# THE ENERGY OF HARMONIC MAPS OBTAINED BY THE TWISTOR CONSTRUCTION

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Harmonic maps may be viewed as the critical points of the energy functional. It is natural to ask, then, whether these critical points have some geometrical description, what critical values are possible, and whether there exist Morse-theoretic results concerning the cohomology of the critical sets. We present some preliminary calculations which answer these questions in a simple case: the maps will be the inclusions

$$G.X \longrightarrow \mathbb{C}P^n$$

of the orbits of points $X$ in complex projective space, under the linear action of some compact Lie group $G$.

In chapter I we describe the harmonic maps in terms of the weights of the representation of $G$ involved, using the fact that the energy functional here is essentially the "norm squared of the moment map". The Morse-theoretic results obtained by F. Kirwan (see [Ki]) thus carry over to the energy functional. In chapter II we discuss the harmonicity condition in terms of results in the literature (i.e. [BS,BW,ES,EW]) which give a "twistor construction" of harmonic maps from a Riemann surface to compact symmetric spaces. We give an example (following [EW]) where the energy of a harmonic map (i.e. the critical value) has topological significance. This is extended to the case of an

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arbitrary harmonic map of the type under consideration in chapter III.

It will be apparent to the reader that these results generalize to linear actions on other homogeneous spaces. In fact it is also possible to say something about maps which are not the inclusions of orbits of such an action; the "homogeneous case" discussed here can be regarded as a formalism in which the derivatives $\partial/\partial z$ , $\partial/\partial \bar{z}$ have been replaced by the actions of elements $e_\alpha$ , $e_{-\alpha}$ of the Lie algebra.

CHAPTER I

# HARMONIC ORBITS OF A PROJECTIVE REPRESENTATION

Let $G$ be a compact simple Lie group, and let $\theta : G \longrightarrow SU_{n+1}$ be a homomorphism. Thus, $G$ acts on $\mathbb{C}^{n+1}$, unitarily with respect to the standard Hermitian form $\langle\langle\ ,\ \rangle\rangle$, via the representation $\theta$. In this chapter we study properties of the orbits of the induced action of $G$ on $\mathbb{C}P^n$. If $[v] \in \mathbb{C}P^n$ is the line through $v \in \mathbb{C}^{n+1}$, the orbit $G.[v]$ is the image of the map

$$f_v : G \longrightarrow \mathbb{C}P^n \quad , \quad f_v(g) = [\theta(g)v] \; .$$

With respect to a standard Kähler metric $h$ on $\mathbb{C}P^n$ and the metric $\langle\ ,\ \rangle$ on $G$ given by minus the Killing form, the <u>energy</u> $E(f_v)$ of $f_v$ is easily calculated. Recall that

$$E(f_v) = (1/2)\int_G |df_v|^2$$

where $|df_v|$ is calculated using the given metrics. We have $|df_v|^2 = \operatorname{tr} f_v^*h$ ; this is a constant function on $G$. If $e_1,\ldots,e_m$ is an orthonormal basis of the Lie algebra $L(G) = T_eG$, then $|df_v|^2 = \sum_{i=1}^{m} \langle\langle p_v^\perp\theta(e_i)v, p_v^\perp\theta(e_i)v\rangle\rangle$, where $p_v^\perp : \mathbb{C}^{n+1} \longrightarrow [v]^\perp$ is orthogonal projection. Here $\theta$ denotes the Lie algebra homomorphism $\theta : L(G) \longrightarrow L(SU_{n+1})$ associated to the Lie group homomorphism $\theta : G \longrightarrow SU_{n+1}$, and $L(SU_{n+1})$ is identified with the subspace of traceless skew-Hermitian elements of $\operatorname{End}(\mathbb{C}^{n+1})$ as usual. This choice of metric on $\mathbb{C}P^n$ is a multiple of the metric $\langle\ ,\ \rangle$ given by minus the Killing form (of $SU_{n+1}$). Thus

$$E(f_v) = K \sum_{i=1}^{m} \sum_{j=1}^{n} |\langle\langle \theta(e_i)v, v_j \rangle\rangle|^2$$

where $v_1, \dots, v_n$ is an orthonormal basis of $[v]^\perp$, and $K$ ( $= \mathrm{vol}(G)/2$ ) is a positive constant.

The critical points of the real valued function

$$E : \mathbb{C}P^n \longrightarrow \mathbb{R} \quad , \quad E([v]) = E(f_v)$$

may be found in the following elementary manner. We use the fact that the unitary group $U_{n+1}$ may be identified with the set of all ordered orthonormal bases $u_0, \dots, u_n$ of $\mathbb{C}^{n+1}$ and that there is then a submersion

$$\pi : U_{n+1} \longrightarrow \mathbb{C}P^n \quad , \quad \pi(u_0, \dots, u_n) = [u_0] \ .$$

The critical points $[v]$ of $E$, with $|v| = 1$, are given by those $v$ for which every ordered orthonormal basis of the form $v, v_1, \dots, v_n$ (of $\mathbb{C}^{n+1}$ ) is critical for $\pi \cdot E$. Now, the critical points of $\pi \cdot E$ may be found by the method of Lagrange multipliers: we want the extrema of

$$\mathbb{C}^{n+1} \times \dots \times \mathbb{C}^{n+1} \longrightarrow \mathbb{R} \quad , \quad (u_0, \dots, u_n) \longmapsto \sum_{i=1}^{m} \sum_{j=1}^{n} |\langle\langle \theta(e_i)u_0, u_j \rangle\rangle|^2$$

subject to the constraints

$$\langle\langle u_i, u_j \rangle\rangle = \delta_{ij} \ .$$

The condition turns out to be that the transformation

$$\sum_{i=1}^{m} \theta(e_i)p_v^\perp \theta(e_i)p_v + \theta(e_i)p_v \theta(e_i)p_v^\perp$$

is Hermitian, where $p_v$, $p_v^\perp$ are respectively the orthogonal projections to $[v]$, $[v]^\perp$. Since $\theta(e_i)^* = -\theta(e_i)$ and $p_v$, $p_v^\perp$ are Hermitian,

$$\sum_{i=1}^{m} p_v \theta(e_i)p_v^\perp \theta(e_i)p_v + p_v^\perp \theta(e_i)p_v \theta(e_i)p_v^\perp$$

is Hermitian. Hence, on subtracting these two transformations, we find the condition for an extremum is that

$$\sum_{i=1}^{m} p_V^{\perp}\theta(e_i)p_V^{\perp}\theta(e_i)p_V + p_V\theta(e_i)p_V\theta(e_i)p_V^{\perp}$$

be Hermitian. This is so if and only if

$$(*) \qquad \sum_{i=1}^{m} p_V^{\perp}\theta(e_i)(p_V^{\perp} - p_V)\theta(e_i)p_V = 0 .$$

