c) Y_0 is locally given by $z_{i_1} \cdots z_{i_k} = 0$, where z_{i_k} are local coordinates on the components D_j of Y_0

Let $Y_t = \pi^{-1}(t)$. In [Cl 77], Clemens constructed a map $h_t : Y_t \to Y_0$ for each $t \in \mathcal{D}^*$ with the following properties.

a) For $z \in Y_0$ with $z \in \mathcal{D}_{i_0} \cap \cdots \cap D_{i_k}$, we have $h_t^{-1}(z) \cong (S^1)^k$;
b) $h_t : Y_t \backslash h_t^{-1}(\mathrm{Sing}(Y_0)) \cong Y_0 \backslash \mathrm{Sing}(Y_0)$; in other words, h_t is a diffeomorphism away from the singular set of Y_0.

With this, we can apply the main theorem [To 76] to obtain a description of the topology of a degenerating Kulikov family of type III.

Theorem 5.1. *Let $\pi : Y \to \mathcal{D}$ degenerating Kulikov family of type III and write $Y_0 = Y_0^{(1)} \cup \cdots \cup Y_0^{(n)}$.*

a) *Let G be the Gysin map*

$$G : \oplus_i H_3(Y_0^{(i)}, \mathbf{Z}) \to \oplus_{i<j} H_1(Y_0^{(i)} \cap Y_0^{(j)}, \mathbf{Z})$$

defined by

$$G(\mu) = \sum_{i \neq j} \mu \cdot [Y_0^{(i)} \cap Y_0^{(j)}],$$

where $\mu \cdot [Y_0^{(i)} \cap Y_0^{(j)}]$ means intersection of the cycles on Y_0. Let

$$\gamma \in \oplus_{i<j} H_i(Y_0^{(i)} \cap Y_0^{(j)}, \mathbf{Z})$$

be such that $\gamma \in \mathrm{Im}(G)$. *Then* $h_t^{-1}(\gamma) \cong (S^1)^2$ *and* $h_t^{-1}(\gamma) \in H_2(Y_t, \mathbf{Z})$ *is homologically non-zero. In addition, let* $\tau \in H_2(Y_t, \mathbf{Z})$ *such that*

$$\tau \cdot h_t^{-1}(\gamma) = 1.$$

Then $T(h_t^{-1}(\gamma)) = h_t^{-1}(\gamma)$ *and* $T(\tau) = \tau + h_t^{-1}(\gamma)$.

b) *Let* $\Gamma(Y_0)$ *be the graph associated with* Y_0, *and suppose* $H_2(\Gamma(Y_0), \mathbf{Q}) \neq 0$. *Then for each triple point* $p = Y_0^{(i_0)} \cap Y_0^{(i_1)} \cap Y_0^{(i_2)}$, *the 2-cycle* $h_t^{-1}(p) = (S^1)^2$ *is non-zero in* $H_2(Y_t, \mathbf{Z})$. *Further, there exist cycles* $\gamma_1, \gamma_2 \in H_2(Y_t, \mathbf{Z})$ *such that*

$$T(h_t^{-1}(p)) = h_t^{-1}(p),$$

$$T(\gamma_1) = \gamma_1 + h_t^{-1}(p),$$

and

$$T(\gamma_2) = \gamma_2 + \gamma_1 + \frac{1}{2} h_t^{-1}(p).$$

The cycle $h_t^{-1}(p)$ is called the **vanishing cycle**, and the cycle γ_1 is called the associated **transverse cycle** of the degenerating Kulikov family of type III.

Remark 1. Recall that the graph $\Gamma(Y_0)$ associated to the singular $K3$ surface Y_0 is constructed as follows. To each triple point $Y_0^{(i)} \cap Y_0^{(j)} \cap Y_0^{(k)}$ we associate a vertex. If two triple points coincide in $Y_0^{(i)} \cap Y_0^{(j)}$, we will join the vertices by a segment. If three triple points lie on the same component $Y_0^{(i)}$ of Y_0, then we connect these three points with a two dimension simplex (triangle).

