Discrete group of integrable mappings as a foundation of the theory of integrable systems

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MPI 96-40

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Abstract

We discuss the general properties of discrete transformation which leavs integrable systems invariant. A group-theoretical interpretion for this transformation is proposed. It allows to describe and understand all essential properties of integrable systems as a direct corollary of a representation theory of discrete groups of integrable mappings.

1 Introduction

Liouville has introduced the term "integrability" with respect to dynamical systems. He proved that if a dynamical system possesses a sufficiently large number of integrals of motion in involution then such a system is integrable. But neither general methods for constructing solution in an explicit form nor any mention of the symmetry of the system under consideration are contained in the Liouville's criterion.

In the case of Lie symmetries the theorem of E.Noether fills this gap and teaches us that the number of conservation laws coincides with the dimension

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of the Lie group and gives the possibility (in the case of a Lagrange theory) of obtaining explicit expressions for integrals of motion.

Roughly speaking the modern theory of integrable systems up to now has maintained the Liouville definition (an integrable system have to possess an infinite number of integrals of motion in involution) and many people have found various consequences which follow from this fact.

The goal of this paper is to show in a deductive way that the theory of integrable systems may be understood as a theory of linear representations of discrete groups of integrable mappings [1, 2].

This does not mean that at the moment we can propose a complete mathematical theory of this connection. Our aim is to show that all known results of the theory of integrable systems do not contradict to this hypothesis.

2 Discrete transformation of integrable systems and its general properties

Let us consider a local invertible transformation described by the substitution

$$\tilde{u} = \phi(u, u', u'', \ldots, u^{(r)}) \equiv \phi(u), \tag{2.1}$$

where u is an s-dimensional vector function and $u', u'', \ldots$ are its derivatives of the corresponding order with respect to "space" coordinates (the dimension of the space may be arbitrary).

At first, we want to enumerate the most important general properties of substitutions (2.1) which result from observation of the sufficiently large number of integrable systems [3].

1. All equations of a given hierarchy are invariant with respect to the same discrete transformation.

2. The substitution (2.1) is invertible: this means that equations (2.1) may be resolved with respect to the "old" variables u which may be expressed as functions of the "new" variables $\tilde{u}$ and their derivatives.

3. The substitution (2.1) is canonical [4, 5]. This fact can be expressed in two equivalent forms. There exists a single generating function from

which by the rules of the theory of canonical transformations it is possible to obtain the explicit form of substitution (2.1). In other words this means that the substitution (2.1) may be related to some Poisson structure [7] (and not single) which is invariant with respect to transformations described by substitution (2.1).

4. The conserved quantities of the theory are shifted by the divergence (with respect to space coordinates) under the transformation (2.1) [7].

5. The substitution (2.1) may be rewritten in the form of infinite chain of equations
$$\tilde{u}_{n+1} = \phi(u_n, u'_n, u''_n, \ldots), \tag{2.2}$$
where u_n denotes the result of n-time application of transformation (2.1) to some initial function u_0 (a possible solution of some integrable system). The general property of chains (2.2) consists in their integrability. This means that it is possible to obtain exact general solution of these chains under appropriate way of interrupting of the chain (by using some "good" boundary conditions) on its both ends. For all integrable systems with a rational spectral parameter (in old inverse scattering method terminology) the chains (2.2) coincide with equations of Toda lattice (the Darboux transformation) or of its generalizations. About situation in the case of elliptic spectral parameter see [9, 10].

6. Substitution (2.1) may be generalized for the case of non-commutative variables. For instance, the function u may be considered as matrix-valued or as operator-valued function in the corresponding representation space [8].

3 The problems which may be solved with the help of discrete transformations.

Now we enumerate the most important results which may be obtained with the help of discrete transformation (2.1).

1. It is possible to obtain the wide class of explicit solutions of integrable systems in a determinant (Hirota) form. These solutions depend on some number of arbitrary functions [3].

2. By an appropriate choice of these functions it is possible to extract those solutions which either are invariant with respect to some inner automorphism of dynamical system under consideration (in particular, multi-soliton solutions) or satisfy some other boundary conditions [8, 9].

3. The condition of invariance of Poisson structures with respect to transformation (2.1)
$$\phi'(u)J_n(u)\phi'^T(u) = J_n(\phi(u))$$
($\phi'(u)$ is the Frechet derivative) allows to obtain the explicit form of nonlocal Hamiltonian operators of an arbitrary order and to construct the whole hierarchy of integrable systems with given discrete substitution [4]–[6].

4. It is possible to obtain the equations of (1+2) integrable hierarchies corresponding to a given integrable substitution (2.1) in two-dimensional space [7].