This is precisely the condition for $f_V$ to be a harmonic map (i.e. a local extremum for the energy, when variations through <u>arbitrary</u> smooth maps are allowed). To explain this, we recall (lemma 2.1 of [Gu1]) that if $\theta : G \to G'$ is a homomorphism of compact Lie groups, and if $\theta(H) \subseteq H'$ for certain subgroups $H \subseteq G$, $H' \subseteq G'$, then the second fundamental form of the "homogeneous" map

$$f_\theta : G/H \longrightarrow G'/H'$$

(with respect to the metrics $\langle\ ,\ \rangle$) is given by:

$$\nabla(df_\theta)(x,y) = ([\theta(x)_{L(H')},\theta(y)_{L(G'/H')}] - [\theta(x)_{L(G'/H')},\theta(y)_{L(H')}])/2 .$$

If $e_1,\ldots,e_m$ is an orthonormal basis for $L(G/H) = L(H)^{\perp} \cap L(G)$, the condition for harmonicity of $f_\theta$ (namely $\operatorname{tr} \nabla(df_\theta) = 0$) becomes:

$$\sum_{i=1}^{m} [\theta(e_i)_{L(H')},\theta(e_i)_{L(G'/H')}] = 0 .$$

When $H = \{e\}$ and $G'/H' = SU_{n+1}/S(U_1 \times U_n) = \mathbb{C}P^n$, this reduces to the condition (*) above (see theorem 3.4 of [Gu1]). The discussion so far may be summarized as follows.

THEOREM 1.1. *For the map* $f_v : G \to \mathbb{C}P^n$ *we have*:

(1) $E(f_v) = -K \langle\langle (\sum_{i=1}^{m} \theta(e_i) p_v^{\perp} \theta(e_i)) v, v \rangle\rangle$

(2) $f_v$ *is harmonic*
*iff* $[v]$ *is a critical point of* $E : \mathbb{C}P^n \to \mathbb{R}$
*iff* $v$ *is an eigenvector of the transformation* $\sum_{i=1}^{m} \theta(e_i)(p_v^{\perp} - p_v)\theta(e_i)$ . □

While orbits of a linear action are evidently rather special, the next observation shows that, even in this case, the harmonicity condition is interesting.

PROPOSITION 1.2. *If* $\mu : \mathbb{C}P^n \to L(G)^*$ *is the moment map for the action of* $G$ *on* $\mathbb{C}P^n$ *via* $\theta$ , *then*

$$(1/K)E(f_v) + 4\pi^2 |\mu([v])|^2 = -\langle\langle (\sum_{i=1}^{m} \theta(e_i)^2 v, v \rangle\rangle .$$

*Proof*. By the moment map we mean the composition $\mu = \theta^* \circ (i/2\pi)p$ , where

$$(i/2\pi)p : \mathbb{C}P^n \longrightarrow L(U_{n+1})^* \quad , \quad [v] \longmapsto (i/2\pi)p_v$$

is the inclusion of $\mathbb{C}P^n$ in $L(U_{n+1})^*$ as a coadjoint orbit, and $\theta^* : L(U_{n+1})^* \to L(G)^*$ is the dual of $\theta$ . Strictly speaking, $(i/2\pi)p$ is a map from $\mathbb{C}P^n$ to $L(U_{n+1})$ , because of our identification of $L(U_{n+1})$ with the skew-Hermitian transformations of $\mathbb{C}^{n+1}$ . However, we shall identify $L(U_{n+1})$ and $L(U_{n+1})^*$ in the usual way, via the invariant metric $(X,Y) \mapsto -\operatorname{tr} XY$ on $L(U_{n+1})$ .

Observe first that $\mu([v])(x) = -\operatorname{tr}(i/2\pi)p_v\theta(x) = -(i/2\pi)\langle\langle\theta(x)v,v\rangle\rangle$ for any $x \in L(G)$ . Hence $|\mu([v])|^2 = \sum_{j=1}^{m} \mu([v])(e_j)^2 = -(1/4\pi^2) \sum_{j=1}^{m} \langle\langle\theta(e_j)v,v\rangle\rangle^2 = (1/4\pi^2) \sum_{j=1}^{m} |p_v\theta(e_j)v|^2$ . But we have seen earlier that $E(f_v) = K \sum_{j=1}^{m} |p_v^{\perp}\theta(e_j)v|^2$ . Hence the result. □

COROLLARY 1.3. If $\theta$ is irreducible, $f_v$ is harmonic if and only if $[v]$ is a critical point of $|\mu|^2$.

Proof. The transformation $\sum_{i=1}^{m} \theta(e_i)^2$ is the Casimir operator for $\theta$ (with respect to $< , >$). If $\theta$ is irreducible, it is well known that this is a scalar operator (see [Hu], 6.2). Hence $|\mu|^2$ and $E$ have the same critical points. □

The critical point theory of $|\mu|^2$ is discussed in detail in [Ki], in the more general context of symplectic group actions on manifolds. It is shown that $|\mu|^2$ is an "equivariantly perfect Morse-Bott function" in the sense that the Morse-Bott inequalities hold (relating the indices of the critical points to the Betti numbers of the manifold), providing G-equivariant cohomology is used. Hence the same is true of $E$, when $\theta$ is irreducible (by the proof of 1.3).

In order to make this more explicit, we shall give the description of the critical sets in our case (i.e. for $\mathbb{C}P^n$), following [Ki]. From the formula

$$|\mu([v])|^2 = -(1/4\pi^2) \sum_{i=1}^{m} <<\theta(e_i)v,v>>^2 ,$$

a Lagrange multiplier argument shows that $[v]$ is critical for $|\mu|^2$ if and only if $v$ is an eigenvector of the transformation $\sum_{i=1}^{m} \theta(e_i)p_v\theta(e_i)$. It is useful to consider also the moment map $\mu_T : \mathbb{C}P^n \longrightarrow L(T)^*$ for the action of a maximal torus $T$ (of $G$) on $\mathbb{C}P^n$ (via $\theta$). If $e_1,\ldots,e_h$ are chosen to form an orthonormal basis for $L(T)$, then one has

$$|\mu_T([v])|^2 = -(1/4\pi^2) \sum_{i=1}^{h} <<\theta(e_i)v,v>>^2 ,$$

and $[v]$ is critical for $|\mu_T|^2$ if and only if $v$ is an eigenvector of the transformation $\sum_{i=1}^{h} \theta(e_i)p_v\theta(e_i)$. The formula for $|\mu_T|^2$ may easily be written in terms of homogeneous coordinates $z_0,\ldots,z_n$ on $\mathbb{C}P^n$: if we write $v = \sum_{j=0}^{n} z_j v_j$ where $v_0,\ldots,v_n$ is an orthonormal basis of $\mathbb{C}^{n+1}$ consisting of weight vectors of $\theta$, with corresponding real weights