We shall now construct a special family of $K3$ surfaces of type III. Let us start with the Legrendre family of elliptic curves $\{E_\lambda\}$, defined by

$$E_\lambda = \{y^2 = x(x-1)(x-\lambda) : 0 < |\lambda| < 1\}.$$

We let $A_\lambda = E_\lambda \times E_\lambda$, and we let K_λ be the associated Kummer surface.

Proposition 5.2. *The family of Kummer surfaces* $\{K_\lambda\}$ *is a degenerating Kulikov family of* $K3$ *surfaces of type III.*

Proof. For the Legrendre family of elliptic curves, there exists a marking of the homology group $H_1(E_\lambda, \mathbf{Z})$ given by the canonical basis γ_1, γ_2 such that the monodromy operator T of this family of elliptic curves acts by

$$T(\gamma_1) = \gamma_1 \quad \text{and} \quad T(\gamma_2) = \gamma_2 + \gamma_1$$

(see chapter 3 of [Fa 73]). Let us now compute the monodromy operator T of the associated family of Kummer surfaces $\{K_\lambda\}$, with respect to the basis of homology given by the (powers of the) 16 exceptional curves and the cycles $\gamma_i \times \gamma_j'$, with $i, j = 1, 2$, where γ_1, γ_2 is a canonical basis of homology of the first factor E_λ of A_λ, and γ_1', γ_2' is a canonical basis of homology of the second factor E_λ of A_λ. By direct calculation, one can show that the monodromy operator T of this family of Kummer surfaces acts by

$$T(2(\gamma_1 \times \gamma_1')) = 2(\gamma_1 \times \gamma_1'),$$

$$T(\gamma_1 \times \gamma_2' + \gamma_2 \times \gamma_1') = (\gamma_1 \times \gamma_2' + \gamma_2 \times \gamma_1') + 2(\gamma_1 \times \gamma_1'),$$

and

$$T(\gamma_2 \times \gamma_2^1) = \gamma_2 \times \gamma_2' + (\gamma_2 \times \gamma_1' + \gamma_1 \times \gamma_2') + \gamma_1 \times \gamma_1'.$$

Hence, T satisfies $(T - I)^3 = 0$ and $(T - I)^2 \neq 0$, which shows that the family of Kummer surfaces $\{K_\lambda\}$ is a Kulikov family of type III. □

Let $\mathcal{X} \to \mathcal{M}^d_{\mathrm{mp}}$ be the universal family of marked, polarized $K3$ surfaces of degree d. As stated above, the existence of a universal family implies that the sheaf $\pi_*\mathcal{K}_{X/\mathcal{M}^d_{\mathrm{mp}}}$ is a trivial. Hence, there exists a non-vanishing global section $\tilde{\eta} \in H^0(\mathcal{M}^d_{\mathrm{mp}}, \pi_*\mathcal{K}_{X/\mathcal{M}^d_{\mathrm{mp}}})$. This observation now completes Step 1 stated above, namely the existence of a non-zero holomorphically varying family of holomorphic 2-forms.

In order to prove Theorem 4.3, we need to properly scale the section $\tilde{\eta}$ obtained in Step 1. We do so by studying the asymptotics of the period of $\tilde{\eta}$ on the vanishing cycle of a Kulikov family of type III. For this, we need to define what is meant by the canonical sheaf on a singular $K3$ surface which is the limit of a degenerating Kulikov family of type III.

Definition 5.3. *Let $X \to \mathcal{D}$ be a semi-stable, degenerating Kulikov family of type II or type III. We define* **the canonical sheaf** *of X_0 over $\mathcal{D}$ by*

$$\mathcal{K}_{X/\mathcal{D}} = \mathcal{K}_{X^*/\mathcal{D}^*}\langle \log X_0 \rangle.$$

By definition, this means:

a) *If $z \in X_0$ is a triple point, then $\mathcal{K}_{X/\mathcal{D}}$ is locally generated by any one of the forms*

$$\frac{dz_i \wedge dz_j}{z_i z_j} \quad \text{for } i \neq j \text{ and } i,j = 1,2,3;$$

b) *If $z \in X_0$ is a double point, then $\mathcal{K}_{X/\mathcal{D}}$ is locally generated by any one of the forms*

$$\frac{dz_i \wedge dz_j}{z_i} \quad \text{for } i \neq j \text{ and } i,j = 1,2,3;$$

Theorem 5.4. *Let $\pi : X \to \mathcal{D}$ be a degenerating family of Kulikov surfaces of type III with vanishing cycle γ. Then the function*

$$\phi(t) = \int_\gamma \tilde{\eta}_t$$

is non-zero for all $t \in \mathcal{D}^*$.