4 Equation determining the discrete substitution and its group-theoretical interpretation

As discussed above, knowledge of a discrete substitution allows to give a solution for many problems of the theory of integrable systems. The only "small" problem is how to choose an appropriate substitution from the infinite set of possible ones?

Below we give the equation solution of which is exactly the mapping (2.1) satisfying all conditions necessary to exploit it as a discrete symmetry of some integrable system [1, 2]. This equation is obtained under the assumption of locality of a substitution.

Let $\phi'(u)$ be a Frechet derivative corresponding to substitution (2.1)

$$\phi'(u) = \frac{\partial\phi}{\partial u} + \frac{\partial\phi}{\partial u'}D + \frac{\partial\phi}{\partial u''}D^2 + \cdots .$$

Next, we denote by $F(u)$ the vector column function components of which are some (may be nonlocal) functions of dynamical variables u and of its derivatives, namely

$$F(u) \equiv F(u, u', u'', \dots, u^{(n)}).$$

Then every solution of the functional differential equation with shifted arguments

$$F(\phi(u)) = \phi'(u)F(u) \tag{4.1}$$

may be related to an evolution-type equation

$$u_t = F(u)$$

which is invariant with respect to the discrete transformation $\tilde{u} = \phi(u)$.

Equation (4.1) is a generalization of the well known condition of integrability in the theory of differential equations. Indeed let us differentiate substitution (2.1) with respect to some parameter on which "initial" function u depends and denote $\dot{\phi}(u) \equiv F(\phi(u)), \dot{u} \equiv F(u)$ then (with all necessary words) we come to (4.1).

So if a mapping (substitution) is integrable (in the above sense), then it is possible to consider it as a discrete symmetry of some integrable system.

Let us now compare the equation (4.1) with a definition of a linear representation $T(g)$ of some group (for instance, Lie group)

$$\Phi(gx) = T(g)\Phi(x), \tag{4.2}$$

where g is a group element, $T(g)$ is the group operator of representation, $\Phi(x)$ is an element of a basis of the corresponding representation space.

Comparing (4.2) with (4.1) we arrive at the obvious correspondence

$$\Phi(x) \to F_n(u), \quad T(g) \to \phi'(u).$$

Let us give a group-theoretical interpretation of equation (4.1) using this correspondence. We have some discrete group of transformation the group element of which is acting exactly as substitution $u \to \phi(u)$. $\phi'(u)$ (a Frechet derivative) is a linear representation of a group element. At last, $F_n(u)$ (the equations of hierarchy) form a basis in a representation space. If this representation is irreducible (this fact should be checked by independent methods), then all possible bases of this representation (solutions of equation (4.1) with different n) must be connected by some operator $W_{n,n'}$

$$F_n(u) = W_{n,n'}F_{n'} \tag{4.3}$$

Certainly the same situation takes place in the theory of $(1+1)$ integrable systems. All equations of the same hierarchy are connected by the "raising" operators constructed from the skew symmetrical (nonlocal) Hamiltonian operators $J_n = -J_n^T$

$$W_{n,n'} = J_n J_{n'}^{-1} \tag{4.4}$$

Two equations (which are typical for a group representation theory) will be important for further considerations

$$\phi'(u)W(u)\phi'(u)^{-1} = W(\phi(u)), \quad \phi'(u)J(u)\phi'(u)^T = J(\phi(u)), \tag{4.5}$$

where $\phi'(u)^T = \phi_u^T - D\phi_{u'}^T + D^2\phi_{u''}^T - \cdots$, and $W(u), J(u)$ are unknown $s \times s$ matrix operators the matrix elements of which are polynomials of some finite order with respect to positive and negative degrees of the operator of differentiation D.

From (4.5) and (4.1) it follows immediately that if $F_n(u)$ is some solution of main equation (4.1), then $W^p(u)F_n(u)$ (p is an arbitrary natural number) will be some other solution of the same equation.

A solution of the second equation (4.5) under additional condition of its skew symmetry may be connected to a Poisson structure which is invariant with respect to a discrete symmetry transformation. Skew symmetric operators $J(u)$ are known as Hamiltonian ones. Two different solutions of the second equation from (4.5), say $J_1(u)$ and $J_2(u)$, in combination $J_1J_2^{-1}$ satisfy the first equation from (4.5). The operator $J_1J_2^{-1}J_1(u)$ is again the solution of the second equation from (4.5) and so on. This is the way how Hamiltonian operators arise in the theory of integrable systems. It is necessary to find two different Poisson structures by independent methods and after this fulfill the above described procedure. In this respect the equations (4.5) were used in [5].

