Max-Planck-Institut für Mathematik
Bonn

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Max-Planck-Institut für Mathematik
Preprint Series 2014 (16)

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MPIM 14-16

FEYNMAN CATEGORIES

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Abstract. In this paper we give a new foundational categorical formulation for operations and relations and objects parameterizing them. This generalizes operads and all their cousins including but not limited to PROPs, modular operads, twisted (modular) operads as well as algebras over operads and an abundance of other related structures, such as FI–algebras.

The usefulness of this approach is that it allows us to handle all the classical as well as more esoteric structures under a common framework and we can treat all the situations simultaneously. Many of the known constructions simply become Kan extensions.

In this common framework, we also derive universal operations, such as those underlying Deligne's conjecture, construct Hopf algebras as well as perform resolutions, (co)bar transforms and Feynman transforms which are related to master equations. For these applications, we construct the relevant model category structures.

Introduction

We introduce Feynman categories as a universal foundational framework for treating operations and their relations. It unifies a plethora of theories which have previously each been treated individually. We give the axiomatics, the essential as well as novel universal constructions which provide a new conceptual level to a sweeping set of examples. In this sense Feynman categories present a new *lingua* or *characteristica universalis* for objects governing operations. We use a categorical formulation in which all constructions become Kan extensions, true to the dictum of MacLane [ML98].

The history of this problem goes back to Leibniz and in modern times it was first considered by Whitehead [Whi98]. Depending on the situation, the special types of operations that have appeared were encoded by various objects fitting the setup. Classically these came from the PROPs of [ML63] and their very successful specialization to operads [May72]. As the subject entered its renaissance in the 90s [LSV97] fueled by the connection to physics, especially (string) field theory and correlation functions, more constructions appeared, such as cyclic operads [GK95], modular & twisted modular operads [GK98] and even in the 2000s there are additions such as Properads [MV09b], wheeled versions [MMS09] and many more, see e.g. [MSS02, KWZ12], Table 1 and §2. There are still more examples to which our theory applies, such as FI-algebras [CEF12], which appear in representation theory, as well as operations coming from considering open/closed field theories or those in homological mirror symmetry [FOOO09] where the relevant framework has perhaps not even been defined yet axiomatically.

The beauty of Feynman categories is that all these examples —and many more— can be treated on equal footing. Although differing greatly in their details the above structures do have a common fabric, which we formalize by the statement that these are all functors from Feynman categories. This enables us to replace "looks like", "feels like", "operad–like" and "is similar/kind of analogous to" by just the the statement that all examples are examples of functors from Feynman categories. The upshot being that there is no need to perform constructions and prove theorems in every example separately, as they can now be established in general once and for all.

What we mean by operations and relations among them can be illustrated by considering a group (which is a trivial example of a Feynman category) and its representations. There are the axioms of a group, the abstract group itself, representations in general and representation of a particular group. The operations when defining a group are given by the group multiplication and inverses, the relations are given by associativity and unit relations. Thus the operations and relations deal with groups and representations in general. These specialize to those of a particular group. Universal constructions in this setting are for instance the free group, resolutions, but also induction and restriction of representations. This situation may be interpreted categorically by viewing a group as a groupoid with one object. Here the morphisms are the elements of the group, with the group operation being concatenation. The fact that there are inverses makes the category into a groupoid by definition. The representations then are functors from this groupoid.

In many situations however, there are many operations with many relations. One main aim in the application of the theory is to gather information about the object which admits these operations. A classical paradigmatic topological example is for instance the recognition theorem for loop spaces, which states that a connected space is a loop space if and only if it has all the operations of a topological homotopy associative algebra [Sta63]. A more modern example of operations are Gromov–Witten invariants [Man99] and string topology [CS99] which are both designed to study the geometry of spaces via operations and compare spaces/algebraic structures admitting these operations. It then becomes necessary to study the operations/relations themselves as well as the objects and rules governing them.

After giving the definition of a Feynman category, we give an extensive list of examples. This can be read in two ways. For those familiar with particular examples they can find their favorite in the list and see that the theory applies to it. Importantly, for those not familiar with operads and the like, this can serve as a very fast and clear definition. Some, but not all, examples are built on graphs. This gives a connection to Feynman diagrams which is responsible for the terminology. In our setup the graphs are —crucially— the morphisms and not the objects.

Having encoded the theories in this fashion, we can make use of categorical tools, such as Kan-extensions. This enables us to define push–forwards, pull–backs, extension by zero etc.. Indeed we show that "all" classical constructions, such as free objects, the PROP generated by an operad, the modular envelope, etc., all push–forwards, pull–backs, are Kan extensions in general. To further extend the applicability, we also consider the enriched version of the theory.

Among the universal constructions are universal operations, Hopf algebras, model structures, and bar/co-bar and Feynman transforms. Examples of universal operations are Gerstenhaber's pre–Lie algebra structure for operads and its cyclic generalization, as well as BV operators etc. The classical examples are collected in Table 1. We show that given any Feynman category there is a universal construction for this type of universal operation, which is actually highly calculable.

The reason for giving the bar/co-bar and Feynman transforms is two-fold. On one hand, they give resolutions. For this to make sense, we need a model category structure, which we provide. The construction of this model structure unifies and generalizes a number of examples found in the literature (see section 8). On the other hand, said transforms give rise to master equations as we discuss. A case-by-case list of these was given in [KWZ12] and here we show that they all fit into the same framework.

The Hopf algebra structures give a new source for such algebras and go back to an observation of Dirk Kreimer. They are expounded in [GCKT] where we treat the Hopf algebra structures of Connes–Kreimer [CK98], Goncharov [Gon05] and Baues [Bau81] as examples.

Our new axiomatization rests on the shoulders of previous work, as mentioned above, but it was most influenced by [BM08] who identified Markl's [Mar08] hereditary condition as essential. For the examples based on graphs, we also use their language. Our construction, however, is one category level higher. With hindsight there are many similar but different generalizations and specializations like [Get09b], [BM07],[EM09] and others, such as Lavwere theories [Law63] and FI–algebras [CEF12] and crossed–simplicial groups [FL91], whose relationship to Feynman categories we discuss in §1.2.

THE PAPER IS ORGANIZED AS FOLLOWS: In §1 we introduce the main character, that is the definition of a Feynman category. The first set of results then deals with free constructions and an equivalence of functors from a Feynman category and algebras over a triple given by the free constructions. This is known as monadicity and gives two equivalent ways of characterizing the operations. §2 contains many examples. They basically come in four types. The first are ones based on graphs and give the usual suspects, operads, cyclic/modular operads, PROPs etc, by varying the types of graphs. The second type is the enriched version for these, which are introduced in §4. This is an essential addition to the theory. It allows us for the first time to give a definition of a twisted modular operads on par with their non–twisted versions. The third type of example is the one in which the symmetries of the Feynman category are trivial. We classify these examples, which include crossed–simplicial groups, the simplicial category and the category FI of finite sets with injection as well as the category of finite sets with surjections. The forth type are free constructions on these. Before giving the enriched theory in §4, we provide several essential general constructions which produce Feynman categories from Feynman categories, such as iterations of Feynman categories, free Feynman categories, level Feynman categories and hyper Feynman categories. The latter are essential for enrichment in linear categories. We also construct a Connes–Kreimer type Hopf algebra out of any $\mathcal{A}b$ enriched Feynman category. In §5 we give a third way to define Feynman categories and their functors, via generators and relations. For the special example of (pseudo)-operads this boils down to the fact that an operad structure can be given by the compositions and the associativity equations. In general, we need such a presentation to define the transforms. In §6 we universal operations on colimits and limits. The paradigmatic example being pre–Lie operations on Operads and Brace operations on operads with multiplication. This is generalized to any Feynman category. §7 introduces the (co)bar and Feynman transforms for well–presented (the technical details are to be found in §6) Feynman categories. The examples encompass all graph examples and the theory is set up in great generality. §8 Then contains the homotopy theory needed to state that the transforms give resolutions, that is free or cofibrant objects. Here the case of topological spaces is especially tricky and we study it in detail. We conclude with two appendices for the convenience of the reader. The first is on graphs and the second on the topological model category structure.

ACKNOWLEDGMENTS

We would like to thank Yu. Manin, D. Kreimer, M. Kontsevich, C. Berger, D. Borisov, B. Fresse, M. Fiore and M. Batanin and M. Markl for enlightening discussions.

RK gratefully acknowledges support from NSF DMS-0805881, the Humboldt Foundation and the Simons Foundation. He also thanks the Institut des Hautes Etudes Scientifiques, the Max–Planck–Institute for Mathematics in Bonn and the Institute for Advanced Study for their support and the University of Hamburg for its hospitality. RK also thankfully acknowledges the Newton Institute where a large part extending the original scope of the paper came into existence during the activity on "Grothendieck-Teichmüler Groups, Deformation and Operads".

Part of this work was conceptualized when RK visited the IAS in 2010. At that time at the IAS RKs work was supported by the NSF under agreement DMS–0635607. It was written in large parts while staying at the aforementioned locations. The final version of this paper was finished at the IAS during a second stay which is supported by a Simons Foundation Fellowship.

Any opinions, findings and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

BW gratefully acknowledges support from the Purdue Research Foundation as well as from the Simons Center for Geometry and Physics.

Conventions and notations

We will use the notion of disjoint union of sets. This gives a monoidal structure to sets. It is not strict, but sometimes it is useful to use a strict monoidal structure. If this is the case we will say so and tacitly use MacLane's Theorem [ML98] to do this.

We will denote the comma categories by $(\downarrow)$. If a functor, say $\imath : \mathcal{V} \to \mathcal{F}$ is fixed, we will just write $(\mathcal{F} \downarrow \mathcal{V})$, and given a category $\mathcal{G}$ and an object X of $\mathcal{G}$, we denote by $(\mathcal{G} \downarrow X)$ the respective comma category. I.e. objects are morphisms $\phi : Y \to X$ with Y in $\mathcal{G}$ and morphisms are morphisms over X, that is morphisms $Y \to Y'$ in $\mathcal{G}$ which commute with the base maps to X.

For any category $\mathcal{E}$ we denote by $Iso(\mathcal{E})$ the category whose objects are those of $\mathcal{E}$, but whose morphisms are only the isomorphisms of $\mathcal{E}$.

For a morphism ϕ in a given category, we will use the notation $s(\phi)$ for the source of ϕ and $t(\phi)$ for the target of ϕ.

For a natural number n, we define the sets $\bar{n} = \{1, \dots, n\}$, $[n] = \{0, \dots, n\}$.

Finally, we will consider categories enriched over a monoidal category $\mathcal{E}$. Unless otherwise stated, we assume that the monoidal product of $\mathcal{E}$ preserves colimits in each variable and that the objects of $\mathcal{E}$ have underlying elements. The first assumption is often essential while the latter is for the sake of exposition.

1. Feynman categories: Definition and Properties

1.1. **Feynman categories.** Fix a symmetric monoidal category $\mathcal{F}$ and let $\mathcal{V}$ be a category that is a groupoid, that is $\mathcal{V} = Iso(\mathcal{V})$. Denote the free symmetric monoidal category on $\mathcal{V}$ by $\mathcal{V}^{\otimes}$. Furthermore let $\imath\colon \mathcal{V} \to \mathcal{F}$ be a functor and let $\imath^{\otimes}$ be the induced monoidal functor $\imath^{\otimes}\colon \mathcal{V}^{\otimes} \to \mathcal{F}$.

Definition 1.1. A triple $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \imath)$ of objects as above is called a Feynman category if

- (i) (Isomorphism condition) The monoidal functor $\imath^{\otimes}$ induces an equivalence of symmetric monoidal categories between $\mathcal{V}^{\otimes}$ and $Iso(\mathcal{F})$.
- (ii) (Hereditary condition) The monoidal functor $\imath^{\otimes}$ induces an equivalence of symmetric monoidal categories between $Iso(\mathcal{F} \downarrow \mathcal{V})^{\otimes}$ and $Iso(\mathcal{F} \downarrow \mathcal{F})$.

(iii) (Size condition) For any $* \in \mathcal{V}$, the comma category $(\mathcal{F} \downarrow *)$ is essentially small, viz. it is equivalent to a small category.

1.1.1. **Essential remarks about the definition.**

1. (Change of base) Due to the condition (i) for each $X \in \mathcal{F}$ there exists an isomorphism
$$\phi_X \colon X \overset{\sim}{\to} \otimes_{v \in I} \imath(*_v) \text{ with } *_v \in \imath(\mathcal{V}) \tag{1.1}$$
for a finite index set I. Moreover, fixing a functor $\jmath : Iso(F) \to \mathcal{V}$ which yields the equivalence, we fix a decomposition for each X. We will call this a choice of basis. The decomposition (1.1) has the following property: For any two such isomorphisms (choices of basis) there is a bijection of the two index sets $\psi \colon I \to J$ and a diagram
$$\begin{tikzcd} & \bigotimes_{v \in I} \imath(*_v) \arrow[dd, "\simeq\bigotimes \imath(\phi_v)"] \\ X \arrow[ur, "\phi_X", "\simeq"'] \arrow[dr, "\simeq", "\phi'_X"'] & \\ & \bigotimes_{w \in J} \imath(*'_w) \end{tikzcd} \tag{1.2}$$
where $\phi_v \in Hom_{\mathcal{V}}(*_v, *'_{\psi(v)})$ are isomorphisms. We call the unambiguously defined value $|I|$ the *length of* X.
2. The hereditary condition means that the comma category $(\mathcal{F} \downarrow \mathcal{V})$ generates the morphisms in the following way. Any morphism $X \to X'$ is part of a commutative diagram
$$\begin{tikzcd} X \arrow[rr, "\phi"] \arrow[d, "\simeq"'] & & X' \arrow[d, "\simeq"] \\ \bigotimes_{v \in I} X_v \arrow[rr, "\bigotimes_{v \in I} \phi_v"] & & \bigotimes_{v \in I} \imath(*_v) \end{tikzcd} \tag{1.3}$$
where $*_v \in \mathcal{V}$, $X_v \in \mathcal{F}$ and $\phi_v \in Hom(X_v, *_v)$.

   Notice that if in (1.3) the vertical isomorphisms are fixed, then so is the lower morphism. Hence, a choice of basis also fixes a particular diagram of type (1.3) where the X_v are now each a tensor product of elements of $\imath(V)$.

   Furthermore, given *any* two decompositions of a morphism according to (ii), it follows from the previous remark that there is a unique isomorphism in $(\imath^{\otimes} \downarrow \imath^{\otimes})$ giving an isomorphism between the two decompositions, that is between the lower rows.

   The condition of equivalence of categories furthermore implies that (1) for any two such decompositions $\bigotimes_{v \in I} \phi_v$ and $\bigotimes_{v' \in I'} \phi'_{v'}$ there is a bijection $\psi : I \to I'$ and isomorphisms $\sigma_v : X_v \to X'_{\psi(v)}$ s.t. $P_{\psi}^{-1} \circ \bigotimes_v \sigma_v \circ \phi_v = \bigotimes \phi'_{v'}$ where P_ψ is the permutation corresponding to ψ. And (2) that these are the only isomorphisms between morphisms.

   Often, for instance in $(\mathcal{S}et, \amalg)$ the conditions (i) and (1.3) imply (ii), that is (1) and (2) above follow (see below).
3. There is a version, where we replace symmetric monoidal with just monoidal. We will call these non-symmetric Feynman categories.
4. These is an enriched version; see §4.

Lemma 1.2. *If* $(\mathcal{F}, \otimes)$ *has a fully faithful strong symmmetric monoidal functor to* $(\mathcal{S}et, \amalg)$*, then (i) and (1.3) imply (ii).*

Proof. Given two decompositions $\otimes_{v\in I}\phi_v : \bigotimes_{v\in I} X_v \to \bigotimes_{v\in I} \imath(*_v)$ and $\otimes_{v\in I}\phi'_v : \bigotimes_{v\in I} X'_v \to \bigotimes_{v\in I} \imath(*_v)$ completing (1.3), we have that after passing to $\mathcal{S}et$: $\phi_v^{-1}(*_v) = X_v$ and $\phi_v'^{-1}(*_v) = X'_v$. Then we can decompose further $X_v \simeq \bigotimes_{w_v\in I_v} \imath(*_{w_v})$ and $X_v \simeq \bigotimes_{w'_{v'}\in I'_v} \imath(*_{w_{v'}})$. By (i) the isomorphism between the full decomposition is a permutation followed by an isomorphism of the individual $*_{v_w}$. This means in particular that under the isomorphism given by the decompositions: $\bigotimes_{w_w\in I_v} \imath(*_{v_w}) \simeq \bigotimes_{w'_w\in I'_v} \imath(*_{w'_v})$ since both are the pre–images of $\imath(*_v)$. It also follows that all isomorphisms are of this type. □

Given a Feynman category $\mathfrak{F}$ we will sometimes write $\mathcal{V}_{\mathfrak{F}}$ and $\mathcal{F}_{\mathfrak{F}}$ for the underlying groupoid and monoidal category and often take the liberty of dropping the subscripts if we have already fixed $\mathfrak{F}$.

1.1.2. **An alternative formulation of condition (ii).** There is another more highbrow way to phrase the condition on morphisms. We claim that the condition (ii) can equivalently be stated as

$$\imath^{\otimes\wedge}Hom_{\mathcal{F}}(\,\cdot\,, X\otimes Y) := Hom_{\mathcal{F}}(\imath^{\otimes}\,\cdot\,, X\otimes Y) = \imath^{\otimes\wedge}Hom_{\mathcal{F}}(\,\cdot\,, X)\circledast\imath^{\otimes\wedge}Hom_{\mathcal{F}}(\,\cdot\,, Y) \quad (1.4)$$

where $\circledast$ is the Day convolution of presheaves on $\mathcal{V}^{\otimes}$, that is for functors in $[\mathcal{V}^{\otimes op}, \mathcal{S}et]$ and $\imath^{\otimes\wedge}$ is the pull–back induced by $\imath^{\otimes}$. The Yoneda embedding for $\mathcal{F}$ is a strong monoidal functor, i.e. $Hom_{\mathcal{F}}(\,\cdot\,, X\otimes Y) = Hom_{\mathcal{F}}(\,\cdot\,, X) \circledast Hom_{\mathcal{F}}(\,\cdot\,, Y)$, where now the convolution is of representable pre–sheaves on $\mathcal{F}$. Thus (1.4) can be cast into

(ii') The pull-back of presheaves $\imath^{\otimes\wedge}\colon [\mathcal{F}^{op}, Set] \to [\mathcal{V}^{\otimes op}, Set]$
restricted to representable pre-sheaves is monoidal.

Lemma 1.3. $\mathfrak{F}$ *is a Feynman category if and only (i), (ii') and (iii) hold.*

Proof. The condition (ii') is equivalent to (1.4) for any choice of X and Y. Using the definition of the Day convolution the right hand side of (1.4) becomes

$$\begin{aligned}\imath^{\otimes\wedge}Hom_{\mathcal{F}}(\,\cdot\,, X) \circledast \imath^{\otimes\wedge}Hom_{\mathcal{F}}(\,\cdot\,, Y) &= Hom_{\mathcal{F}}(\imath^{\otimes}\,\cdot\,, X) \circledast Hom_{\mathcal{F}}(\imath^{\otimes}\,\cdot\,, Y)\\ &= \int^{Z,Z'} Hom_{\mathcal{F}}(\imath^{\otimes}Z, X) \times Hom_{\mathcal{F}}(\imath^{\otimes}Z', Y) \times Hom_{\mathcal{V}^{\otimes}}(\,\cdot\,, Z\otimes Z') \quad (1.5)\end{aligned}$$

Now since $\mathcal{V}^{\otimes}$ was a groupoid, deciphering the co-end, there are contributions for each decomposition of the argument which decompose any chosen morphism into two pieces. The whole situation is precisely equivariant with respect to pairs of isomorphisms and, due to the definition of a co-end, changing the isomorphisms gives the same morphism. Fix $\jmath\colon Iso(\mathcal{F}) \to \mathcal{V}^{\otimes}$ as before. Since any X can be decomposed completely into $X \simeq \bigotimes_v \imath(*_v)$, we can apply (1.4) iteratively to arrive exactly at (ii). □

1.1.3. **More Remarks.**

1. Notice that in (1.3), we can further decompose the X_v as $X_v \simeq \bigotimes_{w\in I_v} *_w$. If $J = \amalg_{v\in I} I_v$ then $X \simeq \bigotimes_{w\in J} *_w$. It may happen that $|J| = |I|$, but this is usually not the case.
2. Since we will be mainly considering categories of functors from Feynman categories to other monoidal categories, up to equivalence of categories one can assume that $\mathcal{V}$ is a subcategory of $\mathcal{F}$, that is $\imath$ is injective on objects and hence we will identify objects of $\mathcal{V}$ with objects of $\mathcal{F}$.

3. Given such a triple we will say that $\mathcal{V}$ is a *one–comma generating subcategory*, and call its objects stars or vertices. The objects of $\mathcal{V}^\otimes$ or $\imath^\otimes(\mathcal{V}^\otimes)$ are sometimes called clusters or aggregates (of stars or vertices).

The following strictification is sometimes useful.

Definition 1.4. Let $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \iota)$ be a Feynman category. The reduction of $\mathfrak{F}$ is defined to be the triple $\tilde{\mathfrak{F}} = (\iota(\mathcal{V}), \tilde{\mathcal{F}}, in)$, where $\tilde{\mathcal{F}}$ is defined to be the full symmetric monoidal subcategory of $\mathcal{F}$ generated by $\iota(\mathcal{V})$ and where in is the inclusion $\iota(\mathcal{V}) \to \tilde{\mathcal{F}}$. We call a Feynman category strict if it is equal to its reduction.

Lemma 1.5. *The reduction of a Feynman category is itself a Feynman category, and these two Feynman categories are equivalent.*

Proof. In the above notation, $in^\otimes \colon \iota(\mathcal{V})^\otimes \to Iso(\tilde{\mathcal{F}})$ is simply the identity functor, and is fully faithful since ι is, and thus is an equivalence. Moreover, applying the hereditary condition for $\mathfrak{F}$ twice gives the necessary decomposition of morphisms in $\tilde{\mathfrak{F}}$.

To see that $\mathfrak{F}$ and $\tilde{\mathfrak{F}}$ are equivalent we may construct an explicit equivalence. Let $\pi \colon \tilde{\mathcal{F}} \to \mathcal{F}$ be the inclusion functor. Let $\eta \colon Iso(\mathcal{F}) \to \mathcal{V}^\otimes$ be a quasi-inverse to $\iota^\otimes$. Then define a symmetric monoidal functor $\bar{\eta} \colon \mathcal{F} \to \tilde{\mathcal{F}}$ by taking $X \mapsto \iota^\otimes\eta(X)$ on objects, and by taking $\phi \colon X \to X'$ to the composite

$$\iota^\otimes\eta(X) \overset{\nu_X}{\to} X \overset{\phi}{\to} X' \overset{\nu^{-1}_{X'}}{\to} \iota^\otimes\eta(X') \tag{1.6}$$

where ν, ν^{-1} are natural transformations giving the isomorphism of functors $id_{Iso(\mathcal{F})} \cong \iota^\otimes\eta$. Then one easily sees that π and $\bar{\eta}$ form an equivalence by using the natural transformations ν^{-1} and ν. □

1.1.4. **Units.** Notice that the empty product yields the monoidal unit $\mathbb{1}_\mathcal{F}$ of $\mathcal{F}$ which has length 0. It is possible with the above definition to have morphisms in $\mathcal{F}(\mathbb{1}_\mathcal{F}, *)$ for $*$ in $\mathcal{V}$. Such a ψ can modify any morphism $\phi \colon X \to Y$ via $X \to X \otimes \mathbb{1}_\mathcal{F} \xrightarrow{\phi\otimes\psi} Y \otimes *$. This will play a role in relations to the category of finite sets and FI–algebras. In particular, in the decomposition (1.3) there can be several factors of length 0.

Co-units are another matter. The axiom (ii) says that any morphism with target $\mathbb{1}_\mathcal{F}$ has $\mathbb{1}_\mathcal{F}$ as the source.

1.2. Relation to other structures.

1.2.1. **Connection to graphs and physics.** The intuition for Feynman categories (FCs) comes from quantum field theory and Feynman graphs, whence the name. The physical terminology is meant as follows. Feynman graphs for a quantum field theory contain two pieces of vital information. First there are fixed vertex types encoding the possible interactions (this role is played $\mathcal{V}$) and secondly these vertices are connected by edges of the graph, which are the propagators or field lines. The morphisms in the category $\mathcal{F}$ are analogous to an S matrix. That is we fix the external legs as the target of a morphism and then the possible morphisms with this target are all possible graphs that we can put inside this effective scattering vertex. Each such graph gives rise to a morphism, where the source is given by specifying the collection of vertices and the morphism is nailed down by putting in the propagation lines.

We wish to point out two things: First, the definition of a Feynman category itself *does not need any underlying graphs.* And secondly, if graphs are present, they are the *morphisms* not the objects. FCs using graphs are formalized below as FCs indexed over a category of graphs see Definition 1.13 and section §A.1.

1.2.2. **Connection to multi-categories and colored operads.** There are several connections to colored operads. First, our framework includes the notions of operads and colored operads, see §2.2

Secondly, for those already familiar with colored operads it is useful to note the following. Consider a discrete $\mathcal{V}$, then as a subcategory of $\mathcal{F}$, $\mathcal{V}$ is the set of colors of a colored operad defined by $\mathcal{F}$. For this set $\mathcal{O}(*_1, \dots, *_n; *_m) := \mathcal{F}(*_1 \otimes \dots \otimes *_n, *_m)$ and let the composition of $\mathcal{O}$ be given by $\circ$ and the monoidal structure in $\mathcal{F}$. This is of course the same way to obtain a multicategory from a tensor category. What the axiom (ii) is saying is that these "multicategory morphisms" are "all the morphisms".

In this light, our setup generalizes the observation of [KS00, BM07] that a non-symmetric operad is an algebra over a certain colored operad in several ways. First, we now look at all possible generalized operadic gadgets that can be formed this way and secondly, we include a precise axiomatization of the possible isomorphims of the colors needed e.g. in the symmetric case, that is we enlarge the set of colors to a groupoid of colors.

Procedurally, note that if we want to avoid circularity in the definition of concepts, we *cannot define* a Feynman category using colored operads, as these are secondary and are defined as functors from a particular Feynman category, see §2.2.

1.2.3. **Manin-Borisov version.** As we will discuss below, graphs are an important source of examples. Graphs and their morphisms form a category which was nicely analyzed and presented in [BM08]. We will review this below and use this category. Whereas in their generalization of operads Borisov and Manin used graphs as objects, we will take graphs as *morphisms*. Thus one could say that we go up one categorical level.

1.2.4. **Getzler's version.** As we realized after having done our construction, in [Get09b] Getzler considers a related notion of operads starting from patterns. The condition (i) appears in this context as the notion of regularity. A regular pattern would then also require the stronger condition (iv) that $\imath^{\otimes\wedge}$ is monoidal on all presheaves.

So in one sense a pattern is more general as one may omit (i) and in another sense it is more restrictive since (iv) is stronger than (ii) and might be hard to check.

An advantage here is that our axiom (ii) is easily checked in all the examples.

1.3. **(Re)–Construction.** One way to construct a Feynman category is to fix a groupoid $\mathcal{V}$. Then the inclusion of $\imath : \mathcal{V} \to \mathcal{V}^{\otimes}$ already trivially yields a Feynman category structure by setting $\mathcal{F} = \mathcal{V}^{\otimes}$. To get to any other Feynman category structure with this $\mathcal{V}$ up to equivalence, one can now select new morphisms for $\mathcal{V}^{\otimes} \downarrow \mathcal{V}$ and generate the sets of morphisms of $\mathcal{F}$ by taking tensor products of these. Then one has to define a composition and check that it is associative. For compositions, one only needs to define the compositions for diagrams of the type:

$$\phi_1 : X \xrightarrow{\otimes_v(\phi_v)} Y = \bigotimes_v *_v \xrightarrow{\phi_0} *$$

with $* \in \mathcal{V}$. Usually this amounts to showing that the natural candidate for ϕ_1 is again of the chosen type of morphism.

Concretely, for instance in the case of underlying graphs, inserting a graph of a certain type (e.g. a tree) with tails into a vertex of another graph of the same type which has the same number of flags as legs of the first graph yields a graph, again of the chosen type. See A.1 for examples.

Finally, one has to check the compatibility with the isomorphisms that are already present transform into each other under isomorphisms and in particular they carry an action of the automorphisms. Often these new morphisms can be defined by using generators and relations. This will be discussed in detail in §5. Another approach is to use a discrete $\mathcal{V}$ and use known (colored) operads for the morphisms - together with the isomorphism operations mentioned above. This is done in §4.5.1. We should already warn that this theory will then yield algebras over these operads, not the operads themselves.

Vice-versa, given a Feynman category, the data of $\imath$ and $(\mathcal{F} \downarrow \mathcal{V})$ is enough to reconstruct it, by using the procedure above.

This also means that to test if $\mathcal{F}$ gives rise to a Feynman category, one has to find a $\mathcal{V}$ such that $\mathcal{V}^{\otimes}$ is equivalent to $Iso(\mathcal{F})$, and then test whether morphisms to the included $\mathcal{V}$ one–comma generate.

1.4. Functors as a generalization of operads and $\mathbb{S}$–modules: $\mathcal{O}ps$ and $\mathcal{M}ods$.

Definition 1.6. Let $\mathcal{C}$ be a symmetric monoidal category and $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \imath)$ be a Feynman category. Consider the category of strong symmetric monoidal functors $\mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}} := Fun_{\otimes}(\mathcal{F}, \mathcal{C})$ which we will call $\mathcal{F}$–$\mathcal{O}ps$ in $\mathcal{C}$ and a particular element will be called an $\mathcal{F}$-$\mathcal{O}p$ in $\mathcal{C}$. The category of functors $\mathcal{V}\text{-}\mathcal{M}ods_{\mathcal{C}} := Fun(\mathcal{V}, \mathcal{C})$ will be called $\mathcal{V}$-modules in $\mathcal{C}$ with elements being called a $\mathcal{V}$–mod in $\mathcal{C}$.

If $\mathcal{C}$ and $\mathfrak{F}$ are fixed, we will only write $\mathcal{O}ps$ and $\mathcal{M}ods$.

Notice that since $Iso(\mathcal{F})$ is equivalent to the free symmetric monoidal category on $\mathcal{V}$ we have an equivalence of categories between $Fun(\mathcal{V}, \mathcal{C})$ and $Fun_{\otimes}(Iso(\mathcal{F}), \mathcal{C})$.

Example 1.7. In the theory of (pseudo)-operads, $\mathcal{O}ps$ is the category of (pseudo)-operads and $\mathcal{M}ods$ is the category of $\mathbb{S}$–modules. A longer list of classical notions is given in §2.

1.4.1. Symmetric monoidal structure.

There is the naive tensor product in $\mathcal{O}ps$ and $\mathcal{M}ods$ given by pulling back this structure from $\mathcal{C}$. That is $\mathcal{O} \otimes \mathcal{O}'(X) = \mathcal{O}(X) \otimes \mathcal{O}'(X)$ the identity being the constant functor $\mathbb{1}(X) = \mathbb{1}_{\mathcal{C}}$.

There is a a natural forgetful functor *forget* : $\mathcal{O}ps \to \mathcal{M}ods$.

In the following we will often have to assume that the monoidal product $\otimes$ in $\mathcal{C}$ preserves colimits in each variable. This is automatic if $\mathcal{C}$ is cocomplete and closed monoidal.

Theorem 1.8. *Let $\mathcal{C}$ be a cocomplete monoidal category such that $\otimes$ preserves colimits in each variable, then there is a left adjoint (free) functor $F = free$ to the forgetful functor $G = forget$. Moreover, these functors are even adjoint functors for the symmetric monoidal categories of $\mathcal{M}ods$ and $\mathcal{O}ps$.*

Proof. First by the remark above $Fun(\mathcal{V}; \mathcal{C}), Fun_{\otimes}(\mathcal{V}^{\otimes}; \mathcal{C})$ and $Fun_{\otimes}(Iso(\mathcal{F}); \mathcal{C})$ are all equivalent. Since $\mathcal{C}$ is cocomplete, the left Kan extension along $\imath^{\otimes}$ exists. When starting with a monoidal functor, we need to check that its Kan extension is also monoidal. By definition of the Kan extension it is clear that the functor is co-monoidal, since $\otimes$ has been assumed to preserves colimits in each variable. This structure is strong because of the hereditary condition (ii) and so the functor is strong monoidal. The commutativity constraint actions are natural and hence the functor is symmetric monoidal. The adjunction follows immediately. For the second statement the same condition shows that the natural induced co-monoidal structure for F is strong and hence by [Kel74] it is also an adjoint for the symmetric monoidal categories.

□

Remark 1.9. In fact this is just one instance of the more general Theorem 1.15 below. The theorem also works in the enriched case. We leave the technical details to the interested reader.

1.4.2. **Proof details.** We would like to provide the details of the arguments above for reference and to expose the inner workings of our concepts. The expert may want to skip this section.

Given a $\mathcal{V}$–module Φ, we extend Φ to all objects of $\mathcal{F}$ by picking a functor $\jmath$ which yields the equivalence of $\mathcal{V}^{\otimes}$ and $Iso(\mathcal{F})$. Then, if $\jmath(X) = \bigotimes_{v\in I} *_v$, we set

$$\Phi(X) := \bigotimes_{v\in I} \Phi(*_v) \tag{1.7}$$

Now, for any $X \in \mathcal{F}$ we set

$$F(\Phi)(X) = \mathrm{colim}_{Iso(\mathcal{F}\downarrow X)} \Phi \circ s \tag{1.8}$$

where s is the source map in $\mathcal{F}$ from $Hom_{\mathcal{F}} \to Obj_{\mathcal{F}}$ and on the right hand side, we mean the underlying object. For a given morphism $X \to Y$ in $\mathcal{F}$, we get an induced morphism of the colimits and it is straightforward that this defines a functor.

We claim that the functor is monoidal. On objects this means that $F(\Phi)(X) \simeq \bigotimes_{v\in I} F(\Phi)(*_v)$. Let $(C(X), \psi^X_\phi : \Phi \circ s(\phi) \to C(X))$ with the ϕ running through the objects of $(\mathcal{F} \downarrow X)$ be a collection defining a colimit $F(\Phi(X))$. Then we can decompose each ϕ into $\bigotimes_v \phi_v : v \in I$ using the condition (ii) and the functor $\jmath$. Due to condition (ii) the index category $(\mathcal{F} \downarrow X)$ is equivalent to $\times_v(\mathcal{F} \downarrow \imath(*_v))$ Now, since we assumed that tensor preserves colimits in each variable, the tensor of the colimits $\bigotimes_v C(*_v)$ is a colimit and assembling the maps ϕ_v into a map ϕ we get a morphism. There is a morphism going the other way around, since any morphism decomposes as in (ii) up to isomorphism. These maps are then easily checked to be inverses.

To check the monoidal structure on morphisms, we note that for a morphism $\phi \in Hom_{\mathcal{F}}(X, Y)$ the map $F(\Phi)(\phi)$ is the map induced by composing with ϕ. Namely, since $(C(Y), \psi^Y_{\phi\circ\tilde{\phi}})$ is also a co–cone over $Iso(\mathcal{F} \downarrow X)$ and hence by the universal property of the colimit we have a map: $C(X) \to C(Y)$. The monoidal structure on maps is now easily checked using this description. The symmetric monoidal structure is then again given by the universal properties of the co-cones and that tensor preserves colimits in each variable.

The adjointness of the functors follows from the following pair of adjunction morphisms. For $X \in \mathcal{F}$ and $\mathcal{O} \in \mathcal{O}ps$ let $\sigma(X)$: $(FG\mathcal{O})(X) \to \mathcal{O}(X)$ be the morphism induced by the universal property of the co-cone $(C(\mathcal{O}(X))$ applied to $\mathcal{O}(X)$, which is made into a co-cone by mapping for each $\Gamma \in (\mathcal{F} \downarrow X)$ each $\mathcal{O}(s(\Gamma))$ with $\mathcal{O}(\Gamma)$ to $\mathcal{O}(X)$. For $* \in \mathcal{V}$ and $\Phi \in \mathcal{M}ods$ let $\tau : \Phi(*) \to (GF\Phi)(*)$ be the morphism given by the inclusion of the identity, that is the morphism $\psi^*_{id_*}$ in the notation above.

It is straightforward to check that these morphisms satisfy the needed conditions using the assumptions. This boils down to the fact that composing or precomposing with the identity leaves any map invariant.

Corollary 1.10. *The triple generated by* F = *free and* G = *forget in* $\mathcal{M}ods$ *given by* $\mathbb{T} = GF$ *has the following explicit description: On objects it is given by* $\mathbb{T}\Phi(*_v) = \mathrm{colim}_{(\mathcal{F}\downarrow *_v)}\Phi \circ s$ *and on morphisms* $\phi \in Hom_{\mathcal{V}}(*_v, *_v)$, $\mathbb{T}$ *acts by composing with* ϕ.

Furthermore the monoidal multiplication map for the triple $\mu : \mathbb{T} \circ \mathbb{T} \to \mathbb{T}$ *is given by composition.* □

Proof. The map $\mu\colon G(FG)F \to GF$ is defined by the adjoint map above. We will make this explicit and show how the conditions of Definition 1.1 are needed.

Let us compute $FGF(\Phi)(*)$

$$FGF(\Phi)(*) = \mathrm{colim}_{Iso(\mathcal{F},*)}(GF\Phi \circ s) = \mathrm{colim}_{Iso(\mathcal{F},*)}(\mathrm{colim}_{Iso(\Gamma,s(\,.\,))}\Phi \circ s) \tag{1.9}$$

where $\mathrm{colim}_{Iso(\Gamma,s(\,.\,))}\Phi\circ s$ is the functor that sends an object ϕ of $(\mathcal{F} \downarrow *)$ to $\mathrm{colim}_{Iso(\Gamma,s(\phi))}$ $\Phi \circ s$. Choosing a representation, we see that the double colimit is indexed by elements $(\phi, \amalg_{v\in I}\phi_v)$, where $s(\phi) \simeq \bigotimes_{v\in I} *_v$ and the ϕ_v are morphisms $X_v = s(\phi_v) \to *_v$. Composing the morphisms to $\phi' := (\amalg_v \phi_v)\circ\phi : \bigotimes X_v \to *$ yields a co-cone (by commutativity of tensor and colimits) and in turn a map $\mu'\colon FGF(\Phi)(*) \to F(*)$ such that $\mu = G(\mu')$. □

From this description, we immediately get a monadicity theorem.

Theorem 1.11. *There is a equivalence of categories between $\mathcal{O}ps$ and algebras over the triple $\mathbb{T}$.* □

Proof. Given an element $\mathcal{O} \in \mathcal{O}ps$ we use the $\mathcal{O}(\phi)$ to define maps $\mathbb{T}(\Phi(*)) \to \Phi(*)$ and vice–versa, for an algebra over $\mathbb{T}$, for $\phi \in Hom_{\mathcal{F}}(X,*)$, we let $\mathcal{O}(\phi)$ be the component corresponding to ϕ of the given map $\mathbb{T}(\Phi(*)) \to \Phi(*)$. The fact that this yields an equivalence in now straightforward. □

Notation 1.12. If $\mathcal{C}$ is cocomplete, then for $\mathcal{O} \in \mathcal{O}ps$ we let $\mathcal{O}^{\oplus} = colim_{\mathcal{V}}\mathcal{O}$.

1.5. **Morphisms of Feynman categories and indexed Feynman categories.** Morphisms of Feynman categories will explain many of the standard operations, such as the PROP generated by an operad, the modular envelope or the operad contained in a PROP.

Many of the familiar examples involve some sort of graphs with extra structure. To capture this, we look at indexed Feynman categories. The paradigmatic example for the indexing category will then be the Feynman category $\mathfrak{G}$ introduced in §2.1.

Definition 1.13. A *morphism of Feynman* categories $(\mathcal{V},\mathcal{F},\imath)$ and $(\mathcal{V}',\mathcal{F}',\imath')$ is a pair (v,f) of a functor $v\colon \mathcal{V} \to \mathcal{V}'$ and monoidal functor $f\colon \mathcal{F} \to \mathcal{F}'$ which preserves all the structures, in particular they commute with $\imath$ and $\imath'$, the induced functor $v^{\otimes} : \mathcal{V}^{\otimes} \to \mathcal{V}'^{\otimes}$ is compatible with f and the decompositions of (ii) are preserved.

In particular in

$$\begin{tikzcd}
\mathcal{V} \arrow[r] \arrow[rr, bend left, "\imath"] \arrow[d, "v"'] & \mathcal{V}^{\otimes} \arrow[r, "\imath^{\otimes}"] \arrow[d, "v^{\otimes}"'] & \mathcal{F} \arrow[d, "f"'] \\
\mathcal{V}' \arrow[r] \arrow[rr, bend right, "\imath'"'] & \mathcal{V}'^{\otimes} \arrow[r, "\imath'^{\otimes}"] & \mathcal{F}'
\end{tikzcd}$$

where $v^{\otimes}, \imath^{\otimes}.\imath'^{\otimes}$ are defined by the universal property of $\mathcal{V}^{\otimes}$ respectively $\mathcal{V}'^{\otimes}$ the right square, and hence the outer square, 2–commute, which means that the two compositions of functors are isomorphic.

Definition 1.14. Let $\mathcal{B}$ be a Feynman category. A *Feynman category $\mathfrak{F}$ indexed over $\mathcal{B}$* is a morphism of Feynman categories from $\mathcal{F}$ to $\mathcal{B}$, which is surjective on objects. We will write B for the underlying functor: $\mathcal{F} \to \mathcal{B}$.

1.5.1. **Pull-backs and push-forwards.** Given a morphism (v, f) from one Feynman category $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \imath)$ to another $\mathfrak{F}' = (\mathcal{V}', \mathcal{F}', \imath')$, there is a pull-back $f^* : \mathcal{F}'\text{–}\mathcal{O}ps_{\mathcal{C}} \to \mathcal{F}\text{–}\mathcal{O}ps_{\mathcal{C}}$ given by sending $\mathcal{O}$ to $\mathcal{O} \circ f$. The same is true for $\mathcal{M}ods$ using v. These are compatible by the conditions on morphisms of FCs.

Suppose that the category $\mathcal{C}$ is cocomplete Then it is possible to construct a left Kan extension along a functor $f \colon \mathcal{F} \to \mathcal{F}'$. Further assuming that the monoidal product in $\mathcal{C}$ preserves colimits in both variables we will show that this transforms symmetric monoidal functors to symmetric monoidal functors.

The value on $X' \in \mathcal{F}'$ is then

$$f_*\mathcal{O}(X') = \mathrm{colim}_{(f \downarrow X')}\mathcal{O} \circ P \tag{1.10}$$

where P is the projection $P(Y, \phi : f(Y) \to X) = Y$ [ML98].

Theorem 1.15. *$Lan_f\mathcal{O}$ is a monoidal functor which we denote as push–forward $f_*\mathcal{O} := Lan_f\mathcal{O}$. Moreover f_*, f^* form an adjunction of symmetric monoidal functors between symmetric monoidal categories.*

Proof. We first check that the functor is indeed monoidal. Let $X' \simeq \bigotimes_{v' \in V} *'_v$ then each $\phi : f(Y) \to X$ splits into $\phi_v \colon f(Y)_v \to *'_v$ since $\mathcal{F}'$ is a Feynman category. Decomposing $Y \simeq \bigotimes_{w \in W} *_w$ and using the fact that f is a morphism of Feynman categories, we have $f(Y) \simeq \bigotimes f(*_w) \simeq \bigotimes f(Y)_v$. Now since the $f(*_w)$ have length one, we see that there is an isomorphism $f(Y)_v \simeq \bigotimes_{w \in W_v} f(*_w)$ for each v such that $\amalg W_v = W$. Summing up we showed that $f(Y)_v \simeq f(Y_v)$ with $Y_v = \bigotimes_{w \in W_v} *_w$. Thus we decomposed the objects (Y, ϕ) in the comma category into factors of the form (Y_v, ϕ_v). The morphisms in the comma category factor likewise. Taking the colimits we get the desired result by using that the monoidal product respects colimits in each variable and again using composition of morphisms in one directions and decomposition according to condition (ii) in the other direction. The symmetric structure is straightforward. For the adjointness, we notice that f^* is surely a symmetric monoidal functor. Now f_* is by construction adjoint as a functor. This can also be checked explicitly. The morphisms ϵ and η again are by inclusion of the identity component and composition of morphism. Like before writing out the adjunction maps one can see that the induced comonoidal structure is indeed strong monoidal due to the fact that we assumed tensor preserves colimits in each variable. □

Remark 1.16. In this language the free functor can be understood as follows. The identity $id_{\mathcal{V}}$ together with inclusion functor $inc : Iso(\mathcal{F}) \to \mathcal{F}$ is a morphism of Feynman categories $(\mathcal{V}, Iso(\mathcal{F}), \imath^{\otimes}) \overset{i=(id_{\mathcal{V}}, inc)}{\longrightarrow} (\mathcal{V}, \mathcal{F}, \imath)$. Noticing that $\mathrm{colim}_{Iso(\mathcal{F}\downarrow *)}\mathcal{O} \circ s = \mathrm{colim}_{(i \downarrow *)}\mathcal{O} \circ P$, one obtains that the pull-back i^* is the forgetful functor *forget* and the push-forward i_* is the free functor *free*.