The proof will be established through several lemmas.

Lemma 5.5. *Let $\pi : X \to \mathcal{D}$ be a degenerating Kulikov family of $K3$ surfaces. Then the vector space $H^0(X, \mathcal{K}_{X/\mathcal{D}})$ is a free $\Gamma(\mathcal{D}, \mathcal{O}_\mathcal{D})$ module of rank 1.*

Proof. Consider the exact sequence

$$0 \to \mathcal{O}_X \stackrel{t}{\hookrightarrow} \mathcal{O}_X \to \mathcal{O}_{X_0} \to 0,$$

where the inclusion map is given by multiplication by the local parameter $t \in \mathcal{D}$. We then obtain the sequence

$$0 \to \mathcal{K}_{X/\mathcal{D}} \stackrel{t}{\to} \mathcal{K}_{X/\mathcal{D}} \to \mathcal{K}_{X/\mathcal{D}}\Big|_{X_0} \to 0,$$

from which we have the corresponding long exact sequence

$$0 \to H^0(X, \mathcal{K}_{X/\mathcal{D}}) \stackrel{t}{\to} H^0(X, \mathcal{K}_{X/\mathcal{D}}) \to H^0(X, \mathcal{K}_{X/\mathcal{D}}\Big|_{X_0}) \to H^1(X, \mathcal{K}_{X/\mathcal{D}}) \stackrel{t}{\to} \cdots .$$

We shall first prove

$$H^1(X, \mathcal{K}_{X/\mathcal{D}}) = 0$$

by considering the Leray spectral sequence

$$H^p(\mathcal{D}, R^q\pi_*\mathcal{K}_{X/\mathcal{D}}) \Longrightarrow H^{p+q}(X, \mathcal{K}_{X/\mathcal{D}}).$$

From the Grauert direct image theorem, we have that the sheaf $R^q\pi_*\mathcal{K}_{X/\mathcal{D}}$ is coherent over $\mathcal{D}$. Since $\mathcal{D}$ is a Stein manifold, we have

$$H^p(\mathcal{D}, R^q\pi_*\mathcal{K}_{X/\mathcal{D}}) = 0 \quad \text{for } p \geq 2,$$

so it remains to show that we have

$$H^0(\mathcal{D}, R^1\pi_*\mathcal{K}_{X/\mathcal{D}}) = 0$$

Steenbrink [St 76] proved that the sheaf $R^p\pi_*\Omega_{X/\mathcal{D}}$ is a locally free sheaf, so, in particular $R^1\pi_*\mathcal{K}_{X/\mathcal{D}}$ is locally free. Since $H^1(X_t, \Omega^2_{X_t}) = 0$, it follows that $R^1\pi_*\mathcal{K}_{X|\mathcal{D}} = 0$.

Hence, from the long exact sequence, we obtain that the restriction map

$$H^0(X, \mathcal{K}_{X/\mathcal{D}}) \to H^0(X_0, \mathcal{K}_{X/\mathcal{D}}\Big|_{X_0})$$

is surjective. Also, from Serre duality we have the isomorphism

$$H^0(X_0, \mathcal{K}_{X/\mathcal{D}}\Big|_{X_0}) \cong H^2(X_0, \mathcal{O}_{X_0}).$$

We can now put everything together to prove the lemma. From the exact sequence

$$0 \to \mathcal{O}_X \stackrel{t}{\hookrightarrow} \mathcal{O}_X \to \mathcal{O}_{X_0} \to 0,$$

we have the induced exact sequence

$$H^1(X_0, \mathcal{O}_{X_0}) \to H^2(X, \mathcal{O}_X) \stackrel{t}{\to} H^2(X, \mathcal{O}_X) \to H^2(X_0, \mathcal{O}_{X_0}) \to 0.$$