5 Some additional consequences of the main equation

Let us differentiate the main equation (4.1) with respect to some parameter p considering it as one of arguments of the function u. The following formula for differentiation of an arbitrary s-th component vector functional $\Phi(u)$ takes place

$$\frac{\partial \Phi(u)}{\partial p} = \Phi'(u)\frac{\partial u}{\partial p},$$

where $\Phi'(u)$ is $s \times s$ integro-differential operator (the operator of variational derivative). In the case of a local functional it coincides with a Frechet derivative operator corresponding to $\Phi(u)$. Differentiating of the main equation with respect to some parameter p and applying the last formula we obtain

$$(F_{n_1}(\phi(u)))'\phi'(u)u_p = (\phi'(u))'u_p F_{n_1}(u) + \phi'(u)F'_{n_1}(u)u_p.$$

The above equality is the identity with respect to the function u_p. Let us substitute the equality into it $u_p = F_{n_2}(u)$, where the last function is some solution of our main equation different from $F_{n_1}(u)$. It is not difficult to understand that the first term in the right hand side of the last equation is symmetric with respect to interchanging n_1 to n_2. Composing the same equation with interchanged indexes and subtracting the last expression from the previous one we obtain

$$(F_{n_1}(\phi(u)))'F_{n_2}(\phi(u)) - (F_{n_2}(\phi(u)))'F_{n_1}(\phi(u)) =$$

$$\phi(u)[(F_{n_1}(u))'F_{n_2}(u) - (F_{n_2}(u))'F_{n_1}(u)]$$

Thus the combination $[(F_{n_1}(u))'F_{n_2}(u) - (F_{n_2}(u))'F_{n_1}(u)]$ satisfies our main equation. For all integrable systems known to us this combination is equal to zero. So we can suppose that this is some additional condition (apart from invariance with respect to a discrete transformation group) which chooses integrable systems from the set of all partial differential equations.

Let us consider the conclusions which follow from this condition.

An equation of integrable hierarchy has the form

$$u_t = F_n(u)$$

the corresponding symmetry equation is the following one

$$U_t = F'_n(u)U$$

and we see that each functions $F_s(u)$ satisfying the main equation is a solution of the symmetry equation if the above additional condition is satisfied.

The additional condition allows to introduce a self-consistent multi-time formalism in the sense that system of equation

$$u_{t_n} = F_n(u)$$

is consistent. Using this language we can say that the additional condition guarantee the equality of the second partial derivatives.

6 The general hypothesis

As a conclusion of the previous considerations it is possible to formulate the following general hypothesis about the structure of a future theory of integrable systems:

- the problem of classification and solution of integrable systems is equivalent to the theory of representations of the discrete group of integrable mappings.

Indeed, if from independent considerations it turns out to be possible to obtain a solution of our main equation (4.1), then we automatically produce an integrable evolution-type equation (4.2) and each space of irreducible representation of (4.3) will give us the exact solution of it. We are well aware of the fact that our main equation (4.1) in its present form is not very suitable for obtaining direct conclusions from it. In this connection, we can notice by analogy with the 'distance' between the original definition of semi-simple algebras (in the sense of an absence of nontrivial ideals) and the Cartan classification into A, B, C, D, E, F, G and E that there may be comparable 'distance' between the problem of classification of the solutions of our main equation as it is formulated here and its possible solution.

We hope that something alike the Cartan's classification will be achieved in the case of representation theory of discrete groups of integrable mappings.

7 Conclusion

The author doesn't insist on the mathematical regorouse of the present paper. The number of arising questions is much more then regorous mathematical output.

The main equation (4.1) will provide the answers to two most important questions of the theory of integrable systems. The first question is a 'quantization' of substitution, i.e., the choosing from the infinite number of invertible substitutions the ones which will be integrable in the above sense. Except for the obvious remark that this will depend essentially upon the dimensions of the spaces involved, the author knows almost nothing about how to solve this problem and thinks that it not going to be resolved quickly.

The second more tractable problem is solution of the main equation (4.1) for a given (ad hoc) integrable substitution $\phi(u)$ [7]. The author is convinced of that the solution to this problem is closely connected with the theory of representations of discrete groups of integrable mappings. From known examples of integrable systems it follows that discrete groups of integrable mapping possess rich storage of different irreducible representations. With each of these representations it may be connected a definite class of exact solutions of corresponding integrable system. In some sense the soliton-like solutions correspond to finite-dimensional representations of such groups.

Acknowledgments

The authors wishes to acknowledge the hospitality of the Max-Planck Institute.

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