Max-Planck-Institut für Mathematik Bonn

Reconstructing GKZ via topological recursion

by

Hiroyuki Fuji
Kohei Iwaki
Masahide Manabe
Ikuo Satake

Max-Planck-Institut für Mathematik
Preprint Series 2017 (52)

Reconstructing GKZ via topological recursions

Hiroyuki Fuji
Kohei Iwaki
Masahide Manabe
Ikuo Satake

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Faculty of Education
Kagawa University
Takamatsu 760-8522
Japan

QGM, Department of Mathematics
Aarhus University
8000 Aarhus C
Denmark

Graduate School of Mathematics
Nagoya University
Nagoya 464-8602
Japan

Faculty of Physics
University of Warsaw
ul. Pasteura 5
02-093 Warsaw
Poland

MPIM 17-52

RECONSTRUCTING GKZ VIA TOPOLOGICAL RECURSION

HIROYUKI FUJI, KOHEI IWAKI, MASAHIDE MANABE, AND IKUO SATAKE

Abstract. In this article, a novel description of the hypergeometric differential equation found from Gel'fand-Kapranov-Zelevinsky's system (referred to *GKZ equation*) for Givental's J-function in the Gromov-Witten theory will be proposed. The GKZ equation involves a parameter $\hbar$, and we will reconstruct it as the WKB expansion from the classical limit $\hbar \to 0$ via the topological recursion. In this analysis, the spectral curve (referred to *GKZ curve*) plays a central role, and it can be defined as the critical point set of the mirror Landau-Ginzburg potential. Our novel description is derived via the duality relations of the string theories, and various physical interpretations suggest that the GKZ equation is identified with the quantum curve for the brane partition function in the cohomological limit. As an application of our novel picture for the GKZ equation, we will discuss the Stokes matrix for the equivariant $\mathbb{C}\mathbf{P}^1$ model and the wall-crossing formula for the total Stokes matrix will be examined. And as a byproduct of this analysis we will study Dubrovin's conjecture for this equivariant model.

Contents

Acknowledgments: The authors thank Prof. Hiroshi Iritani who suggests his idea on the equivariant version of the Dubrovin's conjecture. KI also thanks Dr. Fumihiko Sanda for fruitful discussion. HF and MM thank Prof. Piotr Sułkowski for stimulating discussions and useful comments. The research of HF and IS is supported by the Grant-in-Aid for Challenging Research (Exploratory) [# 17K18781]. The research of HF is also supported by the Grant-in-Aid for Scientific Research(C) [# 17K05239], and Grant-in-Aid for Scientific Research(B) [# 16H03927] from the Japan Ministry of Education, Culture, Sports, Science and Technology, and Fund for Promotion of Academic Research from Department of Education in Kagawa University. The research of KI is supported by the Grant-in-Aid for JSPS KAKENHI KIBAN(S) [# 16H06337], Young Scientists Grant-in-Aid for (B) [# 16K17613] from the Japan Ministry of Education, Culture, Sports, Science and Technology. The work of MM is supported by the ERC Starting Grant no. 335739 "Quantum fields and knot homologies" funded by the European Research Council under the European Union's Seventh Framework Programme. The work of MM is also supported by Max-Planck-Institut für Mathematik in Bonn.

1. Introduction

Let X be a smooth projective variety. The (small) quantum cohomology ring $QH^*(X)$ is a generalization of the ordinary cohomology ring $H^*(X)$ arising from a deformation of the cup product referred to *quantum-cup product*. The quantum cup product is specified by the intersection indices of holomorphic curves in X with cycles which are Poincaré dual to elements in $H^*(X)$, and such indices are known as *Gromov-Witten invariants* of X. In the physics language (see a pedagogical expositions in [64]), the quantum cup product is realized by correlation functions for the cohomology elements of X.

Definition 1.1. *Let $\overline{\mathcal{M}}_{g,n}(X,\beta)$ denote the moduli space of stable maps from connected genus g curves C with n-marked points $p_1,\ldots,p_n$ to X representing the class $\beta \in H_2(X)$. It carries a virtual fundamental class denoted by $\left[\overline{\mathcal{M}}_{g,n}(X,\beta)\right]^{\mathrm{vir}}$. Given classes $\gamma_1,\ldots,\gamma_n \in H^*(X)$, the correlation function $\langle\gamma_1,\cdots,\gamma_n\rangle_{g,\beta}$ is defined by*

$$\langle\gamma_1,\cdots,\gamma_n\rangle_{g,\beta} = \int_{\left[\overline{\mathcal{M}}_{g,n}(X,\beta)\right]^{\mathrm{vir}}} \mathrm{ev}_1^*(\gamma_1)\cup\cdots\cup\mathrm{ev}_n^*(\gamma_n), \tag{1.1}$$

where $\mathrm{ev}_i : \overline{\mathcal{M}}_{g,n}(X,\beta) \to X$ $(i=1,\ldots,n)$ denotes the evaluation map at the i-th marked point such that $\mathrm{ev}_i(C,p_1,\ldots,p_n,\phi)=\phi(p_i)$. Let $\mathcal{L}_i$ $(i=1,\ldots,n)$ be the corresponding tautological line bundles over $\overline{\mathcal{M}}_{g,n}(X,\beta)$. The correlation function for the gravitational descendants $\tau_k(\gamma)$ $(k\geq 0)$ is defined by

$$\langle\tau_{k_1}\gamma_1,\cdots,\tau_{k_n}\gamma_n\rangle_{g,\beta} = \int_{\left[\overline{\mathcal{M}}_{g,n}(X,\beta)\right]^{\mathrm{vir}}} \prod_{i=1}^{n} c_1(\mathcal{L}_i)^{k_i}\cup\mathrm{ev}_i^*(\gamma_i). \tag{1.2}$$

In celebrated works [53, 54, 30] by A. Givental, an elegant framework to uncover profound aspects of the Gromov-Witten theory and mirror symmetry was proposed on basis of the concept of "quantization". In this framework, a generating function of the genus $g=0$ correlation functions with $n=1$ marked point for the gravitational descendants referred to *J-function* plays an important role. For our purpose, we investigate the restriction of the J-function to $H^2(X)\subset H^*(X)$, called *small J-function*. Taking a generator $\beta_1,\ldots,\beta_r\in H_2(X,\mathbb{Z})$, we identify $H_2(X,\mathbb{Z})\cong\mathbb{Z}^r$ and denote its elements by $\boldsymbol{d}=(d_1,\ldots,d_r)$. We also take a basis $T_0=1,T_1,\ldots,T_m\in H^*(X)$ such that $T_1,\ldots,T_r$ give a basis of $H^2(X)$ and $T^0,\ldots,T^m\in H^*(X)$ are the dual basis with respect to the Poincaré pairing.

Definition 1.2 ([53, 54, 30])**.** *The (small) J-function of the smooth projective variety X is the $H^*(X)\otimes\mathbb{C}[[\hbar^{-1}]]$-valued formal series defined as the generating function of the correlation function* *for the $g=0$ gravitational descendants:*

$$\begin{aligned} J_X(\boldsymbol{x}) &= \mathrm{e}^{(t_1T_1+\cdots t_rT_r)/\hbar}\left(1+\sum_{\boldsymbol{d}}\boldsymbol{x}^{\boldsymbol{d}}\sum_{i=0}^{m}\sum_{k=0}^{\infty}\hbar^{-k-1}\left\langle\tau_kT_i\right\rangle_{0,\boldsymbol{d}}\cdot T^i\right)\\ &= \mathrm{e}^{(t_1T_1+\cdots t_rT_r)/\hbar}\left(1+\sum_{\boldsymbol{d}}\boldsymbol{x}^{\boldsymbol{d}}\mathrm{ev}_{1*}^{\boldsymbol{d}}\left(\frac{1}{\hbar-c_1(\mathcal{L}_1)}\right)\right). \end{aligned} \tag{1.3}$$

Here $t_1,\ldots,t_r$ denotes the linear coordinate of $H^2(X)$ with respect to a basis element $T_1,\ldots,T_r$, $\boldsymbol{d}=(d_1,\ldots,d_r)$ runs all $d_i\geq 0$, and $\boldsymbol{x}^{\boldsymbol{d}}=x_1^{d_1}\cdots x_r^{d_r}$ where $x_i=\mathrm{e}^{t_i}$. Also $\mathrm{ev}_1^{\boldsymbol{d}}:\overline{\mathcal{M}}_{0,2}(X,\sum_{i=1}^r d_i\beta_i)\to X$ is the evaluation map at the 1st marked point.

In the following we will consider the smooth Fano complete intersection $X=X_{\boldsymbol{l}}$ of hypersurfaces in the projective space $\mathbb{C}\mathbf{P}^{N-1}$ given by n equations of the degrees $(l_1,\ldots,l_n)$ with $l_1+\cdots+l_n<N$. According to [53, 54], the ($H^*(\mathbb{C}\mathbf{P}^{N-1})$-valued) J-function is defined as

$$\begin{aligned} J_{X_{\boldsymbol{l}}}(x) &= \mathrm{e}^{tp/\hbar}\sum_{d=0}^{\infty}x^d\,\mathrm{ev}_{1*}\big(S_d(\hbar)\big), \qquad x=\mathrm{e}^t\in\mathbb{C}^*,\\ S_d(\hbar) &= \frac{E_d}{\hbar-c_1(\mathcal{L}_1)}\in H^*(\overline{\mathcal{M}}_{0,2}(\mathbb{C}\mathbf{P}^{N-1},d\beta)). \end{aligned} \tag{1.4}$$

Here $p\in H^2(\mathbb{C}\mathbf{P}^{N-1})$, $\beta\in H_2(\mathbb{C}\mathbf{P}^{N-1})$ with $\langle\beta,p\rangle=1$, E_d denotes the Euler class of the vector bundle over $\overline{\mathcal{M}}_{0,2}(\mathbb{C}\mathbf{P}^{N-1},d\beta)$ with the fiber $H^0(C,\phi^*H^{\otimes l_1}\oplus\cdots\oplus\phi^*H^{\otimes l_n})$, where $H^{\otimes l}$ is the l-th tensor power of the hyperplane line bundle over $\mathbb{C}\mathbf{P}^{N-1}$. Also $\mathrm{ev}_1:\overline{\mathcal{M}}_{0,2}(\mathbb{C}\mathbf{P}^{N-1},d\beta)\to\mathbb{C}\mathbf{P}^{N-1}$ is the evaluation map at the 1st marked point.

As a generalization, the equivariant counterpart to the Gromov-Witten theory was also considered in [54]. For the N-dimensional torus T, we consider the natural T-action on $\mathbb{C}\mathbf{P}^{N-1}$. Then we have the T-equivariant cohomology algebra

$$H_T^*(\mathbb{C}\mathbf{P}^{N-1})\cong\mathbb{C}[p,\boldsymbol{w}]/((p-w_0)\cdots(p-w_{N-1})) \tag{1.5}$$

over $H^*(BT)=\mathbb{C}[w_0,\ldots,w_{N-1}]$. In addition, the n-dimensional torus T' action on the vector bundle $\oplus_{i=1}^n H^{\otimes l_i}$ with the equivariant parameters $\boldsymbol{\lambda}=(\lambda_1,\ldots,\lambda_n)$ provides the T'-equivariant Euler class $e_{T'}$ such that

$$e_{T'}\left(\oplus_{i=1}^n H^{\otimes l_i}\right)=(l_1p-\lambda_1)\cdots(l_np-\lambda_n). \tag{1.6}$$

By replacing p, E_d, and $c_1(\mathcal{L}_1)$ in (1.4) to their equivariant partners, we find the J-function for the equivariant Gromov-Witten theory.

By means of the localization of $S_d(\hbar)$ to the fixed point set of the torus action on the moduli space $\overline{\mathcal{M}}_{0,2}(\mathbb{C}\mathbf{P}^{N-1},d\beta)$, the J-functions of the projective space $\mathbb{C}\mathbf{P}^{N-1}$ and the smooth complete intersection of hypersurfaces in the projective space $\mathbb{C}\mathbf{P}^{N-1}$ are computed manifestly:

Proposition 1.3 ([54])**.** *As the equivariant cohomology valued function with $p\in H_T^*(\mathbb{C}\boldsymbol{P}^{N-1})$ (i.e. $\prod_{i=0}^{N-1}(p-w_i)=0$), the J-function $J_{\mathbb{C}\boldsymbol{P}_{\boldsymbol{w}}^{N-1}}(x)$ for the equivariant Gromov-Witten theory of the projective space $X=\mathbb{C}\boldsymbol{P}_{\boldsymbol{w}}^{N-1}$ is given by*

$$J_{\mathbb{C}\boldsymbol{P}_{\boldsymbol{w}}^{N-1}}(x)=\mathrm{e}^{tp/\hbar}\sum_{d=0}^{\infty}\frac{x^d}{\prod_{m=1}^d(p-w_0+m\hbar)\cdots\prod_{m=1}^d(p-w_{N-1}+m\hbar)}. \tag{1.7}$$

And the J-function $J_{X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}}(x)$ for the equivariant Gromov-Witten theory of the smooth Fano complete intersection of hypersurfaces given by n equations of the degrees $(l_1,\ldots,l_n)$ with $l_1+\cdots+l_n<N$ in the projective space $\mathbb{C}\boldsymbol{P}^{N-1}$ is given by

$$J_{X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}}(x)=\mathrm{e}^{tp/\hbar}\sum_{d=0}^{\infty}x^d\frac{\prod_{m=0}^{dl_1}(l_1p-\lambda_1+m\hbar)\cdots\prod_{m=0}^{dl_n}(l_np-\lambda_n+m\hbar)}{\prod_{m=1}^d(p-w_0+m\hbar)\cdots\prod_{m=1}^d(p-w_{N-1}+m\hbar)}. \tag{1.8}$$

Indeed, the J-functions found in Proposition 1.3 obey the hypergeometric differential equations.

Proposition 1.4. *The J-function $J_{\mathbb{C}\mathbf{P}_{\boldsymbol{w}}^{N-1}}(x)$ for the equivariant Gromov-Witten theory of the projective space $X = \mathbb{C}\boldsymbol{P}_{\boldsymbol{w}}^{N-1}$ obeys the hypergeometric differential equation:*

$$\widehat{A}_{\mathbb{C}\boldsymbol{P}_{\boldsymbol{w}}^{N-1}}(\widehat{x},\widehat{y})J_{\mathbb{C}\boldsymbol{P}_{\boldsymbol{w}}^{N-1}}(x) = 0, \tag{1.9}$$

where

$$\widehat{A}_{\mathbb{C}\boldsymbol{P}_{\boldsymbol{w}}^{N-1}}(\widehat{x},\widehat{y}) = \prod_{i=0}^{N-1}(\widehat{y}-w_i) - \widehat{x}.$$

And the J-function $J_{X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}}(x)$ for the equivariant Gromov-Witten theory of the smooth Fano complete intersection $X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}$ obeys the hypergeometric differential equation:

$$\widehat{A}_{X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}}(\widehat{x},\widehat{y})J_{X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}}(x) = 0, \tag{1.10}$$

where

$$\widehat{A}_{X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}}(\widehat{x},\widehat{y}) = \prod_{i=0}^{N-1}(\widehat{y}-w_i) - \widehat{x}\prod_{m=1}^{l_1}(l_1\widehat{y}-\lambda_1+m\hbar)\cdots\prod_{m=1}^{l_n}(l_n\widehat{y}-\lambda_n+m\hbar).$$

Here operators $\widehat{x}$ and $\widehat{y}$ act on $J_X(x)$ as

$$\widehat{x}J_X(x) = xJ_X(x), \qquad \widehat{y}J_X(x) = \hbar x\frac{d}{dx}J_X(x).$$

We call the differential equation $\widehat{A}_X(\widehat{x},\widehat{y})J_X(x) = 0$ *Gel'fand-Kapranov-Zelevinsky (GKZ) equation*. It is a Schrödinger-type differential equation, whose classical limit is given by

$$\hbar \to 0, \qquad (\widehat{x},\widehat{y}) \to (x,y), \qquad \widehat{A}_X(\widehat{x},\widehat{y}) \to A_X(x,y) \in \mathbb{C}[x,y].$$

We call the algebraic curve defined by $\Sigma_X = \{(x,y) \in \mathbb{C}^2 \mid A_X(x,y) = 0\}$ *GKZ* curve.[1]

On the other hand, via the *mirror symmetry*, the Landau-Ginzburg model with suitable pair of a (multi-valued) potential function $W_X(\cdot;x) : (\mathbb{C}^*)^k \to \mathbb{C}$ and a k-form ζ (called the *primitive form* [90]) gives a description of the equivariant Gromov-Witten theory of X [71, 29]. More precisely, we consider the following integral called "*equivariant oscillatory integral*":

$$\mathcal{I}_X(x) = \int_\Gamma \mathrm{e}^{\frac{1}{\hbar}W_X(u_1,\ldots,u_k;x)}\zeta(u_1,\ldots,u_k),$$

where $\zeta(u_1,\ldots,u_k)$ denotes a differential k-form defined on $(\mathbb{C}^*)^k$, and $\Gamma \subset (\mathbb{C}^*)^k$ is a Lefschetz thimble. It is known that the equivariant oscillatory integral satisfies a differential equation, called *Gauss-Manin system*. The Gauss-Manin system is regarded as the *Dubrovin's connection* for the Frobenius structure (flat structure) associated with W_X; see [91] for example.

The mirror symmetry claims that the Gauss-Manin system satisfied by the equivariant oscillatory integral $\mathcal{I}_X(x)$ coincides with the GKZ equations (1.9) and (1.10) for J-functions (Proposition 2.4).[2] This implies that the components of the J-function (defined through the Poincaré pairing $\langle J_X(x),\omega\rangle$ with some cohomology class $\omega \in H^*(X)$) are written in terms of the equivariant oscillatory integrals. To describe precise relation between these objects, we need the notion of the *Gamma class* introduced by [70, 77]; see [51, 52] for details.

Our goal is to reconstruct the GKZ equation, or WKB expansion of its solution, from the GKZ curve Σ_X by the *topological recursion* [42, 18]. In other words, we will show that the Gauss-Manin connections arising from the Landau-Ginzburg B-model (and hence, the GKZ equations (1.9) and (1.10) satisfied by J-functions) are reconstructed as the *quantum curves* via the topological recursion applied to the spectral curve Σ_X arising from the A-model Gromov-Witten theory with target space $X = \mathbb{C}\mathbf{P}_{\boldsymbol{w}}^{N-1}$ or $X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}$. Note that the GKZ curve has an alternative description in terms of the singularity theory for W_X (Proposition 2.5).

[1] In Corollary 2.5 of [69], it is shown that the classical limit of the quantum differential operator $\lim_{\widehat{x}\to x,\,\widehat{y}\to y,\,\hbar\to 0}\widehat{A}_X(\widehat{x},\widehat{y})$ defines a small quantum cohomology.

[2] In general, we can identify these differential equations after some coordinate change through the *mirror map*.

In the topological recursion we need a spectral curve $\Sigma = (C, x, y)$ as an initial input; where C is a compact Riemann surface, $x, y : C \to \mathbb{P}^1$ are meromorphic functions satisfying some condition. Then we can define the correlators $\omega_n^{(g)}$, which are meromorphic differential forms in $\Lambda^1(C)^{\otimes n}$ on C^n determined by the topological recursion relation [42, 18]. In [56, 86, 19] the quantum curve for a spectral curve Σ is constructed via the *wave function* $\psi(D)$ defined by (Definition 3.11)

$$\psi(D) = \exp\left[\frac{1}{\hbar}\int_D \widehat{\omega}_1^{(0)} + \frac{1}{2}\int_D\int_D \widehat{\omega}_2^{(0)} + \sum_{\substack{g\geq 0, n\geq 1\\ g,n\neq (0,1),(0,2)}} \frac{\hbar^{2g+n-2}}{n!}\int_D \cdots \int_D \omega_n^{(g)}\right], \tag{1.11}$$

where $\widehat{\omega}_1^{(0)}$ and $\widehat{\omega}_2^{(0)}$ are some modification of correlators, and D denotes the integration divisor on C. We denote by $\psi_X(D)$ the wave function defined from the GKZ curve Σ_X.

In this article we will show the following theorem:

Theorem 1.5. *Consider the equivariant Gromov-Witten theory of the Fano complete intersection $X = X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}$ of degree $l_i = 1$ $(i = 1, \ldots, n)$ hypersurfaces with $n < N$ in $\mathbb{C}\boldsymbol{P}^{N-1}$. (We regard $X = \mathbb{C}\boldsymbol{P}^{N-1}$ in the case $n = 0$.) The GKZ curve Σ_X is given by*

$$A_X(x, y) = \prod_{i=0}^{N-1}(y - w_i) - x\prod_{a=1}^{n}(y - \lambda_a) = 0, \qquad x \in \mathbb{C}^*, \quad y \in \mathbb{C},$$

and can be described by a local coordinate z as

$$x(z) = \frac{\prod_{i=0}^{N-1}(z - w_i)}{\prod_{a=1}^{n}(z - \lambda_a)}, \qquad y(z) = z.$$

Then by choosing the integration divisor as $D = [z] - [\infty]$, the quantum curve reconstructed for the wave function $\psi_X(x) = \psi_X(D)$ agrees with the GKZ equation (1.10) which is satisfied by the J-function J_X and mirror equivariant oscillatory integral $\mathcal{I}_X$.

This theorem implies that the saddle point (or WKB) expansion of the equivariant oscillatory integral $\mathcal{I}_X(x)$ and the wave function $\psi_X(x)$ with the integration divisor $D = [z] - [\infty]$ are related up to some factor which is independent of x. For oscillatory integrals associated with critical points satisfying a condition given in Section 4.3, we can specify the factor and obtain

$$\mathcal{I}_X(x) \sim C\,(-2\pi\hbar)^{\frac{N+n-1}{2}}\,\psi_X(x) \tag{1.12}$$

where C is a numerical constant which is independent of x and $\hbar$.

Our result is similar, and closely related to the Bouchard-Klemm-Marino-Pasquetti's *remodeling conjecture* [22]. We explain the background of our research from a mathematical-physics perspective.

From the point of view of the string theory, a quantum structure behind the Gromov-Witten theory has been considered in different way [3, 68, 34, 35]. The string theoretical quantum structure emerges in the higher genus (open) string free energy in the topological A-model. The physical definition of the open string free energy is given as follows.

Definition 1.6 ([85])**.** *Let L be a Lagrangian submanifold in local toric Calabi-Yau 3-fold Y, and assume that $b_1(L) = 1$. Consider the holomorphic map from a world-sheet Riemann surface $C_{g,n}$ to Y, and $N_{\boldsymbol{p},\beta}^{(g)}$ denote the open Gromov-Witten invariants enumerating the maps in the topological class labeled by genus g, the class $\beta \in H_2(Y, L)$, and the finding numbers $p_i \in \mathbb{Z}$ $(i = 1, \ldots, n)$ specifying how many times the i-th boundary of $C_{g,n}$ wraps around the one-cycle in L. The generating function referred to the open string free energy is defined by*

$$F_n^{(g)}(\mathsf{x}_1, \ldots, \mathsf{x}_n) = \sum_{\beta\in H_2(Y,L)}\sum_{\boldsymbol{p}\in\mathbb{Z}^n} N_{\boldsymbol{p},\beta}^{(g)} \mathrm{e}^{-\beta\cdot t}\mathsf{x}_1^{p_1}\cdots\mathsf{x}_n^{p_n}, \tag{1.13}$$

where $t^i \in \mathbb{C}$ $(i = 1, \dots, \dim H_2(Y))$ *denotes the Kähler moduli parameters of* Y*. The brane partition function* $Z^Y_{\text{A-brane}}(\mathsf{x})$ *is defined in terms of the open string free energy as*

$$Z^Y_{\text{A-brane}}(\mathsf{x}) = \exp\left[\sum_{g\geq 0, n\geq 1} g_s^{2g+n-2}\frac{1}{n!}F_n^{(g)}(\mathsf{x},\dots,\mathsf{x})\right], \tag{1.14}$$

where the parameter $g_s \in \mathbb{C}$ *is called string coupling.*

For a special Lagrangian submanifold L, it is argued that the brane partition function is annihilated by a q-difference operator $\widehat{A}^K_Y$ [3, 2]:

$$\widehat{A}^K_Y(\widehat{\mathsf{x}}, \widehat{\mathsf{y}})Z^Y_{\text{A-brane}}(\mathsf{x}) = 0, \tag{1.15}$$

which is made of non-commutative operators $\widehat{\mathsf{x}}$ and $\widehat{\mathsf{y}}$:

$$\widehat{\mathsf{x}}\, Z^Y_{\text{A-brane}}(\mathsf{x}) = \mathsf{x}\, Z^Y_{\text{A-brane}}(\mathsf{x}), \qquad \widehat{\mathsf{y}}\, Z^Y_{\text{A-brane}}(\mathsf{x}) = Z^Y_{\text{A-brane}}(q\mathsf{x}),$$

where $q = \mathrm{e}^{-g_s}$. These operators obeys the q-Weyl relation $\widehat{\mathsf{y}}\widehat{\mathsf{x}} = q\widehat{\mathsf{x}}\widehat{\mathsf{y}}$, and such q-difference equation (1.15) is interpreted as a quantum curve (see section 5.2.3 as an example) [35, 34]. We then find that the quantum curve arises from the hidden quantum mechanical system behind the topological string.

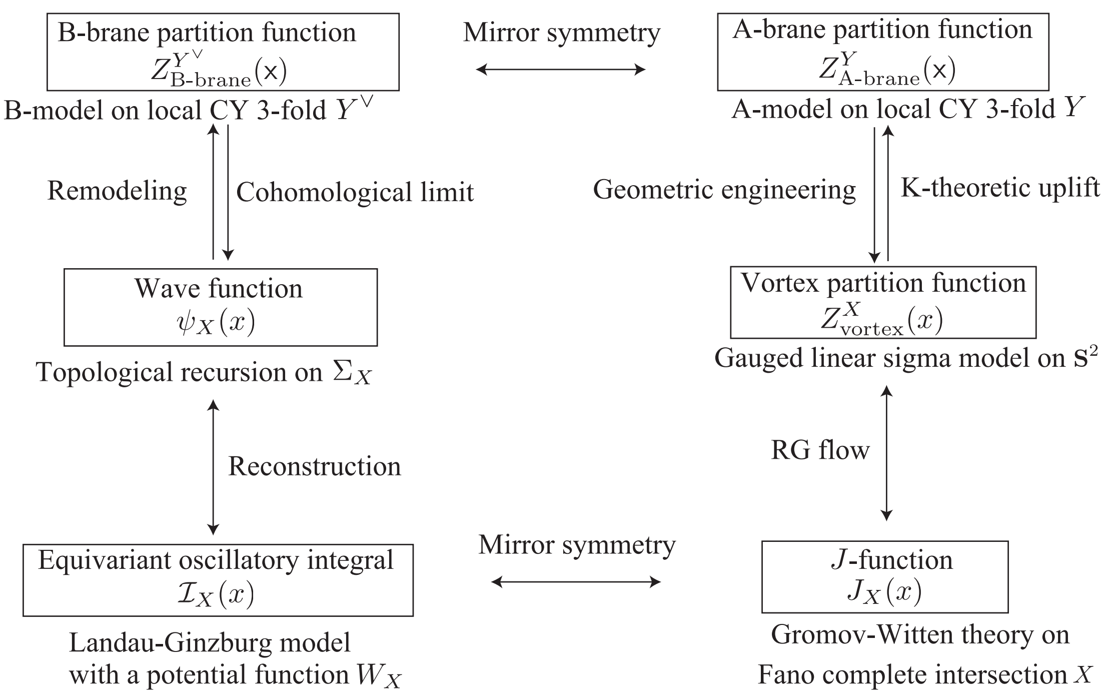


FIGURE 1. String dualities relate the oscillatory integral with the open topological string partition function.

At this stage, we find two kinds of hidden quantum mechanical system behind the Givental's formulation of the Gromov-Witten theory on the compact smooth Fano manifold X and open topological A-model on local toric Calabi-Yau 3-fold Y. The quantum structure of these two theories are reflected in the GKZ equation for the J-function and the q-difference equation for the brane partition function, and the quantum deformation parameter appears as the *Planck's constant* $\hbar$ and string coupling g_s, respectively. Although we find some nice similarities[3] between these two theories, some crucial discrepancies also exist between them. In the physical perspective,

[3]Such similarities were also pointed out in [87].

the Givental's formulation is considered essentially as the closed string theory on X. In this sense, the J-function is the generating function of the genus zero correlation functions and $\hbar$ is associated with the degrees of freedom of the gravitational descendants. On the other hand, the latter theory is considered as the open string theory on Y, and the brane partition function $Z^{Y}_{\text{A-brane}}$ is the generating function of the all genus open free energy and g_s is associated with the coupling to 2-dimensional gravity [15]. Thus the J-function and the brane partition function are essentially different objects in a sense of the string theory.

To overcome such discrepancies, we will employ the physical idea of *string dualities* depicted in Figure 1. The punchline of the string dualities shown in the right hand side of this figure is found as follows. Considering the gauged linear sigma model [100] which describes the Gromov-Witten theory on X, we can reinterpret the J-function as the vortex partition function on $\mathbb{S}^2$ [36, 16]. Furthermore, for the special case $X = \mathbb{C}\mathbf{P}^{N-1}$ and smooth Fano complete intersection of hypersurfaces with degrees $l_1 = \ldots = l_n = 1$, the vortex partition function of the gauged linear sigma model is realized as the brane partition function $Z^{Y}_{\text{A-brane}}$ on a class of local toric Calabi-Yau 3-fold referred to *strip geometry* [67] (see Figure 6 for the toric diagram of a strip geometry) by the geometric engineering [36]. More precisely, the brane partition function $Z^{Y}_{\text{A-brane}}$ realizes the K-theoretic version of the J-function, and we need to take a cohomological limit $\beta \to 0$ with reparametrization

$$g_s = \beta\hbar, \qquad \mathsf{x} = \beta^{N-n} x, \qquad \widetilde{Q}_{w,i} = \mathrm{e}^{-\beta(w_0 - w_i)}, \qquad \widetilde{Q}_{\lambda,a} = \mathrm{e}^{-\beta(w_0 - \lambda_a)}.$$

Here $\widetilde{Q}_{w,i}$ $(i = 0, \ldots, N-1)$ and $\widetilde{Q}_{\lambda,a}$ $(a = 1, \ldots, n)$ denote $\mathrm{e}^{t_{w,i}}$ and $\mathrm{e}^{t_{\lambda,a}}$ with the (reorganized) Kähler parameters $t_{w,i}$ and $t_{\lambda,a}$ of the strip geometry Y, respectively. In fact for these examples, by direct computations, the equivariant J-function of X and the brane partition function of Y agrees in this limit, and the q-difference equation (quantum curve) for the brane partition function reduces to the GKZ equation for the J-function. Thus, as a result of string theoretical discussions, a novel interpretation of the GKZ equation as the quantum curve is uncovered.

Now we will further proceed with the string dualities by applying the mirror symmetry. Via the local mirror symmetry, the open topological A-model on local toric Calabi-Yau 3-fold Y turns to the open topological B-model on a local Calabi-Yau 3-fold $Y^\vee$:

$$Y^\vee = \left\{ \, (\omega_+, \omega_-, \mathsf{x}, \mathsf{y}) \in \mathbb{C}^2 \times (\mathbb{C}^*)^2 \;\middle|\; \omega_+\omega_- = A^K_Y(\mathsf{x}, \mathsf{y}) \, \right\}, \tag{1.16}$$

where $A^K_Y(\mathsf{x}, \mathsf{y}) \in \mathbb{C}[\mathsf{x}, \mathsf{y}]$ is given by the classical limit of the q-difference operator $\widehat{A}^K_Y(\widehat{\mathsf{x}}, \widehat{\mathsf{y}})$ such that

$$q \to 1 \ (g_s \to 0), \qquad (\widehat{\mathsf{x}}, \widehat{\mathsf{y}}) \to (\mathsf{x}, \mathsf{y}), \qquad \widehat{A}^K_Y(\widehat{\mathsf{x}}, \widehat{\mathsf{y}}) \to A^K_Y(\mathsf{x}, \mathsf{y}) \in \mathbb{C}[\mathsf{x}, \mathsf{y}].$$

For the topological B-model on this local geometry, the remodeling conjecture [22] proposed by V. Bouchard, A. Klemm, M. Mariño, and S. Pasquetti is applicable, and the open topological B-model is studied systematically on basis of the formalism of the topological recursion. The key ingredient of this formalism is the spectral curve $\Sigma^K_Y = \left\{ (\mathsf{x}, \mathsf{y}) \in (\mathbb{C}^*)^2 \middle| A^K_Y(\mathsf{x}, \mathsf{y}) = 0 \right\}$, and higher genus open free energies of the topological B-model are evaluated recursively. Then via the local mirror symmetry, one can (re)construct the brane partition function $Z^{Y}_{\text{A-brane}}(\mathsf{x})$ in (1.14). Therefore, by chasing the web of dualities in Figure 1, we perceive that the GKZ equation would be reconstructible as the quantum curve by the topological recursion.

As an application of our novel description of the GKZ equation, we will also study the Stokes structure of the GKZ equation for $X = \mathbb{C}\mathbf{P}^1_{\boldsymbol{w}}$

$$\left[\left(\hbar x \frac{d}{dx} - w_0 \right) \left(\hbar x \frac{d}{dx} - w_1 \right) - x \right] \psi = 0 \tag{1.17}$$

from the view point of WKB analysis. It is known that the WKB expansion (1.11) or (1.12) is usually divergent and just a formal solution of the quantum curve in general. Therefore, we employ the *Borel summation method* which allows us to construct an analytic solution Ψ of the quantum curve (i.e. an exact wave function) whose asymptotic expansion when $\hbar \to 0$ in a certain sectorial domain coincides with the given formal WKB expansion (see [32]). This framework is called the *exact WKB analysis* developed in [99, 33, 78] etc. The exact WKB analysis is a powerful tool to study the global property of exact wave functions which satisfy differential equations containing a small parameter $\hbar$ like the quantum curve (1.17).

For a divergent series, its Borel sum depends on the choice of direction θ around $\hbar = 0$ which is regarded as the bisecting direction of the sector where the Borel sum is asymptotically expanded to the divergent series. The Borel sum is invariant under small variation of θ, and "jumps" discontinuously when θ crosses a critical direction. This phenomenon is so-called *Stokes phenomenon*, and the critical directions are called *Stokes directions*. The jump of Borel sum when θ crosses a Stokes direction is described by an invertible matrix called *Stokes matrix*; such matrix exists since the Borel sum in different directions are solutions of (1.17), which is a linear ODE. The theory of exact WKB analysis allows us to find such Stokes directions by looking the configuration of *Stokes curves* of (1.17); the leaves of a foliation defined by meromorphic 1-form, which is closely related to $\widehat{\omega}_1^{(0)}$ in the topological recursion, on the GKZ curve. We will also see that the integral (or residue) of $\widehat{\omega}_1^{(0)}$ on the GKZ curve and holomorphicity of the correlation functions $\omega_n^{(g)}$ for $2g-2+n \geq 0$ are crucially important in the computation of the Stokes matrices (see [78, Section 3]).

In particular, using the exact WKB method, we will compute the *"total" Stokes matrix*, which captures the Stokes matrices of the WKB solution associated with all Stokes directions sitting in the semi-closed upper half part $0 \leq \theta < \pi$ of the $\hbar$-plane. Note that, since the Borel sum is defined for fixed x (points on the GKZ curve), the Stokes directions and the total Stokes matrix a priori depend on x. One of our results in Section 6 is that, the total Stokes matrix is locally constant when x varies (as long as x lies in a Stokes region). This is straightforward since the Borel sum is also locally constant when x varies, but the consequence of invariance of the total Stokes matrix involves a non-trivial identity of infinite product of Stokes matrices. This is regarded as an example of the *2d-4d wall-crossing formula* established by D. Gaiotto, G.W. Moore and A. Neitzke in [49] (see Remark 6.13), where a loop-type Stokes curve plays the role of the wall of marginal stability. We also note that the wall-crossing identity is only observed in the equivariant case; in the non-equivariant situation $w_0 - w_1 = 0$ a 4d-type wall-crossing (which is associated with a loop-type Stokes curve) never happens.