Remark 1.17. One could also consider right Kan extensions, but they are not always well behaved with respect to the monoidal structure. They do provide an “extension by zero” in some cases, we will denote these by $i_!$.

2. EXAMPLES

In this section, we give old and new examples of Feynman categories. We start with those indexed over a basic category $\mathfrak{G}$ and is variations. These yield all the usual suspects, like operads, colored operads, PROPs, properads, cyclic operads, modular operads, etc..

Furthermore, many of the constructions like the modular envelope or the PROP generated by an operad can be understood via push–forwards and pull–backs. This also allows us to give not so classical notions and their relations. Such as ungraded-modular operads and nc-modular and ungraded nc-modular operads and their relation to di–operads, PROPs, etc., which are for instance helpful in string topology constructions, see e.g. [Kau07a].

The next set of examples we treat are algebras over (colored) operads. In our theory these are on the same footing. So in particular, we get free algebras this way.

Notation 2.1. In the following if $\mathcal{V}$ is actually a subcategory of $\mathcal{F}$, we just write $\imath$ for the inclusion without further ado. Also, since it often happens that the categories we regard are subcategories which contain *all objects*, we recall that the standard nomenclature for these subcategories is *wide* subcategories.

All the Feynman categories appearing here have a fully faithful functor to $\mathcal{S}et, \amalg$ and hence Lemma 1.2 applies.

2.1. The Feynman category $\mathfrak{G} = (\mathcal{C}rl, \mathcal{A}gg, \imath)$ and categories indexed over it.

Definition 2.2. Let $\mathcal{G}raphs$ be the category whose objects are abstract graphs and whose morphisms are the morphisms described in Appendix A. We consider it to be a monoidal category with monoidal product $\amalg$ (see Remark A.3).

We let $\mathcal{A}gg$ be the full subcategory of disjoint unions (aggregates) of corollas. And we let $\mathcal{C}rl$ be the groupoid subcategory whose objects are corollas and whose morphisms are the isomorphisms between them. We denote the inclusion of $\mathcal{C}rl$ into $\mathcal{A}gg$ by $\imath$.

The main observation for "graph based operad like things" is the following.

Proposition 2.3. *The triple* $\mathfrak{G} = (\mathcal{C}rl, \mathcal{A}gg, \imath)$ *is a Feynman category.*

Proof. By construction $Iso(\mathcal{A}gg)$ is equivalent to $\mathcal{C}rl^{\otimes}$ where the morphisms $\imath^{\otimes}$ map the abstract $\otimes$ to $\amalg$. The condition (i) is obviously satisfied on objects. For the equivalence, we first check the condition (ii). A morphism from one aggregate of corollas $X = \amalg_{w \in W} *_w$ to another $Y = \amalg_{v \in V} *_v$ is given by $\phi = (\phi^F, \phi_V, i_\phi)$. We let $\phi_v : \amalg_{w \in \phi^{-1}(v)} *_w \to *_v$ be given by the restrictions (see Appendix A). This yields the needed decomposition. Now any isomorphism thus also decomposes in this fashion and in this case we see that the decomposition yields a decomposition into isomorphisms of corollas and hence $\imath^{\otimes}$ induces and equivalence of $\mathcal{C}rl^{\otimes}$ with $Iso(\mathcal{A}gg)$. □

2.1.1. **Morphisms in $\mathcal{A}gg$ and graphs: a secondary structure.** As many users are used to graphs appearing in operadic contexts, we wish to make their role clear in the current setup. The objects in $\mathcal{A}gg$ are by definition just aggregates of corollas, so they are very simple graphs. Notice we do not take any other graphs as objects. All the structure is in the morphisms. These however have underlying graphs, which are up to *important details* the graphs readers might be expecting to appear. Since this might be a source of confusion, we review the details here carefully.

(1) A morphism ϕ in $\mathcal{A}gg$ from an aggregate $X = \amalg_{v \in V} *_v$ to another aggregate $Y = \amalg_w *_w$ gives rise to a (non–connected) graph $\mathbb{\Gamma}(\phi)$, viz. its ghost graph (see Appendix A). This is the graph $\mathbb{\Gamma}(\phi) = (V_X, F_X, \hat{\imath}_\phi, \partial_X)$ where $\hat{\imath}_\phi$ is the extension of i_ϕ to all flags of X by being the identity on all flags in the image of ϕ^F. This is the graph obtained from X by using orbits of i_ϕ as edges.

The "extreme" cases occur when either $\mathbb{\Gamma}(\phi)$ is connected, in which case the morphism is a called a *contraction*, or when $\mathbb{\Gamma}(\phi)$ has no edges in which case the morphism is called a merger. Here we follow the terminology of [BM08]. For a merger, ϕ^F is a bijection and i_ϕ is hence the unique map from $\emptyset$ to itself. Following the proof of Proposition 2.3 shows that in $\mathcal{Agg}$ the associated graph $\mathbb{\Gamma}(\phi)$ is the disjoint union of the $\mathbb{\Gamma}_v(\phi) := \mathbb{\Gamma}(\phi_v)$, we call these the associated graphs.

This graph however, *does not* (even up to isomorphism) determine the morphism ϕ. One crucial piece of missing information is the map ϕ_V which tells which components need to be merged. That is, we need to know which components of $\mathbb{\Gamma}(\phi_v)$ correspond to the map ϕ_v.

More precisely, to reconstruct the map ϕ we need to have (a) the graph $\mathbb{\Gamma}(\phi)$, (b) the decomposition $\mathbb{\Gamma}(\phi) = \amalg_{v \in V_Y} \mathbb{\Gamma}_v(\phi)$ (notice the $\mathbb{\Gamma}_v$ do not have to be connected), and (c) the isomorphism ϕ^F restricted to its image. The data (b) then is equivalent to the map ϕ_V which is the map sending all vertices in ϕ_v to v. The map i_ϕ is just the restriction of $\hat{i}_{\mathbb{\Gamma}(\phi)}$ to its elements which have 2 element orbits. If we only have (a) and (b), then the data fixes a map up to the isomorphism ϕ^F restricted onto its image which is the same as a morphism up to isomorphism on the target and source. Fixing (a) only we do not even get a class up to isomorphism, since the morphism ϕ_V is unrecoverable.

(2) If $Y = *_v$ is a corolla, then the situation is slightly better. Namely there is only one associated graph and this is just the underlying graph $\mathbb{\Gamma}(\phi) = \mathbb{\Gamma}_v(\phi)$. So that although on all of $\mathcal{Agg}$ morphisms are not determined just by their underlying graph, up to isomorphism this is possible for the comma category $(\mathcal{Agg} \downarrow \mathcal{Crl})$. This fact has the potential to cause confusion and it is the basic reason why, in many descriptions of operad like structures, graphs arise as objects rather than morphisms.

2.1.2. **Compositions in $\mathcal{Agg}$ compared to morphisms of the underlying graphs.** This section can be skipped. It is intended for the reader that is used to dealing with graphs as objects not as underlying objects of morphisms and gives details to clarify the relationship and differences. The morphisms of graphs of this section appear in $\mathcal{Agg}$ as compositions in the following way. Let ϕ_0, ϕ_1 and ϕ_2 be morphisms such that $\phi_0 = \phi_1 \circ \phi_2$

$$\begin{tikzcd}
X \arrow[r, "\phi_2"] \arrow[rr, bend right, "\phi_0"'] & Y \arrow[r, "\phi_1"] & *
\end{tikzcd}$$

Now let $\mathbb{\Gamma}_v$ be the associated graphs of ϕ_2, and $\mathbb{\Gamma}_0$ and $\mathbb{\Gamma}_1$ be the associated graphs to ϕ_0 and ϕ_1 respectively.

So picking a basis, $Y \simeq \amalg_{v \in V} *_v$ and $X_v \simeq \amalg_{w \in V_v} *_{v_w}$ up to isomorphism the diagram above corresponds to the diagram

$$\begin{tikzcd}
\amalg_v \amalg_{w \in V_v} *_w \arrow[r, "\amalg_v \mathbb{\Gamma}_v"] \arrow[rr, bend right, "\mathbb{\Gamma}_0"'] & \amalg_v \arrow[r, "\mathbb{\Gamma}_1"] & *
\end{tikzcd} \text{ up to isomorphisms.}$$

Now going through the definitions, we see that there is a morphism of graphs $\psi : \mathbb{\Gamma}_0 \to \mathbb{\Gamma}_1$ in $\mathcal{Graphs}$, such that for $v \in V(\mathbb{\Gamma}_1)$ the $\mathbb{\Gamma}_v$ are the inverse images of the respective vertices, that is the given restrictions of ψ to the vertices v.

Under the caveat that all statements about the associated graphs are only well defined up to isomorphisms, the composition of graphs in $\mathcal{A}gg$ is related to morphisms of graphs in $\mathcal{G}raphs$ as follows. To compose morphisms of aggregates, we *insert* a collection of graphs into the vertices, that is we blow up the vertex v into the graph $\mathbb{\Gamma}_v$. Reinterpreting the graphs $\mathbb{\Gamma}(\phi)$ as objects, *as one actually should not*, one can view the composition as a morphism from Γ_1 to Γ_0 as being given by the inverse operation, that is contracting the subgraphs Γ_v of Γ_1.

2.1.3. **Physics parlance.** The composition by insertion is again related to the physics terminology. Here we are thinking of the vertices as effective vertices, that is, we can expand them by inserting interactions. As Dirk Kreimer pointed out this leads to universal transfer of Connes–Kreimer [CK98] theory to Feynman categories; see §4.6.

2.1.4. **Directed, oriented and ordered versions of $\mathfrak{G}$.** Graphs may come with extra structures. A list of the most commonly used ones is included in Appendix A. Adding these structures as extra data to the morphisms, we arrive at Feynman categories indexed over $(\mathcal{C}rl, \mathcal{A}gg, \imath)$. Of course we then have to define how to *compose* the extra structure. Moreover, via pull–back, for any Feynman category indexed over $(\mathcal{C}rl, \mathcal{A}gg, \imath)$ we can add the extra structure to the morphisms as we will explain.

The most common indexing is over directed graphs. We will also introduce the relevant categories with orders on the edges and orientations on the edges. The latter two can be understood as analogous to ordered and oriented simplices. We will consider other extra structures as needed.

1. $\mathfrak{G}^{dir} = (\mathcal{C}rl^{dir}, \mathcal{A}gg^{dir}, \imath^{dir})$. $\mathcal{C}rl^{dir}$ has directed corollas as its objects and isomorphisms of directed corollas as its morphisms. $\mathcal{A}gg^{dir}$ is the full subcategory of aggregates of directed corollas in the category of directed graphs; see Appendix A.
2. $\mathfrak{G}^{ord} = (\mathcal{C}rl^{ord}, \mathcal{A}gg^{ord}, \imath^{ord})$ and $\mathfrak{G}^{or} = (\mathcal{C}rl^{or}, \mathcal{A}gg^{or}, \imath^{or})$. The underlying objects in both cases are the same as in $\mathcal{A}gg$ and $\mathcal{C}rl$. A morphism is a pair of a morphism in $\mathcal{A}gg$ or $\mathcal{C}rl$ *together* with an orientation or order of the set of all the edges of the associated graphs.
3. We of course can have both, direction of edges and an orientation/order of all edges.

Definition 2.4. Given a Feynman category $(\mathcal{V}, \mathcal{F}, \jmath)$ indexed over $(\mathcal{C}rl, \mathcal{A}gg, \imath)$ with base functor B, we define the oriented/ordered version $(\mathcal{V}, \mathcal{F}^{or/ord}, \jmath)$ as the Feynman category whose objects are the same, but whose morphisms are pairs (ϕ, ord) where ϕ is a morphisms in $\mathcal{F}$ and ord is an order/orientation on the edges of $\mathbb{\Gamma}(B(\phi))$, and composition on the ord factors is the one given above.

For $\mathcal{F}^{dir}$ and $\mathcal{V}^{dir}$ we take as objects pairs $(X, F_{B(X)} \to \{in, out\})$ with X an original object. Morphisms $(X, F_{B(X)} \to \{in, out\})$ to $(Y, F_{B(Y)} \to \{in, out\})$ are now those morphisms, for which $B(\phi)$ is a morphism in $\mathfrak{G}^{dir}$ from $(B(X), F_{B(X)} \to \{in, out\})$ to $(B(Y), F_{B(Y)} \to \{in, out\})$.

Notice that this is just the pull–back or change of base in the category of Feynman categories.

2.2. **Feynman categories indexed of $\mathfrak{G}^{dir}$.** The most commonly known types of structures are operads, PROPs and their variations. These are all indexed over directed graphs, and so we will start with these, although the more basic ones have less structure. They are however all based on subcategories of $\mathfrak{G}^{dir}$. Of course $\mathfrak{G}^{dir}$ is indexed over $\mathfrak{G}$, so that these examples are also indexed over $\mathfrak{G}$. For operads the subcategories of $\mathfrak{G}^{dir}$ are actually the most restricted.

2.2.1. **Operads.** The Feynman category for operads $\mathfrak{O} = (\mathcal{C}rl^{rt}, \mathcal{O}pd, \imath)$ is given as follows: $\mathcal{C}rl^{rt}$ are directed corollas with exactly one "out" flag (root) and their isomorphisms as directed graphs, that is permutations of the "in" flags. $\mathcal{O}pd$ has as objects aggregates (disjoint unions) of such corollas and as morphisms those morphisms of oriented graphs whose associated graphs $\mathbb{\Gamma}(\phi_v)$ are (connected) trees. Notice these trees are actually oriented and rooted. The root of $\mathbb{\Gamma}(\phi_v)$ is given by the image of the root flag of v.

Classical notation. If $\mathcal{O}$ is a strong monoidal functor $\mathcal{O}pd \to \mathcal{C}$ then the classical notation is $\mathcal{O}(S) := \mathcal{O}(*_{S+})$ where $S+ = S \amalg 0$ and 0 is the root. A basic morphism is a morphism ϕ_s from $*_{S+} \amalg *_{T+}$ which connects the flag $s \in S$ to the flag $0 \in T+$ and contracts it yielding $*_{(S \setminus s \amalg T)+}$ with ϕ^F being id. The underlying graph $\mathbb{\Gamma}(\phi_s)$ is the tree with one edge consisting of the flags s and 0. The usual notation is $\circ_i := \mathcal{O}(\phi)$. The usual associativity and $\mathbb{S}$ equivariance (see e.g. [MSS02]) are now *consequences* of the structure of $\mathfrak{O}$. Another fact which is due to the structure of $\mathfrak{O}$ is that the $\circ_s$ and the isomorphisms $\sigma \in Aut(S)$ actually generate all morphisms. For a thorough discussion see Lemma 5.2.

Notice that $\mathfrak{O}$–$\mathcal{O}ps$ are sometimes called non–unital pseudo–operads.

We will call operads in the old nomenclature May operads. The Feynman category for these again has $\mathcal{V} = \mathcal{C}rl^{rt}$ but the category $\mathcal{F}$ is the wide subcategory of $\mathcal{O}pd$ whose underlying graphs are level trees, with all inputs on one level; see [Mar08, MSS02].

Based version. Let $\mathcal{C}rl^{rt}_{\mathbb{N}}$ be the directed corollas with index sets $[n] = \{0, 1, 2, \dots, n\}$, $n \in \mathbb{N}_0$ with 0 being the "out" flag. The automorphism group of such a corolla is naturally identified with $\mathbb{S}_n$. Here we can choose the isomorphisms $\sigma \in \mathbb{S}_n$ and the ϕ_i as generators. Notice that in the unbiased version, to obtain $\mathcal{O}(\phi)$ the composition along $\mathbb{\Gamma}(\phi)$ is followed by an isomorphism.

Restricting $\mathcal{O}$ to $\mathcal{C}rl^{rt}$ we get a collection of elements $\mathcal{O}(n)$ of $\mathcal{C}$ with an $\mathbb{S}_n$ action. This identifies $\mathcal{C}rl^{rt}$–$\mathcal{M}ods$ with so–called $\mathbb{S}$ or Σ modules.

Given a morphism $\phi \in Hom_{\mathcal{O}pd}(*_{[n_1]} \amalg \cdots \amalg *_{[n_k]}, *_{[m]})$ the morphism $\mathcal{O}(\phi)$ from $\mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_k) \to \mathcal{O}(m)$ is the composition along the tree $\mathbb{\Gamma}(\phi)$.

Functors The natural inclusion map $i \colon \mathcal{A}gg^{rt}_{\mathbb{N}} \to \mathcal{A}gg^{rt}$ and its restriction to $\mathcal{C}rl^{rt}_{\mathbb{N}}$ is a morphism of Feynman categories. The pull–back i^* is just the restriction and the push–forward i_* if it exists yields the usual extension via colimits, e.g. $\mathcal{O}(S) = \bigoplus_{S \leftrightarrow \bar{n}} \mathcal{O}(n)_{\mathbb{S}_n}$.

If $\mathcal{C}$ is cocomplete and tensor preserves colimits in each variable, then i^* and i_* give an equivalence of categories, both between the $\mathcal{M}ods$ and the $\mathcal{O}ps$.

2.2.2. ***(Wheeled) PROPs.*** Let $\mathcal{A}gg^{dir,nl}$ be the wide subcategory whose morphisms satisfy the condition that the graphs $\mathbb{\Gamma}(\phi_v)$ associated to ϕ do not contain any oriented cycles. Set $\mathfrak{P} = (\mathcal{C}rl^{dir}, \mathcal{A}gg^{dir,nl}, \imath)$.

Strictly speaking $\mathfrak{P}$–$\mathcal{O}ps$ could be called non–unital pseudo–PROPs. For the classical version of non–unital PROPs one has to require that the underlying graphs of the morphisms have levels and all input tails are on one level and all output tails are one level; see e.g. [Mar08]. We will call the classical version strict PROPs and call the $\mathfrak{P}$–$\mathcal{O}ps$ simply PROPs.

Wheeled version. Dropping the condition about oriented cycles, we obtain the Feynman category underlying the notion of wheeled (non–unital) PROPs, which is simply $\mathfrak{G}^{dir}$.

Based Versions. In the above two constructions, we can again replace the categories by equivalent small categories $(\mathcal{V},\mathcal{F},\imath)$ as follows. For $\mathcal{V}$ these are given by oriented corollas indexed by $\bar{n} := \{1,\dots,n\}$ on the inputs and $\bar{m}$ on the outputs. Strictly speaking we use $S = \overline{n+m} \simeq \bar{n} \amalg \bar{m}$ together with the function that is *in* on the $i \leq n$ and *out* on the rest. The automorphism groups are then canonically isomorphic to $\mathbb{S}_n \times \mathbb{S}_m$ and we again get the usual picture. For $\mathcal{F}$ one has to take the full subcategory of the respective categories whose objects are aggregates of these corollas.

The generators are morphisms from two corollas to one. If we are in the strict case they are given by pairing the inputs of one corolla with the outputs of the other or a merger. In the non-strict case, there are given by partial pairings or mergers.

Functors Besides the biased/non–biased inclusion, which again yields an equivalence of $\mathcal{O}ps_{\mathcal{C}}$ for suitable $\mathcal{C}$, there are other interesting morphisms. Inclusion gives rise to a morphism of Feynman categories $i\colon \mathfrak{O} \to \mathfrak{P}$. The pull–back i^* is again restriction. This is the operad contained in a PROP. If it exists $i_*\mathcal{O}$ is the PROP generated by an operad and if $\mathcal{C}$ is say abelian complete monoidal with the product preserving limits in both variables, then $i_!$ is the extension by 0 of the operad to a PROP.

2.2.3. **(Wheeled) properads.** For properads, one uses the $\mathcal{C}rl^{dir}$ and the wide subcategory of $\mathcal{A}gg^{dir,nl}$ whose morphisms have connected associated graphs. For the wheeled version, one uses the same condition in $\mathcal{A}gg^{dir}$.

2.2.4. **Dioperads.** Here as above we use $\mathcal{C}rl^{dir}$ and the wide subcategory of $\mathcal{A}gg^{dir}$ whose morphisms are connected trees and call the resulting Feynman category $\mathfrak{D}$.

2.2.5. **More classical examples.** Furthermore, one can quickly write down the Feynman categories for other species such as $\frac{1}{2}$-PROPs and so on.

2.3. Feynman categories indexed over $\mathfrak{G}$.

2.3.1. **Cyclic operads.**

Unbiased version. Consider the Feynman category given by $\mathfrak{C} = (\mathcal{C}rl, \mathcal{C}yc, \imath)$ where $\mathcal{C}yc$ is the wide subcategory of $\mathcal{A}gg$ whose morphisms are only those morphisms ϕ for which each $\mathbb{\Gamma}(\phi_v)$ in the decomposition of condition (ii) of Definition 1.1 is a tree and hence by definition connected. In this definition it is clear that this Feynman category is indexed over $\mathcal{A}gg$.

Classical notation. If $\mathcal{O}$ is a functor from $\mathcal{C}yc$ to $\mathcal{C}$ then set $\mathcal{O}(S) := \mathcal{O}(*_S)$. The basic morphisms are those from two corollas $*_S$ and $*_T$ to a corolla $*_U$ where the ghost tree has a single edge (s,t). If S and T are disjoint and $U = S \setminus \{s\} \cup T \setminus \{t\}$, then we can use $id = \phi^F$ and denoting this morphism by $\phi_{s,t}$, usually one sets $\mathcal{O}(\phi_{s,t}) = {}_s\circ_t$.

Biased version. Let $\mathcal{C}rl_{\mathbb{N}_0}$ be the corollas indexed by the sets $[n]$ and let $\mathcal{C}yc_{\mathbb{N}}$ be the full subcategory of $\mathcal{C}yc$ whose objects are aggregates of objects of $\mathcal{C}rl_{\mathbb{N}}$. Then $\mathfrak{C}_{\mathbb{N}_0} = (\mathcal{C}rl_{\mathbb{N}_0}, \mathcal{C}yc_{\mathbb{N}_0}, \imath)$ is a Feynman category. The classical notations are $\mathcal{O}(n) := \mathcal{O}((n+1)) := \mathcal{O}([n])$. Notice there is now an $\mathbb{S}_{n+} = Aut([n]) \simeq \mathbb{S}_{n+1}$ action on $\mathcal{O}(n)$ and furthermore the $\circ_s := {}_0\circ_s$ for $s \neq 0$ do generate with the associativity and $\mathbb{S}$ equivariance, but due to the $\mathbb{S}_{n+}$ action there is an additional relation, which is the well known relation for cyclic operads $(a \circ_1 b)^* = b^* \circ_n a^*$, where $*$ is the action of $(01\dots n) \in \mathbb{S}_{n+}$ and where a is in $\mathcal{O}(n)$. Again in this formalism, this is a *consequence*.

Pull–back and push–forward functors There is an obvious inclusion functor of Feynman categories $i : \mathfrak{C}_{\mathbb{N}_0} \to \mathfrak{C}$. Now i^* is just the restriction and i_* is the usual colimit formula $\mathcal{O}(S) = \bigoplus_{S \leftrightarrow [|S-1|]} \mathcal{O}((|S|))$ for target categories $\mathcal{C}$, where these exist. In that case i^* and i_* are not only adjoint, but yield an equivalence.

There is also a forgetful functor $inc : \mathfrak{D} \to \mathfrak{C}$ which forgets the direction. Now inc^* is the underlying operad of the cyclic operad. This is often used in the biased version. Furthermore, there is the push–forward, which is the free cyclic operad on an operad. on $\mathcal{V}$. This is induction from $\mathbb{S}_n$ to $\mathbb{S}_{n+}$.

2.3.2. **Ungraded (nc)–modular operads.** We call $\mathfrak{G}$-$\mathcal{O}ps_{\mathcal{C}}$ ungraded nc–modular operads in $\mathcal{C}$. Here "nc" stands for non–connected and ungraded for the fact that there is no genus or Euler characteristic labeling.

Connected version. Let $\mathcal{A}gg^{ctd}$ be the wide subcategory of $\mathcal{A}gg$ whose morphisms are morphisms whose associated graphs $\mathbb{\Gamma}_v(\phi)$ are all connected. Then $\mathfrak{G}^{ctd} = (\mathcal{C}rl, \mathcal{A}gg^{ctd}, \imath)$ is a Feynman category and we refer to $\mathfrak{G}^{ctd}$-$\mathcal{O}ps_{\mathcal{C}}$ as the category of ungraded modular operads.

Biased version/MOs. For the biased version we consider $\mathcal{C}rl_{\mathbb{N}}$, the full subcategory of $\mathcal{C}rl$ with objects $*_{\bar{n}}$ and $\mathcal{A}gg_{\mathbb{N}}$ the full subcategory of $\mathcal{A}gg$ with objects aggregates of $\mathcal{C}rl_{\mathbb{N}}$ then $\mathfrak{G}_{\mathbb{N}} = (\mathcal{C}rl_{\mathbb{N}}, \mathcal{A}gg_{\mathbb{N}}, \imath)$.

Some variant of these objects has previously been defined by Schwarz in [Sch98] and were called MOs. This identification uses the fact that all morphisms in $\mathcal{A}gg$ are generated by merging two corollas and contracting one small ghost loop on a corolla, see 5.1. One has to be careful, since one can only write down generators using indexing sets which is why MOs were basically defined on invariants, see §[Sch98] and [HVZ10].

Functors. Via the inclusion the biased and unbiased versions give equivalent $\mathcal{O}ps$, when the push–forwards exist.

There are also inclusion functors $\mathfrak{C} \xrightarrow{\imath} \mathfrak{G}^{ctd} \xrightarrow{\jmath} \mathfrak{G}$. The pull–backs are restrictions, if they exist i_* is the ungraded modular envelope and $i_!$ the extension by zero. The push–forward j_* is analogous to the free PROP construction, it produces the free ungraded nc–modular operad. Here mergers just go to tensor products. Again, if it exists $j_!$ is the extension by zero.

2.3.3. **Modular and nc–modular operads.**

NC modular operads. Let $\mathcal{C}rl^{\gamma}$ be γ labelled corollas with their isomorphisms. A γ labelling in this case is just a number, that is the label of the lone vertex. The category $\mathcal{A}gg^{\gamma}$ has as objects aggregates of γ labelled corollas. A morphism $\phi \in \mathcal{A}gg^{g}(X,Y)$ is a morphism of the underlying corollas, with the restriction that for each vertex $v \in Y$, $\mathbb{\Gamma}(\phi_v)$ has $\gamma(\mathbb{\Gamma}(\phi_v)) = \gamma(v)$. Here the genus labeling of $\mathbb{\Gamma}(\phi_v)$ is inherited from X and γ of a graph is defined in (A.2). We let $\mathfrak{M}^{nc} = (\mathcal{C}rl^{\gamma}, \mathcal{A}gg^{\gamma}, \imath)$ and call $\mathfrak{M}^{nc}$–$\mathcal{O}ps_{\mathcal{C}}$ nc modular operads in $\mathcal{C}$.

Forgetting the marking gives a morphism to $\mathfrak{G}$ and hence an indexing.

Modular operads.

The category $\mathcal{A}gg^{\gamma,ctd}$ is the wide subcategory of $\mathcal{A}gg^{\gamma}$, with the restriction on morphisms ϕ that each associated graph $\mathbb{\Gamma}(\phi_v)$ is connected. Set $\mathfrak{M} = (\mathcal{C}rl^{\gamma}, \mathcal{A}gg^{\gamma,ct}, \imath)$ and call $\mathfrak{M}$–$\mathcal{O}ps_{\mathcal{C}}$ modular operads in $\mathcal{C}$.

Biased versions. There are biased versions built on $\mathcal{C}rl^{\gamma}_{\mathbb{N}_0}$. As above this yields the standard notation $\mathcal{O}((g,n)) := \mathcal{O}(g,n-1) := \mathcal{O}(*_{[n]}, \gamma(*_{[}n]))$ for functors $\mathcal{O}$. Via the inclusion the biased and unbiased versions give equivalent $\mathcal{O}ps$, when the push–forwards exist.

Functors.

Assigning 0 as the value of Γ to each unlabelled corolla yields an inclusion functors $\mathfrak{C} \xrightarrow{i} \mathfrak{M}$ and then there is the natural inclusion $\mathfrak{M} \xrightarrow{j} \mathfrak{M}^{nc}$. The pull–backs are restrictions, if they exist i_* is modular envelope and $i_!$ the extension by zero.

2.3.4. **Not so classical examples.** Other structures, which were up to now not ordered under some overall algebraic structure, but can be neatly expressed in terms of Feynman categories are the *C/O structures* of [KP06], and one can now readily define *NC C/O structures* by allowing disconnected surfaces.

Another relevant example are the algebraic structures underlying the paper [HVZ10]. Here we have the Feynman category for two–colored (NC) modular operads which has additional morphisms changing the color. These cannot be handled in any previous framework.

Finally, in homological mirror symmetry several other moduli spaces occur, which give rise to Feynman categories. These notions have not been formalized yet, but once this is done, our full theory applies to them.

2.3.5. **Functors.** There are now several functors between Feynman categories that one can write down which make some constructions categorical and illuminate the relation between their respective categories of $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$.

First there is the forgetful functor $\imath\colon \mathfrak{M}^{nc} \to \mathfrak{G}$ which forgets the genus marking. Secondly there is a forgetful functor $\jmath\colon \mathfrak{G}^{dir} \to \mathfrak{G}$ which forgets the direction. So given a wheeled PROP, that is a functor $\mathcal{O}\colon \mathfrak{G}^{dir} \to \mathcal{C}$, we get an associated modular operad $j^*i_*\mathcal{O}$. Vice–versa given an nc–modular operad, we get an associated wheeled PROP.

If we consider modular operads, then there is a natural map from wheeled properads to unmarked modular operads, given by pushing along the forgetful map $\mathfrak{P}^{ctd} \to \mathfrak{G}^{ctd}$ and pulling along the forgetful map $\mathfrak{M} \to \mathfrak{G}^{ctd}$. Vice–versa, we get a properad and even a PROP from a modular operad. This is for instance the case in string topology and the Hochschild action of [Kau07a, Kau08a].

Other interesting relations exist along these lines between operads, dioperads, properads, PROPs and modular operads.

2.4. **Colored versions.** We obtain the colored versions if we use colored corollas and morphisms of graphs, which preserve the coloring in the sense that the flags of the ghost edges have the same color.

2.5. **Planar versions.** Here we will use the special planar subcategories given in Appendix A §A.2.4.

2.5.1. **Non–Σ cyclic operads.** The relevant Feynman category is $\mathfrak{C}^{pl} = (\mathcal{C}rl^{pl}, \mathcal{C}yc^{pl}, \imath)$ There is a biased version using only $\mathcal{C}rl^{pl,dir}_{\mathbb{N}}$. Forgetting the planar structure gives a morphism $(v, f) : \mathfrak{C} \mapsto \mathfrak{C}^{pl}$ and hence this is again a Feynman category indexed over $\mathfrak{G}$. There is a biased version using only $\mathcal{C}rl^{pl,dir}_{\mathbb{N}_0}$.

2.5.2. **Non–Σ operads.** The relevant Feynman category is $\mathfrak{O}^{pl} = (\mathcal{C}rl^{pl,dir}, \mathcal{O}pd^{pl}, \imath)$. There is a biased version using only $\mathcal{C}rl^{pl,dir}_{\mathbb{N}}$. Here again there is a forgetful morphism $\mathfrak{O}^{pl} \to \mathfrak{C}$ forgetting the root.

2.6. $\mathcal{O}ps$ **with special elements: units, unit elements and multiplication.** Often, we have the situation that there are some special elements, which one would like to take into consideration. These are allowed, since we have the freedom to have morphisms in $\mathcal{F}$ from the unit element to any other element. If we already have a Feynman category, we can simply adjoin these morphisms to the monoidal category.

2.6.1. $\mathcal{O}ps$ **for an added multiplication.** A multiplication is generally an element in $\mathcal{O}(2)$. Given any of the Feynman categories above, we define its version with multiplication as follows. Add a new morphism from $\mathbb{1}_\emptyset$ to $*_{[2]}$, for a corolla of the correct type and complete the morphisms by including this element.

Let us be explicit for OPERADS WITH A MULTIPLICATION. Here we adjoin a new arrow $m : \mathbb{1}_\emptyset \to *_{[2]}$ monoidally. Call the resulting category $\mathfrak{O}_m$. This will have arrows which are composable words in old arrows and in $id \amalg \cdots \amalg id \amalg m \amalg id \amalg id$. We can depict such a word by a rooted b/w tree, where the black vertices are trivalent —we can think of them as decorated by m— and the white vertices of arity k are decorated by a morphism from $\amalg_1^k *_{S_i}$ to $*_T$.

We can think of the underlying tree as the ghost tree of a particular morphism w' as follows. If there are k occurrences of m in a morphism from X to Y, consider the factorization of w

$$X \simeq X \amalg \mathbb{1}_\emptyset \amalg \cdots \amalg \mathbb{1}_\emptyset \overset{id \amalg m^k}{\to} X \amalg *_{[2]} \amalg \cdots \amalg *_{[2]} \overset{w'}{\to} Y$$

where now w' sends flags at the black vertices get sent to the new copies of $*_{[2]}$.

Picking a functor $\mathcal{O}$, the morphism m gets sent to $\mathcal{O}(m) : \mathbb{1} \to \mathcal{O}(2)$, which is nothing but an element of $\mathcal{O}(2)$.

ASSOCIATIVE/COMMUTATIVE MULTIPLICATION. If the multiplication is supposed to be commutative, we take m to be appropriately symmetric. If it is to be associative, we quotient by the relation that $id \amalg m \circ m = m \amalg id \amalg m$ from $\mathbb{1}_\emptyset \to *_{[2]} \amalg *_{[2]}$. We can then collapse the black subtrees and get trees which are almost bi–partite as is [Kau07b, Kau05].

2.6.2. $\mathcal{O}ps$ **with a unary morphism.** This is the same procedure as above, using a corolla with 2 flags of the appropriate type. The underlying graphs will then again be b/w with black vertices of valence 2.

DIFFERENTIAL If we are adding a differential, we impose that $m^2 = 0$, which can only be done in the enriched version. This means that we set to 0 any tree with more that one black vertex inserted into an edge.

OPERADIC UNIT. If we are adding a unit, we impose $\phi \circ m = (id \amalg \cdots \amalg id \amalg m \amalg id \amalg id) \circ \phi = \phi$, for $\phi \in (\mathcal{F} \downarrow \mathcal{V})$. In a graph picture the black vertices, can be erased. Erasing the black vertex from a corolla with 2 flags, leaves a degenerate graph which is just a lone flag/edge. This is the presentation for graphs that Markl has used in [Mar08]. Formally, it is actually a colored empty graph.

2.6.3. $\mathcal{O}ps$ **with A_∞ multiplication.** Another example are operads with an A_∞ multiplication. In this case we adjoin morphisms $m_n : \mathbb{1}_\emptyset \to *_{[n]}$ for $n \geq 1$ and then impose the standard relations.

2.7. **Truncation, stability and the role of $0, 1, 2$ flag corollas.** In all of the above pictures, one can truncate by omitting corollas with $0, 1$ or 2 flags or any unstably labeled corollas for the Feynman categories. We call these the truncated versions. Truncation is the just pull–back under the natural inclusion of Feynman categories.

There are several versions of this type of truncation. If one wishes to have a only a unit, one can omit $0, 1$ flag corollas, adjoin a unit in $\mathcal{O}((2))^1$ and restrict to functors, for which $\mathcal{O}((2))$ is restricted. E.g. a point or k in the augmented case.

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[1] Recall that $\mathcal{O}(n-1) = \mathcal{O}((n))$ in the operad case. So that the operadic identity is in $\mathcal{O}(1)$

2.7.1. **Ground monoid: $\mathcal{O}((0))$.** There are two ways to get a ground monoid. The first is to have corollas with no flags. These do appear in the non–directed versions, say by contracting two flags of a corolla with two flags.

If there are no mergers, then there is only the identity map on these corollas and inclusion is benign. If there are mergers then their inclusion introduces a ground monoid $S = \mathcal{O}(*_\emptyset)$ the associative monoidal product on S is given by the merger and all the $\mathcal{O}(*_T)$ become S–modules. If one wants to forget this structure, one can truncate or restrict to functors for which $\mathcal{O}(*_\emptyset) = \mathbb{1}$.

2.7.2. **Algebra elements/units $\mathcal{O}((1))$.** We have allowed corollas with only one flag. Given a functor $\mathcal{O}$, we hence automatically have elements in $\mathcal{O}((1))$ in the classic notation. If the category $\mathcal{C}$ has an initial object that is the monoidal unit then in lieu of truncating, we can restrict to functors $\mathcal{O}$ such that $\mathcal{O}((1))$ is that unit in order to essentially eliminate these types of units. It however turns out that sometimes this is not a good idea.

For instance this is part of the data when one wants to have $\mathcal{O}ps$ with multiplication *and* a unit for the multiplication. Or if one wants to work with pointed spaces.

2.7.3. **Semi–simplicial and simplicial structures.** If we allow 1–flag corollas and pick a specific element —we will call this the pointed case—, in all the cases above, we get a semi–simplicial structure by using the morphism $\delta_i : *_{[n]} \to *_{[n]} \amalg \emptyset \to *_{[n]} \amalg *_{[0]} \to *_{[n-1]}$ where the first morphism is the monoidal unit, the second is given by the chosen element and the third morphism has one virtual edge connecting 0 to the flag i.

If one has an associative multiplication, that is a morphism $\emptyset \to *_{[2]}$, a unit $\emptyset \to *_{[1]}$ and an algebra unit $\emptyset \to *_{[0]}$ which is a unit for the multiplication one gets a simplicial structure. Here associative means that one imposes the relation that two maps $\emptyset \amalg \emptyset \to *_{[2]} \amalg *_{[2]} \to *_{[3]}$ given by the morphisms with virtual edge with vertices 0, 1 and the one with virtual edge with flags 0, 2 are the same. And unit means that one imposes the relation that the two morphisms $\emptyset \to \emptyset \amalg \emptyset \to *_{[2]} \amalg *_{[0]} \to *_{[1]}$ given by virtual edges with flags 0,1 and 0,2 coincide with $\emptyset \to *_{[1]}$.

2.7.4. *K*–**collections $\mathcal{O}((2))$.** If one includes two–flag corollas, which are not represented by special elements as above, then one obtains a second ring structure $K := \mathcal{O}((1))$: with multiplication $\mu = \circ_1 : \mathcal{O}(1) \otimes \mathcal{O}(1) \to \mathcal{O}(1)$. Now the other $\mathcal{O}(n)$ are left monoid modules, but have a more complicated right multiplication over these. This was formalized as K–collections in the terminology of [GK94]. If one does not want to *a priori discard* this information, one can put this information into the target category *loc. cit.*.

2.7.5. **Orders and Orientations: The main examples.** We obtain another class of examples from the above, by using extra data on the morphisms. These will be main examples we will deal with as their $\mathcal{O}ps$ carry the algebraic structures discussed [KWZ12].

Ordered cyclic operads. In this case the category $\mathcal{V}$ is again $\mathcal{C}rl$ and the objects of $\mathcal{F}$ are the disjoint union of corollas. The new input is that a morphism is a morphism ϕ in $\mathcal{C}yc$ together with an order on the edges of the graphs Γ_v. Since there are no mergers or loops this is the same as a choice of decomposition of ϕ into no–loop edge contractions. The composition of morphisms is then just the composition of the two factorizations. In terms of the edges this is the lexicographical order obtained from the two morphisms.

Oriented cyclic operads. Analogous to the above, but only retaining an orientation of the edges, that is a class total orders under the equivalence relation of even permutations.

Ordered/oriented NC modular operads. Again using orders/orientations on the set of *all edges* we obtain ordered/oriented versions of these operads.

We will use these Feynman categories for instance to define $\mathfrak{K}$–modular operads.

2.8. **Algebra type examples.** We can also consider smaller examples having algebras as their $\mathcal{O}ps$. Here the category $\mathcal{V}$ has finitely many elements. This “explains” the appearance of trees/graphs in defining the relevant structures in these algebras.

2.8.1. **Algebras given by graphs.** There are several types of algebras encoded by graphs. These now naturally yield examples of Feynman categories indexed over graphs.

ASSOCIATIVE AND COMMUTATIVE ALGEBRAS. For instance for associative algebras, we let $\mathcal{V}$ be planar corollas with one output. There is then one morphism for each set of n such corollas to a given corolla with the sum of the number of outputs. This is the morphism given by the planted rooted tree with $n+1$ vertices where each vertex represented by the source corollas is attached by exactly one edge to the root.

Remark 2.5. Many other examples such as Lie or Pre-Lie are not of pure combinatorial type and thus need the enriched setup given in the next paragraph.

2.9. **Feynman categories with trivial $\mathcal{V}$.** We consider $\mathcal{V}$ the category with one object 1 with $Hom_{\mathcal{V}}(1,1) = id_1$. Then let $\mathcal{V}^{\otimes}$ be the free symmetric category and let $\bar{\mathcal{V}}^{\otimes}$ be its strict version. It has objects $n := 1^{\otimes n}$ and each of these has an $\mathbb{S}_n$ action: $Hom(n,n) \simeq \mathbb{S}_n$. Let $\bar{\imath}$ be the functor from $\mathcal{V}^{\otimes} \to \bar{\mathcal{V}}^{\otimes}$. $\mathcal{F}$ will be a category with $Iso(\mathcal{F})$ equivalent to $\bar{\mathcal{V}}^{\otimes}$, which itself is equivalent to the skeleton of $Iso(\mathcal{F}in\mathcal{S}et)$ where $\mathcal{F}in\mathcal{S}et$ is the category of finite sets with $\amalg$ as monoidal structure. Here, for convenience, we do use the strictification of $\amalg$.

2.9.1. **Finite sets with surjections.** Here we take $\mathcal{F}$ to be the wide subcategory $Surj$ of $\mathcal{F}in\mathcal{S}et$ with morphisms being only surjections. The inclusion $\imath$ is $\bar{\imath}$ followed by the identification of n with the set $\bar{n}$. Then $\mathfrak{F}$ is naturally a Feynman category, since $f^{-1}(X) = \amalg_{x \in X} f^{-1}x$. This Feynman category is a basic building block and we denote it by $\mathfrak{Sur}$. We may of course also use the skeleton $\aleph_0$ of $\mathcal{F}in\mathcal{S}et$ and surjections $\bar{m} \to \bar{n}$.

2.9.2. $\mathcal{F} = \boldsymbol{\Delta} S_{+}$**.** Another possibility is to use $\boldsymbol{\Delta} S$ as defined by Loday [Lod98] and augment it by adding a monoidal unit (the empty set). This is the category with objects $[n]$, $Aut([n]) \simeq \mathbb{S}_{n+1}$ and morphisms that decompose as $\phi \circ f$ where ϕ is in $Aut([n])$ and f is a non–decreasing map.

2.9.3. FI**–algebras.** In [CEF12] the wide category FI of $\aleph_0$ was considered, which has only the injections as morphisms. This again yields a Feynman category $\mathfrak{F}$. Here we start with $\bar{\mathcal{V}}^{\otimes}$ and adjoin the morphism $j : 1^{\otimes 0} = \emptyset \to 1$ to define $\mathcal{F}$, then $\mathcal{F}$ is equivalent to FI. Indeed let $i \colon X \to X'$ be an injection and let $I = \overline{|X \setminus X'|}$, then we can write it as $X \simeq X \amalg \emptyset \amalg \cdots \amalg \emptyset \xrightarrow{id \amalg j^I} X \amalg I \xrightarrow{i \amalg id} Im(i) \amalg I \simeq X'$. Now FI–algebras are regular functors from FI. This can be accommodated by passing to a free monoidal construction, see §3.1.

2.9.4. $\mathcal{F} = \mathcal{F}in\mathcal{S}et$**.** In order to incorporate injections, just like above, we adjoin a morphism $1^{\otimes 0} = \emptyset \to 1$ to the morphisms of $\mathcal{S}urj$ and obtain a category equivalent (even isomorphic) to $\mathcal{F}in\mathcal{S}et$. To show this notice that any given map of sets $f = i \circ s$ can be decomposed into the surjection s onto its image followed by the injection of the image into the target.

2.9.5. **Non–sigma versions, $\boldsymbol{\Delta}_+$.** If we take the version of Feynman categories, by the above, we can realize non–sigma operads, but also the augmented simplicial category $\boldsymbol{\Delta}_+$. To obtain augmented simplicial objects, we should then use the free monoidal construction, §3.1 to realize these as $\mathfrak{F}^{\otimes}$—$\mathcal{O}ps$.

2.10. **Remarks on relations to similar notions.** Looking at these particular examples, there are connections to PROPs, to Lavwere theories and crossed–simplicial groups.

2.10.1. **Crossed–simplicial groups.** There are two ways crossed simplicial groups can appear in the Feynman category context. First, we found the crossed simplicial group $\boldsymbol{\Delta}S$ above. This is since we started out with the groupoid given by the objects $[n]$ with their automorphisms and added the morphisms of $\boldsymbol{\Delta}$ in a way that the morphisms have the standard decomposition as set forth in [FL91]. We arrived at the objects and automorphisms through a symmetric monoidal category construction on a trivial category. We could likewise arrive at the braid groups if we were to consider braided monoidal categories instead. One could in this way, by altering the background of monoidal categories, achieve some of the symmetries in a free construction. From the fixed setup of symmetric monoidal categories this is not too natural.

