Codimension two immersions of oriented Grassmann manifolds

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Abstract

In this paper we prove that there exist no codiemnsion two immersiona of oriented Grassmann manifolds into Euclidean spaces, except for $\widetilde{G_2(\mathbf{R}^4)}$, $\widetilde{G_2(\mathbf{R}^5)}$, $\widetilde{G_3(\mathbf{R}^6)}$ and spheres.

1. Introduction

For $1 \le k < n$, let $\widetilde{G_k(\mathbf{R}^n)}$ denote the oriented Grassmann manifold of oriented k-dimensional vector subspaces of $\mathbf{R}^n$. $\widetilde{G_k(\mathbf{R}^n)}$ is a smooth manifold of dimension $k(n-k)$. Note that $\widetilde{G_1(\mathbf{R}^n)} \cong S^{n-1}$, the $(n-1)$-sphere, and that $\widetilde{G_k(\mathbf{R}^n)} \cong \widetilde{G_{n-k}(\mathbf{R}^n)}$ under the diffeomorphism that sends an oriented $k-$plane V to $V^\perp$ together with that orientation on $V^\perp$ which induces the standard orientation on $V \oplus V^\perp = \mathbf{R}^n$. The question of stable parallelizability for the oriented Grassmann manifolds was solved in [7] and [8]. Since $\widetilde{G_k(\mathbf{R}^n)}$ is orientable, the stable parallelizability for $\widetilde{G_k(\mathbf{R}^n)}$ is equivalent to the existence of a codimension one immersion of $\widetilde{G_k(\mathbf{R}^n)}$ into Euclidean space. In this paper, we investigate the existences of codimension two immersions of $\widetilde{G_k(\mathbf{R}^n)}$ into Euclidean spaces. Since $\widetilde{G_k(\mathbf{R}^n)} \cong \widetilde{G_{n-k}}(\mathbf{R}^n)$, we assume, without loss of generality, that $2k \le n$. Our main result is

Theorem 1.1 *Let* $2 \le k \le n/2$. *Then* $\widetilde{G_k(\mathbf{R}^n)}$ *immerses into* $\mathbf{R}^{k(n-k)+2}$ *if and only if* $(n,k) = (4,2), (5,2)$ *or* $(6,3)$.

Let $\gamma = \gamma_{n,k}$ denote the canonical k-plane bundle over $\widetilde{G_R(\mathbf{R}^n)}$, and let $\beta = \beta_{n,k}$ be its orthogonal complement, whose fiber over a $V \in \widetilde{G_k(\mathbf{R}^n)}$ is the vector space $V^\perp \subset \mathbf{R}^n$. We have bundle equivalence

$$(1.2) \qquad \gamma_{n,k} \oplus \beta_{n,k} \cong n\varepsilon ,$$

where ε denotes a trivial line bundle.

It is well known that the tangent bundle $\tau\widetilde{G_k(\mathbf{R}^n)}$ of $\widetilde{G_k(\mathbf{R}^n)}$ has the following description ([6]):

$$(1.3) \qquad \tau\widetilde{G_k(\mathbf{R}^n)} \cong \gamma_{n,k} \otimes \beta_{n,k}$$

For a topological space X, let $r : K(X) \to KO(X)$ denote the homomorphism of Abelian groups gotten by restriction of scalars to $\mathbf{R}$, and let $c : KO(X) \to K(X)$ denote the complexification, $c[\xi] = [\xi \otimes_{\mathbf{R}} \mathbb{C}]$, which is a ring homomorphism.

We have the following identities:

$$(1.4) \qquad rc(x) = 2x \qquad \forall x \in KO(X)$$

$$(1.5) \qquad cr(y) = y + \overline{y} \qquad \forall y \in K(X) ,$$

where $\overline{y}$ stands for complex conjugation of y.

2. K-theory of complex projective spaces

Let η denote the Hopf complex line bundle over $\mathbb{C}P^n$, $\sigma = \eta - 1 \in \widetilde{K}(CP^n)$, $\xi = r\eta$, $y = r\sigma = \xi - 2 \in \widetilde{KO}(\mathbb{C}P^n)$.

Proposition 2.1 ([1], [3])

(i) the ring $K(\mathbb{C}P^n)$ is a truncated polynomial ring over the integers generated by σ, i.e.,

$$K(\mathbb{C}P^n) \cong \mathbf{Z}[\sigma]/\langle\sigma^{n+1}\rangle ;$$

(ii) the ring $KO(\mathbb{C}P^n)$ is a truncated polynomial ring over the integers generated by y, with the following relations:

$$\begin{array}{rl} y^{t+1} = 0 & \text{if} \quad n = 2t (t \geq 0) \\ 2y^{2s+1} = 0,\ y^{2s+2} = 0 & \text{if} \quad n = 4s+1 (s \geq 0) \\ y^{2s+2} = 0 & \text{if} \quad n = 4s+3 (s \geq 0); \end{array}$$

(iii) the complexification $c : KO(\mathbb{C}P^n) \to K(\mathbb{C}P^n)$ is a monomorphism if $n \not\equiv 1 \bmod 4$.