Another important claim in Section 6 is an *equivariant version of the Dubrovin's conjecture* for the equivariant $\mathbb{C}\mathbf{P}^1$. That is, our explicit computation of the Stokes matrices implies that a Stokes multiplier (i.e. an off-diagonal entry of the total Stokes matrix) of (1.17) around $\hbar = 0$ coincides with the equivariant Euler pairing of the coherent sheaves $\mathcal{O}$ and $\mathcal{O}(1)$ on equivariant $\mathbb{C}\mathbf{P}^1$ (regarded as an equivariant sheaves in an appropriate manner). Our claim also includes (an alternative proof based on exact WKB method) the original Dubrovin's conjecture for $\mathbb{C}\mathbf{P}^1$ which claims that the Stokes multiplier of the quantum differential equation associated with the Gromov-Witten theory for $\mathbb{C}\mathbf{P}^1$ is described by Euler pairing of coherent sheaves on $\mathbb{C}\mathbf{P}^1$. We expect that we can generalize these results for general equivariant projective spaces or complete intersections if the higher order exact WKB theory is rigorously established. (Stokes graph for the higher order differential equation is introduced in [14, 9], but the Borel summability theorem, a counterpart of Theorem 6.6, is not established in full generality so far. See also [8, 61].) Note also that the (higher order version of) Stokes graphs are called the *spectral networks* [50].

The organization of this paper is as follows. In Section 2 the GKZ curve for the mirror Landau-Ginzburg model is derived for the equivariant Gromov-Witten theory on the projective space $\mathbb{C}\mathbf{P}^{N-1}$ and the smooth Fano complete intersection of hypersurfaces in the projective space $\mathbb{C}\mathbf{P}^{N-1}$. In Section 3 we will summarize the necessary ingredients of the topological recursion and quantum curve in a nut-shell. In Section 4 the WKB reconstruction of the GKZ equation is discussed. If the GKZ equation is the second order differential equation, the (local) topological recursion [42] is applicable and the quantum curve is found manifestly by employing the method developed by the work of M. Mulase and P. Sułkowski [86]. In more general case, we need to use the "*global topological recursion*" [20, 18, 19], and the quantum curve is reconstructible only for the spectral curve which satisfies the admissibility condition considered in the work of V. Bouchard and B. Eynard [19]. Among the GKZ curves, the GKZ equation is reconstructible for the projective space $\mathbb{C}\mathbf{P}^{N-1}$ and the Fano complete intersection of hypersurfaces with degree $l_i = 1$ ($i = 1, \ldots, n < N$) in the projective space $\mathbb{C}\mathbf{P}^{N-1}$, and we will show Theorem 1.5. In Section 5 the string dualities behind our proposal will be discussed. In particular we will focus on 3 different vantage points of the string theories, and the J-function is regarded as the vortex partition function and brane partition function of the open topological A-model and B-model. As a result of the string dualities, we will find that the GKZ curves considered in Section 4 are obtained as the mirror curve in the open topological B-model, and the GKZ equations are found as the quantum curves for the brane partition functions. In Section 6 we compute the total Stokes matrices for the quantum curve arising from equivariant $\mathbb{C}\mathbf{P}^1$ by using the exact WKB method. We also examine a wall-crossing formula and equivariant version of the Dubrovin's conjecture in this particular case.

In Appendix A we will summarize the J-functions for the projective space $\mathbb{C}\mathbf{P}^{N-1}$ and the smooth Fano complete intersection of hypersurfaces in the projective space. And as a side remark of our proposal, we will derive the GKZ curve from the saddle point approximation of the J-function in the similar spirit as the generalized volume conjecture. In Appendix B.1 the GKZ equations for the equivariant oscillatory integrals will be given for the mirror Landau-Ginzburg models of the equivariant Gromov-Witten theory on the projective space $X = \mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}$ and the Fano complete intersection $X = X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}$, and we prove Proposition 2.4. In Appendix C we will show some explicit computational results on the asymptotic solutions of GKZ equations. In particular, for the equivariant $\mathbb{C}\mathbf{P}^1$ model we also compute the free energies via the topological recursion, and directly check the agreement with the asymptotic solutions of the GKZ equation.

2. Oscillatory integral and GKZ equation

2.1. **Oscillatory integrals.**

In the mirror theorem (see pedagogical expositions in [64]), a correspondence between the Gromov-Witten theory for smooth Fano manifold X and the Landau-Ginzburg model with potential function

$$W_X(\cdot,\boldsymbol{x}) : (\widetilde{\mathbb{C}^*})^k \longrightarrow \mathbb{C}, \quad (u_1,\ldots,u_k;\boldsymbol{x}) \mapsto W_X(u_1,\ldots,u_k;\boldsymbol{x}), \tag{2.1}$$

is considered. Here $\widetilde{\mathbb{C}^*}$ is the universal covering of $\mathbb{C}^*$, and $\boldsymbol{x}$ is a deformation parameter of the potential function. For the equivariant Gromov-Witten theory on $X = \mathbb{C}\mathbf{P}^{N-1}$ ($X = \mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}$ in short) and the smooth Fano complete intersection of degrees l_i ($i = 1, \ldots, n$) hypersurfaces in $\mathbb{C}\mathbf{P}^{N-1}$ with $l_1 + \cdots + l_n < N$ ($X = X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}$ in short), the mirror Landau-Ginzburg potential W_X is given as follows.

Definition 2.1 ([30, 54])**.** *For $X = \mathbb{C}\boldsymbol{P}^{N-1}_{\boldsymbol{w}}$, the mirror Landau-Ginzburg potential $W_{\mathbb{C}\boldsymbol{P}^{N-1}_{\boldsymbol{w}}}(\cdot;x) : (\widetilde{\mathbb{C}^*})^{N-1} \to \mathbb{C}$ is given by*

$$W_{\mathbb{C}\boldsymbol{P}^{N-1}_{\boldsymbol{w}}}(u_1,\ldots,u_{N-1};x) = \sum_{i=1}^{N-1}(u_i + w_i \log u_i) + \frac{x}{u_1\cdots u_{N-1}} + w_0 \log\left(\frac{x}{u_1\cdots u_{N-1}}\right), \tag{2.2}$$

where $x = \mathrm{e}^t \in \mathbb{C}^*$. For $X = X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}$, the mirror Landau-Ginzburg potential $W_{X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}}(\cdot;x)$: $(\widetilde{\mathbb{C}^*})^{N-1} \times (\widetilde{\mathbb{C}^*})^{n} \to \mathbb{C}$ is given by

$$\begin{aligned}&W_{X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}}(u_1,\ldots,u_{N-1},v_1,\ldots,v_n;x)\\&= \sum_{i=1}^{N-1}(u_i + w_i\log u_i) - \sum_{a=1}^{n}(v_a + \lambda_a \log v_a) + \frac{v_1^{l_1}\cdots v_n^{l_n}}{u_1\cdots u_{N-1}}x + w_0\log\left(\frac{v_1^{l_1}\cdots v_n^{l_n}}{u_1\cdots u_{N-1}}x\right).\end{aligned} \tag{2.3}$$

Here we introduce the notion of critical set.

Definition 2.2. *Let* Crit $\subset (\widetilde{\mathbb{C}^*})^k \times \mathbb{C}^*$ *be the* critical set *of W_X defined by*

$$\mathrm{Crit} = \left\{ (u_1^{(\mathrm{c})},\ldots,u_k^{(\mathrm{c})};x) \in (\widetilde{\mathbb{C}^*})^k \times \mathbb{C}^* \;\middle|\; \frac{\partial}{\partial u_i}W_X(u_1^{(\mathrm{c})},\ldots,u_k^{(\mathrm{c})};x) = 0 \ (i=1,\cdots,k) \right\}.$$

The image $\boldsymbol{u}^{(\mathrm{c})} = (u_1^{(\mathrm{c})}, \dots, u_k^{(\mathrm{c})})$ *of a point* $(u_1^{(\mathrm{c})}, \dots, u_k^{(\mathrm{c})}; x) \in \mathrm{Crit}$ *by the projection* $\mathrm{Crit} \to (\widetilde{\mathbb{C}^*})^k$ *is called a* critical point *of* $W_X(\cdot; x)$*. The value* $W_X(\boldsymbol{u}^{(\mathrm{c})}; x)$ *at a critical point* $\boldsymbol{u}^{(\mathrm{c})}$ *is called a* critical value*.*

In what follows, for any fixed x, we assume that the parameters w_i $(i = 0, \dots, N-1)$ and λ_a $(a = 1, \dots, n)$ are on a domain where the all critical points of W_X are non-degenerate. Then, for each $\hbar$, we associate the critical point $\boldsymbol{u}^{(c)}$ with the *Lefschetz thimble* Γ: It is a relative k-cycles in $(\widetilde{\mathbb{C}^*})^k$ defined as the real 1-parameter family of $(k-1)$-cycles (called *vanishing cycles*) in the Milnor fiber of $W_X^{-1}(w)$, where w lies on the half-line $\{W_X(\boldsymbol{u}^{(c)}; x) + r\, \mathrm{e}^{i(\pi + \arg \hbar)} \mid r \geq 0\}$ emanating from the critical value $W_X(\boldsymbol{u}^{(c)}; x)$, and the $(k-1)$-cycle tends to a point when $w \to W_X(\boldsymbol{u}^{(c)}; x)$ along the half line. In this section we assume that x and $\hbar$ are chosen so that the half line $\{W_X(\boldsymbol{u}^{(c)}; x) + r\, \mathrm{e}^{i(\pi + \arg \hbar)} \mid r \geq 0\}$ associated with a critical point $\boldsymbol{u}^{(c)}$ never hits critical values of $W_X(\cdot; x)$; then the (equivariant) oscillatory integral defined below has the so-called saddle point expansion of the form (2.9) below.

For a Lefschetz thimble Γ we consider the (equivariant) oscillatory integral $\mathcal{I}_X(\boldsymbol{x})$ of the type

$$\mathcal{I}_X(\boldsymbol{x}) = \int_\Gamma \mathrm{e}^{\frac{1}{\hbar} W_X(u_1, \dots, u_k; \boldsymbol{x})}\, \zeta(u_1, \dots, u_k), \tag{2.4}$$

where $\zeta(u_1, \dots, u_k)$ denotes a k-form on $(\mathbb{C}^*)^k$.

Definition 2.3 ([30, 54])**.** *The oscillatory integral* $\mathcal{I}_{\mathbb{C}\boldsymbol{P}^{N-1}_{\boldsymbol{w}}}(x)$ *for the projective space* $X = \mathbb{C}\boldsymbol{P}^{N-1}_{\boldsymbol{w}}$ *is given by*

$$\mathcal{I}_{\mathbb{C}\boldsymbol{P}^{N-1}_{\boldsymbol{w}}}(x) = \int_\Gamma \prod_{i=1}^{N-1} \frac{du_i}{u_i}\, \mathrm{e}^{\frac{1}{\hbar} W_{\mathbb{C}\boldsymbol{P}^{N-1}_{\boldsymbol{w}}}(u_1, \dots, u_{N-1}; x)}. \tag{2.5}$$

The oscillatory integral $\mathcal{I}_{X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}}(x)$ *for the Fano complete intersection in the projective space* $X = X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}$ *is given by the Laplace transform of the oscillatory integral for* $X = \mathbb{C}\boldsymbol{P}^{N-1}_{\boldsymbol{w}}$*.*

$$\begin{aligned}
\mathcal{I}_{X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}}(x) &= \int_0^\infty dv_1 \cdots \int_0^\infty dv_n\, \mathrm{e}^{-\frac{\sum_{a=1}^n (v_a + \lambda_a \log v_a)}{\hbar}} \mathcal{I}_{\mathbb{C}\boldsymbol{P}^{N-1}_{\boldsymbol{w}}}(v_1^{l_1} \cdots v_n^{l_n} x) \\
&= \int_{\Gamma \times (\mathbb{R}_{\geq 0})^n} \prod_{i=1}^{N-1} \frac{du_i}{u_i} \prod_{a=1}^n dv_a\, \mathrm{e}^{\frac{1}{\hbar} W_{X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}}(u_1, \dots, u_{N-1}, v_1, \dots, v_n; x)}.
\end{aligned} \tag{2.6}$$

The mirror symmetry between the J-function $J_{X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}}(x)$ for the Gromov-Witten theory and the oscillatory integral $\mathcal{I}_{X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}}(x)$ for the mirror Landau-Ginzburg model is given by the proposition below:

Proposition 2.4. *The oscillatory integrals* $\mathcal{I}_X(x)$ *for* $X = \mathbb{C}\boldsymbol{P}^{N-1}_{\boldsymbol{w}}$ *and* $X = X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}$ *obey the GKZ equations (1.9) and (1.10), respectively:*

$$\widehat{A}_X(\widehat{x}, \widehat{y})\, \mathcal{I}_X(x) = 0, \tag{2.7}$$

where operators $\widehat{x}$ *and* $\widehat{y}$ *act on* $\mathcal{I}_X(x)$ *as*

$$\widehat{x} \mathcal{I}_X(x) = x \mathcal{I}_X(x), \quad \widehat{y} \mathcal{I}_X(x) = \hbar x \frac{\partial}{\partial x} \mathcal{I}_X(x). \tag{2.8}$$

The claim of Proposition 2.4 is essentially found in [54] for the equivariant $\mathbb{C}\mathbf{P}^{N-1}$ and in [30] for non-equivariant models, although the claim for $X = X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}$ is not mentioned manifestly. We will give a proof of this Proposition in Appendix B.1.

2.2. **GKZ curves from the classical limit.** For the purpose of the reconstruction of the GKZ equation, we will derive the spectral curve Σ_X (referred to GKZ curve) from the leading behavior of saddle point approximation of the equivariant oscillatory integral when $\hbar \to 0$:

$$\mathcal{I}_X(x) \sim \exp\left(\frac{1}{\hbar} W_X(\boldsymbol{u}^{(\mathrm{c})})\right) \frac{(-2\pi\hbar)^{k/2}\, g(\boldsymbol{u}^{(\mathrm{c})})}{\sqrt{\mathrm{Hess}(\boldsymbol{u}^{(\mathrm{c})})}} \left(1 + \sum_{m=1}^\infty \hbar^m \mathcal{I}_m(x)\right), \tag{2.9}$$

where $\boldsymbol{u}^{(\mathrm{c})}=(u_1^{(\mathrm{c})},\dots,u_k^{(\mathrm{c})})$ is the critical point of $W_X(\cdot;x)$ for the Lefschetz thimble Γ, $\mathrm{Hess}(\boldsymbol{u}^{(\mathrm{c})})=\det(\partial_{u_i}\partial_{u_j}W_X(\boldsymbol{u};x))|_{\boldsymbol{u}=\boldsymbol{u}^{(\mathrm{c})}}$ is the Hessian of $W_X(\cdot;x)$ at $\boldsymbol{u}^{(\mathrm{c})}$, and we wrote $\zeta=g(u_1,\dots,u_k)du_1\cdots du_k$. Note that the right hand side of (2.9) is usually divergent, so this is understood as an asymptotic expansion. We can also arrange the right hand side of (2.9) into a WKB form as

$$\mathcal{I}_X(x)\sim\exp\left(\sum_{m=0}^{\infty}\hbar^{m-1}S_m(x)\right) \tag{2.10}$$

(where overall factor $(-2\pi\hbar)^{k/2}$ is omitted). The leading term

$$S_0(x)=W_X(\boldsymbol{u}^{(\mathrm{c})};x). \tag{2.11}$$

is the critical value of W_X. Adopting (2.10) to $\widehat{A}_X(x,\hbar x\partial_x)\mathcal{I}_X(x)=0$, then we find the semi-classical limit of the differential operator $\widehat{A}_X(\widehat{x},\widehat{y})$ [41]:

$$0=\lim_{\hbar\to 0}\left(\mathrm{e}^{-\frac{1}{\hbar}S_0(x)}\widehat{A}_X(\widehat{x},\widehat{y})\mathrm{e}^{\frac{1}{\hbar}S_0(x)}\mathrm{e}^{\sum_{m\geq 1}\hbar^{m-1}S_m(x)}\right)=A_X\left(x,x\frac{d}{dx}S_0(x)\right)\mathrm{e}^{S_1(x)}. \tag{2.12}$$

For the critical value $S_0(x)=W_X(\boldsymbol{u}^{(\mathrm{c})};x)$, we obtain an equation:

$$A_X\left(x,y(x)\right)=0,\qquad y(x)=x\frac{d}{dx}W_X(\boldsymbol{u}^{(\mathrm{c})};x). \tag{2.13}$$

From this observation we find an alternative definition of the GKZ curve:

Proposition 2.5. *The GKZ curve Σ_X coincides with the image $\mathrm{Im}\,\iota$ of critical set, where*

$$\iota:\mathrm{Crit}\to\mathbb{C}^*\times\mathbb{C},\quad (u_1^{(\mathrm{c})},\dots,u_k^{(\mathrm{c})};x)\mapsto\left(x,x\frac{d}{dx}W_X(u_1^{(\mathrm{c})},\dots,u_k^{(\mathrm{c})};x)\right).$$

We give a proof of this Proposition for the cases of the projective space $X=\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}$ and Fano complete intersection $X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}$.

(1) Projective space $\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}$:
The GKZ curve $A_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(x,y)=0$ is found from the relations

$$\begin{aligned}
y(x)&=x\frac{\partial W_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(u_1,\dots,u_{N-1};x)}{\partial x}=\frac{x}{u_1\cdots u_{N-1}}+w_0,\\
0&=\frac{\partial W_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(u_1,\dots,u_{N-1};x)}{\partial u_i}=1+\frac{w_i}{u_i}-\frac{1}{u_i}\frac{x}{u_1\cdots u_{N-1}}-\frac{w_0}{u_i}.
\end{aligned} \tag{2.14}$$

By eliminating u_i $(i=1,\dots,N-1)$ from the above relations, we obtain a polynomial constraint equation

$$A_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(x,y)=\prod_{i=0}^{N-1}(y-w_i)-x=0. \tag{2.15}$$

(2) Fano complete intersection $X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}$:
The GKZ curve $A_{X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}}(x,y)=0$ is found from the relations

$$\begin{aligned}
y(x)&=x\frac{\partial W_{X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}}(u_1,\dots,u_{N-1},v_1,\dots,v_n;x)}{\partial x}=x\frac{v_1^{l_1}\cdots v_n^{l_n}}{u_1\cdots u_{N-1}}+w_0,\\
0&=\frac{\partial W_{X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}}(u_1,\dots,u_{N-1},v_1,\dots,v_n;x)}{\partial u_i}=1+\frac{w_i}{u_i}-\frac{1}{u_i}\frac{v_1^{l_1}\cdots v_n^{l_n}}{u_1\cdots u_{N-1}}x-\frac{w_0}{u_i},\\
0&=\frac{\partial W_{X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}}(u_1,\dots,u_{N-1},v_1,\dots,v_n;x)}{\partial v_a}=-1-\frac{\lambda_a}{v_a}+\frac{l_a}{v_a}\frac{v_1^{l_1}\cdots v_n^{l_n}}{u_1\cdots u_{N-1}}x+l_a\frac{w_0}{v_a}.
\end{aligned} \tag{2.16}$$

Eliminating u_i $(i = 1, \dots, N-1)$ and v_a $(a = 1, \dots, n)$ from the above relations, we obtain a polynomial constraint equation

$$A_{X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}}(x,y) = \prod_{i=0}^{N-1}(y - w_i) - x\prod_{a=1}^{n}(l_a y - \lambda_a)^{l_a} = 0. \tag{2.17}$$

2.3. **Asymptotics of the coefficients.** In Section 4 we will compare the asymptotic expansion of oscillatory integral to a wave function constructed via topological recursion applied to the GKZ curve $A_X(x,y) = 0$, for $X = \mathbb{CP}^{N-1}_{\boldsymbol{w}}$ and $X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}$ with $l_1 = \cdots = l_n = 1$. For the purpose, we investigate the asymptotic behavior of the coefficients $\mathcal{I}_m(x)$ in the expansion (2.9) when x tends to ∞.

In this subsection we consider the case $X = X_{\boldsymbol{l}=\boldsymbol{1};\boldsymbol{w},\boldsymbol{\lambda}}$:

$$W_X = \sum_{i=1}^{N-1}(u_i + w_i \log u_i) - \sum_{a=1}^{n}(v_a + \lambda_a \log v_a) + \frac{v_1 \cdots v_n}{u_1 \cdots u_{N-1}}x + w_0 \log\left(\frac{v_1 \cdots v_n}{u_1 \cdots u_{N-1}}x\right).$$

($X = \mathbb{CP}^{N-1}_{\boldsymbol{w}}$ is included as the case of $n = 0$.) It is easy to check that, at a critical point $(\boldsymbol{u}^{(\mathrm{c})}, \boldsymbol{v}^{(\mathrm{c})}) = (u_1^{(\mathrm{c})}, \cdots, u_{N-1}^{(\mathrm{c})}, v_1^{(\mathrm{c})}, \dots, v_n^{(\mathrm{c})})$,

$$u_i^{(\mathrm{c})} + w_i - w_0 = v_a^{(\mathrm{c})} + \lambda_a - w_0 = \frac{v_1^{(\mathrm{c})} \cdots v_n^{(\mathrm{c})}}{u_1^{(\mathrm{c})} \cdots u_{N-1}^{(\mathrm{c})}}x \tag{2.18}$$

holds for any i and a. Since the right hand side of (2.18) is independent of i and a, we can write all $u_i^{(\mathrm{c})}$ and $v_a^{(\mathrm{c})}$ in terms of $u_1^{(\mathrm{c})}$:

$$u_i^{(\mathrm{c})} = u_1^{(\mathrm{c})} + w_1 - w_i, \quad v_a^{(\mathrm{c})} = u_1^{(\mathrm{c})} + w_1 - \lambda_a \quad (i = 1, \dots, N-1,\ a = 1, \dots, n).$$

Therefore, $u_1^{(\mathrm{c})}$ must be a solution of the algebraic equation

$$\prod_{i=0}^{N-1}(u_1^{(\mathrm{c})} + w_1 - w_i) - x\prod_{a=1}^{n}(u_1^{(\mathrm{c})} + w_1 - \lambda_a) = 0. \tag{2.19}$$

Hence, for generic w_i and λ_a, there are exactly N critical points of W_X. These critical points define N Lefschetz thimbles, and hence we have N independent solutions of the GKZ equation.

Lemma 2.6. *The asymptotic behavior of a critical point of W_X for large x is given by one of the following:*

(i) *For any fixed $p = 1, \dots, N-n$, there exists a critical point $(\boldsymbol{u}^{(\mathrm{c})}, \boldsymbol{v}^{(\mathrm{c})})$ of W_X behaves as*

$$u_i^{(\mathrm{c})} = \zeta^p x^{\frac{1}{N-n}}(1 + O(x^{-\frac{1}{N-n}})), \quad v_a^{(\mathrm{c})} = \zeta^p x^{\frac{1}{N-n}}(1 + O(x^{-\frac{1}{N-n}})) \tag{2.20}$$

when $x \to \infty$. Here $\zeta = \exp(2\pi i/(N-n))$ is the primitive $(N-n)$-th root of unity.

(ii) *For any fixed $b = 1, \dots, n$, there exists a critical point $(\boldsymbol{u}^{(\mathrm{c})}, \boldsymbol{v}^{(\mathrm{c})})$ of W_X behaves as*

$$u_i^{(\mathrm{c})} = (\lambda_b - w_i) + c_b x^{-1} + O(x^{-2}), \quad v_a^{(\mathrm{c})} = (\lambda_b - \lambda_a) + c_b x^{-1} + O(x^{-2}) \tag{2.21}$$

when $x \to \infty$. Here

$$c_b = \frac{\prod_{i=0}^{N-1}(\lambda_b - w_i)}{\prod_{\substack{a=1 \\ a \neq b}}^{n}(\lambda_b - \lambda_a)}.$$

The claim follows easily from the equations (2.18) and (2.19) for the critical points.

Let us look at the asymptotic expansion (2.9) of $\mathcal{I}_X$ for our case:

$$\mathcal{I}_X(x) \sim \exp\left(\frac{1}{\hbar}W_X(\boldsymbol{u}^{(\mathrm{c})}, \boldsymbol{v}^{(\mathrm{c})}; x)\right) \frac{(-2\pi\hbar)^{\frac{N+n-1}{2}}}{u_1^{(\mathrm{c})} \cdots u_{N-1}^{(\mathrm{c})}\sqrt{\mathrm{Hess}(\boldsymbol{u}^{(\mathrm{c})}, \boldsymbol{v}^{(\mathrm{c})})}} \left(1 + \sum_{m=1}^{\infty}\hbar^m \mathcal{I}_m(x)\right). \tag{2.22}$$

Using Lemma 2.6, we can prove

Proposition 2.7.

(i) *The factor* $(u_1^{(c)} \cdots u_{N-1}^{(c)})^{-1}(\mathrm{Hess}(\boldsymbol{u}^{(c)}, \boldsymbol{v}^{(c)}))^{-1/2}$ *in* (2.22) *behaves when* $x \to \infty$ *as*

$$\frac{1}{u_1^{(c)} \cdots u_{N-1}^{(c)} \sqrt{\mathrm{Hess}(\boldsymbol{u}^{(c)}, \boldsymbol{v}^{(c)})}} = \begin{cases} O(x^{-\frac{N-n-1}{2(N-n)}}) & \text{in the case of } (2.20), \\ O(x^{-1}) & \text{in the case of } (2.21). \end{cases}$$

(ii) *For* $m \geq 1$*, the coefficient* $\mathcal{I}_m(x)$ *in the asymptotic expansion* (2.22) *behaves as*

$$\mathcal{I}_m(x) = \begin{cases} O(x^{-\frac{1}{2(N-n)}}) & \text{in the case of } (2.20) \\ O(1) & \text{in the case of } (2.21) \end{cases} \tag{2.23}$$

when $x \to \infty$.

We will give a proof of Proposition 2.7 in Appendix B.2.

Proposition 2.7 (i) implies that, if we arrange the asymptotic expansion (2.22) into WKB form as (2.10), then

$$\lim_{x\to\infty} S_m(x) = 0 \quad \text{for } m \geq 2 \tag{2.24}$$

if the corresponding critical point behaves as (2.20) when $x \to \infty$. This property is used to compare the wave function and oscillatory integral.

3. Quantum curves and topological recursion

Quantization of a spectral curve Σ (or a quantum curve, for short) is formulated as an $\hbar$-deformed differential equation, whose semi-classical limit $\hbar \to 0$ yields Σ [40, 41]. In this section, under a general setting we will briefly summarize how to construct quantum curves by the topological recursion.

3.1. Spectral curves and quantum curves.

Definition 3.1 (Spectral curve [42, 19])**.** *A* spectral curve *is a triple* $\Sigma = (C, x, y)$*, where* C *is a Torelli marked compact Riemann surface and* $x, y : C \to \mathbb{C}\boldsymbol{P}^1$ *are meromorphic functions, such that the zeroes of* dx *do not coincide with the zeroes of* dy.

The meromorphic functions x, y must satisfy an absolutely irreducible equation of the form $A(x, y) = \sum_{i,j} A_{i,j} x^i y^j = 0$. We will just simply denote a spectral curve by

$$\Sigma = \left\{ (x, y) \in \mathbb{C}^2 \mid A(x, y) = 0 \right\}, \tag{3.1}$$

after the parameterization $(x, y) = (x(z), y(z))$ of Σ by $z \in C$ is fixed. We will show the meromorphic functions $x(z)$ and $y(z)$ which parametrize the GKZ curves in Section 4.

Remark 3.2. From the view point of the quantum curve or the WKB analysis, it is natural to regard that (x, y) is an affine coordinate of the cotangent bundle of $\mathbb{C}$, where x (resp. y) represents the coordinate of the base (resp. fiber) of the cotangent bundle (e.g. [40]). In particular, the spectral curve Σ is equipped with the 1-form

$$\omega(x) = y(x)\, dx, \tag{3.2}$$

which is the restriction of the canonical 1-form in the cotangent bundle $T^*\mathbb{C} \cong \mathbb{C}^2$.

Definition 3.3 (Quantum curve)**.** *A quantum curve is a triple* $(\widehat{A}(\widehat{x}, \widehat{y}), A(x, y), \psi(x))$ *where* $\psi(x)$ *is a function,* $\widehat{A}(\widehat{x}, \widehat{y})$ *is a differential operator with*

$$\widehat{x}\, \psi(x) = x\, \psi(x), \qquad \widehat{y}\, \psi(x) = \hbar \frac{d}{dx}\, \psi(x), \tag{3.3}$$

for the operators $\widehat{x}$ *and* $\widehat{y}$,[4] *and* $A(x,y)$ *is an irreducible polynomial of* x, y *satisfying the following conditions:*

- $\widehat{A}$ *is a (possibly infinite-order) differential operator such that*
$$\widehat{A}(\widehat{x},\widehat{y})\,\psi(x) = 0. \tag{3.4}$$
- $\psi(x)$ *has the following expression (WKB solution):*
$$\psi(x) = \exp\left(\frac{1}{\hbar}\int^x \omega(x) + O(\hbar^0)\right). \tag{3.5}$$
- *By taking the semi-classical limit* $\hbar \to 0$, *we have* $A(x,y)$*:*
$$\hbar \to 0, \qquad (\widehat{x},\widehat{y}) \to (x,y), \qquad \widehat{A}(\widehat{x},\widehat{y}) \to A(x,y) \in \mathbb{C}[[x,y]].$$

We call the function $\psi(x)$ *a wave function. We also call the equation (3.4) a quantum curve for short.*

In Section 3.3, following [19] we will define the wave function associated with a spectral curve Σ (Definition 3.11) defined by the topological recursion in Section 3.2, and give a construction of a quantum curve which annihilates it.

Remark 3.4. For our purpose, we will regard x as a coordinate of $\mathbb{C}^*$:
$$\Sigma = \{\,(x,y) \in \mathbb{C}^* \times \mathbb{C} \mid A(x,y) = 0\,\}, \tag{3.6}$$
since we use $x = e^t \in \mathbb{C}^*$ as a coordinate when we regard GKZ curve Σ_X as a spectral curve. The spectral curve is a subset of $T^*\mathbb{C}^* \cong \mathbb{C}^* \times \mathbb{C}$. Then we use
$$\omega(x) = y(x)\frac{dx}{x} \tag{3.7}$$
as the counterpart of (3.2), and the quantum curve is similarly defined as Definition 3.3 by[5]
$$\widehat{x}\,\psi(x) = x\,\psi(x), \qquad \widehat{y}\,\psi(x) = \hbar x \frac{d}{dx}\,\psi(x).$$

Remark 3.5. On the one hand, in the local mirror symmetry discussed in Section 5, spectral curves which are algebraic in exponentiated variables $\mathsf{x}, \mathsf{y} \in \mathbb{C}^*$ appear. Then the corresponding quantum curve is defined by[6]
$$\widehat{\mathsf{x}}\,\psi(\mathsf{x}) = \mathsf{x}\,\psi(\mathsf{x}), \qquad \widehat{\mathsf{y}}\,\psi(\mathsf{x}) = \psi(\mathrm{e}^{\hbar}\mathsf{x}),$$
with the commutation relation $\widehat{\mathsf{y}}\,\widehat{\mathsf{x}} = \mathrm{e}^{\hbar}\,\widehat{\mathsf{x}}\,\widehat{\mathsf{y}}$. Here the counterpart of (3.2) in this case is given by
$$\omega(\mathsf{x}) = \log \mathsf{y}(\mathsf{x})\frac{d\mathsf{x}}{\mathsf{x}}. \tag{3.8}$$

3.2. **Topological recursion.** For a spectral curve $\Sigma = (C, x, y)$ one can (re)construct the wave function as the WKB expansion via the topological recursion. Before describing the reconstruction we will firstly review the (local) topological recursion defined for Σ with only simple ramification points [42], and also review the (global) topological recursion defined for Σ with arbitrary ramification points [20, 18, 19]. In the following we use

- $\omega(x(z)) = y(z)dx(z)$ if (x,y) is a coordinate of $\mathbb{C}^2$.
- $\omega(x(z)) = y(z)\frac{dx(z)}{x(z)}$ if (x,y) is a coordinate of $\mathbb{C}^* \times \mathbb{C}$ (see Remark 3.4).
- $\omega(x(z)) = \log y(z)\frac{dx(z)}{x(z)}$ if (x,y) is a coordinate of $(\mathbb{C}^*)^2$ (see Remark 3.5).

---

[4] We remark that the operators $\widehat{x}$ and $\widehat{y}$ obey the commutation relation
$$[\widehat{y},\widehat{x}] = \hbar.$$

[5] The commutation relation becomes $[\widehat{y},\widehat{x}] = x\hbar$ which is sightly modified from the previous one.

[6] Using the operators $\widehat{x}$ and $\widehat{y}$ in (3.3) one can represent operators $\widehat{\mathsf{x}}$ and $\widehat{\mathsf{y}}$ as $\widehat{\mathsf{x}} = \mathrm{e}^{\widehat{x}}$and $\widehat{\mathsf{y}} = \mathrm{e}^{\widehat{y}}$.

3.2.1. *Local topological recursion for simple ramified spectral curves.* Let Σ be a spectral curve whose all branch points (zeros of $dx = 0$) on the x-plane are simple. Let R be the set of all ramification points in C. Then near each ramification point $q \in R \subset C$ one can take a local coordinate $z \in C$ and find a unique conjugate point $\overline{z} = \sigma_q(z) \neq z$, where σ_q is the local Galois conjugation of Σ near q.

Definition 3.6 ([42]). *For a simple ramified spectral curve Σ, the symmetric meromorphic differentials $\omega_n^{(g)}$ ($g \in \mathbb{Z}_{\geq 0}$) on C^n for $(g,n) \neq (0,1)$, $(0,2)$ are recursively defined by the local topological recursion*

$$\begin{aligned}\omega_{n+1}^{(g)}(z, \boldsymbol{z}_N) = \sum_{q\in R} \operatorname*{Res}_{w=q} \frac{\frac{1}{2}\int_{\overline{w}}^{w} B(\cdot, z)}{\omega(x(w)) - \omega(x(\overline{w}))} \Bigg(& \omega_{n+2}^{(g-1)}(w, \overline{w}, \boldsymbol{z}_N) \\ & + \sum_{\ell=0}^{g} \sum_{\emptyset = J \subseteq N} \omega_{|J|+1}^{(g-\ell)}(w, \boldsymbol{z}_J)\omega_{|N|-|J|+1}^{(\ell)}(\overline{w}, \boldsymbol{z}_{N\setminus J})\Bigg),\end{aligned} \tag{3.9}$$

with initial inputs

$$\omega_1^{(0)}(z) = 0, \qquad \omega_2^{(0)}(z_1, z_2) = B(z_1, z_2),$$

in addition to the 1-form $\omega(x(z))$ on C, where $\boldsymbol{z}_N = \{z_1, \ldots, z_n\}$, $N = \{1, \ldots, n\} \supset J = \{i_1, \ldots, i_j\}$, $N\backslash J = \{i_{j+1}, \ldots, i_n\}$. Here $B(z_1, z_2)$ is the Bergman kernel on C^2, which is a bi-differential and holomorphic except $z_1 = z_2$, defined uniquely by

- $B(z_1, z_2) = B(z_2, z_1)$,
- $B(z_1, z_2) \underset{z_1 \to z_2}{\sim} \dfrac{dz_1 dz_2}{(z_1 - z_2)^2} + reg.$
- $\displaystyle\oint_{A_i} B(z_1, z_2) = 0,$

where A_i ($i = 1, \ldots$, genus of C) are canonical A-cycles (recall that C is Torelli marked).