The second way crossed simplicial groups can appear is in the wider context of non–trivial $\mathcal{V}$. For instance the datum of objects $[n]$ with automorphism groups G_n^{op} gives rise to a groupoid $\mathcal{V}$. We can then use the category $\boldsymbol{\Delta}G$ as $(\imath \downarrow \imath)$. This gives a set of basic morphisms, in general we may still choose morphisms for all of $(\mathcal{F} \downarrow \imath)$. The existence of these implements both these symmetries *and* a co–simplicial structure on $\mathcal{O}ps$. This is most naturally expressed as follows. Let $\mathfrak{F}_G = (\mathcal{V}, \mathcal{F}, \imath)$ be given by letting $\mathcal{V}$ be as above, and $\mathcal{F}$ be the category with objects $\mathcal{V}^{\otimes}$ and morphisms monoidally generated by $\boldsymbol{\Delta}G$. That is any morphism $Hom_{\mathcal{F}}(X, Y)$ is not empty only if both X and Y have the same length and in that case the decomposition of the morphism is given by $\phi = \amalg_i \phi_i$ with $\phi_i : [n_i] \to [m_i]$ for appropriate decompositions. $X \simeq \amalg_i [n_i]$ and $Y \simeq \amalg_i [m_i]$. Since $\mathcal{O}ps$ are strong monoidal functors, we see that $\mathfrak{F}_{\boldsymbol{\Delta}G}$–$\mathcal{O}ps_{\mathcal{C}}$ is isomorphic to ordinary functors in $Fun(\boldsymbol{\Delta}G, \mathcal{C})$ and the $\mathcal{O}ps^{co}$ are functors in $Fun(\boldsymbol{\Delta}G^{op}, \mathcal{C})$.

We can implement these symmetries and the (co)–simplicial structure by looking at Feynman categories $\mathfrak{F}$ which have a faithful functor $\mathfrak{F}_{\boldsymbol{\Delta}G} \to \mathfrak{F}$ and we may also consider the strong version, namely that the functor is essentially surjective.

2.10.2. **Lavwere theories.** For these theories, the first formal similarity is that we have a functor $\mathcal{V}^{\otimes} \to \mathcal{F}$ whereas for a Lavwere theory we have a functor $\aleph_0^{op} \to L$. Apart from that similarity there are marked differences. First, the basic datum is $\mathcal{V}$ not $\mathcal{V}^{\otimes}$, and $\mathcal{V}$ and $\mathcal{V}^{\otimes}$ have to be groupoids. Secondly $\mathcal{V}$ is not fixed, but variable.

A second deeper level of similarity is given by regarding functors of Feynman categories. Since $\mathcal{F}in\mathcal{S}et$ is a part of Feynman category $\mathfrak{F}_{\mathcal{F}in\mathcal{S}et}$, we can consider morphisms of Feynman categories $\mathfrak{F}_{\mathcal{F}in\mathcal{S}et} \to \mathfrak{F}$ which are identity on objects. Now this gives functors $\mathfrak{F}_{\mathcal{F}in\mathcal{S}et}^{op} \to \mathfrak{F}^{op}$ and one can ask if $\mathcal{F}^{op}$ is a Lavwere theory. This is the case if $\mathcal{F}$ has a co–product. The requirement of being a Feynman category is then an extra requirement. If $\otimes$ in $\mathcal{F}$ is a co–product, which it is in $\mathcal{F}_{\mathcal{F}in\mathcal{S}et}$, then $\mathfrak{F}$–$\mathcal{O}ps_{\mathcal{C}}^{op}$ for a $\mathcal{C}$ with $\otimes$ a product are models, e.g. $\mathcal{C} = (\mathcal{T}op, \times)$ or $(\mathcal{A}b, \otimes)$. Thus, we can say that some special Feynman categories give rise to some special Lavwere theories and models.

3. General constructions

In this section, we gather several constructions which turn a given Feynman category into another one.

3.1. **Free monoidal construction $\mathcal{F}^{\otimes}$.** Sometimes it is convenient to construct a new Feynman category from a given one whose vertices are the elements of $\mathfrak{F}$. Formally, we set $\mathfrak{F}^{\boxtimes} = (\mathcal{V}^{\otimes}, \mathcal{F}^{\boxtimes}, \imath^{\otimes})$ where $\mathcal{F}^{\boxtimes}$ is the free monoidal category on $\mathcal{F}$ and we denoted the "outer" free monoidal structure by $\boxtimes$. This is again a Feynman category. There is the functor $\mu : \mathcal{F}^{\boxtimes} \to \mathcal{F}$ which sends $\boxtimes_i X_i \mapsto \bigotimes_i X_i$ and by definition $Hom_{\mathcal{F}^{\boxtimes}}(\mathbf{X} = \boxtimes_i X_i, \mathbf{Y} = \boxtimes_i Y_i) = \bigotimes_i Hom_{\mathcal{F}}(X_i, Y_i)$. The only way that the index sets can differ, without the Hom–sets being empty, is if some of the factors are $\mathbb{1} \in \mathcal{F}^{\boxtimes}$. Thus the one–comma generators are simply the elements of $Hom_F(X, Y)$. Using this identification $Iso(\mathcal{F}^{\boxtimes}) \simeq Iso(\mathcal{F})^{\boxtimes} \simeq (\mathcal{V}^{\otimes})^{\boxtimes}$. The factorization and size axiom follow readily from this description.

Since $\mathcal{F}^{\boxtimes}$ is the free symmetric monoidal category on $\mathcal{F}$, $\mathcal{F}^{\boxtimes} - \mathcal{O}ps_{\mathcal{C}}$ is equivalent to the category of functors (not necessarily monoidal) $Fun(\mathcal{F}, \mathcal{C})$.

Examples are now FI algebras and (crossed) simplicial objects for the free monoidal Feynman categories for FI and $\mathbf{\Delta}$ where for the latter one uses the non–symmetric version.

3.2. **NC–construction.** For any Feynman category one can define its non–connected (nc) version. The terminology has its origin in the graph examples.

This plays a crucial role in physics and mathematics and manifests itself through the BV equation [KWZ12]. Let $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \imath)$ then we set $\mathfrak{F}^{nc} = (\mathcal{V}^{\otimes}, \mathcal{F}^{nc}, \imath^{\otimes})$ where $\mathcal{F}^{nc}$ has objects $\mathcal{F}^{\boxtimes}$, the free monoidal product. We however add more morphisms. The one–comma generators will be $Hom_{\mathcal{F}^{nc}}(\mathbf{X}, Y) := Hom_F(\mu(\mathbf{X}), Y)$, where $\mathbf{X} = \boxtimes_{i \in I} X_i$. This means that for $\mathbf{Y} = \boxtimes_{j \in J} Y_j$ $Hom_{\mathcal{F}}(\mathbf{X}, \mathbf{Y}) \subset Hom_{\mathcal{F}}(\mu(\mathbf{X}), \mu(\mathbf{Y}))$, but only those morphisms for which there is a partition $I_j, j \in J$ of I such that the morphisms factors through $\bigotimes_{j \in J} Z_j$ where $Z_j \overset{\sigma_j}{\to} \bigotimes_{k \in I_j} X_k$ is an isomorphism. That is $\psi = \bigotimes_{j \in J} \phi_j \circ \sigma_j$ with $\phi_j : Z_j \to Y_j$. Notice that there is a map of "disjoint union" or "exterior multiplication" given by $X_1 \boxtimes X_2 \to X_1 \otimes X_2$ given by $id \otimes id$.

Examples can be found in [KWZ12], where also a box–picture for graphs is presented.

3.3. **Iterating Feynman categories.** Given a Feynman category $\mathfrak{F}$ there is a simplicial tower of iterated categories built on the original one.

Let $\mathcal{V}'$ be the groupoid $\mathcal{V}' = (\imath^{\otimes} \downarrow \imath)$ and let $\mathcal{F}' = (id_{\mathcal{F}} \downarrow \imath^{\otimes})$. There is an obvious inclusion $\imath' \colon \mathcal{V}' \to \mathcal{F}'$ and $\mathcal{V}'^{\otimes} \simeq Iso(\mathcal{F}')$, since $\mathfrak{F}$ was a Feynman category and condition (i) holds. The difference between the two is just the choice of the representation of the source and target. $\mathcal{F}'$ is also equivalent to $(id_F \downarrow i)$ where $i \colon \imath^{\otimes}(\mathcal{V}^{\otimes}) \to \mathcal{F}$ is the inclusion of the image. The difference being the choice of a representation of the target. The morphisms in that category are given by diagrams

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "\psi"'] & Y \arrow[d, "\simeq\sigma"] \\ X' \arrow[r, "\phi'"] & Y' \end{tikzcd} \tag{3.1}$$

and these morphisms can be factored into two morphisms, which we will call type I and type II:

$$\begin{tikzcd}
X \arrow[r, "\phi"] \arrow[d, "\psi"'] & Y \arrow[d, equal] \\
X' \arrow[r, "\sigma^{-1}\circ\phi'"] \arrow[d, equal] & Y \arrow[d, "\simeq\sigma"] \\
X' \arrow[r, "\phi'"] & Y'
\end{tikzcd} \tag{3.2}$$

and similarly there is a factorization in $\mathcal{F}'$.

Notice that $Hom_{\mathcal{F}'}(\phi_0, \phi_1)$ is empty if $t(\phi_0) \not\simeq t(\phi_1)$. Also, if we are indexed over $\mathfrak{G}$, the underlying graphs satisfy $\mathbb{\Gamma}(\phi) = \mathbb{\Gamma}(\phi') \circ \mathbb{\Gamma}(\phi)$.

There are also the two standard source and target functors $s, t : \mathcal{F}' \to \mathcal{F}$, which also restrict to $s, t : \mathcal{V}' \to \mathcal{V}$. They are just the restrictions of the functors on the full arrow category. Explicitly any object $\phi \in \mathcal{F}'$ is sent to its source or its target, and s sends a morphism (3.1) to ϕ and t sends it to σ.

Definition-Proposition 3.1. *The iterate of a Feynman category $\mathfrak{F}$ is the Feynman category $\mathfrak{F}' = (\mathcal{V}', \mathcal{F}', \imath')$. We set $\mathfrak{F}^{(k)} = \mathfrak{F}^{(k-1)\prime}$. The morphisms (s, s) and (t, t) are morphisms of Feynman categories from $\mathfrak{F}' \to \mathfrak{F}$, which by abuse of notation we will just call s and t.*

Proof. We need to show that $\mathfrak{F}'$ is indeed a Feynman category. Since $\mathfrak{F}$ was a Feynman category, the condition (ii) for $\mathcal{F}$ guarantees that indeed $\imath'^{\otimes}$ is an equivalence of $\mathcal{V}'^{\otimes}$ and $\mathcal{F}'$. For condition (ii) for $\mathcal{F}'$, one uses (ii) on composition in $\mathcal{F}$. Given $\phi_1 \colon X \overset{\psi}{\to} Y \overset{\phi_0}{\to} Z$ we can first decompose $Z \simeq \amalg_v *_v$ and hence $\phi_0 \simeq \amalg_v \phi_v^0$ for the decomposition $Y \simeq \amalg_v Y_v$ and $\phi_0 \simeq \amalg_v \phi_v^1$ for the decomposition $X \simeq \amalg_v X_v$. Further decomposing the $Y_v \simeq \amalg_{w_v \in I_v} *_{w_v}$ we get the desired decomposition of ψ by first splitting into the various ψ_{w_v} and then assembling them into the $\amalg_{w_v \in I_v} \psi_w$, where now up to choosing isomorphisms $\psi_w \colon X_v \to Y_v$. The assertion about (s, s) is straightforward. □

Proposition 3.2. *The Feynman categories $\mathfrak{F}^{(k)}$ form a simplicial object in the category of Feynman categories with the simplicial morphisms given by composition, the source and target maps and identities.*

Proof. For this one just uses the nerve functor for categories and notices that this is compatible with the Feynman category structure. □

Lemma 3.3. *The Feynman category $\mathfrak{F}'$ splits into subcategories or slices $\mathfrak{F}'_{[X]}$ whose objects have targets isomorphic to $[X]$. Restricting to $* \in \mathcal{V}$ we even obtain full Feynman subcategories $\mathfrak{F}'|_{[\imath(*)]}$ for $* \in \imath(*)$. Here splits means that $Hom_{\mathfrak{F}'}(\phi, \phi') = \emptyset$ unless ϕ and ψ are in the same subcategory.*

Proof. The first statement is true by definition of the morphisms in $\mathfrak{F}'$: there are only morphisms between ϕ and ϕ' if they have isomorphic targets. The second assertion then follows from the same fact restricted to $\mathcal{V}'$. □

Lemma 3.4. *If $\mathcal{O} \in \mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$ then $FG\mathcal{O} = t_* s^* \mathcal{O}$.*

Proof. On the l.h.s. evaluating at Z, we get $colim_{(\imath^{\otimes} \downarrow Z)} \mathcal{O} \circ s$. On the r.h.s. we get $colim_{(t \downarrow Z)} \mathcal{O} \circ s \circ P$. We see that for a given object $(\phi \colon X \to Y,\ \chi \colon Y \to Z)$, we can use morphisms of type I with $\psi = \phi$ to reduce the colimit to just a colimit over objects $(id_Y, \chi\colon Y \to Z)$ and then use morphism of type II to reduce to the colimit over automorphisms of these objects. Now $\mathcal{O} \circ s \circ P(id_Y, \chi : Y \to Z) = \mathcal{O}(Y) = \mathcal{O}(s\chi)$. The claim then follows, since we get an isomorphism between the two universal co-cones. □

3.4. **Arrow category.** For a category $\mathcal{F}$ we set $Ar(\mathcal{F})$ to be the category $(\mathcal{F} \downarrow \mathcal{F})$ which has as objects morphisms in $\mathcal{F}$ and as morphisms commutative diagrams in $\mathcal{F}$. It is clear that any functor $\imath$ of categories induces a functor $Ar(\imath)$ on the level or arrow categories.

Proposition 3.5. *If* $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \imath)$ *is a Feynman category, so is* $Ar(\mathfrak{F}) = ((\mathcal{V}^{\otimes} \downarrow \mathcal{V}), Ar(\mathcal{F}), Ar(\imath))$.

Proof. It is clear that $(\mathcal{V}^{\otimes} \downarrow V)$ is a groupoid and straightforward that since $\mathcal{V}^{\otimes} \simeq Iso(\mathcal{F})$ that $Iso(Ar(\mathcal{F})) \simeq (\mathcal{V}^{\otimes} \downarrow V)^{\otimes}$. The condition (ii) guarantees that $Ar(\mathcal{F})$ is one–comma generated. □

3.5. **Feynman level category $\mathfrak{F}^+$.** Given a Feynman category $\mathfrak{F}$, and a choice of basis for it, we will define its Feynman level category $\mathfrak{F}^+ = (\mathcal{V}^+, \mathcal{F}^+, \imath^+)$ as follows. The underlying objects of $\mathcal{F}^+$ are the morphisms of $\mathcal{F}$. The morphisms of $\mathcal{F}^+$ are given as follows: given ϕ and ψ, consider their decompositions

$$\begin{tikzcd}
X \arrow[r, "\phi"] \arrow[d, "\sigma"', "\simeq"] & Y \arrow[d, "\hat{\sigma}", "\simeq"'] & X' \arrow[r, "\psi"] \arrow[d, "\tau"', "\simeq"] & Y' \arrow[d, "\hat{\tau}", "\simeq"'] \\
\bigotimes_{v\in I} \bigotimes_{w\in I_v} *_w \arrow[r, "\bigotimes_{v\in I} \phi_v"] & \bigotimes_{v\in I} *_v & \bigotimes_{v'\in I'} \bigotimes_{w'\in I'_{v'}} *'_w \arrow[r, "\bigotimes_{v'\in I'} \psi_{v'}"] & \bigotimes_{v'\in I'} *_{v'}.
\end{tikzcd}$$

(3.3)

where we have dropped the $\imath$ from the notation, $\sigma, \hat{\sigma}, \tau$ and $\hat{\tau}$ are given by the choice of basis and the partition I_v of the index set for X and $I'_{v'}$ for the index set of Y is given by the decomposition of the morphism.

A morphism from ϕ to ψ is a two level partition of $I : (I_{v'})_{v'\in I'}$, and partitions of $I_{v'} : (I^1_{v'} \dots, I^{k_{v'}}_{v'})$ such that if we set $\phi^i_{v'} := \bigotimes_{v\in I^i_{v'}} \phi_v$ then $\psi_{v'} = \phi^k_{v'} \circ \dots \circ \phi^1_{v'}$.

To compose two morphisms $f\colon \phi \to \psi$ and $g\colon \psi \to \chi$, given by partitions of I : $(I_{v'})_{v'\in I'}$ and of the $I_{v'}$: $(I^1_{v'} \dots, I^{k_{v'}}_{v'})$ respectively of I' : $(I'_{v''})_{v''\in I''}$ and the $I_{v''}$: $(I'^1_{v''} \dots, I'^{k_{v''}}_{v''})$, where I'' is the index set in the decomposition of χ, we set the compositions to be the partitions of I : $(I_{v''})_{v''\in I''}$ where $I_{v''}$ is the set partitioned by $(I_{v'})_{v'\in I'^j_{v''}, j=1,\dots,k_{v''}}$. That is, we replace each morphism $\psi_{v'}$ by the chain $\phi^{v'}_1 \circ \dots \circ \phi^{v'}_k$.

Morphisms alternatively correspond to rooted forests of level trees thought of as flow charts. Here the vertices are decorated by the ϕ_v and the composition along the rooted forest is ψ. There is exactly one tree $\tau_{v'}$ per $v' \in I'$ in the forest and accordingly the composition along that tree is ψ'_v.

Technically, the vertices are the $v \in I$. The flags are the union $\amalg_v \amalg_{w\in I_v} *_w \amalg \amalg_{v\in I} *_v$ with the value of ∂ on $*_w$ being v if $w \in I_v$ and v on $*_v$ for $v \in I$. The orientation at each vertex is given by the target being out. The involution $\imath$ is given by matching source and target objects of the various ϕ_v. The level structure of each tree is given by the partition $I_{v'}$. The composition is the composition of rooted trees by gluing trees at all vertices — that is we blow up the vertex marked by $\psi_{v'}$ into the tree $\tau_{v'}$.

The groupoid $\mathcal{V}^+$ are morphisms in $(\mathcal{F} \downarrow \mathcal{V})$, with the one level trees as morhpisms, i.e. (I) is the partition into the set itself and the second level partitions are given by the elements of I. Notice that in this case $\phi \simeq \psi$ even in the comma-category via

$(\tau^{-1}\sigma, \hat{\tau}^{-1} \circ \hat{\sigma})$. We also recall that since we are indexing our tensor products by sets, there is always the permutation action of the automorphisms of this set.

The one–comma generating maps are those corresponding to the morphisms from any morphism ϕ to a morphism $\chi \in (\mathcal{F} \downarrow \mathcal{V})$. The other axioms of a Feynman category are easily checked.

3.5.1. **$\mathcal{F}^+$–$\mathcal{O}ps$.** After passing to the equivalent strict Feynman category, an element $\mathcal{D}$ in $\mathcal{F}^+$-$\mathcal{O}ps$ is a symmetric monoidal functor that has values on each morphisms $\mathcal{D}(\phi) = \bigotimes \mathcal{D}(\phi_v)$ and has composition maps $\mathcal{D}(\phi_0 \otimes \phi) \to \mathcal{D}(\phi_1)$ for each decomposition $\phi_1 = \phi \circ \phi_0$. Further decomposing $\phi = \bigotimes \phi_v$ where the decomposition is according to the target of ϕ_0, we obtain morphisms

$$\mathcal{D}(\phi_0) \otimes \bigotimes_v \mathcal{D}(\phi_v) \to \mathcal{D}(\phi_1) \tag{3.4}$$

It is enough to specify these functors for $\phi_1 \in (\mathcal{F} \downarrow \mathcal{V})$ and then check associativity for triples.

Example 3.6. If we start from the trivial Feynman category $\mathfrak{F} = (1, 1^{\otimes}, \imath)$ then $\mathfrak{F}^+$ is the Feynman category $\mathfrak{Sur}$ of surjections. Indeed the possible trees are all linear, that is only have 2–valent vertices, and there is only one decoration. Such a rooted tree is specified by its total length n and the permutation which gives the bijection of its vertices with the set n_i. Looking at a forest of these trees we see that we have the natural number as objects with morphisms being surjections.

Example 3.7. We also have $\mathfrak{Sur}^+ = \mathfrak{O}_{May}$, which is the Feynman category for May operads. Indeed the basic maps (3.4) are precisely the composition maps γ. If one is very careful, these are May operads without units.

3.6. **Feynman hyper category $\mathfrak{F}^{hyp}$.** There is a "reduced" version of $\mathfrak{F}^+$ which is central to our theory of enrichment. This is the universal Feynman category through which any functor $\mathcal{D}$ factors, which satisfies the following restriction $\mathcal{D}(\sigma) \simeq \mathbb{1}$ *for any isomorphism* σ *where* $\mathbb{1}$ *is the unit of the target category* $\mathcal{C}$*.*

For this we invert the morphisms corresponding to composing with isomorphisms. That is for all $\phi\colon X \to Y$ and $\sigma \in Aut(X)$ or respectively $\sigma' \in Aut(Y)$, we invert the morphisms $\phi \otimes \sigma \to \phi \circ \sigma$ and respectively $\sigma' \otimes \phi \to \sigma' \circ \phi$ corresponding to the partitions yielding the concatenation. And we furthermore add isomorphisms $id_\emptyset \xrightarrow{\sim} \sigma$ for any $\sigma \in \mathcal{V}$ where $id_\emptyset$ is the identity unit of $\mathcal{F}$ which is the empty word in $\mathcal{V}$. This gives us an action of the isomorphisms of $(\mathcal{F} \downarrow \mathcal{F})$. Let $\boldsymbol{\sigma} = (\sigma^{-1}, \hat{\sigma}^{-1})$ be such an isomorphism from ϕ to ${}^{\boldsymbol{\sigma}}\phi = \sigma \circ \phi \circ \hat{\sigma}$. Decomposing say $\sigma \otimes \hat{\sigma} = \bigotimes_v \sigma_v$ as usual and denoting the tree corresponding to $\hat{\sigma} \circ \phi \circ \sigma$ this gives us the following diagram

$$\begin{tikzcd}
\phi \otimes \bigotimes_v \sigma_v \arrow[r, "\mathcal{D}(\tau)", "\sim"'] & {}^{\boldsymbol{\sigma}}\phi_0 \\
\phi \otimes \bigotimes_v \emptyset \arrow[u, "\sim"] & \phi \arrow[u, "\boldsymbol{\sigma}"'] \arrow[l, "id\otimes\bigotimes_v r_\emptyset"', "\sim"]
\end{tikzcd} \tag{3.5}$$

where the right morphisms are given by the new isomorphisms, the top morphism is invertible by construction and the bottom morphism is the unit constraint.

3.6.1. **$\mathcal{F}^{hyp}$-$\mathcal{O}ps$.** An element $\mathcal{D} \in \mathfrak{F}^{hyp}$-$\mathcal{O}ps$ corresponds to the data of functors from $Iso(\mathcal{F} \downarrow \mathcal{F}) \to \mathcal{C}$ together with morphisms (3.4) which are associative and satisfy the equality induced by (3.5), i.e. the following diagram commutes:

$$\begin{tikzcd}
\mathcal{D}(\phi)\otimes\bigotimes_v \mathcal{D}(\sigma_v) \arrow[r, "\mathcal{D}(\tau)", "\sim"'] & \mathcal{D}({}^{\boldsymbol{\sigma}}\phi) \\
\mathcal{D}(\phi)\otimes\bigotimes_v \mathbb{1} \arrow[u, "\sim"] & \mathcal{D}(\phi) \arrow[l, "id\otimes\bigotimes_v r_{\mathbb{1}}^{-1}"', "\sim"] \arrow[u, "\mathcal{D}(\boldsymbol{\sigma})"']
\end{tikzcd} \tag{3.6}$$

Example 3.8. The paradigmatic examples are hyper–operads in the sense of [GK98]. Here $\mathfrak{F}=\mathfrak{M}$ and $\mathfrak{F}^{hyp}$ is the Feynman category for hyper–operads.

3.6.2. **Relation of $\mathfrak{F}'^{op}$–$\mathcal{O}ps$ and $\mathfrak{F}^{hyp}$-$\mathcal{O}ps$.** Any $\mathcal{D}\in\mathfrak{F}^{hyp}$-$\mathcal{O}ps$ can be viewed as a universal $\mathfrak{F}'$–$\mathcal{O}ps$ in the following way.

Looking at isomorphisms only, the two categories $Iso(\mathcal{F}')$ and $Iso(\mathcal{F}^{hyp})$ coincide and give the same data. The isomorphism of $\mathbb{1}\to\mathcal{D}(\sigma)$ comes from the square (3.1) with top and right arrow σ and the other two arrows being id. With source $\mathcal{F}'^{op}$ the type I morphisms give maps $\mathcal{D}(\psi,id)\in Hom(D(\phi'),\mathcal{D}(\phi))$ where $\phi=\psi\circ\phi'$. Now on the other hand $\mathcal{D}(\psi,id):\mathbb{1}\to\mathcal{D}(\psi)$ by picking the top arrow in the square (3.1) to be ψ as well. Thus to give a functor from $\mathcal{F}'^{op}$ we have to give a collection of elements in $Hom(\mathcal{D}(\psi),Hom(\mathcal{D}(\phi'),\mathcal{D}(\phi)))\simeq Hom(\mathcal{D}(\psi)\otimes\mathcal{D}(\phi'),\mathcal{D}(\phi))$ that is consistent with concatenation and compatible with isomorhpisms. A functor from $\mathcal{F}^{hyp}$ provides exactly this type of universal information.

3.6.3. **A reduced version $\mathfrak{F}^{hyp,rd}$.** One may define $\mathfrak{F}^{hyp,rd}$, a Feynman subcategory of $\mathfrak{F}^{hyp}$ which is equivalent to it by letting $\mathcal{F}^{hyp,rd}$ and $\mathcal{V}^{hyp,rd}$ be the respective subcategories whose objects are morphisms that do not contains isomorphisms in their decomposition. In view of the isomorphisms $\emptyset\to\sigma$ this is clearly an equivalent subcategory. In particular the respective categories of $\mathcal{O}ps$ and $\mathcal{M}ods$ are equivalent.

The morphisms are described by rooted forests of trees whose vertices are decorated by the ϕ_v as above –none of which is an isomorphism–, with the additional decoration of an isomorphism per edge and tail. Alternatively one can think of the decoration as a black 2-valent vertex. Indeed, using maps from $\emptyset\to\sigma$, we can introduce as many isomorphisms as we wish. These give rise to 2–valent vertices, which we mark black. All other vertices remain labeled by ϕ_v. If there are sequences of such black vertices, the corresponding morphism is isomorphic to the morphism resulting from composing the given sequence of these isomorphisms.

Example 3.9. For $\mathfrak{F}^{hyp,rd}_{Surj}=\mathfrak{O}_0$, the Feynman category whose morphisms are trees with at least trivalent vertices (or identities) and whose $\mathcal{O}ps$ are operads whose $\mathcal{O}(1)=\mathbb{1}$. Indeed the basic non–isomorphism morphisms are the surjections $\bar{n}\to\bar{1}$, which we can think of as rooted corollas. Since for any two singleton sets there is a unique isomorphism between them, we can suppress the black vertices in the edges. The remaining information is that of the tails, which is exactly the map ϕ^F in the morphism of graphs.

Example 3.10. For the trivial Feynman category, we obtain back the trivial Feynman category, since the trees all collapse to a the tree with one black vertex.

4. Enriched version, (odd) twists and Hopf algebras

4.1. **Enriched version.** We will also consider the enriched version of Feynman categories. Much like in the case of simplicial (co)homology, this allows us to pass from orientations to Abelian groups, where an orientation change induces a minus sign. Besides being a useful generalization, it in particular also makes it possible to treat algebras over $\mathcal{O} \in \mathfrak{F}^{hyp}$–$\mathcal{O}ps$ as $\mathcal{O}ps$ for an $\mathcal{O}$ based–enriched version of $\mathfrak{F}$. The prime examples being algebras over operads. Another application are twisted modular operads, in their traditional form [GK98], here $\mathfrak{F} = \mathfrak{M}$.

Definition 4.1. Let $\mathcal{E}$ be a monoidal category that is Cartesian. A Feynman category $\mathfrak{F}$ enriched over $\mathcal{E}$ is a triple $(\mathcal{F}, \mathcal{V}, \imath)$ of a category $\mathcal{F}$ enriched over $\mathcal{E}$ and an enriched category $\mathcal{V}$ which satisfy the enriched version of the axioms of Definition 1.1. That is (i),(ii') as above and

(iii') All indexing functors $\tilde{\imath}^{\otimes}(X) := Hom_{\mathcal{F}}(\imath^{\otimes}X, -)$ are essentially small.

Notice that in particular, $\mathcal{V}$ being a groupoid means that the $Hom_{\mathcal{V}}(X, Y)$ are $\otimes$–invertible in $\mathcal{E}$. We also remark that (iii') implies (iii).

Fixing a target category $\mathcal{C}$, which is also enriched over $\mathcal{E}$, we define the categories $\mathcal{O}ps$ and $\mathcal{M}ods$ as before, but insisting that the functors are functors of enriched categories.

This definition allows one to enrich over $\mathcal{T}op$ or simplicial sets. To enrich over the standard Abelian categories, there are two choices. First one can simply weaken the conditions, see definition below; this is very close to [Get09b].

Secondly, one can use an ordinary Feynman category as an indexing system, which we develop in this section. This is guided by the free construction and is more important for our current purposes. It will allow us to treat twisted modular operads or more generally twisted Feynman categories and give a second approach to algebras.

Definition 4.2. A weak Feynman category is a triple $(\mathcal{W}, \mathcal{F}, \imath)$, both $\mathcal{W}$ and $\mathcal{F}$ enriched over $\mathcal{E}$ and $\mathcal{V}$ symmetric monoidal tensored over $\mathcal{E}$ satisfying: (i') $\imath^{\otimes}$ is essentially surjective, and (ii') and (iii') as above.

Notice, we dropped the condition on $\mathcal{V}$ that it is a groupoid.

Remark 4.3. For $\mathcal{A}b$ or $\mathcal{V}ect$ (ii') translates to the hereditary condition that each ϕ can be decomposed into summands. In general, proceeding as in Lemma 1.3 the analogue of (ii) in the enriched case is that for each choice of base for $X \simeq \bigotimes_{w \in W} \imath(*_w)$ and $X' = \bigotimes_{v \in V} \imath(*_v)$ there is an induced isomorphism

$$Hom_{\mathcal{F}}(X, Y) \simeq \bigoplus_{(W_v)_{v \in V} : \amalg_v W_v = W} \bigotimes_v Hom_{\mathcal{F}}(\bigotimes_{w \in W_v} \imath(*_w), \imath(*_v))$$

where the coproduct is over all partitions and the isomorphism is given by the monoidal structure, viz. the product and commmutativity/associativity constraints.

4.2. **Freely enriched Feynman categories.** We recall from [Kel82] that if $\mathcal{E}$ is a symmetric monoidal closed category with the underlying $\mathcal{E}_0$ being locally small, complete and cocomplete, that there is a left adjoint functor $(-)_{\mathcal{E}}$ to the underlying category functor $(-)_0$. We will assume these conditions on $\mathcal{E}$ from now on.

A freely enriched Feynman category is then the a triple $\mathfrak{F}_{\mathcal{E}} := (\mathcal{V}_{\mathcal{E}}, \mathcal{F}_{\mathcal{E}}, \imath_{\mathcal{E}})$ where $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \imath)$ is a Feynman category.

4.3. **Indexed aka twisted versions.** We now also fix that the coproduct in the enrichment category $\mathcal{E}$, which we will denote by $\oplus$, is distributive with respect to the monoidal structure $\otimes$ in both variables: $(X \oplus Y) \otimes Z \simeq (X \otimes Z) \oplus (Y \otimes Z)$ and similarly in the other variable.

The cleanest way to define an indexed enrichment is using 2–categories. We will disentangle the definition below. First, we can consider any category $\mathcal{F}$ to be a 2–category with the two morphisms generated by triangles of composable morphisms. If $\mathcal{E}$

is a monoidal category, let $\underline{\mathcal{E}}$ be the corresponding 2–category with one object. I.e. the 1-morphisms of $\underline{\mathcal{E}}$ are the objects of $\mathcal{E}$ with the composition being $\otimes$, the monoidal structure of $\mathcal{E}$. The 2–morphisms are then the 2–morphisms of $\mathcal{E}$, their horizontal composition being $\otimes$ and their vertical composition being $\circ$.

Definition 4.4. Let $\mathfrak{F}$ be a Feynman category. An enrichment functor is a lax 2–functor $\mathcal{D} : \mathcal{F} \to \underline{\mathcal{E}}$ with the following properties

1. $\mathcal{D}$ is strict on compositions with isomorphisms.
2. $\mathcal{D}(\sigma) = \mathbb{1}_{\mathcal{E}}$ for any isomorphism.
3. $\mathcal{D}$ is monoidal, that is $\mathcal{D}(\phi \otimes_{\mathcal{F}} \psi) = \mathcal{D}(\phi) \otimes_{\mathcal{E}} \mathcal{D}(\psi)$

What this means is that for any morphisms ϕ we have an object $\mathcal{D}(\phi) \in \mathcal{E}$ and for any two composable morphisms ϕ and ψ there is a morphism $\mathcal{D}(\psi) \otimes \mathcal{D}(\phi) \overset{\mathcal{D}(\circ)}{\to} \mathcal{D}(\phi \circ \psi)$. Condition (1) states that this morphism is an isomorphism whenever ϕ or σ is an isomorphism. It follows that any $\mathcal{D}(\sigma)$ is invertible for an isomorphism σ. (2) then fixes that this invertible element is $\mathbb{1}_{\mathcal{E}}$.

Lemma 4.5. *An enrichment functor $\mathcal{D}$ for a Feynman category $\mathcal{F}$ corresponds 1–1 to $\tilde{\mathcal{D}} \in \mathfrak{F}^{hyp}$-$\mathcal{O}ps_{\mathcal{E}}$. Thus these are equivalent concepts.*

Proof. We define $\tilde{\mathcal{D}}(\phi) := \mathcal{D}(\phi)$. Decomposing the morphism ϕ, we get the composition morphisms

$$\mathcal{D}(\psi) \otimes \bigotimes_v \mathcal{D}(\phi_v) \to \mathcal{D}(\phi \circ \psi) \tag{4.1}$$

The condintion (1)–(3) yield an action of isomorphisms in $(\mathcal{F} \downarrow \mathcal{F})$ given exactly by the diagram (3.6) and hence functors $\tilde{\mathcal{D}}$ from $Iso(\mathcal{F} \downarrow \mathcal{F})$ which are automatically compatible. According to §3.6.1 this data fixes $\tilde{\mathcal{D}}$. The other direction of the construction is similar. ☐

4.3.1. **Indexed enrichment.** Given a monoidal category $\mathcal{F}$ considered as a 2–category and lax 2–functor $\mathcal{D}$ to $\underline{\mathcal{E}}$ as above, we define an enriched monoidal category $\mathcal{F}_{\mathcal{D}}$ as follows. The objects of $\mathcal{F}_{\mathcal{D}}$ are those of $\mathcal{F}$. The morphisms are given as

$$Hom_{\mathcal{F}_{\mathcal{D}}}(X,Y) := \bigoplus_{\phi \in Hom_{\mathcal{F}}(X,Y)} \mathcal{D}(\phi) \tag{4.2}$$

The composition is given by

$$\begin{aligned} Hom_{\mathcal{F}_{\mathcal{D}}}(X,Y) \otimes Hom_{\mathcal{F}_{\mathcal{D}}}(Y,Z) &= \bigoplus_{\phi \in Hom_{\mathcal{F}}(X,Y)} \mathcal{D}(\phi) \otimes \bigoplus_{\psi \in Hom_{\mathcal{F}}(Y,Z)} \mathcal{D}(\psi) \\ \simeq \bigoplus_{(\phi,\psi) \in Hom_{\mathcal{F}}(X,Y) \times Hom_{\mathcal{F}}(Y,Z)} \mathcal{D}(\phi) \otimes \mathcal{D}(\psi) &\overset{\bigoplus \mathcal{D}(\circ)}{\longrightarrow} \bigoplus_{\chi \in Hom_{\mathcal{F}}(X,Z)} \mathcal{D}(\chi) = Hom_{\mathcal{F}_{\mathcal{D}}}(X,Z) \end{aligned} \tag{4.3}$$

The image lies in the components $\chi = \phi \circ \psi$. Using this construction on $\mathcal{V}$, pulling back $\mathcal{D}$ via $\imath$, we obtain $\mathcal{V}_{\mathcal{D}} = \mathcal{V}_{\mathcal{E}}$. The functor $\imath$ then is naturally upgraded to an enriched functor $\imath_{\mathcal{E}} : \mathcal{V}_{\mathcal{D}} \to \mathcal{F}_{\mathcal{D}}$.

Definition 4.6. Let $\mathfrak{F}$ be a Feynman category and let $\mathcal{D}$ be an enrichment functor. We call $\mathfrak{F}_{\mathcal{D}} := (\mathcal{V}_{\mathcal{E}}, \mathcal{F}_{\mathcal{D}}, \imath_{\mathcal{E}})$ a Feynman category enriched over $\mathcal{E}$ indexed by $\mathcal{D}$.

Theorem 4.7. *$\mathfrak{F}_{\mathcal{D}}$ is a weak Feynman category and moreover the forgetful functor from $\mathfrak{F}_{\mathcal{D}}$-$\mathcal{O}ps$ to $\mathcal{V}_{\mathcal{E}}$-$\mathcal{M}ods$ has an adjoint and more generally push–forwards among indexed enriched Feynman categories exist. Finally there is an equivalence of categories between algebras over the triple GF and $\mathcal{F}_{\mathcal{D}}$–$\mathcal{O}ps$.*

Proof. This is a generalization of the arguments of [GK98]. (i') holds by construction and (iii) for $\mathfrak{F}$ implies (iii') for $\mathfrak{F}_{\mathcal{D}}$. It remains to check (ii'). For this we notice that

$$\begin{aligned}
&\int^{Z,Z'} Hom_{\mathcal{F}_{\mathcal{D}}}(\imath^{\otimes}_{\mathcal{E}}(Z),X)\otimes_{\mathcal{E}} Hom_{\mathcal{F}_{\mathcal{D}}}(\imath^{\otimes}_{\mathcal{E}}(Z'),Y)\otimes_{\mathcal{E}} Hom_{\mathcal{V}_{\mathcal{E}}}(W,Z\otimes Z')\\
&= \int^{Z,Z'} \bigoplus_{\phi\in Hom_{\mathcal{F}}(\imath^{\otimes}(Z),X)} \mathcal{D}\phi \otimes_{\mathcal{E}} \bigoplus_{\phi\in Hom_{\mathcal{F}}(\imath^{\otimes}(Z'),Y)} \mathcal{D}\psi \otimes_{\mathcal{E}} \bigoplus_{\phi\in Hom_{\mathcal{V}}(W,\imath^{\otimes}(Z)\otimes\imath^{\otimes}(Z'))} \mathcal{D}\sigma\\
&\quad = \int^{Z,Z'} \bigoplus_{(\phi,\psi,\sigma)\in Hom_{\mathcal{F}}(\imath^{\otimes}(Z),X)\times Hom_{\mathcal{F}}(\imath^{\otimes}(Z'),Y)\times Hom_{\mathcal{V}}(W,Z\otimes Z')} \mathcal{D}(\phi\otimes\psi\otimes\sigma)\\
&\qquad = \bigoplus_{\chi\in Hom_{\mathcal{F}}(W,X\otimes Y)} D(\chi) = Hom_{\mathcal{F}_{\mathcal{D}}}(W,X\otimes Y) \qquad (4.4)
\end{aligned}$$

where we have used the conditions (1)-(3) on $\mathcal{D}$ to obtain the first equalities and furthermore used (ii') for $\mathcal{F}$ to perform the co–end which replaces the discrete category over which the inner ordinary colimit runs by a representative. Then the analoge of Theorem 1.8, Theorem 1.15 and Theorem 1.11 follow readily. □

Proposition 4.8. *If $\mathcal{C}$ is tensored over $\mathcal{E}$ then the triple $\mathbb{T}_{\mathcal{D}}$ on $\mathfrak{F}_{\mathcal{D}}$–$\mathcal{M}ods_{\mathcal{C}}$ for $\mathfrak{F}_{\mathcal{D}}$–$\mathcal{O}ps_{\mathcal{C}}$ satisfies $\mathbb{T}_{\mathcal{D}} = \mathbb{T}\otimes\mathcal{D} := \mathrm{colim}_{Iso(\imath^{\otimes}\downarrow *)}\mathcal{D}\otimes\mathcal{O}\circ s$, where $\mathbb{T}$ is the triple for $\mathfrak{F}$–$\mathcal{O}ps_{\mathcal{C}}$.*

Proof. We have to show that the triple $\mathbb{T}$ for the Feynman category given by $\mathcal{D}$ has the form given in [GK98]. We denoted $G\mathcal{O}$ simply by $\mathcal{O}$ and as above let B be the base functor $B:\mathcal{F}_{\mathcal{D}}\to\mathcal{F}$ which is identity on objects. Then,

$$\begin{aligned}
\mathbb{T}_{\mathcal{D}}(*) = \mathcal{E}(\imath^{\otimes}_{\mathcal{D}}-,*) &= \int^{X\in\imath^{\otimes}_{\mathcal{D}}\mathcal{V}^{\otimes}} \mathcal{E}(\imath^{\otimes}_{\mathcal{D}}X,*)\otimes\mathcal{O}(X)\\
&= \int^{X\in\imath^{\otimes}_{\mathcal{D}}\mathcal{V}^{\otimes}} \bigoplus_{\phi\in(B(X),B(*))} \mathcal{D}(\phi)\otimes\mathcal{O}(B(X)) = \mathrm{colim}_{Iso(\imath^{\otimes}\downarrow *)}\mathcal{D}\otimes\mathcal{O}\circ s
\end{aligned}$$

where in the last line, we consolidated the colimits and $\mathcal{D}$ is considered as a functor by restricting it to the subcategory with fixed target and isomorphism of the source; see also below. □

Remark 4.9. This can be rephrased without the assumption of being tensored over $\mathcal{E}$ in terms of weighted colimits as follows: Consider $Hom_{\mathcal{B}}(X,Y)$ as a discrete sub–category of $Iso(\mathcal{B}\downarrow\mathcal{B})$ with the trivial indexing $\mathcal{I}\colon Hom_{\mathcal{B}}(X,Y)^{op}\to\mathcal{E}$ given by $\phi\mapsto\mathbb{1}_{\mathcal{E}}$ then the definition postulates functors $\mathcal{D}_{X,Y}\colon Hom_{B(\mathcal{F})}(B(X),B(Y))\to\mathcal{E}$ such that $Hom_{\mathcal{F}}(X,Y)=\mathcal{I}\star\mathcal{D}$, where $Hom_{B(\mathcal{F})}(B(X),B(Y))$ is the full subcategory of $(\mathcal{F}\downarrow\mathcal{F})$ with objects $B(X)$ and $B(Y)$ and we use the notation of [Kel82]. The formula above then corresponds to:

$$Lan_{\imath^{\otimes}}\mathcal{O} = \mathcal{E}(\imath^{\otimes}_{\mathcal{D}}-,-)\star\mathcal{O} = (\mathcal{I}\star\mathcal{D})\star\mathcal{O} = \mathcal{I}\star(\mathcal{D}\star\mathcal{O})$$

4.3.2. **$\mathcal{V}$–twists.** Given a functor $\mathfrak{L}\colon \mathcal{V} \to Pic(\mathcal{E})$, that is the full subcategory of $\otimes$-invertible elements of $\mathcal{E}$, we can define a twist of a Feynman category indexed by $\mathcal{D}$ by setting the new twist-system to be $\mathcal{D}_{\mathfrak{L}}(\phi) = \mathfrak{L}(t(\phi))^{-1} \otimes \mathcal{D}(\phi) \otimes \mathfrak{L}(s(\phi))$.

For the composition $\phi \circ \psi$ we just use $\mathfrak{L}(t(\psi))^{-1} \otimes \mathfrak{L}(s(\phi)) = \mathfrak{L}(t(\psi))^{-1} \otimes \mathfrak{L}(t(\psi)) \xrightarrow{\sim} \mathbb{1}$ and the unit morphisms in $\mathcal{E}$. Associativity and compatibility with isomorphisms is clear.

Proposition 4.10. *If $\mathcal{C}$ is tensored over $\mathcal{E}$ there is an equivalence of categories between $\mathcal{F}_{\mathcal{D}}$-$\mathcal{O}ps_{\mathcal{C}}$ and $\mathcal{F}_{\mathcal{D}_{\mathfrak{L}}}$-$\mathcal{O}ps_{\mathcal{C}}$ given by tensoring with $\mathfrak{L}$. That is, $\mathcal{O}(-) \mapsto \mathfrak{L}^{-1} \otimes \mathcal{O}(-)$.*

Proof. We see that $\mathcal{O}(X) \mapsto \mathfrak{L}(X)^{-1} \otimes \mathcal{O}(X)$ and hence for $\phi\colon X \to Y$, $\mathcal{O}_{\mathcal{D}}(\phi) \mapsto \mathfrak{L}(Y)^{-1} \otimes \mathcal{D}(\phi) \otimes \mathfrak{L}(X) = \mathcal{O}_{\mathcal{D}_{\mathfrak{L}}}(\phi)$.