The vector space $H^2(X, \mathcal{O}_X)$ is a free modulo over $\Gamma(\mathcal{D}, \mathcal{O}_\mathcal{D})$ of rank 1. In [To 76] it is shown that $H^1(X_0, \mathcal{O}_{X_0}) = 0$. By applying Serre duality, as above, we have that the vector space

$$H^0(X, \mathcal{K}_{X/\mathcal{D}})$$

is locally free over $\Gamma(\mathcal{D}, \mathcal{O}_\mathcal{D})$ and necessarily of rank 1, which was to be shown. □

Lemma 5.6. *Let $X \to \mathcal{D}$ be a degenerating Kulikov family of polarized $K3$ surfaces of type III. Let $\gamma_t \in H_2(X_t, \mathbf{Z})$ be the vanishing cycle. From Lemma 5,5, let*

$$\tilde{\omega} \in H^0(X, \mathcal{K}_{X/\mathcal{D}});$$

so, over $\mathcal{D}^*$, $\{\tilde{\omega}_t\}$ is a family of holomorphically varying holomorphic 2-forms. Then for all $t \in \mathcal{D}^*$, we have

$$\int_{\gamma_t} \tilde{\omega}_t \neq 0.$$

Proof. From the Clemens map, we have

$$\gamma = h_t^{-1}(\mathcal{D}_{i_1} \cap \mathcal{D}_{i_2} \cap \mathcal{D}_{i_3}).$$

Let $p \in \mathcal{D}_{i_1} \cap \mathcal{D}_{i_2} \cap \mathcal{D}_{i_3}$ be the triple point, and let γ_1 be a small circle around p in $\mathcal{D}_{i_0} \cap \mathcal{D}_{i_1}$. Let $N(\mathcal{D}_{i_0} \cap \mathcal{D}_{i_1})$ be a tubular neighborhood of $\mathcal{D}_{i_0} \cap \mathcal{D}_{i_1}$ in $\mathcal{D}_{i_0}$ with radius ϵ. Let $\gamma_0 = \partial\pi^{-1}(\gamma_1)$ where

$$\pi : N(\mathcal{D}_{i_0} \cap \mathcal{D}_{i_1}) \to \mathcal{D}_{i_0} \cap \mathcal{D}_{i_1}$$

and ∂ is the boundary operator. It is immediate that $\gamma_0 = S^1_\epsilon \times S^1_\epsilon$ in $\mathcal{D}_{i_0}$. It is proved in [To 76] that the cycle $h_t^{-1}(p)$ is homologically equivalent to $h_t^{-1}(\gamma_0)$ in $H_2(X_t, \mathbf{Z})$. Hence,

$$\int_{\gamma} \tilde{\omega}_t = \int_{h_t^{-1}(\gamma_0)} \tilde{\omega}_t.$$

Since $\tilde{\omega}_t \in H^0(X, \mathcal{K}_{X|\mathcal{D}})$, then near the triple point p we have

$$\tilde{\omega}_t = f(t)\frac{dz_1 \wedge dz_2}{z_1 z_2} = f(t)\tilde{\omega}_0$$

where $f(t) \neq 0$ for $t \neq 0$, and $\tilde{\omega}_0$ is as in Definition 5.3. From this we have

$$\int_{\gamma} \tilde{\omega}_t = \int_{h_t^{-1}(p)} \tilde{\omega}_t = \int_{h_t^{-1}(\gamma_0)} \tilde{\omega}_t = f(t)(2\pi i)^2 \mathrm{Res}_p(\tilde{\omega}_0) = \pm f(t)(2\pi i)^2 \neq 0,$$

which completes the proof of the lemma. □

Remark 2. From Lemma 5.5, any two elements of $H^0(X, \mathcal{K}_{X/\mathcal{D}})$ differ by a non-zero multiplicative holomorphic function on $\mathcal{D}$, which will appear as a multiplicative factor in the calculation of the period over γ. So, since the period over γ is non-zero and holomorphic in t, one can divide by this period. Further, the result after dividing is also an element of $H^0(X, \mathcal{K}_{X/\mathcal{D}})$ which and is independent of the initial choice of section.