Example 3.7. For the case $C = \mathbb{C}\mathbf{P}^1$, the Bergman kernel is given by

$$B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}. \tag{3.10}$$

3.2.2. *Global topological recursion for arbitrary ramified spectral curves.* The local topological recursion in Definition 3.6 is applicable only for simple ramified spectral curves. In [20, 18, 19] it was proposed the global topological recursion which is also applicable for arbitrary ramified spectral curves. Consider a spectral curve with degree r of x defined by

$$A(x, y) = \sum_{k=0}^{r} a_{r-k}(x) y^k = 0, \qquad x, y \in \mathbb{C} \text{ or } \mathbb{C}^*, \tag{3.11}$$

where $a_k(x)$ are polynomials of x. Let R be the set of all ramification points on the x-plane, and σ_q be the local Galois conjugation of Σ near $q \in R \subset C$. For a local coordinate $z \in C$ one finds a set $\sigma_q(z)$ of $r-1$ points near each ramification point q.

Definition 3.8 ([20, 18, 19]). *For a multi-ramified spectral curve Σ defined by (3.11), the symmetric meromorphic differentials $\omega_n^{(g)}$ ($g \in \mathbb{Z}_{\geq 0}$) on C^n for $(g,n) \neq (0,1)$, $(0,2)$ are recursively defined by the global topological recursion*

$$\begin{aligned}\omega_{n+1}^{(g)}(z, \boldsymbol{z}_N) = \sum_{q\in R} \operatorname*{Res}_{w=q} \Bigg(\sum_{k=1}^{r-1} \sum_{\beta(w) \subseteq_k \sigma_q(w)} & \frac{(-1)^{k+1}\int_{w_*}^{w} B(\cdot, z)}{\prod_{b_w \in \beta(w)} (\omega(x(w)) - \omega(x(b_w)))} \\ & \times \mathcal{R}^{(k+1)} \omega_{n+1}^{(g)}(w, \beta(w); \boldsymbol{z}_N)\Bigg),\end{aligned} \tag{3.12}$$

with initial inputs

$$\omega_1^{(0)}(z) = 0, \qquad \omega_2^{(0)}(z_1, z_2) = B(z_1, z_2),$$

in addition to the 1-form $\omega(x(z))$ on C, where

$$\mathcal{R}^{(k)}\omega^{(g)}_{n+1}(\boldsymbol{w}_K;\boldsymbol{z}_N) = \sum_{\mu\in\mathcal{P}(K)}\ \sum_{\uplus_{i=1}^{\ell(\mu)} J_i=N}\ \sum_{\sum_{i=1}^{\ell(\mu)} g_i=g+\ell(\mu)-k} \left(\prod_{i=1}^{\ell(\mu)} \omega_{g_i,|\mu_i|+|J_i|}(\boldsymbol{w}_{\mu_i},\boldsymbol{z}_{J_i})\right).$$

Here $\boldsymbol{z}_N = \{z_1,\dots,z_n\}$ (resp. $\boldsymbol{w}_K = \{w_1,\dots,w_k\}$) for $N = \{1,\dots,n\}$ (resp. $K = \{1,\dots,k\}$), $\beta(w) \subseteq_k \sigma_q(w)$ means $\beta(w) \subseteq \sigma_q(w)$ with $|\beta(w)| = k$, $\mathcal{P}(K)$ is the set of partitions of K, $\ell(\mu)$ is the number of subsets in the set partition μ, and the symbol $\uplus$ means the pairwise disjoint union. w_* is a reference point on Σ and we see that $\omega^{(g)}_n$'s do not depend on it.

Remark 3.9. For simple ramified spectral curves, the global topological recursion in Definition 3.8 is equivalent to the local topological recursion in Definition 3.6.

3.3. **Reconstruction of quantum curves by topological recursion.** By the symmetric meromorphic differentials $\omega^{(g)}_n$ on C^n defined by the topological recursion (3.9) or (3.12), one can (re)construct the wave function as the WKB expansion. For the reconstruction we define the divisor D for the integration contour of $\omega^{(g)}_n$'s as follows.

Definition 3.10. *Let D denote the degree 0 divisor on C with $q_k \in C$ such that*

$$D = \sum_k d_k[q_k], \quad \deg D = \sum_k d_k = 0.$$

For the degree 0 divisor D, an integration of a meromorphic 1-form α on C is defined by

$$\int_D \alpha = \sum_k d_k \int_b^{q_k} \alpha,$$

where $b \in C$ is an arbitrary reference point, and the integration contours are assumed to not intersect with the homology 1-cycles of C. In fact, each integral $\int_D \alpha$ does not depend on the choice of the reference point b, because the integration divisor D obeys $\deg D = 0$.

On basis of this notation the wave function is reconstructed as:

Definition 3.11 (Reconstructing WKB)**.** *The wave function $\psi(D)$ associated with a spectral curve Σ and degree 0 divisor D is defined by*

$$\begin{aligned}\psi(D) = \exp\Bigg[&\frac{1}{\hbar}\int_D \widehat{\omega}^{(0)}_1(z_1) + \frac{1}{2}\int_D\int_D \widehat{\omega}^{(0)}_2(z_1,z_2) \\ &+ \sum_{(g,n)\neq(0,1),(0,2)} \frac{1}{n!}\hbar^{2g-2+n}\int_D\cdots\int_D \omega^{(g)}_n(z_1,\dots,z_n)\Bigg],\end{aligned} \tag{3.13}$$

where $z \in C$ is away from ramification points. Here we have defined

$$\widehat{\omega}^{(0)}_1(z_1) = \omega(x(z_1)), \qquad \widehat{\omega}^{(0)}_2(z_1,z_2) = B(z_1,z_2) - \frac{dx(z_1)dx(z_2)}{(x(z_1)-x(z_2))^2}.$$

In this reconstruction the leading and subleading integrals $\int_D \widehat{\omega}^{(0)}_1(z_1)$ and $\int_D\int_D \widehat{\omega}^{(0)}_2(z_1,z_2)$ should be regularized so as to remove divergence by an overall normalization factor for $\psi(D)$.

In [19], by the global topological recursion (3.12) the quantum curve which annihilates the wave function in (3.13) was reconstructed systematically for a special class of the spectral curve. Here we will review their elegant results of the WKB reconstruction of the quantum curve. In the following we consider the spectral curve in $\mathbb{C}^2$ as

$$\Sigma = \left\{ (x,y) \in \mathbb{C}^2 \;\middle|\; P(x,y) = \sum_{k=0}^{r} p_k(x) y^{r-k} = 0 \right\}, \tag{3.14}$$

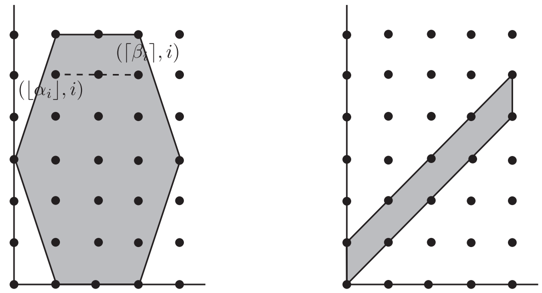


FIGURE 2. The Newton polygon for 2-parameter polynomial $P(x, y)$. The right figure satisfies the admissibility condition.

where $p_k(x)$ are polynomials of x. The special class of the spectral curve Σ discussed in [19] is referred to be *admissible*. The admissibility condition is defined by the Newton polygon Δ for the defining polynomial $P(x, y)$ of the spectral curve Σ:

$$P(x,y) = \sum_{k=0}^{r} p_k(x) y^{r-k} = \sum_{(k,i)\in\mathbb{Z}^2} p_{k,i} x^k y^i.$$

A Newton polygon Δ for $P(x, y)$ is then a convex hull of the set S_P such that

$$S_P = \left\{ (k,i) \in \mathbb{Z}^2 \;\middle|\; p_{k,i} \neq 0 \right\}.$$

For each level set labelled by $m \in \mathbb{Z}$ in a Newton polygon Δ, we define

$$\alpha_i = \inf\left\{ k \,|\, (k,i) \in \Delta \right\}, \qquad \beta_i = \sup\left\{ k \,|\, (k,i) \in \Delta \right\}.$$

The number of interior integral points of a Newton polygon Δ is given by $\sum_{i\in\mathbb{Z}}(\lceil\beta_i\rceil - \lfloor\alpha_i\rfloor - 1)$, and the Newton polygon Δ has no interior point, if S_P satisfies

$$\lceil\beta_i\rceil - \lfloor\alpha_i\rfloor = 1, \quad \text{for all } i.$$

On basis of the above notions of the Newton polygon, the admissibility condition of the spectral curve Σ is given as follows.

Definition 3.12 ([19])**.** *The spectral curve $\Sigma = \{(x, y) \in \mathbb{C}^2 | P(x, y) = 0\}$ is admissible if the following two conditions are satisfied:*

(1) The Newton polygon Δ associated with Σ has no interior point.
(2) If Σ contains the origin $(x, y) = (0, 0) \in \mathbb{C}^2$, then the curve is smooth at this point.

The quantum curve which annihilates the wave function $\psi(D)$ with $D = [z]-[z_*]$ is reconstructed manifestly for the admissible spectral curve.

Proposition 3.13 (Lemma 5.14 in [19])**.** *Let D_k be the differential operators*

$$D_k = \hbar \frac{x^{\lfloor\alpha_k\rfloor}}{x^{\lfloor\alpha_{k-1}\rfloor}} \frac{d}{dx}, \quad k = 1, \ldots, r. \tag{3.15}$$

For the degree 0 divisor $D = [z] - [z_*]$ with a simple pole z_* of x as reference point, the wave function $\psi(D)$ satisfies the order r ordinary differential equation

$$\Bigg[D_1 D_2 \cdots D_{r-1} \frac{p_0(x)}{x^{\lfloor\alpha_r\rfloor}} D_r + D_1 D_2 \cdots D_{r-2} \frac{p_1(x)}{x^{\lfloor\alpha_{r-1}\rfloor}} D_{r-1} + \cdots + \frac{p_{r-1}(x)}{x^{\lfloor\alpha_1\rfloor}} D_1 + \frac{p_r(x)}{x^{\lfloor\alpha_0\rfloor}}$$
$$- \hbar C_1 D_1 D_2 \cdots D_{r-2} \frac{x^{\lfloor\alpha_{r-1}\rfloor}}{x^{\lfloor\alpha_{r-2}\rfloor}} - \hbar C_2 D_1 D_2 \cdots D_{r-3} \frac{x^{\lfloor\alpha_{r-2}\rfloor}}{x^{\lfloor\alpha_{r-3}\rfloor}} - \cdots - \hbar C_{r-1} \frac{x^{\lfloor\alpha_1\rfloor}}{x^{\lfloor\alpha_0\rfloor}} \Bigg] \psi(D) = 0. \tag{3.16}$$

Here the coefficients C_k's ($k = 1, \ldots r-1$) are given by

$$C_k = \lim_{z \to z_*} \frac{P_{k+1}(x, y(x))}{x^{\lfloor \alpha_{r-k} \rfloor + 1}}, \qquad P_{k+1}(x, y) = \sum_{i=1}^{k} p_{k-i}(x) y^i, \tag{3.17}$$

where $y(x)$ obeys $P(x, y(x)) = 0$.

Remark 3.14 (Remark 5.12 in [19]). Even when the reference point z_* is a higher order pole of x, if the integrals in the WKB reconstruction (3.13) converge, Proposition 3.13 is correct.

4. GKZ equations as quantum curves

In this section we will reconstruct GKZ equations as quantum curves of GKZ curves for the equivariant Gromov-Witten theory on the projective space $\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}$ and the Fano complete intersection $X_{\boldsymbol{l}=\mathbf{1};\boldsymbol{w},\boldsymbol{\lambda}}$. For this purpose we employ two methods developed in works by Mulase-Sułkowski [86] and Bouchard-Eynard [19].

The former method uses the local topological recursion and the recursion relation for S_m's obtained from the quantum curve is reconstructed manifestly for two examples: (1) $\mathbb{C}\mathbf{P}^1_{\boldsymbol{w}}$ and (2) the Fano hypersurface $X_{l_1=1;\boldsymbol{w},\lambda_1}$ in $\mathbb{C}\mathbf{P}^1_{\boldsymbol{w}}$. Such manifest results are helpful to find the relation between the wave function and the WKB solution which will be studied in Section 6 (e.g. Lemma 6.4 below).

The latter method uses the global topological recursion and we can reconstruct more general class of X such that $X = \mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}$ and the Fano complete intersection $X = X_{\boldsymbol{l}=\mathbf{1};\boldsymbol{w},\boldsymbol{\lambda}}$. For these class of X the admissibility condition in Definition 3.12 is satisfied, and we can reconstruct the GKZ equation as the quantum curve for the wave function $\psi_X(x)$ which admits the expansion at $x = \infty$ (i.e. $z_* = \infty$ which gives $x = \infty$ is chosen to be the endpoint in the integration divisor D).

4.1. The GKZ equation from the local topological recursion à la Mulase-Sułkowski.

In [86] the second order ordinary differential equation for the wave function $\psi(x)$ was derived via the local topological recursion (3.9) in Definition 3.6. In the following we will reconstruct the GKZ equation from the data of the GKZ curve. Among GKZ equations discussed in this article, this derivation is applicable to the following two models.

(1) Equivariant Gromov-Witten theory on the projective space $\mathbb{C}\mathbf{P}^1_{\boldsymbol{w}}$:

$$\left(\hbar x \frac{d}{dx} - w_0\right)\left(\hbar x \frac{d}{dx} - w_1\right)\psi(x) = x\psi(x). \tag{4.1}$$

(2) Equivariant Gromov-Witten theory on the degree 1 hypersurface $X_{l_1=1;\boldsymbol{w},\boldsymbol{\lambda}}$ in $\mathbb{C}\mathbf{P}^1_{\boldsymbol{w}}$:

$$\left(\hbar x \frac{d}{dx} - w_0\right)\left(\hbar x \frac{d}{dx} - w_1\right)\psi(x) = x\left(\hbar x \frac{d}{dx} - \lambda_1 + \hbar\right)\psi(x). \tag{4.2}$$

The defining equation of the GKZ curve Σ_X is directly found by replacements $x \to x$, $\hbar x d/dx \to y$, and $\hbar \to 0$ in the GKZ equation, and indeed, one finds the GKZ curves for these models

$$\Sigma_{\mathbb{C}\mathbf{P}^1_{\boldsymbol{w}}} = \left\{ (x, y) \in \mathbb{C}^* \times \mathbb{C} \;\middle|\; y^2 - (w_0 + w_1)y + w_0 w_1 - x = 0 \right\}, \tag{4.3}$$

$$\Sigma_{X_{l_1=1;\boldsymbol{w},\boldsymbol{\lambda}}} = \left\{ (x, y) \in \mathbb{C}^* \times \mathbb{C} \;\middle|\; y^2 - (w_0 + w_1 + x)y + w_0 w_1 + \lambda_1 x = 0 \right\}. \tag{4.4}$$

To apply the topological recursion to the above GKZ curves, we need to use appropriate local coordinates[7] to pick up residues in the topological recursion (3.9) systematically. In the following for the case (1) we will introduce a local coordinate z as

$$x(z) = z^2 - \Lambda, \qquad y(z) = z + \frac{1}{2}(w_0 + w_1), \qquad \Lambda = \frac{1}{4}(w_0 - w_1)^2, \tag{4.5}$$

[7] In [86], such local coordinates are specified by the Lagrangian singularity of Σ_X.

and for the case (2) we will introduce a local coordinate z via the Zhukovsky coordinate u as

$$\begin{aligned} x(z) &= \frac{\alpha+\beta}{2} + \frac{\beta-\alpha}{4}\left(u(z) + \frac{1}{u(z)}\right), \\ y(z) &= \frac{w_0+w_1+x(z)}{2} + \frac{\beta-\alpha}{8}\left(u(z) - \frac{1}{u(z)}\right), \\ \alpha+\beta &= -2(w_0+w_1-2\lambda_1), \quad \beta-\alpha = 4\sqrt{(\lambda_1-w_0)(\lambda_1-w_1)}, \quad u(z) = \frac{z+1}{z-1}. \end{aligned} \tag{4.6}$$

We assume that the parameters w_i and λ_a are generic so that the dx and dy do not have common zero. Then we can identify the GKZ curve Σ_X with the spectral curve $(C = \mathbb{C}\mathbf{P}^1, x(z), y(z))$ in the sense of Definition 3.1. For both of these local coordinates, we see that the local Galois conjugation σ near the branch point $z = 0 \in \Sigma_X$ acts as $\sigma(z) = -z$.

The basic building blocks of the topological recursion (3.9) are given in the above local coordinates as follows:

$$\omega(x(z)) = y(z)\frac{dx(z)}{x(z)} = \frac{y(z)}{x(z)}\frac{dx(z)}{dz}dz, \qquad B(z_1,z_2) = \frac{dz_1dz_2}{(z_1-z_2)^2}.$$

And we obtain

$$\begin{aligned} \frac{\frac{1}{2}\int_{-z}^{z} B(\cdot, z_1)}{\omega(x(z)) - \omega(x(-z))} &= p(z)\left(\frac{1}{z+z_1} + \frac{1}{z-z_1}\right)\frac{dz_1}{dz}, \\ p(z) &= -\frac{x(z)}{2\big(y(z)-y(-z)\big)\frac{dx(z)}{dz}}, \end{aligned}$$

where for the above two models the function $p(z)$ is given by

$$\begin{aligned} p(z) &= -\frac{z^2-\Lambda}{8z^2}, \qquad \text{for } X = \mathbb{C}\mathbf{P}^1_{\boldsymbol{w}}, \\ p(z) &= \frac{(z^2-1)^2\left((w_0+w_1-2\lambda_1)(1-z^2) + 2(1+z^2)\sqrt{(\lambda_1-w_0)(\lambda_1-w_1)}\right)}{64z^2(\lambda_1-w_0)(\lambda_1-w_1)}, \quad \text{for } X = X_{l_1=1;\boldsymbol{w},\boldsymbol{\lambda}}. \end{aligned}$$

Both of these functions do not have poles except for $z = 0, \infty$.

Adopting the ingredients one finds that the topological recursion (3.9) is rewritten as follows:

$$\begin{aligned} &\omega_n^{(g)}(z_1,\ldots,z_n) \\ &= \frac{1}{2\pi \mathrm{i}}\int_\gamma p(z)\left(\frac{1}{z+z_1} + \frac{1}{z-z_1}\right)\frac{dz_1}{dz} \\ &\quad \times \Bigg[\sum_{j=2}^{n}\Big(B(z,z_j)\omega_{n-1}^{(g)}(-z,z_2,\ldots,\widehat{z}_j,\ldots,z_n) + B(-z,z_j)\omega_{n-1}^{(g)}(z,z_2,\ldots,\widehat{z}_j,\ldots,z_n)\Big) \\ &\qquad + \omega_{n+1}^{(g-1)}(z,-z,z_2,\ldots,z_n) + \sideset{}{'}\sum_{\substack{g_1+g_2=g \\ I\sqcup J=\{2,3,\ldots,n\}}} \omega_{|I|+1}^{(g_1)}(z,\boldsymbol{z}_I)\omega_{|J|+1}^{(g_2)}(-z,\boldsymbol{z}_J)\Bigg], \end{aligned} \tag{4.7}$$

where the contour γ encloses an annulus bounded by two concentric circles centered at the origin encircles $z = \pm z_i$ $(i = 1,\ldots,n)$ as depicted in Figure 3. The prime in the last summation means that $\omega_1^{(0)}$ and $\omega_2^{(0)}$ are excluded from the summation. Proceeding along the same line of the proof for Theorem 4.1 in [86], we can show the following lemma.

Lemma 4.1. *Let $F_n^{(g)}(z_1,\ldots,z_n;z_*)$ $(2g-2+n>0)$ denote a function such that*

$$F_n^{(g)}(z_1,\ldots,z_n;z_*) = \int_{z_*}^{z_1}\cdots\int_{z_*}^{z_n}\omega_n^{(g)}(z_1',\ldots,z_n'), \tag{4.8}$$

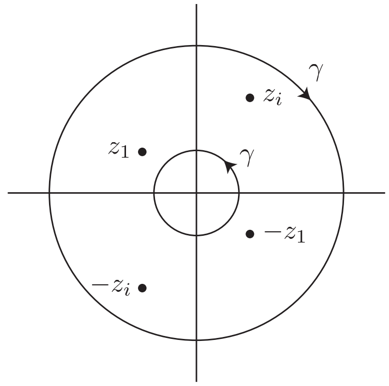


FIGURE 3. The contour γ in z-plane consists of two concentric circles centered at the origin. The inner circle which encircles the origin has an infinitesimally small radius with the positive orientation. The outer circle which encircles $z = \pm z_i$ $(i = 1, \ldots, n)$ is oriented negatively.

where z_* denotes a reference point of the integration. If the meromorphic function $p(z)$ does not have any poles except for $z = 0, \infty$, the topological recursion (4.7) leads to the following recursion relation:

$$\begin{aligned}\frac{\partial}{\partial z_1} F_n^{(g)}(\boldsymbol{z}_{[n]}; z_*) = &\sum_{j=2}^{n} \frac{z_j}{z_1^2 - z_j^2} \left(4p(z_1) \frac{\partial}{\partial z_1} F_{n-1}^{(g)}(\boldsymbol{z}_{[\hat{j}]}; z_*) - 4p(z_j) \frac{\partial}{\partial z_j} F_{n-1}^{(g)}(\boldsymbol{z}_{[\hat{1}]}; z_*) \right) \\ &- \sum_{j=2}^{n} \frac{z_*}{z_1^2 - z_*^2} 4p(z_1) \frac{\partial}{\partial z_1} F_{n-1}^{(g)}(\boldsymbol{z}_{[\hat{j}]}; z_*) \\ &+ 2p(z_1) \left[\frac{\partial^2}{\partial u_1 \partial u_2} F_{n+1}^{(g)}(u_1, u_2, \boldsymbol{z}_{[\hat{1}]}; z_*) \right] \Bigg|_{u_1 = u_2 = z_1} \\ &+ 2p(z_1) \sideset{}{'}\sum_{\substack{g_1 + g_2 = g \\ I \sqcup J = \{2,3,\ldots,n\}}} F_{|I|+1}^{(g_1)}(z_1, \boldsymbol{z}_I; z_*) F_{|J|+1}^{(g_2)}(z_1, \boldsymbol{z}_J; z_*),\end{aligned} \tag{4.9}$$

where $[n] = \{1, 2, \ldots, n\}$ and $[\hat{j}] = \{1, 2, \ldots, \hat{j}, \ldots, n\}$.

In addition to (4.8) we also define

$$\begin{aligned} F_1^{(0)}(z_1; z_*) &= \int_{z_*}^{z_1} \omega(x(z_1')) = \int_{z_*}^{z_1} y(z_1') \frac{dx(z_1')}{x(z_1')}, \\ F_2^{(0)}(z_1, z_2; z_*) &= \int_{z_*}^{z_1} \int_{z_*}^{z_2} \left(B(z_1', z_2') - \frac{dx(z_1') dx(z_2')}{(x(z_1') - x(z_2'))^2} \right) = \int_{z_*}^{z_1} \int_{z_*}^{z_2} \frac{dz_1' dz_2'}{(z_1' + z_2')^2}, \end{aligned} \tag{4.10}$$

where in the second equality of $F_2^{(0)}(z_1, z_2; z_*)$ we have used the local coordinates (4.5) and (4.6). Note that these integrals should be regularized so that the integrals converge by adding certain constants which depend on the reference point z_*. Following the WKB reconstruction (3.13), we define the wave function $\psi_X(x)$ such that

$$\psi_X(x) = \exp\left(\sum_{m=0}^{\infty} \hbar^{m-1} F_m(x) \right), \qquad F_m(x(z)) = \sum_{\substack{g \geq 0,\, n \geq 1 \\ 2g-1+n=m}} \frac{1}{n!} F_n^{(g)}(z, \ldots, z; z_*). \tag{4.11}$$

Adopting technical identities developed in Lemma A.1 of [86] to the recursion relation (4.9) we arrive at the following lemma.

Lemma 4.2. *The functions F_m $(m \geq 2)$ obey the following recursion relation:*

$$\frac{d}{dz}F_{m+1} = 2p(z)\left(\frac{d^2}{dz^2}F_m + \sum_{\substack{a+b=m+1\\ a,b\geq 2}} \frac{dF_a}{dz}\frac{dF_b}{dz}\right) + \left(2\frac{dp(z)}{dz} - 4\frac{z_*}{z^2-z_*^2}p(z)\right)\frac{d}{dz}F_m. \tag{4.12}$$

Now we will show that the above recursion relation (4.12) agrees with the recursion relation for F_m's found from the GKZ equation for two models.

Proposition 4.3. *For $X = \mathbb{C}\boldsymbol{P}^1_{\boldsymbol{w}}$, the wave function $\psi_X(x)$ defined by (4.11) with the integration divisor $D = [z] - [\infty]$ in the local coordinate (4.5) satisfies the GKZ equation (4.1) for the equivariant Gromov-Witten theory of $\mathbb{C}\boldsymbol{P}^1_{\boldsymbol{w}}$. Here we choose the reference point $z_* = \infty$ so that $x(z_*) = \infty$ holds.*

Proof. Adopt the WKB expansion

$$\psi(x) = \exp\left(\sum_{m=0}^{\infty} \hbar^{m-1} S_m(x)\right) \tag{4.13}$$

into the GKZ equation (4.1), then one finds a hierarchy of differential equations for S_m's:

$$x\left(\frac{d^2}{dx^2}S_m + \sum_{a+b=m+1}\frac{dS_a}{dx}\frac{dS_b}{dx}\right) - (w_0+w_1)\frac{d}{dx}S_{m+1} + \frac{d}{dx}S_m = 0. \tag{4.14}$$

The remaining $\hbar^0$-terms are treated separately as follows:

$$\left(x\frac{dS_0}{dx}\right)^2 - (w_0+w_1)x\frac{dS_0}{dx} + w_0w_1 - x = 0. \tag{4.15}$$

By definitions (4.10) and (4.11) we see that $F_0(x(z)) = F_1^{(0)}(z; z_*)$ obeys the differential equation (4.15) by $S_0 = F_0$, and one obtains the GKZ curve (4.3) by a replacement $xdS_0/dx = y$.

Next we will consider the relation between S_1 and F_1. The subleading term S_1 of the WKB expansion (4.13) is computed from the recursion relation (4.14) for $m = 0$.

$$\frac{dS_1}{dx} = -\frac{1}{4x+(w_0-w_1)^2} = -\frac{1}{4z^2},$$

where in the second equality we have used the local coordinate (4.5). On the other hand, by definitions (4.10) and (4.11) one finds

$$\begin{aligned} F_1(x(z)) &= \frac{1}{2}F_2^{(0)}(z,z;z_*) = \frac{1}{2}\log\frac{(z+z_*)^2}{4zz_*}, \\ \left.\frac{dF_1}{dx}\right|_{x=x(z)} &= \frac{1}{2}\left(\frac{dx}{dz}\right)^{-1}\left(\frac{2}{z+z_*} - \frac{1}{z}\right). \end{aligned} \tag{4.16}$$

By comparison of these results, we see that for the specialization $z_* = \infty$ with $x(z_*) = \infty$, dF_1/dx in (4.16) agrees with dS_1/dx.

Now rewriting the second term in (4.14):

$$\sum_{a+b=m+1}\frac{dS_a}{dx}\frac{dS_b}{dx} = 2\frac{dS_0}{dx}\frac{dS_{m+1}}{dx} + 2\frac{dS_1}{dx}\frac{dS_m}{dx} + \sum_{\substack{a+b=m+1\\ a,b\geq 2}}\frac{dS_a}{dx}\frac{dS_b}{dx}, \tag{4.17}$$

one obtains

$$-\left(2x\frac{dS_0}{dx} - (w_0+w_1)\right)\frac{d}{dx}S_{m+1} = x\frac{d^2}{dx^2}S_m + \left(2x\frac{dS_1}{dx} + 1\right)\frac{d}{dx}S_m + x\sum_{\substack{a+b=m+1\\ a,b\geq 2}}\frac{dS_a}{dx}\frac{dS_b}{dx}. \tag{4.18}$$

To switch x-derivatives in the above recursion to z-derivatives, one can use

$$\frac{d}{dx} = \left(\frac{dx}{dz}\right)^{-1} \frac{d}{dz}, \qquad \frac{d^2}{dx^2} = \left(\frac{dx}{dz}\right)^{-2} \frac{d^2}{dz^2} - \left(\frac{dx}{dz}\right)^{-3} \frac{d^2x}{dz^2} \frac{d}{dz}. \tag{4.19}$$

Plugging $xdS_0/dx = y$ and (4.19) into (4.18), one gets the recursion relation for S_m's

$$\frac{d}{dz} S_{m+1} = c_1(z) \left(\frac{d^2}{dz^2} S_m + \sum_{\substack{a+b=m+1 \\ a,b\geq 2}} \frac{dS_a}{dz} \frac{dS_b}{dz} \right) + c_2(z) \frac{d}{dz} S_m, \tag{4.20}$$

where

$$c_1(z) = \frac{-x(z)}{(2y(z) - (w_0 + w_1)) \frac{dx}{dz}},$$
$$c_2(z) = \frac{-1}{2y(z) - (w_0 + w_1)} \left(-x(z) \left(\frac{dx}{dz}\right)^{-2} \frac{d^2x}{dz^2} + 2x(z) \frac{dS_1}{dx}\bigg|_{x=x(z)} + 1 \right).$$

Using the local coordinate (4.5), after some short computations, one obtains

$$c_1(z) = -\frac{z^2 - \Lambda}{4z^2} = 2p(z), \qquad c_2(z) = -\frac{\Lambda}{2z^3} = 2\frac{dp(z)}{dz}.$$

As a consequence, it is found that the recursion relation (4.20) agrees with the recursion relation (4.12) for F_m's of $X = \mathbb{C}\mathbf{P}^1_{\boldsymbol{w}}$ under the choice $z_* = \infty$. □

Proposition 4.4. *For the degree 1 hypersurface $X = X_{l_1=1;\boldsymbol{w},\boldsymbol{\lambda}}$ in $\mathbb{C}\boldsymbol{P}^1_{\boldsymbol{w}}$, the wave function $\psi_X(x)$ defined by (4.11) with the integration divisor $D = [z] - [-1]$ in the local coordinate (4.6) satisfies the GKZ equation (4.2) for the equivariant Gromov-Witten theory of $X_{l_1=1;\boldsymbol{w},\boldsymbol{\lambda}}$. Here we choose the reference point $z_* = -1$ so that $x(z_*) = \infty$ holds.*

Proof. Adopt the WKB expansion (4.13) into the GKZ equation (4.2), then one finds a hierarchy of differential equations for S_m's:

$$-\left(2x\frac{dS_0}{dx} - (w_0 + w_1 + x)\right) \frac{d}{dx} S_{m+1} = x\frac{d^2}{dx^2} S_m + \left(2x\frac{dS_1}{dx} + 1\right) \frac{d}{dx} S_m + x \sum_{\substack{a+b=m+1 \\ a,b\geq 2}} \frac{dS_a}{dx} \frac{dS_b}{dx}. \tag{4.21}$$

In particular, S_0 obeys

$$\left(x\frac{dS_0}{dx}\right)^2 - (w_0 + w_1 + x)x\frac{dS_0}{dx} + w_0 w_1 + \lambda_1 x = 0, \tag{4.22}$$

and this differential equation gives the GKZ curve (4.4) by $y = xdS_0/dx$.

From the recursion relation (4.21) for $m = 0$, it is found that

$$\begin{aligned}\frac{dS_1}{dx} &= -\frac{w_0 + w_1 + x - 2\lambda_1 + \sqrt{x^2 + 2(w_0 + w_1 - 2\lambda_1)x + (w_0 - w_1)^2}}{2(x^2 + 2(w_0 + w_1 - 2\lambda_1)x + (w_0 - w_1)^2)} \\ &= -\frac{(z+1)^3(z-1)}{16z^2\sqrt{(\lambda_1 - w_0)(\lambda_1 - w_1)}},\end{aligned}$$

where in the second equality the local coordinate (4.6) is adopted. On the other hand, by definitions (4.10) and (4.11) one obtains (4.16). We see that for the specialization $z_* = -1$ which corresponds to $x(-1) = \infty$, dF_1/dx in (4.16) agrees with dS_1/dx.

Switching from x-coordinate to the local z-coordinate, one finds that the recursion relation (4.21) is rewritten as

$$\frac{d}{dz} S_{m+1} = c_1(z) \left(\frac{d^2}{dz^2} S_m + \sum_{\substack{a+b=m+1 \\ a,b\geq 2}} \frac{dS_a}{dz} \frac{dS_b}{dz} \right) + c_2(z) \frac{d}{dz} S_m, \tag{4.23}$$

where

$$c_1(z) = \frac{-x(z)}{(2y(z)-(w_0+w_1+x(z)))\frac{dx}{dz}},$$
$$c_2(z) = \frac{-1}{2y(z)-(w_0+w_1+x(z))}\left(-x(z)\left(\frac{dx}{dz}\right)^{-2}\frac{d^2x}{dz^2}+2x(z)\frac{dS_1}{dx}\bigg|_{x=x(z)}+1\right).$$

Using the local coordinate (4.6), after some short computations we obtain

$$c_1(z) = 2p(z), \qquad c_2(z) = 2\frac{dp(z)}{dz}+\frac{4}{z^2-1}p(z).$$

Thus if one chooses $z_* = -1$ s.t. $x(-1)=\infty$, the recursion relation (4.23) agrees with the recursion relation (4.12) for F_m's for $X = X_{l_1=1;\boldsymbol{w},\boldsymbol{\lambda}}$ under the choice $z_*=\infty$. □

Propositions 4.3 and 4.4 show that the GKZ equations (4.1) and (4.2) are reconstructible as the quantum curve on the GKZ curve (4.3) and (4.4), respectively.

4.2. **The GKZ equation from the global topological recursion à la Bouchard-Eynard.** In Section 4.1, it was proven for the two models (4.1) and (4.2) that their GKZ equations are reconstructible by the local topological recursion (3.9), if endpoints of integrals of $\omega_n^{(g)}(z_1,\ldots,z_n)$ in the WKB reconstruction (3.13) are chosen to be at $x(z_*)=\infty$ in the global coordinate $x=x(z)$. To generalize these results, we will apply the consequences of the WKB reconstruction of the quantum curve in [19] (summarized shortly in Section 3.3) to the GKZ curve (2.17) for the equivariant Gromov-Witten theory on $\mathbb{CP}^{N-1}_{\boldsymbol{w}}$ and the complete intersection $X_{\boldsymbol{l}=\boldsymbol{1};\boldsymbol{w},\boldsymbol{\lambda}}$ of the degree $l_i=1$ ($i=1,\ldots,n$) hypersurfaces in $\mathbb{CP}^{N-1}_{\boldsymbol{w}}$. The defining polynomial $A_{X_{\boldsymbol{l}=\boldsymbol{1};\boldsymbol{w},\boldsymbol{\lambda}}}$ of the GKZ curve for this model is

$$A_{X_{\boldsymbol{l}=\boldsymbol{1};\boldsymbol{w},\boldsymbol{\lambda}}}(x,y) = \prod_{i=0}^{N-1}(y-w_i) - x\prod_{a=1}^{n}(y-\lambda_a), \qquad x\in\mathbb{C}^*,\ \ y\in\mathbb{C}, \tag{4.24}$$

and the GKZ equation is reconstructed subsequently.