□

Example 4.11. We recover the original definitions for hyperoperads and twisted modular operads of [GK98], if we use $\mathcal{E} = \mathcal{C} = \mathcal{V}ect^{\mathbb{Z}}$ and $\mathfrak{F} = \mathfrak{M}$.

In particular, since any morphism ϕ is determined (up to isomorphism) by the $\mathbb{\Gamma}_v(\phi)$ we get the form (equivariant with respect to isomorphism)

$$\bigotimes_v \mathcal{D}(\mathbb{\Gamma}_v) \otimes \mathcal{D}(\mathbb{\Gamma}_0) \to \mathcal{D}(\mathbb{\Gamma}_1) \tag{4.5}$$

just like above, the functor is determined on the one–comma generators, restricts to isomorphisms, is associative and unital.

Indeed $\mathcal{O} \in \mathfrak{M}_{\mathcal{D}}$–$\mathcal{O}ps$ is an algebra over the $\mathcal{D}$-twisted modular operad in the sense of [GK98].

Example 4.12. If we start with $\mathfrak{F}_{surj}$ then $\mathfrak{F}^{hyp,rd} = \mathfrak{O}_0$, that is the Feynman category for operads with trivial $\mathcal{O}(1)$ and hence an indexed enrichment is equivalent to a choice of such an operad $\mathcal{O}$. As we show below, these are exactly all enriched indexed Feynman categories with trivial $\mathcal{V}_{\mathcal{E}}$. The $\mathfrak{F}_{surj,\mathcal{O}}$–$\mathcal{O}ps$ are just algebras over operads with trivial $\mathcal{O}(1)$.

Definition 4.13. Let $\mathfrak{F}$ be a Feynman category and $\mathfrak{F}^{hyp,rd}$ its reduced hyper category, $\mathcal{O}$ an $\mathfrak{F}^{hyp,rd}$-$\mathcal{O}p$ and $\mathcal{D}_{\mathcal{O}}$ the corresponding enrichment functor. Then we define an $\mathcal{O}$–algebra to be a $\mathfrak{F}_{\mathcal{D}_{\mathcal{O}}}$-$\mathcal{O}p$.

4.4. **Indexed Feynman $\mathcal{A}b$-Categories:Orientations and Odd $\mathcal{O}ps$.** Besides the free construction there are standard constructions for the oriented and ordered versions which given an indexing for an $\mathcal{A}b$ enrichment. These are obtained from the free construction by a relative construction.

4.4.1. **Oriented/ordered versions.** In the edge oriented examples of Feynman categories enriched over $\mathfrak{G}$ we define the associated indexed $\mathcal{A}b$-structure as follows: Use the quotient category of the free $\mathcal{A}b$-construction above with the relations $\sigma \sim -\bar{\sigma}$. Here σ is an orientation and $\bar{\sigma}$ is the opposite orientation.

Alternatively this means that one takes $\mathbb{Z}[Hom(X,Y)] \otimes_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$, with $\mathbb{Z}/2\mathbb{Z}$ acting by orientation reversal, as the enriched Hom. In the ordered version we can take $\bigoplus_k (\mathbb{Z}[Hom(X,*)_k]) \otimes_{\mathbb{S}_k} \mathbb{Z}/2\mathbb{Z}$ where $Hom(X,*)_k$ is the degree k part, that is those ϕ with $|E_{\mathbb{\Gamma}(\phi)}| = k$ and $\mathbb{S}_k$ acting by permuting the edges and the action on $\mathbb{Z}/2\mathbb{Z}$ is the sign representation. Both the ordered and the oriented construction lead to the same Feynman Ab-category.

Here the relevant functor $\mathcal{D}$ on the underlying non-edge oriented category is $\mathcal{D}(\phi) = sign(E_{\mathbb{\Gamma}(\phi)})$, the sign representation on the set of edges of the ghost graph.

This works the same way in $\mathcal{V}ect$ and $\mathcal{V}ect^{\mathbb{Z}}$. These constructions factor through the smaller Ab-enrichment.

Definition 4.14. Given a Feynman category $\mathfrak{F}$ indexed over $\mathfrak{G}$, we define the odd version $\mathfrak{F}^{odd}$ to be the indexed Feynman $\mathcal{A}b$-category resulting from the above operation on $\mathfrak{F}^{or}$ or equivalently on $\mathfrak{F}^{ord}$.

This allows us in one fell swoop to define odd versions for any of the graph examples such as odd operads, odd PROPs, odd modular operads, odd NC modular operads, etc. that were treated in [KWZ12], which are responsible for Lie brackets and BV operators, etc.. This realization allows us to define a Feynman transform/cobar construction in great generality. See §7.

4.5. **Examples.** We will discuss two types of examples. First the algebra type examples given in §2.8 can be generalized to algebras over operads. We can also consider Lavwere theories, which amounts to adding degeneracies.

Both constructions are reminiscent of the original definition of a PROP [ML65]. We consider $\mathcal{V}$ a trivial category, that is a category with one object 1, enriched over $\mathcal{C}$, i.e. $Hom_{\mathcal{V}}(1,1) = \mathbb{1}_{\mathcal{C}}$. Then let $\mathcal{V}^{\otimes}$ be the free symmetric category and let $\bar{\mathcal{V}}^{\otimes}$ be its strict version. It has objects $n := 1^{\otimes n}$ and each of these has an $\mathbb{S}_n$ action. Let $\imath$ be the functor from $\mathcal{V}^{\otimes} \to \bar{\mathcal{V}}^{\otimes}$, and $\mathcal{F}$ a category with $Iso(\mathcal{F}) = \bar{\mathcal{V}}^{\otimes}$.

Remark 4.15. The possible category structures with underlying $\mathcal{V}^{\otimes}$ from above as an underlying category are by their original definition PROPs $P(n,m) = Hom(n,m)$, see [ML65]. Additionally imposing the conditions of a Feynman category, we restrict to only those PROPs which are actually generated by operads with $\mathcal{O}(1)$ having only constants as units as we explain in the next sections.

4.5.1. **Trivial $\mathcal{V}$: aka algebras over operads.** To give a detailed example, we consider the possible weak Feynman categories $(\mathcal{V}, \mathcal{F}, \imath)$ with $\mathcal{V}$ a trivial category over $\mathcal{C}$ and the objects of $\bar{\mathcal{V}}^{\otimes} \to \mathcal{F}$ essentially surjective. Given such a category, if we set $\mathcal{O}_{\mathcal{F}}(n) := Hom_{\mathcal{F}}(n,1)$ then this is a (unital) operad. Indeed, composition $\gamma : Hom_{\mathcal{F}}(n,k) \otimes Hom_{\mathcal{F}}(k,1) \to Hom_{\mathcal{F}}(n,1)$ gives the structure of a May operad due to the hereditary condition and the identity in $Hom_{\mathcal{F}}(1,1)$ gives a unit. The $\mathbb{S}_n$ action is given by composing with $Aut(n) \simeq \mathbb{S}_n$. If $\mathcal{O}(1) \neq \mathbb{1}$ then the colimits defining the Kan extensions will not be monoidal. Thus we see that the invertible elements in $\mathcal{O}(1)$ are only the constants.

Vice-versa given an operad $\mathcal{O}$ in $\mathcal{C}$ with trivial $\mathcal{O}(1)$, that is $\mathcal{O} \in \mathfrak{O}\text{–}\mathcal{O}ps_{\mathcal{C}}$, let $\mathcal{V}$ be the trivial category enriched over $\mathcal{C}$ and $\mathcal{F} = \bar{\mathcal{V}}^{\otimes}$ which is additionally one–comma generated by $Hom_{\mathcal{F}_{\mathcal{O}}}(n,1) := \mathcal{O}(n)$ and the composition given by the operadic composition $\gamma\colon \mathcal{O}(n_1) \otimes \dots \mathcal{O}(n_k) \otimes \mathcal{O}(k) \to \mathcal{O}(\sum_{i=1}^{k} n_k)$. Notice that we need the $\mathbb{S}_n$ action on $\mathcal{O}(n)$, in order to accommodate the action by precomposing with isomorphism. $\mathfrak{F}_{\mathcal{O}} := (\mathcal{V}, \mathcal{F}_{\mathcal{O}}, \imath)$ is a Feynman category and more precisely the Feynman category $\mathfrak{Sur}_{\mathcal{O}}$.

Now $\mathcal{F}$–$\mathcal{O}ps$ are exactly algebras over the operad. Explicitly, let ρ be a monoidal functor, then it is fixed by $\rho(1) = X$ to $\rho(n) = X^{\otimes n}$. For the morphisms we get $\rho : \mathcal{O}(n) \to Hom_{\mathcal{C}}(X^{\otimes n}, X)$. Now since the functor is symmetric monoidal, then we get the $\mathbb{S}_n$ action on $\mathcal{O}(n)$ to be compatible with the permutation action on the factors of $X^{\otimes n}$. We can also disregard the $\mathbb{S}_n$ action by no longer using the symmetric monoidal structure and obtain algebras over non–Σ operads.

When we consider the lax 2–functor $\mathcal{D}$, we see that we can also get operads which are a) non–unital and b) have non–trivial $\mathcal{O}(1)$. Likewise, we obtain non–trivial $\mathcal{O}(1)$ from considering $\mathcal{S}urj^{+}$.

Example 4.16.

- Associative algebras in $\mathcal{C}$. $\mathcal{F}(n,1) = \mathbb{S}_n = Aut(\mathbb{1}^{\otimes n})$ and the composition is given by the inclusion of products of symmetric groups. We will call the Feynman category $\mathfrak{F}_{Assoc}$. Notice that this is defined without enrichment or equivalently with the trivial enrichment.
- Commutative algebras in $\mathcal{C}$. In this case, $\mathcal{F}(n,1) = \mathbb{1}$ with the trivial $\mathbb{S}_n$ action.
- Lie and pre-Lie algebras. Notice that these are only defined for categories which are enriched over $\mathcal{A}b$. We will call the Feynman categories $\mathfrak{F}_{Lie}$ and $\mathfrak{F}_{pre-Lie}$.

4.5.2. **Graph insertion.** Other examples that are useful as indexing Feynman categories are those coming from graph insertions. The standard one is given by trivial $\mathcal{V}$, $Iso(\mathcal{F}) = \bar{\mathcal{V}}^{\otimes}$ as above and the underlying operad given by $\mathcal{O}(n)$ graphs without tails and n–labelled vertices up to isomorphism. The $\mathbb{S}_n$ action permutes the labels. The composition is given by the insertion composition for unlabelled graph — see A.3.5. This example can be altered/modified by choosing different types of graphs, putting restrictions on the grafting or the labeled vertices. For example the Feynman category $\mathfrak{F}_{pre-Lie}$ is the one obtained by choosing rooted trees [CL01].

4.6. **A Connes–Kreimer style bi-algebra structure.** The following observation is essentially due to D. Kreimer. Applications of this theory and further details can be found in [GCKT]. Let $\mathfrak{F}$ be enriched over $\mathcal{A}b$ and consider the direct sum of $Hom_{\mathcal{F}}(X,Y)$ over a monoidal skeleton $sk(\mathcal{F})$ of $\mathcal{F}$, denoted $\hat{H} = \coprod_{X,Y \in sk(\mathcal{F})} Hom(X,Y)$. Assume that $\mathcal{F}$ is composition finite, that is given any morphism ϕ there are only finitely many pairs (ψ, ϕ_0) s.t. $\phi = \psi \circ \psi$.

Proposition 4.17. *$\hat{H}$ has the structure of a bi-algebra, with multiplication given by the monoidal structure $\otimes$ of $\mathcal{F}$ and the coproduct given by*

$$\Delta(\phi) = \sum_{(\psi,\phi_0):\phi=\psi\circ\phi_0} \phi_0 \otimes \psi \tag{4.6}$$

where the sum is over all decompositions and $\psi = \bigotimes_v \phi_v$ in a suitable choice of skeleton.

Proof. The fact that the coproduct is co-associative follows from associativity of the composition. The fact that it is compatible with the product follows from the hereditary property (ii). □

Note that if we have a grading and the bi-algebra is connected, there is an automatic antipode and the bi-algebra becomes Hopf. Such a grading can come from a graded Feynman category cf. §7.2.

4.6.1. **Connes–Kreimer on coinvariants.** Since the above sum tends to be large and a bit redundant, since the isomorphisms are still part of the structure, one can opt for a smaller model. To this end let $H = colim_{Iso(\mathcal{F})} Hom(-,-)$ and call two pairs of composable morphisms (ψ, ϕ_0) and (ψ', ϕ'_0) equivalent if there exist isomorphisms $\sigma, \sigma', \sigma''$ such that $\psi \circ \phi_0 = \sigma\psi'\sigma'\phi'_0\sigma''$. For an element/equivalence class $[\phi] \in H$,

$$\Delta([\phi]) = \sum_{[(\psi,\phi_0)]:[\phi]=[\psi\circ\phi_0]} [\phi_0] \otimes [\psi] \tag{4.7}$$

where the sum is over a system of representatives of decomposition classes.

When we mod out by isomorphisms, we have to also alter the product structure, as the commutativity constraints now act as identities. Therefore we are forced to consider symmetric products. If we decompose $\psi = \bigotimes_v \phi_v$, then we obtain that the class $[\psi]$ is actually the symmetric product $\bigodot_v [\phi_v]$.

Proposition 4.18. *Together with the symmetric product as multiplication, the coproduct above gives H the structure of a bi-algebra.*

Proof. As above. $\square$

Remark 4.19. With hindsight, we discovered that the co–algebra structure can be traced back to [JR79]. There one can also find the idea to quotient out by co–ideals, say those generated by the equivalence under isomorphisms.

Remark 4.20. If we are in the graded Feynman category situation, then the Hopf algebra structure on the quotient by the co–ideal generated by isomorphisms is connected provided all non–isomorphism have degree greater that 0.

4.6.2. **Connes–Kreimer on Graphs.** If we look at Feynman categories indexed over $\mathfrak{G}$ we see that the decomposition is basically in terms of graphs and subgraphs. But we have to be a little careful, see §2.1. This nicely illustrates the difference between the two constructions above. For instance the ϕ_v are not just the subgraphs $\mathbb{\Gamma}_v(\phi)$ but also carry a labelling of the vertices via their source and target maps. Forgetting this forces one to work with symmetric products.

Proposition 4.21. *Considering ghost graph morphisms for $\mathfrak{G}^{ctd}$, Δ induces a bi-algebra structure on graphs, which is that of Connes–Kreimer and by grading with the number of edges this bi-algebra is connected and hence Hopf with the unique antipode.*

Proof. Indeed as discussed in §2.1, the extra data needed to get from $\mathbb{\Gamma}(\phi)$ to ϕ, knowing the source and target are the isomorphism of ϕ^F restricted to its image. Since we are connected, the morphism ϕ_V is automatic. One lift for the restriction of ϕ^F is to fix strict identity for the isomorphism. This however, still leaves the order of the vertices. Recall that in $\mathcal{V}^{\otimes}$ the order matters. We can kill this by formally using symmetric products of graphs as done in Connes–Kreimer. Indeed we then lift a symmetric product to a morphism by using identity for the isomorphism, use Δ as above and project back to the symmetrized ghost graphs. The monoidal structure then becomes the symmetric product. Since we kill isomorphisms this way, the bi-algebra graded by the number of edges is indeed connected.

$\square$

Remark 4.22.

(1) Notice that the above Proposition easily translates to the indexed case.
(2) If we use $\mathfrak{F}_{\mathcal{O}}$, with $\mathcal{O}$ the operad of trees with labeled leaves and gluing at leaves, we arrive precisely at the Hopf algebra of rooted trees of Connes–Kreimer [CK98]. This can be seen as coming naturally from the construction above to the Feynman category of $\mathfrak{D}^{pl}$ or $\mathfrak{D}$, see [GCKT].

5. Feynman categories given by generators and relations

The monadicity theorem gives two ways of defining $\mathcal{F}$-$\mathcal{O}ps$. Often there is a third way, given by generators and relations. For example operads can be defined by the $\circ_i$ operations and $\mathbb{S}_n$ actions along with associativity and compatibility relations. In this paragraph, we consider the general setup for this. Besides being of separate interest this type of presentation is what we need to define ordered/oriented/odd versions of Feynman categories that are essential for the definition of (co)bar/Feynman transforms.

5.1. **Structure of $\mathfrak{G}$.** We will first study $\mathfrak{G}$ as an archetypical example. It is basically generated by four types of operations: simple edge contractions, simple loop contraction, simple mergers and isomorphisms.

5.1.1. **Generators.** There are useful numerical invariants for $\mathfrak{G}$: Let $\deg(*_v) = \mathrm{wt}(*_v) = |F(*_v)|$ be the degree and weight of $*_v$ and if $X \simeq \amalg_w *_w$ set $\mathrm{wt}(X) = \sum_w (\mathrm{wt}(*_w) + 1)$ and $\deg(X) = \sum_w \deg(*_w)$ that is just the number of flags. The degree and weight of ϕ are then defined as

$$\deg(\phi) = \frac{1}{2}(\deg(X) - \deg(Y)) \quad \mathrm{wt}(\phi) = \mathrm{wt}(X) - \mathrm{wt}(Y) \tag{5.1}$$

It is clear that the degree and weight are additive under concatenation. Note that the degree is actually an integer and $\deg(\phi) = |E(\mathbb{\Gamma}(\phi))|$.

Proposition 5.1. *All morphisms ϕ from the comma category $(\mathcal{A}gg \downarrow \mathcal{C}rl)$ can be factored into morphisms of the following four types, which we call simple.*

(1) Simple edge contraction. *The complement of the image ϕ^F is given by two flags s, t, which form a unique ghost edge, and the two flags are not adjacent to the same vertex. This has degree 1 and weight 3. We will denote this by ${}_s\circ_t$.*
(2) Simple loop contraction. *As above, but the two flags of the ghost edge are adjacent to the same vertex, this is called a simple loop contraction and it has degree 1 and weight 2. We will denote this by $\circ_{st}$.*
(3) Simple merger. *This is a merger in which ϕ_V only identifies two vertices v and w. Its degree is 0 and the weight is 1. We will denote this by ${}_v\boxminus_w$.*
(4) Isomorphism. *The pure isomorphisms are of degree and weight 0.*

This factorization nor its length are in general unique, but there a minimal number $|\phi|$ of morphisms a factorization contains. If ϕ is an isomorphism this number is 1 and if it is as above and not an isomorphism this number is:

$$|\phi| = \sum_{v \in V} (|E(\mathbb{\Gamma}_v)| + |\pi_0(\mathbb{\Gamma}_v)| - 1) \tag{5.2}$$

(6) *Any such minimal factorization is obtained by first contracting all ghost edges of ϕ step by step and then performing mergers step by step and then performing an isomorphism.*
(7) *Any such minimal factorization is uniquely determined by an order on the ghost edges of ϕ and an order on the components of $\mathbb{\Gamma}(\phi)$.*

Proof. Given a morphism ϕ as above, we choose an order for the vertices $v \in V$, an order for the components of $\mathbb{\Gamma}_v := \mathbb{\Gamma}_v(\phi)$ and finally an order for the edges in each component. This gives an order on all edges and all components of the disjoint union of the $\mathbb{\Gamma}_v$. Then a factorization is given by contracting the edges in the given order and then merging the components in their order. Note that we need one less merger than components of $\mathbb{\Gamma}_v$ since we are left with one component $*_v$. Note during the last operation, we can simultaneously perform the needed isomorphism, unless there is only an isomorphism. This establishes the existence of factorizations and an upper bound. The non–uniqueness is clear by the choices made. Also note that one can replace a non–loop edge contraction by a merger and a loop contraction. Loop contractions and mergers commute as classes, but commutation with non–loop contractions is delicate.

However, since the weight is fixed, we obtain the least number of factors precisely if we have the maximal number of non–loop edge contractions. This is guaranteed by the given order and hence indeed this is the lower bound.

If we use a merger and contraction in lieu of a contraction, we only get a longer word. Thus we see that in order to have a minimal factorization, we have to contract the ghost edges, the choice here is an order of these edges, and then merge the components (now contracted to a vertex), where now the choice is again an order of the mergers. □

5.1.2. **Relations.** All relations among morphisms in $\mathfrak{G}$ are homogeneous in both weight and degree. We will not go into the details here, since they follow directly from the description in the appendix. There are the following types.

1. *Isomorphisms.* Isomorphisms commute with any ϕ in the following sense. For any ϕ and any isomorphisms σ there are unique ϕ' and σ' with $\mathbb{\Gamma}(\phi\circ\sigma) = \mathbb{\Gamma}(\phi')$ such that
$$\phi\circ\sigma = \sigma'\circ\phi' \tag{5.3}$$
2. *Simple edge/loop contractions.* All edge contractions commute in the following sense: If two edges do not form a cycle, then the simple edge contractions commute on the nose ${}_s\circ_t\,{}_{s'}\circ_{t'} = {}_{s'}\circ_{t'}\,{}_s\circ_t$. The same is true if one is a simple loop contraction and the other a simple edge contraction: ${}_s\circ_t\circ_{s't'} = \circ_{s't'}\,{}_s\circ_t$. If there are two edges forming a cycle, this means that ${}_s\circ_t\circ_{s't'} = {}_{s'}\circ_{t'}\circ_{st}$.
3. *Simple mergers.* Mergers commute amongst themselves ${}_v\boxminus_w\,{}_{v'}\boxminus_{w'} = {}_{v'}\boxminus_{w'}\,{}_v\boxminus_w$. If $\{\partial(s),\partial(t)\}\neq\{v,w\}$ then
$${}_s\circ_t\,{}_v\boxminus_w = {}_v\boxminus_w\,{}_s\circ_t, \quad \circ_{st}\,{}_v\boxminus_w = {}_v\boxminus_w\circ_{st} \tag{5.4}$$
If $\partial(s) = v$ and $\partial(t) = w$ then for a simple edge contraction, we have the following relation
$${}_s\circ_t = \circ_{st}\,{}_v\boxminus_w \tag{5.5}$$

5.1.3. **$\mathfrak{G}^{or}$ and $\mathfrak{G}^{odd}$.** Notice that in the case of words in a standard viz. minimal form, two words differ only by permutations on edges and components, see Proposition 5.1 (7). Let $Or(\phi)$ be the set of standard words for ϕ and $Odd(\phi)$ be Abelian group subgroup of $Hom(W(\phi),\mathbb{Z}/2\mathbb{Z})$ for whose elements a switch in order of the edges induces a minus sign and is invariant with respect to switches of orders of components. For the composition we use the same technique as above. We then get the natural and twisted versions discussed previously.

5.1.4. **$\mathfrak{F}^{ord}$, $\mathfrak{F}^{or}$ and $\mathfrak{F}^{odd}$ in the case $\mathfrak{F}$ is indexed over $\mathfrak{G}$.** We can adapt the situation above for Feynman categories indexed over $\mathfrak{G}$ just like in §2.1.4.

5.2. **Feynman categories with ordered presentations.** The aim of this section is to find a more abstract setting for the constructions above. In particular, we want to generalize $\mathfrak{F}^{ord}$ and $\mathfrak{F}^{odd}$ for Feynman categories not necessarily indexed over $\mathfrak{G}$. We say that in a Feynman category a set of morphisms $\Phi\subset(\mathcal{F}\downarrow\imath(\mathcal{V}))$ one–comma generates if any morphism $\psi\in Hom_{\mathcal{F}}(X,*)$, with $*\in\imath(\mathcal{V})$ factors as $\psi=\sigma\phi_0\dots\phi_k\sigma'$ with the $\phi_i\in\Phi'$, $\sigma\in Aut(X)$ and $\sigma'\in Aut(*)$, where Φ' is the trivial extension of Φ by identities. That is the set of morphisms of the type $p'\circ\phi\otimes id\otimes\dots\otimes id\circ p$ with $\phi\in\Phi$ and p,p' any commutativity/associativity isomorphisms. We say that ϕ is n-ary if $\phi\in(\imath(\mathcal{V})^{\otimes n}\downarrow\mathcal{V})$.

We say that Φ is of crossed type, if for any $\phi\in\Phi,\phi\in Hom_{\mathcal{F}}(X,*)$ and any $\sigma\in Iso\mathcal{F}(X,X)$ there is a commutative square

$$\begin{tikzcd} X \arrow[r, "\phi'"] \arrow[d, "\sigma"] & * \arrow[d, "\sigma'"] \\ X \arrow[r, "\phi"] & * \end{tikzcd} \tag{5.6}$$

with $\phi' \in \Phi, \sigma' \in Iso(\mathcal{F})$ being unique.

For example:

Lemma 5.2. *In $\mathfrak{G}$, let Ψ consist of the morphisms ${}_i\circ_j : *_{[n]} \amalg *_{[m]} \to *_{[n+m]}$, $i \in [n], j \in [m]$, $\circ_{ii'} : *_{[n]} \to *_{[n-2]}, i \neq i' \in [n]$ and ${}_n\boxminus_m : *_{[n]} \amalg *_{[m]} \to *_{[n+m+1]}$, which is given by the merger and enumerating the flags of $*_{[n]}$ before those of $*_{[m]}$ from 0 up to $n+m+1$.*

Let Φ consist of the morphisms ${}_0\circ_0 : *_{[n]} \amalg *_{[m]} \to *_{[n+m]}$, $\circ_{01} : *_{[n]} \to *_{[n-2]}$ and ${}_n\boxminus_m : *_{[n]} \amalg *_{[m]} \to *_{[n+m+1]}$, which is given by the merger as above.

Then both Φ and Ψ one–comma generate and Φ is of crossed type.

Proof. Indeed, any simple edge contraction is conjugate by elements in $Iso(\mathcal{F})$ to the first type of morphism, any simple loop contraction to a morphism of the second type and finally any simple merger is conjugate to the third type. Since any morphism can be factored into these three types of morphisms (after extending by identities) the first statement follows.

The fact that Φ is of crossed type is straightforward. We will deal with the second case. Fix $*_S$ and an iso $\sigma : *_{[n]} \to *_{[n]}$, Let s_0 and s_1 be the flags mapped to 0 and 1 and set $S' = S \setminus \{0,1\}$ then $\phi' = {}_{s_0}\circ_{s_1}$ and σ' is the restriction of σ to S' (or strictly speaking the restriction of σ to $*_{S'}$ followed by the renumbering as before).

□

By restriction, we get that

Corollary 5.3. *In $\mathfrak{O}$ the morphisms $\circ_1$ generate. In $\mathfrak{M}$ the morphisms $\circ_1$ and $\circ_{01}$ generate (cf.*[GK98, Sch98]*, where $\circ_{nn+1}$ was chosen).*

In particular we find the $\mathbb{S}_n$ equivariance of the $\circ_i$ products equivalent to the property of the restriction of Φ being crossed.

$$\sigma_n(a) \circ_{\sigma_n(i)} \sigma_n b = (\sigma_n \circ_i \sigma_m) a \circ_i b \tag{5.7}$$

5.2.1. $\mathfrak{G}^{ord}$.

There are actually two ordered versions one could consider. The first is the naive one, where a morphism is a pair of a morphism and a word in the chosen set of generators. The second is a pair of a word and a word in a standard form. For the second case, we can and will for instance choose that given Φ, we only use words of minimal length, e.g. using Lemma 5.1 with all the isomorphisms at the end.

To compose two such minimal words, we use the relations explained above to move all the mergers and isomorphisms to the right and to obtain the order as described in §A.2.3.

For the Feynman category of $\mathfrak{M}$, $\mathfrak{F}^{odd}$ is exactly the Feynman category for $\mathfrak{K}$–twisted modular operads and for twisted modular operads the indexing adds the $\mathfrak{K}$ twist. The "classical" versions of $\mathfrak{F}^{odd}$ are considered in depth in [KWZ12].

5.3. $\mathfrak{F}^{ord}$, $\mathfrak{F}^{or}$ and $\mathfrak{F}^{odd}$ in the case of an ordered presentation.

More generally, given generators and relations, we can try to emulate the above. To this end, we will make several definitions.

5.3.1. $\mathfrak{F}^{ord}$ for a choice of a one–comma generating set.

Let Φ be a one–comma generating set, then $\mathfrak{F}^{ord}$ is the indexed version over $\mathfrak{F}$ where the morphisms are morphisms of $\mathcal{F}$ together with a decomposition into elements of Φ and isomorphisms. If the generating set is crossed, then we can only retain the decomposition into elements of Φ, the isomorphism at the end being fixed.

5.3.2. **$\mathfrak{F}^{or}$,$\mathfrak{F}^{odd}$ for an ordered presentation.** Note that if we give a category by generators and relations, the relations can be depicted as polygons with oriented edges, which form two simple edge paths to which every path belongs and which start and end at the same vertices. If two edges are marked by the same morphism we will identify them and also their vertices. Thus we have chains of polygons whose sides are indexed by different morphisms. We can of course combine relations by gluing along sides, extending them by morphisms on both sides and concatenating these. We call a polygon decomposable if it is the combination of two polygon relations, an extension or gluing of two relations. A non-decomposable polygon is called simple.

Definition 5.4. We call a presentation of a category ordered if for each relation, i.e. polygon chain, as above we can assign a value $+1, -1$ which is multiplicative under decomposition into two polygons. Relations involving isomorphisms are fixed to be $+1$, that is composition with isomorphisms and composition of isomorphisms.

It is clear that such an order is fixed on simple polygons, if they exist.

Definition 5.5. Given an ordered presentation with one-comma generators and isomorphisms, we define $\mathfrak{F}^{or}$ to be the Feynman category with unchanged $\mathcal{V}$ and objects of $\mathcal{F}$, but with morphisms being a pair of a morphism and a class of representation, where two representations are equivalent if they from part of a relation with value 1. The choice that isomorphisms have only $+1$ relations guarantees that $\mathcal{V}^{or} = \mathcal{V}$ is well defined and $\mathfrak{F}^{or}$ is still a Feynman category.

Likewise we define $\mathfrak{F}^{odd}$, where now $\mathcal{F}^{odd}$ is the free Abelian construction, quotiented out by the relations with the given sign. That is each morphism is a morphism of $\mathcal{F}$ plus a presentation modulo changing the presentation and simultaneously changing the sign.

5.3.3. **Example: quadratic relations.** The first example is if we have quadratic equations among the elements of Φ. In this case, we can associate to each square not involving isomorphisms the sign -1. All relations, after canceling isomorphisms, have polygons which are cubical. That is, a relation involving n generators can be obtained from a n–cube, by comparing the $n!$ ways of representing the diagonal by edge morphisms. It is now clear that the induced signs are compatible. The total sign is $(-1)^f$ where f is the number of edge flips.

5.3.4. **Example: multi–linear–quadratic relations.** A more complex example is if we have homogeneous relations that are quadratic and linear (or higher order) and we can (as in $\mathfrak{G}$) search for certain standard forms. Here we would need something like the length. In the case of $\mathfrak{G}$ we used the grading in which contractions were odd and mergers are even.

The general theory is best expressed in 2–categories, see below.

Definition 5.6. A set of one–comma generators Φ is k at most quadratic, if the set of generators splits into k disjoint sets, such that

(1) every morphism has an expression of in these generators of the form $\Phi_1 \circ \cdots \circ \Phi_k$ where the Φ_k are expressed in generators of the k–th set.
(2) Relations among the sets separately are quadratic. And relations among the sets Φ_i and Φ_j are triangular for $i > j$. That is of the type $\phi_j \phi_i = \phi'_j$.

Proposition 5.7. *In the situation above, the presentation can be ordered by assigning -1 to each quadratic relation and $+1$ to each triangular relations.*

Proof. Using the triangular relations, we see that there is a well defined minimal word length and we need to define the signs in relations only involving these. The relations there are now diagonal paths on cubes stuck together along the diagonal, and the same reasoning as above applies. □

The archetypical example of course is $\mathfrak{G}$ where Φ_1 is generated by simple edge/loop contractions and Φ_2 is generated by simple mergers.

5.3.5. **2–category construction for $\mathfrak{F}^{odd}$ with an ordered presentation.** We can also consider the assignment of $+1, -1$ as the additional structure of 2–morphisms for a presentation $(\Phi, \mathcal{R})$ of a category. More precisely, given a presentation of a category, we can write the category as the category obtained from the 2–category whose 1–morphisms are the free category in generators and whose 2–morphisms are the relations generated by $\mathcal{R}$ which are treated as identity 2–morphisms. The category is then obtained by reducing to a 1–category with morphism being isomorphism classes under 2–morphisms.

In case we are enriched over $\mathcal{A}b$, we can extend this to 2–morphisms that are $\pm id$. We can then take the $\mathcal{A}b$ category quotient. As additional data we specify 2–morphism to be $\pm id$ for each 2–morphism in $\mathcal{R}$. Such data is admissible if it is compatible with horizontal and vertical compositions, that is every relation 2–morphism has a well defined sign.

An *ordered presentation* is then a presentation together with an admissible assignment of signs.

This consideration then easily gives rise to further generalizations, where for instance 2–morphisms take values in any Abelian group and there is a (double) groupoid morphism induced by horizontal and vertical composition etc.. We will not use this in the following as we are mainly interested in differentials arising from this situation and this is tied to introducing only signs and not other characters or in physics parlance other statistics such as parafermions.

6. Universal Operations

In this section, for a given Feynman category, we constuct a new Feynman category of universal operations. This is what conceptually explains the constructions of Gerstenhaber and the algebraic half of Deligne's conjecture by rendering them as the outcome of a calculable construction. This of course now translates to all contexts.

6.1. **Cocompletion and the universal Feynman category.** Given a Feynman category $\mathfrak{F}$ and an $\mathcal{O} \in \mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$, with $\mathcal{C}$ a cocomplete monoidal category, there are natural operations on $\mathrm{colim}_{\mathcal{V}}\mathcal{O}$. The example par excellence being the generalized Deligne conjecture, which states that for an operad with multiplication in $dg\mathcal{V}ect$ there is a Gerstenhaber up to homotopy structure on $\bigoplus_n \mathcal{O}(n)$. The *ur*–operation of this type is the structure of pre–Lie algebra on $\bigoplus_n \mathcal{O}(n)$ found by Gerstenhaber. Here we show that this is a universal feature. Any Feynman category $\mathfrak{F}$ gives rise to a new Feynman category $\mathfrak{F}_{\mathcal{V}}$ of coinvariants. This new Feynman category is of operadic type, i.e. its underlying $\mathcal{V}_{\mathcal{V}}$ is trivial. If $\mathcal{C}$ is cocomplete then it naturally acts on the appropriate colimit in $\mathcal{C}$, that is the colimit of any $\mathcal{O} \in \mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$ forms an element of $\mathcal{F}_{\mathcal{V}}\text{-}\mathcal{O}ps_{\mathcal{C}}$.

A Feynman category $\mathcal{F}$ need not be cocomplete itself. Of course if as previously we map to a cocomplete category, this map factors through its cocompletion $\hat{\mathcal{F}}$. Recall that $\hat{\mathcal{F}}$ is the category of accessible functors $\hat{F} \subset [\mathcal{F}^{op}, \mathcal{S}et]$, and in the case that $\mathcal{F}$ is small, $\hat{\mathcal{F}}$ is the category of all presheaves $Fun(\mathcal{F}^{op}, \mathcal{S}et)$. This is true as well in the enriched case, where $\mathcal{S}et$ is replaced by $\mathcal{E}$. Here one assumes that the underlying category $\mathcal{E}_0$ of $\mathcal{E}$ is cocomplete, [Day70] — see also Remark 6.6 below.

Since $\mathcal{F}$ was monoidal its category of presheaves is also monoidal with the Day convolution product. For two presheaves F and G

$$F \circledast G = \int^{X,Y\in\mathcal{F}} F(X)\otimes G(Y)\otimes Hom_{\mathcal{F}}(-,X\otimes Y)$$

We denote the Yoneda embedding to the cocompletion by $y:\mathcal{F}\to\hat{\mathcal{F}}$.

Lemma 6.1. *The Day convolution of $\hat{\mathcal{F}}$ preserves colimits in each variable if $\otimes$ does in $\mathcal{E}$.*

Proof.

$$\begin{aligned}
\int^{Z} F(Z)\circledast\int^{Z'} G(Z') &= \int^{X,Y\in\mathcal{F}}\int^{Z} F(Z)\times\int^{Z'} G(Z')\times Hom_{\mathcal{F}}(-,X\otimes Y)\\
&= \int^{X,Y\in\mathcal{F}}\int^{Z,Z'\in F} F(Z)\times G(Z')\times Hom_{\mathcal{F}}(-,X\otimes Y)\\
&= \int^{Z,Z'\in\mathcal{F}}\int^{X,Y\in F} F(Z)\times G(Z')\times Hom_{\mathcal{F}}(-,X\otimes Y)\\
&= \int^{Z,Z'\in\mathcal{F}} F(Z)\circledast G(Z')
\end{aligned}$$

□

Definition 6.2. We define the symmetric monoidal category $\mathcal{F}_{\mathcal{V}}$ of coinvariants of a Feynman category to be the full symmetric monoidal subcategory of $([\mathcal{F}^{op},\mathcal{S}et],\circledast)$ generated by $1:=\mathrm{colim}_{\mathcal{V}}\,(y\circ\imath)$.

Lemma 6.3. *If $\otimes$ in $\mathcal{E}$ preserves colimits in each variable, which we have assumed, then $\mathcal{F}_{\mathcal{V}}$ is a subcategory of $\hat{\mathcal{F}}$. In particular $n := 1^{\circledast n} = \mathrm{colim}_{\mathcal{V}^{\times n}}(y\circ\imath^{\otimes n})$ and $Hom_{\mathcal{F}_{\mathcal{V}}}(n,m) := \lim_{\mathcal{V}^{\times n}}\mathrm{colim}_{\mathcal{V}^{\times m}} Hom_{\mathcal{F}}((\imath)^{\otimes n},(\imath)^{\otimes m})$.*

Proof. We will prove the case for 2, the general case being analogous. Using the coend formula for the colimits and co–Fubini:

$$\begin{aligned}
1\circledast 1 &= \int^{Z,Z'}\int^{X} Hom_{\mathcal{F}}(Z,X)\times\int^{Y} Hom_{\mathcal{F}}(Z',Y)\times Hom_{\mathcal{F}}(-,Z\times Z')\\
&= \int^{Z,Z'}\int^{X,Y} Hom_{\mathcal{F}}(Z,X)\times Hom_{\mathcal{F}}(Z',Y)\times Hom_{\mathcal{F}}(-,Z\times Z')\\
&= \int^{X,Y}\int^{Z,Z'} Hom_{\mathcal{F}}(Z,X)\times Hom_{\mathcal{F}}(Z',Y)\times Hom_{\mathcal{F}}(-,Z\times Z')\\
&= \mathrm{colim}_{\mathcal{V}\times\mathcal{V}}\, y\circ\imath\circledast y\circ\imath\\
&= \mathrm{colim}_{\mathcal{V}\times\mathcal{V}}\, y\circ(\imath\otimes\imath) = 2
\end{aligned}$$

and the penultimate equation follows, since y is strong monoidal. The general case follows in the same fashion. The last statement follows from the fact that colimits in the first variable pull out as limits and in the second variable the colimit comes from the fact that in $\hat{\mathcal{F}}$ the colimits are computed pointwise. □

The lemma above lets us characterize the morphisms in $\mathcal{F}_{\mathcal{V}}$ in a practical manner. We first reduce to a skeleton $sk(\mathcal{V})$ of $\mathcal{V}$, that is pick representatives $[*_i], i\in I$ for each of the equivalence classes, and $X_{\mathbf{j}},\mathbf{j}=(j_1,\dots,j_n)\in J_n\subset I^n$ of representatives of $\imath(*_{i_1})\otimes\cdots\otimes\imath(*_{i_n})$. For $\mathcal{E}$, we denote the product by $\prod$, the coproduct by $\coprod$, the tensor structure by $\otimes$, coinvariants by a subscript and invariants by a superscript.

Lemma 6.4.

$$Hom_{\mathcal{F}_{\mathcal{V}}}(n,m) = \prod_{\mathbf{j}\in J_n} \coprod_{\mathbf{k}\in J_m} \bigotimes_{i=1}^{m} Hom_{\mathcal{F}}(X_{\mathbf{j}}, \imath(*_{k_i}))_{Aut(*_{k_i})}^{Aut(*_{j_1})\times\cdots\times Aut(*_{j_n})} \tag{6.1}$$

Proof. This follows by unraveling the definitions and using the fact that $\mathfrak{F}$ is a Feynman category. □

6.2. **Enriched Versions.** This construction readily generalizes to both enriched settings, Cartesian or indexed enriched. In this case, we have to replace the colimit by the appropriate indexed colimit as in the previous paragraph. In particular we are index–enriched over $\mathcal{A}b$ then this simply becomes

$$Hom_{\mathcal{F}_{\mathcal{V}}}(n,m) = \prod_{\mathbf{j}\in J_n} \bigoplus_{\mathbf{k}\in J_m} \bigotimes_{i=1}^{m} Hom_{\mathcal{F}}(X_{\mathbf{j}}, \imath(*_{k_i}))_{Aut(*_{k_i})}^{Aut(*_{j_1})\times\cdots\times Aut(*_{j_n})} \tag{6.2}$$

In this case, each morphism to 1 has components $[\phi_{X_{\mathbf{j}},*_i}]$ of classes of coinvariants of morphisms $\phi_{X_{\mathbf{j}},i} \in Hom_{\mathcal{F}}(X_{\mathbf{j}}, \imath(*_i))^{Aut(*_{j_1})\times\cdots\times Aut(*_{j_n})} \subset Hom_{\mathcal{F}}(X_{\mathbf{j}}, \imath(*_i))$ and any morphism has components $[\phi_{X_{\mathbf{j}},X_{\mathbf{k}}}]$ which are tensor products of these. By abuse of language, we will call a choice of representatives in (6.1) "components" also in the general case.

Let $\mathcal{V}_{\mathcal{V}}$ be the subcategory of $\mathcal{F}_{\mathcal{V}}$ given by the object $1 := \text{colim}_{\mathcal{V}}\imath$ with only the identity morphism and $\imath_{\mathcal{V}}$ be the inclusion.

Theorem 6.5. *The category $\mathfrak{F}_{\mathcal{V}} = (\mathcal{V}_{\mathcal{V}}, \mathcal{F}_{\mathcal{V}}, \imath_{\mathcal{V}})$ is a Feynman category. This also holds in the enriched and enriched indexed case.*

Proof. First notice that $\mathcal{V}_{\mathcal{V}}$ is indeed a groupoid and by construction all elements of $\mathcal{F}_{\mathcal{V}}$ are even equal to tensors of 1, so that the inclusion is essentially surjective. The isomorphisms from $\mathcal{V}_{\mathcal{V}}^{\otimes}$ (up to equivalence) are the commutativity constrains and identities, and these are preserved yielding that the functor to $Iso(\mathcal{F}_{\mathcal{V}})$ is faithful. To show that it is full, we check that these are the only isomorphisms.

If there would be any extra isomorphism in $\mathcal{F}_{\mathcal{V}}$, then the components $\phi_{X_{\mathbf{j}},X_{\mathbf{k}}}$ of the limit/colimit would need to be isomorphisms. But these are taken care of by the colimit/choice of representative, since (i) holds for $\mathcal{F}$, leaving only the identity and the permutations. The condition (ii) holds by Lemma 6.4, and the condition (iii) holds, because it did in $\mathfrak{F}$.

The enriched versions follow analogously using the constructions of §4. □

Remark 6.6. We could have alternatively defined $\mathcal{F}_{\mathcal{V}}$ abstractly to have objects given by the natural numbers with + as a tensor product and morphisms given by the formula of Lemma 6.3. The onus would then be to show that this is indeed a tensor category.

6.3. **Cocompletion for $\mathcal{O}ps$.** Given $\mathcal{O} \in \mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$ where $\mathcal{C}$ is cocomplete by its universal property, $\mathcal{O}$ factors through the cocompletion $\mathcal{O} = y \circ \hat{\mathcal{O}}$; $\hat{\mathcal{O}}$ being the cocontinuous extension. Since $\mathcal{F}_{\mathcal{V}}$ is a subcategory of $\hat{\mathcal{F}}$, we can restrict $\hat{\mathcal{O}}$ to $\mathcal{F}_{\mathcal{V}}$. This is just the pull back under the inclusion. We call the resulting functor $\mathcal{O}_{\mathcal{V}}$ and this is then an element of $\mathcal{F}_{\mathcal{V}}\text{-}\mathcal{O}ps_{\mathcal{C}}$.

6.4. **Generators and weak generators.** From the theorem above, we see that the morphisms in $\mathcal{F}_{\mathcal{V}}$ are one-comma generated by the morphisms $\phi \in Hom_{\mathcal{F}_{\mathcal{V}}}(n,1)$, as they should be. But even if $\mathcal{F}$ is generated in low arity, this might not be the case for $\mathcal{F}_{\mathcal{V}}$. In fact this is seldom the case.

If $\mathcal{E}$ is Abelian, we say $\mathfrak{F}_{\mathcal{V}}$ is weakly generated by morphisms $\phi \in \Phi$, if the components $[\phi_{X_{\mathbf{j}},i}]$ generate the morphisms of $\mathfrak{F}_{\mathcal{V}}$.