With the above lemmas, we can complete the proof of Theorem 4.3 as follows. Let

$$\pi_{\mathrm{unm}} : \mathcal{M}^d_{\mathrm{mp}} \to \mathcal{M}^d_{\mathrm{p}} = \Gamma_d \backslash \mathcal{M}^d_{\mathrm{mp}}$$

be the projection obtained by ignoring the marking. For any point $\tau \in \mathcal{M}^d_{\mathrm{mp}}$, we will construct a disc $\mathcal{D} \subset \bar{\mathcal{M}}^d_{\mathrm{p}}$ that contains $\pi_{\mathrm{unm}}(\tau)$ and the point $\mathrm{per}(X_0)$, where $p : X \to \mathcal{D}$ is the degenerating Kulikov family of type III in Proposition 5.2, and $X_0 = p^{-1}(0)$. Next, we prove that over $\mathcal{D}$ we have a Kulikov family of type III, say $\pi : Y \to \mathcal{D}$ which, by Theorem 5.1(b), can be marked so that there exists a vanishing cycle γ that is necessarily invariant with respect to the action by the monodromy operator. Let $\tilde{\omega}_t \in H^0(Y, \mathcal{K}_{Y/\mathcal{D}})$ be a non-zero element. By Lemma 5.6, the period of $\tilde{\omega}_t$ relative to the vanishing cycle is non-zero for all $t \neq 0$.

Now in $\mathcal{D}$ we can choose a sector of form

$$\mathcal{D}_\alpha = \{t \in D : 0 < \arg(t) < \alpha < 2\pi\}$$

with $\pi_{\mathrm{unm}}(\tau) \in \mathcal{D}_\alpha$, and let $\mathcal{D}^0_\alpha$ be the component of $\pi^{-1}_{\mathrm{unm}}(\mathcal{D}_\alpha)$ that contains τ. Note that, by definition, $\mathcal{D}^0_\alpha \subset \mathcal{M}^d_{\mathrm{mp}}$. If we restrict

$$\tilde{\eta}_t \in H^0(\mathcal{M}^d_{\mathrm{mp}}, \pi^*_{\mathrm{unm}}\mathcal{K}_{X/\mathcal{M}^d_{\mathrm{mp}}})$$

to $\mathcal{D}^0_\alpha$, we will get a family of non-zero holomorphic 2-forms with $\tilde{\eta}_t = g(t)\tilde{\omega}_t$ which, by Lemma 5.6, is such that $g(t) \neq 0$ for $t \neq 0$. We can then scale $\tilde{\eta}$ by the period with respect to the vanishing cycle, which is given by $(2\pi i)^2 g(t)$ times the period of $\tilde{\omega}_t$, which is non-zero and holomorphic in t. In this way, we have appropriately scaled $\tilde{\eta}$ by dividing by the non-zero holomorphic function given by the period of the vanishing cycle, thus obtaining a new element

$$\eta \in H^0(\mathcal{M}^d_{\mathrm{mp}}, \pi^*_{\mathrm{unm}}\mathcal{K}_{X/\mathcal{M}^d_{\mathrm{mp}}}).$$

By direct calculation, we will prove that the new family of holomorphically varying holomorphic 2-forms $\{\eta\}$ satisfies Theorem 4.3, thus completing our proof.

We shall now construct the family $Y \to \mathcal{D}$ described above. Let $\bar{\mathcal{M}}^d_{\mathrm{p}}$ be the Baily-Borel compactification of $\mathcal{M}^d_{\mathrm{p}}$, and let $\mathrm{per} : D \to \bar{\mathcal{M}}^d_{\mathrm{p}}$ be the period map of the the degenerating family of Kummer surfaces constructed in Proposition 5.2. Let $\tilde{\mathcal{M}}^d_{\mathrm{p}}$ be the resolution of the singularities of $\bar{\mathcal{M}}^d_{\mathrm{p}}$, so

$$\tilde{\mathcal{M}}^d_{\mathrm{p}} \setminus \mathcal{M}^d_{\mathrm{p}} = D_\infty$$

is a divisor with normal crossing. Let

$$\pi_{\mathrm{res}} : \tilde{\mathcal{M}}^d_{\mathrm{p}} \to \bar{\mathcal{M}}^d_{\mathrm{p}}$$

be the map of the resolution of the singularities, which is guaranteed by Hironaka's work. Consider an embedding

$$\tilde{\mathcal{M}}^d_{\mathrm{p}} \hookrightarrow \mathbf{P}^N,$$

and let $\mathbf{P}^2$ be a plane in $\mathbf{P}^N$ with the following properties:

a) $\mathbf{P}^2$ intersects $\tilde{\mathcal{M}}^d_{\mathrm{p}}$ transversely;
b) $\pi_{\mathrm{unm}}(\tau) \in \mathbf{P}^2$;
c) $\mathbf{P}^2$ intersects $h^{-1}(\mathrm{per}(X_0))$ transversely.