$\lambda_0,\ldots,\lambda_n \in L(T)^*$ , then $\langle\langle\theta(e_k)v,v\rangle\rangle = 2\pi i\langle\langle \sum_{j=0}^{n} \lambda_j(e_k)z_j v_j , \sum_{j=0}^{n} z_j v_j\rangle\rangle = 2\pi i \sum_{j=0}^{n} \lambda_j(e_k)|z_j|^2$ . Hence

$$\begin{aligned}
\mu_T([v]) &= \sum_{k=1}^{h} \mu_T([v])(e_k)e_k^* \\
&= -(i/2\pi) \sum_{k=1}^{h} \langle\langle\theta(e_k)v,v\rangle\rangle e_k^* \\
&= \sum_{k=1}^{h} \sum_{j=0}^{n} |z_j|^2\lambda_j(e_k)e_k^* \\
&= \sum_{j=0}^{n} |z_j|^2\lambda_j .
\end{aligned}$$

LEMMA 1.4. *For* $\beta \in L(T)^*$ , *let* $Z_\beta = \{[v] : z_i = 0$ *if* $\langle\lambda_i-\beta,\beta\rangle \neq 0 \}$ . *Then* $[v]$ *is critical for* $|\mu_T|^2$ *if and only if* $[v] \in Z_\beta \cap \mu_T^{-1}(\beta)$ *for some* $\beta$ . *The set* $Z_\beta \cap \mu_T^{-1}(\beta)$ *is non-empty if and only if* $\beta$ *is the closest point to* $0$ *of the convex hull of some non-empty subset of the weights* $\lambda_0,\ldots,\lambda_n$ .

*Proof*. We have seen that $[v]$ is critical for $|\mu_T|^2$ if and only if $\sum_{i=1}^{h} \theta(e_i)p_v\theta(e_i)v = Cv$ for some constant $C$ . Taking the inner product of both sides with $v$ gives $C = \sum_{i=1}^{h} \langle\langle\theta(e_i)p_v\theta(e_i)v,v\rangle\rangle$ . Since $p_v x = \langle\langle x,v\rangle\rangle v$ , the condition is

$$\sum_{i=1}^{h} \langle\langle\theta(e_i)v,v\rangle\rangle\theta(e_i)v = -\sum_{i=1}^{h} |\langle\langle\theta(e_i)v,v\rangle\rangle|^2 v$$

i.e. for each $r = 0,\ldots,n$

$$z_r \sum_{i=1}^{h} (\lambda_r(e_i) \sum_{j=0}^{n} \lambda_j(e_i)|z_j|^2) = z_r \sum_{i=1}^{h} ( \sum_{j=0}^{n} \lambda_j(e_i)|z_j|^2 )^2$$

This reduces to

$$z_r\langle\lambda_r, \sum_{j=0}^{n} \lambda_j|z_j|^2 \rangle = z_r\langle \sum_{j=0}^{n} \lambda_j|z_j|^2 , \sum_{j=0}^{n} \lambda_j|z_j|^2 \rangle$$

i.e. for each $r = 0,\ldots,n$ , *either* $z_r = 0$ *or* $\lambda_r - \mu_T([v]) \perp \mu_T([v])$ .

Hence if $[v]$ is critical for $|\mu_T|^2$ , we see that $\mu_T([v])$ is necessarily the closest point to $0$ of the convex hull of those weights $\lambda_r$

for which $\lambda_r - \mu_T([v]) \perp \mu_T([v])$ . So $[v] \in Z_\beta \cap \mu_T^{-1}(\beta)$ for $\beta = \mu_T([v])$ . Conversely, for any $\beta \in L(T)^*$ , assume that there exists $[v] \in Z_\beta \cap \mu_T^{-1}(\beta)$ . Then $[v]$ is a critical point of $|\mu_T|^2$ and $\beta$ is of the required type. □

THEOREM 1.5. The critical points of $|\mu|^2$ are $\bigcup G.(Z_\beta \cap \mu^{-1}(\beta))$ , where $\beta$ varies over closest points to 0 of convex hulls of subsets of the weights of $\theta$ .

Proof. An essential property of $\mu : \mathbb{C}P^n \longrightarrow L(G)^*$ is that it is G-equivariant, with respect to the given action on $\mathbb{C}P^n$ and the co-adjoint action on $L(G)^*$ . This may be verified directly from the definition of $\mu$ given earlier, or from the formula

$$\mu([v]) = \sum_{k=1}^{m} \mu([v])(e_k)e_k^* = -(i/2\pi) \sum_{k=1}^{m} \langle\langle \theta(e_k)v, v\rangle\rangle e_k^* .$$

From now on we shall identify $L(G)$ and $L(G)^*$ by means of $\langle \, , \, \rangle$ , so that $\mu([v]) = -(1/2\pi) \sum_{k=1}^{m} \langle\langle \theta(e_k)v, v\rangle\rangle e_k \in L(G)$ .

If $[v]$ is critical for $|\mu|^2$ , so is $g.[v] = [\theta(g)v]$ for any $g \in G$ , by G-equivariance of $|\mu|^2$ . Since $\mu(g.[v]) = Ad(g)\mu([v])$ we may choose $g$ such that $Ad(g)\mu([v]) \in L(T)$ . From the criteria of the paragraph preceeding lemma 1.4, if $\mu([w]) \in L(T)$ (i.e. $\mu_T([w]) = \mu([w])$), then $[w]$ is critical for $|\mu_T|^2$ if and only if it is critical for $|\mu|^2$ . Putting $[w] = g.[v]$ and applying lemma 1.4 gives $[v] \in G.(Z_\beta \cap \mu^{-1}(\beta))$ for some $\beta$ . This argument in reverse shows that every point of $G.(Z_\beta \cap \mu^{-1}(\beta))$ is critical for $|\mu|^2$ . □

Examples.

1. If $\beta$ is a non-zero weight of $\theta$ , then $Z_\beta$ is the (projectivized) weight space of $\beta$ . From the formula for $\mu$ above one sees that $Z_\beta \subseteq \mu^{-1}(\beta)$ . Hence the corresponding set of critical points for $|\mu|^2$ is just $G.Z_\beta$ , i.e. the union of the projective weight orbits with weight $\beta$ .

If $\beta = 0$ , then $Z_\beta = \mathbb{C}P^n$ . Hence the corresponding set of critical points for $|\mu|^2$ is $\mu^{-1}(0)$ , i.e. the set of minima. The quotient space $\mu^{-1}(0)/G$ is the symplectic quotient or Marsden-Weinstein reduction (of $\mathbb{C}P^n$ ) ; see [Ki].

2. Let $G = SU_2$ , and let $\theta = S^n\sigma$ be the representation of $SU_2$ on the space $P_n$ ( $\cong \mathbb{C}^{n+1}$ ) of homogeneous polynomials in two variables of degree $n$ . This is irreducible, and the weights (via an identification $L(T)^* \cong \mathbb{R}$ ) are $n - 2i$ , $i = 0,\ldots,n$ ; each weight space has dimension 1 . The non-zero weights give rise to non-minimal critical submanifolds for $|\mu|^2$ (or $E$ ) , namely the projective orbits of the corresponding weight vectors. For such a weight vector $v$ , the projective orbit is isomorphic to $\mathbb{C}P^1$ , being the image of the harmonic map $f_v : SU_2 \to \mathbb{C}P^n$ . One also has the critical set $\mu^{-1}(0)$ . If 0 is a weight, its weight vector gives a distinguished point of $\mu^{-1}(0)$ , whose orbit is isomorphic to $\mathbb{R}P^2$ . The remaining orbits in $\mu^{-1}(0)$ are 3 dimensional.