6.5. **Feynman categories indexed over $\mathfrak{G}$.** We will now treat the standard examples, those indexed over $\mathfrak{G}$, in some more detail. We assume that we are enriched at least over $\mathcal{A}b$. There are several levels of Feynman categories which are obtained from $\mathfrak{F}$. All of them are of operadic type. The first is $\mathfrak{F}_{\mathcal{V}}$, this is described by insertion operads for graphs with tails. Forgetting tails, using the operator $trun$,[2] $\mathcal{F}_{\mathcal{V}}$ is weakly indexed over $\mathcal{F}_{\mathcal{V}}^{nt}$ which is based on graphs without tails and insertion. Here by weakly indexed, we mean that $trun(\phi\circ\psi)$ is a summand of $trun(\phi)\circ trun(\psi)$.

More importantly, there is a *bona fide* inclusion of Feynman categories, given by $leaf$ and the components of the section weakly generate.

Analyzing $\mathcal{F}_{\mathcal{V}}^{nt}$, we see that it is usually quite complex and many times not generated by its binary morphisms. However, one can often find sub–Feynman categories that correspond to known operads which again weakly generate $\mathcal{F}_{\mathcal{V}}$ after applying $leaf$. Examples for $\mathcal{F}_{\mathcal{V}}^{nt}$ are the operads used in [CL01, KS00, Kau08b, Kau07b, KS10] The morphisms are given by the relevant type of trees and the objects of $\mathcal{V}$ are the (white) vertices.

If $\mathcal{O}$ is in $\mathcal{F}-\mathcal{O}ps_{\mathcal{C}}$ then there is an induced $\mathcal{O}_{\mathcal{V}}$ in $\mathcal{F}_{\mathcal{V}}-\mathcal{O}ps_{\mathcal{C}}$ and pulling back under the inclusion gives the usual structures, such as pre–Lie or Gerstenhaber.

Before giving a tabular theorem, we will analyze the classical cases of an operad and less classically an operad with multiplication in detail. We will refrain from adding the additional technical definitions needed to give the most general framework in which the transition $\mathcal{F}_{\mathcal{V}} \rightsquigarrow \mathcal{F}_{\mathcal{V}}^{nt}$ can be done; although this is certainly possible. We do wish to point out that $\mathcal{F}_{\mathcal{V}}^{nt}$ is canonical.

6.6. Gerstenhaber's construction and its generalizations in terms of Feynman categories.

6.6.1. **Operads and pre–Lie in the Feynman category setup.** Let us compute the standard example which is the Feynman category of operads $\mathfrak{O}_{\mathcal{V}}$.

Let us first look at the unenriched case. In this situation, we can calculate the morphisms. Let $1 = \mathrm{colim} y\circ\imath = \mathrm{colim}_n \mathrm{colim}_{\mathbb{S}_n} y(*_n)$. It is easy to see that the outside colimit lets us consider the components $\lim_{\mathbb{S}_{n_1}\times\cdots\times\mathbb{S}_{n_m}} \mathrm{colim}_{\mathbb{S}_k} Hom(*_{n_1}\amalg\cdots\amalg *_{n_m}, *_k)$. As a sample calculation let us consider a 2–ary morphism. This will lift to a morphism $\circ_i$. Now acting on the left by $\mathbb{S}_n\times\mathbb{S}_m$, and using equation (5.7) we see that $\circ_i$ after acting on it by any element $\mathbb{S}_n\times\mathbb{S}_m$ is equivalent under the action on the right by $\mathbb{S}_{n+m-1}$ to $\circ_j$. Thus there are no morphisms unless we consider enrichment over $\mathcal{A}b$.

In that case, however, by the same reasoning the coefficients in the lift $\phi = \sum_i a_i \circ_i$ must all coincide.

Proposition 6.7. *In the case of enrichment over* $\mathcal{A}b$ *the components of the one-comma generating morphisms of* $\mathfrak{O}_{\mathcal{V}}$ *are in 1–1 correspondence with* $\mathbb{S}_k$ *coinvariant ghost trees of* $\phi \in Hom(*_{n_1}\amalg\cdots\amalg *_{n_m}, *_k)$*. Furthermore* $\mathfrak{F}_{\mathcal{V}}^{nt} = \mathfrak{F}_{pre-Lie}$ *and under the inclusion* $\mathfrak{F}_{pre-Lie} \hookrightarrow \mathfrak{O}_{\mathcal{V}}$ *the components of* $\mathfrak{F}_{pre-Lie}$ *generate.*

Proof. Let us pick an m–ary morphism. Using the lim colim formula, means that we lift to an invariant morphism and consider it up to coinvariants of the target. Picking a lift, we can consider the components in $Hom(*_{n_1}\amalg\cdots\amalg *_{n_m}, *_k)_{\mathbb{S}_k}^{\mathbb{S}_{n_1}\times\cdots\times\mathbb{S}_{n_m}}$ as classes of invariant maps. Now due to the description of morphisms in $\mathfrak{G}$ we see that a class of a morphism up to the $\mathbb{S}_k$ corresponds in a 1–1 fashion to a ghost tree. Since Φ is of crossed type, generating as above the invariance up to $\mathbb{S}_k$ action under the $\mathbb{S}_{n_1}\times\cdots\times\mathbb{S}_{n_m}$ action means that the coefficients of all rooted trees that are isomorphic must be equal. That

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[2] See §A.3.

is we can identify a component with a class of a ghost tree under $\mathbb{S}_k$ invariance, which means that the flags are unlabelled.

If we apply *trun*, we end up with rooted trees without tails. These define a Feynman category under the insertion operad structure, see e.g. 4.5.2. Then *leaf* gives a functor that is identity on objects and sends a tree to the direct sum of components. Here each summand is considered to have source $\amalg_v *_{F_v}$.

The fact that $\mathfrak{F}^{nt}_{\mathcal{V}} = \mathfrak{F}_{pre-Lie}$ is proved in [CL01]. The last statement amounts to the fact that $\mathfrak{O}$ is generated by 2–ary operations, and hence in the iterated pre-Lie structure every possible tree appears at one point. ☐

The proposition lets us recover the results of [KM01]

Corollary 6.8. *For any Abelian category $\mathcal{C}$ and any operad $\mathcal{O}$ there is a pre-Lie and, by taking the commutator, a Lie structure on* $\bigoplus_n \mathcal{O}(n)_{\mathbb{S}_n} = \mathrm{colim}(\mathcal{O})$.

Proof. Just pull back with respect to the inclusion $\mathcal{F}^{nt}_{\mathcal{V}} \to \mathcal{F}_{\mathcal{V}}$ above. ☐

6.6.2. **Non-Σ Operads and the classical bracket.** For the Feynman category of non-Σ operads, we see that any conjugate of any morphism is a component of a universal morphism, since here $\mathcal{V}$ is discrete. A more interesting question is given by the inclusions.

$$\begin{tikzcd}
\mathfrak{O}^{pl} \arrow[r] \arrow[d] & \hat{\mathfrak{O}}^{pl} \arrow[d] & \mathfrak{O}^{pl}_{\mathcal{C}rl^{pl,rt}} \arrow[l] \arrow[d] \\
\mathfrak{O} \arrow[r] & \hat{\mathfrak{O}} & \mathfrak{O}_{\mathcal{C}rl^{rt}} \arrow[l]
\end{tikzcd}$$

and is: can we lift the universal operations, that is find a pre-image of the pre-Lie structure in $\mathfrak{O}^{pl}_{\mathcal{C}rl^{pl,dir}}$ Practically, this is the question, if we can lift the bracket from the coinvariants to the direct sum $\bigoplus \mathcal{O}(n)$. The answer is known to be "yes" and indeed, we can, by using Gerstenhaber's classical formula [Ger63] without signs as in [KM01].

6.6.3. **The cyclic case and the non-Σ version.** [KWZ12] In the cyclic case, we have a similar story. However the universal operations of $\mathcal{C}yc^{nt}_{\mathcal{C}rl}$ are now given by an odd Lie algebra. This is explained in detail in [KWZ12] and has predecessors in [Gin01, BLB02]. The reason is that if we proceed as above, again, we can calculate that there is a universal operation for each now (non-rooted tree) with unlabelled leaves. This operation is not pre-Lie in the usual sense, but we can shift signs. The lifting question was answered in [KWZ12] as well. It is important to emphasize here the lift is only to $\mathfrak{C}_{\mathcal{C}rl^{pl}}$, which means to C_n invariants.

One can further inquire if there is a lift for the inclusion $\mathfrak{O} \to \mathfrak{C}$. Again the answer is "yes" as given in [KWZ12].

6.6.4. **Operads with an (A_∞)–multiplication.** In the case of operads with associative multiplication, the universal operations given by the *nt* version are given by the operad of rooted bi-partite trees first defined in [Kau07b] called the Tamarkin operad in [BBM13]. The foliage operator again provides a lift to the non–Σ case. For an A_∞–multiplication, the relevant operad is the one of [KS00]. In all these examples, the situation is analogous to that of operads, these operads include into the universal operations via the foliage operator and their components generate.

The differential, which is an additional structure can be viewed as follows. Either from coming from a general Maurer–Cartan element for the bracket (a view that will be taken in [War]) or from the general differential d_{Φ^1} which inserts an edge, see [KS00, Kau07b, KS10]

6.6.5. **Di–operads.** For di–operads, we can do a similar calculation. It is similar to the operad case, but the result is slightly different. The $\mathcal{F}_{\mathcal{V}}^{nt}$ Feynman category is that of directed graphs without loops, and again there is weak generation under *leaf*. This Feynman category is no longer generated by its 2–ary part. The subcategory one–comma generated by it is the Feynman category for Lie admissible algebras [Fio]. However, since every graph is an iterate of inserting one edge this subcategory still weakly generates.

6.6.6. **Properads.** For properads, there is an inclusion of Lie–admissible into the universal operations, which was found in [MV09b]. Here the Lie–admissible structure is not equal to the 2–ary generated one, but does weakly generate.

6.6.7. **PROPs.** Similarly for PROPs, in [KM01] a structure of associative algebra was found, which gives rise to an inclusion of Feynman categories $\mathfrak{F}_{Assoc} \to \mathfrak{P}_{\mathcal{C}rl^{dir}}$. Again the components do weakly generate.

6.6.8. **Signs.** It is sometimes useful to consider the odd versions of the classical structures to find the correct signs. This is done in [KWZ12] and is summarized in the following section.

6.7. **Collecting results.** We will assume that we are enriched over $\mathcal{A}b$. In order to phrase the results in general terms, given a morphism, we will call the underlying ghost tree, the topological type. We will call the binary generated part of $\mathfrak{F}_{\mathcal{V}}$ the sub–Feynman category one–comma generated by the binary morphisms and denote it by $\mathfrak{F}_{\mathcal{V}}^{bin}$. Finally we will call an inclusion of Feynman categories to a Feynman category type surjective if each combinatorial type appears (as a summand) in the binary generated part.

We can now rephrase the results of [Ger63, Bar07, CL01, KM01, MV09b, Fio, KWZ12] in this language.

Theorem 6.9. *The Table 1 holds where $\mathfrak{O}$, $\mathfrak{O}_{mult}$, etc are the Feynman categories for operads, operads with multiplication, etc.. With the exception of PROPs and properads the weakly generating suboperad is the suboperad generated by the binary morphisms. In the last two cases, one sums over all binary operations for the inclusion.*

6.8. **Dual construction $\mathcal{F}^{\mathcal{V}}$.** Dual to the Feynman category $\mathcal{F}_{\mathcal{V}}$ which acts on $colim_{\mathcal{V}}(\mathcal{Q})$ for any $\mathcal{Q} \in \mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$, one can define a Feynman category $\mathcal{F}^{\mathcal{V}}$ which acts on $lim_{\mathcal{V}}(\mathcal{Q})$ for any $\mathcal{Q} \in \mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$. In particular if we are index–enriched over $\mathcal{A}b$, we have:

$$Hom_{\mathcal{F}^{\mathcal{V}}}(n,1) := \prod_{v \in \mathcal{V}} \bigoplus_{|X|=n} Hom_{\mathcal{F}}(X, \imath(*_v))_{Aut(X)}^{Aut(*_v)} \tag{6.3}$$

which is suitably dual to the morphisms of $\mathcal{F}_{\mathcal{V}}$; compare equation 6.2. To specify the action on $lim_{\mathcal{V}}(\mathcal{Q})$, it suffices to describe the action on a target factor, which is given by projecting $lim_{\mathcal{V}}(Q)^{\otimes n} \to \otimes Q(v_i) \cong Q(X)$ and then composing with the given morphism $X \to \imath(*_v)$.

6.9. **Infinitesimal automorphism group, graph complex and $\mathfrak{grt}$.** Another nice example is given by $\mathfrak{G}ra^{odd}$. This is the odd version of a Feynman category with trivial $\mathcal{V}$. The morphisms of this Feynman category are the elements of a graph operad, that is graphs/automorphisms with numbered vertices and substitution at the vertices. There are several versions depending on whether one allows symmetries and flags, which are easy modifications. The most natural being $\mathfrak{G}$ itself. The odd version $\mathfrak{G}ra^{odd}$ is then given by graphs together with an orientation of the vertices.

$\mathfrak{F}$	*Feynman category for*	$\mathfrak{F}_{\mathcal{V}}$,$\mathfrak{F}_{\mathcal{V}}^{nt}$	*weakly gen. subcat.*
$\mathfrak{O}$	*Operads*	*rooted trees*	$\mathfrak{F}_{pre-Lie}$
$\mathfrak{O}^{odd}$	*odd operads*	*rooted trees + orientation of set of edges*	*odd pre-Lie*
$\mathfrak{O}^{pl}$	*non-Sigma operads*	*planar rooted trees*	*all* $\circ_i$ *operations*
$\mathfrak{O}_{mult}$	*Operads with mult.*	*b/w rooted trees*	*pre-Lie + mult.*
$\mathfrak{C}$	*cyclic operads*	*trees*	*commutative mult.*
$\mathfrak{C}^{odd}$	*odd cyclic operads*	*trees + orientation of set of edges ++ orientation of the set of edges*	*odd Lie*
$\mathfrak{G}^{odd}$	*unmarked modular*	*connected graphs*	*BV*
$\mathfrak{M}^{odd}$	$\mathfrak{K}$*–modular*	*connected + orientation on set of edges + genus marking*	*odd dg Lie*
$\mathfrak{M}^{nc,odd}$	*nc* $\mathfrak{K}$*-modular*	*orientation on set of edges + genus marking*	*BV*
$\mathfrak{D}$	*Dioperads*	*connected directed graphs w/o directed loops or parallel edges*	*Lie–admissible*
$\mathfrak{P}$	*PROPs*	*directed graphs w/o directed loops*	*associative*
$\mathfrak{P}^{ctd}$	*properads*	*connected directed graphs w/o directed loops*	*Lie–admissible*
$\mathfrak{D}^{\circlearrowleft odd}$	*odd wheeled dioperads*	*directed graphs w/o parallel edges + orientations of edges*	*BV*
$\mathfrak{P}^{\circlearrowleft,ctd,odd}$	*odd wheeled properads*	*connected directed graphs w/o parallel edges + orientations of edeges*	*odd Lie admissible +extra differential*
$\mathfrak{P}^{\circlearrowleft,odd}$	*odd wheeled props*	*directed graphs w/o parallel edges + orientations of edeges*	*BV*

TABLE 1. Here $\mathfrak{F}_{\mathcal{V}}$ and $\mathfrak{F}_{\mathcal{V}}^{nt}$ are given as $\mathcal{F}_{\mathcal{O}}$ for the insertion operad. The former for the type of graph with unlabelled tails and the latter for the version with no tails.

$\mathfrak{G}ra^{\mathcal{V}}$ has a subspace GC_2 consisting of graphs with at least trivalent vertices, which is Kontsevich's graph complex The universal operations are generated by the one vertex loop graph, which gives rise to a BV operator Δ (if one allows disconnected graphs) and the two vertex graph with one edge which gives to a Lie algebra bracket and a differential as discussed in general above. It is a theorem of [Wil13] that $H^0(GC_2) = \mathfrak{grt}$ and if one extends coefficients to $k[[\hbar]]$ and adds the BV operator $D = d + \hbar\Delta$ then also $H^0(GC_2[[\hbar]]) = \mathfrak{grt}$ [MW13]. Now one can pull back or push–forward this construction with morphisms of Feynman categories.

6.10. **Universal operations in iterated Feynman categories.** Let us consider $\mathfrak{F}'^{op}_{\mathcal{V}'}$. Here any morphism $\psi \in Hom_{\mathcal{F}}(Y, X)$ is a component of a universal morphism in $Hom^{op}_{\mathcal{F}'_{\mathcal{V}'}}$ given by precomposition with ψ. It is precisely a component with source any morphism $\phi_0 \colon X \to X_0$ and target $\psi \circ \phi_0$, so that this is inside the slice $\mathfrak{F}|'^{op}_{[X_0]}$. In particular, this is also true for the slices $\mathfrak{F}|'^{op}_{[\imath(*)]}$ for any $* \in \mathcal{V}$, that is we obtain components in $Hom_{\mathcal{F}'_{\mathcal{V}'}}(1,1)^{op}$. If we are enriched over $\mathcal{A}b$, then we can simply sum over these components and obtain an endomorphism $d_\psi \in Hom_{\mathcal{F}'^{op}_{\mathcal{V}'}}(n, n)$, if $|X| = n$. This only depends on the isomorphism class of ψ. We will in particular be interested in $Hom_{\mathcal{F}'^{op}_{\mathcal{V}'}}(1, 1)$.

6.10.1. **A universal morphism in the case of generators.** If we have a set of morphisms Φ, such as a (sub)set of one–comma generators, we can consider $d_\Phi = \sum_{\phi\in\Phi} d_\phi$.

Here d_ϕ acts on χ with $s(\chi) = \bigotimes_v \imath(*_v)$ as $\sum_v id \otimes \cdots \otimes id \otimes \phi \otimes id \otimes \cdots \otimes id \circ \chi$ where ϕ is in the v–th position if $t(\phi) \simeq *_v$ and 0 else. Hence d_Φ gives a morphism on $H = \mathrm{colim}_{\mathcal{F}} Hom(-,-)$ and also gives a morphism in any $Hom_{\mathcal{F}'^{op}_{\mathcal{V}'}}(n,n)$ in particular in $Hom_{\mathcal{F}'^{op}_{\mathcal{V}'}}(1,1)$.

We call a set Φ *resolving* if $d^2_\Phi = 0$ as a morphism in $Hom_{\mathcal{F}'^{op}_{\mathcal{V}'}}(1,1)$, moreover this morphism respects the slices $\mathfrak{F}_{\mathcal{V}}|_{[\imath(*)]}$ and can be restricted to these.

6.10.2. **Differential in the odd case.** The morphism d_Φ is particularly interesting in the case of an ordered presentation, when we are dealing with $\mathfrak{F}^{odd}$.

Proposition 6.10. *Let Φ be a set of one–comma generators, and let Φ^1 be a subset of generators such that its elements are pairwise non–isomorphic and there is an involution $\sim$ on Φ^1 such that for any two elements ϕ, ϕ' if they form a side of a square relation (after tensoring with id), then there is a unique other side of the square given by $\tilde{\phi}, \tilde{\phi}'$*

$$\begin{tikzcd}
X_1 \otimes X_2 \arrow[r, "\phi\otimes id"] \arrow[d, "id\otimes\tilde{\phi}'"] & *_1 \otimes X_2 \arrow[d, "\phi'"] \\
X_1 \otimes *_2 \arrow[r, "\tilde{\phi}"] & *''
\end{tikzcd} \tag{6.4}$$

for suitable objects and where X_1 or X_2 is allowed to be $\mathbb{1}$ if one of the morphisms is unary. This in particular this holds for an ordered presentation.

In the situation above $d^2_{\Phi^1} = 0$, i.e. Φ^1 is a resolving set.

Proof. This is the usual proof. If we apply d_{Φ^1} twice, we obtain pairwise summands that differ by the order of two elements that anti–commute in the above sense or are zero. ☐

Example 6.11. The prime example comes from Feynman categories indexed over $\mathfrak{G}$. For instance for connected graphs, $\Phi = \Phi^1$ is a set of edge insertions considered in Lemma 5.2.

This gives the usual differential for trees and graphs with orientation on the set of edges. In fact this gives the usual differential for the cobar construction and the Feynman transform, see below.

An example where $\Phi^1 \neq \Phi$ is the non–connected case. Here Φ^1 are the edge insertions above, while Φ also contains the mergers ${}_n\boxminus_m$. This will allow us to define Feynman transforms which are partial resolutions.

Remark 6.12. Combining this result with Lemma 3.4, we arrive at the following observation. For any free Feynman categories $\mathfrak{F}^{odd}$ indexed over $\mathfrak{G}$ and enriched over $\mathcal{A}b$. This gives co–differentials $d := d_{\Phi^1}$ on any $\imath_*\imath^*\mathcal{O}(\imath(*))$, where $\imath$ is the inclusion for $\mathfrak{F}^{odd}$.

In particular, if $\mathcal{C} = \mathcal{V}ect^{dg}$, we get a new dg–structure by using the degree given by counting edges and the sum of the internal and new external differential d.

Things turn out to be more interesting, if there is an additional "dual" structure, which allows one to turn the co–differentials into differential. This observation is key to the definition of the transforms of the next section.

7. Feynman transform, the (co)bar construction and Master Equations

In this section we study and generalize a family of constructions known alternately as the bar/cobar construction or the Feynman transform. These constructions have been exhibited for particular Feynman categories in [GK94], [GK95],[GK98], [Gan03], [Val07], [MMS09] and have two important properties:

(1) For well chosen $\mathcal{C}$, double iteration gives a functorial cofibrant replacement thus (often) describing a duality (by single application) on their respective categories of $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$.

(2) These constructions produce objects which represent solutions to the Maurer-Cartan/quantum master equation in associated dg Lie and BV algebras.

The necessary input for the Feynman transform and the (co)bar construction is a Feynman category along with an ordered presentation, which permits the corresponding odd notion, and a resolving subset, which permits the construction of the differential (c.f. above). In practice, one way to establish these structures is to assign an appropriate notion of degree to the morphisms of $\mathfrak{F}$. To this end we introduce the notion of a graded Feynman category as a basis for applying the Feynman transform and the (co)bar construction. The output of these constructions is a quasi-free object (and even cofibrant for well chosen $\mathcal{C}$; see Section 8) whose algebras are solutions to a general master equation that generalizes the usual Maurer-Cartan/quantum master equations mentioned above in property (2). In particular, we recover the equations studied in depth in [KWZ12].

Considering property (1) in this generalized context is more subtle. In particular a double iteration may be a partial or a full resolution depending on the nature of the associated degree function, see Theorem 7.12 and Corollary 8.36.

7.1. **Preliminaries.**

7.1.1. **Quasi-free $\mathcal{O}ps$.** Recall that given a Feynman category $(\mathcal{V}, \mathcal{F}, \imath)$ an $\mathcal{O}p$ $\mathcal{O}$ is free if it is in the image of $\imath_*\colon \mathcal{V}\text{-}\mathcal{M}ods_{\mathcal{C}} \to \mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$. Informally, we say an $\mathcal{O}p$ is quasi-free if it is free after forgetting the differential. To make this precise, suppose that $\mathcal{C}$ is an additive category and let $Kom(\mathcal{C})$ be the category of complexes on $\mathcal{C}$ and let $\mathcal{C}^{\mathbb{Z}}$ be the category of $\mathbb{Z}$ indexed sequences of objects/morphisms of $\mathcal{C}$. Then there are standard inclusion/forgetful functors:

$$\mathcal{C} \leftrightarrows \mathcal{C}^{\mathbb{Z}} \leftrightarrows Kom(\mathcal{C}) \tag{7.1}$$

It is apparent that the functors in 7.1 are strict symmetric monoidal functors between symmetric monoidal categories, and this allows us to move between their respective categories of $\mathcal{F}$-$\mathcal{O}ps$.

Definition 7.1. Let $\mathcal{C}$ be an additive category. An $\mathcal{O}p$ $\mathcal{O} \in \mathcal{F}\text{-}\mathcal{O}ps_{Kom(\mathcal{C})}$ is quasi-free if the push-forward of $\mathcal{O}$ along the functor $Kom(\mathcal{C}) \to \mathcal{C}^{\mathbb{Z}}$ is free in $\mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}^{\mathbb{Z}}}$.

7.1.2. **Dualizing.** For a groupoid $\mathcal{V}$ there is an equivalence, even isomorphism, of categories $\mathcal{V}$ and $\mathcal{V}^{op}$. The functor is just given by identity on objects and inversion on morphisms. This means that there is a functorial isomorphism between $Fun(\mathcal{V}, \mathcal{C})$ and $Fun(\mathcal{V}^{op}, \mathcal{C})$. Combining it with the usual isomorphism $Fun(\mathcal{V}^{op}, \mathcal{C})$ and $Fun(\mathcal{V}, \mathcal{C})$ we get an isomorphism between $Fun(\mathcal{V}, \mathcal{C})$ and $Fun(\mathcal{V}, \mathcal{C}^{op})$. More generally for an additive category $\mathcal{C}$ we get an isomorphism between $Fun(\mathcal{V}, Kom(\mathcal{C}))$ and $Fun(\mathcal{V}, Kom(\mathcal{C}^{op}))$ using the opposite grading and opposite morphisms to interpolate between $Kom(\mathcal{C})$ and $Kom(\mathcal{C}^{op})$ and the aforementioned isomorphisms on morphisms of complexes. We denote the image of a functor $\mathcal{O}$ under this isomorphism by $\mathcal{O}^{op}$ (in both directions).

Definition 7.2. A duality for $\mathcal{C}$ is a contravariant functor $\vee\colon \mathcal{C} \to \mathcal{C}$ such that $\vee\vee\colon \mathcal{C} \to \mathcal{C}$ is equivalent to the identity. Note that a duality for $\mathcal{C}$ induces a equivalence $Kom(\mathcal{C}) \cong Kom(\mathcal{C}^{op})$ by taking the opposite complex grading when passing to the opposite. These equivalences will also be denoted by $\vee$.

7.2. Graded Feynman categories.

Definition 7.3. A degree function on a Feynman category $\mathfrak{F}$ is a map $\deg\colon Mor(\mathcal{F}) \to \mathbb{N}_0$, such that:

- $\deg(\phi \circ \psi) = \deg(\phi) + \deg(\psi)$
- $\deg(\phi \otimes \psi) = \deg(\phi) + \deg(\psi)$
- Every morphism is generated under composition and monoidal product by those of degree 0 and 1.

In addition a degree function is called proper if

- $\deg(\phi) = 0 \Leftrightarrow \phi$ is an isomorphism.

Given a graded Feynman category $\mathfrak{F}$ admitting a degree function and objects $A, B \in \mathcal{F}$ we define $C_n(A,B)$ to be the set of sequences of m composible morphisms for $m \geq n$, such that exactly n of the morphisms have nonzero degree and such that the source of the composition is A and the target of the composition is B, modulo the equivalence relation given by

$$\Big[A \to \cdots \to X_{i-1} \xrightarrow{f} X_i \xrightarrow{g} X_{i+1} \to \cdots \to B\Big] \sim \Big[A \to \cdots \to X_{i-1} \xrightarrow{g\circ f} X_{i+1} \to \cdots \to B\Big] \tag{7.2}$$

if f or g is of degree 0. Note that there is a natural equivalence relation on the set $C_n(A,B)$ given by equating sequences whose composition is equivalent. We call the equivalences classes of this relation composition classes.

Definition 7.4. A graded Feynman category $\mathfrak{F}$ is a Feynman category with a degree function such that there is a free $\mathbb{S}_n$ action on the set $C_n(A,B)$ for every A and B in $\mathcal{F}$ which satisfies the following additional criteria:

- The $\mathbb{S}_n$ action preserves composition class.
- If ϕ is a sequence with no morphisms of degree > 1, the $\mathbb{S}_n$ action is transitive on the representatives of the composition class of ϕ.
- The $\mathbb{S}_n$ action respects composition of sequences, considering $\mathbb{S}_n \times \mathbb{S}_m \subset \mathbb{S}_{n+m}$.

A graded Feynman category is called quadratic if the underlying degree function is proper.

Let us make several remarks about graded Feynman categories. First, note that graded Feynman categories are common; the Feynman categories for non-unital operads, cyclic operads, modular operads are all graded, in fact quadratic. The unital and non-connected versions are no longer quadratic but are still graded Feynman categories.

Second, if $\mathfrak{F}$ is graded, then for $\mathcal{O} \in \mathcal{V}\text{-}\mathcal{M}ods_{\mathcal{C}}$ we may regard the free $\mathcal{O}p\ F(\mathcal{O})$ as being graded, i.e. $F(\mathcal{O}) \in \mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}^{\mathbb{Z}}}$, with the grading induced from that of $Mor(\mathcal{F})$. This grading will support a differential in the (co)bar construction and the Feynman transform.

Third, any graded Feynman category $\mathfrak{F}$ can be given an ordered presentation as follows. Since $\mathfrak{F}$ is graded, any polygonal relation can be expressed via two classes of chains of morphisms $[f], [g] \in C_n(A,B)$ for some n, with no morphisms in the chain of degree > 1. Now, again since $\mathfrak{F}$ is graded, there is a unique $\sigma \in \mathbb{S}_n$ such that $\sigma([f]) = [g]$. We define the value of the polygonal relation to be $(-1)^{|\sigma|}$. Note that this value is multiplicative since the symmetric group action respects composition by assumption, and so determines an ordered presentation. As a consequence, in the $\mathcal{A}b$ enriched context, we can consider the corresponding odd version of a graded Feynman category which takes the twisted $\mathbb{S}_n$ representation on morphism chains of degree n. Note the degree function on $\mathfrak{F}$ allows us to define the degree of morphisms in $\mathcal{F}^{odd}$, and there is a correspondence between the degree 1 morphisms in $\mathcal{F}$ and the degree 1 morphisms in $\mathcal{F}^{odd}$.

Finally, any graded Feynman category has a resolving subset of the generators by taking Φ^1 to be the one-comma generators of degree 1.

7.3. **The differential.** In this subsection we will give a description of what will be the differentials in the (co)bar construction and the Feynman transform using the universality of the involved colimits.

Definition 7.5. Let Y be an object of $\mathcal{F}$ and define e_Y to be the category whose objects are degree 1 morphisms $X \to Y$ and whose morphisms are isomorphisms between the sources. A graded Feynman category is said to be of finite type if the category e_Y has finitely many isomorphism classes of objects for each object Y.

We now assume that $\mathfrak{F}$ is an $\mathcal{A}b$ enriched graded Feynman category of finite type and that $\mathcal{C}$ is an additive category which is complete and cocomplete.

Fix $\mathcal{O} \in \mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$ and $*_v \in \mathcal{V}$. First define $\mathsf{B} := \operatorname{colim}_{(\mathcal{F}^{odd}\downarrow \imath_{\mathfrak{F}^{odd}}(*_v))}(\imath \circ \mathcal{O})^{op} \circ s$. Next define a functor $L\colon Iso(\mathcal{F}^{odd} \downarrow \imath_{\mathfrak{F}^{odd}}(*_v)) \to \mathcal{C}^{op}$ by $\phi \mapsto lim_{e_{s(\phi)}} \imath(\mathcal{O})^{op} \circ s$ and then define $\mathsf{A} := \operatorname{colim}(L)$. We will construct a square zero operator on B as a composite $\mathsf{B} \to \mathsf{A} \to \mathsf{B}$.

To construct these maps first observe that, using the $\mathcal{F}$-$\mathcal{O}p$ structure of $\mathcal{O}$, $\mathcal{O}^{op}(Y)$ admits an obvious cone over the functor $\mathcal{O}^{op} \circ s\colon e_Y \to \mathcal{C}^{op}$ and hence there is a map $\mathcal{O}^{op}(Y) \to L(Y)$ for every object $Y \in \mathcal{F}$.

Next note that A admits a cocone over the functor $(\mathcal{F}^{odd} \downarrow \imath_{\mathfrak{F}^{odd}}(*_v))\colon (\imath \circ \mathcal{O})^{op} \circ s \to \mathcal{C}^{op}$, for if $\phi\colon Y \to \imath_{\mathfrak{F}^{odd}}(*_v)$, then as a colimit A induces a map $L(Y) \to \mathsf{A}$, and hence a map $\mathcal{O}^{op}(Y) \to \mathsf{A}$, by composition. Thus by the universality of B we get a map $\mathsf{B} \to \mathsf{A}$.

Notice that in the composition $\mathcal{O}^{op}(Y) \to L(Y) \to \mathsf{A}$ the first morphism came from something even and the second morphism came from something odd, so the picture to keep in mind in the $\mathfrak{G}$ indexed case is that a term in the image of the map $\mathsf{B} \to \mathsf{A}$ is an even graph inside an odd graph.

For the second morphism we show that B admits a cocone over the functor L, using the finiteness assumption. Again let $\phi\colon Y \to \imath_{\mathfrak{F}^{odd}}(*_v)$. As mentioned above the degree 1 maps in $\mathcal{F}$ correspond to the degree 1 maps in $\mathcal{F}^{odd}$, and such a map $\rho\colon X \to Y$ induces a map $X \to *_v$ and hence a sequence $L(Y) \to \mathcal{O}^{op}(X) \to \mathsf{B}$. Moreover the composite of this sequence is invariant under the action of $Mor(e_Y)$, (recall e_Y is a groupoid), and so each isomorphism class $[\rho]$ in e_Y gives us a map $L(Y) \to \mathsf{B}$. Since e_Y has finitely many isomorphism classes, and since $\mathcal{C}$ is additive, we may sum these maps to give a natural map $L(Y) \to \mathsf{B}$, and hence a cocone of B under L, whence a map $\mathsf{A} \to \mathsf{B}$.

Definition 7.6. We define $d_{\Phi^1}\colon \mathsf{B} \to \mathsf{B}$ via the composition $\mathsf{B} \to \mathsf{A} \to \mathsf{B}$ defined above.

Lemma 7.7. $d^2_{\Phi^1} = 0$

Proof. For any morphism $Y \to *_v$, the induced composite $\mathcal{O}^{op}(Y) \to \mathsf{B} \overset{d^2}{\to} \mathsf{B}$ is a sum over the odd degree 2 morphisms with target Y. Since $\mathfrak{F}$ is graded, the odd degree 2 morphisms are indexed in pairs over the even degree 2 morphisms and each pair adds to 0, hence the claim. □

Remark 7.8. One can define a square zero operator in an analogous way for the cobar construction, and we will also refer to this operator as d_{Φ^1}. It should be noted however that the fact that d_{Φ^1} squares to zero in the bar case comes from the fact that we took a free odd construction, whereas the fact that it squares to zero in the cobar case comes from the fact that we started with an odd $\mathcal{O}p$.

Remark 7.9. The above construction of d_{Φ^1} also works if we replace the categories $\mathcal{C}$ and $\mathcal{C}^{op}$ with the categories $Kom(\mathcal{C})$ and $Kom(\mathcal{C}^{op})$. In this case the operator d_{Φ^1} has degree 1 with respect to the induced grading on the free $\mathcal{O}p$ mentioned above. In particular we will employ the total differential $d_{\mathcal{O}^{op}} + d_{\Phi^1}$.

7.4. **The (Co)bar construction and the Feynman transform.** In this section, we will define the named transformations in the case that we have an ordered presentation and a resolving subset, e.g. if we are indexed over $\mathfrak{G}$.

Definition 7.10. Let $\mathfrak{F}$ be a Feynman category enriched over $\mathcal{A}b$ and with an ordered presentation and let $\mathfrak{F}^{odd}$ be its corresponding odd version. Furthermore let Φ^1 be a resolving subset of one-comma generators and let $\mathcal{C}$ be an additive category. Then:

(1) The bar construction is a functor

$$\mathsf{B}\colon \mathcal{F}\text{-}\mathcal{O}ps_{Kom(\mathcal{C})} \to \mathcal{F}^{odd}\text{-}\mathcal{O}ps_{Kom(\mathcal{C}^{op})}$$

defined by

$$\mathsf{B}(\mathcal{O}) := \imath_{\mathfrak{F}^{odd}}{}_*(\imath_{\mathfrak{F}}^*(\mathcal{O}))^{op}$$

together with the differential $d_{\mathcal{O}^{op}} + d_{\Phi^1}$.

(2) The cobar construction is a functor

$$\Omega\colon \mathcal{F}^{odd}\text{-}\mathcal{O}ps_{Kom(\mathcal{C}^{op})} \to \mathcal{F}\text{-}\mathcal{O}ps_{Kom(\mathcal{C})}$$

defined by

$$\Omega(\mathcal{O}) := \imath_{\mathfrak{F}}{}_*(\imath_{\mathfrak{F}^{odd}}^*(\mathcal{O}))^{op}$$

together with the co-differential $d_{\mathcal{O}^{op}} + d_{\Phi^1}$.

(3) Assume there is a duality equivalence $\vee\colon \mathcal{C} \to \mathcal{C}^{op}$. The Feynman transform is a pair of functors, both denoted FT,

$$\mathsf{FT}\colon \mathcal{F}\text{-}\mathcal{O}ps_{Kom(\mathcal{C})} \leftrightarrows \mathcal{F}^{odd}\text{-}\mathcal{O}ps_{Kom(\mathcal{C})} \colon \mathsf{FT}$$

defined by

$$\mathsf{FT}(\mathcal{O}) := \begin{cases} \vee \circ \mathsf{B}(\mathcal{O}) & \text{if } \mathcal{O} \in \mathcal{F}\text{-}\mathcal{O}ps_{Kom(\mathcal{C})} \\ \vee \circ \Omega(\mathcal{O}) & \text{if } \mathcal{O} \in \mathcal{F}^{odd}\text{-}\mathcal{O}ps_{Kom(\mathcal{C})} \end{cases}$$

Several remarks about this definition are in order. First, for certain Feynman categories there is an equivalence of categories between $\mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$ and $\mathcal{F}^{odd}\text{-}\mathcal{O}ps_{\mathcal{C}}$. For example this is the case for $\mathfrak{O}$ and $\mathfrak{C}$, but not for $\mathfrak{M}$. The general construction of the bar and cobar construction above agrees with the examples in the literature up to this equivalence. These equivalences are given by various operadic suspensions and degree shifts, and were studied in detail in [KWZ12].

Second, note that there are natural transformations $\Omega\mathsf{B} \Rightarrow id$ and $id \Rightarrow \mathsf{B}\Omega$ coming from the naturality of the left Kan extensions involved. Moreover:

Lemma 7.11. *The bar and cobar construction form an adjunction:*

$$\Omega\colon \mathcal{F}^{odd}\text{-}\mathcal{O}ps_{Kom(\mathcal{C}^{op})} \rightleftarrows \mathcal{F}\text{-}\mathcal{O}ps_{Kom(\mathcal{C})} \colon\!\mathsf{B}$$

Proof. This follows from Theorem 1.15. In particular this theorem tell us that there is an adjunction induced by $\imath_{\mathfrak{F}^{odd}}$ between $\overline{\mathcal{V}}^{\otimes}\text{-}\mathcal{O}ps_{Kom(\mathcal{C}^{op})}$ and $\mathcal{F}^{odd}\text{-}\mathcal{O}ps_{Kom(\mathcal{C}^{op})}$. Composing this adjunction with the equivalence of categories $\overline{\mathcal{V}}^{\otimes}\text{-}\mathcal{O}ps_{Kom(\mathcal{C}^{op})} \cong \overline{\mathcal{V}}^{\otimes}\text{-}\mathcal{O}ps_{Kom(\mathcal{C})}$ we still have an adjunction, switching left and right. Call this adjunction 'A'. Also, this theorem tells us that we have an adjunction induced by $\imath$ between $\overline{\mathcal{V}}^{\otimes}\text{-}\mathcal{O}ps_{Kom(\mathcal{C}^{op})}$ and $\mathcal{F}\text{-}\mathcal{O}ps_{Kom(\mathcal{C}^{op})}$, call this adjunction 'B'. Now by definition the bar construction is the composite of the right adjoint of A after the right adjoint of B, and the cobar construction is the composite of the left adjoint of B after the left adjoint of A, whence the claim. □

In light of the previous lemma and the terminology "bar/cobar" it is natural to ask if $\Omega\mathsf{B} \Rightarrow id$ gives a resolution. The answer in general depends on the nature of $\mathfrak{F}$.

Theorem 7.12. *Let $\mathfrak{F}$ be a quadratic Feynman category and $\mathcal{O} \in \mathcal{F}\text{-}\mathcal{O}ps_{Kom(\mathcal{C})}$. Then the counit $\Omega\mathsf{B}(\mathcal{O}) \to \mathcal{O}$ of the above adjunction is a levelwise quasi-isomorphism.*

Proof. We follow the proof in the case of classic operads given in [GK94] Theorem 3.2.16. Fix $*_v \in \imath(\mathcal{V})$. Assume that the internal differential is zero, and thus it is enough to show that $\Omega\mathsf{B}(\mathcal{O})(*_v)$ is acyclic.

First, note that $\Omega\mathsf{B}(\mathcal{O})(*_v)$ is a colimit of $\mathcal{O} \circ s$ over isomorphism classes of triangles:

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "\phi"'] & & Y \arrow[dl] \\ & *_v & \end{tikzcd}$$

and it is enough to show that the colimit over such triangles is acyclic with respect to the differential taking

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "\phi"'] & & Y \arrow[dl] \\ & *_v & \end{tikzcd} \mapsto \sum_{\substack{\rho\colon Y\to Y' \\ \deg(\rho)=1}} \begin{tikzcd} X \arrow[rr] \arrow[dr, "\phi"'] & & Y' \arrow[dl] \\ & *_v & \end{tikzcd}$$

Note that the complex associated to all such triangles splits over the isomorphism class of the source X, and as such we may restrict our attention to the subcomplex $\mathrm{colim}_{X\to Y\to *_v}\mathcal{O}(X)$. Using the fact that colimits commute with $\otimes$ in each variable, we may factor out $\mathcal{O}(X)$ to write said complex as a tensor product of $\mathcal{O}(X)$ with a purely combinatorial complex.

We now use the necessary assumption that $\mathfrak{F}$ is quadratic. Since $\mathfrak{F}$ is quadratic, isomorphism classes of such triangles with source X are exactly the set $\coprod_{n\leq 2} C_n(X, *_v)$. The fact that our Feynman category is graded gives this set the natural structure of a semi-simplicial set which recovers the differential. We will show that this complex is acyclic. In particular we will show that $\coprod_{n\leq 2} C_n(X, *_v)$ splits over composition class as a coproduct of semi-simplicial sets with associated chain complex equal to the augmented chain complex of a simplex of degree 1 less than the degree of the composition class of the morphism.

Let $\phi\colon X \to *_v$ of degree n. Define $\Delta_\phi(m)$ to be generated by the set of triangles

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "\phi"'] & & Y \arrow[dl, "\psi"] \\ & *_v & \end{tikzcd}$$

such that $\deg(\psi) = m+1$. In particular $\Delta_\phi(m) = 0$ if $m \geq n$. Since $\mathfrak{F}$ is graded, the morphism ϕ can be written as a composition of degree 1 morphisms in exactly $n!$ distinct ways up to isomorphism. Pick such a decomposition of ϕ into degree 1 morphisms, labeled as:

$$X =: Y_\emptyset \xrightarrow{e_1^{id}} Y_{\{1\}} \xrightarrow{e_2^{id}} Y_{\{1,2\}} \xrightarrow{e_3^{id}} \cdots \xrightarrow{e_n^{id}} Y_{\{1,\dots,n\}} := *_v$$

For $\sigma \in \mathbb{S}_n$, label the action by σ on the above sequence as:

$$X = Y_\emptyset \xrightarrow{e^\sigma_{\sigma(1)}} Y_{\sigma(\{1\})} \xrightarrow{e^\sigma_{\sigma(2)}} Y_{\sigma(\{1,2\})} \xrightarrow{e^\sigma_{\sigma(3)}} \cdots \xrightarrow{e^\sigma_{\sigma(n)}} Y_{\sigma(\{1,\dots,n\})} = *_v \tag{7.3}$$

Considering all such sequences simultaneously, given a set $S \subset \{1,\dots,n\}$ of size $n-r$, there are exactly r morphisms emanating from Y_S and this set of morphisms has a total order induced by the order on $\{1,\dots,n\} \setminus S$. Define r face maps by sending a sequence $X \to Y_S \to *_v$ to $X \to Y_{S\cup j} \to *_v$ for $j \in \{1,\dots,n\} \setminus S$, along with the appropriate compositions of morphisms $Y_S \to Y_{S\cup j}$. This gives Δ_ϕ the structure of a semi-simplicial set whose chain complex is the augmented chain complex of an $n-1$ simplex.

We thus conclude that the combinatorial chain complex associated to $\coprod_{n\leq 2} C_n(X, *_v)$ is acyclic, from which the claim follows. □

Remark 7.13. The previous theorem can be used to show that when $\mathcal{C} = dg\mathcal{V}ect_k$, under the above hypotheses, the counit of the adjunction is a cofibrant replacement in a model category structure on $\mathcal{F}$-$\mathcal{O}ps_\mathcal{C}$, see Corollary 8.36.

7.5. **A general master equation.** In this subsection let $\mathcal{C}$ be the category $dg\mathcal{V}ect_k$ over a field of characteristic 0. Since the Feynman transform is quasi-free, a map from the underlying $\mathcal{V}$-module has only one obstruction to inducing a map from the Feynman transform, namely that the induced map respects the differentials on the respective sides. In [KWZ12], the following tabular theorem was compiled which states that in various studied cases this obstruction is measured by associated master equations.