In order to apply the consequences of the WKB reconstruction directly to the GKZ curve, we will change the presentation of the GKZ curve (4.24):

$$\Sigma_{X_{\boldsymbol{l}=\boldsymbol{1};\boldsymbol{w},\boldsymbol{\lambda}}} = \left\{(x,y)\in\mathbb{C}^*\times\mathbb{C}\ \middle|\ A_{X_{\boldsymbol{l}=\boldsymbol{1};\boldsymbol{w},\boldsymbol{\lambda}}}(x,y)=0\right\}, \qquad \omega(x) = y(x)\,\frac{dx}{x},$$

since the spectral curve Σ considered in [19] is defined as a Lagrangian (3.14) in $\mathbb{C}^2$:

$$\Sigma = \left\{(x,y)\in\mathbb{C}^2\ \middle|\ P(x,y)=\sum_{k=0}^{r}p_k(x)y^{r-k}=0\right\}, \qquad \omega(x)=y(x)dx.$$

The WKB reconstruction of quantum curves in [19] is based only on the global topological recursion (3.12), and what we need are the 1-form $\omega(x)$ and the Bergman kernel on Σ as the inputs. Therefore, changing the presentation of the GKZ curve $\Sigma_{X_{\boldsymbol{l}=\boldsymbol{1};\boldsymbol{w},\boldsymbol{\lambda}}}$ with

$$y = xY, \quad A_{X_{\boldsymbol{l}=\boldsymbol{1};\boldsymbol{w},\boldsymbol{\lambda}}}(x,xY) = P_{X_{\boldsymbol{l}=\boldsymbol{1};\boldsymbol{w},\boldsymbol{\lambda}}}(x,Y),$$

we can utilize the remarkable results in [19]. In the following we will consider a local coordinate z defined by

$$x(z) = \frac{\prod_{i=0}^{N-1}(z-w_i)}{\prod_{a=1}^{n}(z-\lambda_a)}, \qquad Y(x(z)) = \frac{\prod_{a=1}^{n}(z-\lambda_a)}{\prod_{i=0}^{N-1}(z-w_i)}\,z. \tag{4.25}$$

Now we will prove Theorem 1.5 by applying this presentation of the GKZ curve to Proposition 3.13. At first we will show it for the equivariant Gromov-Witten theory of the Fano complete intersection of degree $l_a=1$ ($a=1,\ldots,n<N$) hypersurfaces in $\mathbb{CP}^{N-1}_{\boldsymbol{w}}$.

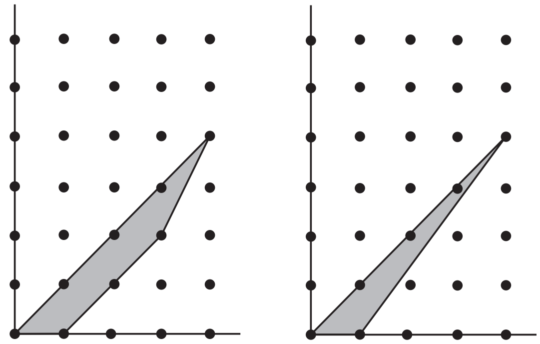


FIGURE 4. The Newton polygons for the defining polynomial $P_X(x,Y)$. Left: X is the complete intersection of two degree 1 hypersurfaces in $\mathbb{C}\mathbf{P}^3_{\boldsymbol{w}}$. Right: $X = \mathbb{C}\mathbf{P}^3_{\boldsymbol{w}}$

Theorem 4.5. *The GKZ equation (1.10) for the equivariant Gromov-Witten theory of the Fano complete intersection of degree $l_a = 1$ $(a = 1, \ldots, n < N)$ hypersurfaces in $\mathbb{C}\boldsymbol{P}^{N-1}_{\boldsymbol{w}}$ is reconstructible as the quantum curve on the GKZ curve $\Sigma_{X_{\boldsymbol{l}=\mathbf{1};\boldsymbol{w},\boldsymbol{\lambda}}}$ by specifying the integration divisor D to be*

$$D = [z] - [\infty],$$

in the local coordinate (4.25). Here by $n < N$ the reference point $z_* = \infty$ corresponds to $x(z_*) = \infty$ in the global coordinate $x = x(z)$.[8]

Proof. Consider the defining polynomial $P_{X_{\boldsymbol{l}=\mathbf{1};\boldsymbol{w},\boldsymbol{\lambda}}}(x,Y) = A_{X_{\boldsymbol{l}=\mathbf{1};\boldsymbol{w},\boldsymbol{\lambda}}}(x,xY)$ of the GKZ curve for the Fano complete intersection $X_{\boldsymbol{l}=\mathbf{1};\boldsymbol{w},\boldsymbol{\lambda}}$ of degree $l_i = 1$ $(i = 1, \ldots, n < N)$ hypersurfaces in $\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}$:

$$P_{X_{\boldsymbol{l}=\mathbf{1};\boldsymbol{w},\boldsymbol{\lambda}}}(x,Y) = \prod_{i=0}^{N-1}(xY - w_i) - x\prod_{a=1}^{n}(xY - \lambda_a) = \sum_{k=0}^{N} p_k(x) Y^{N-k}, \tag{4.26}$$

where

$$p_k(x) = \begin{cases} (-1)^k e_k(\boldsymbol{w}) x^{N-k}, & \text{for } 0 \le k \le N-n-1, \\ (-1)^k e_k(\boldsymbol{w}) x^{N-k} - (-1)^{k-N+n} e_{k-N+n}(\boldsymbol{\lambda}) x^{N-k+1}, & \text{for } N-n \le k \le N. \end{cases}$$

Here $e_k(\boldsymbol{w})$ and $e_k(\boldsymbol{\lambda})$ denote the elementary symmetric polynomials of degree k in $\boldsymbol{w} = \{w_0, \ldots, w_{N-1}\}$ and $\boldsymbol{\lambda} = \{\lambda_1, \ldots, \lambda_n\}$, respectively. Clearly this defining polynomial satisfies the admissibility condition in Definition 3.12 (see Figure 4), and one finds

$$\lfloor \alpha_r \rfloor = r, \qquad r = 0, \ldots, N.$$

For the spectral curve $\Sigma_{X_{\boldsymbol{l}=\mathbf{1};\boldsymbol{w},\boldsymbol{\lambda}}}$ with the defining polynomial (4.26), the differential operators D_k $(k = 1, \ldots, N)$ in (3.15) become

$$D_k = \hbar x \frac{d}{dx}, \tag{4.27}$$

and the coefficients $p_{N-k}(x)/x^{\lfloor \alpha_k \rfloor}$ in the quantum curve (3.16) are given by

$$\frac{p_{N-k}(x)}{x^{\lfloor \alpha_k \rfloor}} = \begin{cases} (-1)^{N-k} e_{N-k}(\boldsymbol{w}), & \text{for } n+1 \le k \le N, \\ (-1)^{N-k} e_{N-k}(\boldsymbol{w}) - (-1)^{n-k} e_{n-k}(\boldsymbol{\lambda}) x, & \text{for } 0 \le k \le n. \end{cases} \tag{4.28}$$

The coefficients C_k in (3.17) for the integration divisor $D = [z] - [\infty]$ are evaluated as follows:

$$C_k = \lim_{z \to \infty} \frac{P_{k+1}(x, Y(x))}{x^{\lfloor \alpha_{N-k} \rfloor + 1}}$$

[8] The convergence in Remark 3.14 is confirmed by looking at the $z \to \infty$ behavior of the building blocks of the topological recursion (3.12) such as $Y(x(z))$ and $\int_{w_*}^{w} B(\cdot, z)$.

$$
\begin{aligned}
&= \lim_{z\to\infty} \left(\frac{P_{k+1}(x,Y)}{x^{N-k+1}} - \frac{P_{X_{\boldsymbol{l}=\mathbf{1};\boldsymbol{w},\boldsymbol{\lambda}}}(x,Y)}{x^{N-k+1}Y^{N-k}} \right) \Bigg|_{Y=Y(x)} \\
&= - \lim_{z\to\infty} \left(\frac{p_k(x)}{x^{N-k+1}} + \frac{p_{k+1}(x)}{x^{N-k+1}Y} + \cdots + \frac{p_N(x)}{x^{N-k+1}Y^{N-k}} \right) \Bigg|_{Y=Y(x)} \\
&= \begin{cases} 0, & \text{for } 1 \le k \le N-n-1, \\ (-1)^{k-N+n} e_{k-N+n}(\boldsymbol{\lambda}), & \text{for } N-n \le k \le N-1, \end{cases}
\end{aligned}
\tag{4.29}
$$

where $Y(x)$ obeys $P_{X_{\boldsymbol{l}=\mathbf{1};\boldsymbol{w},\boldsymbol{\lambda}}}(x, Y(x)) = 0$. To rewrite the differential equation further, use a key identity for $\ell \ge 1$:

$$
x \left(\hbar x \frac{d}{dx} + \hbar \right)^{\ell} f(x) = \left(\hbar x \frac{d}{dx} \right)^{\ell-1} \left(\hbar x^2 \frac{d}{dx} f(x) \right) + \hbar \left(\hbar x \frac{d}{dx} \right)^{\ell-1} \big(x f(x) \big). \tag{4.30}
$$

This identity is proven by induction with respect to ℓ. Adopt the equations (4.27) – (4.30) to the equation (3.16), then one finds

$$
\begin{aligned}
0 &= \sum_{k=0}^{N} (-1)^{N-k} e_{N-k}(\boldsymbol{w}) \left(\hbar x \frac{d}{dx} \right)^{k} \psi_{X_{\boldsymbol{l}=\mathbf{1};\boldsymbol{w},\boldsymbol{\lambda}}}(x) - \sum_{\ell=0}^{n} (-1)^{n-\ell} e_{n-\ell}(\boldsymbol{\lambda}) x \left(\hbar x \frac{d}{dx} + \hbar \right)^{\ell} \psi_{X_{\boldsymbol{l}=\mathbf{1};\boldsymbol{w},\boldsymbol{\lambda}}}(x) \\
&= \prod_{i=0}^{N-1} \left(\hbar x \frac{d}{dx} - w_i \right) \psi_{X_{\boldsymbol{l}=\mathbf{1};\boldsymbol{w},\boldsymbol{\lambda}}}(x) - x \prod_{a=1}^{n} \left(\hbar x \frac{d}{dx} - \lambda_a + \hbar \right) \psi_{X_{\boldsymbol{l}=\mathbf{1};\boldsymbol{w},\boldsymbol{\lambda}}}(x).
\end{aligned}
$$

Thus the GKZ equation (1.8) for the equivariant Gromov-Witten theory of the Fano complete intersection of degree $l_i = 1$ ($i = 1, \ldots, n$) hypersurfaces in $\mathbb{C}\mathbf{P}^{N-1}$ is correctly reconstructed from the GKZ curve (4.26). □

Specialized the number n of hypersurfaces to be zero in the above proof, we immediately find that the GKZ equation for the equivariant $\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}$ theory is also reconstructible.

Theorem 4.6. *The GKZ equation (1.9) for the equivariant Gromov-Witten theory of the projective space $\mathbb{C}\boldsymbol{P}^{N-1}_{\boldsymbol{w}}$ is reconstructible as the quantum curve on the GKZ curve (2.15) by specifying the integration divisor D to be $D = [z] - [\infty]$ in the local coordinate (4.25). Here the reference point $z_* = \infty$ corresponds to $x(z_*) = \infty$ in the global coordinate $x = x(z)$.*

4.3. **Relation to the oscillatory integrals.** As a corollary of results proved in this section, we can find an explicit relation between the oscillatory integral $\mathcal{I}_X$ and the wave function ψ_X for a complete intersection $X = X_{\boldsymbol{l}=\mathbf{1};\boldsymbol{w},\boldsymbol{\lambda}}$ of degree $l_i = 1$ ($i = 1, \ldots, n$) hypersurfaces with $n < N$ in $\mathbb{C}\mathbf{P}^{N-1}$ (we regard $X = \mathbb{C}\mathbf{P}^{N-1}$ for the case $n = 0$).

Recall that, for generic w_i and λ_a, there are N critical points $(\boldsymbol{u}_1^{(c)}, \boldsymbol{v}_1^{(c)}), \ldots, (\boldsymbol{u}_N^{(c)}, \boldsymbol{v}_N^{(c)})$ of W_X (after taking the projection $(\widetilde{\mathbb{C}^*})^{N+n-1} \to (\mathbb{C}^*)^{N+n-1}$) which give N solutions $\mathcal{I}_1, \ldots, \mathcal{I}_N$ of GKZ equation (1.8) as oscillatory integrals over the associated Lefschetz thimbles. On the other hand, Theorem 4.5 shows that the topological recursion constructs N formal solutions $\psi_1, \ldots, \psi_N$ of (1.8) as the wave function with the integration divisors $D_1, \ldots, D_N$ specified as follows. For a fixed $x \in \mathbb{C}^*$ away from branch points, we can find $z_1(x), \ldots, z_N(x) \in \mathbb{C}\mathbf{P}^1$ (or Σ_X) satisfying $x(z_i(x)) = x$ ($i = 1, \ldots, N$), where $x(z)$ is given by (4.25). Then, we define $D_i = [z_i(x)] - [\infty]$ ($i = 1, \ldots, N$).

The correspondence between the critical points and points on the GKZ curve Σ_X is given in Proposition 2.5. More explicitly, we can choose a label so that

$$
x \frac{d}{dx} W_X(\boldsymbol{u}_i^{(c)}, \boldsymbol{v}_i^{(c)}; x) = z_i(x), \quad i = 1, \ldots, N \tag{4.31}
$$

holds. In view of (4.25), for sufficiently large x, we can arrange the label[9] so that

$$z_i(x) \sim \begin{cases} \zeta^i x^{\frac{1}{N-n}} & \text{for } i = 1, \ldots, N-n, \\ \lambda_{a_i} & \text{for } i = N-n+1, \ldots, N, \end{cases} \tag{4.32}$$

when $x \to \infty$. Here $\zeta = \exp(2\pi \mathrm{i}/(N-n))$ and $\{a_{N-n+1}, \ldots, a_N\} = \{1, \ldots, n\}$. This is consistent with Lemma 2.6, where $z_1(x), \ldots, z_{N-n}(x)$ correspond to the critical points satisfying (2.20) while $z_{N-n+1}(x), \ldots, z_N(x)$ correspond to the rest n critical points satisfying (2.21).

The following claim shows that the oscillatory integrals associated with the critical points satisfying (2.20) coincides with the wave functions (up to some numerical factor) after taking the asymptotic expansion:

Corollary 4.7. *Let $\psi_i(x)$ be the wave function (3.13) for $X = X_{\boldsymbol{l}=\boldsymbol{1};\boldsymbol{w},\boldsymbol{\lambda}}$ with the integration divisor $D_i = [z_i(x)] - [\infty]$ where the point $z_i(x)$ is specified as (4.32). Also, let $\mathcal{I}_i$ be the oscillatory integral (2.4) for the mirror Landau Ginzburg potential W_X defined over the Lefschetz thimble associated with a critical point $(\boldsymbol{u}_i^{(\mathrm{c})}, \boldsymbol{v}_i^{(\mathrm{c})})$ specified by (4.31). Then, for $i = 1, \ldots, N-n$, these (formal) solutions of the GKZ equation (1.9) are related through the asymptotic expansion for $\hbar \to 0$:*

$$\mathcal{I}_i(x) \sim C_i \, (-2\pi\hbar)^{\frac{N+n-1}{2}} \, \psi_i(x). \tag{4.33}$$

Here the constant C_i is determined by

$$C_i = \lim_{x\to\infty} \frac{u_{i,1}^{(\mathrm{c})} \cdots u_{i,N-1}^{(\mathrm{c})} \sqrt{\mathrm{Hess}(\boldsymbol{u}_i^{(\mathrm{c})}, \boldsymbol{v}_i^{(\mathrm{c})})}}{\exp(F_1)}, \tag{4.34}$$

where we write $(\boldsymbol{u}_i^{(\mathrm{c})}, \boldsymbol{v}_i^{(\mathrm{c})}) = (u_{i,1}^{(\mathrm{c})}, \ldots, u_{i,N-1}^{(\mathrm{c})}, v_{i,1}^{(\mathrm{c})}, \ldots, v_{i,n}^{(\mathrm{c})})$.

This follows from the fact that the coefficients $S_m(x)$ of WKB expansion are uniquely determined up to an additive constant (cf. (4.17)) and the behavior of (2.24) when $x \to \infty$ which is valid if we choose a critical point satisfying (2.20). In particular, for $X = \mathbb{C}\mathbf{P}_{\boldsymbol{w}}^{N-1}$, the relation (4.33) is valid for all N solutions because $n = 0$ in this case.

5. Several different vantage points of the J-function

In this section we will give a physical derivation of Theorem 1.5 (referred to *reconstruction theorem*) by reinterpreting the equivariant J-functions as the brane partition functions in topological strings on local Calabi-Yau 3-folds. In the subsequent sections we will discuss the following vantage points of the J-function:

1. J-function as the vortex partition function,
2. J-function as the brane partition function in the local A-model,
3. J-function as the brane partition function in the local B-model.

At first we will summarize the physical interpretation of the equivariant J-function as the vortex partition function in the $\mathcal{N} = (2,2)$ gauged linear sigma model (GLSM) on $\mathbb{S}^2$ [36, 16] (Section 5.1). Next we will reconsider it via the geometric engineering as a particular type of vortex partition function obtained from a brane partition function in the topological A-model on a local toric Calabi-Yau 3-fold Y [36] (Section 5.2). And then, we will move to the local B-model picture via the local mirror symmetry, and give yet another description of the brane partition function as the wave function via the topological recursion on a mirror curve residing in the mirror local Calabi-Yau 3-fold $Y^\vee$ on basis of remodeling conjecture [21] (Section 5.3). As a consequence of physical discussions, we will find the reconstruction theorem.

[9]Precisely speaking, these labels are not well-defined since the labels exchange if x move around a ramification point (∞ is a ramification point). Here we consider a situation that x moves along a straight path to infinity.

Remark 5.1. Via the string dualities, we find a novel picture of the J-function and GKZ equation. One of the most curious but interesting aspects of this picture is the following point. Originally the J-function is defined in regard to the genus 0 closed string theory on Fano manifold X, and the variable $x = \mathrm{e}^t$ denotes the *closed string modulus* which measures the area of the closed string worldsheet around the 2-cycle in X. On the other hand, the brane partition function is defined for all genus open string theory on local toric Calabi-Yau 3-fold Y involving a special Lagrangian submanifold L, and the variable $x = \mathrm{e}^u$ denotes the *open string modulus* which measures the area of the open string worldsheet ending on $L \subset Y$.

5.1. **Vantage point 1: J-function as the vortex partition function.** In [36, 16] it was argued that the equivariant J-function is reinterpreted as the vortex partition function in the $\mathcal{N} = (2,2)$ GLSM on $\mathbb{S}^2$ [100]. The GLSM consists of gauge multiplet $\mathcal{V}$ with a gauge group G and matter chiral multiplets Φ_i's with some representations of G. The vacuum moduli space of the GLSM is defined by D- and F-term equations. More precisely, the D-term contains Fayet-Iliopoulos (FI) parameters $\boldsymbol{\xi}$ (and theta-angles $\boldsymbol{\theta}$) associated with the generators of the center of the gauge group G, and the F-term is described by a gauge invariant function $W(\boldsymbol{\Phi})$ of matter multiplets Φ_i's called *superpotential*.

Via the renormalization flow, the GLSM flows into the geometric regime $\xi \gg 0$, and one finds the non-linear sigma model with a target space X defined by D- and F-term equations, if X is the Fano or Calabi-Yau variety. Indeed the FI parameters in the D-term are associated with the Kähler moduli of X, and the solutions of the F-term equation are associated with the complex structure moduli of X. In addition, the $U(1)$ equivariant parameter $\hbar$ on $\mathbb{S}^2$ is introduced, if we consider the A-twisted $\mathcal{N} = (2,2)$ GLSM on the Ω-deformed sphere which has the generator of $\mathbb{S}^1$ acting on $\mathbb{S}^2$ [28] (see also [13]). The Ω-deformation parameter is given by $\hbar$ and the $\mathbb{S}^1$ action has two fixed points at the north and south poles on $\mathbb{S}^2$.

Field	$U(1)$	Twisted mass	$U(1)_V$
Φ_i	$+1$	$-w_i$	0
P_a	$-l_a$	λ_a	2

TABLE 1. Matter content for the complete intersection $X_{\boldsymbol{l}} \subset \mathbb{C}\mathbf{P}^{N-1}$. Here $i = 0, \dots, N-1$ and $a = 1, \dots, n$.

In the following we will focus mainly on the smooth complete intersection $X = X_{\boldsymbol{l}}$, defined by homogeneous degree $l_{a=1,\dots,n}$ polynomial equations $F_a(\boldsymbol{\phi}) = 0$ in $\mathbb{C}\mathbf{P}^{N-1} \ni (\phi_0 : \dots : \phi_{N-1})$ with $l_1 + \cdots + l_n \le N$. The equivariant Gromov-Witten theory of the complete intersection $X_{\boldsymbol{l}}$ corresponds to the $G = U(1)$ GLSM with the matter contents listed in Table 1 and a superpotential $W(P_a, \boldsymbol{\Phi}) = \sum_{a=1}^n P_a F_a(\boldsymbol{\Phi})$. The D-term equation has the FI parameter ξ and the theta-angle θ, and it realizes $\mathbb{C}\mathbf{P}^{N-1}$ as the moduli space. On the other hand, twisted masses $-w_i$ (resp. λ_a) for the matter multiplets Φ_i (resp. P_a) are identified with the equivariant parameters of the Gromov-Witten theory.

The GLSM also has the vector $U(1)_V$ R-symmetry and the superpotential needs to have R-charge 2. The R-charge of matter multiplets Φ_i (resp. P_a) are assigned to be 0 (resp. 2). In [28] (see also [13]) it is found that the A-twisted correlator for a function (operator) $\mathcal{O}^{(\mathrm{N})}(\sigma)$ (resp. $\mathcal{O}^{(\mathrm{S})}(\sigma)$) of the complex scalar field σ in the $U(1)$ gauge multiplet inserted at the north (resp. south) pole of $\mathbb{S}^2$, is given exactly by

$$\begin{aligned}&\left\langle \mathcal{O}^{(\mathrm{N})}(\sigma)\mathcal{O}^{(\mathrm{S})}(\sigma)\right\rangle \\ &= \sum_{d=0}^{\infty} x^d \oint_\gamma \frac{d\sigma}{2\pi i} \frac{\prod_{a=1}^n \prod_{m=0}^{l_a d} \left(l_a\sigma - \lambda_a - \frac{l_a d}{2}\hbar + m\hbar\right)}{\prod_{i=0}^{N-1}\prod_{m=0}^{d}\left(\sigma - w_i - \frac{d}{2}\hbar + m\hbar\right)}\, \mathcal{O}^{(\mathrm{N})}\big(\sigma - \frac{d}{2}\hbar\big)\mathcal{O}^{(\mathrm{S})}\big(\sigma + \frac{d}{2}\hbar\big),\end{aligned} \tag{5.1}$$

where the contour $\gamma \subset \mathbb{C}$ encloses the poles $\sigma = w_i - \frac{d}{2}\hbar + p\hbar$ ($i = 0, \ldots, N-1$, $p = 0, \ldots, d$) of the integrand. Here

$$x = \mathrm{e}^{-2\pi\xi + \mathrm{i}\theta},$$

and x must be replaced with $\mu^{N-(l_1+\cdots+l_n)}x$ which is modified by the RG invariant energy scale μ for the Fano ($l_1 + \cdots + l_n < N$) case, because the FI parameter ξ runs under the renormalization group (RG) flow. In the following we will use the same symbol x for the modified one. Actually it is found that the correlator (5.1) is factorized as [98]:

$$\left\langle \mathcal{O}^{(\mathrm{N})}(\sigma)\mathcal{O}^{(\mathrm{S})}(\sigma) \right\rangle = \sum_{i=0}^{N-1} \oint_{\sigma=w_i} \frac{d\sigma}{2\pi i}\, Z_{\text{1-loop}}(\sigma; \boldsymbol{w}, \boldsymbol{\lambda}) Z^{(\mathrm{N})}_{\mathrm{vortex}}(\sigma; x, \boldsymbol{w}, \boldsymbol{\lambda}, \hbar) Z^{(\mathrm{S})}_{\mathrm{vortex}}(\sigma; x, \boldsymbol{w}, \boldsymbol{\lambda}, -\hbar),$$

where

$$Z_{\text{1-loop}}(\sigma; \boldsymbol{w}, \boldsymbol{\lambda}) = \frac{\prod_{a=1}^{n}(l_a\sigma - \lambda_a)}{\prod_{i=0}^{N-1}(\sigma - w_i)},$$

$$Z^{(\mathrm{N,S})}_{\mathrm{vortex}}(\sigma; x, \boldsymbol{w}, \boldsymbol{\lambda}, \hbar) = \sum_{d=0}^{\infty} x^d \frac{\prod_{a=1}^{n}\prod_{m=1}^{l_a d}(l_a\sigma - \lambda_a + m\hbar)}{\prod_{i=0}^{N-1}\prod_{m=1}^{d}(\sigma - w_i + m\hbar)}\, \mathcal{O}^{(\mathrm{N,S})}(\sigma - d\hbar).$$

In this factorization the factor $Z^{(\mathrm{N})}_{\mathrm{vortex}}$ (resp. $Z^{(\mathrm{S})}_{\mathrm{vortex}}$) is interpreted as "off-shell" vortex partition function with the operator $\mathcal{O}^{(\mathrm{N})}(\sigma)$ (resp. $\mathcal{O}^{(\mathrm{S})}(\sigma)$) at the north (resp. south) pole of $\mathbb{S}^2$. Excluded the operators $\mathcal{O}^{(\mathrm{N,S})}(\sigma)$, this vortex partition function agrees with the I-function for the complete intersection $X_{\boldsymbol{l}}$. In particular for the Fano ($l_1 + \cdots + l_n < N$) case this agrees with the J-function. By taking the residue at $\sigma = w_0$ ($\sigma = w_i$ in general) in the above correlator, we obtain the ("on-shell") vortex partition function [93, 36, 101, 17]:

$$Z^{X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}}_{\mathrm{vortex}}(x) = \sum_{d=0}^{\infty} \frac{x^d}{d!\hbar^d} \frac{\prod_{a=1}^{n}\prod_{m=1}^{l_a d}(l_a w_0 - \lambda_a + m\hbar)}{\prod_{i=1}^{N-1}\prod_{m=1}^{d}(w_0 - w_i + m\hbar)}. \tag{5.2}$$

This ("on-shell") vortex partition function agrees with the ("on-shell") equivariant J-function introduced in Appendix A, and obeys the GKZ equation (see Lemma A.2):

$$\widehat{A}_{X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}}(\widehat{x}, \widehat{y})\big(x^{w_0/\hbar} Z^{X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}}_{\mathrm{vortex}}(x)\big) = 0. \tag{5.3}$$

Remark 5.2. In [11, 38] the $\mathcal{N} = (2,2)$ GLSM partition function on $\mathbb{S}^2$ is computed exactly, and the factorization into the vortex partition functions is shown. In [16] under the identification of the inverse radius r^{-1} of $\mathbb{S}^2$ with the equivariant parameter $\hbar$, this vortex partition function is reinterpreted as the $I(J)$-function. In [65] it is shown that the GLSM partition on $\mathbb{S}^2$ is also factorized into two hemisphere partition functions and one annulus partition function, where the hemisphere partition function is shown to give the D-brane central charge (see also [94, 62]).

Remark 5.3. In [13] the twisted partition function of the 3d $\mathcal{N} = 2$ gauge theory on $\mathbb{S}^2 \times \mathbb{S}^1$ with the Ω-deformation is exactly computed, and shown to be factorized into the K-theoretic vortex partition functions (see also [12, 48]). In [13] it is also discussed the factorization into the elliptic vortex partition functions of the twisted partition function of the 4d $\mathcal{N} = 1$ gauge theory on $\mathbb{S}^2 \times T^2$ with the Ω-deformation (see also [102, 89]).

5.2. **Vantage point 2: J-function as the brane partition function in the local A-model.** Here we will consider the 4d $\mathcal{N} = 2$ gauge theory on Ω-deformed $\mathbb{R}^4 \cong \mathbb{C}^2 \ni (z_1, z_2)$ with the gauge group $G = SU(N)$. In this gauge theory we can put a half-BPS surface operator [57] as the codimension 2 defect along the z_1-plane D at $z_2 = 0$. For the equation of motion of the gauge theory with a surface operator, the solitonic solutions which are the composite of 4d instantons and 2d vortices can be found, and they are called *ramified instantons* [81, 82]. The moduli space for the ramified instantons is characterized by the flag manifold $G/\mathbb{L}$ where $\mathbb{L} = S\left[U(n_1) \times U(n_2) \times \cdots \times U(n_M)\right]$ with $N = n_1 + n_2 + \cdots + n_M$ denotes the Levi subgroup of $G = SU(N)$ [25, 74] (see also [23, 24, 6, 80]).

The generating function of the number of ramified instantons is called the *ramified instanton partition function*. If we take a (decoupling) limit for the instanton counting parameter in the ramified instanton partition function and suppress the counting of the 4d instantons (i.e. focus only on the ramified instantons with the instanton number zero), the generating function reduces to the vortex partition function in an $\mathcal{N}=(2,2)$ GLSM described by the map:

$$\overline{D}=\mathbb{C}\mathbf{P}^1 \xrightarrow{\boldsymbol{d}} G/\mathbb{L}, \qquad \boldsymbol{d}\in H_2(G/\mathbb{L},\mathbb{Z}), \tag{5.4}$$

where $\overline{D}$ denotes a one-point compactification of the z_1-plane D.

In particular, the surface operator is referred to *simple type*, if it has the Levi subgroup $\mathbb{L}=U(1)\times SU(N-1)$. Here we will consider the surface operator of the simple type and $D=\mathbb{R}^2$ (i.e. $\overline{D}=\mathbb{S}^2$). In this case we find $G/\mathbb{L}\cong\mathbb{C}\mathbf{P}^{N-1}$.

If the 4d gauge theory does not involve any matter fields (i.e. pure Yang-Mills theory), the resulting GLSM on $\overline{D}=\mathbb{S}^2$ consists of a $U(1)$ vector multiplet and the matter multiplets Φ_i's listed in Table 1, and the superpotential is absent in this GLSM. By taking the decoupling limit of the ramified instanton partition function, we obtain the ("on-shell") vortex partition function $Z_{\text{vortex}}^{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}$ for this GLSM, and it is given by the specialization $n=0$ of (5.2) because multiplets P_a's are absent in this case:

$$Z_{\text{vortex}}^{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(x)=\sum_{d=0}^{\infty}\frac{x^d}{d!\hbar^d}\frac{1}{\prod_{i=1}^{N-1}\prod_{m=1}^{d}(w_0-w_i+m\hbar)}. \tag{5.5}$$

If the 4d $U(N)$ gauge theory involves n ($n\le N$) matter hypermultiplets in the fundamental representation, we will find the same GLSM that we have considered in the vantage point 1. The decoupling limit of the ramified instanton partition function agrees with the ("on-shell") vortex partition function $Z_{\text{vortex}}^{X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}}(x)$ in (5.2).

Subsequently we will survey on the punchline of the geometric engineering which realizes the ("on-shell") vortex partition function as the brane partition function in the open topological A-model. And then we will see how the GKZ equation appears in the open topological A-model on the strip geometry.

5.2.1. *Geometric engineering of the ("on-shell") equivariant J-function for $\mathbb{C}\boldsymbol{P}^{N-1}_{\boldsymbol{w}}$.* Our starting point is the open topological A-model on the local toric Calabi-Yau 3-fold $Y_{\boldsymbol{l}}$ which is specified by charge vectors $\boldsymbol{l}_i\in\mathbb{Z}^{m+3}$ ($i=1,\ldots,m$). The local toric Calabi-Yau 3-fold $Y_{\boldsymbol{l}}$ is the quotient such that

$$Y_{\boldsymbol{l}}=\left\{\ (X_1,\ldots,X_{m+3})\in\mathbb{C}^{m+3}\ \middle|\ \sum_{\alpha=1}^{m+3}l_{i,\alpha}|X_\alpha|^2=\mathrm{Re}\,(\log Q_i)\ \right\}/U(1)^m, \tag{5.6}$$

where $U(1)$ charge vectors $\boldsymbol{l}_i=(l_{i,1},\ldots,l_{i,m+3})$, ($i=1,2,\ldots,m$) obey the Calabi-Yau condition $\sum_{\alpha=1}^{m+3}l_{i,\alpha}=0$. Here Q_i's denote $Q_i=\mathrm{e}^{t_i}$ with the Kähler parameters t_i, and $U(1)^m$ acts on X_α as $X_\alpha\to\mathrm{e}^{\mathrm{i}\sum_{i=1}^{m}\epsilon_i l_{i,\alpha}}X_\alpha$.

If $Y_{\boldsymbol{l}}$ is chosen to be the A_{N-1}-fibration over $\mathbb{C}\mathbf{P}^1$, the physical spectra (i.e. vector multiplets, hypermultiplets, etc.) of the 4d $U(N)$ gauge theory are realized from the topological A-model in the string theoretical way [75, 76]. Such a realization of the 4d gauge theory is known as the *geometric engineering*.

In the framework of the geometric engineering, the surface operator of the simple type in the 4d gauge theory is realized from the topological A-model on $Y_{\boldsymbol{l}}$ by introducing the 3d object of the topological A-model referred to *toric brane*, which wraps around the special Lagrangian submanifold $L\in Y_{\boldsymbol{l}}$ [36]. In a local atlas of $Y_{\boldsymbol{l}}$ which covers $X_\alpha=X_\beta=X_\gamma=0$, the special Lagrangian submanifold $L\cong\mathbb{C}\times\mathbb{S}^1$ is found as the following locus (see Theorem 3.1 in [59]):

$$|X_\alpha|^2-|X_\gamma|^2=\mathrm{Re}\,(\log\mathsf{x}), \quad |X_\beta|^2-|X_\gamma|^2=0, \quad \mathrm{Im}(X_\alpha X_\beta X_\gamma)=0, \quad \mathrm{Re}(X_\alpha X_\beta X_\gamma)\ge 0, \tag{5.7}$$

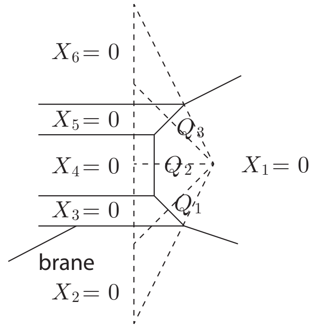


FIGURE 5. Strip geometry Y_3 of three $(-2, 0)$ curves. The dashed line and solid line describe the toric diagram and the dual web diagram, respectively. In this diagram a toric brane is inserted at $X_2 = 0$, and this gives a Lagrangian submanifold (5.7) with $\alpha = 1, \beta = 3, \gamma = 2$.

where $\mathsf{x} = \mathrm{e}^u$ denotes a open string modulus of the toric brane. The toric brane is represented by a ray attached on one of the lines in the web diagram of the toric variety [1]. In Figure 5 a toric brane insertion at the lowest leg in the web diagram is depicted.

On basis of the string theoretical discussions[10], the brane partition $Z^{Y_{\boldsymbol{l}}}_{\text{A-brane}}(\mathsf{x})$ is defined as the generating function for the number of the holomorphic embedding maps of the open Riemann surface (referred to the world-sheet) which ends on the toric brane $L \in Y_{\boldsymbol{l}}$ [88]. For the case that $Y_{\boldsymbol{l}}$ is the A_{N-1}-fibration over $\mathbb{C}\mathbf{P}^1$, the brane partition function is computed by various physical techniques such as the topological vertex [4], the open BPS state counting [88] and the open BPS wall-crossing [5]. From various observations (see e.g. [10]), it is proposed that the brane partition function $Z^{Y_{\boldsymbol{l}}}_{\text{A-brane}}(\mathsf{x})$ agrees with the K-theoretic generalization of the ramified instanton partition function.