It is immediate that $\mathbf{P}^2 \cap \tilde{\mathcal{M}}_{\mathrm{p}}^d = C$ is a non-singular curve, and we can find a domain $\mathcal{D} \subset C$ with the following properties:

a) $\pi_{\mathrm{unm}}(\tau) \in \mathcal{D}$;
b) $\mathcal{D}$ contains only one of the points in $\pi_{\mathrm{res}}^{-1}(\mathrm{per}(X_0)) \cap \mathbf{P}^2 = 0$, say q;
c) $\mathcal{D}$ is holomorphically equivalent to $\{t : |t| < 1\}$, so $\mathcal{D}\setminus\{q\}$ is holomorphically equivalent to the punctured unit disc.

Now let $Y^* \to \mathcal{D}^*$ be the family of polarized $K3$ surfaces of degree d over the punctured disc $\mathcal{D}^* \subset \mathcal{M}_{\mathrm{p}}^d$. By Theorem 4.2, we can compactify this family to a family $Y \to \mathcal{D}$ which is degenerating Kulikov family.

Lemma 5.7. *The family $\pi : Y \to \mathcal{D}$ is of type III, and $Y^* \to \mathcal{D}^*$ is diffeomorphic to the family $\{K_\lambda\} \to D^*$ from Proposition 5.3.*

Proof. Let U be a polycylinder in $\tilde{\mathcal{M}}_{\mathrm{p}}^d$ intersecting $\pi_{\mathrm{res}}^{-1}(\mathrm{per}(X_0))$, and let $X_U \to U$ be the corresponding family of polarized $K3$ surfaces. In fact, one can construct the family $X_U \to U$ in the following manner.

Let

$$Z \to \mathrm{Hilb}^{ss}_{X/\mathbf{P}^{n_0}}$$

be the family of $K3$ surfaces over the semi-stable points of the Hilbert scheme $\mathrm{Hilb}_{X/\mathbf{P}^{n_0}}$, which is the Hilbert scheme of $K3$ surfaces imbedded by the linear system $|3L|$, where L is the polarization class. From the global Torelli theorem and geometric invariant theory [MF 82] we have that the space

$$\mathrm{Hilb}^{ss}_{X/\mathbf{P}^{n_0}}/SL_{n_0+1}(\mathbf{C})$$

is a projective variety and we have the universal family

$$\tilde{\mathcal{X}} \to \mathrm{Hilb}^{ss}_{X/\mathbf{P}^{n_0}}/SL_{n_0+1}(\mathbf{C}).$$

By applying the global Torelli theorem, we obtain an embedding

$$\mathcal{M}_{\mathrm{p}}^d \hookrightarrow \mathrm{Hilb}^{ss}_{X/\mathbf{P}^{n_0}}/SL_{n_0+1}(\mathbf{C}) = \hat{\mathcal{M}}_{\mathrm{p}}^d.$$

We can resolve the singularities of $\hat{\mathcal{M}}_{\mathrm{p}}^d$ along $\tilde{D}_\infty = \hat{\mathcal{M}}_{\mathrm{p}}^d \setminus \mathcal{M}_{\mathrm{p}}^d$ obtaining new space $\tilde{Z}$, and there exists a holomorphic map

$$\tilde{\pi}_{\mathrm{unm}} : \tilde{Z} \to \hat{M}_{\mathrm{p}}^d$$

such that $\tilde{Z} \setminus \hat{M}_{\mathrm{p}}^d$ is a divisor with normal crossing. With all this, it is immedate that $\tilde{\pi}_{\mathrm{unm}}(\tilde{\mathcal{X}}) \to \tilde{Z}$ is a family of $K3$ surfaces over $\tilde{Z}$. From Borel's theorem, one has the existence of a holomorphic map