Proposition 2.2 For arbitrary real 2-plane bundle ζ over $\mathbb{C}P^2$, there exists $s \in \mathbf{Z}$, such that $\zeta - 2 = s^2(\xi - 2) \in \widetilde{KO}(\mathbb{C}P^2)$.

Proof: Since $\mathbb{C}P^2$ is one-connected, all real bundles over $\mathbb{C}P^2$ are orientable. Observe that ζ may arise from the realification of a complex line bundle over $\mathbb{C}P^2$, $\left(SO(2) \cong U(1)\right)$, but the complex line bundles over $\mathbb{C}P^2$ are in bijection with $H^2(\mathbb{C}P^2; \mathbf{Z}) \cong \mathbf{Z}$.. Therefore we get

$$\begin{array}{ll} \zeta = r\eta^2 = r(\eta \otimes \eta \otimes \cdots \otimes \eta), & \text{or} \\ \zeta = r\overline{\eta}^s = r(\overline{\eta} \otimes \overline{\eta} \otimes \cdots \otimes \overline{\eta}) & \text{for some} \quad s \in \mathbf{Z}^+ = \{n \geq 0, n \in \mathbf{Z}\} \end{array}$$

Let us consider first the case $\zeta = r\eta^s$. Now, by proposition 2.1,

$$\begin{aligned} \zeta = r\eta^s = r(\sigma+1)^s &= r\left(1 + s\sigma + \binom{s}{2}\sigma^2\right) \\ &= 2 + s(\xi - 2) + \binom{s}{2} r\sigma^2 \end{aligned} \tag{2.3}$$

and we have to compute $r\sigma^2 \in KO(\mathbb{C}P^2)$. Note that $\eta\overline{\eta} = 1$, so $(1+\sigma)(1+\overline{\sigma}) = 1$, it follows that $\overline{\sigma} = -\sigma + \sigma^2$. By (1.5), we have

$$cr\sigma^2 = \sigma^2 + \overline{\sigma^2} = \sigma^2 + \overline{\sigma}^2 = \sigma^2 + \left(-\sigma + \sigma^2\right)^2 = 2\sigma^2$$

On the other hand, $c(\xi - 2) = cr\sigma = \sigma + \overline{\sigma} = \sigma + \left(-\sigma + \sigma^2\right) = \sigma^2$. So $c(2(\xi - 2)) = 2\sigma^2 = cr\sigma^2$. By proposition 2.1 (iii), we finally obtain $r\sigma^2 = 2(\xi - 2)$. Now, (2.3) implies that $\zeta - 2 = s(\xi - 2) + 2\binom{s}{2}(\xi - 2) = s^2(\xi - 2)$. For the other case $\zeta = r\overline{\eta}^s$, the proof is similar.

3. Proof of theorem

Proposition 3.1 For $2 \leq k < n, \quad n \neq 2k, \quad n \geq 6, \quad \widetilde{G_k(\mathbf{R}^n)}$ has not codimension two immersion into Euclidean space.

Proof: Without loss of generality we assume that $2k \leq n$. It follows that $n - k \geq 4 = \dim \mathbf{C}P^2$. Thus every real orientable k-plane bundle α over $\mathbf{C}P^2$ can be classified by a map $f : \mathbf{C}P^2 \to \widetilde{G_k(\mathbf{R}^n)}$ so that $f^*(\gamma) \cong \alpha$. Taking $\alpha = \xi \oplus (k-2)\varepsilon$, where ξ is the underlying real 2-plane bundle of the canonical complex line bundle over $\mathbf{C}P^2$, we obtain the following equalities in $KO(\mathbf{C}P^2)$:

$$\begin{aligned} f^*(\gamma) &\cong \xi \oplus (k-2)\varepsilon \ , \\ f^*(\beta) &\cong f^*(n\varepsilon - \gamma) \\ &\cong (n-k+2)\varepsilon - \xi . \end{aligned}$$