Here we will consider the decoupling limit of the 4d instantons at the level of the toric geometry. Using the dictionary of the geometric engineering, we find that the decoupling limit of the 4d instantons corresponds to the large volume limit of the base $\mathbb{C}\mathbf{P}^1$ in A_{N-1}-fibration over $\mathbb{C}\mathbf{P}^1$. After taking this large volume limit, $Y_{\boldsymbol{l}}$ reduces to a local toric Calabi-Yau 3-fold Y_{N-1} which consists of the $N-1$ copies of the local Calabi-Yau 3-fold: $\mathcal{O}(-2) \oplus \mathcal{O}(0) \to \mathbb{C}\mathbf{P}^1$ (i.e. $(-2, 0)$ curve). More precisely Y_{N-1} is defined by $N-1$ charge vectors $\boldsymbol{l}_{i=1,\ldots,N-1}$:

$$\begin{aligned}
\boldsymbol{l}_1 &= (0, 1, -2, 1, 0, 0, 0, \ldots, 0, 0, 0), \\
\boldsymbol{l}_2 &= (0, 0, 1, -2, 1, 0, 0, \ldots, 0, 0, 0), \\
\boldsymbol{l}_3 &= (0, 0, 0, 1, -2, 1, 0, \ldots, 0, 0, 0), \\
&\vdots \\
\boldsymbol{l}_{N-1} &= (0, 0, 0, 0, 0, 0, 0, \ldots, 1, -2, 1).
\end{aligned} \tag{5.8}$$

Such a local toric Calabi-Yau 3-fold Y_{N-1} is known as the *strip geometry* [67] of the $(-2, 0)$ curves, and the web diagram of Y_3 is depicted in Figure 5.

Now we will see the brane partition function $Z^{Y_{N-1}}_{\text{A-brane}}(\mathsf{x})$ for the strip geometry Y_{N-1} of the $(-2, 0)$ curves. As a result of topological vertex computations [67, 36], we obtain the manifest

[10]There are no mathematically rigorous definition for the brane partition function as the generating function of the open Gromov-Witten invariants in general. But switched to the type IIA superstring picture, we can find it as the enumeration of degeneracies of the open BPS states [88] which arise from D0-D2-D4 brane bound states for the case of local toric Calabi-Yau 3-fold. In this sense we have only the string theoretical definition of the brane partition function.

form of the brane partition function which is normalized s.t. $Z^{Y_{N-1}}_{\text{A-brane}}(\mathsf{x}=0)=1$:

$$Z^{Y_{N-1}}_{\text{A-brane}}(\mathsf{x})=\sum_{d=0}^{\infty}\frac{1}{\prod_{i=0}^{N-1}\prod_{m=1}^{d}(1-\widetilde{Q}_i q^m)}\,(q^{1/2}\mathsf{x})^d, \tag{5.9}$$

up to the framing ambiguity (see Remark 5.4). Here

$$\widetilde{Q}_0=1,\qquad \widetilde{Q}_i=q^{-1}\prod_{1\le j\le i}Q_j,\qquad q=\mathrm{e}^{-g_s},$$

and g_s is the topological string coupling constant.

By construction, the brane partition function $Z^{Y_{N-1}}_{\text{A-brane}}(\mathsf{x})$ should agree with the K-theoretic version of the vortex partition function. To obtain the vortex partition function, we need to take a cohomological limit. For this purpose, we will reparameterize parameters in $Z^{Y_{N-1}}_{\text{A-brane}}(\mathsf{x})$:

$$g_s=\beta\hbar,\qquad \mathsf{x}=\beta^N x,\qquad \widetilde{Q}_i=\mathrm{e}^{-\beta(w_0-w_i)}.$$

After taking the cohomological limit $\beta\to 0$ we find that the brane partition function (5.9) reduces to (5.5) [36, 17]:

$$Z^{Y_{N-1}}_{\text{A-brane}}(\mathsf{x})\quad\xrightarrow{\beta\to 0}\quad Z^{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}_{\text{vortex}}(x).$$

Thus we see that the vortex partition function $Z^{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}_{\text{vortex}}(x)$ is found from the open topological A-model on the strip geometry. Combing with the consequences in the vantage point 1, we find yet another realization of the ("on-shell") equivariant J-function for $\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}$ as the brane partition function in the topological A-model on the local toric Calabi-Yau 3-fold Y_{N-1} defined by the charge vectors (5.8) [36].

Remark 5.4. In the computation (5.9), there is a framing ambiguity $f\in\mathbb{Z}$ of the brane at infinity as

$$\mathsf{x}^d\quad\longrightarrow\quad(-1)^{fd}q^{fd(d-1)/2}\,\mathsf{x}^d.$$

But this ambiguity becomes irrelevant under the cohomological limit $\beta\to 0$.

5.2.2. *Geometric engineering of the ("on-shell") equivariant J-functions for degree* 1 *complete intersections in* $\mathbb{C}\boldsymbol{P}^{N-1}_{\boldsymbol{w}}$. Next we will consider the geometric engineering of the 4d $SU(N)$ gauge theory with n ($n\le N$) matter hypermultiplets in the fundamental representation. For this purpose we choose the local toric Calabi-Yau 3-fold $Y_{\boldsymbol{l}}$ to be the A_{N-1}-fibration over $\mathbb{C}\mathbf{P}^1$ with blow-ups at n points [75, 76].

For a particular case $n=N$, after taking the large volume limit of the base $\mathbb{C}\mathbf{P}^1$ (which corresponds to the decoupling limit of the 4d instantons), $Y_{\boldsymbol{l}}$ reduces to the strip geometry $Y_{N-1,N}$ which consists of $2N-1$ copies of the local Calabi-Yau $\mathcal{O}(-1)\oplus\mathcal{O}(-1)\to\mathbb{C}\mathbf{P}^1$ ($(-1,-1)$ curves). Such a strip geometry $Y_{N-1,N}$ is defined by $2N-1$ charge vectors $\boldsymbol{l}_{\lambda,i=1,\dots,N}$ and $\boldsymbol{l}_{w,i=1,\dots,N-1}$ [67]:

$$\begin{aligned}
\boldsymbol{l}_{\lambda,1}&=(1,-1,-1,1,0,0,0,\dots,0,0,0,0),\\
\boldsymbol{l}_{w,1}&=(0,1,-1,-1,1,0,0,\dots,0,0,0,0),\\
\boldsymbol{l}_{\lambda,2}&=(0,0,1,-1,-1,1,0,\dots,0,0,0,0),\\
&\ \ \vdots\\
\boldsymbol{l}_{w,N-1}&=(0,0,0,0,0,0,0,\dots,-1,-1,1,0),\\
\boldsymbol{l}_{\lambda,N}&=(0,0,0,0,0,0,0,\dots,1,-1,-1,1).
\end{aligned} \tag{5.10}$$

According to (5.6), the Kähler moduli parameters $Q_{\lambda,i}=\mathrm{e}^{t_{\lambda,i}}$ and $Q_{w,i}=\mathrm{e}^{t_{w,i}}$ are associated with the charge vectors $\boldsymbol{l}_{\lambda,i}$ and $\boldsymbol{l}_{w,i}$, respectively.

Now we will introduce a toric brane which warps around the Lagrangian submanifold (5.7) in $Y_{N-1,N}$. In Figure 6, a toric brane wrapping around the Lagrangian submanifold with $\alpha=2,\beta=$

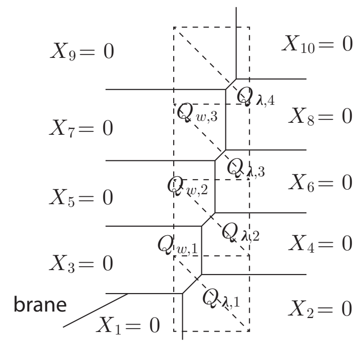


FIGURE 6. Strip geometry $Y_{3,4}$ consisting of seven $(-1,-1)$ curves ($N = 4$). The dashed line and solid line describe the toric diagram and the dual web diagram, respectively. In the diagram a toric brane is inserted at $X_1 = 0$, and this gives a Lagrangian submanifold (5.7) with $\alpha = 2, \beta = 3, \gamma = 1$.

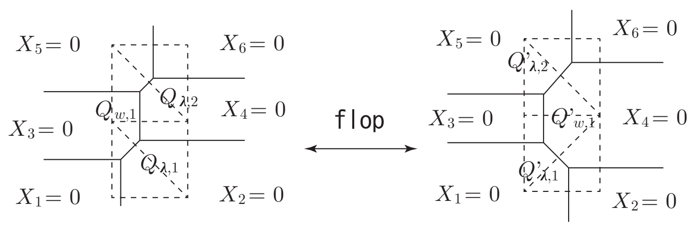


FIGURE 7. A flop of strip consisting of three $(-1,-1)$ curves gives a strip made of one $(-2,0)$ curve and two $(-1,-1)$ curves.

$3, \gamma = 1$ is depicted as an insertion in the lowest leg in the web diagram. For this geometric set-up, the brane partition function $Z^{Y_{N-1,N}}_{\text{A-brane}}(\mathsf{x})$ is computed in [67, 36]:

$$Z^{Y_{N-1,N}}_{\text{A-brane}}(\mathsf{x}) = \sum_{d=0}^{\infty} \frac{\prod_{i=1}^{N} \prod_{m=1}^{d} (1 - \widetilde{Q}_{\lambda,i} q^{m-1})}{\prod_{i=0}^{N-1} \prod_{m=1}^{d} (1 - \widetilde{Q}_{w,i} q^{m})} \, (q^{1/2}\mathsf{x})^d, \tag{5.11}$$

up to the framing ambiguity, where

$$\widetilde{Q}_{w,0} = 1, \qquad \widetilde{Q}_{w,i} = q^{-1} \prod_{1 \le j \le i} Q_{\lambda,j} Q_{w,j}, \qquad \widetilde{Q}_{\lambda,i} = Q_{\lambda,1} \prod_{1 \le j \le i-1} Q_{w,j} Q_{\lambda,j+1}, \qquad q = \mathrm{e}^{-g_s}.$$

Since the brane partition function on the strip geometry realizes the K-theoretic version of the vortex partition function, we adopt the following reparametrizations:

$$g_s = \beta\hbar, \qquad \mathsf{x} = x, \qquad \widetilde{Q}_{w,i} = \mathrm{e}^{-\beta(w_0 - w_i)}, \qquad \widetilde{Q}_{\lambda,i} = \mathrm{e}^{-\beta(w_0 - \lambda_i + \hbar)},$$

and take the cohomological limit. In $\beta \to 0$ we find that the brane partition function (5.11) reduces to the vortex partition function (5.2) for the GLSM on $\mathbb{S}^2$ with $n = N$:

$$Z^{Y_{N-1,N}}_{\text{A-brane}}(\mathsf{x}) \quad \overset{\beta \to 0}{\longrightarrow} \quad Z^{X_{\boldsymbol{l}=\boldsymbol{1};\boldsymbol{w},\{\lambda_1,\ldots,\lambda_N\}}}_{\text{vortex}}(x).$$

Thus we also find the realization of the vortex partition function $Z^{X_{\boldsymbol{l}=\boldsymbol{1};\boldsymbol{w},\{\lambda_1,\ldots,\lambda_N\}}}_{\text{vortex}}(x)$ from the open topological A-model on the strip geometry $Y_{N-1,N}$

For the case $n < N$ we obtain the the strip geometry $Y_{N-1,n}$ by taking the large volume limit of the base $\mathbb{C}\mathbf{P}^1$ in $Y_{\boldsymbol{l}}$. Such a strip geometry $Y_{N-1,n}$ is made of $N-1$ copies of $(-2,0)$ and n copies of $(-1,-1)$ curves, and such local toric Calabi-Yau 3-fold is found by acting "flops" and "decouplings" repeatedly to the strip geometry $Y_{N-1,N}$. To see the actions of "flops" and "decouplings" manifestly, we will focus on the $N=2$ case:

$$\begin{aligned}\boldsymbol{l}_{\lambda,1} &= (1,-1,-1,1,0,0),\\ \boldsymbol{l}_{w,1} &= (0,1,-1,-1,1,0),\\ \boldsymbol{l}_{\lambda,2} &= (0,0,1,-1,-1,1).\end{aligned} \tag{5.12}$$

The Kähler moduli parameters $Q_{\lambda,1}$, $Q_{w,1}$, and $Q_{\lambda,2}$ are associated with the charge vectors $\boldsymbol{l}_{\lambda,1}$, $\boldsymbol{l}_{w,1}$, and $\boldsymbol{l}_{\lambda,2}$, respectively. By the flop transitions described in Figure 7, we obtain a strip geometry made of one $(-2,0)$ curve and two $(-1,-1)$ curves, and it is given by the charge vectors

$$\begin{aligned}\boldsymbol{l}'_{\lambda,1} &= (-1,1,1,-1,0,0) = -\boldsymbol{l}_{\lambda,1},\\ \boldsymbol{l}'_{w,1} &= (1,0,-2,0,1,0) = \boldsymbol{l}_{\lambda,1}+\boldsymbol{l}_{w,1},\\ \boldsymbol{l}'_{\lambda,2} &= (0,0,1,-1,-1,1) = \boldsymbol{l}_{\lambda,2}.\end{aligned} \tag{5.13}$$

The Kähler moduli parameters $Q'_{\lambda,1}$, $Q'_{w,1}$, and $Q'_{\lambda,2}$ are associated with the charge vectors $\boldsymbol{l}'_{\lambda,1}$, $\boldsymbol{l}'_{w,1}$, and $\boldsymbol{l}'_{\lambda,2}$, respectively, and they are related with $Q_{\lambda,1}$, $Q_{w,1}$, and $Q_{\lambda,2}$ by

$$(Q'_{\lambda,1}, Q'_{w,1}, Q'_{\lambda,2}) = (Q_{\lambda,1}^{-1}, Q_{\lambda,1}Q_{w,1}, Q_{\lambda,2}). \tag{5.14}$$

After taking the decoupling limit $Q'_{\lambda,1}\to 0$ or $Q'_{\lambda,2}\to 0$, we obtain the strip geometry made of one $(-2,0)$ curve and one $(-1,-1)$ curve.

Similarly for general N one can find a strip geometry $Y_{N-1,n}$ with $n<N$. As a consequence, the brane partition function for a toric brane in $Y_{N-1,n}$ takes the form [67, 36]:

$$Z_{\text{A-brane}}^{Y_{N-1,n}}(\mathsf{x}) = \sum_{d=0}^{\infty} \frac{\prod_{a=1}^{n}\prod_{m=1}^{d}(1-\widetilde{Q}_{\lambda,a}q^{m-1})}{\prod_{i=0}^{N-1}\prod_{m=1}^{d}(1-\widetilde{Q}_{w,i}q^{m})}\,(q^{1/2}\mathsf{x})^d, \qquad \widetilde{Q}_{w,0}=1. \tag{5.15}$$

up to the framing ambiguity and the normalization which is independent of the open string moduli. After taking the cohomological limit $\beta\to 0$ under the reparametrizations:

$$g_s=\beta\hbar, \qquad \mathsf{x}=\beta^{N-n}x, \qquad \widetilde{Q}_{w,i}=\mathrm{e}^{-\beta(w_0-w_i)}, \qquad \widetilde{Q}_{\lambda,a}=\mathrm{e}^{-\beta(w_0-\lambda_a+\hbar)},$$

the brane partition function (5.15) reduces to the vortex partition function $Z_{\text{vortex}}^{X_{\boldsymbol{l}=\boldsymbol{1};\boldsymbol{w},\boldsymbol{\lambda}}}$ in (5.2) [36, 17]:

$$Z_{\text{A-brane}}^{Y_{N-1,n}}(\mathsf{x}) \quad \stackrel{\beta\to 0}{\longrightarrow} \quad Z_{\text{vortex}}^{X_{\boldsymbol{l}=\boldsymbol{1};\boldsymbol{w},\boldsymbol{\lambda}}}(x).$$

Combining this result with the vantage point 1 again for $n<N$, this shows a realization of the ("on-shell") equivariant J-function for the Fano complete intersection of degree $l_{a=1,\dots,n}=1$ hypersurfaces in $\mathbb{C}\mathbf{P}^{N-1}$ as the brane partition function in the topological A-model on the strip geometry [36].

5.2.3. *q-difference equation for the brane partition function.* The brane partition function obeys a q-difference equation known as the Schrödinger equation or the quantum curve[11] [35, 34]. For the brane partition function $Z_{\text{A-brane}}^{Y_{N-1,n}}(\mathsf{x})$ in (5.15), one finds the q-difference equation (c.f. Remark 3.5)

$$\widehat{A}_Y^K(\widehat{\mathsf{x}},\widehat{\mathsf{y}})Z_{\text{A-brane}}^{Y_{N-1,n}}(\mathsf{x}) = \left[\prod_{i=0}^{N-1}\left(1-\widetilde{Q}_{w,i}\,\widehat{\mathsf{y}}\right) - \widehat{\mathsf{x}}\prod_{a=1}^{n}\left(1-\widetilde{Q}_{\lambda,a}\,\widehat{\mathsf{y}}\right)\right] Z_{\text{A-brane}}^{Y_{N-1,n}}(\mathsf{x}) = 0, \tag{5.16}$$

where the operators $\widehat{\mathsf{x}}$ and $\widehat{\mathsf{y}}$ obey the relation $\widehat{\mathsf{y}}\widehat{\mathsf{x}} = q\,\widehat{\mathsf{x}}\widehat{\mathsf{y}}$, and act on the brane partition function as

$$\widehat{\mathsf{x}}\,Z_{\text{A-brane}}^{Y_{N-1,n}}(\mathsf{x}) = \mathsf{x}\,Z_{\text{A-brane}}^{Y_{N-1,n}}(\mathsf{x}), \qquad \widehat{\mathsf{y}}\,Z_{\text{A-brane}}^{Y_{N-1,n}}(\mathsf{x}) = Z_{\text{A-brane}}^{Y_{N-1,n}}(q\mathsf{x}).$$

Let us consider the cohomological limit of this q-difference equation. For this purpose we will replace the operators $\widehat{\mathsf{x}}$ and $\widehat{\mathsf{y}}$ by $\widehat{\mathsf{x}}=\beta^{N-n}x$ and $\widehat{\mathsf{y}}=\mathrm{e}^{-\beta\hbar x\frac{d}{dx}}$ respectively, and reparameterize

[11] This q-difference equation is defined simply as the annihilating equation of the brane partition function. Via the mirror symmetry (see discussions in Section 5.3), the brane partition function is regarded as the wave function [3], and in this sense, we can identify this q-difference equation as the quantum curve. (See Remark 3.5.)

$\widetilde{Q}_{w,i}$ and $\widetilde{Q}_{\lambda,a}$ as $\widetilde{Q}_{w,i} = \mathrm{e}^{\beta \widetilde{w}_i}$ and $\widetilde{Q}_{\lambda,a} = \mathrm{e}^{\beta(\widetilde{\lambda}_a - \hbar)}$. Then taking the cohomological limit $\beta \to 0$ for the quantum curve (5.16), we obtain a differential equation:

$$\left[\left(\hbar x \frac{d}{dx}\right) \prod_{i=1}^{N-1} \left(\hbar x \frac{d}{dx} - \widetilde{w}_i\right) - x \prod_{a=1}^{n} \left(\hbar x \frac{d}{dx} - \widetilde{\lambda}_a + \hbar\right)\right] Z^{\mathrm{coh}}_{\text{A-brane}}(x) = 0. \tag{5.17}$$

Using this differential equation, we find that $\mathrm{e}^{w_0/\hbar} Z^{\mathrm{coh}}_{\text{A-brane}}(x)$ obeys the GKZ equation (1.10) for the equivariant J-function.

The classical limit of this q-difference equation is found from the WKB expansion of the brane partition function:

$$Z^{Y_{N-1,n}}_{\text{A-brane}}(\mathsf{x}) \sim \exp\left(\sum_{m=0}^{\infty} g_s^{m-1} F^K_m(\mathsf{x})\right).$$

Denoting $\log \mathsf{y} = \mathsf{x}\partial_{\mathsf{x}} F^K_0(\mathsf{x})$, we find that the q-difference equation (5.16) for the brane partition function reduces to the defining equation of a classical curve $\Sigma^K_{Y_{N-1,n}}$:

$$\Sigma^K_{Y_{N-1,n}} = \left\{ (\mathsf{x}, \mathsf{y}) \in \mathbb{C}^* \times \mathbb{C}^* \;\middle|\; A^K_Y(\mathsf{x}, \mathsf{y}) = \prod_{i=0}^{N-1} \left(1 - \widetilde{Q}_{w,i}\, \mathsf{y}\right) - \mathsf{x} \prod_{a=1}^{n} \left(1 - \widetilde{Q}_{\lambda,a}\, \mathsf{y}\right) = 0 \right\}, \tag{5.18}$$

under the classical limit $g_s \to 0$. In the next subsection we will see that this classical curve agrees with the mirror curve in the mirror B-model picture.

5.3. **Vantage point 3: J-function as the brane partition function in the local B-model.** Via the local mirror symmetry, we will study the brane partition function $Z^Y_{\text{A-brane}}(x)$ in the topological A-model in terms of the topological B-model on the mirror local Calabi-Yau 3-fold $Y^\vee_{\boldsymbol{l}}$. The mirror local Calabi-Yau 3-fold $Y^\vee_{\boldsymbol{l}}$ corresponding to the local toric Calabi-Yau 3-fold $Y_{\boldsymbol{l}}$ of (5.6) is defined with a substitution by $|x_\alpha| = \mathrm{e}^{|X_\alpha|^2}$ [66, 63]:

$$Y^\vee_{\boldsymbol{l}} = \left\{ (\omega_+, \omega_-, x_1, \ldots, x_{m+3}) \in \mathbb{C}^2 \times (\mathbb{C}^*)^{m+3} \;\middle|\; \omega_+\omega_- = \sum_{\alpha=1}^{m+3} x_\alpha,\ \prod_{\alpha=1}^{m+3} x_\alpha^{l_{i,\alpha}} = z_i \right\}, \tag{5.19}$$

where z_i's parametrize the complex moduli space. We eliminate local coordinates X_δ ($\delta = 1, \ldots, m+3$) in the defining equation of (5.19) except for $\delta = \alpha, \beta, \gamma$ and fix $X_\gamma = 0$. By choosing local coordinates in such a way, the Lagrangian submanifold (5.7) defined on the local atlas which covers $X_\alpha = X_\beta = X_\gamma = 0$ can be described well. As a consequence, the defining equation (5.19) of the mirror Calabi-Yau 3-fold $Y^\vee_{\boldsymbol{l}}$ is rewritten as the hypersurface in $\mathbb{C}^2 \times (\mathbb{C}^*)^2$:

$$Y^\vee_{\boldsymbol{l}} = \left\{ (\omega_+, \omega_-, x, y) \in \mathbb{C}^2 \times (\mathbb{C}^*)^2 \;\middle|\; \omega_+\omega_- = A^K_{Y^\vee_{\boldsymbol{l}}}(x, y) \right\},$$

where $x = x_\alpha$ and $y = x_\beta$, and the open string modulus x of the toric brane is mapped to x. *Mirror curve* $\Sigma^K_{Y^\vee_{\boldsymbol{l}}}$ is defined as the complex 1 dimensional submanifold which resides in $Y^\vee_{\boldsymbol{l}}$:

$$\Sigma^K_{Y^\vee_{\boldsymbol{l}}} = \left\{ (x, y) \in (\mathbb{C}^*)^2 \;\middle|\; A^K_{Y^\vee_{\boldsymbol{l}}}(x, y) = 0 \right\} \subset Y^\vee_{\boldsymbol{l}}. \tag{5.20}$$

Remark 5.5. For the mirror curve $\Sigma^K_{Y^\vee_{\boldsymbol{l}}}$ one can consider degrees of freedom for the framing $f \in \mathbb{Z}$ of the brane mentioned in Remark 5.4 by an $SL(2, \mathbb{Z})$ transformation $x \to xy^f$ and $y \to y$, which preserves the symplectic form $d\omega = d\log x \wedge d\log y$ on $\Sigma^K_{Y^\vee_{\boldsymbol{l}}}$ [21].

Definition 5.6 (Mirror map [26, 83]). *The mirror map between Q_i's in (5.6) and z_i's in (5.19) is given by the logarithmic solutions to the Picard-Fuchs equations $\mathcal{D}_i f(\boldsymbol{z}) = 0$ for periods of the holomorphic 3-form on $Y^\vee_{\boldsymbol{l}}$. Here*

$$\mathcal{D}_i = \prod_{l_{i,\alpha}>0} \left(\frac{\partial}{\partial x_\alpha}\right)^{l_{i,\alpha}} - \prod_{l_{i,\alpha}<0} \left(\frac{\partial}{\partial x_\alpha}\right)^{-l_{i,\alpha}},$$

and local coordinates x_α's are related with z_i's by $\prod_{\alpha=1}^{m+3} x_\alpha^{l_{i,\alpha}} = z_i$ in (5.19). Explicitly the inverse mirror map between Q_i's and z_i's (i.e. the logarithmic solution of the Picard-Fuchs equations) is given by

$$\log Q_i = \log z_i - \sum_{\gamma,(\mathfrak{m}_\gamma>0)} l_{i,\gamma} \sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_{\geq 0}^m \\ \boldsymbol{n}\neq(0,\dots,0)}} (-1)^{\mathfrak{m}_\gamma} \frac{(\mathfrak{m}_\gamma - 1)!}{\prod_{\alpha\neq\gamma}\left(\sum_j l_{j,\alpha} n_j\right)!} z_1^{n_1}\cdots z_m^{n_m}, \tag{5.21}$$

where $\mathfrak{m}_\gamma = -\sum_j l_{j,\gamma} n_j$.

The mirror map between open string moduli x in (5.7) and x in (5.20) is also obtained by extending the charge vectors $\boldsymbol{l}_i$ to $(\boldsymbol{l}_i;0,0)$ and adding one more charge $\boldsymbol{l}_0 = (\underbrace{\dots};1,-1)$, where the underbrace means $l_{0,\alpha}=1$, $l_{0,\gamma}=-1$, and $l_{0,\beta}=0$ for $\beta\neq\alpha,\gamma$. Explicitly the inverse mirror map between x and x is given by

$$\log \mathsf{x} = \log x - \sum_{\gamma,(\mathfrak{m}_\gamma>0)} l_{0,\gamma} \sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_{\geq 0}^m \\ \boldsymbol{n}\neq(0,\dots,0)}} (-1)^{\mathfrak{m}_\gamma} \frac{(\mathfrak{m}_\gamma - 1)!}{\prod_{\alpha\neq\gamma}\left(\sum_j l_{j,\alpha} n_j\right)!} z_1^{n_1}\cdots z_m^{n_m}. \tag{5.22}$$

5.3.1. *Geometric engineering of the ("on-shell") equivariant J-function for $\mathbb{C}\boldsymbol{P}_{\boldsymbol{w}}^{N-1}$.* Consider the local toric Calabi-Yau 3-fold defined by charge vectors (5.8), and a Lagrangian submanifold (5.7) with $\alpha=1,\beta=3,\gamma=2$ as depicted in Figure 5 for example. From the vantage point 2 this brane partition function gives the ("on-shell") equivariant J-function for $\mathbb{C}\mathbf{P}_{\boldsymbol{w}}^{N-1}$. For the mirror Calabi-Yau 3-fold (5.19), by taking local coordinate $x_1=x$, $x_2=1$, $x_3=y$ respecting the Lagrangian submanifold L in the A-model, we find a defining equation of the mirror curve (5.20):

$$A^K_{Y_{\boldsymbol{l}}^\vee}(x,y) = \sum_{i=1}^{N-1} \widetilde{z}_i y^{i+1} + y + x + 1 = 0, \qquad \widetilde{z}_i = \prod_{1\leq j\leq i} z_j^{i-j+1}. \tag{5.23}$$

Adopting the mirror maps (5.21) and (5.22) to this geometry, we see that for the local coordinate $x_1=x$, $x_2=1$, $x_3=y$ the quantum corrections are absent for x, i.e. $x=\mathsf{x}$. On the other hand there are quantum corrections for z_i's such that

$$Q_i = \mathrm{e}^{-g_{i-1}+2g_i-g_{i+1}} z_i.$$

Here $g_0=g_N=0$, and $g_i=g_i(\boldsymbol{z})$ for $i=1,2,\dots,N-1$ are defined by

$$g_i(\boldsymbol{z}) = \sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_{\geq 0}^{N-1} \\ \boldsymbol{n}\neq(0,\dots,0)}} (-1)^{n_{i-1}+n_{i+1}} \frac{(-n_{i-1}+2n_i-n_{i+1}-1)!(n_{i-1}-2n_i+n_{i+1})!}{\prod_{j=0}^N (n_{j-1}-2n_j+n_{j+1})!} z_1^{n_1}\cdots z_{N-1}^{n_{N-1}},$$

where $n_{-1}=n_0=n_N=n_{N+1}=0$. For functions $f_i(\boldsymbol{Q})$ of Q_i such that

$$\log f_i(\boldsymbol{Q}) = g_i\big(\boldsymbol{z}(\boldsymbol{Q})\big), \qquad f_0(\boldsymbol{Q}) = f_N(\boldsymbol{Q}) = 1,$$

we obtain the inverse of the mirror map (5.21):

$$z_i = \frac{f_{i-1}(\boldsymbol{Q}) f_{i+1}(\boldsymbol{Q})}{f_i(\boldsymbol{Q})^2} Q_i. \tag{5.24}$$

We find that such functions $f_a(\boldsymbol{Q})$ are given by[12]

$$f_i(\boldsymbol{Q}) = \frac{1}{\prod_{j=1}^{i-1}\widetilde{Q}_j} \sum_{0\leq t_1<\dots<t_i\leq N-1} \widetilde{Q}_{t_1}\cdots\widetilde{Q}_{t_i}, \qquad \widetilde{Q}_0 = 1, \quad \widetilde{Q}_i = \prod_{1\leq j\leq i} Q_j.$$

[12]We have directly checked this up to some orders.

Example 5.7. In the case of $N=4$ in Figure 5 we obtain

$$\begin{aligned} z_1 &= \frac{Q_1(1+Q_2+Q_1Q_2+Q_2Q_3+Q_1Q_2Q_3+Q_1Q_2^2Q_3)}{(1+Q_1+Q_1Q_2+Q_1Q_2Q_3)^2}, \\ z_2 &= \frac{Q_2(1+Q_1+Q_1Q_2+Q_1Q_2Q_3)(1+Q_3+Q_2Q_3+Q_1Q_2Q_3)}{(1+Q_2+Q_1Q_2+Q_2Q_3+Q_1Q_2Q_3+Q_1Q_2^2Q_3)^2}, \\ z_3 &= \frac{Q_3(1+Q_2+Q_1Q_2+Q_2Q_3+Q_1Q_2Q_3+Q_1Q_2^2Q_3)}{(1+Q_3+Q_2Q_3+Q_1Q_2Q_3)^2}. \end{aligned}$$

By the mirror map (5.24) the defining equation of the mirror curve (5.23) yields

$$A^{K}_{Y_{\boldsymbol{l}}^{\vee}}(x,y) = \prod_{i=0}^{N-1}\left(1+f_1(\boldsymbol{Q})^{-1}\widetilde{Q}_i y\right)+x=0. \tag{5.25}$$

After a change of variables:

$$y \;\to\; -f_1(\boldsymbol{Q})y, \qquad x \;\to\; -x,$$

we find that the mirror curve (5.25) agrees with the classical curve (5.18) for $Z^{Y_{N-1}}_{\text{A-brane}}(\mathsf{x})$ in the A-model.

5.3.2. *Geometric engineering of the ("on-shell") equivariant J-functions for degree* 1 *complete intersections in* $\mathbb{C}\boldsymbol{P}^{N-1}_{\boldsymbol{w}}$. Next we will consider the local toric Calabi-Yau 3-fold defined by charge vectors (5.10), and a Lagrangian submanifold (5.7) with $\alpha=2,\beta=3,\gamma=1$ as depicted in Figure 6. For the mirror Calabi-Yau 3-fold (5.19), by taking local coordinate $x_1=1$, $x_2=x$, $x_3=y$ we find a mirror curve (5.20) that describes this brane

$$A^{K}_{Y_{\boldsymbol{l}}^{\vee}}(x,y) = \sum_{i=1}^{N-1}\widetilde{z}_{\lambda,i}\widetilde{z}_{w,i}y^{i+1}+y+x+1+\sum_{i=1}^{N}\widetilde{z}_{\lambda,i}\widetilde{z}_{w,i-1}xy^{i}=0, \tag{5.26}$$

where

$$\widetilde{z}_{\lambda,i=1,\dots,N}=\prod_{1\le j\le i} z_{\lambda,j}^{i-j+1}, \qquad \widetilde{z}_{w,0}=0, \qquad \widetilde{z}_{w,i=1,\dots,N-1}=\prod_{1\le j\le i} z_{w,j}^{i-j+1}.$$

For this mirror curve the open string modulus x receives no quantum corrections, namely $x=\mathsf{x}$, and the mirror map (5.21) is given by

$$Q_{\lambda,i}=\mathrm{e}^{-g_{w,i-1}+g_{\lambda,i-1}+g_{w,i}-g_{\lambda,i}}z_{\lambda,i}, \qquad Q_{w,i}=\mathrm{e}^{-g_{\lambda,i-1}+g_{w,i}+g_{\lambda,i}-g_{w,i+1}}z_{w,i}.$$

Here $g_{w,0}=g_{\lambda,0}=g_{w,N}=g_{\lambda,N}=0$, and $g_{w,i}=g_{w,i}(\boldsymbol{z})$, $g_{\lambda,i}=g_{\lambda,i}(\boldsymbol{z})$ for $i=1,2,\dots,N-1$ are defined by

$$\begin{aligned} g_{w,i}(\boldsymbol{z}) = &\sum_{\substack{\boldsymbol{n}\in\mathbb{Z}^{2N-1}_{\ge 0}\\ \boldsymbol{n}\neq(0,\dots,0)}}(-1)^{n_{w,i-1}+n_{w,i}+n_{\lambda,i}+n_{\lambda,i+1}}\; z_{\lambda,1}^{n_{\lambda,1}}z_{w,1}^{n_{w,1}}\cdots z_{w,N-1}^{n_{w,N-1}}z_{\lambda,N}^{n_{\lambda,N}} \\ &\times\frac{(-n_{w,i-1}+n_{\lambda,i}+n_{w,i}-n_{\lambda,i+1}-1)!(n_{w,i-1}-n_{\lambda,i}-n_{w,i}+n_{\lambda,i+1})!}{\prod_{j=0}^{N}(n_{w,j-1}-n_{\lambda,j}-n_{w,j}+n_{\lambda,j+1})!(n_{\lambda,j}-n_{w,j}-n_{\lambda,j+1}+n_{w,j+1})!}, \\ g_{\lambda,i}(\boldsymbol{z}) = &\sum_{\substack{\boldsymbol{n}\in\mathbb{Z}^{2N-1}_{\ge 0}\\ \boldsymbol{n}\neq(0,\dots,0)}}(-1)^{n_{\lambda,i}+n_{\lambda,i+1}+n_{w,i}+n_{w,i+1}}\; z_{\lambda,1}^{n_{\lambda,1}}z_{w,1}^{n_{w,1}}\cdots z_{w,N-1}^{n_{w,N-1}}z_{\lambda,N}^{n_{\lambda,N}} \\ &\times\frac{(-n_{\lambda,i}+n_{w,i}+n_{\lambda,i+1}-n_{w,i+1}-1)!(n_{\lambda,i}-n_{w,i}-n_{\lambda,i+1}+n_{w,i+1})!}{\prod_{j=0}^{N}(n_{w,j-1}-n_{\lambda,j}-n_{w,j}+n_{\lambda,j+1})!(n_{\lambda,j}-n_{w,j}-n_{\lambda,j+1}+n_{w,j+1})!}, \end{aligned}$$

where $n_{w,-1}=n_{\lambda,0}=n_{w,0}=n_{w,N}=n_{\lambda,N+1}=n_{w,N+1}=0$. For functions $f_{w,i}(\boldsymbol{Q})$ and $f_{\lambda,i}(\boldsymbol{Q})$ of $Q_{\lambda,i}$ and $Q_{w,i}$ such that

$$\begin{aligned} &\log f_{w,i}(\boldsymbol{Q})=g_{w,i}\big(\boldsymbol{z}(\boldsymbol{Q})\big), \qquad \log f_{\lambda,i}(\boldsymbol{Q})=g_{\lambda,i}\big(\boldsymbol{z}(\boldsymbol{Q})\big), \\ &f_{w,0}(\boldsymbol{Q})=f_{\lambda,0}(\boldsymbol{Q})=f_{w,N}(\boldsymbol{Q})=f_{\lambda,N}(\boldsymbol{Q})=1, \end{aligned}$$

we obtain the inverse of the mirror map (5.21):

$$z_{\lambda,i} = \frac{f_{w,i-1}(\boldsymbol{Q}) f_{\lambda,i}(\boldsymbol{Q})}{f_{\lambda,i-1}(\boldsymbol{Q}) f_{w,i}(\boldsymbol{Q})} Q_{\lambda,i}, \qquad z_{w,i} = \frac{f_{\lambda,i-1}(\boldsymbol{Q}) f_{w,i+1}(\boldsymbol{Q})}{f_{w,i}(\boldsymbol{Q}) f_{\lambda,i}(\boldsymbol{Q})} Q_{w,i}. \tag{5.27}$$

We find that such functions $f_{w,i}(\boldsymbol{Q})$ and $f_{\lambda,i}(\boldsymbol{Q})$ are given by[13]

$$f_{w,i}(\boldsymbol{Q}) = \frac{1}{\prod_{j=1}^{i-1} \widetilde{Q}_{w,j}} \sum_{0 \le t_1 < \ldots < t_i \le N-1} \widetilde{Q}_{w,t_1} \cdots \widetilde{Q}_{w,t_i}, \qquad \widetilde{Q}_{w,0} = 1, \ \ \widetilde{Q}_{w,i} = \prod_{1 \le j \le i} Q_{\lambda,j} Q_{w,j},$$

$$f_{\lambda,i}(\boldsymbol{Q}) = \frac{1}{\prod_{j=1}^{i} \widetilde{Q}_{\lambda,j}} \sum_{1 \le t_1 < \ldots < t_i \le N} \widetilde{Q}_{\lambda,t_1} \cdots \widetilde{Q}_{\lambda,t_i}, \qquad \widetilde{Q}_{\lambda,i} = Q_{\lambda,1} \prod_{1 \le j \le i-1} Q_{w,j} Q_{\lambda,j+1}.$$

By the mirror map (5.27) the mirror curve (5.26) yields

$$A^K_{Y_{\boldsymbol{l}}^\vee}(x,y) = \prod_{i=0}^{N-1} \Big(1 + f_{w,1}(\boldsymbol{Q})^{-1} \widetilde{Q}_{w,i} y \Big) + x \prod_{i=1}^{N} \Big(1 + f_{w,1}(\boldsymbol{Q})^{-1} \widetilde{Q}_{\lambda,i} y \Big) = 0. \tag{5.28}$$

After a change of variables:

$$y \ \to \ -f_{w,1}(\boldsymbol{Q}) y, \qquad x \ \to \ -x,$$

we find that the mirror curve (5.28) agrees with the classical curve (5.18) for $Z^{Y_{N-1,N}}_{\text{A-brane}}(\mathsf{x})$ in the A-model.