3. If $H$ is any subgroup of $G$ which leaves $[v] \in \mathbb{C}P^n$ fixed, the map $f_v : G \to \mathbb{C}P^n$ factors through $G/H$ . If in theorem 1.1 we replace $f_v : G \to \mathbb{C}P^n$ by the induced map $f_v : G/H \to \mathbb{C}P^n$ , the proof goes through in exactly the same way, providing $e_1,\ldots,e_m$ is now taken to be an orthonormal basis of $L(G/H) = L(H)^\perp \cap L(G)$ . (Note that if $x \in L(H)$ , $p_v^\perp\theta(x)v = 0$ .) The metric on $G/H$ is that obtained from $<\ ,\ >$ on $G$ , and $K = \mathrm{vol}(G/H)/2$ .

For the remainder of this article we shall concentrate on the situation in example 3, with $H$ a maximal torus $T$ . Thus $v$ is a weight vector of $\theta$ (with respect to $T$ ) and we have

$$f_v : G/T \longrightarrow \mathbb{C}P^n \quad .$$

The fact that $G/T$ admits complex structures makes this case of special interest. For background information we refer to [BH,Gu1]; in particular we recall that a homogeneous complex structure on $G/T$ corresponds to a choice $\Delta^+$ of positive roots of $G$ (with respect to $T$ ). Given $\Delta^+$ , the decomposition $(TG/T)\otimes\mathbb{C} = (TG/T)_{1,0} \oplus (TG/T)_{0,1}$ may be written

$$L(G/T)\otimes\mathbb{C} = \sum_{\alpha\in\Delta^+} \mathbb{C}e_\alpha \oplus \sum_{\alpha\in\Delta^+} \mathbb{C}e_{-\alpha}$$

in terms of which an orthonormal basis of $L(G/T) \subsetneq L(G/T)\otimes\mathbb{C}$ consists of the vectors $(1/\sqrt{2})(e_\alpha - e_{-\alpha})$, $(i/\sqrt{2})(e_\alpha + e_{-\alpha})$, $\alpha \in \Delta^+$, and $\langle e_\alpha, e_{-\alpha}\rangle = -1$ for all $\alpha$. Thus, if $J \in \mathrm{End}(L(G/T))$ is the corresponding operator, $Je_\alpha = ie_\alpha$ and $Je_{-\alpha} = -ie_{-\alpha}$ for all $\alpha \in \Delta^+$.

PROPOSITION 1.6. Consider $f_v : G/T \to \mathbb{C}P^n$, where $v$ is a weight vector of $\theta$ of unit length.

(1) With respect to the complex structure on $G/T$ given by $\Delta^+$ and the complex structure on $\mathbb{C}P^n$ defined via the identification $(T_{[v]}\mathbb{C}P^n)_{1,0} = \mathrm{Hom}([v]^\perp,[v])$, $f_v$ is holomorphic if and only if $\theta(e_\alpha)v = 0$ for all $\alpha \in \Delta^+$.

(2) $f_v$ is harmonic if and only if $v$ is an eigenvector of $\sum_{\alpha\in\Delta} \theta(e_\alpha)\theta(e_{-\alpha})$.

(3) $E(f_v) = E'(f_v) + E''(f_v)$ where

$$E'(f_v) = K \sum_{\alpha\in\Delta^+} \langle\langle \theta(e_\alpha)\theta(e_{-\alpha})v,v\rangle\rangle$$

$$E''(f_v) = K \sum_{\alpha\in\Delta^+} \langle\langle \theta(e_{-\alpha})\theta(e_\alpha)v,v\rangle\rangle .$$

(4) If $\omega$ is the Kähler form of $\mathbb{C}P^n$ and $\kappa$ is the 2-form on $G/T$ defined by $\kappa(x,y) = \langle x,Jy\rangle$, then

$$E'(f_v) - E''(f_v) = \langle f_v^*\omega,\kappa\rangle = 4\pi^2K\langle\lambda, \sum_{\alpha\in\Delta^+}\alpha\rangle$$

where $\lambda$ is the weight of $v$.

Proof. (1) follows from the definitions of the complex structures. For (2) and (3), use the identity

$$\sum_{i=1}^{m} \theta(e_i)p_v\theta(e_i) = -\sum_{\alpha\in\Delta} \theta(e_\alpha)p_v\theta(e_{-\alpha})$$

where $e_1,\ldots,e_m$ is the basis of $L(G/T)$ described above. Since $v$ is a weight vector, $p_v\theta(e_\alpha)v = 0$ for all $\alpha \in \Delta$. (4) is proved by calculating the first two quantities. First, we recall that for $\alpha \in \Delta^+$, $[e_\alpha, e_{-\alpha}] = h_\alpha \in L(T) \otimes \mathbb{C}$, where (with our conventions) $\gamma(h_\alpha) = -2\pi i\langle\gamma,\alpha\rangle$ for all $\gamma \in L(T)^*$. Thus

$$\begin{aligned} E'(f_v) - E''(f_v) &= K \sum_{\alpha\in\Delta^+} \langle\langle\theta([e_\alpha, e_{-\alpha}])v, v\rangle\rangle \\ &= K \sum_{\alpha\in\Delta^+} \langle\langle 2\pi i\lambda(h_\alpha)v, v\rangle\rangle \\ &= 4\pi^2 K \sum_{\alpha\in\Delta^+} \langle\lambda,\alpha\rangle \quad . \end{aligned}$$

Next we must calculate $\omega$ and $\kappa$. If $a,b \in L(\mathbb{C}P^n) \subseteq L(SU_{n+1})$,

$$\begin{aligned} \omega(a,b) &= \operatorname{Im} \langle\langle p_v av, p_v bv\rangle\rangle \\ &= -(i/2)\langle\langle [a,b]v, v\rangle\rangle \quad . \end{aligned}$$