$$\tilde{p} : \tilde{Z} \to \bar{\mathcal{M}}_{\mathrm{p}}^d$$

where $\bar{\mathcal{M}}_{\mathrm{p}}^d$ is the Baily-Borel compactification. Hence, we may assume $\tilde{Z} = \hat{\mathcal{M}}_{\mathrm{p}}^d$, which is the resolution of singularities of the Baily-Borel compactification. Therefore, we may assume $U \subset \hat{\mathcal{M}}_{\mathrm{p}}^d$, from which we have the existence of the family $X_U \to U$.

From the construction of the degenerating family of Kummer surfaces given in Proposition 5.2, it follows that we have $D^* \subset \hat{\mathcal{M}}^d_{\rm p}$, hence $D \subset \hat{\tilde{\mathcal{M}}}^d_{\rm p}$. Further, by the construction of the family $Y \to \mathcal{D}$, we may assume that the base discs D and $\mathcal{D}$ only intersect at the point $q = \text{per}(X_0)$. Since both discs are subsets of the polycylinder $U \setminus (U \cap \tilde{D}_\infty)$, it follows that we can deform the family of Kummer surfaces diffeomorphically to the family $Y^* \to \mathcal{D}^*$, which implies that the monodromy operator T of $Y^* \to \mathcal{D}^*$ has the same properties as that of the family of Kummer surfaces. This completes the proof of the lemma. □

The deformation of the family $\pi : Y \to \mathcal{D}$ to the degenerating Kulikov family of Kummer surfaces of type III necessarily maps the vanishing cycle of one family to the vanishing cycle of the other family. Hence, we can extend the period mapping $\phi(\tilde{\eta})$ of $\tilde{\eta}$ with respect to the vanishing cycle of the family of Kummer surfaces from Proposition 5.2 to all $\mathcal{M}^d_{\rm mp}$. This function is then holomorphic and, by Lemma 5.6, is also non-zero. Therefore, we can consider the family of holomorphically varying holomorphic 2-forms given by $\eta = \tilde{\eta}/\phi(\tilde{\eta})$.

Remark 3. As emphasized above, the construction of the family of forms η involves scaling a choice of family of forms $\tilde{\eta}$ by the periods along the vanishing cycle. It should be noted that since

$$\dim H^0(\mathcal{M}^d_{\rm p}, \pi_* \mathcal{K}_{X/\mathcal{M}^d_{\rm p}}) = 1,$$

where the dimension is over $H^0(\mathcal{M}^d_{\rm p}, \mathcal{O})$, the family of forms η is independent of the initital choice of family of forms.

All that remains is to show that the family of forms $\{\eta\}$ coincides with the family of 2-forms given in §3 in the case that the underlying $K3$ surface is a Kummer surfaces. Since the above proof is constructive, we shall follow the above set-up to prove this last point.

In the notation of §3, let $K(\Omega)$ be the Kummer surface associated to the marked abelian surface $A(\Omega)$. We can form a degenerating Kulikov family of type III by first deforming the matrix Ω in the Siegel upper half space of dimension 2 to a diagonal matrix of the form

$$\Omega = \begin{pmatrix} \tau & 0 \\ 0 & \tau \end{pmatrix}.$$

Hence, $K(\tau)$ is the special Kummer surface associated to $E_\tau \times E_\tau$. In the notation of Proposition 5.2, the vanishing invariant cycle γ is given by

$$\gamma = 2(A \times A'),$$

where A and B denotes a canonical basis of $H_1(E_\tau, \mathbf{Z})$. Recall that any two holomorphic 2-forms on $K(\tau)$ differ by a non-zero multiplicative scalar, and the period of the form $dz_1 \wedge dz_2$ on $A(\Omega)$ along $A \times A'$ is 1. Hence, two forms on K_τ coincide if their periods along the cycle associated to $A \times A'$ coincide. Since the period of η along $A \times A'$ is one, we conclude that the pullback of η is indeed $dz_1 \wedge dz_2$.

With all this, the proof of Theorem 4.3 is complete.

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