Theorem 7.14. *(*[Bar07],[MV09b],[MMS09],[KWZ12]*) Let $\mathcal{O} \in \mathcal{F}$-$\mathcal{O}ps_\mathcal{C}$ and $\mathcal{P} \in \mathcal{F}^{odd}$-$\mathcal{O}ps_\mathcal{C}$ for an $\mathcal{F}$ represented in Table* 2*. Then there is a bijective correspondence:*

$$Hom(\mathsf{FT}(\mathcal{P}), \mathcal{O}) \cong ME(lim(\mathcal{P} \otimes \mathcal{O}))$$

Since we now have described the Feynman transform as a general construction, we can give the general version of the tabular Theorem 7.14.

Fix a Feynman category $\mathfrak{F}$ which permits the Feynman transform (as in Definition 7.10). By definition, the differential d specifies a map

$$Hom_\mathcal{F}(Y, \imath(*_v))^{Aut(*_v)}_{Aut(Y)} \to \coprod_{X \xrightarrow{1} Y} Hom_\mathcal{F}(X, \imath(*_v))^{Aut(*_v)}_{Aut(X)}$$

and by pushing forward the identity map of $\imath(*_v)$, when $Y = \imath(*_v)$, we get a distinguished element on the right hand side, which we can interpret as $Hom_{\mathcal{F}^\mathcal{V}}(1,1)$; cf. 6.8. Similarly for each n the set $Hom_{\mathcal{F}^\mathcal{V}}(n,1)$ has a distinguished element by pushing forward the identity as in the above procedure for all $v \in \mathcal{V}$. The action then specifies an n-cochain in $Hom(lim_\mathcal{V}(\mathcal{Q})^{\otimes n}, lim_\mathcal{V}(\mathcal{Q}))$. Call this cochain $\Psi_{\mathcal{Q},n}$.

Definition 7.15. For a Feynman category $\mathfrak{F}$ admitting the Feynman transform and for $\mathcal{Q} \in \mathcal{F}$-$\mathcal{O}ps_\mathcal{C}$ we define the formal master equation of $\mathfrak{F}$ with respect to $\mathcal{Q}$ to be the completed cochain $\Psi_\mathcal{Q} := \prod \Psi_{\mathcal{Q},n}$. If there is an N such that $\Psi_{\mathcal{Q},n} = 0$ for $n > N$, then we define the master equation of $\mathfrak{F}$ with respect to $\mathcal{Q}$ to be the finite sum:

$$d_\mathcal{Q} + \sum_n \Psi_{\mathcal{Q},n} = 0$$

We say $\alpha \in lim(\mathcal{Q})$ is a solution to the master equation if $d_\mathcal{Q}(\alpha) + \sum_n \Psi_{\mathcal{Q},n}(\alpha^{\otimes n}) = 0$, and we denote the set of such solutions as $ME(lim(\mathcal{Q}))$.

With the groundwork now laid, the tabular Theorem 7.14 can be stated in a general form:

Name of $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$	$\lim(\mathcal{P}\otimes\mathcal{O})$	Algebraic Structure	Master Equation (ME)
operad [GK94],[GJ94]	$\prod_n (\mathcal{P}(n)\otimes\mathcal{O}(n))^{S_n}$	odd pre-Lie	$d(-)+-\circ-=0$
cyclic operad [GK95]	$\prod_n (\mathcal{P}(n)\otimes\mathcal{O}(n))^{S_n^+}$	odd Lie	$d(-)+\frac{1}{2}[-,-]=0$
modular operad [GK98]	$\prod_{(n,g)} (\mathcal{P}(n,g)\otimes\mathcal{O}(n,g))^{S_n^+}$	odd Lie + Δ	$d(-)+\frac{1}{2}[-,-]+\Delta(-)=0$
properad [Val07]	$\prod_{(n,m)} (\mathcal{P}(n,m)\otimes\mathcal{O}(n,m))^{S_n\times S_m}$	odd Lie-admissible	$d(-)+-\circ-=0$
wheeled properad [MMS09]	$\prod_{(n,m)} (\mathcal{P}(n,m)\otimes\mathcal{O}(n,m))^{S_n\times S_m}$	odd Lie ad. + Δ	$d(-)+-\circ-+\Delta(-)=0$
wheeled prop [KWZ12]	$\prod_{(n,m)} (\mathcal{P}(n,m)\otimes\mathcal{O}(n,m))^{S_n\times S_m}$	dgBV	$d(-)+\frac{1}{2}[-,-]+\Delta(-)=0$

TABLE 2. The $\mathcal{O}ps$ in column 1 have a notion of Feynman transform. A citation for the Feynman transform or closely related construction is also given in column 1. In columns 2 and 3, we suppose $\mathcal{P}\in\mathcal{F}^{odd}$-$\mathcal{O}ps_{\mathcal{C}}$ and $\mathcal{O}\in\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$ and give the colimit of the product in column 2 and the algebraic structure that this dg vector space necessarily has. Note that starting with $\mathcal{P}$ odd results in the Lie brackets being odd. The final column gives the master equation relevant to Theorem 7.14. Since the Lie brackets are odd, solutions to these master equations are of degree 0.

Theorem 7.16. *Let $\mathcal{O}\in\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$ and $\mathcal{P}\in\mathcal{F}^{odd}$-$\mathcal{O}ps_{\mathcal{C}}$ for an $\mathcal{F}$ admitting a Feynman transform and master equation. Then there is a bijective correspondence:*

$$Hom(\mathsf{FT}(\mathcal{P}),\mathcal{O})\cong ME(lim(\mathcal{P}\otimes\mathcal{O}))$$

Proof. First, there is a forgetful map $Hom_{dg}(\mathsf{FT}(\mathcal{P}),\mathcal{O})\to Hom_{gr}(F(\mathcal{P}^*),\mathcal{O})$ given by forgetting the differential. By adjunction this specifies a map in $Hom_{\mathcal{V}\text{-}\mathcal{M}ods_{\mathcal{C}}}(\mathcal{P}^*,\mathcal{O})$, which in turn specifies an element of $lim_V(\mathcal{P}\otimes\mathcal{O})$ via the natural isomorphism

$$Hom_{Aut(v)}(\mathcal{P}(v)^*,\mathcal{O}(v))\cong\mathcal{P}(v)\otimes^{Aut(v)}\mathcal{O}(v)$$

Let $\eta\in Hom_{gr}(F(\mathcal{P}^*),\mathcal{O})$ and let $\bar{\eta}$ be the corresponding element in $lim(\mathcal{P}\otimes\mathcal{O})$. It remains to show that η induces a dg map if and only if $\bar{\eta}\in ME(lim(\mathcal{P}\otimes\mathcal{O}))$. To see this fix $v=\imath(*_v)$. Then the pushforward of id_v as above via the differential produces finitely many isomorphism classes of degree 1 morphisms $X\to v$ which we label $\{\gamma_i\}_{i\in I}$. Now each map γ_i contributes to $\Psi_{\mathcal{Q},|X|}$ via the map $Hom(\mathcal{P}^*(X),\mathcal{O}(X))\to Hom(\mathcal{P}^*(v),\mathcal{O}(v))$ given by convolution:

$$\mathcal{P}^*(v)\overset{\gamma_i^*}{\to}\mathcal{P}^*(X)\cong\otimes\mathcal{P}^*(v_j)\overset{\otimes\eta_{v_j}}{\to}\otimes\mathcal{O}(v_j)\cong\mathcal{O}(X)\overset{\gamma_i}{\to}\mathcal{O}(v)\tag{7.4}$$

and the fact that $\mathcal{F}^{\mathcal{V}}$ acts by first projecting tells us that this convolution is equal to the γ_i action on $\bar{\eta}$, from which the claim follows.

□

Remark 7.17. Interpreting master equation solutions as morphisms suggests a notion of homotopy equivalence of such solutions: namely two solutions are homotopic if the associated morphisms are homotopic. However, in order to make sense of the notion of homotopy classes of morphisms we need to study the homotopical algebra of $\mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$. This is done in Section 8; see Theorem 8.39.

8. HOMOTOPY THEORY OF $\mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$.

In this section we give conditions on a symmetric monoidal category and model category $\mathcal{C}$ which permit the construction of a model category structure on $\mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$. In so doing we generalize prior work done in particular Feynman categories including operads [Spi01], [Hin97b], [BM03], props [Fre10], properads [MV09c], and colored props [JY09].

After establishing the main theorem (Theorem 8.15) we consider several implications. For example our perspective of $\mathcal{O}ps$ as symmetric monoidal functors allows for consideration of the relationships between these model categories under the adjunctions induced by either morphisms of Feynman categories or adjunctions of the base categories. As another example we show that, in a dg context, applying the bar construction/Feynman transform returns a cofibrant $\mathcal{F}\text{-}\mathcal{O}p$, and as a result the bar-cobar construction/double Feynman transform gives a functorial cofibrant replacement in the above model structure when $\mathfrak{F}$ is quadratic. The fact that the bar construction/Feynman transform is cofibrant gives us, via the general theory, the notion of homotopy classes of maps from the bar construction/Feynman transform. As seen above, such maps are given by Maurer Cartan elements in a certain dg Lie algebra [KWZ12]. There is then a notion of homotopy equivalence on both sides of this correspondence and we show these notions coincide.

Throughout this section we let $\mathcal{C}$ denote a category which is a model category and a closed symmetric monoidal category. We do not assume any compatibility between the model and monoidal structures *a priori*, but we will make such assumptions as necessary.

8.1. **Preliminaries.**

We will begin the section by establishing some preliminaries needed to prove this section's main result, Theorem 8.15.

8.1.1. **Reduction.**

In order to endow the category $\mathcal{O}ps$ over a Feynman category $\mathfrak{F}$ with a model structure it will be enough to consider what the reduction of $\mathfrak{F}$. Recall (cf. Lemma 1.5) that the category $\mathfrak{F}$–$\mathcal{O}ps$ over a Feynman category is equivalent to the category $\tilde{\mathfrak{F}}\text{-}\mathcal{O}ps$ of its reduction. In what follows we will establish a model structure in the context of strict Feynman categories, and the following lemma tells us that this is sufficient to establish a model structure on the categories $\mathcal{F}\text{-}\mathcal{O}ps$ over all Feynman categories.

Lemma 8.1. *Let $\mathcal{C}$ be a model category and let $F : C \rightleftarrows D : G$ be an equivalence of categories. Then D is a model category by defining ϕ to be a weak equivalence/fibration/cofibration if and only if $G(\phi)$ is.*

Proof. We prove this by direct verification of the axioms as enumerated in [Hir03].

M1: (limit axiom) The functor F is both a left adjoint and a right adjoint of G. Thus it preserves all colimits and limits. In particular D is complete and cocomplete since C is.

M2: (two out of three axiom) Let $f, g \in Mor(D)$ be composible with two of f, g, fg weak equivalences. Then the corresponding two of the three of $G(f), G(g), G(fg) = G(f)G(g)$ are weak equivalences in C, and hence so is the third. Thus by definition of the weak equivalences in D, the third of the list f, g, fg is also a weak equivalence.

M3: (retract axiom) Suppose f is a retract of g in D. This means that there is a commutative diagram in D:

$$\begin{tikzcd}
d_1 \arrow[r] \arrow[d, "f"] & d_2 \arrow[r] \arrow[d, "g"] & d_1 \arrow[d, "f"] \\
d_3 \arrow[r] & d_4 \arrow[r] & d_3
\end{tikzcd}$$

such that the top and bottom lines are the identity. Applying G to the diagram we see that $G(f)$ is a retract of $G(g)$. If g is a weak equivalence/fibration/cofibration then so is $G(g)$, then so is $G(f)$ by **M3** in C, then so is f by definition. Hence **M3** holds in D.

M4: (lifting axiom) Consider the following diagram in D

$$\begin{tikzcd}
d_1 \arrow[r] \arrow[d, "i"] & d_2 \arrow[d, "p"] \\
d_3 \arrow[r] & d_4
\end{tikzcd}$$

Suppose that either i is a cofibration and p is an acyclic fibration, or that i is an acyclic cofibration and p is a fibration. Applying G to the diagram we get a commutative diagram in C which admits a lift due to **M4** in D. This lift is in the image of G since the functor G is full. Moreover, the fact that the lift commutes in C means that the preimage commutes in D since G is faithful. Thus **M4** holds in D.

M5: (factorization axiom) Let f be a morphism in D. Then $G(f)$ can be written as a cofibration followed by an acyclic fibration, say $G(f) = rs$. Let c be the source of r and the target of s. Since G is essentially surjective, there exists a c' in the image of G such that $c \cong c'$. Notice that by the lifting characterizations of acyclic fibrations and acyclic cofibrations (in the model category C), every isomorphism in C is both an acyclic fibration and an acyclic cofibration. Thus by composing with these isomorphisms, $G(f)$ can be written as a cofibration followed by an acyclic fibration as $G(f) = r's'$ where the source of r' and the target of s' are c'. Then since c' is in the image of G, and since G is full and faithful, this factorization is in the image of G, hence f factors as a cofibration followed by an acyclic fibration. Similarly f factors as an acyclic cofibration followed by a fibration. ☐

8.1.2. **Limits and colimits in $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$.** Let $\mathfrak{F}$ be a strict Feynman category.

We make the following definition for convenience.

Definition 8.2. Given a Feynman category $(\mathcal{V}, \mathcal{F}, \imath)$ we define $\mathcal{V}_{id}$ to be the category with the same objects as $\mathcal{V}$ and only identity morphisms. A $\mathcal{V}$ sequence in $\mathcal{C}$ is defined to be a functor $\mathcal{V}_{id} \to \mathcal{C}$. The category of such is denoted $\mathcal{V}$-Seq$_{\mathcal{C}}$.

Lemma 8.3. *The forgetful functor $\mathcal{V}$-$\mathcal{M}ods_{\mathcal{C}} \to \mathcal{V}$-$Seq_{\mathcal{C}}$ creates all limits and colimits.*

Proof. Let $H\colon \mathcal{V}$-$\mathcal{M}ods_{\mathcal{C}} \to \mathcal{V}$-Seq$_{\mathcal{C}}$ be the forgetful functor. Then let $\alpha\colon J \to \mathcal{V}$-$\mathcal{M}ods_{\mathcal{C}}$ be a functor such that $colim(H \circ \alpha)$ exists. Define $L := colim(H \circ \alpha)$. Then we will show that L is naturally a $\mathcal{V}$-module. Let $\psi\colon v \to w$ be a morphism in $\mathcal{V}$. Then $L(w)$ naturally forms a cocone over the functor $H\alpha(-)(v)\colon J \to C$, and the colimit of this functor is $L(v)$. As such we get a map $L(v) \to L(w)$ which we define to be $L(\psi)$. Then L naturally forms a cocone over α, which is limiting since its image under H is. A similar argument show that H creates limits. ☐

Lemma 8.4. *The forgetful functor $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}} \to \mathcal{V}$-$\mathcal{M}ods_{\mathcal{C}}$ creates all limits, filtered colimits, and reflexive coequalizers.*

Proof. Let $G\colon \mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}} \to \mathcal{V}\text{-}\mathcal{M}ods_{\mathcal{C}}$ be this forgetful functor. First we consider limits. Consider a diagram

$$J \xrightarrow{\alpha} \mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}} \xrightarrow{G} \mathcal{V}\text{-}\mathcal{M}ods_{\mathcal{C}}$$

such that the limit $lim(G \circ \alpha)$ exists in $\mathcal{V}\text{-}\mathcal{M}ods_{\mathcal{C}}$. Call this limit L. Now L is *a priori* a $\mathcal{V}$-module, but it has a natural $\mathcal{F}\text{-}\mathcal{O}p$ structure as follows. First extend L to $Iso(\mathcal{F})$ strict monoidally. Let $\lambda\colon \otimes_{i=1}^{n} v_i \to v$ be a generating morphism in $\mathcal{F}$. Then for every morphism $a \xrightarrow{f} b$ in J we have the diagram:

$$\begin{tikzcd}
\otimes_{i=1}^{n}\alpha(a)(v_i) \arrow[rr, "\alpha(a)(\lambda)"] & & \alpha(a)(v) \arrow[rr, "\alpha(f)(v)"] & & \alpha(b)(v) & & \otimes_{i=1}^{n}\alpha(b)(v_i) \arrow[ll, "\alpha(b)(\lambda)"'] \\
& & & L(v) \arrow[ul] \arrow[ur] & & & \\
& & & \otimes_{i=1}^{n}L(v_i) \arrow[uulll] \arrow[uurrr] \arrow[u, dotted, "\exists!"] & & &
\end{tikzcd}$$

where the diagonal arrows come from the cone morphisms in $\mathcal{V}\text{-}\mathcal{M}ods_{\mathcal{C}}$. Using the universality of the limit the diagram gives us the morphism $\otimes_{i=1}^{n}L(v_i) \to L(v)$ which we define to be $L(\lambda)$. This makes L an $\mathcal{F}\text{-}\mathcal{O}p$, after extending monoidally, and makes the cone maps into morphisms in $\mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$. Thus L admits a cone over α. Finally, the fact that this cone is limiting follows from the fact that the corresponding cone over $G \circ \alpha$ is. Hence G creates limits.

Now let $\beta\colon K \to \mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$ be a functor such that $colim(G \circ \beta)$ exists and such that K is either a filtered category or a reflexive coequalizing category. Define C to be this colimit. Now C is *a priori* a $\mathcal{V}$-module, and we will show it is naturally an $\mathcal{F}\text{-}\mathcal{O}p$. First extend C monoidally to $Iso(\mathcal{F})$. On morphisms it is enough to define the image by C of the generating morphisms. Let $\psi\colon \otimes_{i=1}^{n} v_i \to v_0$ be such a morphism (so $v_i \in \mathcal{V}$). Notice that for each $v \in \mathcal{V}$, $colim(\beta(-)(v)) \cong colim(G \circ \beta)(v)$, and hence $\times_i colim(\beta(-)(v_i))$ exists in $\mathcal{C}^{\times n}$ and is isomorphic to the colimit of the functor

$$K \xrightarrow{\Delta} K^{\times n} \xrightarrow{\times\beta(-)(v_i)} \mathcal{C}^{\times n}$$

which for some $x \in K$ sends x to $\times_i \beta(x)(v_i)$. Since $\mathcal{C}$ is monoidally closed, $\otimes$ preserves colimits in each variable separately, and since the category K is supposed to be either filtered or reflexive coequalizing, we have that $\otimes$ preserves colimits in all variables (see [Fre09] section 1.2 or [Rez96] lemma 2.3.2), that is:

$$\otimes_i colim(\beta(-)(v_i)) \cong colim(\otimes_i \beta(-)(v_i))$$

and using ψ to construct a cocone, the universality of this colimit produces a map to $colim(\beta(-))(v_0)$. Thus,

$$\begin{aligned} C(\otimes_i v_i) = \otimes_i colim(G \circ \beta)(v_i) &\cong \otimes_i colim(\beta(-)(v_i)) \\ &\cong colim(\otimes_i \beta(-)(v_i)) \to colim(\beta(-)(v_0)) \cong colim(G \circ \beta)(v_0) = C(v_0) \end{aligned}$$

gives the image of ψ. Extending monoidally C may be viewed as an object in $\mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$ and it is immediate from the construction that the cocone maps for C in $\mathcal{V}\text{-}\mathcal{M}ods_{\mathcal{C}}$ lift to maps in $\mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$.

It thus remains to show that as a cocone over β in $\mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$, C is universal. If Z is any cocone over β in $\mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$ then by universality in $\mathcal{V}\text{-}\mathcal{M}ods_{\mathcal{C}}$ there is a unique morphism of underlying $\mathcal{V}$-modules $C \to Z$. Using this morphism of $\mathcal{V}$-modules and given a generating morphism $\phi\colon X \to v$, it is possible to write $Z(v)$ as a cocone over $\beta(-)(X)$

in two *a priori* distinct ways, but the uniqueness of the morphism $C(X) \to Z(v)$ given by the universality of $C(X)$ in $\mathcal{C}$ ensures that these two morphisms are in fact the same. As such this morphism $C \to Z$ lifts to a morphism in $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$, completing the proof. □

Lemma 8.5. *Let $\mathcal{C}$ be a category with all small limits and colimits. Then the categories $\mathcal{V}$-Seq$_{\mathcal{C}}$, $\mathcal{V}$-$\mathcal{M}ods_{\mathcal{C}}$, and $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$ have all small limits and colimits.*

Proof. Since $\mathcal{C}$ has all small limits and colimits, the category $\mathcal{V}$-Seq$_{\mathcal{C}}$ has all small limits and colimits levelwise. Thus by Lemma 8.3 the category $\mathcal{V}$-$\mathcal{M}ods_{\mathcal{C}}$ has all small limits and colimits. Then applying Lemma 8.4 we see that the category $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$ has all small limits as well. Note in addition that by Lemma 8.4 $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$ also has all reflexive coequalizers. It thus remains to show that $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$ has all small colimits. In order to show this we generalize an argument of Rezk in ([Rez96] proposition 2.3.5).

As above, let F and G be the free and forgetful functors respectively between $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$ and $\mathcal{V}$-$\mathcal{M}ods_{\mathcal{C}}$. Let $\alpha\colon J \to \mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$ be a functor from a small category J. For a given $\mathcal{F}$-$\mathcal{O}p$ $\mathcal{P}$ we define two functors $\Phi, \Psi\colon J^{op} \to \mathcal{V}$-$\mathcal{M}ods_{\mathcal{C}}$ as follows.

$$\Phi := Hom_{\mathcal{V}\text{-}\mathcal{M}ods_{\mathcal{C}}}(GFG\alpha(-), G(\mathcal{P})) \quad \text{and} \quad \Psi := Hom_{\mathcal{V}\text{-}\mathcal{M}ods_{\mathcal{C}}}(G\alpha(-), G(\mathcal{P}))$$

There exist two natural transformations $\Psi \rightrightarrows \Phi$ defined as follows:

1. precompose $G\alpha(-) \to \mathcal{P}$ with the natural transformation $G(FG)\alpha(-) \to G\alpha(-)$
2. take the image of $G\alpha(-) \to G(\mathcal{P})$ by GF and postcompose with the natural map $FG(\mathcal{P}) \to \mathcal{P}$.

These two natural transformations give us the following:

$$\begin{array}{rcl} lim(\Psi) & \rightrightarrows & lim(\Phi) \\ lim\left(Hom_{\mathcal{V}\text{-}\mathcal{M}ods_{\mathcal{C}}}(G\alpha(-), G(\mathcal{P}))\right) & \rightrightarrows & lim\left(Hom_{\mathcal{V}\text{-}\mathcal{M}ods_{\mathcal{C}}}(GFG\alpha(-), G(\mathcal{P}))\right) \\ Hom_{\mathcal{V}\text{-}\mathcal{M}ods_{\mathcal{C}}}\left(colim(G\alpha), G(\mathcal{P})\right) & \rightrightarrows & Hom_{\mathcal{V}\text{-}\mathcal{M}ods_{\mathcal{C}}}\left(colim(GFG\alpha), G(\mathcal{P})\right) \\ Hom_{\mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}}\left(F(colim(G\alpha)), \mathcal{P}\right) & \rightrightarrows & Hom_{\mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}}\left(F(colim(GFG\alpha)), \mathcal{P}\right) \end{array}$$

Now since the above pair of morphisms is natural in $\mathcal{P}$, we can apply the Yoneda embedding theorem to see that these maps are given by maps of the sources, i.e. maps

$$F(colim(GFG\alpha)) \rightrightarrows F(colim(G\alpha)) \tag{8.1}$$

and these maps have an obvious candidate for a section, namely the map induced by the natural transformation $id \to GF$. Checking this we see that this does give us a section, as an immediate consequence of the fact that the composites $G \to GFG \to G$ and $F \to FGF \to F$ are the identity. Thus line 8.1 can be represented as a reflexive coequalizing diagram. Define Q to be its coequalizer, which exists by Lemma 8.4.

To complete the proof we will show that Q is the colimit of α in $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$. First notice that $G(Q)$ admits a cocone over $G \circ \alpha$ by the composite $colim(G \circ \alpha) \to GF(colim(G \circ \alpha)) \to G(Q)$. Call these cocone morphisms λ. By adjointness, diagram 8.1 gives us morphisms

$$colim(GFG\alpha) \rightrightarrows GFcolim(G\alpha) \to G(Q) \tag{8.2}$$

Examining the natural transformations above we see the two compositions can be described by the following diagram:

$$\begin{tikzcd} GFG\alpha(-) \arrow[r] \arrow[d, "GF(\lambda)"'] & G\alpha(-) \arrow[d, "\lambda"] \\ GFG(Q) \arrow[r] & GQ \end{tikzcd}$$

which commutes since Q is a coequalizer. The fact that this diagram commutes tells us that the λ are actually morphisms of $\mathcal{F}$-$\mathcal{O}ps$, and in particular the cocone $G(Q)$ over $G \circ \alpha(-)$ lifts to a cocone Q over α. The fact that this cocone is limiting follows immediately from the universality of the coequalizer. Thus the colimit $colim(\alpha)$ exists in $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$, completing the proof of the lemma. □

8.1.3. **Monoidal model categories.** We assume the basics of model category theory and refer to [Hir03] and [Hov99] thoroughout. We make several recollections here for future use.

Definition 8.6. A category which is both symmetric monoidal and a model category satisfies the pushout product axiom (PPA) if for any pair of cofibrations $f_1\colon X_1 \hookrightarrow Y_1$ and $f_2\colon X_2 \hookrightarrow Y_2$ the induced map from the pushout

$$X_1 \otimes Y_2 \coprod_{X_1 \otimes X_2} X_2 \otimes Y_1 \to Y_1 \otimes Y_2$$

is a cofibration which is acyclic if f_1 or f_2 is.

Lemma 8.7. *Let $\mathcal{D}$ be a model category and let $\mathcal{E}$ be a small category whose only morphisms are identity morphisms. Then the category of functors $\mathbf{Fun}(\mathcal{E}, \mathcal{D})$ carries a model category structure where a morphism (natural transformation) $\phi\colon \mathcal{P} \to \mathcal{Q}$ is a weak equivalence/fibration/cofibration if and only if $\phi\colon \mathcal{P}(X) \to \mathcal{Q}(X)$ is for each $X \in \mathcal{E}$. Moreover if $\mathcal{D}$ is cofibrantly generated so is $\mathbf{Fun}(\mathcal{E}, \mathcal{D})$.*

Proof. See e.g. [Hir03] proposition 7.1.7 and proposition 11.1.10. □

8.1.4. **Transfer Principle.** The technique that we will use to endow the category of $\mathcal{F}$-$\mathcal{O}ps$ in $\mathcal{C}$ with a model structure is to transfer the model structure across adjunctions from $\mathcal{V}$-sequences to $\mathcal{V}$-modules and then to $\mathcal{F}$-$\mathcal{O}ps$.

Theorem 8.8. *(*[Hir03] *Theorem 11.3.2) Let $\mathcal{C}$ be a cofibrantly generated model category with generating cofibrations I and generating acyclic cofibrations J. Let $\mathcal{D}$ be a category which is complete and cocomplete and let $F: \mathcal{C} \rightleftarrows \mathcal{D} : G$ be an adjunction. Further suppose that*

1. *Both $F(I)$ and $F(J)$ permit the small object argument and*
2. *G takes relative $F(J)$-cell complexes to weak equivalences.*

Then there is a cofibrantly generated model category structure on $\mathcal{D}$ in which $F(I)$ is the set of generating cofibrations, $F(J)$ is the set of generating acyclic cofibrations, and a map is a weak equivalence (resp. fibration) if and only if its image by G is.

In order to apply Theorem 8.8 in our cases of interest we will reformulate the hypotheses to derive the following corollary. This reformulation is inspired by [SS00], [BM03] and [Fre10].

Corollary 8.9. *Let $\mathcal{C}$ be a cofibrantly generated model category with generating cofibrations I and generating acyclic cofibrations J. Let $\mathcal{D}$ be a category which is complete and cocomplete and let $F\colon \mathcal{C} \rightleftarrows \mathcal{D} : G$ be an adjunction. Further suppose the following:*

(i) *All objects of $\mathcal{C}$ are small,*
(ii) *G preserves filtered colimits,*
(iii) *$\mathcal{D}$ has a fibrant replacement functor,*
(iv) *$\mathcal{D}$ has functorial path objects for fibrant objects.*

Then there is a cofibrantly generated model category structure on $\mathcal{D}$ in which $F(I)$ is the set of generating cofibrations, $F(J)$ is the set of generating acyclic cofibrations, and a map is a weak equivalence (resp. fibration) if and only if its image by G is.

Proof. Define in $\mathcal{D}$ a class of weak equivalences and fibrations as in the statement, and define a class of cofibrations as morphisms having the left lifting property with respect to acyclic fibrations. It is enough to show that the conditions $(i-iv)$ in the statement imply conditions (1) and (2) of Theorem 8.8. In particular we show that $(i),(ii)$ imply (1) and $(iii),(iv)$ imply (2).

First, since G preserves filtered colimits, F preserves small objects. Since all objects in $\mathcal{C}$ are small, all objects of the form $F(A)$ are small in $\mathcal{D}$. In particular, given a generating cofibration (resp. acyclic cofibration) $F(A) \to F(B)$ in $F(I)$ (resp. $F(J)$), $F(A)$ is small in $\mathcal{D}$, and so in particular is small relative to the subcategory of $F(I)$-cell (resp. $F(J)$-cell) complexes. Thus $F(I)$ (resp. $F(J)$) permits the small object argument.

Second, let ϕ be a relative $F(J)$-cell complex. Then ϕ is an $F(J)$-cofibration ([Hir03] proposition 10.5.10) in $\mathcal{D}$ and hence has the LLP with respect to $F(J)$-injectives. One quickly sees that every fibration in $F(J)$ is an $F(J)$-injective, and thus ϕ is a cofibration in $\mathcal{D}$. Let R denote the fibrant replacement functor in $\mathcal{D}$ and P denote the path objects for fibrants. Let $\lambda\colon Y \to RX$ denote the resulting lift of ϕ and consider the diagram:

$$\begin{tikzcd}
X \arrow[r, "\phi"] \arrow[d, "\phi"] & Y \arrow[r, "R_Y"] & R(Y) \arrow[r, "\sim"] & P(RY) \arrow[d, two heads] \\
Y \arrow[rrr, "{(R_Y,R(\phi)\circ\lambda)}"'] \arrow[urrr, dotted] & & & RY\prod RY
\end{tikzcd}$$

Since R is functorial, the diagram commutes. Note that since G preserves limits, weak equivalences, and fibrations it also preserves path objects and thus $G(R_Y)$ is right homotopic to $G(R(\phi)\circ\lambda)$ in $\mathcal{C}$. Since $G(R_Y)$ is a weak equivalence it follows ([Hir03] proposition 7.7.6) that $G(R(\phi)\circ\lambda)$ is a weak equivalence. It follows that $G(R(\phi))$ and $G(\lambda)$ induce an isomorphism in the homotopy category of $\mathcal{C}$ and are thus weak equivalences. It follows from the 2-out-of-3 axiom that $G(\phi)$ is a weak equivalence which completes the proof. □

Remark 8.10. The smallness requirement (condition (i)) does not hold in the category of topological spaces, but we will circumvent this problem following [Fre10], see Example 8.26.

8.1.5. **Strictification.** In our arguments below we would like to compose an $\mathfrak{F}$-$\mathcal{O}p$ with a lax monoidal functor. In general however, this composition would no longer be in $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$, and thus we introduce the following notation.

Definition 8.11. Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal categories, let $\mathfrak{F}$ be a strict Feynman category, and let $\gamma\colon \mathcal{C} \to \mathcal{D}$ be a lax symmetric monoidal functor. We define a functor $\hat{\gamma}\colon \mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}} \to \mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{D}}$ as follows. For $\mathcal{O} \in \mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$ we define $\hat{\gamma}(\mathcal{O})$ restricted to $\mathcal{V}$ to be the composition $\gamma \circ \mathcal{O}$. We then extend strict monoidally to all objects,

$$\hat{\gamma}(\mathcal{O})(X) := \bigotimes_{i\in\Lambda} \gamma(\mathcal{O}(v_i)) \quad \text{for } X = \otimes_{i\in\Lambda} v_i$$

Finally for morphisms we define the image of generating morphisms by composing with the monoidal structure maps for the symmetric monoidal functor γ, that is for $\phi\colon X \to v$ we have,

$$\hat{\gamma}(\phi)\colon\ \otimes_{i\in\Lambda}\gamma(\mathcal{O}(v_i)) \to \gamma(\otimes_{i\in\Lambda}\mathcal{O}(v_i)) \overset{\gamma(\phi)}{\to} \gamma(\mathcal{O}(v))$$

and then extend strict monoidally to all morphisms.

8.1.6. **$\otimes$-coherent path objects.**

Definition 8.12. We say that $\mathcal{C}$ has functorial path objects for fibrant objects if there is a functor $P\colon \mathcal{C}^{\text{fib}} \to \mathcal{C}$, and, for each object A in $\mathcal{C}$, factorizations of the diagonal $\Delta_A = \psi_A \circ \phi_A$ as a weak equivalence followed by a fibration, such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "\phi_A", "\sim"'] \arrow[d, "f"] & P(A) \arrow[r, two heads, "\psi_A"] \arrow[d, "P(f)"] & A\coprod A \arrow[d, "f\coprod f"] \\
B \arrow[r, "\phi_B", "\sim"'] & P(B) \arrow[r, two heads, "\psi_B"] & B\coprod B
\end{tikzcd}$$

Recall the notion of a symmetric monoidal natural transformation between symmetric monoidal functors. In particular η being a symmetric monoidal natural transformation requires:

$$\begin{tikzcd}
F(A)\otimes F(B) \arrow[r, "\eta_A\otimes\eta_B"] \arrow[d] & G(A)\otimes G(B) \arrow[d] \\
F(A\otimes B) \arrow[r, "\eta_{A\otimes B}"] & G(A\otimes B)
\end{tikzcd}$$

We will make use of the following example of a symmetric monoidal natural transformation:

Example 8.13. Let $\mathcal{C}$ be symmetric monoidal category with products. Define a symmetric monoidal functor $\Delta_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ by taking $\Delta_{\mathcal{C}}(X) := X\coprod X$, the product on morphisms and the symmetric structure given by

$$(\pi_1\otimes\pi_1)\coprod(\pi_2\otimes\pi_2)\colon (A\coprod A)\otimes(B\coprod B) \to (A\otimes B)\coprod(A\otimes B)$$

There is a canonical natural transformation $\mathrm{id}_{\mathcal{C}} \Rightarrow \Delta_{\mathcal{C}}$, which we call the diagonal transformation, given by the diagonal maps $X \to X\coprod X$, and it is easily checked that this natural transformation is symmetric monoidal.

Definition 8.14. Let $\mathcal{C}$ be a symmetric monoidal category and a model category. We say $\mathcal{C}$ has $\otimes$-coherent path objects if there is a symmetric monoidal functor $P\colon \mathcal{C} \to \mathcal{C}$ along with symmetric monoidal natural transformations

$$\mathrm{id}_{\mathcal{C}} \overset{\phi}{\Rightarrow} P \overset{\psi}{\Rightarrow} \Delta_{\mathcal{C}}$$

which factor the diagonal natural transformation $id_{\mathcal{C}} \Rightarrow \Delta_{\mathcal{C}}$ (see Example 8.13) and such that ϕ_A is a weak equivalence and ψ_A is a fibration for every $A \in \mathcal{C}$. Furthermore, we say $\mathcal{C}$ has $\otimes$-coherent path objects for fibrant objects if there is a symmetric monoidal functor $P\colon \mathcal{C}^{\text{fib}}_{\otimes} \to \mathcal{C}$ (where $\mathcal{C}^{\text{fib}}_{\otimes} := (\mathcal{C}^{\text{fib}})_{\otimes}$) along with symmetric monoidal natural transformations

$$\mathrm{id}_{\mathcal{C}^{\text{fib}}_{\otimes}} \overset{\phi}{\Rightarrow} P \overset{\psi}{\Rightarrow} \Delta_{\mathcal{C}}$$

which factor the diagonal natural transformation $\mathrm{id}_{\mathcal{C}^{\text{fib}}_{\otimes}} \Rightarrow \Delta_{\mathcal{C}}$ (interpreted as a natural transformation of functors $\mathcal{C}^{\text{fib}}_{\otimes} \to \mathcal{C}$) and such that ϕ_A is a weak equivalence and ψ_A is a fibration for every $A \in \mathcal{C}^{\text{fib}}$.

Note that the symmetric monoidal functor P need not be strong monoidal. In this lax case, composition with P does not directly induce an endofunctor of $\mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$. However, using the strictificaion procedure mentioned above we can get an endofunctor $\hat{P}$, which will be the path object functor for $\mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$.

8.2. **The Model Structure.**

Theorem 8.15. *Let $\mathfrak{F}$ be a Feynman category and let $\mathcal{C}$ be a cofibrantly generated model category and a closed symmetric monoidal category having the following additional properties:*

1. *All objects of $\mathcal{C}$ are small.*
2. *$\mathcal{C}$ has a symmetric monoidal fibrant replacement functor.*
3. *$\mathcal{C}$ has $\otimes$-coherent path objects for fibrant objects.*

Then $\mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$ is a model category where a morphism $\phi\colon \mathcal{O} \to \mathcal{Q}$ of $\mathcal{F}\text{-}\mathcal{O}ps$ is a weak equivalence (resp. fibration) if and only if $\phi\colon \mathcal{O}(v) \to \mathcal{Q}(v)$ is a weak equivalence (resp. fibration) in $\mathcal{C}$ for every $v \in \mathcal{V}$.

Proof. Let R denote the symmetric monoidal fibrant replacement functor for $\mathcal{C}$ and let P denote the path object functor for fibrant objects in $\mathcal{C}$. The proof of this theorem will be achieved via:

$$\text{Lemma 8.7} \Rightarrow \left\{ \begin{array}{c} \text{model structure} \\ \text{on } \mathcal{V}\text{-Seq}_{\mathcal{C}} \end{array} \right\} \overset{\substack{\text{transfer} \\ \text{principle}}}{\longrightarrow} \left\{ \begin{array}{c} \text{model structure} \\ \text{on } \mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}} \end{array} \right\}$$

That is, since by Lemma 8.7 the category of $\mathcal{V}$-sequences in $\mathcal{C}$ has a levelwise cofibrantly generated model structure, the proof will amount to a verification that the hypotheses for the transfer principle (Corollary 8.9) are satisfied with respect to the composite adjunction

$$\mathcal{V}\text{-Seq}_{\mathcal{C}} \leftrightarrows \mathcal{V}\text{-}\mathcal{M}ods_{\mathcal{C}} \leftrightarrows \mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$$

First by Lemma 8.5 the category $\mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$ is complete and cocomplete. Next we show that all objects of $\mathcal{V}\text{-Seq}_{\mathcal{C}}$ are small. Pick a cardinal κ_1 such that all objects of $\mathcal{C}$ are κ_1-small (such a cardinal exists; see [Hir03] Lemma 10.4.6). Then pick a cardinal κ which is greater than both κ_1 and the cardinality of $\mathcal{V}$. Let $X \to Z$ be the transfinite composition in $\mathcal{V}\text{-Seq}_{\mathcal{C}}$ of a λ-sequence $X = X_0 \to X_1 \to ..$ for any regular cardinal $\lambda \geq \kappa$. Then for any $\mathcal{V}$-sequence A we have

$$\text{colim}_{\beta<\lambda} Hom_{\mathcal{V}\text{-Seq}_{\mathcal{C}}}(A, X_\beta) = \text{colim}_{\beta<\lambda} \prod_{v\in\mathcal{V}} Hom_{\mathcal{C}}(A(v), X_\beta(v)) \cong \prod_{v\in\mathcal{V}} colim_{\beta<\lambda} Hom_{\mathcal{C}}(A(v), X_\beta(v))$$

using the fact that $\lambda \geq \kappa > card(\mathcal{V})$ to interchange the product and the colimit. Now the fact that $A(v)$ is κ-small for each v tells us that

$$\prod_{v\in V} \text{colim}_{\beta<\lambda} Hom_{\mathcal{C}}(A(v), X_\beta(v)) \cong \prod_{v\in\mathcal{V}} Hom_{\mathcal{C}}(A(v), \text{colim}_{\beta<\lambda} X_\beta(v)) = Hom_{\mathcal{V}\text{-Seq}_{\mathcal{C}}}(A, Z)$$

and hence A is small. Thus condition (i) is satisfied. By Lemmas 8.3 and 8.4 the constituent forgetful functors both preserve filtered colimits. Thus their composite does and condition (ii) of the transfer principle is satisfied.

For condition (iii) of the transfer principle, given $\mathcal{O} \in \mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$ there is a functorial fibrant replacement given by $\hat{R}(\mathcal{O})$, where $\hat{R}$ is as in Definition 8.11. Indeed $\hat{R}(\mathcal{O})(v) = R \circ \mathcal{O}(v)$ is fibrant for each $v \in \mathcal{V}$. Thus it remains to show that condition (iv) of the transfer principle is satisfied, so we now suppose $\mathcal{O}$ to be a fibrant $\mathcal{F}\text{-}\mathcal{O}p$ and take $\hat{P}(\mathcal{O})$

to be our candidate for a path object functor for fibrant $\mathcal{F}$-$\mathcal{O}ps$. Since $\mathcal{O}$ is a fibrant $\mathcal{F}$-$\mathcal{O}p$, $\mathcal{O}(v)$ is a fibrant object in $\mathcal{C}$ for every $v \in \mathcal{V}$. As such there are maps in $\mathcal{C}$

$$\mathcal{O}(v) \xrightarrow{\phi_v} P(\mathcal{O}(v)) = \hat{P}(\mathcal{O})(v) \xrightarrow{\psi_v} \mathcal{O}(v) \prod \mathcal{O}(v)$$

for each $v \in \mathcal{V}$ such that ϕ_v is a weak equivalence and ψ_v is a fibration. Since weak equivalences and fibrations in $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$ are induced levelwise, we can show that $\hat{P}(\mathcal{O})$ gives us functorial path objects in $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$ if we can show that the levelwise diagrams above fit together to give a diagram in $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$. To see this it is enough to see that these levelwise diagrams are compatible for a generating morphism $\alpha\colon X = \otimes_{i\in\Lambda} v_i \to v$, which follows from commutativity of the following diagram:

$$\begin{tikzcd}
\mathcal{O}(X) \arrow[r, "\otimes_i \phi_{v_i}"] \arrow[d, "="] & \hat{P}(\mathcal{O})(X) \arrow[r, "\otimes_i \psi_{v_i}"] \arrow[d] \arrow[dr, dotted] & \otimes_i(\mathcal{O}(v_i) \prod \mathcal{O}(v_i)) \arrow[d, "(\otimes_i \pi_1) \prod (\otimes_i \pi_2)"] \\
\mathcal{O}(X) \arrow[r, "\phi_X"] \arrow[d, "\mathcal{O}(\alpha)"] & P(\mathcal{O}(X)) \arrow[r, "\psi_X"] \arrow[d, "P(\alpha)"] & \mathcal{O}(X) \prod \mathcal{O}(X) \arrow[d, "\alpha \prod \alpha"] \\
\mathcal{O}(v) \arrow[r, "\sim", "\phi_v"'] & \hat{P}(\mathcal{O})(v) \arrow[r, two heads, "\psi_v"'] & \mathcal{O}(v) \prod \mathcal{O}(v)
\end{tikzcd}$$

Note the two bottom squares commute by the assumption that ϕ and ψ are natural transformations and the top two squares commute by the assumption that these natural transformations are symmetric monoidal. Composing to the dotted arrow we get a diagram of $\mathcal{F}$-$\mathcal{O}ps$

$$\mathcal{O} \xrightarrow{\sim} \hat{P}(\mathcal{O}) \twoheadrightarrow \mathcal{O} \prod \mathcal{O}$$

which is a weak equivalence followed by a fibration, because it is levelwise for each $v \in \mathcal{V}$. Thus $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$ has functorial path objects for fibrant objects, which permits this application of the transfer principle and hence completes the proof. $\square$

Remark 8.16. Note that since the category of $\mathcal{V}$-modules is equivalent to the category of $Iso(\mathcal{F})$-$\mathcal{O}ps$, the above theorem also gives $\mathcal{V}$-$\mathcal{M}ods_{\mathcal{C}}$ a transferred model structure when the condition on $\mathcal{C}$ are met.

8.2.1. **Existence of $\otimes$-coherent path objects.** Before considering examples of categories which satisfy the conditions of Theorem 8.15 we will consider how, in practice, one can establish the existence of $\otimes$-coherent path objects. These conditions were inspired by [BM03].

Definition 8.17. We say J is an interval in $\mathcal{C}$ if there is a factorization of the folding map of I as

$$I \sqcup I \hookrightarrow J \xrightarrow{\sim} I$$

where I is the monoidal unit, and $\sqcup$ is the coproduct in $\mathcal{C}$ (we will denote the product in $\mathcal{C}$ by $\sqcap$). We say that J is a cocommutative interval if J is an interval with a cocommutative coassociative counital comultiplication $J \to J \otimes J$, with counit map as above $J \xrightarrow{\sim} I$.

Note that if $\mathcal{C}$ is Cartesian closed then $\mathcal{C}$ necessarily has a cocommutative interval via the diagonal for any factorization of the folding map.