Thus, if $x,y \in L(G/T) \subseteq L(G)$,

$$f_v^*\omega(x,y) = -(i/2)\langle\langle\theta([x,y])v, v\rangle\rangle$$

and so $f_v^*\omega(e_\alpha, e_{-\alpha}) = -2i\pi^2\langle\lambda,\alpha\rangle$. By definition of $\kappa$, $\kappa(e_\alpha, e_{-\alpha}) = \langle e_\alpha, Je_{-\alpha}\rangle = -i\langle e_\alpha, e_{-\alpha}\rangle = i$. Since $f_v^*\omega$ and $\kappa$ respect the decomposition

$$L(G/T) \otimes \mathbb{C} = \sum_{\alpha\in\Delta^+} (\mathbb{C}e_\alpha \oplus \mathbb{C}e_{-\alpha})$$

we can take their inner product on each subspace and add the results. Hence

$$\begin{aligned} \langle f_v^*\omega, \kappa\rangle &= \operatorname{vol}(G/T) \sum_{\alpha\in\Delta^+} -2i\pi^2\langle\lambda,\alpha\rangle i \\ &= 4\pi^2 K \sum_{\alpha\in\Delta^+} \langle\lambda,\alpha\rangle \end{aligned}$$

as required. □

# CHAPTER II

# TWISTOR CONSTRUCTIONS OF HARMONIC MAPS

By a "twistor construction" of a map $f : X \to Y$ of complex manifolds we mean a factorization $f = \pi \circ g$ through an (almost) complex manifold $Z$, where $g : X \to Z$ is holomorphic and $\pi : Z \to Y$ is a fibre bundle. Such a construction was used in [EW] to produce all harmonic maps $\mathbb{C}P^1 \to \mathbb{C}P^n$. A general treatment first appeared in [ES], where $\pi : Z \to Y$ was a twistor fibration over $Y$, and this explains our terminology. We shall consider compositions of the form

$$G/T \xrightarrow[f_\theta]{} F \xrightarrow[\pi_i]{} \mathbb{C}P^n$$

where $F = SU_{n+1}/S(U_1 \times \ldots \times U_1)$ is the full flag manifold of $\mathbb{C}^{n+1}$ (i.e. $G/T$ for $G = SU_{n+1}$) and $f_\theta$ is the map induced by a representation $\theta : G \to SU_{n+1}$, and where $\pi_i$ is the i-th natural projection which associates to a flag $\{0\} = E_0 \subsetneq E_1 \subsetneq \ldots \subsetneq E_{n+1} = \mathbb{C}^{n+1}$ the line $E_{i-1}^{\perp} \cap E_i$. Let us now fix a flag $\{0\} = E_0 \subsetneq E_1 \subsetneq \ldots \subsetneq E_{n+1} = \mathbb{C}^{n+1}$ (denoted $\sigma \in F$) and write $E_{i-1}^{\perp} \cap E_i = [v_i] \in \mathbb{C}P^n$ with $[v_i] = 1$. Then

$$f_{v_i} = \pi_i \circ f_\theta$$

i.e. we are considering maps of the type discussed in chapter I. Our aim is to give results (based on [ES,EW]) which relate harmonicity of $f_{v_i}$ (a second order condition) to holomorphicity of $f_\theta$ (a first order condition); this reflects the relation between $E$ and $|\mu|^2$ described in chapter I, since it is known from [Ki] that arbitrary critical points of $|\mu|^2$ for a symplectic action are related to minima of $|\mu|^2$ for subsidiary symplectic actions. For simplicity we deal only with maps $f_{v_i}$ which are induced by homomorphisms, but (as we shall point out later on)[1] a feature of the approach is that the results extend to more general ("non-homogeneous") maps.

On $G/T$ we fix a complex structure given by a choice of positive roots $\Delta^+$, and on $\mathbb{C}P^n$ we take the standard complex structure compatible with the identification

$$(T_{[v_i]}\mathbb{C}P^n)_{1,0} = \mathrm{Hom}([v_i]^\perp,[v_i])$$

as in chapter I. We have at our disposal various (homogeneous) almost complex structures $J$ on $F$, which may be described via the identification

$$(T_\sigma F)\otimes\mathbb{C} = \sum_{0\le i\ne j\le n} \mathrm{Hom}([v_i],[v_j]) .$$

Following [BS] we define a relation " $\to$ " by writing $i\to j$ if and only if $(T_\sigma F)_{1,0}$ contains $\mathrm{Hom}([v_i],[v_j])$. Homogeneous almost complex structures $J$ correspond to choices of $\to$. It is known that $J$ is integrable if and only if $\to$ is transitive, hence homogeneous complex structures correspond to (total) orderings of the integers $0,\dots,n$. On $G/T$ we take the Riemannian metric $<\,,\,>$, and on $\mathbb{C}P^n$ a multiple of $<\,,\,>$ as in chapter I. Thus $\mathbb{C}P^n$ (unlike $G/T$ !) has a Kähler structure. For further details of almost complex structures and metrics on homogeneous spaces see [BH,Gu1]. A combinatorial discussion of the relation $\to$ is given in [BS].

With respect to an almost complex structure $J$, the map $f_\theta$ is holomorphic (i.e. $df_\theta(TG/T)_{1,0}\subseteq(TF)_{1,0}$) if and only if $p_i\theta(e_\alpha)p_j = 0$ for all $\alpha\in\Delta^+$, and all $i,j$ with $i\to j$, where $p_i:\mathbb{C}^{n+1}\longrightarrow[v_i]$ is orthogonal projection. To simplify notation we shall write

$$P^\alpha_{ij} = p_i\theta(e_\alpha)p_j \quad \text{for } \alpha\in\Delta^+ .$$

and

$$Q^\alpha_{ij} = p_i\theta(e_{-\alpha})p_j \quad \text{for } \alpha\in\Delta^+ .$$

Observe that $(P^{\alpha}_{ij})^* = Q^{\alpha}_{ji}$ . In order to understand the role played by $F$ , we restate theorem 1.1 using this terminology.

<u>THEOREM</u> 2.1. <u>For the map</u> $f_{v_i} : G/T \to \mathbb{C}P^n$ <u>we have</u>:

(1) $E'(f_{v_i}) = K \sum_{\alpha\in\Delta^+} \sum_{j\neq i} |P_{ji}|^2$ , $E''(f_{v_i}) = K \sum_{\alpha\in\Delta^+} \sum_{j\neq i} |Q^{\alpha}_{ji}|^2$

(2) $f_{v_i}$ <u>is harmonic</u>

<u>iff</u> $\sum_{\alpha\in\Delta} \sum_{t\neq i} (P^{\alpha}_{st}Q^{\alpha}_{ti} - P^{\alpha}_{si}Q^{\alpha}_{ii}) = 0$ for all $s \neq i$ . □

<u>COROLLARY</u> 2.2. <u>Assume</u> $f_\theta : G/T \to F$ <u>is holomorphic, with respect to</u> $J$ <u>on</u> $G/T$ <u>and the almost complex structure on</u> $F$ <u>given by</u> $\to$ . <u>The map</u> $f_{v_i} = \pi_i \circ f_\theta$ <u>is harmonic if and only if</u>

$$\sum_{\alpha\in\Delta^+} \sum_{t\to i} P^{\alpha}_{st}Q^{\alpha}_{ti} = 0 \quad \underline{\text{for all}} \quad s \quad \underline{\text{with}} \quad i \to s$$

<u>and</u>

$$\sum_{\alpha\in\Delta^+} \sum_{i\to t} Q^{\alpha}_{st}P^{\alpha}_{ti} = 0 \quad \underline{\text{for all}} \quad s \quad \underline{\text{with}} \quad s \to i .$$