Thus

$$\begin{aligned} f^*\left(\tau\widetilde{G_k}(\mathbf{R}^n)\right) &\cong f^*(\gamma \otimes \beta) \cong f^*(\gamma) \otimes f^*(\beta) \\ &\cong (\xi \oplus (k-2)\varepsilon) \otimes ((n-k+2)\varepsilon - \xi) \\ &\cong (n-2k+4)\xi + (k-2)(n-k+2)\varepsilon - \xi \otimes \xi \end{aligned}$$

Using the relation $\xi \otimes \xi \cong 4\xi - 4$ in $KO(\mathbf{C}P^2)$ (proposition 3.1), we obtain

$$f^*\left(\tau\widetilde{G_k}(\mathbf{R}^n)\right) \cong (n-2k)\xi + ((k-2)(n-k+2)+4)\varepsilon \tag{3.2}$$

Suppose $\widetilde{G_k}(\mathbf{R}^n)$ immerses into $\mathbf{R}^{K(n-k)+2}$, then there exists an orientable 2-plane bundle ζ' over $\widetilde{G_k}(\mathbf{R}^n)$, such that

$$\tau\widetilde{G_k}(\mathbf{R}^n) \oplus \zeta' \cong (k(n-k)+2)\varepsilon .$$

It follows that

$$f^*\left(\tau\widetilde{G_k}(\mathbf{R}^n)\right) \oplus f^*(\zeta') \cong (k(n-k)+2)\varepsilon .$$

Using (3.2), we obtain

$$f^*(\zeta') \oplus (n-2k)\xi \oplus ((k-2)(n-k+2)+4)\varepsilon \cong (k(n-k)+2)\varepsilon .$$

By proposition 3.2, (taking $\zeta = f^*\zeta'$), we obtain

$$\left(n-2k+s^2\right)(\xi - 2) = 0 \quad \text{in} \quad \widetilde{KO}(\mathbf{C}P^2)$$

with $\left(n-2k+s^2\right) > 0$, a contradiction to proposition 3.1.

Proposition 3.3 For $k \geq 4, \quad \widetilde{G_k}\left(\mathbf{R}^{2k}\right)$ has not codimension two immersion into $\mathbf{R}^{k^2+2}$.

Proof: V. Bartik and J. Korbaš [2] have computed $w_i(G_R(\mathbf{R}^n))$ for $1 \leq i \leq 9$. From their results $w_8\left(G_4\left(\mathbf{R}^8\right)\right) = w_2^4 + w_1^2 + w_3^2 \in H^8\left(G_4\left(\mathbf{R}^8\right); \mathbf{Z}_2\right)$. It follows that $w_8\left(\widetilde{G_4}\left(\mathbf{R}^8\right)\right) = (w_2(\gamma_{8,4}))^2 \in H^8\left(\widetilde{G_4}\left(\mathbf{R}^8\right); \mathbf{Z}_2\right)$. We use the Gysin sequence associated to the double covering $\widetilde{G_4}\left(\mathbf{R}^8\right) \to G_4\left(\mathbf{R}^8\right)$ together with cohomology of $G_4\left(\mathbf{R}^8\right)$ to establish that $(w_2(\gamma_{8,4}))^4 \neq 0$ in $H^8\left(\widetilde{G_4}\left(\mathbf{R}^8\right); \mathbf{Z}_2\right)$. It is easy to see that $w_i\left(\widetilde{G_4}\left(\mathbf{R}^8\right)\right) = 0$ for $1 \leq i \leq 7$.

These imply that $w_i\left(\widetilde{G_4}(\mathbf{R}^8)\right) = (w_2(\gamma_{8,4}))^4 \neq 0.$ So $\widetilde{G_4}(\mathbf{R}^8)$ has no codimension two immersion into $\mathbf{R}^{18}$.

In case $n = 2k,\ k > 4$, consider the inclusion $\mathbf{R}^8 \to \mathbf{R}^{k-4} \oplus \mathbf{R}^8 \oplus \mathbf{R}^{k-4}$. This induces an inclusion $j : \widetilde{G_4}(\mathbf{R}^8) \to \widetilde{G}_R(\mathbf{R}^{2k})$ where $j(A) = \tilde{X} + \tilde{A},\ \tilde{X} = \mathbf{R}^{k-4} \oplus \circ \oplus \circ$, and $\tilde{A} = \circ \oplus A \oplus \circ$. It is readily seen taht $j^*\gamma_{2k,k} = \gamma_{8,4} \oplus (k-4)\varepsilon$. Hence

$$\begin{aligned} j^*\left(\tau\widetilde{G_k}\left(\mathbf{R}^{2k}\right)\right) &\cong j^*(\gamma_{2k,k} \otimes \beta_{2k,k}) \\ &\cong j^*(\gamma_{2k,k}) \otimes j^*(\beta_{2k,k}) \\ &\cong (\gamma_{8,4} \oplus (k-4)\varepsilon) \otimes (\beta_{8,4} \oplus (k-4)\varepsilon) \\ &\cong \gamma_{8,4} \otimes \beta_{8,4} \oplus (k-4)\varepsilon \otimes (\beta_{8,4} \oplus \gamma_{8,4}) \oplus (k-4)^2\varepsilon \\ &\cong \tau\left(\widetilde{G_4}(\mathbf{R}^8)\right) \oplus (k^2 - 16)\varepsilon. \end{aligned}$$