To obtain the defining equation of the mirror curve for $Y_{N-1,n}$ with $n < N$ we will act the flop transitions which change the toric charges from (5.12) to (5.13). In this case, by the local mirror symmetry, the complex structure moduli parameters $z_{\lambda,1}$, $z_{w,1}$, and $z_{\lambda,2}$ (resp. $z'_{\lambda,1}$, $z'_{w,1}$, and $z'_{\lambda,2}$) are associated with the charge vectors $\boldsymbol{l}_{\lambda,1}$, $\boldsymbol{l}_{w,1}$, and $\boldsymbol{l}_{\lambda,2}$ (resp. $\boldsymbol{l}'_{\lambda,1}$, $\boldsymbol{l}'_{w,1}$, and $\boldsymbol{l}'_{\lambda,2}$), respectively. Under the flop transitions, the same relation as (5.14) for the Kähler moduli parameters holds for the complex structure moduli parameters:

$$(z'_{\lambda,1}, z'_{w,1}, z'_{\lambda,2}) = (z_{\lambda,1}^{-1}, z_{\lambda,1} z_{w,1}, z_{\lambda,2}). \tag{5.29}$$

Combining this relation (5.29) with the relation (5.14) together, the inverse mirror map can be considered after the flop transitions. As a consequence of "flops" and "decouplings" for some of complex structure moduli parameters in the mirror curve (5.28), we obtain the classical curve $\Sigma^K_{Y_{N-1,n}}$ in (5.18).

5.3.3. *Wave function for the mirror curve and the remodeling conjecture.* Regarding the mirror curve (5.20) as the spectral curve, one can find the wave function $\psi^K_{Y_{\boldsymbol{l}}^\vee}(x)$ for this curve via the topological recursion. The parameters z_i's and x are mapped to Q_i's and x by the inverse mirror maps (5.21) and (5.22). As a pullback of the wave function $\psi^K_{Y_{\boldsymbol{l}}^\vee}(x)$ by the inverse mirror map, we define the wave function $\psi^K_{Y_{\boldsymbol{l}}}(\mathsf{x})$. On the other hand the brane partition function $Z^{Y_{\boldsymbol{l}}^\vee}_{\text{B-brane}}(x)$ in the local B-model is defined as the pullback of $Z^{Y_{\boldsymbol{l}}}_{\text{A-brane}}(\mathsf{x})$ by the mirror map. The relation between these brane partition functions $Z^{Y_{\boldsymbol{l}}}_{\text{A-brane}}(\mathsf{x})$ and $Z^{Y_{\boldsymbol{l}}^\vee}_{\text{B-brane}}(x)$ and wave functions $\psi^K_{Y_{\boldsymbol{l}}^\vee}(x)$ and $\psi^K_{Y_{\boldsymbol{l}}}(\mathsf{x})$ is found from the remodeling conjecture proposed by V. Bouchard, A. Klemm, M. Mariño, and S. Pasquetti [84, 21, 22] (see [103, 43, 46, 47] for proofs and generalizations).

Conjecture 5.8 (Remodeling conjecture)**.** *Consider the topological A-model on a local toric Calabi-Yau 3-fold $Y_{\boldsymbol{l}}$ in (5.6) with a special Lagrangian submanifold L in (5.7). Let $F^{(g)}_n(\mathsf{x}_1,\ldots,\mathsf{x}_n)$ be the generating function of the open Gromov-Witten invariants that enumerate the world sheet instantons for the map from the genus g Riemann surface with n boundaries $\Sigma_{g,n}$ (resp. the*

[13] We have directly checked this up to some orders.

boundaries $\partial\Sigma_{g,n}$ of $\Sigma_{g,n}$) to $Y_{\boldsymbol{l}}$ (resp. L). Via the mirror maps (5.21) and (5.22), the generating function $F_n^{(g)}(\mathsf{x}_1,\dots,\mathsf{x}_n)$ is given by

$$\begin{aligned}
&F_1^{(0)}(\mathsf{x}_1)=\int_{z_1^*}^{z_1}\omega(x(z_1')), \qquad \omega(x(z))=\log y(x(z))\,\frac{dx(z)}{x(z)},\\
&F_2^{(0)}(\mathsf{x}_1,\mathsf{x}_2)=\int_{z_1^*}^{z_1}\int_{z_2^*}^{z_2}\left(B(z_1',z_2')-\frac{dx(z_1')dx(z_2')}{(x(z_1')-x(z_2'))^2}\right),\\
&F_n^{(g)}(\mathsf{x}_1,\dots,\mathsf{x}_n)=\int_{z_1^*}^{z_1}\cdots\int_{z_n^*}^{z_n}\omega_n^{(g)}(z_1',\dots,z_n'), \quad \text{for } (g,n)\neq(0,1),(0,2),
\end{aligned} \tag{5.30}$$

up to the framing ambiguity. Here $B(z_1,z_2)$ is the Bergman kernel and $\omega_n^{(g)}$ ($(g,n)\neq(0,1),(0,2)$) are the multilinear meromorphic differentials recursively defined by the topological recursion (3.9) or (3.12) on a mirror curve $\Sigma^K_{Y_{\boldsymbol{l}}^\vee}$. z_i's denote points on the mirror curve $\Sigma^K_{Y_{\boldsymbol{l}}^\vee}$ in a local coordinate, and z_i^*'s denote reference points in $\Sigma^K_{Y_{\boldsymbol{l}}^\vee}$ so that the integrals converge to 0 at these points.

Following the WKB reconstruction (3.13), we can define the wave function by (5.30) such that

$$\psi^K_{Y_{\boldsymbol{l}}}(\mathsf{x})=\exp\left(\sum_{g=0,n=1}^{\infty}\frac{1}{n!}\hbar^{2g-2+n}F_n^{(g)}(\mathsf{x},\dots,\mathsf{x})\right).$$

From the remodeling conjecture we see that the WKB reconstruction of this wave function $\psi^K_{Y_{\boldsymbol{l}}}(\mathsf{x})$ agrees with the WKB expansion of brane partition function $Z^{Y_{\boldsymbol{l}}}_{\text{A-brane}}(\mathsf{x})$ in the local A-model. In particular for the spectral curve $\Sigma^K_{Y_{N-1,n}}$ in (5.18) the wave function $\psi^K_{Y_{N-1,n}}(\mathsf{x})$ is defined in this way, and it gives the WKB reconstruction of the brane partition function $Z^{Y_{N-1,n}}_{\text{A-brane}}(\mathsf{x})$ in (5.15):

$$Z^{Y_{N-1,n}}_{\text{A-brane}}(\mathsf{x})\sim\psi^K_{Y_{N-1,n}}(\mathsf{x}). \tag{5.31}$$

From the vantage points 1 and 2, for $n<N$, the cohomological limit $\beta\to 0$ of $\psi^K_Y(\mathsf{x})$ gives a WKB reconstruction of the ("on-shell") equivariant J-function for the Fano complete intersection of the degree $l_{a=1,\dots,n}=1$ hypersurfaces in $\mathbb{C}\mathbf{P}^{N-1}$ via the topological recursion.

In summary, from these 3 vantage points, we have found results as follows.

- $Z^{Y_{N-1,n}}_{\text{A-brane}}(\mathsf{x})$ obeys the differential equation (5.17) which comes from the q-difference equation (5.16) in the cohomological limit. This differential equation agrees with the GKZ equation (1.10) compensated by a factor $x^{w_0/\hbar}$. (See Lemma A.2.)
- The classical limit of the q-difference equation (5.16) defines the classical curve (5.18), and it agrees with the mirror curve $\Sigma^K_{Y_{N-1,n}}$ for $Y_{N-1,n}$.
- The wave function $\psi^K_{Y_{N-1,n}}(\mathsf{x})$ found from the topological recursion for the mirror curve $\Sigma^K_{Y_{N-1,n}}$ is regarded as the WKB expansion of the brane partition function $Z^{Y_{N-1,n}}_{\text{A-brane}}(\mathsf{x})$.

At the level of the cohomological limit, the above results suggest that the GKZ equation (1.10) is regarded as the quantum curve for the GKZ curve (2.17), because the differential equation (5.17) for the cohomological limit of the brane partition function satisfies the properties in Definition 3.3. Although some physical (but mathematical obscure) definitions and conjectures are used to obtain the above results, we find the physical derivation of the reconstruction theorem at last.

6. Stokes matrix for $\mathbb{C}\mathbf{P}^1_{\boldsymbol{w}}$

In this section we consider the quantum curve

$$\left[\left(\hbar x\frac{d}{dx}-w_0\right)\left(\hbar x\frac{d}{dx}-w_1\right)-x\right]\psi=0 \tag{6.1}$$

arising from the GKZ curve

$$\Sigma_{\mathbb{C}\mathbf{P}^1_{\boldsymbol{w}}}=\{(x,y)\in\mathbb{C}^*\times\mathbb{C}\mid A_{\mathbb{C}\mathbf{P}^1_{\boldsymbol{w}}}(x,y)=0\}, \quad A_{\mathbb{C}\mathbf{P}^1_{\boldsymbol{w}}}(x,y)=(y-w_0)(y-w_1)-x \tag{6.2}$$

as is discussed in Section 4.1. This equation is also known as the quantum differential equation (Dubrovin's first structure connection) for the equivariant Gromov-Witten theory for $\mathbb{C}\mathbf{P}^1$. The goal of this section is to compute the Stokes matrix for the WKB solution of the equation (6.1) using the exact WKB method (see [78, 72] for the foundation of the exact WKB method). As we will see below, integrals over the GKZ curve play crucially important role in the description of the Stokes matrices.

6.1. **Normalization of the WKB solution.** From the view point of the WKB method, it is convenient to transform (6.1) to the following Schrödinger-type equation:

$$\left(\hbar^2 \frac{d^2}{dx^2} - Q\right)\varphi = 0, \quad Q = Q_0(x) + \hbar^2 Q_2(x) = \frac{4x + (w_0 - w_1)^2}{4x^2} - \hbar^2 \frac{1}{4x^2} \tag{6.3}$$

through the gauge transform

$$\psi = \exp\left(\frac{w_0 + w_1 - \hbar}{2\hbar} \log x\right)\varphi. \tag{6.4}$$

The equation (6.3) has a unique turning point (i.e., the zero of the leading term of Q) at

$$v = -\frac{(w_0 - w_1)^2}{4}.$$

In what follows we assume

$$w_0 - w_1 \neq 0 \tag{6.5}$$

to avoid the case that the turning point coalesces with the pole of Q .

Although the construction of the WKB solution via the topological recursion has given in Section 3, here we reformulate the construction and introduce a "normalized WKB solution at a turning point" to use the so-called *Voros' formula* (see Theorem 6.8 below).

Two independent WKB solutions of (6.3) can be written as

$$\varphi_\pm(x,\hbar) = \exp\left(\int^x P^{(\pm)}(x',\hbar)dx'\right), \tag{6.6}$$

where $P^{(\pm)}(x,\hbar) = \sum_{n=0}^\infty \hbar^{n-1} P_n^{(\pm)}(x)$, are two formal solutions of the Riccati equation $\hbar^2\left(P^2 + \frac{dP}{dx}\right) = Q(x,\hbar)$. That is, $P_n^{(\pm)}$ are determined by solving the recursion relation

$$P_0^{(\pm)}(x) \;=\; \pm\sqrt{Q_0(x)} \;=\; \pm\sqrt{\frac{4x + (w_0 - w_1)^2}{4x^2}}, \tag{6.7}$$

$$P_{n+1}^{(\pm)}(x) \;=\; \frac{1}{2P_0^{(\pm)}}\left(\delta_{n,1} Q_2(x) - \sum_{\substack{n_1+n_2=n+1 \\ n_1,n_2\geq 1}} P_{n_1}^{(\pm)} P_{n_2}^{(\pm)} - \frac{dP_n^{(\pm)}}{dx}\right) \quad (n \geq 0). \tag{6.8}$$

The functions $P_n^{(\pm)}(x)$ are defined on the Riemann surface of $\sqrt{Q_0(x)}$, which can be identified with the spectral curve $\Sigma_{\mathbb{C}\mathbf{P}^1_{\boldsymbol{w}}}$ (see (6.19) below). After fixing a branch cut between the turning point v and ∞, we regard them as meromorphic functions on the cut plane $\mathbb{C}\mathbf{P}^1 \setminus \{\text{cut}\}$. In what follows, we choose the branch which behaves as

$$\sqrt{Q_0(x)} = \frac{w_0 - w_1}{2x}\left(1 + O(x)\right), \quad x \to 0 \tag{6.9}$$

as the branch on the first sheet. We will also regard the coordinate x of $\mathbb{C}\mathbf{P}^1$ (restricted to the cut plane) as that of the first sheet of $\Sigma_{\mathbb{C}\mathbf{P}^1_{\boldsymbol{w}}}$, and use the covering involution $\sigma : \Sigma_{\mathbb{C}\mathbf{P}^1_{\boldsymbol{w}}} \to \Sigma_{\mathbb{C}\mathbf{P}^1_{\boldsymbol{w}}}$ to describe a point on the second sheet.

The following statements are consequence of (6.7), (6.8) and [78, Remark 2.2].

Lemma 6.1.

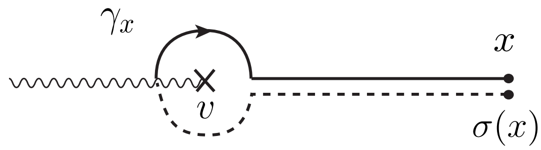


FIGURE 8. For a given x, the path γ_x starts from the point $\sigma(x)$ and ends at x after encircling the turning point v. The wiggly lines designate a branch cut, and the solid (resp. dotted) part represents a part of path on the first (resp. the second) sheet of the spectral curve.

(i) *The asymptotic behavior of* $P_n^{(\pm)}(x)$ *when* x *tends to* 0 *are given as follows:*

$$P_0^{(\pm)}(x) = \pm \frac{w_0 - w_1}{2x}(1 + O(x)), \qquad P_1^{(\pm)}(x) = \frac{1}{2x}(1 + O(x)), \tag{6.10}$$

and $P_n^{(\pm)}(x)$ *for* $n \geq 2$ *are holomorphic at* 0.

(ii) *The asymptotic behavior of* $P_n^{(\pm)}(x)$ *when* x *tends to* ∞ *are given as follows:*

$$P_n^{(\pm)}(x) = O(x^{-\frac{n}{2}-\frac{1}{2}}) \quad (n \geq 0). \tag{6.11}$$

(iii) *If we define*

$$P_{\rm odd}(x,\hbar) = \frac{P^{(+)}(x,\hbar) - P^{(-)}(x,\hbar)}{2}, \quad P_{\rm even}(x,\hbar) = \frac{P^{(+)}(x,\hbar) + P^{(-)}(x,\hbar)}{2}, \tag{6.12}$$

(i.e. $P^{(\pm)} = \pm P_{\rm odd} + P_{\rm even}$*), then we have*

$$P_{\rm even}(x,\hbar) = -\frac{1}{2P_{\rm odd}(x,\hbar)} \frac{dP_{\rm odd}(x,\hbar)}{dx}. \tag{6.13}$$

Here we note that the holomorphicity of $P_n^{(\pm)}(x)$ in (i) and (ii) is a consequence of the topological recursion (correlation functions must be holomorphic except for the ramification point v).

We will use a special normalization of the WKB solution to compute Stokes matrices, following [78, Section 2]. Thanks to (iii) of Lemma 6.1 implies that the WKB solutions can be written in the following form (up to some factor which is independent of x):

$$\varphi_\pm(x,\hbar) = \frac{1}{\sqrt{P_{\rm odd}(x,\hbar)}} \exp\left(\pm \int_v^x P_{\rm odd}(x',\hbar)dx'\right). \tag{6.14}$$

Here the lower end-point v in (6.16) is the unique turning point of (6.3), and the integral is defined in terms of contour integral

$$\int_v^x P_{\rm odd}(x',\hbar)dx' = \frac{1}{2}\int_{\gamma_x} P_{\rm odd}(x',\hbar)dx' \tag{6.15}$$

along the path depicted in Figure 8 (see Remark 6.2). Through the relation (6.4), we also have an expression of the WKB solution of (6.1):

$$\psi_\pm(x,\hbar) = \frac{\exp\left(\frac{w_0+w_1-\hbar}{2\hbar}\log x\right)}{\sqrt{P_{\rm odd}(x,\hbar)}} \exp\left(\pm \int_v^x P_{\rm odd}(x',\hbar)dx'\right). \tag{6.16}$$

Remark 6.2. When we integrate $P_{\rm odd}\,dx$, the path of integration in (6.16) should be taken on the spectral curve $\Sigma_{\mathbb{C}\mathbf{P}^1_w}$. (Although the coefficients of WKB solution are defined on $\Sigma_{\mathbb{C}\mathbf{P}^1_w}$, the Borel sum of WKB solution (defined in Section 6.2) is single-valued around turning points (i.e., well-defined on the x-plane).

Remark 6.3. Since $\sqrt{Q_0(x)}$ has a simple pole at $x = 0$, the path γ_x must avoid the point. If we choose different path from v to x, then the corresponding WKB solutions are modified by diagonal matrix. For example, for the WKB solutions $\psi_\pm$ (resp. $\tilde{\psi}_\pm$) normalized along γ_x (resp. $\tilde{\gamma}_x$) depicted in Figure 9, we have

$$\psi_\pm = \exp\left(\pm\frac{1}{2}V_{\gamma_0}\right)\tilde{\psi}_\pm, \qquad V_{\gamma_0} = \oint_{\gamma_0-\sigma_*\gamma_0} P_{\mathrm{odd}}(x,\hbar)\,dx = \frac{2\pi\mathrm{i}\,(w_0-w_1)}{\hbar}. \tag{6.17}$$

(Cf. (6.9).) Here γ_0 is a positively oriented cycle around $x=0$ on the first sheet of $\Sigma_{\mathbb{C}\mathbf{P}^1_{\boldsymbol{w}}}$, and $\sigma_*\gamma_0$ is the image of γ_0 by σ. The integral V_{γ_0} is called *Voros coefficient* for the closed cycle γ_0 (see [33, 72]), which is important in the exact WKB analysis since it appears in the expression of monodromy or connection matrices of (Borel resumed) WKB solutions ([78, Section 3]). Note also that, although V_{γ_0} is a priori a formal power series, V_{γ_0} only consists of one term in our example thanks to the holomorphicity of $P_n(x)$ for $n\geq 2$ in Lemma 6.1 (again recall that it is a consequence of topological recursion).

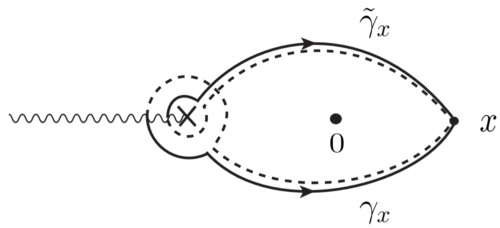


FIGURE 9. The paths γ_x and $\tilde{\gamma}_x$

We also note that the formal series expression of (6.16) can be arranged to

$$\psi_\pm(x,\hbar) = \exp\left(\frac{1}{\hbar}S_0^{(\pm)}(x)\right)\sum_{n=0}^{\infty}\hbar^{n+\frac{1}{2}}\psi_n^{(\pm)}(x), \tag{6.18}$$

where

$$S_0^{(\pm)}(x) = \frac{w_0+w_1}{2}\log x \pm \int_v^x \sqrt{Q_0(x')}\,dx' \tag{6.19}$$

(which coincides with the one computed in Table 3 in Appendix C up to an additive constant) and $y_\pm(x) = x(dS_0^{(\pm)}/dx)$ satisfies the equation $A_{\mathbb{C}\mathbf{P}^1_{\boldsymbol{w}}}(x, y_\pm(x)) = 0$ for the GKZ curve $\Sigma_{\mathbb{C}\mathbf{P}^1_{\boldsymbol{w}}}$.

Lemma 6.4. *The coefficients in the expansion* (6.18) *satisfy* $\lim_{x\to\infty}\psi_n^{(\pm)}(x) = 0$ *for* $n\geq 0$.

Proof. The term $\psi_0^{(\pm)}$ (which coincides with $\exp(S_1)$ in Table 3 in Appendix C) is given by

$$\psi_0^{(\pm)}(x) = \frac{\exp(-\frac{1}{2}\log x)}{Q_0(x)^{1/4}} = \frac{\sqrt{2}}{(4x+(w_0-w_1)^2)^{1/4}} = O(x^{-1/4}) \tag{6.20}$$

(which is independent of $\pm$). The behavior of subsequent terms can be derived from the estimate (6.11) and the equality:

$$\int_v^x P_{2m}^{(\pm)}(x')dx' = \int_\infty^x P_{2m}^{(\pm)}(x')dx' = O(x^{-\frac{1}{2}}) \quad (m\geq 1).$$

The first equality holds since there is only one branch point v on the spectral curve $\Sigma_{\mathbb{C}\mathbf{P}^1_{\boldsymbol{w}}}$, and $P_{2m}^{(\pm)}(x)dx$ has no residue at $x=0$ for $m\geq 1$ (see (i) in Lemma 6.1). $\square$

The above lemma and Proposition 4.3 imply the relation between the WKB solution constructed here and the wave function constructed through the topological recursion at the level of formal power series.

Proposition 6.5. *The WKB solution* (6.18) *agrees with the wave function* (4.11) *(up to the overall factor $\hbar^{1/2}$) constructed through the topological recursion for the GKZ curve $\Sigma_{\mathbb{C}\mathbf{P}^1_{\boldsymbol{w}}}$ with the integration divisor $D=[z]-[\infty]$ (i.e., the reference point is chosen as $z_*=\infty$).*

6.2. **Borel summation and the Stokes graph.** The expansion (6.18) is a divergent series of $\hbar$. To give an analytic interpretation for (6.18), we employ the *Borel summation method* (for a formal series of $\hbar$). For the convenience of the readers, here we briefly recall the Borel summation method (see [32] for details.)

For fixed $\theta \in \mathbb{R}$ and $x_0 \in \mathbb{C}$ satisfying $x_0 \neq 0, v$, the WKB solution $\psi_\pm$ is said to be *Borel summable in the direction* θ near x_0 if the following conditions are satisfied (see [78, Definition 1.3]):

- The *Borel transform*
$$\mathcal{B}\psi_\pm(x,y) = \psi_{\pm,B}(x,y) = \sum_{n=0}^{\infty} \frac{\psi_{\pm,n}(x)}{\Gamma(n+\frac{1}{2})}\big(y - a_\pm(x)\big)^{n-\frac{1}{2}} \tag{6.21}$$
of $\psi_\pm$ is holomorphic on a domain
$$D = \{(x,y) \in U \times \mathbb{C} \mid -\epsilon < \mathrm{Im}\big(\mathrm{e}^{-\mathrm{i}\theta}(y - a_\pm(x))\big) < +\epsilon\}$$
with a sufficiently small $\epsilon > 0$. Here U is a neighborhood of x_0 and $a_\pm(x) = -S_0^{(\pm)}(x)$. (The convergence of the Borel transform $\psi_{\pm,B}(x,y)$ near $y = a_\pm(x)$ is always true; see [78, Lemma 2.5].)
- $|\psi_{\pm,B}(x,y)| < C_1 \mathrm{e}^{C_2|y|}$ holds with some $C_1, C_2 > 0$ on the above domain D.

If $\psi_\pm$ is Borel summable in the direction θ, then the following Laplace integral defines a holomorphic function of both x and $\hbar$ on $\{(x,\hbar) \in \mathbb{C}^2 \mid x \in U,\ |\arg\hbar - \theta| < \pi/2,\ |\hbar| \ll 1\}$:
$$\Psi_\pm^{(\theta)} = \int_{\ell_\theta} \mathrm{e}^{-y/\hbar}\, \psi_{\pm,B}(x,y)dy, \tag{6.22}$$
where $\ell_\theta = \{y = a_\pm(x) + r\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} \mid r \geq 0\}$. The function (6.22) is called the *Borel sum* of $\psi_\pm$ in the direction θ. If $\psi_\pm$ is Borel summable in the direction θ, then the Borel sum recovers the WKB solution as its asymptotic expansion (for any fixed $x \in U$):
$$\Psi_\pm^{(\theta)} \sim \psi_\pm \quad \text{when } \hbar \to 0 \text{ with } |\arg\hbar - \theta| < \pi/2.$$
Moreover, for any fixed $\hbar$ satisfying $|\arg\hbar - \theta| < \pi/2$ and $|\hbar| \ll 1$, the Borel sum $\Psi_\pm^{(\theta)}$ is a holomorphic solution of the equation (6.1) on U. Singularities of $\psi_{\pm,B}$ on y-plane (Borel-plane) spoils the Borel summability of the WKB solutions, and hence causes the Stokes phenomenon. The Stokes multipliers of the WKB solutions are discussed in [99, 78] for example (see also Section 6.4).

To discuss the Borel summability and the Stokes phenomenon for the WKB solutions, let us recall the notion of the Stokes graph[14] for a fixed phase $\theta \in \mathbb{R}$ (see [78, Definition 2.6]).

- A *Stokes curve* of (6.1) of phase θ is a real one-dimensional integral curve of the direction field
$$\mathrm{Im}\left(\mathrm{e}^{-\mathrm{i}\theta}\int^x \sqrt{Q_0(x)}\,dx\right) = \text{const.}$$
emanating from a turning point.
- A *saddle connection* of phase θ is a Stokes curve of phase θ which connects turning points.
- The *Stokes graph* of (6.1) of phase θ is defined as a graph on x-plane whose vertices are zeros and poles of $Q_0(x)dx^2$, and whose edges are Stokes curves emanating from turning points.

We will use the notation G_θ for the Stokes graph of (6.1) of phase θ. Figure 10 depicts G_θ for several θ between 0 and π where the equivariant parameters are chosen as $(w_0, w_1) = (1, 0)$.

A sufficient condition for the Borel summability is given as follows.

Theorem 6.6 ([79])**.** *Fix $\theta \in \mathbb{R}$. The WKB solution (6.16) is Borel summable in the direction θ near any point on each face of the Stokes graph G_θ when the following conditions are satisfied:*

[14]Stokes graph is also known as an example of spectral networks [49].

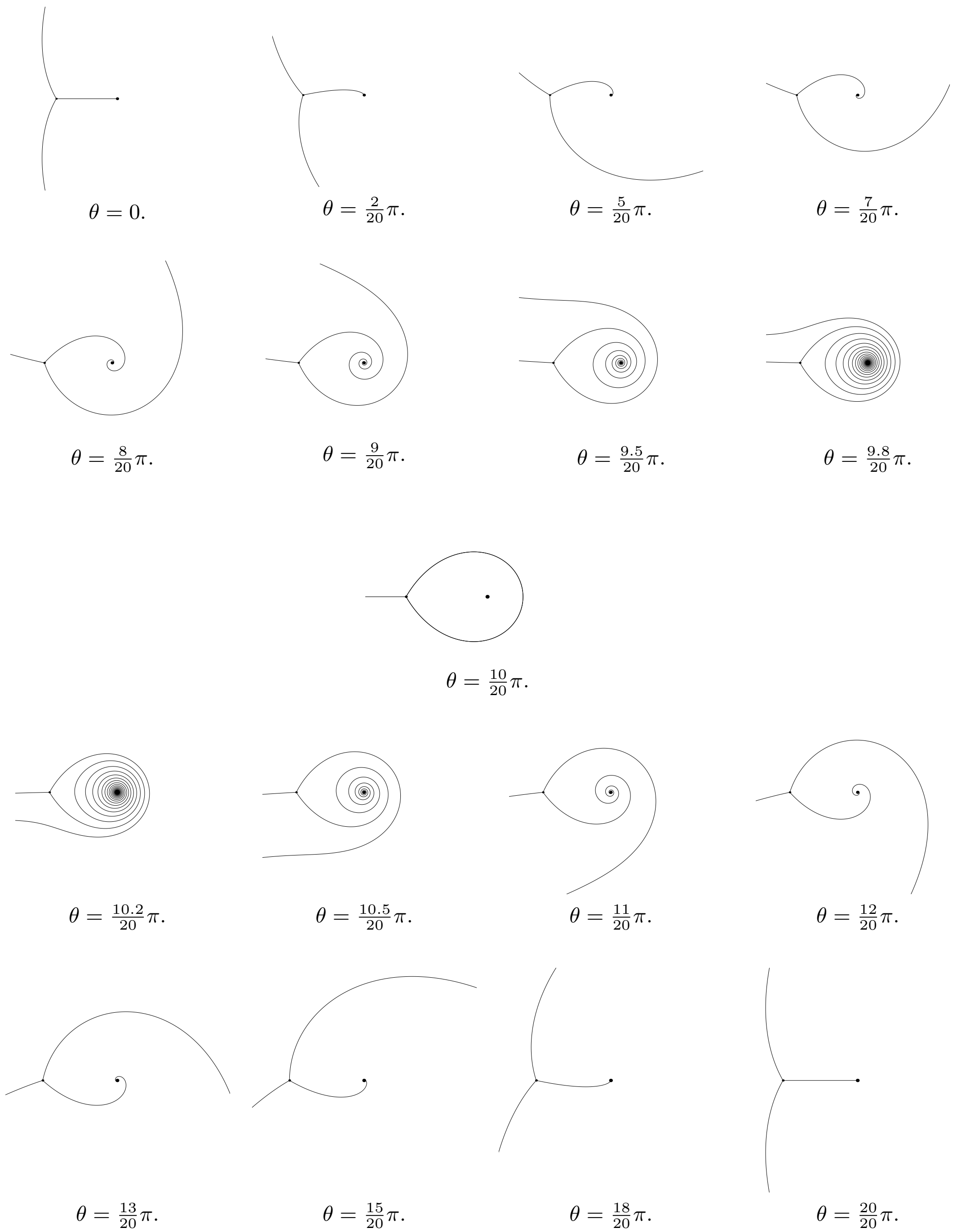


FIGURE 10. Stokes graphs of the equation (6.1) for $w_0 - w_1 = 1$. A loop-type saddle connection appears when $\theta = \pi/2$.

(i) *The upper end-point x of the integral in* (6.16) *does not lie on G_θ.*
(ii) *The path γ_x of integration in* (6.15) *can be deformed in the spectral curve $\Sigma_{\mathbb{C}\mathbf{P}^1_{\boldsymbol{w}}}$ so that its projection by $\pi : \Sigma_{\mathbb{C}\mathbf{P}^1_{\boldsymbol{w}}} \to \mathbb{C}\mathbf{P}^1$ never intersects with saddle connections in G_θ.*

A proof of Theorem 6.6 will be given in forthcoming paper [79]. See also [95, Section 3.1].

6.3. **Oscillatory integral and the Borel resummed WKB solution.** As is mentioned in Proposition 2.4, the GKZ equation (6.1) is satisfied by the oscillatory integral

$$\mathcal{I}_{\pm}^{(\theta)}(x,\hbar) = \int_{\Gamma_{\pm}^{(\theta)}} \exp\left(\frac{W(u;x)}{\hbar}\right)\frac{du}{u}, \quad W(u;x) = W_{\mathbb{C}\mathbf{P}^1_{\boldsymbol{w}}}(u;x) = u + \frac{x}{u} + w_0 \log u + w_1 \log\left(\frac{x}{u}\right). \tag{6.23}$$

Here, for a fixed $\theta \in \mathbb{R}$, $\Gamma_{\pm}^{(\theta)}$ is the Lefschetz thimble of the phase θ (i.e. the steepest descent path for the function $\mathrm{Re}(\mathrm{e}^{-\mathrm{i}\theta}W)$) associated with the critical point

$$u_{\pm} = \frac{w_1 - w_0}{2} \pm \frac{\sqrt{4x + (w_0 - w_1)^2}}{2}$$

of W. Precisely speaking, since the function W contains the logarithm, we regard $u_{\pm}$ as a point on the universal cover $\widetilde{\mathbb{C}^*}$ of $\mathbb{C}^*$. Thus, any lift $\mathrm{e}^{2k\pi\mathrm{i}}u_{\pm}$ ($k \in \mathbb{Z}$) of $u_{\pm}$ onto $\widetilde{\mathbb{C}^*}$ is a critical point of W, and the corresponding critical values satisfy $W(\mathrm{e}^{2k\pi\mathrm{i}}u_{\pm};x) = W(u_{\pm};x) + 2k\pi\mathrm{i}(w_0 - w_1)$. Note also that these critical points are non-degenerate as long as $x \neq v$.

Using Corollary 4.7, let us compare the oscillatory integral $\mathcal{I}_{\pm}^{(\theta)}$ and the Borel resummed WKB solution $\Psi_{\pm}^{(\theta)}$. For the purpose, we should know the well-definedness of the Lefschetz thimbles; that is, a sufficient condition which guarantees that the image $W(\Gamma_{\pm}^{(\theta)};x)$ of the Lefschetz thimble never hits another critical values of W.