<u>Proof</u>. Let

$$P_s = \sum_{\alpha\in\Delta^+} \sum_{t\neq i} (P^{\alpha}_{st}Q^{\alpha}_{ti} - Q^{\alpha}_{si}P^{\alpha}_{ii}) , \quad Q_s = \sum_{\alpha\in\Delta^+} \sum_{t\neq i} (Q^{\alpha}_{st}P^{\alpha}_{ti} - P^{\alpha}_{si}Q^{\alpha}_{ii})$$

so that $f_{v_i}$ is harmonic if and only if $P_s + Q_s = 0$ for $s \neq i$ . Now,

$$P_s - Q_s = p_s\Big( \sum_{\alpha\in\Delta^+} (\theta(e_\alpha)\theta(e_{-\alpha}) - \theta(e_{-\alpha})\theta(e_\alpha)) \Big)p_i$$

$$= p_s\Big( \sum_{\alpha\in\Delta^+} \theta([e_\alpha, e_{-\alpha}]) \Big)p_i$$

and this is zero for $s \neq i$ , since $[e_\alpha, e_{-\alpha}] \in L(T)\otimes\mathbb{C}$ and $v_i$ is a weight vector. Therefore, a necessary and sufficient condition for harmonicity of $f_{v_i}$ is that for each $s$ ( $\neq i$ ) , either $P_s = 0$ or $Q_s = 0$ . Since $f_\theta$ is holomorphic, $P^{\alpha}_{st} = 0$ (i.e. $Q^{\alpha}_{ts} = 0$ ) for each $\alpha \in \Delta^+$

and for all $s$, $t$ with $s \to t$. Combining these conditions proves the corollary. □

From this, the following sufficient condition for harmonicity is immediate.

COROLLARY 2.3. Let $f_\theta : G/T \to F$ be holomorphic, with respect to $J$ on $G/T$ and the almost complex structure on $F$ given by $\to$. Assume in addition that $P^\alpha_{st} = 0$ and $Q^\alpha_{st} = 0$ for all $\alpha \in \Delta^+$ and all $s$, $t$ with $s \to i \to t$ or $t \to i \to s$. Then $f_{v_i} = \pi_i \circ f_\theta$ is harmonic. □

With respect to the map $f_{v_i}$ and the almost complex structure given by $\to$ we define associated maps

$$f_i^- : G/T \longrightarrow Gr_a(\mathbb{C}^{n+1})$$

$$f_i^+ : G/T \longrightarrow Gr_b(\mathbb{C}^{n+1})$$

where $a + b = n$, by taking the orbits of the subspaces $A = \bigoplus_{j \to i} [v_j]$, $B = \bigoplus_{i \to j} [v_j]$ $(\dim A = a,\ \dim B = b)$. In other words, $f_i^-(gT) = \bigoplus_{j \to i} [\theta(g)v_j]$, and similarly for $f_i^+$. The Grassmannian $Gr_k(\mathbb{C}^{n+1})$ has a standard homogeneous metric (analogous to the one on $\mathbb{C}P^n$ used in chapter I) which is defined on $T_X Gr_k(\mathbb{C}^{n+1}) \cong Hom(X, X^\perp)$ by $(S,T) \longmapsto tr\, S^*T$. Using this metric on $Gr_a(\mathbb{C}^{n+1})$, $Gr_b(\mathbb{C}^{n+1})$ we have:

PROPOSITION 2.4. Assume that $P^\alpha_{st} = 0$ and $Q^\alpha_{st} = 0$ for all $\alpha \in \Delta^+$ and all $s$, $t$ with $s \to i \to t$ or $t \to i \to s$. Then

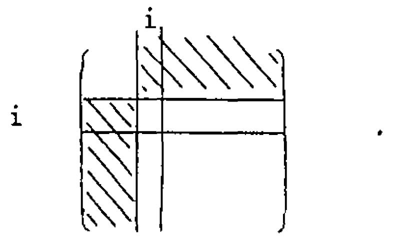
$$E(f_{v_i}) = E(f_i^-) + E(f_i^+) .$$

Moreover, if $f_\theta : G/T \to F$ is holomorphic with respect to $J$ and $\to$ (i.e. we are in the situation of 2.3), then $f_i^+$ and $f_i^-$ are respectively holomorphic and anti-holomorphic with respect to the homogeneous complex structures specified on the Grassmannians by $(T_A Gr_a(\mathbb{C}^{n+1}))_{1,0} = Hom(A^\perp, A)$, $(T_B Gr_b(\mathbb{C}^{n+1}))_{1,0} = Hom(B^\perp, B)$.

Proof. Using the definition of the metric on the Grassmannians, one obtains the formula

$$E(f_i^-) = K \sum_{\alpha\in\Delta^+} \sum |P^\alpha_{jk}|^2$$

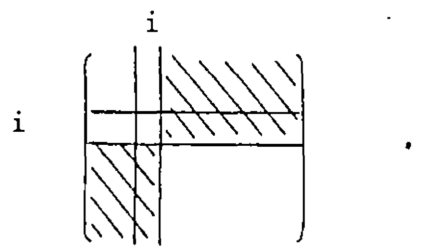
where the second sum is over $(j,k)$ with $j \to i$, $k \not\to i$ or $j \not\to i$, $k \to i$ (c.f. (1) of theorem 2.1). These values of $(j,k)$ are represented by the shaded area of the matrix below:



Similarly

$$E(f_i^+) = K \sum_{\alpha\in\Delta^+} \sum |P^\alpha_{jk}|^2$$

where the second sum is over $(j,k)$ with $i \to j$, $i \not\to k$ or $i \not\to j$, $i \to k$ :



Since by hypothesis all terms outside the i-th row and i-th column are zero, we obtain

$$E(f_i^-) + E(f_i^+) = K \sum_{\alpha\in\Delta^+} \sum_{j\neq i} (|P^\alpha_{ij}|^2 + |P^\alpha_{ji}|^2) = E(f_{v_i})$$

(using 2.1).

The map $f_i^+$ is holomorphic if and only if $P^\alpha_{st} = 0$ for all $\alpha \in \Delta^+$

and all $s, t$ with $i \not\to s$, $i \to t$. Now, $i \not\to s$ if and only if $i = s$ or $s \to i$. If $s \to i$, $P^{\alpha}_{st} = 0$ by the first hypothesis. If $i = s$, $P^{\alpha}_{it} = 0$ since $f_{\theta}$ is holomorphic. Similarly, $f_i^-$ is anti-holomorphic. □

The condition in corollary 2.3 is essentially the horizontality condition of §3 of [EW]: the induced map $G/T \to F(i-1,i,n+1)$ into the space of flags of the form $\{0\} = E_0 \subsetneq E_{i-1} \subsetneq E_i \leq E_{n+1} = \mathbb{C}^{n+1}$ is horizontal with respect to the projection $F(i-1,i,n+1) \longrightarrow \mathbb{C}P^n$. (Proposition 2.4 corresponds then to (ii) of proposition 7.1 of [EW].) It is a strong condition if rank $G > 1$, implying in particular that, for each $\alpha \in \Delta^+$, the map

$$\mathbb{C}P^1 \xrightarrow[\alpha]{} G/T \xrightarrow[f_{v_i}]{} \mathbb{C}P^n$$

is harmonic, where $\alpha : \mathbb{C}P^1 \to G/T$ is induced by the inclusion $L(SU_2) \cong [e_\alpha, e_{-\alpha}, h_\alpha] \to L(G)$.