Suppose $\widetilde{G_k}(\mathbf{R}^{2k})$ immerses into $\mathbf{R}^{k^2+2}$, then there exists an orientable 2-plane bundle ζ over $\widetilde{G_k}(\mathbf{R}^{2k})$, such that $\tau\widetilde{G_k}(\mathbf{R}^{2k}) \oplus \zeta \cong (k^2+2)\varepsilon$, thus

$$\tau\left(\widetilde{G_4}(\mathbf{R}^8)\right) \oplus (k^2 - 16)\varepsilon \oplus j^*(\zeta) \cong (k^2+2)\varepsilon.$$

By Hirsch theory ([4]), we obtain that $\widetilde{G_4}(\mathbf{R}^8)$ immerses into $\mathbf{R}^{18}$, a contradiction to the conclusion above. This concludes the proof of the proposition.

Proposition 3.4 $\widetilde{G_2}(\mathbf{R}^5)$ immerses into $\mathbf{R}^8$.

Proof: An investigation similar to lemma 3.2 in [5] yields: the quotient space $\widetilde{G_2}(\mathbf{R}^5)/\widetilde{G}_2(\mathbf{R}^4)$ is homeomorphic to the Thom space $T(3\varepsilon)$ of a 3-plane trivial bundle over $\widetilde{G_1}(\mathbf{R}^4) \cong S^3$ (since S^3 is parallelizable. Obstruction theory establishes:

$$\widetilde{KO}\left(\widetilde{G_2}(\mathbf{R}^5)/_{\widetilde{G}_2(\mathbf{R}^4)}\right) \cong \widetilde{KO}(T(3\varepsilon)) \cong \widetilde{KO}\left(S^3 \wedge (S^3 \cup \infty)\right) \cong 0\,.$$

This yields that the injectivity of i^* in the exact KO sequence of the cofibration: $S^2 \times S^2 \cong \widetilde{G_2}(\mathbf{R}^4) \xrightarrow{i} \widetilde{G_2}(\mathbf{R}^5) \to \widetilde{G_2}(\mathbf{R}^5)/_{\widetilde{G_2}(\mathbf{R}^4)}$

$$i^* : \widetilde{KO}\left(\widetilde{G_2}(\mathbf{R}^5)\right) \to \widetilde{KO}(S^2 \times S^2) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}.$$

It is easy to see that $2i^*(\gamma - 2) = 0$ in $\widetilde{KO}(S^2 \times S^2)$, since $i^*(\gamma)$ is the canonical 2-plane bundle over $S^2 \times \{x_0\} \subset S^2 \times S^2$. So we have

$$2(\gamma - 2) = 0 \quad \text{in} \quad \widetilde{KO}\left(\widetilde{G_2}(\mathbf{R}^5)\right). \tag{3.5}$$

On the other hand, using λ^2-construction (second exterior power) we obtain

$$\binom{5}{2}\varepsilon \cong \lambda^2(\gamma \oplus \beta) \cong \lambda^2(\gamma) \oplus \lambda^2(\beta) \oplus \gamma \otimes \beta \cong \gamma \otimes \beta \oplus \beta \oplus \varepsilon. \tag{3.6}$$

Combining (3.5) and (3.6), (1.2), we may obtain

$$\tau\widetilde{G_2}(\mathbf{R}^5) \oplus \gamma \cong 8\varepsilon.$$

Together with Hirsch theory ([4]), we know at once that $\widetilde{G_2}(\mathbf{R}^5)$ immerses into $\mathbf{R}^8$.

Proof of theorem. The "only if" part of the theorem comes from proposition 3.1, 3.3. Then it suffices to show that $\widetilde{G_2}\left(\mathbf{R}^4\right)$ immerses into $\mathbf{R}^6$ and $\widetilde{G_3}(\mathbf{R}^6)$ immerses in $\mathbf{R}^{11}$. But it is well known that $\widetilde{G_2}(\mathbf{R}^4) \cong S^2 \times S^2$, and $\widetilde{G_3}(\mathbf{R}^6)$ is parallelizable [8].

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