- If $\mathrm{e}^{-\mathrm{i}\theta}\,2\pi\mathrm{i}(w_0 - w_1) \in \mathbb{R}$, then the image $W(\Gamma_{\pm}^{(\theta)};x)$ of Lefschetz thimbles hits the critical value $W(\mathrm{e}^{2\pi\mathrm{i}}u_{\pm};x)$. Therefore, we assume that the phase θ satisfies
$$\mathrm{e}^{-\mathrm{i}\theta}\,2\pi\mathrm{i}(w_0 - w_1) \notin \mathbb{R}. \tag{6.24}$$
Note that the above condition is satisfied if there is no loop-type saddle connection in the Stokes graph G_θ (see (6.9)).
- Since $u_+ = u_-$ at $x = v$, we can verify $W(u_+;x) - W(u_-;x) = 2\int_v^x \sqrt{Q_0(x')}dx' + 2k\pi\mathrm{i}(w_0 - w_1)$, where k is an integer specified by the branch of logarithm at the critical points. In view of (6.9), we can take an appropriate path γ_k from v to x (which turns around $x = 0$ several times depending on k) satisfying $W(u_+;x) - W(u_-;x) = 2\int_{\gamma_k}\sqrt{Q_0(x')}dx'$. Therefore, if we assume
$$\mathrm{e}^{-\mathrm{i}\theta}\int_{\gamma_k}\sqrt{Q_0(x')}dx' \notin \mathbb{R} \quad \text{for any } k \in \mathbb{Z}, \tag{6.25}$$
then the image $W(\Gamma_{\pm}^{(\theta)};x)$ of Lefschetz thimbles never intersect with each other. Note that the condition is satisfied if x does not lie on Stokes curves of the phase θ.

In summary, we can show that the Lefschetz thimbles are well defined if the conditions (6.24) and (6.25) are satisfied[15]. The discussion given here also implies that the WKB solutions are Borel summable under these conditions, and hence, the Borel resummed WKB solutions $\Psi_{\pm}^{(\theta)}$ are well-defined.

Let us compare the asymptotic expansions (saddle point approximation). As is mentioned in Section 2.2, the asymptotic expansion of $\mathcal{I}_{\pm}^{(\theta)}$ when $\hbar \to 0$, $|\arg\hbar - \theta| < \pi/2$ is given by

$$\mathcal{I}_{\pm}^{(\theta)}(x,\hbar) \sim \exp\left(\frac{1}{\hbar}W(u_{\pm};x)\right)\frac{(-2\pi\hbar)^{1/2}}{u_{\pm}\sqrt{\mathrm{Hess}(u_{\pm})}}\left(1 + \sum_{n=1}^{\infty}\mathcal{I}_n^{(\pm)}(x)\hbar^n\right), \tag{6.26}$$

where $\mathrm{Hess}(u_{\pm}) = W''(u_{\pm};x)$ is the Hessian of W at the critical point $u_{\pm}$. We can verify that $S_0^{(\pm)} = W(u_{\pm};x)$ (we fix the ambiguity in the branch of logarithm in $S_0^{(\pm)}$ so that this equality

[15] These conditions are sufficient conditions because, even if the image of Lefschetz thimbles intersects, the oscillatory integral is well-defined and the saddle point approximation is valid if the corresponding vanishing cycles never intersect.

holds), and

$$\frac{(-2\pi)^{1/2}}{u_\pm\sqrt{\mathrm{Hess}(u_\pm)}} = \frac{(\pm1)^{-1/2}(-2\pi)^{1/2}}{(4x+(w_0-w_1)^2)^{1/4}} = (\pm1)^{-1/2}(-\pi)^{1/2}\psi_0^{(\pm)}(x). \tag{6.27}$$

This equality together with Corollary 4.7 and Proposition 6.5 show that

$$\mathcal{I}_\pm^{(\theta)} \sim (\pm1)^{-1/2}(-\pi)^{1/2}\psi_\pm \tag{6.28}$$

holds when $\hbar \to 0$, $|\arg\hbar - \theta| < \pi/2$. Comparing the asymptotic expansion, we obtain the relationship between exact solutions of the GKZ equation (6.1) (which is a refinement of Proposition 4.3):

Proposition 6.7. $\mathcal{I}_\pm^{(\theta)}(x,\hbar) = (\pm1)^{-1/2}(-\pi)^{1/2}\Psi_\pm^{(\theta)}(x,\hbar)$ *holds if the conditions* (6.24) *and* (6.25) *are satisfied.*

Hence, the computation of Stokes matrices in the subsequent sections can be translated to results for the oscillatory integrals.

6.4. **Stokes matrices for WKB solutions normalized at the turning point.** Suppose that, in a direction θ_0, the one of the following conditions is satisfied:

(i) A Stokes curve of the phase θ_0 hits the point x.
(ii) A saddle connection appears in G_{θ_0}, and it intersects with the path γ_x.

The WKB solution is not Borel summable in these cases because certain singularities appear on the ray $\{y = a_\pm(x) + re^{\mathrm{i}\theta_0} \in \mathbb{C} \mid r \geq 0\}$ in the Borel-plane. Let us describe the Stokes matrices for both cases.

6.4.1. *The case* (i). First, we recall the Voros' connection formula which describes the Stokes phenomenon of the type (i) for the Borel resummed WKB solutions.

Let us specify the situation to state the connection formula. Take any point x, and suppose that there exists a direction θ_0 and a sufficiently small number $\varepsilon > 0$ satisfying the following conditions:

- The Stokes graphs G_θ have no saddle connection for any θ satisfying $\theta_0 - \varepsilon \leq \theta \leq \theta_0 + \varepsilon$.
- The point x lies on a Stokes curve C of the phase θ_0 emanating from the turning point v. Note that this assumption implies $e^{-\mathrm{i}\theta_0}\int_v^x \sqrt{Q_0(x')}\,dx' \in \mathbb{R}_{\neq 0}$ (where the integral is taken along C) by the definition of Stokes curves.
- The point x does not lie on G_θ for any θ satisfying $\theta_0 - \varepsilon \leq \theta < \theta_0$ or $\theta_0 < \theta \leq \theta_0 + \varepsilon$.

Let $\psi_\pm$ be the WKB solution (6.16) normalized at v along the Stokes curve C of phase θ_0. Denote by $\Psi_\pm^{(\theta_0-\varepsilon)}$ (resp. $\Psi_\pm^{(\theta_0+\varepsilon)}$) the Borel sum of $\psi_\pm$ in the direction $\theta_0 - \varepsilon$ (resp. $\theta_0 + \varepsilon$). Then, we have the following statement.

Theorem 6.8 ([99, Section 6]; see also [78, Theorem 2.23])**.** *In the situation above, the Borel transformed WKB solution $\psi_{+,B}$ (resp. $\psi_{-,B}$) has the singular point at $y = a_-(x)$ (resp. at $y = a_+(x)$). Moreover, the following equalities hold:*

(i) *When* $e^{-\mathrm{i}\theta_0}\int_v^x \sqrt{Q_0(x)}dx > 0$ *on* C*, then*

$$(\Psi_+^{(\theta_0-\varepsilon)}, \Psi_-^{(\theta_0-\varepsilon)}) = (\Psi_+^{(\theta_0+\varepsilon)}, \Psi_-^{(\theta_0+\varepsilon)})\begin{pmatrix} 1 & 0 \\ -\mathrm{i} & 1 \end{pmatrix}. \tag{6.29}$$

(ii) *When* $e^{-\mathrm{i}\theta_0}\int_v^x \sqrt{Q_0(x)}dx < 0$ *on* C*, then*

$$(\Psi_+^{(\theta_0-\varepsilon)}, \Psi_-^{(\theta_0-\varepsilon)}) = (\Psi_+^{(\theta_0+\varepsilon)}, \Psi_-^{(\theta_0+\varepsilon)})\begin{pmatrix} 1 & -\mathrm{i} \\ 0 & 1 \end{pmatrix}. \tag{6.30}$$

The lower/upper triangular matrices in (6.29) and (6.30) are called the *Stokes matrices* associated with the direction θ_0.

6.4.2. *The case* (ii). Next let us show the formula for the Stokes phenomenon of the type (ii) caused by the loop-type saddle connection.

Suppose that a direction θ_0 and a point x satisfy the following conditions:

- The Stokes graph G_{θ_0} has a loop-type saddle connection around 0.
- The point x does not lie on the Stokes graph G_{θ_0}.

Set

$$\delta = \oint_{\gamma_0 - \sigma_* \gamma_0} \sqrt{Q_0(x)}dx = 2\pi \mathrm{i}(w_0 - w_1). \tag{6.31}$$

The first assumption implies that $\mathrm{e}^{-\mathrm{i}\theta_0}\delta \in \mathbb{R}_{\neq 0}$. To specify the situation, we further assume:

- The real part of $\mathrm{e}^{-\mathrm{i}\theta_0}\delta$ is positive.

The second assumption implies x lies on one of connected components D_0 and D_∞ of $\mathbb{C}\mathbf{P}^1 \setminus \{\text{loop}\}$, where D_0 (resp. D_∞) contains $x = 0$ (resp. $x = \infty$).

Let $\psi_\pm$ be the WKB solution (6.16) normalized at v. Note that there is an ambiguity in the choice of the path from v to x (see Remark 6.3), but the following formula holds for arbitrary choice.

Theorem 6.9 ([7]). *In the situation above, the following statements hold:*

(i) *If $x \in D_\infty$, then the WKB solutions $\psi_\pm$ are Borel summable in the direction θ_0. In particular, the Borel sum of the WKB solutions satisfy*

$$(\Psi_+^{(\theta_0-\varepsilon)}, \Psi_-^{(\theta_0-\varepsilon)}) = (\Psi_+^{(\theta_0+\varepsilon)}, \Psi_-^{(\theta_0+\varepsilon)}), \tag{6.32}$$

where $\Psi_\pm^{(\theta_0-\varepsilon)}$ (resp. $\Psi_\pm^{(\theta_0+\varepsilon)}$) are the Borel sum of $\psi_\pm$ in the direction $\theta_0 - \varepsilon$ (resp. $\theta_0 + \varepsilon$) for sufficiently small $\varepsilon > 0$.

(ii) *If $x \in D_0$, then the Borel transformed WKB solution $\psi_{\pm,B}$ has singular points at $y = a_\pm(x) + m\delta$ with $m \in \mathbb{Z}_{\neq 0}$ (and hence $\psi_\pm$ is not Borel summable in the direction θ_0). The Borel sum of the WKB solutions satisfy*

$$(\Psi_+^{(\theta_0,R)}, \Psi_-^{(\theta_0,R)}) = (\Psi_+^{(\theta_0,L)}, \Psi_-^{(\theta_0,L)}) \begin{pmatrix} 1 - \mathrm{e}^{-V_{\gamma_0}} & 0 \\ 0 & (1 - \mathrm{e}^{-V_{\gamma_0}})^{-1} \end{pmatrix}, \tag{6.33}$$

where $\Psi_\pm^{(\theta_0,R)}$ (resp. $\Psi_\pm^{(\theta_0,L)}$) is the Borel sum of $\psi_\pm$ in the direction θ_0 defined as the Laplace integral as (6.22) along the path $\ell_{\theta_0,R}$ (resp. $\ell_{\theta_0,L}$) depicted in Figure 11.

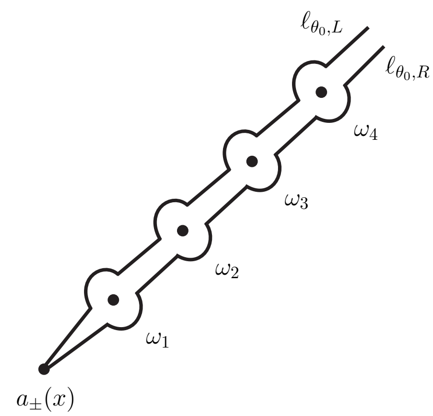


FIGURE 11. The paths $\ell_{\theta_0,R}$ and $\ell_{\theta_0,L}$. These paths are obtained by deforming the path ℓ_{θ_0}; $\ell_{\theta_0,R}$ and $\ell_{\theta_0,L}$ avoid the singular point $\omega_m = a^{(\pm)}(x) + m\delta$ from the right and the left, respectively.

We also call the diagonal matrix in (6.33) the *Stokes matrix* for the direction θ_0. This kind of Stokes phenomenon is never observed in the non-equivariant case (i.e. $w_0 - w_1 = 0$). The condition for δ implies that the quantity $\mathrm{e}^{-V_{\gamma_0}} = \mathrm{e}^{-\delta/\hbar}$ in the Stokes matrix is exponentially small when $\hbar \to 0$, $\arg \hbar = \theta_0$.

Remark 6.10. In the situation $x \in D_0$, for any sufficiently small $\varepsilon > 0$, there are infinitely many directions between $\theta_0 - \varepsilon$ and $\theta_0 + \varepsilon$ where the WKB solution is not Borel summable. Hence we employ the slightly different version of the Borel sum in the relation (6.33). The effect of infinitely many Stokes phenomenon will be discussed in next subsection.

6.5. **Computation of the total Stokes matrix.** In this subsection, for a fixed x, we compute the *"total" Stokes matrix* defined as

$$S_{\mathrm{tot}} = \overleftarrow{\prod_{0 \le \theta < \pi}} S_\theta. \tag{6.34}$$

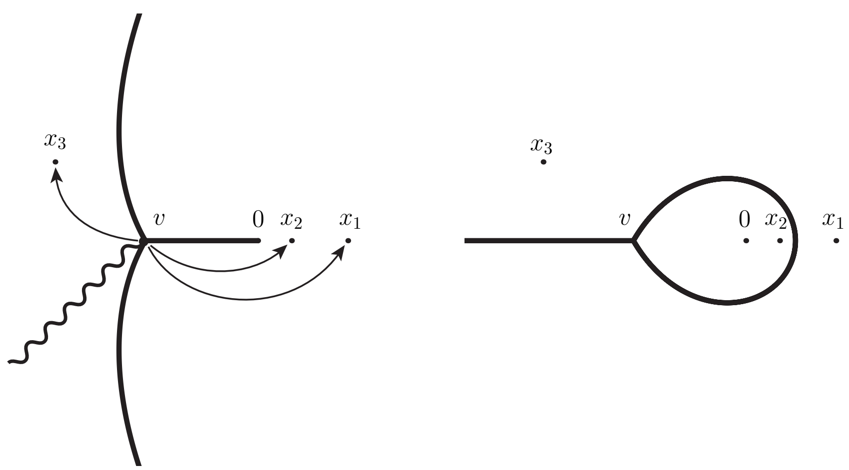
Here S_θ is the Stokes matrix for the WKB solution in the direction θ, and they are multiplied from the left as θ increases. We regard $S_\theta = \mathrm{Id}$ when $\psi_\pm$ is Borel summable in the direction θ. Therefore, S_{tot} relate the Borel sum of WKB solutions in opposite directions:

$$(\Psi_+^{(0)}, \Psi_-^{(0)}) = (\Psi_+^{(\pi)}, \Psi_-^{(\pi)}) S_{\mathrm{tot}}. \tag{6.35}$$

In what follows, we consider the following situation:

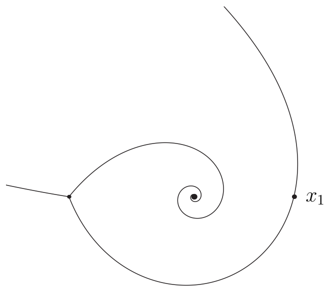
- $w_0 - w_1 \in \mathbb{R}_{>0}$; that is, δ defined in (6.31) satisfies $\mathrm{e}^{-\pi \mathrm{i}/2}\delta \in \mathbb{R}_{>0}$.
- x is fixed at one of x_1, x_2 or x_3 satisfying the following conditions (see Figure 12 (a) and (b) which depicts the Stokes graph G_0 for $\theta = 0$ and $G_{\pi/2}$ for $\theta = \pi/2$, respectively):
  - x_1 and x_2 are points on the same Stokes region, which has the point $x = 0$ on its boundary in G_0. We choose $x_1 \in D_\infty$ and $x_2 \in D_0$.
  - x_3 is on the other Stokes region in G_0, which does not contain x_1 and x_2.
- We use the WKB solution normalized at the turning point v, whose normalization path from v to x_i ($i = 1, 2, 3$) is chosen as indicated in Figure 12 (a).

In this situation we can verify that $\int_v^x \sqrt{Q_0(x)}dx < 0$ on the Stokes curve of the phase $\theta = 0$ which flows into the origin. We will employ the technique developed in [78, Section 3] to compute the Stokes matrix for the WKB solutions.



(a) : Stokes graph for $\theta = 0$. (b) : Stokes graph for $\theta = \pi/2$.

FIGURE 12. The points x_1, x_2 and x_3. Note that this figure depicts the specific situation where $w_0 - w_1 = 1$, $x_1 = 0.8$, $x_2 = 0.3$ and $x_3 = -1.8 + 0.7i$. However, the formulas obtained in this subsection holds in general (under the assumption above).



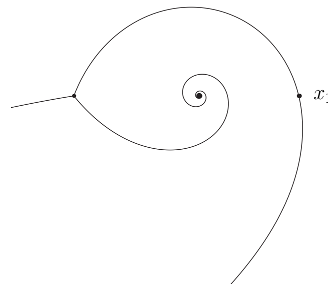


FIGURE 13. The point x_1 is hit by Stokes curves. The situation (a) (resp. (b)) occurs when $\theta_1 = \arg\bigl(\int_{\gamma_{x_1}} \sqrt{Q_0(x)}dx\bigr)$ (resp. $\theta_2 = \arg\bigl(\int_{\tilde{\gamma}_{x_1}} \sqrt{Q_0(x)}dx\bigr)$) where γ_{x_1} and $\tilde{\gamma}_{x_1}$ are depicted in Figure 9.

6.5.1. *Stokes matrix at x_1.* Since the point x_1 is contained in D_∞, Figure 10 shows that x_1 is hit by Stokes curves twice when we vary θ from 0 to π. The situation for the first hit (resp. the second hit) is depicted in Figure 13 (a) (resp. Figure 13 (b)), and it happens at some θ_1 satisfying $0 < \theta_1 < \pi/2$ (resp. at some θ_2 satisfying $\pi/2 < \theta_2 < \pi$). These two Stokes curves causes Stokes phenomena for the WKB solutions, and each of contribution to $S_{\rm tot}$ is given as follows:

- For the first hit depicted in Figure 13 (a), since $\mathrm{e}^{-\mathrm{i}\theta_1} \int_{\gamma_{x_1}} \sqrt{Q_0(x)}dx > 0$ holds in this situation, Theorem 6.8 (i) implies that the corresponding Stokes matrix is given by
$$(\Psi_+^{(\theta_1-\varepsilon)}, \Psi_-^{(\theta_1-\varepsilon)}) = (\Psi_+^{(\theta_1+\varepsilon)}, \Psi_-^{(\theta_1+\varepsilon)}) S_{\theta_1}, \quad S_{\theta_1} = \begin{pmatrix} 1 & 0 \\ -\mathrm{i} & 1 \end{pmatrix}. \tag{6.36}$$
- For the second hit depicted in Figure 13 (b), we need to care about the normalization of the WKB solutions. Since $\psi_\pm$ is not normalized along the Stokes curve which hits x_1 in Figure 13 (b), we cannot apply Theorem 6.8 directly. However, Theorem 6.8 can be applied to the WKB solution $\tilde{\psi}_\pm$ which is normalized along the path $\tilde{\gamma}_x$ in Figure 9. Since the WKB solutions $\psi_\pm$ and $\tilde{\psi}_\pm$ are related as (6.17), we can compute the Stokes matrix for $\psi_\pm$ and obtain the Stokes matrix (note that $\mathrm{e}^{-\mathrm{i}\theta_2} \int_{\tilde{\gamma}_{x_1}} \sqrt{Q_0(x)}dx < 0$ in this case):
$$(\Psi_+^{(\theta_2-\varepsilon)}, \Psi_-^{(\theta_2-\varepsilon)}) = (\Psi_+^{(\theta_2+\varepsilon)}, \Psi_-^{(\theta_2+\varepsilon)}) S_{\theta_2}, \quad S_{\theta_2} = \begin{pmatrix} 1 & -\mathrm{i}\,\mathrm{e}^{-V_{\gamma_0}} \\ 0 & 1 \end{pmatrix}. \tag{6.37}$$

Although the loop-type saddle connection appears in $\theta = \pi/2$, WKB solution is Borel summable since $x_1 \in D_\infty$ (see Theorem 6.9 (i)), and hence no Stokes phenomenon occurs in $\theta = \pi/2$. Therefore, we have

Theorem 6.11. *The total Stokes matrix at x_1 is given by*
$$S_{\rm tot} = S_{\theta_2} S_{\theta_1} = \begin{pmatrix} 1 - \mathrm{e}^{-V_{\gamma_0}} & -\mathrm{i}\,\mathrm{e}^{-V_{\gamma_0}} \\ -\mathrm{i} & 1 \end{pmatrix}. \tag{6.38}$$

6.5.2. *Stokes matrix at x_2.* From Figure 10, we can observe that, for $0 \le \theta < \pi/2$ and for $\pi/2 < \theta < \pi$, Stokes curves hit x_2 infinitely many times. Stokes directions (i.e. directions with non-trivial S_θ) accumulates to $\theta = \pi/2$ due to the spiral behavior of the Stokes curve. In fact, such situation has been analyzed in [72, Appendix B] and the total Stokes matrix which relate $\Psi_\pm^{(0)}$ and $\Psi_\pm^{(\pi)}$ is given by a convergent infinite product of matrices as follows.

- Let us describe the Stokes matrix between $\Psi_\pm^{(0)}$ and $\Psi_\pm^{(\frac{\pi}{2},R)}$ which includes all contribution from the Stokes curve which hits x_2 infinitely many times in $0 < \theta < \pi/2$. For each hit, the computation of the Stokes matrix can be done in a similar manner as above. Since $e^{-i\theta}\int_v^x \sqrt{Q_0(x)}dx < 0$ along the Stokes curve which hits x_2, each of Stokes matrix is upper triangular due to Theorem 6.8. The result of the computation in [72, Appendix B] shows

$$(\Psi_+^{(0)}, \Psi_-^{(0)}) \quad = \quad (\Psi_+^{(\frac{\pi}{2},R)}, \Psi_-^{(\frac{\pi}{2},R)}) \prod_{n\geq 1} \begin{pmatrix} 1 & -\mathrm{i}\, e^{-nV_{\gamma_0}} \\ 0 & 1 \end{pmatrix}. \tag{6.39}$$

The infinite product converges as long as $\hbar$ lies on the upper half plane.

- For $\theta = \pi/2$, we can use Theorem 6.9 (ii):

$$(\Psi_+^{(\frac{\pi}{2},R)}, \Psi_-^{(\frac{\pi}{2},R)}) = (\Psi_+^{(\frac{\pi}{2},L)}, \Psi_-^{(\frac{\pi}{2},L)}) \begin{pmatrix} 1 - e^{-V_{\gamma_0}} & 0 \\ 0 & (1 - e^{-V_{\gamma_0}})^{-1} \end{pmatrix}. \tag{6.40}$$

- For $\pi/2 < \theta < \pi$, a Stokes curve hits x_2 infinitely many times again. Similarly to the case $0 < \theta < \pi/2$, we have

$$(\Psi_+^{(\frac{\pi}{2},L)}, \Psi_-^{(\frac{\pi}{2},L)}) \quad = \quad (\Psi_+^{(\pi)}, \Psi_-^{(\pi)}) \prod_{n\geq 0} \begin{pmatrix} 1 & 0 \\ -\mathrm{i}\, e^{-nV_{\gamma_0}} & 1 \end{pmatrix}. \tag{6.41}$$

The total Stokes matrix is the product of the matrices computed in (6.39), (6.40) and (6.41):

Theorem 6.12. *The total Stokes matrix at $x = x_2$ is given by*

$$\begin{aligned} S_{\mathrm{tot}} &= \left(\prod_{n\geq 0} \begin{pmatrix} 1 & 0 \\ -\mathrm{i}\, e^{-nV_{\gamma_0}} & 1 \end{pmatrix}\right) \begin{pmatrix} 1 - e^{-V_{\gamma_0}} & 0 \\ 0 & (1 - e^{-V_{\gamma_0}})^{-1} \end{pmatrix} \left(\prod_{n\geq 1} \begin{pmatrix} 1 & -\mathrm{i}\, e^{-nV_{\gamma_0}} \\ 0 & 1 \end{pmatrix}\right) \\ &= \begin{pmatrix} 1 - e^{-V_{\gamma_0}} & -\mathrm{i}\, e^{-V_{\gamma_0}} \\ -\mathrm{i} & 1 \end{pmatrix}. \end{aligned} \tag{6.42}$$

Remark 6.13. We have observed that the total Stokes matrix at x_1 and at x_2 are the same matrix. This is because x_1 and x_2 lies on the same Stokes region in G_0 and the Borel resummed WKB solutions at x_1 and x_2 give the same basis of the space of the solution of the equation (6.1). However, this is a consequence of a non-trivial identity of infinite product of matrices:

$$\begin{aligned} &\begin{pmatrix} 1 & -\mathrm{i}\, e^{V_{\gamma_0}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\mathrm{i} & 1 \end{pmatrix} \\ &\quad = \left(\prod_{n\geq 0} \begin{pmatrix} 1 & 0 \\ -\mathrm{i}\, e^{-nV_{\gamma_0}} & 1 \end{pmatrix}\right) \begin{pmatrix} 1 - e^{-V_{\gamma_0}} & 0 \\ 0 & (1 - e^{-V_{\gamma_0}})^{-1} \end{pmatrix} \left(\prod_{n\geq 1} \begin{pmatrix} 1 & -\mathrm{i}\, e^{-nV_{\gamma_0}} \\ 0 & 1 \end{pmatrix}\right). \end{aligned} \tag{6.43}$$

Note that, the identity (6.43) is an example of the *2d/4d wall-crossing formula* in the sense of [49] by D. Gaiotto, G.W. Moore and A. Neitzke. Each upper and lower triangular matrix (contribution from the situation where x lies on a Stokes curve) captures 2d-BPS states, while the diagonal matrix (contribution from the loop-type saddle connection) captures a 4d-BPS state. Here the loop-type Stokes curve plays the role of wall, and the above identity describes how BPS indices "jump" when x crosses the wall. The 2d/4d wall-crossing formula in $\mathbb{CP}^1$ sigma model[16] has already been mentioned in [49, Section 8.2], and we have given an exact WKB theoretic interpretation of the wall-crossing formula in this special case.

6.5.3. *Stokes matrix at $x = x_3$ and Dubrovin's conjecture.* As well as the case of $x = x_1$, Figure 10 shows that there are only two Stokes directions in this case. We can verify that the two Stokes matrices are both lower triangular, and the result is given as follows:

Theorem 6.14. *The total Stokes matrix at $x = x_3$ is given by*

$$S_{\mathrm{tot}} = \begin{pmatrix} 1 & 0 \\ -\mathrm{i}\, e^{V_{\gamma_0}} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\mathrm{i} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\mathrm{i}\,(1 + e^{V_{\gamma_0}}) & 1 \end{pmatrix}.$$

[16] In this context, the GKZ curve appears as the chiral ring studied by N. Dorey [37].

Using the relation proved in Proposition 6.7, we obtain the Stokes matrix for the oscillatory integral solutions:

$$(I_+^{(0)}, I_-^{(0)}) = (I_+^{(\pi)}, I_-^{(\pi)}) \begin{pmatrix} 1 & 0 \\ 1+\mathrm{e}^{V_{\gamma_0}} & 1 \end{pmatrix}. \tag{6.44}$$

In the non-equivariant limit $w_0 - w_1 \to 0$ the non-trivial Stokes multiplier $1+\mathrm{e}^{V_{\gamma_0}}$ tends to 2, and this coincides with the Euler pairing $\chi(\mathcal{O}, \mathcal{O}(1)) = 2$ on the derived category $D^b\mathrm{Coh}(\mathbb{C}\mathbf{P}^1)$. This is consistent with the Dubrovin's conjecture [39] for the Stokes matrix of the quantum cohomology of $\mathbb{C}\mathbf{P}^1$. The conjecture was proved by D. Guzzetti [58] for projective spaces, K. Ueda [96, 97] for Grassmannians and smooth cubic surfaces; see also [73, 51, 52, 92] for related works on the conjecture.

Here we propose a statement which suggests that there is an "equivariant-version" of the Dubrovin's conjecture. We will verify that the above Stokes multiplier $1+\mathrm{e}^{V_{\gamma_0}}$ for the quantum differential equation (6.1) of the equivariant $\mathbb{C}\mathbf{P}^1$ can be identified with the equivariant Euler pairing of equivariant coherent sheaves on $\mathbb{C}\mathbf{P}^1$, following the idea of H. Iritani.

Let $T = (\mathbb{C}^*)^2$ be the algebraic torus. We regard $\mathbb{C}\mathbf{P}^1$ as a T-space by the action

$$T \times \mathbb{C}\mathbf{P}^1 \to \mathbb{C}\mathbf{P}^1, \quad (t_0, t_1)\cdot[x_0 : x_1] = [t_0^{-1}x_0 : t_1^{-1}x_1],$$

where $[x_0 : x_1]$ is the homogeneous coordinate of $\mathbb{C}\mathbf{P}^1$. Let us also regard $\mathcal{O}$ and $\mathcal{O}(1)$ as T-equivariant coherent sheaves (see [27, Section 5] for example) on $\mathbb{C}\mathbf{P}^1$ as follows:

- $\mathcal{O}$ is equipped with the trivial T-action,
- To equip a T-action on $\mathcal{O}(1)$, use the expression

  $$\mathcal{O}(1) = \big((\mathbb{C}^2 \setminus \{0\}) \times \mathbb{C}\big)/\mathbb{C}^*, \quad (x_0, x_1, s) \ \sim\ (\lambda x_0, \lambda x_1, \lambda s) \ \ (\lambda \in \mathbb{C}^*),$$

  and denote by $[x_0 : x_1 : s]$ the homogeneous coordinate of $\mathcal{O}(1)$. We introduce a T-action on $\mathcal{O}(1)$ by

  $$T \times \mathcal{O}(1) \to \mathcal{O}(1), \quad (t_0, t_1)\cdot[x_0 : x_1 : s] = [t_0^{-1}x_0 : t_1^{-1}x_1 : t_0 s]$$

  which gives a T-equivariant structure on $\mathcal{O}(1)$.

Our goal is to compute the $\hbar$-modified T-equivariant Euler pairing of $\mathcal{O}$ and $\mathcal{O}(1)$:

$$\chi_T^{\hbar}(\mathcal{O}, \mathcal{O}(1)) = \sum_i (-1)^i \mathrm{ch}_T^{\hbar}([H^i(\mathbb{C}\mathbf{P}^1, \mathcal{O}(1))]),$$

where the right hand-side takes value in $\mathbb{Z}[T] = \mathbb{Z}[\mathrm{e}^{\pm 2\pi \mathrm{i} w_0/\hbar}, \mathrm{e}^{\pm 2\pi \mathrm{i} w_1/\hbar}]$, and $\mathrm{ch}_T^{\hbar}$ is the $\hbar$-modified Chern character map (introduced in [31, Section 3.1]) from the T-representation ring $R[T]$ to $\mathbb{Z}[T]$. Note that $R[T] = \mathbb{Z}[\mathrm{e}^{\pm\mu_0}, \mathrm{e}^{\pm\mu_1}]$ (which can be identified with T-equivariant K-group $K_T^0(\mathrm{pt})$ of a point) consists of the class of irreducible representations of T; the symbol $\mathrm{e}^{m\mu_0 + n\mu_1}$ for $(m,n) \in \mathbb{Z}^2$ is the class of the representation spanned by a weight vector v satisfying $(t_0, t_1)\cdot v = t_0^m\, t_1^n\, v$ for $(t_0, t_1) \in T$. The map $\mathrm{ch}_T^{\hbar}$ sends $\mathrm{e}^{m\mu_0+n\mu_1} \in R[T]$ to $\mathrm{e}^{2\pi \mathrm{i}(m w_0 + n w_1)/\hbar} \in \mathbb{Z}[T]$.

There are two independent global sections

$$s_i : \mathbb{C}\mathbf{P}^1 \to \mathcal{O}(1), \quad [x_0 : x_1] \mapsto [x_0 : x_1 : x_i] \quad (i = 0, 1).$$

of $\mathcal{O}(1)$. The T-actions on these global sections are given by

$$(t_0, t_1)\cdot s_0 = s_0, \quad (t_0, t_1)\cdot s_1 = t_0\, t_1^{-1}\, s_1 \qquad (t_0, t_1) \in T.$$

Therefore, the global section s_0 (resp. s_1) gives a weight vector of the weight $(0,0)$ (resp. weight $(1,-1)$), and hence

$$[H^0(\mathbb{C}\mathbf{P}^1, \mathcal{O}(1))] = [\Gamma(\mathbb{C}\mathbf{P}^1, \mathcal{O}(1))] = 1 + \mathrm{e}^{\mu_0 - \mu_1} \tag{6.45}$$

holds in $R[T]$. The weight decomposition (6.45) (together with the fact that $H^i(\mathbb{C}\mathbf{P}^1, \mathcal{O}(1)) = 0$ for $i \neq 0$) implies the following:

Proposition 6.15. *The $\hbar$-modified T-equivariant Euler pairing of $\mathcal{O}$ and $\mathcal{O}(1)$ (regarded as T-equivariant coherent sheaves as above) is given by*

$$\chi_T^{\hbar}(\mathcal{O}, \mathcal{O}(1)) = 1 + \mathrm{e}^{2\pi\mathrm{i}(w_0 - w_1)/\hbar}, \tag{6.46}$$

and this coincides with the Stokes multiplier $1 + \mathrm{e}^{V_{\gamma_0}}$ in the total Stokes matrix at x_3.

Thus, we conclude that the Stokes multiplier of the quantum differential equation (6.1) for the equivariant $\mathbb{C}\mathbf{P}^1$ model can be identified with the equivariant Euler pairing of equivariant coherent sheaves on $\mathbb{C}\mathbf{P}^1$. This observation was pointed to the authors by H. Iritani. We expect that similar coincidence (between Stokes multiplies and equivariant Euler pairings) hold for wider class of quantum differential equations for equivariant target spaces. We also expect that the coincidence follows from the categorical equivalence discussed in [44, 45] etc.

Appendix A. GKZ curve from the J-function

In this appendix we will discuss a heuristic derivation of the GKZ curves from the J-functions for the projective space and complete intersections. For this purpose, we will introduce the *on-shell equivariant J-function.*[17] Let X be the projective space $\mathbb{C}\mathbf{P}_{\boldsymbol{w}}^{N-1}$ or the complete intersection $X = X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}$ in $\mathbb{C}\mathbf{P}^{N-1}$. Let $\ell : H_T^*(\mathbb{C}\mathbf{P}^{N-1}) \to \mathbb{C}$ be a $\mathbb{C}$-linear map. For the equivariant J-function $J_X(x)$, the composite map $(\ell \circ J_X)(x)$ is a $\mathbb{C}$-valued function satisfying GKZ equation. We assume that ℓ is a $\mathbb{C}$-algebra homomorphism. Then by

$$H_T^*(\mathbb{C}\mathbf{P}^{N-1}) \cong \mathbb{C}[p]/(\prod_{i=0}^{N-1}(p - w_i)), \tag{A.1}$$

$\ell(p)$ must be one of equivariant parameters w_i $(i = 0, \ldots, N-1)$.

Definition A.1. *The function $\mathcal{J}_X(x)$ is named on-shell equivariant J-function, if the second equivariant cohomology element $p \in H_T^2(X)$ in Proposition 1.3 is replaced with one of equivariant parameters w_i $(i = 0, \ldots, N-1)$:*

$$\mathcal{J}_X(x) = J_X(x)|_{p=w_i}. \tag{A.2}$$

By the construction, we have the following lemma:

Lemma A.2. *The on-shell equivariant J-function obeys the GKZ equation.*

$$\widehat{A}_X(\widehat{x}, \widehat{y})\mathcal{J}_X(x) = 0. \tag{A.3}$$

We remark that if $w_i \neq w_j$ for $i \neq j$, then $\mathbb{C}[p]/(\prod_{i=0}^{N-1}(p - w_i)) \cong (\mathbb{C}[p]/(p - w_0)) \times \cdots \times (\mathbb{C}[p]/(p - w_{N-1}))$ is a product of N-copies of the $\mathbb{C}$-algebra $\mathbb{C}$. Then the $\mathbb{C}$-algebra homomorphisms $H_T^*(\mathbb{C}\mathbf{P}^{N-1}) \to \mathbb{C}, p \mapsto w_i$ give a $\mathbb{C}$-basis of the space of $\mathbb{C}$-linear maps: $H_T^*(\mathbb{C}\mathbf{P}^{N-1}) \to \mathbb{C}$. Thus the on-shell J-functions give basis of the solution of GKZ equation.