If $\omega_k$ is the Kähler form of $Gr_k(\mathbb{C}^{n+1})$, and $f_X : G/T \to Gr_k(\mathbb{C}^{n+1})$ is defined by $f_X(gT) = \theta(g)X$, then a calculation similar to that in (4) of proposition 1.6 gives

$$E'(f_X) - E''(f_X) = \langle f_X^* \omega_k, \kappa \rangle .$$

In the situation of 2.4, $E''(f_i^+) = 0$ and $E'(f_i^-) = 0$, hence

$$E(f_{v_i}) = (E'(f_i^+) - E''(f_i^+)) - (E'(f_i^-) - E''(f_i^-))$$

$$= \langle (f_i^+)^* \omega_a, \kappa \rangle - \langle (f_i^-)^* \omega_b, \kappa \rangle .$$

We may write this in terms of the maps $f_{v_i} \circ \alpha$ as follows. Using a suffix to denote the relevant group, we have

$$\langle \, , \, \rangle_G \big|_{L(SU_2)} = (1/8\pi^2 \langle \alpha, \alpha \rangle_G) \langle \, , \, \rangle_{SU_2}$$

and so

$$E(f_{v_i}) = (8\pi^2 K_G/K_{SU_2}) \sum_{\alpha\in\Delta^+} \langle\alpha,\alpha\rangle E(f_{v_i} \cdot \alpha) \ .$$

By proposition 2.4, applied to each $f_{v_i} \circ \alpha$ ,

$$E(f_{v_i} \cdot \alpha) = \langle (f_i^+ \cdot \alpha)^*\omega_a , \kappa_{SU_2} \rangle_{SU_2} - \langle (f_i^- \circ \alpha)^*\omega_b , \kappa_{SU_2} \rangle_{SU_2} \ .$$

Now, although $\kappa_G$ is not a closed form if rank $G > 1$ , it is when $G = SU_2$ . So $\langle (f_i^+ \cdot \alpha)^*\omega_b , \kappa_{SU_2} \rangle_{SU_2}$ , $\langle (f_i^- \cdot \alpha)^*\omega_a , \kappa_{SU_2} \rangle_{SU_2}$ represent the <u>topological</u> <u>degrees</u> of $f_i^+ \cdot \alpha$ , $f_i^- \cdot \alpha$ (i.e., up to a constant, the integers given by the induced maps on second cohomology groups); this fact was noticed in a more general context by A. Lichnerowicz - see §9 of [EL]. In chapter III we shall show that $E(f_{v_i})$ may be expressed in terms of degrees of holomorphic and anti-holomorphic maps whenever $f_{v_i}$ is harmonic, i.e. without assuming the full hypotheses of corollary 2.3.

A stronger result than 2.3 (apparently without a simple interpretation of $E(f_{v_i})$ , however), is the following version of proposition 1.6 of [BW].

PROPOSITION 2.5. *Let $f_\theta : G/T \to F$ be holomorphic, with respect to an almost complex structure $J$ which corresponds to the relation $\to$ . Assume in addition that $J$ satisfies the condition*

$$a \to i \ , \ i \to b \ \Rightarrow \ b \to a \quad \text{for all} \quad i,a,b \ .$$

*Then $f_{v_i} = \pi_i \circ f_\theta$ is harmonic.*

*Proof*. Let $\alpha \in \Delta^+$ . If $a \to i \to b$ , then $b \to a$ . Since $f_\theta$ is holomorphic, $P^\alpha_{ba} = 0$ . By corollary 2.2, $f_{v_i}$ is harmonic. □

The hypothesis on J in 2.5 is in an obvious sense the opposite of the integrability condition for an almost complex structure. When $\pi_i$ is neither holomorphic nor anti-holomorphic, such an almost complex structure J is the result of taking a complex structure on F and reversing it on the subbundle which is vertical with respect to $\pi_i$ .

Finally, we note that the hypotheses of corollary 2.3 and proposition 2.5 have the following property, which provides further justification for introducing the "twistor space" F . If $f_\theta$ satisfies either of these hypotheses, then so does P.f for any $(n+1) \times (n+1)$ complex matrix P . Since one can in fact replace $f_\theta$ by any smooth map in 2.3 and 2.5 (c.f. the remarks in the introduction), one deduces that all maps of the form $\pi_i \circ P.f_\theta$ are harmonic. (To be more precise, only those P are allowed for which $\pi_i \circ P.f_\theta$ is defined.) For example, all harmonic maps $\mathbb{C}P^1 \to \mathbb{C}P^n$ arise in this fashion (see [Gu1]).

CHAPTER III

# THE ENERGY OF A PROJECTIVE WEIGHT ORBIT

Let $v$ be a weight vector of unit length with weight $\lambda$ for the representation $\theta$ of $G$. From chapter I we have

$$E(f_v) = K \langle\langle (\sum_{\alpha\in\Delta} \theta(e_\alpha)\theta(e_{-\alpha}))v, v\rangle\rangle ,$$

and $f_v$ is harmonic if and only if $v$ is an eigenvector of $\sum_{\alpha\in\Delta}\theta(e_\alpha)\theta(e_{-\alpha})$. Let $e_1,\ldots,e_m$ be an orthonormal basis of $L(G)$ such that $e_1,\ldots,e_h$ is a basis of $L(T)$ and $e_{h+1},\ldots,e_m$ is a basis of $L(G/T)$. The Casimir operator of $\theta$ with respect to $\langle\, ,\, \rangle$, i.e.

$$\sum_{i=1}^{m} \theta(e_i)^2 = \sum_{i=1}^{h} \theta(e_i)^2 - \sum_{\alpha\in\Delta} \theta(e_\alpha)\theta(e_{-\alpha}) ,$$

is a scalar operator on each irreducible submodule. If the submodule has maximal weight $\tilde{\lambda}$, this constant is $-4\pi^2\langle\tilde{\lambda},\tilde{\lambda}+2\delta\rangle$, where $\delta$ is half the sum of the positive roots (see [Ja]).