Lemma 8.18. *Suppose that $\mathcal{C}$ satisfies PPA, that the monoidal unit is cofibrant, and that $\mathcal{C}$ has a cocommutative interval J. Then $P := hom(J, -)$ defines $\otimes$-coherent path objects for fibrant objects in $\mathcal{C}$.*

Proof. First note J having a cocommutative, coassociative multiplication makes $P\colon \mathcal{C} \to \mathcal{C}$ a symmetric monoidal functor by precomposition $J \to J \otimes J$ (and unit axiom by $I \cong Hom(I, I)$). Second notice that since $\mathcal{C}$ is closed symmetric monoidal, $X \otimes -$ is a left adjoint and so commutes with colimits, and so in particular $X \sqcup X \cong X \otimes (I \sqcup I)$ and hence $hom(I, X) \sqcap hom(I, X) \cong hom(I \sqcup I, X)$.

Note that if X is a fibrant object we may apply the contravariant functor $hom(-, X)$ to the factorization above to get a path object for X which is functorial:

$$X \cong hom(I, X) \overset{\sim}{\to} hom(J, X) \twoheadrightarrow hom(I \sqcup I, X) \cong X \sqcap X$$

The fact that $hom(I, X) \overset{\sim}{\to} hom(J, X)$ is a weak equivalence follows from the PPA and the fact that I, hence J, is cofibrant and X is fibrant (see [BM03] lemma 2.3). The fact that $hom(J, X) \twoheadrightarrow hom(I \sqcup I, X)$ is a fibration follows from the PPA by taking the exponential transpose, and the fact that $I \sqcup I \hookrightarrow J$ is a cofibration (see [Fre10] p.14).

It is then straight forward to check that the induced natural transformations are symmetric monoidal. □

We can formulate adjoint conditions as well:

Definition 8.19. We say K is a cointerval in $\mathcal{C}$ if $K \in \mathcal{C}^{\mathrm{op}}$ is an interval. That is, K is a cointerval if there is a factorization of the diagonal map of I as a weak equivalence followed by a fibration

$$I \overset{\sim}{\to} K \twoheadrightarrow I \sqcap I$$

where I is the monoidal unit. We say that K is a commutative cointerval if K is a cointerval with a commutative associative unital multiplication $K \otimes K \to K$ with the unit map as above $I \overset{\sim}{\to} K$.

Lemma 8.20. *Suppose that $\mathcal{C}$ is not Cartesian closed and that the monoidal product is distributive with respect to the categorical product. Suppose also that $\mathcal{C}$ satisfies the following conditions:*

1. *I is fibrant.*
2. *$\mathcal{C}$ has a commutative cointerval K.*
3. *For any fibrant object X, the functor $- \otimes X$ preserves weak equivalences between fibrant objects and preserves fibrations.*

Then $P := - \otimes K$ defines $\otimes$-coherent path objects for fibrant objects in $\mathcal{C}$.

Proof. Tensoring with the cointerval sequence we have, for any fibrant object X,

$$X \cong X \otimes I \overset{\sim}{\to} X \otimes K \twoheadrightarrow X \otimes (I \sqcap I) \cong X \sqcap X$$

and the remaining details are easily checked. □

8.2.2. **Examples.** We will now consider a short list of the primary examples and nonexamples of categories which satisfy the conditions of Theorem 8.15 that we wish to consider.

Example 8.21. (Simplicial Sets) The category of simplicial sets is a Cartesian closed model category with injections as cofibrations and realization weak equivalences as weak equivalences (e.g. [Hov99] section 3.2). All simplicial sets are small ([Hov99] Lemma 3.1.1). There is a fibrant replacement functor given by taking the singular complex of the realization. All simplicial sets are cofibrant; in particular the monoidal unit is cofibrant, thus Lemma 8.18 applies and simplicial valued $\mathfrak{F}$-$\mathcal{O}ps$ form a model category.

Example 8.22. (Vector Spaces in characteristic 0) Let k be a field of characteristic zero and let $dg\mathcal{V}ect_k$ be the category of differential graded k vector spaces. We consider $dg\mathcal{V}ect_k$ to be a model category with weak equivalences / fibrations / cofibrations given by quasi-isomorphisms / surjections / injections. Note that all objects in this model category are fibrant, so the identity gives a symmetric monoidal fibrant replacement functor. Let $k[t, dt]$ be the unique commutative dga of polynomials in the variables t and dt which satisfies $d(t) = dt$ and $d(dt) = 0$. Consider the sequence

$$k \xrightarrow{\phi} k[t, dt] \xrightarrow{\psi} k \oplus k$$

where $\phi(a) = a$ and $\psi(f(t) + g(t, dt)dt) = f(0) \oplus f(1)$. Note that the composite $\psi \circ \phi$ is the diagonal. Further notice that ψ is surjective and that ϕ is a quasi-isomorphism, since $k[t, dt]$ is acyclic in characteristic 0. Thus $K = k[t, dt]$ is a commutative cointerval object. Notice that the other conditions of Lemma 8.20 are clearly satisfied, thus Theorem 8.15 applies and linear $\mathcal{F}$-$\mathcal{O}ps$ in characteristic 0 form a model category.

Remark 8.23. By the above work, given an $\mathcal{F}$-$\mathcal{O}p$ $\mathcal{O}$ in $dg\mathcal{V}ect_k$ (characteristic 0) we get a factorization of the diagonal in the category of $\mathcal{F}$-$\mathcal{O}ps$:

$$\mathcal{O} \overset{\sim}{\to} \mathcal{O}[t, dt] \twoheadrightarrow \mathcal{O} \oplus \mathcal{O}$$

where $\mathcal{O}[t, dt]$ is the $\mathcal{F}$-$\mathcal{O}p$ given by composition of the symmetric monoidal functors:

$$\mathcal{F} \xrightarrow{\mathcal{O}} \mathcal{C} \xrightarrow{-\otimes k[t,dt]} \mathcal{C}$$

In particular for an object $v \in \mathcal{V}$ we have $\mathcal{O}[t, dt](v) = \mathcal{O}(v) \otimes k[t, dt]$ and for a general object $X \in \mathcal{F}$ with $X \cong \otimes_i v_i$ we define $\mathcal{O}[t, dt](X) = \otimes_i(\mathcal{O}(v_i) \otimes k[t, dt])$. Then for a generating morphism $\phi \colon X \to v$ we have

$$\mathcal{O}[t, dt](X) \cong \otimes_{i \in I}\mathcal{O}(v_i) \otimes k[t, dt]^{\otimes I} \xrightarrow{\phi \otimes \mu} \mathcal{O}(v) \otimes k[t, dt] = \mathcal{O}[t, dt](v)$$

where μ is the commutative associative multiplication in $k[t, dt]$.

It is important to notice that the characteristic zero assumption in the previous example is necessary. In particular the path object given above requires characteristic zero for the map ϕ to be a weak equivalence. More generally, we have the following lemma.

Lemma 8.24. *Let $\mathcal{C}$ be the category of differential graded vector spaces in a field of characteristic $p > 0$. Then $\mathcal{C}$ does not have $\otimes$-coherent path objects for fibrant objects.*

Proof. By contradiction suppose $P(-)$ gives us $\otimes$-coherent path objects in $\mathcal{C}$. Choose a commutative dga A which contains an element x of even degree which represents a nonzero class in homology and such that x^p represents a nonzero class in homology. Suppose we have a factorization of the diagonal as:

$$\begin{tikzcd} A \arrow[r, "\sim"] & P(A) \arrow[r, two heads, "g"] & A \oplus A \end{tikzcd}$$

Then g is surjective so there is a $y \in P(A)$ such that $g(y) = (x, 0)$. Then, since A is commutative and since y is of even degree we have $d(y^p) = py^{p-1} = 0$. Tracing through Definition 8.12 we see that $g(y^p) = (x^p, 0)$. Now taking homology of the above diagram we have:

$$H_*(A) \cong H_*(P(A)) \xrightarrow{g_*} H_*(A) \oplus H_*(A)$$

On the one hand the composite is the diagonal, but on the other hand the element $([x^p], [0])$ is in the image of the composite. This contradiction proves the lemma. $\square$

Remark 8.25. A similar argument can be made for e.g. chain complexes in $\mathbb{Z}$ modules.

Example 8.26. (Topological Spaces) Let **Top** be the category of compactly generated spaces with the Quillen model structure: weak equivalences given by weak homotopy equivalences and fibrations given by Serre fibrations. Then **Top** is a cofibrantly generated symmetric monoidal model category (see e.g. [Hov99]) having all objects fibrant. Since **Top** is Cartesian closed and since the monoidal unit, a point, is cofibrant, Lemma 8.18 applies to endow **Top** with $\otimes$-coherent path objects. Thus conditions (ii),(iii),(iv) of Corollary 8.9 are satisfied. However condition (i) is not satisfied and so we can not directly apply the result. In order to circumvent this problem we follow [Fre10] and use the fact that all objects in $\mathcal{T}op$ are small with respect to topological inclusions. The details are given in the appendix, but we record the result here:

Theorem 8.27. *Let $\mathcal{C}$ be the category of topological spaces with the Quillen model structure. The category $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$ has the structure of a cofibrantly generated model category in which the forgetful functor to $\mathcal{V}$-$Seq_{\mathcal{C}}$ creates fibrations and weak equivalences.*

See Appendix B.

8.3. **Quillen adjunctions from morphisms of Feynman categories.** We assume $\mathcal{C}$ is a closed symmetric monoidal and model category satisfying the assumptions of Theorem 8.15. Let $\mathfrak{E}$ and $\mathfrak{F}$ be Feynman categories and let $\alpha\colon \mathfrak{E} \to \mathfrak{F}$ be a morphism between them. Recall (Theorem 1.15) this morphism induces an adjunction

$$\alpha_L\colon \mathcal{E}\text{-}\mathcal{O}ps_{\mathcal{C}} \leftrightarrows \mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}\colon \alpha_R$$

where $\alpha_R(\mathcal{A}) := \mathcal{A} \circ \alpha$ is the right adjoint and $\alpha_L(\mathcal{B}) := Lan_\alpha(\mathcal{B})$ is the left adjoint. In this section we will see that several prominent examples of such adjunctions are in fact Quillen adjunctions.

Lemma 8.28. *Suppose α_R restricted to $\mathcal{V}_{\mathfrak{F}}$-$\mathcal{M}ods_{\mathcal{C}} \to \mathcal{V}_{\mathfrak{E}}$-$\mathcal{M}ods_{\mathcal{C}}$ preserves fibrations and acyclic fibrations (see Remark 8.16). Then the adjunction (α_L, α_R) is a Quillen adjunction.*

Proof. To show this adjunction is a Quillen adjunction it is enough to show that α_R is a right Quillen functor (see e.g. [Hov99] Lemma 1.3.4), i.e. that α_R preserves fibrations and acyclic fibrations (on the entire domain $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$). This follows by the assumption of the lemma and the commutativity of the following diagram

$$\begin{tikzcd}
\mathcal{E}\text{-}\mathcal{O}ps_{\mathcal{C}} \arrow[d, "G_{\mathfrak{E}}"] & \mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}} \arrow[l, "\alpha_R"'] \arrow[d, "G_{\mathfrak{F}}"] \\
\mathcal{V}_{\mathfrak{E}}\text{-}\mathcal{M}ods_{\mathcal{C}} & \mathcal{V}_{\mathfrak{F}}\text{-}\mathcal{M}ods_{\mathcal{C}} \arrow[l, "\alpha_R"']
\end{tikzcd}$$

along with the fact that $G_{\mathfrak{E}}$ and $G_{\mathfrak{F}}$ preserve and reflect weak equivalences and fibrations. □

Common examples where the conditions of the Lemma 8.28 are satisfied come from the standard adjunctions between various $\mathcal{O}ps$, several of which we now describe.

Example 8.29. Lemma 8.28 immediately implies that the forgetful/free adjunction between $\mathcal{V}$-$\mathcal{M}ods_{\mathcal{C}}$ and $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$ is a Quillen adjunction.

Example 8.30. Recall that $\mathfrak{C}$ and $\mathfrak{M}$ denote the Feynman categories whose $\mathcal{O}ps$ are cyclic and modular operads respectively and that there is a morphism $i\colon \mathfrak{C} \to \mathfrak{M}$ by including as genus zero. As discussed above, this morphism induces an adjunction between cyclic and modular operads

$$i_L\colon \mathfrak{C}\text{-}\mathcal{O}ps_{\mathcal{C}} \leftrightarrows \mathfrak{M}\text{-}\mathcal{O}ps_{\mathcal{C}}\colon i_R$$

and the left adjoint is called the modular envelope of the cyclic operad. The fact that the morphism of Feynman categories is inclusion means that i_R restricted to the underlying $\mathcal{V}$-modules is given by forgetting, and since fibrations and weak equivalences are levelwise, i_R restricted to the underlying $\mathcal{V}$-modules will preserve fibrations and weak equivalences. Thus by Lemma 8.28 this adjunction is a Quillen adjunction.

Example 8.31. In the directed case we have various morphisms of Feynman categories given by inclusion. For example $\mathfrak{O} \to \mathfrak{P}^{ctd} \to \mathfrak{P}$ or $\mathfrak{O} \to \mathfrak{D}$, and, as discussed above, the induced adjunctions recover various free constructions in the literature, (see e.g. [Val07],[MV09a]). Since restriction to the underlying $\mathcal{V}$-modules is the identity or inclusion in each example, Lemma 8.28 applies and each associated adjunction is a Quillen adjunction. In particular we have a Quillen adjunction between the categories of operads and props in such a model category.

Remark 8.32. In this section we have given conditions on an adjunction induced by a morphism of Feynman categories (source categories) to be a Quillen adjunction. Similarly it is possible to give conditions on a symmetric monoidal adjunction of base categories such that the induced adjunction is a Quillen adjunction or Quillen equivalence. In particular one can show that the categories of simplicial and topological $\mathcal{F} - \mathcal{O}ps$ are Quillen equivalent.

8.4. **Cofibrant objects.** We continue to assume $\mathcal{C}$ is a closed monoidal category and a model category such that the conditions of Theorem 8.15 are satisfied. Let $F : \mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}} \rightleftarrows \mathcal{V}\text{-}\mathcal{M}ods_{\mathcal{C}} : G$ be the free and forgetful adjunction.

Proposition 8.33. *Let $\Phi \in \mathcal{V}\text{-}\mathcal{M}ods_{\mathcal{C}}$ be cofibrant. Then $F(\Phi) \in \mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$ is cofibrant.*

Proof. Since F is a left Quillen functor it preserves cofibrations and initial objects, hence preserves cofibrant objects. □

For the remainder of this section we fix $\mathcal{C}$ to be the category of dg vector spaces over a field of characteristic 0. Recall that an object in $\mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$ is quasi-free if its image under the forgetful functor to graded vector spaces is free. We now show that quasi-free $\mathcal{F}\text{-}\mathcal{O}ps$ are cofibrant. The argument is a generalization of that for operads [Fre09] and properads [MV09c].

Theorem 8.34. *Let $\mathcal{Q} = (F(\Phi), \delta)$ be a quasi-free $\mathcal{F}\text{-}\mathcal{O}p$. Furthermore assume that Φ admits an exhaustive filtration*

$$\Phi_0 \subset \Phi_1 \subset \Phi_2 \subset \cdots \subset \Phi$$

in the category of $\mathcal{V}$-modules (so that these inclusions are split injections of $Aut(v)$-modules) such that $\delta(\Phi_i) \subset F(\Phi_{i-1})$. Then $\mathcal{Q}$ is cofibrant in $\mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$.

Proof. The proof follows as in *loc.cit.* The first step is to reduce the problem to the case of quasi-free objects built from free $\mathcal{V}$-modules; to this end we will show that Φ is a retract of a free $\mathcal{V}$-module. Fix $v \in \mathcal{V}$ and let $A := \Phi(v)$, $A_l := \Phi_l(v)$ and $H := Aut(v)$. Then A is an H-representation and we choose an isomorphism of representations $A_l \cong \oplus_{i \in J_l} A_l^i$ where each A_l^i is irreducible. The filtration allows us to consider $J_l \subset J_{l+1}$ and $A = \oplus_{i \in J} A^i$ where J is the union of the J_l. Then define maps

$$A \xrightarrow{\epsilon} k[J] \otimes \mathrm{Reg}(H) \xrightarrow{\eta} A$$

by taking

$$\epsilon(A^i) = i \otimes A_i \quad \text{and} \quad \eta(j \otimes A_i) = \begin{cases} A_i \text{ if } i = j \\ 0 \text{ else} \end{cases}$$

where we have arbitrarily chosen for each $i \in J$ a copy of A_i appearing as a summand of $\mathrm{Reg}(H)$, the regular representation. By construction the maps η and ϵ are H-equivariant and the composition is the identity. Define $\Gamma(v) := k[J]$ and then we have, by taking the above construction at each level, a retraction of $\mathcal{V}$-modules

$$\Phi \to R(\Gamma) \to \Phi$$

where R is the free functor $\mathcal{V}\text{-Seq}_{\mathcal{C}} \to \mathcal{V}\text{-}\mathcal{M}ods$. Finally notice that the filtration on Φ induces a filtration on Γ by $\Gamma_l(v) := k[J_l]$ and if we define $\delta' \colon FR(\Gamma) \to FR(\Gamma)$ by $\delta' = FR(\epsilon) \circ \delta \circ FR(\eta)$ then the filtration on Γ satisfies $\delta'(\Gamma_i) \subset FR(\Gamma_{i-1})$.

By the above construction we have a retraction of quasi-free $\mathcal{F}$-$\mathcal{O}ps$

$$\mathcal{Q} = (F(\Phi), \delta) \to (FR(\Gamma), \delta') \to \mathcal{Q}$$

and so if we can show that the quasi-free $\mathcal{F}$-$\mathcal{O}p$ $(FR(\Gamma), \delta')$ is cofibrant, it will follow that $\mathcal{Q}$ is cofibrant.

Fix a natural number j and define two $\mathcal{V}$-sequences S^j and D^j as follows. For a fixed j and $v \in \mathcal{V}$, choose a basis $\{e_b : b \in B\}$ for the vector space $\Gamma_{j+1}(v) \setminus \Gamma_j(v)$. We then define

$$S^j(v) = \bigoplus_{b \in B} \Sigma^{|e_b|-1} k \quad \text{and} \quad D^j(v) = \bigoplus_{b \in B} (\Sigma^{|e_b|} k \oplus \Sigma^{|e_b|-1} k)$$

as vector spaces. Define x_b to be the generator of $S^j(v)$ corresponding to index b and $y_b \oplus z_b$ to be the generator of $D^j(v)$ corresponding to index b. Define differentials on these vector spaces by $d(x_b) = 0$ and $d(y_b) = z_b$ and define the canonical inclusion $\mathrm{in}_j \colon S^j(v) \to D^j(v)$ by $x_b \mapsto z_b$.

Taking this construction for all $v \in \mathcal{V}$ gives us $\mathrm{in}_j \colon S^j \to D^j$ which is a cofibration in the category of $\mathcal{V}$-sequences. As such $FR(\mathrm{in}_j)$ is a cofibration. Define a map $f \colon FR(S^j) \to (FR(\Gamma_j), \delta')$ to be the image under adjunction of the map $S^j(v) \to (FR(\Gamma_j)(v), \delta')$ taking $x_b \mapsto \delta'(e_b)$. We have the following pushout diagram in $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$:

$$\begin{tikzcd}
FR(S^j) \arrow[r, "f"] \arrow[d, "FR(\mathrm{in}_j)"'] & (FR(\Gamma_j), \delta') \arrow[d, dotted] \\
FR(D^j) \arrow[r, dotted] & \dfrac{(FR(\Gamma_j), \delta') \oplus FR(D^j)}{d(y_b) = \delta'(e_b)}
\end{tikzcd}$$

Then under the identification $e_b \mapsto y_b$, this pushout is isomorphic to $(FR(\Gamma_{j+1}), \delta')$. Moreover, the morphism $(FR(\Gamma_j), \delta') \hookrightarrow (FR(\Gamma_{j+1}), \delta')$ is the pushout of the cofibration $FR(\mathrm{in}_j)$ and hence a cofibration. Taking the colimit over all j we see that $(FR(\Gamma), \delta')$ is cofibrant, hence the proof. ☐

Corollary 8.35. *The Feynman transform (Definition* 7.10*) of a non-negatively graded dg $\mathcal{F}$-$\mathcal{O}p$ is cofibrant.*

Corollary 8.36. *The double Feynman transform of a non-negatively graded dg $\mathcal{F}$-$\mathcal{O}p$ in a quadratic Feynman category is a cofibrant replacement.*

8.5. **Homotopy classes of maps and master equations.** In this section we work over the category $\mathcal{C} = dg\mathcal{V}ect_k$ where k is a field of characteristic 0. By the above work we know that the Feynman transform produces a cofibrant object and that all $\mathcal{F}$-$\mathcal{O}ps$ are fibrant in $dg\mathcal{V}ect_k$. Thus we can consider homotopy classes of maps in this context, whose definition we now recall.

Definition 8.37. Let $\mathcal{M}$ be a cofibrant $\mathcal{F}$-$\mathcal{O}p$, let $\mathcal{O}$ be any $\mathcal{F}$-$\mathcal{O}p$, and let $\gamma_1, \gamma_2 \in Hom(\mathcal{M}, \mathcal{O})$. Then γ_1 and γ_2 are homotopic if there exists a path object $P(\mathcal{O})$ of $\mathcal{O}$ along with a lift of the map $\gamma_1 \oplus \gamma_2$ such that the following diagram commutes:

$$\begin{tikzcd}
 & P(\mathcal{O}) \arrow[d] & \mathcal{O} \arrow[l, "\sim"'] \arrow[dl, "\Delta"] \\
\mathcal{M} \arrow[ur, dotted] \arrow[r, "\gamma_1\oplus\gamma_2"] & \mathcal{O}\oplus\mathcal{O} &
\end{tikzcd} \tag{8.3}$$

The set of homotopy classes of such maps is denoted $[\mathcal{M}, \mathcal{O}]$.

Thus we have a notion of homotopy equivalence of morphisms from the Feynman transform due to Definition 8.37. By the above work (Theorem 7.16) such morphisms can be encoded by solutions to certain master equations. Often these master equations can be interpreted as the Maurer-Cartan equation in a certain dg Lie algebra, in which case there is a notion of homotopy equivalence of MC solutions which we recall below. The main result of this subsection will be to show that the notions of homotopy equivalence on both sides of the bijection of Theorem 7.14 coincide.

8.5.1. **MC simplicial set.** Let Ω_n be the commutative dga of polynomial differential forms on the simplex Δ^n,

$$\Omega_n := \frac{k[t_0, \dots, t_n, dt_0, \dots, dt_n]}{(\sum t_i - 1, \sum dt_i)} \tag{8.4}$$

having $d(t_i) := dt_i$ and with $|t_i| = 0$ and $|dt_i| = 1$. The collection of these commutative dgas $\Omega_\bullet$ is a simplicial object with face and degeneracy maps,

$$\delta_i \colon \Omega_{n+1} \to \Omega_n \quad i = 0, \dots, n+1 \qquad\qquad \sigma_j \colon \Omega_n \to \Omega_{n+1} \quad j = 0, \dots, n$$

$$\delta_i(t_k) = \begin{cases} t_k & \text{if } k < i \\ 0 & \text{if } k = i \\ t_{k-1} & \text{if } k > i \end{cases} \qquad\qquad \sigma_j(t_k) = \begin{cases} t_k & \text{if } k < j \\ t_k + t_{k+1} & \text{if } k = j \\ t_{k+1} & \text{if } k > j \end{cases}$$

Following [Hin97a] and [Get09a], given a dgLa $\mathfrak{g}$ we define a simplicial set

$$MC_\bullet(\mathfrak{g}) := MC(\mathfrak{g} \otimes \Omega_\bullet)$$

where $\mathfrak{g} \otimes \Omega_n$ is a dgLa by taking, for $g, g' \in \mathfrak{g}$ and $f, f' \in \Omega_n$,

$$d(g \otimes f) = dg \otimes f + (-1)^{|g|} g \otimes d(f) \;\text{ and }\; [g \otimes f, g' \otimes f'] = [g, g'] \otimes (-1)^{|g'||f|} f f'$$

and with face and degeneracy maps coming from the functorial image of $- \otimes \Omega_\bullet$. The fact that the image of an MC element is an MC element follows from the fact that the face and degeneracies on $\Omega_\bullet$ are maps of commutative dgas.

In the event that $\mathfrak{g}$ is nilpotent, the simplicial set $MC_\bullet(\mathfrak{g})$ is a Kan complex [Hin97a], [Get09a], i.e. a fibrant simplicial set, and we can consider its homotopy groups $\pi_*(MC_\bullet(\mathfrak{g}))$ (see e.g. [Wei94]). We then define the notion of homotopy equivalence of MC elements by defining $\pi_0(MC_\bullet(\mathfrak{g}))$ to be the set of homotopy classes of MC elements. In particular

$$\pi_0(MC_\bullet(\mathfrak{g})) = MC_\bullet(\mathfrak{g})/\sim$$

where

$$s_0 \sim s_1 \Leftrightarrow \exists\, s \in MC_1(\mathfrak{g}) \text{ such that } \delta_0(s) = s_0 \text{ and } \delta_1(s) = s_1 \tag{8.5}$$

This equivalence relation coincides with the notion of gauge equivalence in the nilpotent case. Without the nilpotence assumption it is not clear that $\sim$ and its higher dimensional brethren are transitive. One common approach to this problem is to tensor the dgLas of interest with the maximal ideal in a local Artin ring. In our context we could also consider truncated operads. However we will show that $\sim$ defined in Equation 8.5 is an equivalence relation in the context of operadic Lie algebras, and thus π_0 is well defined in our contexts of interest. Further study of this simplicial set in the context of generalized operadic Lie algebras and their possible higher homotopy groups is a potentially interesting future direction.

8.5.2. **Homotopy classes theorem.** We can now state and prove the result linking homotopy classes of maps to homotopy classes of MC elements. The proof will make use of the following lemma.

Lemma 8.38. *Let $\mathcal{M}$ be a cofibrant $\mathcal{F}$-$\mathcal{O}p$ and let γ_1, γ_2 be maps $\mathcal{M} \to \mathcal{O}$ for some $\mathcal{F}$-$\mathcal{O}p$ $\mathcal{O}$. Then $\gamma_1 \sim \gamma_2$ if and only if there exist a lift*

$$\begin{tikzcd}
 & \mathcal{O}[t,dt] \arrow[d, "\psi"] & \mathcal{O} \arrow[l, "\phi"', "\sim"] \arrow[dl, "\Delta"] \\
\mathcal{M} \arrow[r, "\gamma_1 \oplus \gamma_2"] \arrow[ur, dotted] & \mathcal{O} \oplus \mathcal{O} &
\end{tikzcd}$$

where ϕ and ψ are induced levelwise as in Remark 8.23. *In particular $\psi = \{\psi_v\}_v$ is given levelwise by $\psi_v(f(t) + g(t, dt)dt) = (f(0), f(1))$.*

Proof. The $\Leftarrow$ direction follows immediately from the definition of $\sim$ and the fact that ϕ and ψ make $\mathcal{O}[t, dt]$ a path object for $\mathcal{O}$ by the above work. The $\Rightarrow$ implication is as follows. If $\gamma_1 \sim \gamma_2$ then there is a $P(\mathcal{O})$ as in diagram 8.3. Since the map $\mathcal{O} \overset{\sim}{\to} P(\mathcal{O})$ is a weak equivalence we may factor it as an acyclic cofibration followed by an acyclic fibration. Calling the object in the center of this factorization $P'(\mathcal{O})$ we have the following commutative diagram:

$$\begin{tikzcd}
 & & \mathcal{O} \arrow[dl, hook, "\sim"'] \arrow[d] \arrow[dd, bend left, "\Delta"] \\
* \arrow[r] \arrow[d, hook] & P'(\mathcal{O}) \arrow[r, dotted] \arrow[d, "\sim"] & \mathcal{O}[t,dt] \arrow[d] \\
\mathcal{M} \arrow[r] \arrow[ur, dotted] \arrow[rr, bend right, "\gamma_1 \oplus \gamma_2"'] & P(\mathcal{O}) \arrow[r] & \mathcal{O} \oplus \mathcal{O}
\end{tikzcd}$$

The two lifts in the diagram combine to give the lift $\mathcal{M} \to \mathcal{O}[t, dt]$ that we require.

□

Theorem 8.39. *For entries in Table* 2*,*

$$[\mathsf{FT}(\mathcal{P}), \mathcal{O}] \cong \pi_0(ME_\bullet(colim(\mathcal{P} \otimes \mathcal{O})))$$

Proof. Let $\mathcal{F}$ be a Feynman category whose category of linear $\mathcal{O}ps$ appears in the table. Let $\mathcal{P}$ be an odd $\mathcal{F}$-$\mathcal{O}p$ and let $\mathcal{O}$ be an $\mathcal{F}$-$\mathcal{O}p$. Then by Theorem 7.14 we know there is a one to one correspondence

$$Hom(\mathsf{FT}(\mathcal{P}),\mathcal{O}) \overset{1 \text{ to } 1}{\longleftrightarrow} ME(colim(\mathcal{P}\otimes\mathcal{O}))$$

Let γ_0 and γ_1 be maps $\mathsf{FT}(\mathcal{P}) \to \mathcal{O}$ and let $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ be their images via this bijection. Then we want to show

$$\gamma_0 \sim \gamma_1 \Leftrightarrow \exists\ \tilde{\gamma} \in ME_1(colim(\mathcal{P}\otimes\mathcal{O})) \text{ such that } \delta_i(\tilde{\gamma}) = \tilde{\gamma}_i \text{ for } i = 0, 1.$$

For the implication $\Rightarrow$, suppose that the maps γ_1 and γ_2 are homotopic. Then, by Lemma 8.38 there is a lift of the sum of these maps, call it γ:

$$\begin{tikzcd}
 & \mathcal{O}[t,dt] \arrow[d, "\psi"] \\
\mathsf{FT}(\mathcal{P}) \arrow[ur, dotted, "\gamma"] \arrow[r, "\gamma_0\oplus\gamma_1"] & \mathcal{O}\oplus\mathcal{O}
\end{tikzcd}$$

where, as above, $\psi(f(t) + g(t,dt)dt) = (f(0), f(1))$. Next define an isomorphism of commutative dgas $\Omega_1 \cong k[t,dt]$ by sending $t_0 \mapsto t$ and $t_1 \mapsto 1 - t$. Since $\mathcal{O}[t,dt]$ is in particular a fibrant odd $\mathcal{F}$-$\mathcal{O}p$ we can apply the above 1 to 1 correspondence to get an element

$$\tilde{\gamma} \in ME(colim(\mathcal{P}\otimes(\mathcal{O}[t,dt])))$$

Using the above isomorphism and the fact that this colimit is just a direct sum of coinvariant spaces, with the automorphism group acting trivially on the Ω_1 factor, we have

$$colim(\mathcal{P}\otimes(\mathcal{O}[t,dt])) \cong colim(\mathcal{P}\otimes\mathcal{O})\otimes\Omega_1$$

as Lie algebras. As such we consider

$$\tilde{\gamma} \in ME(colim(\mathcal{P}\otimes\mathcal{O})\otimes\Omega_1) = ME_1(colim(\mathcal{P}\otimes\mathcal{O}))$$

and it is enough to show that $\delta_i(\tilde{\gamma}) = \tilde{\gamma}_i$ for $i = 0, 1$. To see this note that under the above isomorphism $\Omega_1 \cong k[t,dt]$ the morphism $id_{\mathcal{O}}\otimes\delta_i\colon \mathcal{O}\otimes\Omega_1 \to \mathcal{O}\otimes\Omega_0 \cong \mathcal{O}$ is sent to the morphism $\pi_{i+1}\circ\psi\colon \mathcal{O}[t,dt] \to \mathcal{O}$, where π_1, π_2 are projections. Then the fact that $\pi_{i+1}\circ\psi(\gamma) = \gamma_i$ tells us that $id_{\mathcal{O}}\otimes\delta_i(\tilde{\gamma}) = \tilde{\gamma}_i$, where we have abused notation by identifying elements across the following isomorphisms:

$$\begin{tikzcd}
colim(\mathcal{P}\otimes\mathcal{O})\otimes\Omega_1 \arrow[r, "\delta_i"] \arrow[d, "\cong"] & colim(\mathcal{P}\otimes\mathcal{O}) \arrow[d, "\cong"] \\
\oplus_v Hom_{Aut(v)}(\mathcal{P}^*(v), \mathcal{O}(v)\otimes\Omega_1) \arrow[r, "(id_{\mathcal{O}}\otimes\delta_i)\circ -"] & \oplus_v Hom_{Aut(v)}(\mathcal{P}^*(v), \mathcal{O}(v))
\end{tikzcd}$$

The fact that this diagram commutes then tells us that $\delta_i(\tilde{\gamma}) = \tilde{\gamma}_i$, proving the implication $\Rightarrow$. Taking the argument in reverse yields $\Leftarrow$.

$\square$

The above theorem shows us that the equivalence relations coincide. It was not however clear that the equivalence relation defining π_0 was transitive. The above proof shows that in the case of the operadic Lie algebras, i.e. those arising in Table 2, it is.

8.6. **W Construction.** In this subsection we work in the category $\mathcal{C} = \mathbf{Top}$ and with Feynman categories which we assume for convenience are strict. In [BV73] Boardmann and Vogt give a construction, called the W construction which, under suitable hypotheses, replaces a topological operad whose underlying S-module is cofibrant with a cofibrant operad in a functorial way. In this section we will generalize the W construction to $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$ when $\mathfrak{F}$ is quadratic. We will use the definitions set up in §7.2.

Definition 8.40. For a quadratic Feynman category $\mathfrak{F}$ and an object $Y \in \mathcal{F}$, we define $w(\mathfrak{F}, Y)$ to be the category whose objects are the set $\coprod_n C_n(X, Y) \times [0,1]^n$. An object in $w(\mathfrak{F}, Y)$ will be represented (uniquely up to contraction of isomorphisms) by a diagram

$$X \xrightarrow[f_1]{t_1} X_1 \xrightarrow[f_2]{t_2} X_2 \to \cdots \to X_{n-1} \xrightarrow[f_n]{t_n} Y$$

where each morphism is of positive degree and where $t_1, \dots, t_n$ represents a point in $[0,1]^n$. These numbers will be called weights. Note that in this labeling scheme isomorphisms are always unweighted. The morphisms of $w(\mathfrak{F}, Y)$ are those generated by the following three classes:

(1) Levelwise commuting isomorphisms which fix Y, i.e.:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\cong"] & X_1 \arrow[r] \arrow[d, "\cong"] & X_2 \arrow[r] \arrow[d, "\cong"] & \dots \arrow[r] & X_n \arrow[r] \arrow[d, "\cong"] & Y \\
X' \arrow[r] & X_1' \arrow[r] & X_2' \arrow[r] & \dots \arrow[r] & X_n' \arrow[ur] &
\end{tikzcd}$$

(2) Simultaneous $\mathbb{S}_n$ action.

(3) Truncation of 0 weights: morphisms of the form $(X_1 \overset{0}{\to} X_2 \to \cdots \to Y) \mapsto (X_2 \to \cdots \to Y)$.

(4) Decomposition of identical weights: morphisms of the form $(\cdots \to X_i \overset{t}{\to} X_{i+2} \to \dots) \mapsto (\cdots \to X_i \overset{t}{\to} X_{i+1} \overset{t}{\to} X_{i+2} \to \dots)$ for each (composition preserving) decomposition of a morphism of degree ≥ 2 into two morphisms each of degree ≥ 1.

Note that since $\mathfrak{F}$ is assumed to be quadratic every morphism of degree n is generated by compositions of morphisms of degree 1 uniquely upto $\mathbb{S}_n$ action and composition of isomorphisms. The graph based intuition takes the objects of $w(\mathfrak{F}, v)$ to be graphs with weighted edges along with a total order on the set of edges. Here the total order is necessary to say which weight goes with which edge. However, passing to the colimit, since we allow simultaneous $\mathbb{S}_n$ action, removes the edge ordering, and here the role of an edge is played by the orbit of a morphism under the $\mathbb{S}_n$ action. Furthermore, the truncation of 0 morphisms have the effect in the colimit of identifying an edge of weight 0 with the graph having said edge contracted, as one expects from the classical W-construction. As such we make the following definition.

Definition 8.41. Let $\mathcal{P} \in \mathcal{F}$-$\mathcal{O}ps_{\mathcal{T}op}$. For $Y \in ob(\mathcal{F})$ we define

$$W(\mathcal{P})(Y) := colim_{w(\mathfrak{F},Y)} \mathcal{P} \circ s(-)$$

Proposition 8.42. *$W(\mathcal{P})$ is naturally an object in $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{T}op}$, that is $W(\mathcal{P})(-)$ is a symmetric monoidal functor.*

Proof. This proof follows similarly to the proof of the fact that the free $\mathcal{F}$-$\mathcal{O}p$ is in fact an $\mathcal{F}$-$\mathcal{O}p$. In particular, using the fact that $\mathcal{T}op$ is closed monoidal, for $X = \otimes_{i \in I} v_i$ we have $W(\mathcal{P})(X) :=$

$$colim_{w(\mathfrak{F},\times_{i\in I}v_i)}\mathcal{P}\circ s(-) \cong colim_{\times_{i\in I}w(\mathfrak{F},v_i)}\mathcal{P}\circ s(-) \cong \times_{i\in I}colim_{w(\mathfrak{F},v_i)}\mathcal{P}\circ s(-)$$

For morphisms we define the image of $\phi\colon X \to v$ via the functor $w(\mathfrak{F}, X) \cong \times_{i\in I}w(\mathfrak{F}, v_i) \to w(\mathfrak{F}, v)$ which composes with ϕ and which gives ϕ weight 1. $\square$

Note that there is a functor $Iso(\mathcal{F} \downarrow v) \to w(\mathcal{F}, v)$ which assigns weight 1 to any non-isomorphism $X \to v$. This functor induces a map on the respective colimits of $\mathcal{P} \circ s$ and as a result induces a morphism $F(\mathcal{P})(v) \to W(\mathcal{P})(v)$. The fact that the $\mathcal{F}$-$\mathcal{O}p$ structure on $W(\mathcal{P})$ was defined by giving compositions weight 1 ensures that these maps give a morphism in $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{T}op}$. To show that $W(\mathcal{P})$ is a cofibrant replacement for $\mathcal{P}$, it will ultimately be necessary to determine conditions under which the map $F(\mathcal{P}) \to W(\mathcal{P})$ is a cofibration. To this end we make the following definitions.

Definition 8.43. Let $\gamma\colon X \to v$ be a morphism in $\mathcal{F}$. We define $Aut(\gamma)$ to be the subgroup of $Aut(X)$ such that $\sigma \in Aut(\gamma) \Leftrightarrow \gamma\sigma = \gamma$. We define $\rho(\gamma)$ to be the subgroup of $Aut(v)$ such that $\tau \in \rho(\gamma) \Leftrightarrow \exists\ \sigma \in Aut(X)$ such that $\gamma\sigma = \tau\gamma$. We define $\mathfrak{F}$ to be γ-simple if whenever $\gamma = \gamma'\sigma$ for $\sigma \in Aut(X)$, there exists $\tau \in Aut(v)$ such that $\gamma = \tau\gamma'$. We define a quadratic Feynman category $\mathfrak{F}$ to be simple if it is γ-simple for every γ.

Remark 8.44. The graph based intuition for the above definitions is as follows. The group $Aut(\gamma)$ are the automorphisms of the graph γ which fix the tails of the graph element-wise, where a tail is a flag which is not part of an edge. The group $\rho(\gamma)$ permutes the tails within each vertex, while fixing the non-tail flags. The familiar graph based examples are simple since the condition that $\gamma = \gamma'\sigma$ means that σ doesn't interchange a tail with a non-tail, and thus the permutation of the tails can be performed after contracting edges instead of before.

Lemma 8.45. *Let $\mathcal{P} \in \mathcal{F}$-$\mathcal{O}ps_{\mathcal{T}op}$ and let $\gamma\colon X \to v$. Then $\mathcal{P}(X)_{Aut(\gamma)}$ is naturally a $\rho(\gamma)$-space.*

Proof. For $a \in \mathcal{P}(X)$ and $\tau \in \rho(\gamma)$, define $\tau([a]) := [\mathcal{P}(\sigma)(a)]$ where $\sigma \in Aut(X)$ with $\gamma\sigma = \tau\gamma$ (which exists since $\tau \in \rho(\gamma)$). It remains to show that this assignment is independent of the choice of such a σ and is independent of the choice of representative of $[a]$. These both follow from the fact that if σ_1 and σ_2 satisfy $\gamma\sigma_i = \tau\gamma$, then $\sigma_2^{-1}\sigma_1 \in Aut(\gamma)$. $\square$

Definition 8.46. We say that $\mathcal{P} \in \mathcal{F}$-$\mathcal{O}ps_{\mathcal{T}op}$ is ρ-cofibrant if for each morphism γ, $(\mathcal{P} \circ s(\gamma))_{Aut(\gamma)}$ is cofibrant in the category of $\rho(\gamma)$-spaces.

Remark 8.47. Recall from [BM03] that a (classic) operad is Σ-cofibrant if it is cofibrant in the category of Σ-modules after forgetting the operad structure. When $\mathfrak{F} = \mathfrak{O}$, the Feynman category for operads, the notion of ρ-cofibrancy and Σ-cofibrancy coincide, since the group $Aut(\gamma)$ is trivial. However in a context where there are tail fixing automorphisms, such as in $\mathfrak{M}$, ρ-cofibrancy is stronger than just asking that the image of the forgetful functor be cofibrant. However for modular operads whose $\mathbb{S}_{flag(v)}$ action corresponds to permuting labels, i.e. is free, ρ-cofibrancy is satisfied.

Theorem 8.48. *Let $\mathfrak{F}$ be a simple Feynman category and let $\mathcal{P} \in \mathcal{F}$-$\mathcal{O}ps_{\mathcal{T}op}$ be ρ-cofibrant. Then $W(\mathcal{P})$ is a cofibrant replacement for $\mathcal{P}$ with respect to the above model structure on $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{T}op}$.*

Proof. First, there is a universal map $W(\mathcal{P})(v) \to \mathcal{P}(v)$ for each $v \in V$ and by definition this induces a morphism in $\mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{T}op}$; indeed for any weighted sequence $X \to \cdots \to v$ there is a map $\mathcal{P}(X) \to \mathcal{P}(v)$ given by retaining only the composition type of the sequence and using the $\mathcal{F}\text{-}\mathcal{O}p$ structure of $\mathcal{P}$. These morphisms clearly give a morphism in $\mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{T}op}$.

To show the morphism $W(\mathcal{P}) \to \mathcal{P}$ is a weak equivalence, it is enough to show that for any $v \in \mathcal{V}$, $W(\mathcal{P})(v) \to \mathcal{P}(v)$ is a weak homotopy equivalence. Now taking the cocone morphism corresponding to id_v we get a factorizaion of the identity $\mathcal{P}(v) \overset{i}{\to} W(\mathcal{P})(v) \overset{\pi}{\to} \mathcal{P}(v)$. It thus remains to show that $i \circ \pi \sim id_{W(\mathcal{P})(v)}$, and we may define the homotopy $W(\mathcal{P})(v) \times [0,1] \to W(\mathcal{P})(v)$ as follows.

For $t \in [0,1]$ define $w(\mathcal{F}, v, t) \subset w(\mathcal{F}, v)$ to be the full subcategory whose weights are each less than or equal to t. Define a functor $w(\mathcal{F}, v) \to w(\mathcal{F}, v, t)$ by multiplying all of the weights of a given sequence by t. This functor and the inclusion functor induce maps,

$$\alpha_t \colon colim_{w(\mathcal{F},v)} \mathcal{P} \circ s \leftrightarrows colim_{w(\mathcal{F},v,t)} \mathcal{P} \circ s \colon \beta_t$$

Clearly $\beta_1 \alpha_1$ is the identity. On the other hand, since we can truncate 0 weights, one has $colim_{w(\mathcal{F},v,0)} \mathcal{P} \circ s \cong \mathcal{P}(v)$ and $\beta_0 \alpha_0 = i \circ \pi$. We thus conclude that there is a weak equivalence $W(\mathcal{P}) \overset{\sim}{\to} \mathcal{P}$.

To complete the proof we must show that $W(\mathcal{P})$ is cofibrant. We know, however, that $F(\mathcal{P})$ is cofibrant, and that there is a morphism $F(\mathcal{P}) \to W(\mathcal{P})$ induced by the functors $Iso(\mathcal{F} \downarrow v) \to w(\mathfrak{F}, v)$ taking all non isomorphisms to have weight 1. Thus it suffices to show that this morphism is a cofibration. Using our assumptions, we may follow as in [Vog03], [BM06].