<u>PROPOSITION</u> 3.1. <u>The map $f_v : G/T \to \mathbb{C}P^n$ is harmonic if and only if $\langle\tilde{\lambda},\tilde{\lambda}+2\delta\rangle = \langle\tilde{\mu},\tilde{\mu}+2\delta\rangle$, for all maximal weights $\tilde{\lambda}$, $\tilde{\mu}$ of irreducible submodules in which $v$ has a non-zero component. When this is the case, one has:</u>

$$E(f_v) = 4\pi^2 K(\langle\tilde{\lambda},\tilde{\lambda}+2\delta\rangle - \langle\lambda,\lambda\rangle) .$$

<u>Proof</u>. The first assertion follows from the discussion above. To prove the

second assertion, use the formula above for $E(f_v)$ together with $\sum_{i=1}^{h} \langle\langle\theta(e_i)^2 v,v\rangle\rangle = -4\pi^2 \sum_{i=1}^{h} \lambda(e_i)^2 = -4\pi^2\langle\lambda,\lambda\rangle$ . □

For example, let $G = SU_2$ and let $\theta = S^n\sigma$ (as in example 2 of chapter I). An orthonormal basis for $L(T)$ consists of one element $e_1$ . This may be extended to an orthonormal basis $e_1,e_2,e_3$ of $L(G)$ , with $e_2 = (1/\sqrt{2})(e_\alpha - e_{-\alpha})$ , $e_3 = (i/\sqrt{2})(e_\alpha + e_{-\alpha})$ where $\alpha$ is the single positive root. The action of these elements on an orthonormal basis $v_0,\ldots,v_n$ of weight vectors is easy to write down explicitly (see, for example, [Gu2]). If $v_i$ has weight $n - 2i$ , so that $v_0$ has maximal weight $n$ , one has

$$E'(f_{v_i}) = K\langle\langle\theta(e_\alpha)\theta(e_{-\alpha})v_i,v_i\rangle\rangle = (K/4)(i+1)(n-i)$$

$$E''(f_{v_i}) = K\langle\langle\theta(e_{-\alpha})\theta(e_\alpha)v_i,v_i\rangle\rangle = (K/4)i(n-i+1) \quad .$$

All the maps

$$f_{v_i} : \mathbb{C}P^1 \longrightarrow \mathbb{C}P^n$$

are harmonic, since $\theta$ is irreducible, and

$$E'(f_{v_i}) - E''(f_{v_i}) = (K/4)(n-2i)$$

$$E'(f_{v_i}) + E''(f_{v_i}) = (K/4)(n+2i(n-i))$$

in agreement with 1.6 and 3.1.

In the situation of proposition 3.1 it is possible to interpret the energy as a combination of degrees of holomorphic and anti-holomorphic maps, thus generalizing (the remarks following) proposition 2.4. To do this we shall use a basic result from representation theory.

<u>THEOREM 3.2 (Freudenthal's Recursion Formula). Let $\theta$ be an irreducible unitary representation of $G$ with maximal weight $\tilde{\lambda}$ . Let the multiplicity of a weight $\lambda$ of $\theta$ be $m(\lambda)$ . Then</u>

$$(\langle\tilde{\lambda}+\delta,\tilde{\lambda}+\delta\rangle - \langle\lambda+\delta,\lambda+\delta\rangle)m(\lambda) = \sum_{\alpha\in\Delta^+} \sum_{k\geq 1} m(\lambda+k\alpha)\langle\lambda+k\alpha,\alpha\rangle \quad .$$

*Proof*. See [Ja], chapter VIII. □

If $\phi$ is an irreducible representation of $G$, the composition of $\phi$ with the homomorphism $\alpha : SU_2 \to G$ defined by $\alpha \in \Delta^+$ is a representation of $SU_2$. Since any irreducible representation of $SU_2$ is of the form $S^n\sigma$ for some $n$, we may write

$$\phi \circ \alpha \cong \bigoplus_{i=1}^{r_i^\alpha} S^{n_i^\alpha}\sigma .$$

Let $\lambda$ be a weight of $\phi$. If the corresponding weight of $\phi\circ\alpha$ occurs in the summand $S^{n_i^\alpha}\sigma$, then it may be written in the form $n_i^\alpha - 2k_i^\alpha$ for some $k_i^\alpha$ with $0 \leq k_i^\alpha \leq n_i^\alpha$. The corresponding weight vector gives rise to a harmonic map

$$\mathbb{C}P^1 \longrightarrow \mathbb{C}P^{n_i^\alpha} ,$$

whose energy, $E(n_i^\alpha, k_i^\alpha)$ say, we have calculated above:

$$E(n,k) = (K/4)(n+2k(n-k)) .$$

THEOREM 3.3. *Let $\theta$ be a unitary representation of $G$. Let $v$ be a weight vector of unit length with weight $\lambda$. Assume that the map $f_v : G/T \to \mathbb{C}P^n$ is harmonic. Then*

$$E(f_v) = (8\pi^2 K_G/m(\lambda)K_{SU_2}) \sum_{\alpha\in\Delta^+} (\langle\alpha,\alpha\rangle \sum_i E(n_i^\alpha, k_i^\alpha))$$

*where the integers $n_i^\alpha$, $k_i^\alpha$ are those obtained as explained above from any irreducible submodule in which $v$ has a non-zero component.*

*Proof*. Freudenthal's recursion formula is equivalent to

$$(\langle\tilde\lambda,\tilde\lambda+2\delta\rangle - \langle\lambda,\lambda\rangle)m(\lambda) = \sum_{\alpha\in\Delta} \sum_{k\geq 0} m(\lambda+k\alpha)\langle\lambda+k\alpha,\alpha\rangle$$

(formula (19) of [Ja], chapter VIII), hence one has (using 3.1)

$$E(f_v) = (4\pi^2 K_G/m(\lambda))\left(\sum_{\alpha\in\Delta^+}\sum_{k\geq 1} m(\lambda+k\alpha)<\lambda+k\alpha,\alpha> - \sum_{\alpha\in\Delta^+}\sum_{k\geq 1} m(\lambda-k\alpha)<\lambda-k\alpha,\alpha>\right)$$

where the multiplicities refer to any irreducible submodule in which $v$ has a non-zero component. Let

$$E_\alpha = \sum_{k\geq 1} m(\lambda+k\alpha)<\lambda+k\alpha,\alpha> - \sum_{k\geq 1} m(\lambda-k\alpha)<\lambda-k\alpha,\alpha>$$

for each $\alpha \in \Delta^+$. The weights of the irreducible submodule of the form $\lambda-k\alpha$ occur in a "string" of the form $\lambda-p\alpha,\ldots,\lambda,\ldots,\lambda+q\alpha$ and one has (see [Ja], chapter VIII)

$$<\lambda,\alpha> = ((p-q)/2)<\alpha,\alpha> .$$

Hence

$$\begin{aligned}(2/<\alpha,\alpha>)E_\alpha &= \sum_{k=1}^{q} m(\lambda+k\alpha)(p-q+2k) - \sum_{k=1}^{p} m(\lambda-k\alpha)(p-q-2k) \\ &= \sum_{i=0}^{q-1} m(\lambda+q\alpha-i\alpha)(p+q-2i) - \sum_{i=0}^{p-1} m(\lambda-p\alpha+i\alpha)(-p-q+2i) .\end{aligned}$$

Since

$$\begin{aligned}E(n,k) &= (K_{SU_2}/4)(n+2k(n-k)) \\ &= (K_{SU_2}/4)\left(\sum_{i=0}^{k-1} (n-2i) - \sum_{i=0}^{n-k-1} (-n+2k)\right)\end{aligned}$$

the result follows.□

This should be compared with the formula (from chapter II)

$$E(f_{v_i}) = (8\pi^2 K_G/K_{SU_2}) \sum_{\alpha\in\Delta^+} <\alpha,\alpha>E(f_{v_i} \circ \alpha)$$

which holds under the more restrictive hypotheses of corollary 2.3.

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