For each natural number r we define $wt^r(\mathcal{F}, v)$ to be the subcategory of $w(\mathcal{F}, v)$ which omits the truncation morphisms and which omits those objects $X \to v$ of degree (in $\mathcal{F}$), greater than r. We define $\partial wt^r(\mathcal{F}, v) \subset wt^r(\mathcal{F}, v)$ to be the full subcategory of those objects having at least one weight equal to 0 or 1. Next define a $\mathcal{V}$-module $\Psi_r(\mathcal{P})$ by:

$$\Psi_r(\mathcal{P})(v) := \underset{wt^r(\mathcal{F},v)}{\text{colim}} \mathcal{P} \circ s(-)$$

and a $\mathcal{V}$-module $\partial\Psi_r(\mathcal{P})$ by

$$\partial\Psi_r(\mathcal{P})(v) := \underset{\partial wt^r(\mathcal{F},v)}{\text{colim}} \mathcal{P} \circ s(-)$$

We will show that the inclusion $\partial\Psi_r(\mathcal{P})(v) \hookrightarrow \Psi_r(\mathcal{P})(v)$ is a cofibration in the category of $Aut(v)$-spaces. First, because we have removed truncations, the colimit $\Psi_r(\mathcal{P})(v)$ splits over the isomorphism classes of the source. Now, an isomorphism $X \overset{\cong}{\to} X'$ fixes a bijection between objects in $wt^r(\mathcal{F}, v)$ with source X and with source X', and so when considering the component of the colimit $\Psi_r(\mathcal{P})(v)$ corresponding to the isomorphism class $[X] = [X']$, we may choose a representative, X say, and consider only those sequences with source X.

The assumption that $\mathfrak{F}$ is quadratic means that each composition class $[\gamma]$ contributes a factor of $\mathcal{P}(X)_{Aut(\gamma)} \times [0,1]^{|\gamma|}$ to the colimit. Now $Aut(v)$ acts on the colimit (by postcomposition), and by Lemma 8.45 the action restricted to $\mathcal{P}(X)_{Aut(\gamma)}$ is closed under the elements of $\rho(\gamma)$. Thus the $Aut(v)$ orbit of $\mathcal{P}(X)_{Aut(\gamma)}$ is the $\rho(\gamma)$ coinvariants of the free $Aut(v)$-space on $\mathcal{P}(X)_{Aut(\gamma)}$. Since $\mathcal{P}(X)_{Aut(\gamma)}$ is cofibrant as a $\rho(\gamma)$-space, it follows that the $Aut(v)$ orbit of $\mathcal{P}(X)_{Aut(\gamma)}$ is cofibrant as an $Aut(v)$-space (since it is the image of a cofibrant object by a left adjoint). Then the assumption that $\mathfrak{F}$ is simple tells us that the only identifications in the $[X]$ component of the colimit occur within the $Aut(v)$ orbit of the morphism. By [BM06] lemma 2.5.2, taking the product of this $Aut(v)$-cofibrant space with the topological cofibration $\partial[0,1]^n \to [0,1]^n$ is a cofibration of $Aut(v)$-spaces. Thus we may write $\partial\Psi_r(\mathcal{P})(v) \hookrightarrow \Psi_r(\mathcal{P})(v)$ as a coproduct of cofibrations of $Aut(v)$-spaces; in particular this coproduct is taken over composition classes of morphisms modulo composing with automorphisms of the source. It follows that $\partial\Psi_r(\mathcal{P})(v) \hookrightarrow \Psi_r(\mathcal{P})(v)$ is a cofibration in the category of $Aut(v)$-spaces, and from this we then conclude that $F(\partial\Psi_r(\mathcal{P})) \hookrightarrow F(\Psi_r(\mathcal{P}))$ is a cofibration in $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{T}op}$.

To complete the proof, define a sequence $\dots \to W^r(\mathcal{P}) \to W^{r+1}(\mathcal{P}) \to \dots$ as follows. First, $W^0(\mathcal{P}) := F(\mathcal{P})$, and then inductively $W^r(\mathcal{P})$ is defined to be the pushout:

$$\begin{tikzcd}
F(\partial\Psi_r(\mathcal{P})) \arrow[r] \arrow[d] & F(\Psi_r(\mathcal{P})) \arrow[d] \\
W^{r-1}(\mathcal{P}) \arrow[r] & W^r(\mathcal{P})
\end{tikzcd}$$

Then it is straight forward to show that the direct limit is $F(\mathcal{P}) \to W(\mathcal{P})$, see e.g. [Vog03], [BM06]. Thus $F(\mathcal{P}) \to W(\mathcal{P})$ is a direct limit of pushouts of cofibrations, and hence a cofibration, implying in particular that $W(\mathcal{P})$ is cofibrant, which implies the result. □

Remark 8.49. In analogy with the free operad, a reasonable question would be to ask if $W(\mathcal{P})$ can be given as a Kan extension, and the answer is yes. In particular if one considers $([0,1]^n)_{\mathbb{S}_n} \cong \Delta^n$, we can achieve the trunctation and $\mathbb{S}_n$ identifications by considering the following nerve-like construction

$$|C_*(A,B)| = \coprod_{n\geq 0} C_n(A,B) \times \Delta^n / \sim \tag{8.6}$$

where the equivalence relation identifies $(\delta_i(\gamma), p) \sim (\gamma, d^i(p))$ for $1 \leq i \leq n-1$ for $\gamma \in C_n(A,B)$, and $p \in \Delta^{n-1}$, where δ_i composes morphisms as per usual. Graphically we consider ordering the edges from least to greatest weights, with the identifications accounting for any ambiguities. The spaces $|C_*(A,B)|$ are then the morphisms of a category with composition given by composition of sequences followed by the unique permutation which reorders from least to greatest. If we call this category $W(\mathfrak{F})$, we may define a family of morphisms

$$\mathcal{F} \xrightarrow{\epsilon^t} W(\mathfrak{F})$$

for $t \in \Delta^1$ as by setting ϵ^t to be the identity on objects and, if $A \xrightarrow{f} B$ is a morphism in $\mathcal{F}$ then

$$\epsilon^t(f) = \begin{cases} (f,t) \in C_1(A,B) \times \Delta^1 & \text{if } deg(f) \neq 0 \\ (f,*) \in C_0(A,B) \times \Delta^0 & \text{if } deg(f) = 0 \end{cases}$$

Then one can show

$$W(\mathcal{P}) = \epsilon^1 \circ Lan_{\epsilon_0}(\mathcal{P})$$

Remark 8.50. In assuming that $\mathfrak{F}$ is quadratic we exclude from consideration any non-isomorphisms of the form $k \to X$ where k is the monoidal unit. For the W construction to be cofibrant when such morphisms are included requires additional cofibrancy conditions. For example considering operads with units $k \to \mathcal{P}(1)$ requires the additional cofibrancy condition that $k \to \mathcal{P}(1)$ is a cofibration. Such operads are called well-pointed in [BM03], [Vog03].

Remark 8.51. In [BM06] Berger and Moerdijk give a generalization of the W construction for classical operads in other monoidal model categories equipped with a suitable interval object. Combining their arguments with the one above we expect an even more general notion of cofibrant replacement for $\mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$. In particular one would expect an equivalence between the double Feynman transform and the generalized W-construction in $dg\mathcal{V}ect_k$.

Appendix A. Graph Glossary

A.1. **The category of graphs.** Most of the known examples of Feynman categories used in operadic type theory are indexed over a Feynman category build from graphs. It is important to note that although we will first introduce a category of graphs $\mathcal{G}raphs$, the relevant Feynman category is given by a full subcategory $\mathcal{A}gg$ whose objects are disjoint unions or aggregates of corollas. The corollas themselves play the role of $\mathcal{V}$.

Before giving more examples in terms of graphs it will be useful to recall some terminology. A nice presentation is given in [BM08] which we follow here.

A.1.1. **Abstract graphs.** An abstract graph Γ is a quadruple $(V_\Gamma, F_\Gamma, i_\Gamma, \partial_\Gamma)$ of a finite set of vertices V_Γ a finite set of half edges or flags F_Γ an involution on flags $i_\Gamma\colon F_\Gamma \to F_\Gamma; i_\Gamma^2 = id$ and a map $\partial_\Gamma\colon F_\Gamma \to V_\Gamma$. We will omit the subscripts Γ if no confusion arises.

Since the map i is an involution, it has orbits of order one or two. We will call the flags in an orbit of order one *tails* and denote the set of tails by T_Γ. We will call an orbit of order two an *edge* and denote the set of edges by E_Γ. The flags of an edge are its elements. The function ∂ gives the vertex a flag is incident to. It is clear that the set of vertices and edges form a 1–dim CW complex. The realization of a graph is the realization of this CW complex.

A graph is (simply) connected if and only if its realization is. Notice that the graphs do not need to be connected. Lone vertices, that is vertices with no incident flags are also possible.

We also allow the empty graph. That is the unique graph with $V_\Gamma = \mathbb{1}_\emptyset$. It will serve as the monoidal unit.

Example A.1. A graph with one vertex is called a *corolla*. Such a graph only has tails and no edges. Any set S gives rise to a corolla. Let p be a one point set then the corolla is $*_{p,S} = (p, S, id, \partial)$ where ∂ is the constant map.

Given a vertex v of Γ we set $F_v = F_v(\Gamma) = \partial^{-1}(v)$ and call it *the flags incident to* v. This set naturally gives rise to a corolla. The *tails* at v is the subset of tails of F_v.

As remarked above F_v defines a corolla $*_v = *_{\{v\},F_v}$.

Remark A.2. The way things are set up, we are talking about finite sets, so changing the sets even by bijection changes the graphs.

Remark A.3. The graphs do not need to be connected. Moreover, given two graphs Γ and Γ' we can form their disjoint union:
$\Gamma \amalg \Gamma' = (F_\Gamma \amalg F_{\Gamma'}, V_\Gamma \amalg V_{\Gamma'}, i_\Gamma \amalg i_{\Gamma''}, \partial_\Gamma \amalg \partial_{\Gamma'})$.

One actually needs to be a bit careful about how disjoint unions are defined. Although one tends to think that the disjoint union $X \amalg Y$ is symmetric, this is not the case. This becomes apparent if $X \cap Y \neq \emptyset$ of course there is a bijection $X \amalg Y \xleftrightarrow{1-1} Y \amalg X$. Thus the categories here are symmetric monoidal, but not strict symmetric monoidal. This is important, since we consider functors into other not necessarily strict monoidal categories.

Using MacLane's theorem it is however possible to make a technical construction that makes the monoidal structure (on both sides) into a strict symmetric monoidal structure

Example A.4. An *aggregate of corollas* or aggregate for short is a finite disjoint union of corollas, that is, a graph with no edges.

Notice that if one looks at $X = \amalg_{v \in I} *_{S_v}$ for some finite index set I and finite sets of flags S_v, then the set of flags is automatically the disjoint union of the sets S_v. We will referring to specific elements in this disjoint union by their pre-image, that is just say $s \in F(X)$.

A.1.2. **Category structure; Morphisms of Graphs.**

Definition A.5. [BM08] Given two graphs Γ and Γ' a morphism form Γ to Γ' is a triple (ϕ^F, ϕ_V, i_ϕ) where

(i) $\phi^F \colon F_{\Gamma'} \hookrightarrow F_\Gamma$ is an injection,
(ii) $\phi_V \colon V_\Gamma \twoheadrightarrow V_{\Gamma'}$ and i_ϕ is a surjection and
(iii) i_ϕ is a fixed point free involution on the tails of Γ not in the image of ϕ^F.

One calls the edges and flags that are not in the image of ϕ the contracted edges and flags. The orbits of i_ϕ are called ghost edges and denoted by $E_{ghost}(\phi)$.

Such a triple is *a morphism of graphs* $\phi \colon \Gamma \to \Gamma'$ if

1. The involutions are compatible:
   - (a) An edge of Γ is either a subset of the image of ϕ^F or not contained in it.
   - (b) If an edge is in the image of ϕ^F then its pre–image is also an edge.
2. ϕ^F and ϕ_V are compatible with the maps ∂:
   - (a) Compatibility with ∂ on the image of ϕ^F: If $f = \phi^F(f')$ then $\phi_V(\partial f) = \partial f'$
   - (b) Compatibility with ∂ on the complement of the image of ϕ^F: The two vertices of a ghost edge in Γ map to the same vertex in Γ' under ϕ_V.

If the image of an edge under ϕ^F is not an edge, we say that ϕ grafts the two flags.

The composition $\phi' \circ \phi \colon \Gamma \to \Gamma''$ of two morphisms $\phi \colon \Gamma \to \Gamma'$ and $\phi' \colon \Gamma' \to \Gamma''$ is defined to be $(\phi^F \circ \phi'^F, \phi'_V \circ \phi_V, i)$ where i is defined by its orbits viz. the ghost edges. Both maps ϕ^F and ϕ'^F are injective, so that the complement of their concatenation is in bijection with the disjoint union of the complements of the two maps. We take i to be the involution whose orbits are the union of the ghost edges of ϕ and ϕ' under this identification.

Remark A.6. A *naïve morphism* of graphs $\psi \colon \Gamma \to \Gamma'$ is given by a pair of maps $(\psi_F \colon F_\Gamma \to F_{\Gamma'}, \psi_V \colon V_\Gamma \to V_{\Gamma'})$ compatible with the maps i and ∂ in the obvious fashion. This notion is good to define subgraphs and automorphisms.

It turns out that this data *is not enough* to capture all the needed aspects for composing along graphs. For instance it is not possible to contract edges with such a map or graft two flags into one edge. The basic operations of composition in an operad viewed in graphs is however exactly grafting two flags and then contracting.

For this and other more subtle aspects one needs the more involved definition above which we will use.

Definition A.7. We let $\mathcal{G}raphs$ be the category whose objects are abstract graphs and whose morphisms are the morphisms described in Appendix A. We consider it to be a monoidal category with monoidal product $\amalg$ (see Remark A.3).

A.1.3. **Decomposition of morphisms.** Given a morphism $\phi\colon X \to Y$ where $X = \amalg_{w \in V_X} *_v$ and $Y = \amalg_{v \in V_Y} *_w$ are two aggregates, we can decompose $\phi = \amalg \phi_v$ with $\phi_v\colon X_v \to *_v$ and $\amalg_v X_v = X$. Let X_v be the sub–aggregate of X consisting of the vertices $V_{X_v} = \phi_V^{-1}(v)$ together with all the incident flags $F_{X_v} = \partial_X^{-1}(V_{X_v})$. Let $(\phi_v)_V$, be the restriction of ϕ_V to V_{X_v}. Likewise let ϕ_v^F be the restriction of ϕ^F to $(\phi^F)^{-1}(F_{X_v} \cap \phi^F(F_Y))$. This is still injective. Finally let i_{ϕ_v} be the restriction of i_ϕ to $F_{X_v} \setminus \phi^F(F_Y)$. These restrictions are possible due to the condition (2) above.

A.1.4. **Ghost graph of a morphism.** Given a morphism $\phi\colon \Gamma \to \Gamma'$, the underlying ghost graph of a morphism of the graph is the graph given by $\mathbb{\Gamma}(\phi) = (V(\Gamma), F_\Gamma, \hat{i}_\phi)$ where $\hat{i}_\phi$ is i_ϕ on the complement of $\phi^F(\Gamma')$ and identity on the image of edges of Γ' under ϕ^F. The edges of $\mathbb{\Gamma}(\phi)$ are exactly the ghost edges of ϕ.

A.2. **Extra structures.**

A.2.1. **Glossary.** This section is intended as a reference section. All the following definitions are standard.

Recall that an order of a finite set S is a map $S \to \{1, \dots, |S|\}$. Thus the group $\mathbb{S}_{|S|} = Aut\{1, \dots, n\}$ acts on all orders. An orientation of a finite set S is an equivalence class of orders, where two orders are equivalent if they are obtained from each other by an even permutation.

A tree	is a connected, simply connected graph.
A directed graph Γ	is a graph together with a map $F_\Gamma \to \{in, out\}$ such that the two flags of each edge are mapped to different values.
A rooted tree	is a directed tree such that each vertex has exactly one “out” flag.
A ribbon or fat graph	is a graph together with a cyclic order on each of the sets F_v.
A planar graph	is a a ribbon graph that can be embedded into the plane such that the induced cyclic orders of the sets F_v from the orientation of the plane coincide with the chosen cyclic orders.
A planted planar tree	is a rooted planar tree together with a linear order on the set of flags incident to the root.
An oriented graph	is a graph with an orientation on the set of its edges.
An ordered graph	is a graph with an order on the set of its edges.
A γ labelled graph	is a graph together with a map $\gamma : V_\Gamma \to \mathbb{N}_0$.
A b/w graph	is a graph Γ with a map $V_\Gamma \to \{black, white\}$.
A bipartite graph	is a b/w graph whose edges connect only black to white vertices.
A c colored graph	for a set c is a graph Γ together with a map $F_\Gamma \to c$ s.t. each edge has flags of the same color.

A.2.2. **Remarks and language.**

(1) In a directed graph one speaks about the “in” and the “out” edges, flags or tails at a vertex. For the edges this means the one flag of the edges is an “in“ flag at the vertex. In pictorial versions the direction is indicated by an arrow. A flag is an “in” flag if the arrow points to the vertex.

(2) As usual there are edge paths on a graph and the natural notion of an oriented edge path. An edge path is a (oriented) cycle if it starts and stops at the same vertex and all the edges are pairwise distinct. It is called simple, if each vertex on the cycle has exactly one incoming flag and one outgoing flag belonging to the cycle. An oriented simple cycle will be called a *wheel.* An edge whose two vertices coincide is called a *(small) loop.*

(3) There is a notion of a the genus of a graph, which is the minimal dimension of the surface it can be embedded on. A ribbon graph is planar if this genus is 0.

(4) For any graph, its Euler characteristic is given by

$$\chi(\Gamma) = b_0(\Gamma) - b_1(\Gamma) = |V_\Gamma| - |E_\Gamma|;$$

where b_0, b_1 are the Betti numbers of the (realization of) Γ. Given a γ labelled graph, we define the total γ as

$$\gamma(\Gamma) = 1 - \chi(\Gamma) + \sum_{v \text{ vertex of } \Gamma} \gamma(v) \tag{A.1}$$

If Γ is *connected*, that is $b_0(\Gamma) = 1$ then a γ labeled graph is traditionally called a genus labeled graph and

$$\gamma(\Gamma) = \sum_{v \in V_\Gamma} \gamma(v) + b_1(\Gamma) \tag{A.2}$$

is called the genus of Γ. This is actually not the genus of the underlying graph, but the genus of a connected Riemann surface with possible double points whose dual graph is the genus labelled graph.

A genus labelled graph is called *stable* if each vertex with genus labeling 0 has at least 3 flags and each vertex with genus label 1 has at leas one edge.

(5) A planted planar tree induces a linear order on all sets F_v, by declaring the first flag to be the unique outgoing one. Moreover, there is a natural order on the edges, vertices and flags given by its planar embedding.

(6) A rooted tree is usually taken to be a tree with a marked vertex. Note that necessarily a rooted tree as described above has exactly one "out" tail. The unique vertex whose "out" flag is not a part of an edge is the root vertex. The usual picture is obtained by deleting this unique "out" tail.

A.2.3. **Category of directed/ordered/oriented graphs.**

(1) Define the category of directed graphs $\mathcal{G}raphs^{dir}$ to be the category whose objects are directed graphs. Morphisms are morphisms ϕ of the underlying graphs, which additionally satisfy that ϕ^F preserves orientation of the flags and the i_ϕ also only has orbits consisting of one "in" and one "out" flag, that is the ghost graph is also directed.

(2) The category of edge ordered graphs $\mathcal{G}raphs^{or}$ has as objects graphs with an order on the edges. A morphism is a morphism together with an order ord on all of the edges of the ghost graph.

The composition of orders on the ghost edges is as follows. $(\phi, ord) \circ \amalg_{v\in V}(\phi_v, ord_v) := (\phi \circ \amalg_{v\in V}\phi_v, ord \circ \amalg_{v\in V} ord_v)$ where the order on the set of all ghost edges, that is $E_{ghost}(\phi) \amalg \amalg_v E_{ghost}(\phi_v)$, is given by first enumerating the elements of $E_{ghost}(\phi_v)$ in the order ord_v where the order of the sets $E(\phi_v)$ is given by the order on V, i.e. given by the explicit ordering of the tensor product in $Y = \amalg_v *_v$.[3] and then enumerating the edges of $E_{ghost}(\phi)$ in their order *ord*.

(3) The oriented version $\mathcal{G}raphs^{or}$ is then obtained by passing from orders to equivalence classes.

A.2.4. **Category of planar aggregates and tree morphisms.** Although it is hard to write down a consistent theory of planar graphs with planar morphisms, if not impossible, there does exist a planar version of special subcategory of $\mathcal{G}raphs$.

We let $\mathcal{C}rl^{pl}$ have as objects planar corollas –which simply means that there is a cyclic order on the flags– and as morphisms isomorphisms of these, that is isomorphisms of graphs, which preserve the cyclic order. The automorphisms of a corolla $*_S$ are then isomorphic to $C_{|S|}$, the cyclic group of order $|S|$. Let $\mathfrak{C}^{pl}$ be the full subcategory of aggregates of planar corollas whose morphisms are morphisms of the underlying corollas, for which the ghost graphs in their planar structure induced by the source is compatible with the planar structure on the target via ϕ^F. For this we use the fact that the tails of a planar tree have a cyclic order.

Let $\mathcal{C}rl^{pl,dir}$ be directed planar corollas with one output and let $\mathfrak{D}^{pl}$ be the subcategory of $\mathcal{A}gg^{pl,dir}$ of aggregates of corollas of the type just mentioned, whose morphism are morphisms of the underlying directed corollas such that its associated ghost graphs are compatible with the planar structures as above.

A.3. **Flag killing and leaf operators; insertion operations.**

A.3.1. **Killing tails.** We define the operator *trun*, which removes all tails from a graph. Technically, $trun(\Gamma) = (V_\Gamma, F_\Gamma \setminus T_\Gamma, \partial_\Gamma|_{F_\Gamma \setminus T_\Gamma}, \imath_\Gamma|_{F_\Gamma \setminus T_\Gamma})$.

A.3.2. **Adding tails.** Inversely, we define the formal expression *leaf* which associates to each Γ without tails the formal sum $\sum_n \sum_{\Gamma':trun(\Gamma')=\Gamma; F(\Gamma')=F(\Gamma')\amalg \bar{n}} \Gamma'$, that is all possible additions of tails where these tails are a standard set, to avoid isomorphic duplication. To make this well defined, we can consider the series as a power series in t: $leaf(\Gamma) = \sum_n \sum_{\Gamma':trun(\Gamma')=\Gamma; F(\Gamma')=F(\Gamma')\amalg \bar{n}} \Gamma' t^n$

This is the foliage operator of [KS00, Kau07b] which was rediscovered in [BBM13]. This is the foliage operator of [KS00, Kau07b] which was rediscovered in [BBM13].

A.3.3. **Insertion.** Given graphs, Γ,Γ', a vertex $v \in V_\Gamma$ and an isomorphism ϕ: $F_v \mapsto T_{\Gamma'}$ we define $\Gamma \circ_v \Gamma'$ to be the graph obtained by deleting v and identifying the flags of v with the tails of Γ' via ϕ. Notice that if Γ and Γ' are ghost graphs of a morphism then is just the composition of ghost graphs, with the morphisms at the other vertices being the identity.

A.3.4. **Unlabelled insertion.** If we are considering graphs with unlabelled tails, that is classes $[\Gamma]$ and $[\Gamma']$ of coinvariants under the action of permutation of tails. The insertion naturally lifts as $[\Gamma] \circ [\Gamma'] := [\sum_\phi \Gamma \circ_v \Gamma']$ where ϕ runs through all the possible isomorphisms of two fixed lifts.

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[3] Now we are working with ordered tensor products. Alternatively one can just index the outer order by the set V by using [Del90]

A.3.5. **No–tail insertion.** If Γ and Γ' are graphs without tails and v a vertex of v, then we define $\Gamma \circ_v \Gamma' = \Gamma \circ_v \operatorname{coeff}(leaf(\Gamma'), t^{|F_v|})$, the (formal) sum of graphs where ϕ is one fixed identification of F_v with $\overline{|F_v|}$. In other words one deletes v and grafts all the tails to all possible positions on Γ'. Alternatively one can sum over all $\partial : F_\Gamma \amalg F_{\Gamma'} \to V_\Gamma \setminus v \amalg V_{\Gamma'}$ where ∂ is ∂_G when restricted to $F_w, w \in V_\Gamma$ and $\partial_{\Gamma'}$ when restricted to $F_{v'}, v' \in V_{\Gamma'}$.

A.3.6. **Compatibility.** Let Γ and Γ' be two graphs without flags, then for any vertex v of Γ $leaf(\Gamma \circ_v \Gamma') = leaf(\Gamma) \circ_v leaf(\Gamma')$.

Appendix B. Topological Model Structure

In this appendix we fix $\mathcal{C}$ to be the category of topological spaces with the Quillen model structure. The goal of this section is to show that $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$ has a model structure via Theorem 8.8. The general corollary to this theorem does not apply in this case however because not all objects are small [Hov99]. Thus we will devise a separate argument to show that the conditions of the theorem are satisfied. This work is inspired by [Fre10]. The key to this argument is the fact that all topological spaces are small with respect to toplogical inclusions [Hov99], whose definition we now recall.

Definition B.1. A continuous map $f\colon X \to Y$ is called a topological inclusion if it is injective and if for every open set $U \subset X$ there exists an open set $V \subset Y$ such that $f^{-1}(V) = U$. We will often refer to topological inclusions as simply 'inclusions'.

We will use the following facts about topological inclusions.

Lemma B.2. *Inclusions are closed under pushouts, transfinite compositions, and finite products.*

Proof. This is straight forward. See [Hov99] section 2.4 for the first two statements. For the third statement, note that a product of injective maps is injective, and that every open set in a product can be written as a union of products of open sets, from which the claim quickly follows.

□

Lemma B.3. *Let J be a groupoid, let α, β be functors from J to $\mathcal{C}$ and let $\eta\colon \alpha \Rightarrow \beta$ be a natural transformation such that $\eta_j\colon \alpha(j) \to \beta(j)$ is an inclusion for each $j \in J$. Then the induced map $colim(\alpha) \to colim(\beta)$ is an inclusion.*

Proof. Define ϕ to be the induced map between the colimits. We first show ϕ is injective. Recall that the forgetful functor from spaces to sets preserves colimits and thus as a set,

$$colim(\alpha) = \coprod_{j \in J} \alpha(j)/\sim_\alpha$$

where $\sim_\alpha$ is the equivalence relation defined by $x_1 \sim_\alpha x_2$ if and only if there exists $f \in Mor(J)$ such that $x_2 = \alpha(f)(x_1)$. Note that since J is a groupoid this is an equivalence relation. Similarly we define $colim(\beta)$ via an equivalence relation $\sim_\beta$.

Let $[x_1]$ and $[x_2]$ be elements in $colim(\alpha)$ with $\phi([x_1]) = \phi([x_2])$. Suppose $x_i \in \alpha(a_i)$. Then $\phi([x_i]) := [\eta_{a_i}(x_i)]$, and by assumption $\eta_{a_1}(x_1) \sim_\beta \eta_{a_2}(x_2)$. Thus there exists $f \in Mor(J)$ such that $\beta(f)(\eta_{a_1}(x_1)) = \eta_{a_2}(x_2)$. Since η_{a_2} is an injection by assumption, the naturality of η implies that $\alpha(f)(x_1) = x_2$, and thus $[x_1] = [x_2]$. So ϕ is injective.

Next we must show that any open set in $colim(\alpha)$ is the pullback via ϕ of an open set in $colim(\beta)$. Let $U \subset colim(\alpha)$ be open. Then by definition of the topology of the colimit, $\lambda_a^{-1}(U) =: U_a$ is open in $\alpha(a)$ for every $a \in J$, where $\lambda_\bullet\colon \alpha(-) \to colim(\alpha)$ is the cocone. Now by assumption for each $a \in J$ there exists an open set $V_a \subset \beta(a)$ such that $\eta_a^{-1}(V_a) = U_a$. Define $V := \cup_{a \in J} \gamma_a(V_a)$, where $\gamma_\bullet$ is the cocone for β. This will be our candidate for an open set whose pull back under ϕ is U.

To show V is open it is necessary and sufficient to show that for each $a \in J$, $\gamma_a^{-1}(V) \subset \beta(a)$ is open, and to show this we show

$$\gamma_a^{-1}(V) = \bigcup_{\substack{f \in Mor(J) \\ t(f)=a}} \beta(f)(V_{s(f)}). \tag{B.1}$$

First let $x \in \gamma_a^{-1}(V)$. Then $\gamma_a(x) \in V$ and so there exists $b \in J$ and $y \in V_b$ with $\gamma_a(x) = \gamma_b(y)$, and thus there exists a morphism $f\colon b \to a$ in J with $x = \beta(f)(y)$. As a result, $x \in \beta(f)(V_b)$ and hence the inclusion $\subset$ in equation $B.1$. For the other inclusion, suppose w is an element of the rhs of equation $B.1$. Then $w = \beta(f)(z)$ for some $z \in V_b$ and $f\colon b \to a$. Using commutativity in the cocone this implies $\gamma_a(w) = \gamma_b(z)$ and hence $w \in \gamma_a^{-1}(\gamma_b(V_b)) \subset \gamma_a^{-1}(V)$, from which equation $B.1$ follows. It follows that $V \subset colim(\beta)$ is open.

It remains to show that $\phi^{-1}(V) = U$. For the $\supset$ inclusion: if $[u] \in U$ then there exists $u \in U_a$ such that $\lambda_a(u) = [u]$ and by definition of ϕ, $\gamma_a\eta_a(u) = \phi([u])$. Now $\eta_a(u) \in V_a$ implies that $\phi([u]) \in V$ and hence $[u] \in \phi^{-1}(V)$. For the $\subset$ inclusion: if $[w] \in \phi^{-1}(V)$, where $w \in \alpha(a)$, then there exists $[v] \in V$ with $\phi([w]) = [v]$. Since $[v] \in V$ there exists $b \in J$ with $v \in V_b$ and by commutativity of the relevent diagram we have $[v] = \gamma_b(v) = \gamma_a\eta_a(w)$. Thus there exists a morphism $f\colon a \to b$ such that $v = \beta(f)(\eta_a(w))$, and by naturality, $v = \eta_b(\alpha(f)(w))$ so that there exists a unique $u := \alpha(f)(w) \in U_b$ such that $\eta_b(u) = v$. Then $[w] = [u]$ by definition of $\sim_\alpha$ and the fact that $[u] \in U$ permits the conclusion. □

Corollary B.4. *For $A \in \mathcal{V}$-$Seq_{\mathcal{C}}$, the natural map $A \to GF(A)$ is a levelwise inclusion.*

Proof. Fix $v \in \mathcal{V}$ and let J be the groupoid $Iso(\mathcal{F} \downarrow v)$, let β be the functor $A \circ s(-)(v)$ and let α be the functor which sends an isomorphism object in J to $A(v)$, with morphisms in J sent to the identity, and which sends all other objects in J to $\emptyset$. Note that $colim(\alpha) = A(v)$ and that there is an obvious natural transformation $\alpha \Rightarrow \beta$ which is an inclusion for each $j \in J$. Thus Lemma $B.3$ applies, from which the conclusion follows.

□

Fix a Feynman category $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \imath)$ and let $F\colon \mathcal{V}\text{-Seq}_{\mathcal{C}} \rightleftarrows \mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}} : G$ be the forgetful free adjunction.

Proposition B.5. *Let I be a collection of levelwise inclusions in $\mathcal{V}$-$Seq_{\mathcal{C}}$. Then $F(I)$ permits the small object argument.*

Proof. The proof will be given in several steps.

1. Show the objects of $\mathcal{V}\text{-Seq}_{\mathcal{C}}$ are small with respect to levelwise inclusions.
2. Show GF preserves levelwise inclusions.
3. Show pushouts in $\mathcal{F}$-$\mathcal{O}ps$ preserve levelwise inclusions.
4. Conclude that all objects in $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$ are small relative to $F(I)$, from which the claim follows.

Step 1: This follows exactly as in the first part of the proof of Theorem 8.15.

Step 2: Suppose $\eta\colon A \to B$ is a levelwise inclusion in $\mathcal{V}\text{-Seq}_{\mathcal{C}}$. For a fixed $v \in V$, we will apply Lemma $B.3$ with $J = Iso(\mathcal{F} \downarrow v)$, $\alpha = A \circ s$ and $\beta = B \circ s$. To do so note that the morphism of spaces $F(A)(v) \to F(B)(v)$ is induced by the natural transformation

$A \circ s \to B \circ s$ given by mapping $\times_i A(v_i) \overset{\times_i \eta_{v_i}}{\to} \times_i B(v_i)$, and that $\times_i \eta_{v_i}$ is an inclusion by Lemma $B.2$ since each η_v is. Lemma $B.3$ thus implies $F(A)(v) \to F(B)(v)$ is an inclusion for each $v \in \mathcal{V}$. In particular $GF(\eta)$ is a levelwise inclusion.

Step 3: Let J be the category with three objects whose nonidentity morphisms are $*_A \leftarrow *_C \to *_B$ and let $\alpha\colon J \to \mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$ be a functor whose image is of the form $A \overset{s}{\leftarrow} C \overset{r}{\to} B$, where the induced map $G(C)(v) \to G(B)(v)$ is a topological inclusion for each $v \in V$. In this step we want to show that for a given $v \in \mathcal{V}$, the natural map $\gamma\colon GA(v) \to Gcolim(\alpha)(v)$ is an inclusion. Consider the diagram

$$\begin{tikzcd}
 & colim(G\alpha)(v) \arrow[d, "\beta_1"] & GA(v) \arrow[l, "\beta_2"'] \arrow[dl, "\beta:=\beta_1\circ\beta_2"'] \arrow[d, "\gamma=\lambda\circ\beta"] \\
GFcolim(GFG\alpha)(v) \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & GFcolim(G\alpha)(v) \arrow[l, "h"'] \arrow[r, "\lambda"] & Gcolim(\alpha)(v)
\end{tikzcd} \tag{B.2}$$

where the bottom line is given by taking the image under G of the reflexive coequalizer of equation 8.1. Note that by the construction of colimits in $\mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$ and the fact that G preserves reflexive coequalizers, the bottom line of this diagram is still a reflexive coequalizer. Also by the above construction of $colim(\alpha)$ we know the map γ is equal to $\lambda \circ \beta$. Finally note that β is an inclusion by Lemma $B.2$ and Corollary $B.4$.

Substep 1: Analyze the morphisms f, g, β.

Let us now consider the spaces $GFcolim(GFG\alpha)(v)$ and $GFcolim(G\alpha)(v)$ and the maps between them in more detail. First note that since the map r is injective, we can write $colim(G\alpha)(v)$ as $A(v) \coprod (B(v) \setminus im(r(v)))$. Therefore, writing $B'(v) := B(v) \setminus im(r(v))$, we have

$$GFcolim(G\alpha)(v) = \mathrm{colim}_{Iso(\mathcal{F}\downarrow v)}(A \coprod B') \circ s \tag{B.3}$$

Similarly, if we define $B''(v) := FB(v) \setminus im(Fr(v))$, then

$$GFcolim(GFG\alpha)(v) = \mathrm{colim}_{Iso(\mathcal{F}\downarrow v)}(FA \coprod B'') \circ s \tag{B.4}$$

Now these two spaces are subspaces of slightly less mysterious spaces, namely

$$GFcolim(G\alpha)(v) \subset \mathrm{colim}_{Iso(\mathcal{F}\downarrow v)}(A \coprod B) \circ s \tag{B.5}$$

and

$$\begin{aligned} &GFcolim(GFG\alpha)(v) \subset \mathrm{colim}_{Iso(\mathcal{F}\downarrow v)}(FA \coprod FB) \circ s \\ \cong\ &\mathrm{colim}_{Iso(\mathcal{F}\downarrow v)}(F(A \coprod B)) \circ s \cong \mathrm{colim}_{Iso(\mathcal{F}^2\downarrow v)}(A \coprod B) \circ s \end{aligned} \tag{B.6}$$

So to define maps $GFcolim(GFG\alpha)(v) \to GFcolim(G\alpha)(v)$, it suffices to define maps

$$\begin{tikzcd}
\mathrm{colim}_{Iso(\mathcal{F}^2\downarrow v)}(A \coprod B) \circ s \arrow[r, shift left=2, "g"] \arrow[r, shift right=2, "f"'] & \mathrm{colim}_{Iso(\mathcal{F}\downarrow v)}(A \coprod B) \circ s
\end{tikzcd} \tag{B.7}$$

and to show they land in the appropriate subspace of the target when restricted to the appropriate subspace of the source.

Recall that the objects of $Iso(\mathcal{F}^2 \downarrow v)$ are sequences of morphisms $X \to Y \to v$ and that the morphisms are levelwise isomorphisms. Define g to be the map that is induced on the colimits by forgetting the middle term; $(X \to Y \to v) \mapsto (X \to v)$, and which takes the identity on the argument $(A \coprod B)(X)$. Define f to be the map that is induced on colimits by forgetting the left hand term; $(X \to Y \to v) \mapsto (Y \to v)$ and which uses to operad structure on the argument; $(A \coprod B)(X) \to (A \coprod B)(Y)$. Note that although $FB \setminus im(F(r)) \supset F(B \setminus im(r))$ are not equal, for each point in $im(F(r)(v))$ there is a unique point in $F(A)(v)$ which is equivalent, and so the maps will restrict appropriately via this relabeling.

Now it must be checked that this definition of f and g recover the morphisms defined in the proof of Lemma 8.5. The morphism f is the morphism (1) of the lemma, which is easily seen since contraction via the operad structure. corresponds to the natural transformation $FG \Rightarrow id$. More delicate is to check that the morphism g is the morphism (2) of the lemma. To see this note that when the Yoneda lemma was applied in the proof, we pushed forward the identity morphism of $Fcolim(G\alpha)$. Therefore, taking $G\mathcal{P} := GFcolim(G\alpha)$, the morphism (2) in the lemma can be described as the image of the cocone $GFG\alpha(-) \to colim(GFG\alpha) \to GFcolim(G\alpha)$ by GF along with postcomposition by $FG \Rightarrow id$, i.e.

$$GFcolim(GFG\alpha) \to G(FG)Fcolim(G\alpha) \to GFcolim(G\alpha) \tag{B.8}$$

On the other hand notice that

$$GFGFcolim(G\alpha) = colim_{Iso(\mathcal{F}^2 \downarrow v)}(A \coprod B') \circ s \subset colim_{Iso(\mathcal{F}^2 \downarrow v)}(A \coprod B) \circ s$$

and so g as described above is preciesly restriction of the map $G(FG)Fcolim(G\alpha) \to GFcolim(G\alpha)$ to the subspace $GFcolim(GFG)\alpha \hookrightarrow G(FG)Fcolim(G\alpha)$, and hence the maps coincide.

There are two other maps that we wish to consider via colimits. First, notice that we may write

$$colim(G\alpha)(v) = A(v) \coprod B'(v) = colim_{Iso(\mathcal{V} \downarrow v)}(A \coprod B') \circ s \tag{B.9}$$

and therefore can realize the map β_1 as being induced by the inclusion functor $Iso(\mathcal{V} \downarrow v) \hookrightarrow Iso(\mathcal{F} \downarrow v)$. Second notice that there is a map

$$\epsilon\colon \mathrm{colim}_{Iso(\mathcal{F} \downarrow v)}(A \coprod B) \circ s \to colim_{Iso(\mathcal{V} \downarrow v)}(A \coprod B) \circ s \tag{B.10}$$

induced by composition in the $\mathcal{F}$-$\mathcal{O}ps$ A and B. Moreover $\epsilon\beta_1$ is the identity.

Substep 2: Show γ is injective.

For simplicity of notation we will rewrite diagram $B.2$ as simply

$$\begin{tikzcd}
 & & Z \arrow[dl, hook', "\beta"'] \arrow[d, "\gamma"] \\
X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & Y \arrow[l, "h"'] \arrow[r, "\lambda"] & Y/\sim
\end{tikzcd} \tag{B.11}$$

where the equivalence relation is that generated by setting $f(x) \sim g(x)$ for each $x \in X$.

Let $z_1, z_2 \in Z$ and suppose $\gamma(z_1) = \gamma(z_2)$. Then $\beta(z_1) \sim \beta(z_2)$. Since β is an injection, it is sufficient to show that $\beta(z_1) = \beta(z_2)$. Now since $\beta(z_1) \sim \beta(z_2)$, there exists a finite series of points $x_1, \ldots, x_m \in X$ along with an m-tuple $(l_1, \ldots, l_m) \in (\mathbb{Z}/2\mathbb{Z})^m$ such that:

$$\begin{tikzcd}
x_1 \arrow[d, "f_{l_1}"] \arrow[dr, "f_{l_1+1}"] & x_2 \arrow[d, "f_{l_2}"] \arrow[dr] & \cdots & x_m \arrow[d] \arrow[dr, "f_{l_m+1}"] & \\
\beta(z_1) = f_{l_1}(x_1) & f_{l_1+1}(x_1) = f_{l_2}(x_2) & \cdots & {} & \beta(z_2) = f_{l_m+1}(x_m)
\end{tikzcd} \tag{B.12}$$

where we define $f_1 := f$ and $f_2 = f_0 := g$.

Now clearly any adjacent pair of points in the bottom row have the same image via ϵ and as a result $\epsilon\beta(z_1) = \epsilon\beta(z_2)$, but $\epsilon\beta$ is the identity, from which we conclude $z_1 = z_2$, and hence γ is injective.

Substep 3: Show the injective map γ is an inclusion. Let $U \subset A(v)$ be an open set. We want to show that there is an open set $V \subset G(colim(\alpha))(v)$ such that $\gamma^{-1}(V) = U$. In order to define V, we first define a subspace

$$W \subset GFcolim(G\alpha)(v) = \mathrm{colim}_{Iso(\mathcal{F}\downarrow v)}(A \coprod B') \circ s \tag{B.13}$$

as follows. Let $(A \coprod B') \circ s \overset{\epsilon_\bullet}{\to} GFcolim(G\alpha)(v)$ be the cocone maps. Then $W := \cup W_\phi$ where ϕ runs over the objects in $Iso(\mathcal{F} \downarrow v)$ and where W_ϕ is defined in two cases. First, if ϕ is not an isomorphism then define W_ϕ to be the image of ϵ_ϕ. Second if ϕ is an isomorphism, from $v' \to v$ say, then it induces a homeomorphism $A(v) \cong A(v')$ and we define U' to be the image of U via this morphism. We then define W_ϕ to be the image of $(U' \coprod B') \circ s$ via ϵ_ϕ.

Now to show W is open it suffices to show W_ϕ is open for each ϕ, and to show this it suffices to show $\epsilon_\psi^{-1}(W_\phi)$ is open for every pair of objects ϕ and ψ. However, the fact that $Iso(\mathcal{F} \downarrow v)$ is a groupoid tells us that $\epsilon_\psi^{-1}(W_\phi)$ is empty unless $\phi \cong \psi$, in which case $\epsilon_\psi^{-1}(W_\phi)$ is clearly open. So W is an open set and it was chosen to be large enough to contain the entirety of any of its equivalence classes. Indeed the only potential snag would be having $\beta(z_1) \sim \beta(z_2)$ with $z_1 \in U$ and $z_2 \in A(v) \setminus U$, but this is precluded by the above arguement, thus we may conclude $\lambda^{-1}\lambda(W) = W$.

Define $V := \lambda(W)$ and note V is open by the definition of the topology of the colimit of a reflexive coequalizer. Further note that $\gamma^{-1}(V) = \beta^{-1}(W) = \beta_2^{-1}\epsilon_{id}^{-1}(W) = U$ as desired. We thus conclude that γ is an inclusion.

Step 4: Let $P \to Q$ be a relative $F(I)$-cell complex in $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$. Then by definition there exists a sequence of pushouts:

$$\begin{tikzcd}
F(A_0) \arrow[r, "F(i_0)"] \arrow[d] & F(B_0) \arrow[dr] & F(A_1) \arrow[r, "F(i_2)"] \arrow[d] & F(B_1) \arrow[dr] & F(A_2) \arrow[r] \arrow[d] & \dots \\
P = P_0 \arrow[rr] & & P_1 \arrow[rr] & & P_2 \arrow[r] & \dots
\end{tikzcd} \tag{B.14}$$

whose composition is the morphism $P \to Q$. By the above work we know that the bottom line is a sequence of levelwise inclusions, and it thus follows from Lemma $B.2$ that $GP \to GQ$ is a levelwise inclusion. Thus for any domain of a morphism in $F(I)$, call it $F(A)$, we have

$$\begin{aligned}
Hom(F(A), colim_i P_i) &\cong Hom(A, Gcolim_i P_i) \cong Hom(A, colim_i(GP_i)) \\
&\cong colim_i Hom(A, GP_i) \cong colim_i Hom(F(A), P_i)
\end{aligned}$$

since A is small with respect to levelwise inclusions. Thus, $F(I)$ permits the small object argument.

□

Theorem B.6. *Let $\mathcal{C}$ be the category of topological spaces with the Quillen model structure. The category $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$ has the structure of a cofibrantly generated model category in which the forgetful functor to $\mathcal{V}$-$Seq_{\mathcal{C}}$ creates fibrations and weak equivalences.*

Proof. We will apply Theorem 8.8. Since both the generating cofibrations I and the generating acyclic cofibrations J in $\mathcal{V}$-Seq$_{\mathcal{C}}$ are levelwise inclusions, it follows from Proposition $B.5$ that $F(I)$ and $F(J)$ permit the small object argument. In addition, as discussed above in Example 8.26, conditions (ii),(iii) and (iv) of Corollary 8.9 are satisfied, which imply (as seen in the proof of said corollary) that G takes relative $F(J)$-cell complexes to weak equivalences. Thus the conditions for transfer are verified, and Theorem 8.8 applies to imply the desired result. ☐

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