Now we will consider the asymptotic expansion of the on-shell equivariant J-function.

$$\mathcal{J}_X(x) \sim \exp\left(\sum_{m=0}^{\infty} \hbar^{m-1} S_m(x)\right). \tag{A.4}$$

Using (2.12) and this asymptotic expansion we find the defining equation of the GKZ equation $A_X(x, y) = 0$ in $(x, y) \in \mathbb{C}^* \times \mathbb{C}$ from the relation:

$$y = x\frac{dS_0(x)}{dx} \in \mathbb{C}. \tag{A.5}$$

In the following we will evaluate the saddle point value $S_0(x)$ of the on-shell equivariant J-function $\mathcal{J}_X(x)$ for the projective space $X = \mathbb{C}\mathbf{P}_{\boldsymbol{w}}^{N-1}$ and the smooth Fano complete intersection $X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}$ in

[17] The name *on-shell* is inherited from the ("on-shell") vortex partition function (5.2). Indeed for the choice $p = w_0$, $\mathcal{J}_X(x)$ agrees with $Z_{\mathrm{vortex}}^X(x)$.

a heuristic way. As a consequence we will show that the defining equation $A_X(x, y) = 0$ of the GKZ equation is obtained for these two cases of X.

(1) Projective space $\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}$:
Let p denote one of equivariant parameters w_i $(i = 0, \dots, N-1)$. We focus on the factor $\prod_{m=1}^{d}(p - w + m\hbar)$ to find the saddle point value of the on-shell equivariant J-function $\mathcal{J}_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(x)$ in (1.7). In the $\hbar \to 0$ limit while keeping $d\hbar = z$ finite, we use the Riemann integral as follows:

$$\prod_{m=1}^{d}(p - w + m\hbar) = \exp\left[\sum_{m=1}^{d}\log(p - w + m\hbar)\right] \underset{\substack{\hbar\to 0\\ d\hbar = u:\text{finite}}}{\sim} \exp\left[\frac{1}{\hbar}\int_0^u du' \, \log(p - w + u')\right]$$
$$= \exp\left[\frac{1}{\hbar}\left((u + p - w)\log(u + p - w) - u - (p - w)\log(p - w)\right)\right]. \tag{A.6}$$

Adopting this factor we can approximate $\mathcal{J}_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(x)$ by the integral on z as

$$\mathcal{J}_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(x) \underset{\hbar\to 0}{\sim} \int_\gamma du \, \exp\left[\frac{1}{\hbar}\mathcal{W}_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(u; x)\right], \tag{A.7}$$
$$\mathcal{W}_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(u; x) = (u + p)\log x - \sum_{i=0}^{N-1}\big((u + p - w_i)\log(u + p - w_i) - u - (p - w_i)\log(p - w_i)\big),$$

where we interpret the term $(p - w_i)\log(p - w_i)$ as 0 when $p = w_i$. Here we call $\mathcal{W}_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(u; x)$ *effective superpotential*, and an analytical continuation can be performed by deforming the integration path γ on the complex u plane.[18] Assuming such analytical continuation, we can approximate the integral (A.7) by the saddle point value in $\hbar \to 0$ limit:

$$S_0(x) = \mathcal{W}_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(u_c; x), \qquad \left.\frac{\partial \mathcal{W}_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(u; x)}{\partial u}\right|_{u=u_c} = 0.$$

The saddle point condition is then given by

$$\left.\frac{\partial \mathcal{W}_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(u; x)}{\partial u}\right|_{u=u_c} = \log x - \sum_{i=0}^{N-1}\log(u_c + p - w_i) = 0, \tag{A.8}$$

and by (A.5) one has

$$y = x\left.\frac{\partial \mathcal{W}_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(u; x)}{\partial x}\right|_{u=u_c} = u_c + p. \tag{A.9}$$

By eliminating the variable u_c from these relations (A.8) and (A.9), we find a constraint equation on $(x, y) \in \mathbb{C}^* \times \mathbb{C}$:

$$A_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(x, y) = \prod_{i=0}^{N-1}(y - w_i) - x = 0. \tag{A.10}$$

This polynomial $A_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(x, y)$ agrees with the defining equation of the GKZ curve (2.15).
(2) Complete intersection $X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}$ in $\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}$:
Adopting the similar approximation for the factor (A.6) in $\mathcal{J}_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(x)$, we obtain the effective superpotential $\mathcal{W}_{X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}}(u; x)$ for the on-shell equivariant J-function $\mathcal{J}_{X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}}(x)$ in (1.8) as

$$\mathcal{J}_{X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}}(x) \underset{\hbar\to 0}{\sim} \int_\gamma dz \, \exp\left[\mathcal{W}_{X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}}(u; x)\right], \tag{A.11}$$
$$\mathcal{W}_{X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}}(u; x) = (u + p)\log x - \sum_{i=0}^{N-1}\big((u + p - w_i)\log(u + p - w_i) - u - (p - w_i)\log(p - w_i)\big)$$

[18] In [55, 60] the similar analysis is discussed for the colored Jones polynomial of the knot in $\mathbb{S}^3$.

$$+\sum_{a=1}^{n}\big((l_a u + l_a p - \lambda_a)\log(l_a u + l_a p - \lambda_a) - l_a u - (l_a p - \lambda_a)\log(l_a p - \lambda_a)\big),$$

where p denotes one of equivariant parameters w_i $(i = 0, \ldots, N-1)$ and we interpret the term $(p - w_i)\log(p - w_i)$ as 0 when $p = w_i$. The saddle point condition is then given by

$$\left.\frac{\partial \mathcal{W}_{X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}}(u;x)}{\partial u}\right|_{u=u_c} = \log x - \sum_{i=0}^{N-1}\log(u_c + p - w_i) - \sum_{a=1}^{n}\log(l_a u_c + l_a p - \lambda_a)^{-l_a} = 0, \tag{A.12}$$

and by (A.5) one has

$$y = x\left.\frac{\partial \mathcal{W}_{X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}}(u;x)}{\partial x}\right|_{u=u_c} = u_c + p. \tag{A.13}$$

As the result of the elimination of the variable u_c from these relations (A.12) and (A.13), we find a constraint equation on $(x, y) \in \mathbb{C}^* \times \mathbb{C}$:

$$A_{X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}}(x,y) = \prod_{i=0}^{N-1}(y - w_i) - x\prod_{a=1}^{n}(l_a y - \lambda_a)^{l_a} = 0. \tag{A.14}$$

This polynomial $A_{X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}}(x,y)$ agrees with the defining equation of the GKZ curve (2.17).

Appendix B. GKZ equations for oscillatory integrals

In this appendix we will give a proof of Propositions 2.4 and 2.7.

B.1. **Proof of Propositions 2.4.** We will show that the oscillatory integral $\mathcal{I}_X(x)$ satisfies the GKZ equation for the projective space $X = \mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}$ and the Fano complete intersection $X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}$ separately.

Proposition B.1. *The oscillatory integral $\mathcal{I}_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(x)$ in (2.5) satisfies the GKZ equation (1.9).*

Proof. Act a differential operator $\left(\hbar x\frac{d}{dx} - w_i\right)\left(\hbar x\frac{d}{dx} - w_0\right)$ on the oscillatory integral $\mathcal{I}_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(x)$,

$$\begin{aligned}
&\left(\hbar x\frac{d}{dx} - w_i\right)\left(\hbar x\frac{d}{dx} - w_0\right)\mathcal{I}_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(x)\\
&= \left(\hbar x\frac{d}{dx} - w_i\right)\int_\Gamma \prod_{i=1}^{N-1} du_i\,\frac{x}{(u_1\cdots u_{N-1})^2}\,\mathrm{e}^{\frac{1}{\hbar}W_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(u_1,\ldots,u_{N-1};x)}\\
&= \int_\Gamma \prod_{i=1}^{N-1} du_i\,\frac{x}{(u_1\cdots u_{N-1})^2}\left(\frac{x}{u_1\cdots u_{N-1}} + w_0 - w_i + \hbar\right)\mathrm{e}^{\frac{1}{\hbar}W_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(u_1,\ldots,u_{N-1};x)},
\end{aligned}$$

where $W_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}$ is the Landau-Ginzburg potential given in (2.2). To manipulate further we will use the following integration by parts:

$$\begin{aligned}
0 &= \int_\Gamma \prod_{i=1}^{N-1} du_i\,\hbar\frac{d}{du_i}\left(\frac{1}{u_i}\,\mathrm{e}^{\frac{1}{\hbar}W_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(u_1,\ldots,u_{N-1};x)}\right)\\
&= \int_\Gamma \prod_{i=1}^{N-1} du_i\,\frac{1}{u_i^2}\left[\left(u_i + w_i - w_0 - \hbar - \frac{x}{u_1\cdots u_{N-1}}\right)\right]\mathrm{e}^{\frac{1}{\hbar}W_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(u_1,\ldots,u_{N-1};x)}.
\end{aligned}$$

In this computation, any boundary contributions do not appear, because the image of the Lefschetz thimble Γ is a relative cycle starting from a non-degenerate critical point p_{crit} of the Landau-Ginzburg potential W_X to the infinity. Then one finds that

$$\left(\hbar x\frac{d}{dx} - w_i\right)\left(\hbar x\frac{d}{dx} - w_0\right)\mathcal{I}_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(x) = \int_\Gamma \prod_{i=1}^{N-1} du_i\,\frac{x u_i}{(u_1\cdots u_{N-1})^2}\,\mathrm{e}^{\frac{1}{\hbar}W_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(u_1,\ldots,u_{N-1};x)}.$$

Repeating the above manipulations for $\left[\prod_{i=1}^{N-1}\left(\hbar x\frac{d}{dx}-w_i\right)\right]\left(\hbar x\frac{d}{dx}-w_0\right)$, the following relation is obtained:

$$\begin{aligned}
&\left[\prod_{i=1}^{N-1}\left(\hbar x\frac{d}{dx}-w_i\right)\right]\left(\hbar x\frac{d}{dx}-w_0\right)\mathcal{I}_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(x)\\
&=\int_{\Gamma}\prod_{i=1}^{N-1}du_i\,\frac{xu_1\cdots u_{N-1}}{(u_1\cdots u_{N-1})^2}\,\mathrm{e}^{\frac{1}{\hbar}W_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(u_1,\ldots,u_{N-1};x)}=x\mathcal{I}_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(x).
\end{aligned}\tag{B.1}$$

This differential equation is the same as the GKZ equation (1.9) for the J-function $J_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}$. □

Proposition B.2. *The oscillatory integral $\mathcal{I}_{X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}}(x)$ in (2.6) satisfies the GKZ equation (1.9).*

Proof. Consider the GKZ equation (B.1) for the oscillatory integral $\mathcal{I}_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(x)$ for the mirror Landau-Ginzburg model of the projective space $X=\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}$ denoted by

$$0=\widehat{A}_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(\widehat{x},\widehat{y})\,\mathcal{I}_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(x)=\prod_{i=0}^{N-1}(\widehat{y}-w_i)\,\mathcal{I}_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(x)-\widehat{x}\mathcal{I}_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(x),$$

where $\widehat{x}$ (resp. $\widehat{y}$) acts on $\mathcal{I}_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(x)$ as x (resp. $\hbar x d/dx$). Perform the Laplace transformation of this differential equation:

$$\begin{aligned}
0&=\int_0^\infty dv_1\cdots\int_0^\infty dv_n\,\mathrm{e}^{-\frac{\sum_{i=1}^n(v_i+\lambda_i\log v_i)}{\hbar}}\widehat{A}_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}\left(v_1^{l_1}\cdots v_n^{l_n}\widehat{x},\widehat{y}\right)\mathcal{I}_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(v_1^{l_1}\cdots v_n^{l_n}x)\\
&=\int_0^\infty dv_1\cdots\int_0^\infty dv_n\,\mathrm{e}^{-\frac{\sum_{a=1}^n(v_a+\lambda_a\log v_a)}{\hbar}}\prod_{i=0}^{N-1}\left(\hbar x\frac{d}{dx}-w_i\right)\mathcal{I}_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(v_1^{l_1}\cdots v_n^{l_n}x)\\
&\quad-\int_0^\infty dv_1\cdots\int_0^\infty dv_n\,\mathrm{e}^{-\frac{\sum_{a=1}^n(v_a+\lambda_a\log v_a)}{\hbar}}v_1^{l_1}\cdots v_n^{l_n}x\mathcal{I}_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(v_1^{l_1}\cdots v_n^{l_n}x)\\
&=\prod_{i=0}^{N-1}\left(\hbar x\frac{d}{dx}-w_i\right)\mathcal{I}_{X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}}(x)\\
&\quad-x\int_0^\infty dv_1\cdots\int_0^\infty dv_n\,\mathrm{e}^{-\frac{\sum_{a=1}^n(v_a+\lambda_a\log v_a)}{\hbar}}v_1^{l_1}\cdots v_n^{l_n}\mathcal{I}_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(v_1^{l_1}\cdots v_n^{l_n}x).
\end{aligned}$$

To manipulate further we will use the following integration by parts repeatedly for each v_i's:

$$\begin{aligned}
0&=\int_0^\infty dv_a\,\hbar\frac{d}{dv_a}\left(v_a^m\mathrm{e}^{-\frac{v_a+\lambda_a\log v_a}{\hbar}}\mathcal{I}_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(v_1^{l_1}\cdots v_n^{l_n}x)\right)\\
&=\int_0^\infty dv_a\,(-v_a-\lambda_a+m\hbar)\,v_a^{m-1}\mathrm{e}^{-\frac{v_a+\lambda_a\log v_a}{\hbar}}\mathcal{I}_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(v_1^{l_1}\cdots v_n^{l_n}x)\\
&\quad+\int_0^\infty dv_a v_a^m\mathrm{e}^{-\frac{v_a+\lambda_a\log v_a}{\hbar}}\hbar l_a\frac{x}{v_a}\frac{d}{dx}\mathcal{I}_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(v_1^{l_1}\cdots v_n^{l_n}x)\\
&=-\int_0^\infty dv_a\,v_a^m\mathrm{e}^{-\frac{v_a+\lambda_a\log v_a}{\hbar}}\mathcal{I}_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(v_1^{l_1}\cdots v_n^{l_n}x)\\
&\quad+\left(l_a\hbar x\frac{d}{dx}-\lambda_a+m\hbar\right)\int_0^\infty dv_a\,v_a^{m-1}\mathrm{e}^{-\frac{v_a+\lambda_a\log v_a}{\hbar}}\mathcal{I}_{\mathbb{C}\mathbf{P}^{N-1}_{\boldsymbol{w}}}(v_1^{l_1}\cdots v_n^{l_n}x).
\end{aligned}$$

Then one finds the GKZ equation (1.10) for the J-function $J_{X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}}(x)$:

$$\left[\prod_{i=0}^{N-1}\left(\hbar x\frac{d}{dx}-w_i\right)-x\prod_{a=1}^{n}\prod_{m=1}^{l_a}\left(l_a\hbar x\frac{d}{dx}-\lambda_a+m\hbar\right)\right]\mathcal{I}_{X_{\boldsymbol{l};\boldsymbol{w},\boldsymbol{\lambda}}}(x)=0.\tag{B.2}$$

□

B.2. **Proof of Proposition 2.7.** Here we investigate the behavior of coefficients when $x \to \infty$ in the saddle point approximation (2.22) of the oscillatory integral for

$$W_X = \sum_{i=1}^{N-1}(u_i + w_i \log u_i) - \sum_{a=1}^{n}(v_a + \lambda_a \log v_a) + \frac{v_1 \cdots v_n}{u_1 \cdots u_{N-1}} x + w_0 \log\left(\frac{v_1 \cdots v_n}{u_1 \cdots u_{N-1}} x\right),$$

which is mirror to $X = X_{\boldsymbol{l}=\boldsymbol{1};\boldsymbol{w},\boldsymbol{\lambda}}$. In this subsection we write $W = W_X$ for simplicity.

To prove (i) in Proposition 2.7, it is enough to find an asymptotic behavior of second derivatives of W:

$$\begin{aligned}
\frac{\partial^2 W}{\partial u_i^2} &= -\frac{w_i - w_0}{u_i^2} + \frac{2}{u_i^2}\frac{v_1 \cdots v_n}{u_1 \cdots u_{N-1}} x \\
\frac{\partial^2 W}{\partial u_i \partial u_j} &= \frac{1}{u_i u_j}\frac{v_1 \cdots v_n}{u_1 \cdots u_{N-1}} x \quad (i \neq j) \\
\frac{\partial^2 W}{\partial u_i \partial v_a} &= -\frac{1}{u_i v_a}\frac{v_1 \cdots v_n}{u_1 \cdots u_{N-1}} x \\
\frac{\partial^2 W}{\partial v_a \partial v_b} &= \frac{1}{v_a v_b}\frac{v_1 \cdots v_n}{u_1 \cdots u_{N-1}} x \quad (a \neq b) \\
\frac{\partial^2 W}{\partial v_a^2} &= \frac{\lambda_a - w_0}{v_a^2}.
\end{aligned}$$

At a critical point $(\boldsymbol{u}^{(\mathrm{c})}, \boldsymbol{v}^{(\mathrm{c})})$, we can use

$$\left(\frac{v_1 \cdots v_n}{u_1 \cdots u_{N-1}} x\right)\Bigg|_{(\boldsymbol{u},\boldsymbol{v})=(\boldsymbol{u}^{(\mathrm{c})},\boldsymbol{v}^{(\mathrm{c})})} = u_i^{(\mathrm{c})} + w_i - w_0 = v_a^{(\mathrm{c})} + \lambda_a - w_0.$$

This behaves as $O(x^{\frac{1}{N-n}})$ in the case of (2.20), and as $\lambda_b - w_0 + O(x^{-1})$ in the case of (2.21). Therefore, we can find the behavior

$$\mathrm{Hess}(\boldsymbol{u}^{(\mathrm{c})}, \boldsymbol{v}^{(\mathrm{c})}) = \begin{cases} O(x^{-\frac{N+n-1}{N-n}}) & \text{in the case of (2.20)}, \\ O(x^2) & \text{in the case of (2.21)} \end{cases}$$

of the Hessian when $x \to \infty$. The claim of (i) in Proposition 2.7 follows immediately from this computation.

Let us prove (ii) in Proposition 2.7. We take a coordinate at a critical point $(\boldsymbol{u}^{(\mathrm{c})}, \boldsymbol{v}^{(\mathrm{c})})$ of W:

$$\boldsymbol{\xi} = (\xi_1, \ldots, \xi_{N+n-1}) = (u_1 - u_1^{(\mathrm{c})}, \ldots, u_{N-1} - u_{N-1}^{(\mathrm{c})}, v_1 - v_1^{(\mathrm{c})}, \ldots, v_n - v_n^{(\mathrm{c})}),$$

and consider the Taylor expansion of W at the critical point $\boldsymbol{\xi}^{(\mathrm{c})} = \boldsymbol{0}$:

$$W(\boldsymbol{\xi}; x) = W(\boldsymbol{\xi}^{(\mathrm{c})}; x) + \frac{1}{2!}\sum_{i,j} W_{ij}(\boldsymbol{\xi}^{(\mathrm{c})}; x)\, \xi_i \xi_j + \sum_{m \geq 3} \frac{1}{m!} \sum_{i_1, \ldots, i_m} W_{i_1 \ldots i_m}(\boldsymbol{\xi}^{(\mathrm{c})}; x)\, \xi_{i_1} \cdots \xi_{i_m}, \quad \text{(B.3)}$$

where $W_{i_1 \ldots i_m} = (\partial^m W)/(\partial \xi_{i_1} \cdots \partial \xi_{i_m})$. Since we have chosen generic w_i and λ_a so that the critical points are non-degenerate, the Hesse matrix H has non-zero determinant at $\boldsymbol{\xi}^{(\mathrm{c})}$. We further take a linear transformation $\xi_i = (-\hbar)^{1/2} \sum_j C_i^j(x) s_j$ of the coordinates which transforms the quadratic part of W as

$$\frac{1}{2\hbar}\sum_{i,j} W_{ij}(\boldsymbol{\xi}^{(\mathrm{c})}; x)\, \xi_i \xi_j = -\frac{1}{2}\sum_{i=1}^{N+n-1} s_i^2. \quad \text{(B.4)}$$

We can find these coefficients $C_i^j(x)$ by applying the simultaneous completing the square to the quadratic form in the left hand side of (B.4). Eventually we can find the behavior of the coefficients $C_i^j(x)$ for large x as follows:

- Let us consider the case when the critical point $(\boldsymbol{u}^{(\mathrm{c})}, \boldsymbol{v}^{(\mathrm{c})})$ behaves as (2.20) in Lemma 2.6. Since $W_{ij}(\boldsymbol{\xi}^{(\mathrm{c})}; x)$ behaves as $O(x^{-\frac{1}{N-n}})$ in this case, we can show that
$$C_i^j(x) = O(x^{\frac{1}{2(N-n)}}) \tag{B.5}$$
holds for all $i, j = 1, \ldots, N+n-1$, when $x \to \infty$.
- Let us consider the case when the critical point $(\boldsymbol{u}^{(\mathrm{c})}, \boldsymbol{v}^{(\mathrm{c})})$ behaves as (2.21) in Lemma 2.6. Let $i_b \in \{N, \ldots, N+n-1\}$ be the label of v_b; that is, $\xi_{i_b} = v_b - v_b^{(\mathrm{c})}$. Since we can arrange the quadratic part of W as
$$\begin{aligned}
W_{i_b i_b}(\boldsymbol{\xi}^{(\mathrm{c})}; x)\xi_{i_b}^2 &+ 2\sum_{i \neq i_b} W_{i\, i_b}(\boldsymbol{\xi}^{(\mathrm{c})}; x)\xi_{i_b}\xi_i + \sum_{i,j \neq i_b} W_{ij}(\boldsymbol{\xi}^{(\mathrm{c})}; x)\xi_i\xi_j \\
&= W_{i_b i_b}(\boldsymbol{\xi}^{(\mathrm{c})}; x)\left(\xi_{i_b} + \sum_{i \neq i_b} \frac{W_{i\, i_b}(\boldsymbol{\xi}^{(\mathrm{c})}; x)}{W_{i_b i_b}(\boldsymbol{\xi}^{(\mathrm{c})}; x)}\xi_i\right)^2 \\
&\quad - \frac{1}{W_{i_b i_b}(\boldsymbol{\xi}^{(\mathrm{c})}; x)}\left(\sum_{i \neq i_b} W_{i\, i_b}(\boldsymbol{\xi}^{(\mathrm{c})}; x)\xi_i\right)^2 + \sum_{i,j \neq i_b} W_{ij}(\boldsymbol{\xi}^{(\mathrm{c})}; x)\xi_i\xi_j.
\end{aligned}$$
Thus we can choose
$$s_{i_b} = W_{i_b i_b}^{1/2}(\boldsymbol{\xi}^{(\mathrm{c})}; x)\left(\xi_{i_b} + \sum_{i \neq i_b} \frac{W_{i\, i_b}(\boldsymbol{\xi}^{(\mathrm{c})}; x)}{W_{i_b i_b}(\boldsymbol{\xi}^{(\mathrm{c})}; x)}\xi_i\right)$$
as one of new coordinates, and other s_i's are written in terms of ξ_i's except for ξ_{i_b}. Since $W_{i_b i_b}(\boldsymbol{\xi}^{(\mathrm{c})}; x) = O(x^2)$, $W_{i\, i_b}(\boldsymbol{\xi}^{(\mathrm{c})}; x) = O(x)$ if $i \neq i_b$ and $W_{ij}(\boldsymbol{\xi}^{(\mathrm{c})}; x) = O(1)$ for $i, j \neq i_b$, we can conclude
$$C_i^j(x) = \begin{cases} O(x^{-1}) & \text{if } i = i_b \\ O(1) & \text{otherwise} \end{cases} \tag{B.6}$$
hold when $x \to \infty$.

Let us proceed the computation of saddle point expansion. The above change of the coordinate yields
$$d\xi_1 \cdots d\xi_{N+n-1} = \frac{(-\hbar)^{\frac{N+n-1}{2}}}{\sqrt{\mathrm{Hess}(\boldsymbol{u}^{(\mathrm{c})}, \boldsymbol{v}^{(\mathrm{c})})}}\, ds_1 \cdots ds_{N+n-1}.$$
Then, by the standard argument of the saddle point method, the asymptotic expansion of the oscillatory integral is computed by term-wise integration:
$$\begin{aligned}
\mathcal{I}_X(x) &\sim \exp\left(\frac{1}{\hbar}W(\boldsymbol{\xi}^{(\mathrm{c})}; x)\right) \frac{(-\hbar)^{\frac{N+n-1}{2}}}{u_1^{(\mathrm{c})} \cdots u_{N-1}^{(\mathrm{c})}\sqrt{\mathrm{Hess}(\boldsymbol{\xi}^{(\mathrm{c})})}} \\
&\times \left(1 + \sum_{m=1}^{\infty} \hbar^{\frac{m}{2}} \sum_{i_1, \ldots, i_m} f_{i_1 \ldots i_m} \int_{\mathbb{R}^{N+n-1}} ds_1 \cdots ds_{N+n-1} \exp\left(-\frac{1}{2}\sum_{i=1}^{N+n-1} s_i^2\right) s_{i_1} \cdots s_{i_m}\right),
\end{aligned} \tag{B.7}$$
where $f_{i_1 \ldots i_m}$ is the Taylor coefficient given by
$$\begin{aligned}
\frac{u_1^{(\mathrm{c})} \cdots u_{N-1}^{(\mathrm{c})}}{u_1 \cdots u_{N-1}} \exp\left(\frac{1}{\hbar}\sum_{m \geq 3} \frac{1}{m!} \sum_{i_1, \ldots, i_m} W_{i_1 \ldots i_m}(\boldsymbol{\xi}^{(\mathrm{c})}; x)\xi_{i_1} \cdots \xi_{i_m}\right)\Bigg|_{\xi_i = (-\hbar)^{1/2}\sum_j C_i^j(x) s_j} \\
= 1 + \sum_{m=1}^{\infty} \hbar^{\frac{m}{2}} \sum_{i_1, \ldots, i_m} f_{i_1 \ldots i_m} s_{i_1} \cdots s_{i_m}.
\end{aligned}$$

- If $(\boldsymbol{u}^{(\mathrm{c})}, \boldsymbol{v}^{(\mathrm{c})})$ behaves as (2.20), then $W_{i_1 \dots i_m}(\boldsymbol{\xi}^{(\mathrm{c})}; x) = O(x^{-\frac{m-1}{N-n}})$, and hence,
$$W_{i_1 \dots i_m}(\boldsymbol{\xi}^{(\mathrm{c})}; x)\, C_{i_1}^{j_1} \cdots C_{i_m}^{j_m} = O(x^{-\frac{m-2}{2(N-n)}}) \quad \text{for any } j_1, \dots, j_m.$$
For $m \geq 3$, this tends to 0 when $x \to \infty$.
- If $(\boldsymbol{u}^{(\mathrm{c})}, \boldsymbol{v}^{(\mathrm{c})})$ behaves as (2.21), then $W_{i_1 \dots i_m}(\boldsymbol{\xi}^{(\mathrm{c})}; x) = O(x^{\ell_b})$, where ℓ_b is the number of i_b in the indices $i_1, \dots, i_m$. Therefore,
$$W_{i_1 \dots i_m}(\boldsymbol{\xi}^{(\mathrm{c})}; x)\, C_{i_1}^{j_1} \cdots C_{i_m}^{j_m} = O(1) \quad \text{for any } j_1, \dots, j_m.$$

We can also verify that
$$\frac{u_i^{(\mathrm{c})}}{u_i} = \left(1 + (-\hbar)^{1/2} \sum_i \frac{C_i^j}{u_i^{(\mathrm{c})}} s_j\right)^{-1},$$
and the coefficient satisfies
$$\frac{C_i^j}{u_i^{(\mathrm{c})}} = \begin{cases} O(x^{-\frac{1}{2(N-n)}}) & \text{for the case of (2.20)} \\ O(1) & \text{for the case of (2.21)} \end{cases}$$
when $x \to \infty$. Therefore, we can conclude that
$$f_{i_1 \dots i_m} = \begin{cases} O(x^{-\frac{1}{2(N-n)}}) & \text{for the case of (2.20)} \\ O(1) & \text{for the case of (2.21)} \end{cases} \tag{B.8}$$
for $m \geq 1$. After evaluating the Gaussian integrals in (B.7) by using
$$\int_{\mathbb{R}} ds_i\, e^{-\frac{1}{2} s_i^2}\, s_i^k = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ \sqrt{2\pi}\,(k-1)!! & \text{if } k \text{ is even,} \end{cases}$$
we obtain the saddle point approximation (2.22). In particular, (B.8) proves the claim (ii) in Proposition 2.7.

Appendix C. Computational results by iteration and topological recursion

In this appendix we will firstly give some explicit computational results of the WKB solutions to the GKZ equations. In Appendix C.2 we will explicitly perform the WKB reconstruction (3.13) for the equivariant $\mathbb{C}\mathbf{P}^1$ model, and see agreements with the results in Appendix C.1.

C.1. **Some iterative computations for the GKZ equation.** Assume the saddle point approximation of the oscillatory integral
$$\mathcal{I}_X(x) \sim \exp\left(\sum_{m=0}^{\infty} \hbar^{m-1} S_m(x)\right),$$
one finds a set of the first order differential equations for S_m's by expanding the GKZ equation around $\hbar = 0$.

$\mathbb{C}\mathbf{P}^{N-1}$ model
The GKZ equation for the (non-equivariant) $\mathbb{C}\mathbf{P}^{N-1}$ model is
$$\left[\left(\hbar x \frac{d}{dx}\right)^N - x\right] \mathcal{I}_{\mathbb{C}\mathbf{P}^{N-1}}(x) = 0. \tag{C.1}$$
Some computational results of S_m's for $N = 2, 3, 4$ are listed in table 2.
Equivariant $\mathbb{C}\mathbf{P}^1$ model
The GKZ equation for the equivariant $\mathbb{C}\mathbf{P}^1$ model is
$$\left[\left(\hbar x \frac{d}{dx} - w_0\right)\left(\hbar x \frac{d}{dx} - w_1\right) - x\right] \mathcal{I}_{\mathbb{C}\mathbf{P}^1_{\boldsymbol{w}}}(x) = 0. \tag{C.2}$$

	$N=2$	$N=3$	$N=4$
$S_0(x)$	$2x^{1/2}$	$3x^{1/3}$	$4x^{1/4}$
$S_1(x)$	$\log x^{-1/4}$	$\log x^{-1/3}$	$\log x^{-3/8}$
$S_2(x)$	$x^{-1/2}/16$	$x^{-1/3}/9$	$5x^{-1/4}/32$
$S_3(x)$	$x^{-1}/64$	$x^{-2/3}/54$	$5x^{-1/2}/256$
$S_4(x)$	$25x^{-3/2}/3072$	$x^{-1}/243$	$17x^{-3/4}/24576$

TABLE 2. $S_m(x)$ for the $\mathbb{C}\mathbf{P}^{N-1}$ models ($N = 2, 3, 4$) obtained from the GKZ equation (C.1).

m	$S^{(\pm)}_m(x)$
0	$\pm\sqrt{4x+(w_0-w_1)^2}$ $+w_0\log\left(-w_0+w_1\pm\sqrt{4x+(w_0-w_1)^2}\right)+w_1\log\left(w_0-w_1\pm\sqrt{4x+(w_0-w_1)^2}\right)$
1	$-\frac{1}{4}\log\left(\frac{4x+(w_0-w_1)^2}{4}\right)$
2	$\pm\left(6x-(w_0-w_1)^2\right)/\left(12\left(4x+(w_0-w_1)^2\right)^{3/2}\right)$
3	$x\left(x-(w_0-w_1)^2\right)/\left(4x+(w_0-w_1)^2\right)^3$
4	$\pm\left(1500x^3-3654x^2(w_0-w_1)^2+378x(w_0-w_1)^4+(w_0-w_1)^6\right)/\left(360\left(4x+(w_0-w_1)^2\right)^{9/2}\right)$

TABLE 3. $S^{(\pm)}_m(x)$ for the equivariant $\mathbb{C}\mathbf{P}^1$ model obtained from the GKZ equation (C.2).

For this model there are two solutions which have the formal power series expansion:

$$\mathcal{I}^{(\pm)}_{\mathbb{C}\mathbf{P}^1_{\boldsymbol{w}}}(x) \sim \exp\left(\sum_{m=0}^{\infty}\hbar^{m-1}S^{(\pm)}_m(x)\right).$$

Computational results of $S^{(\pm)}_m$ for $m = 0, 1, 2, 3, 4$ are listed in Table 3 modulo constant shifts.

Degree 1 hypersurface in $\mathbb{C}\mathbf{P}^1$

The GKZ equation for the equivariant $\mathbb{C}\mathbf{P}^1$ model is

$$\left[\left(\hbar x\frac{d}{dx}-w_0\right)\left(\hbar x\frac{d}{dx}-w_1\right)-x\left(\hbar x\frac{d}{dx}-\lambda_1+\hbar\right)\right]\mathcal{I}_{X_{l=1;\boldsymbol{w},\boldsymbol{\lambda}}}(x)=0. \tag{C.3}$$

For this model we also find two solutions which have the formal power series expansion:

$$\mathcal{I}^{(\pm)}_{X_{l=1;\boldsymbol{w},\boldsymbol{\lambda}}}(x) \sim \exp\left(\sum_{m=0}^{\infty}\hbar^{m-1}S^{(\pm)}_m(x)\right).$$

Computational results of $S^{(\pm)}_m$ for $m = 0, 1, 2, 3$ are listed in Table 4 modulo constant shifts. Especially, focusing on the $S^{(\pm)}_1(x)$ we find the following expansion around $x=\infty$:

$$\begin{aligned} \mathrm{e}^{S^{(+)}_1(x)} &= 1+\frac{(w_0-\lambda_1)(w_1-\lambda_1)}{2x^2}+O(x^{-3}),\\ \mathrm{e}^{S^{(-)}_1(x)} &= \frac{1}{x}-\frac{w_0+w_1-2\lambda_1}{x^2}+O(x^{-3}), \end{aligned}$$

and these asymptotic expansions are consistent with Proposition 2.7 (i).

C.2. **Topological recursion for the equivariant $\mathbb{C}\mathbf{P}^1$ model.** In the following, for the equivariant $\mathbb{C}\mathbf{P}^1$ model, we will explicitly recover the computational result in Table 3 by applying the topological recursion (3.9) in [42]. The GKZ curve

$$\Sigma_{\mathbb{C}\mathbf{P}^1_{\boldsymbol{w}}} = \left\{ (x,y)\in\mathbb{C}^*\times\mathbb{C} \;\middle|\; (y-w_0)(y-w_1)-x=0 \right\}$$

m	$S_m^{(\pm)}(x)$
0	$\Big(x \pm \sqrt{x^2+2(w_0+w_1-2\lambda_1)x+(w_0-w_1)^2} \pm (w_0-w_1)\log x + (w_0+w_1)\log x$ $\pm(w_0+w_1-2\lambda_1)\log\left[(x+w_0+w_1-2\lambda_1+\sqrt{x^2+2(w_0+w_1-2\lambda_1)x+(w_0-w_1)^2})\right]$ $\mp(w_0-w_1)\log\Big[(w_0+w_1-2\lambda_1)x+(w_0-w_1)^2$ $+(w_0-w_1)\sqrt{x^2+2(w_0+w_1-2\lambda_1)x+(w_0-w_1)^2}\Big]\Big)/2$
1	$-\frac{1}{4}\log\left(x^2+2(w_0+w_1-2\lambda_1)x+(w_0-w_1)^2\right)$ $\pm\frac{1}{2}\log\left(x+w_0+w_1-2\lambda_1+\sqrt{x^2+2(w_0+w_1-2\lambda_1)x+(w_0-w_1)^2}\right] \mp \log\sqrt{2}$
2	$-x/(2(x^2+2(w_0+w_1-2\lambda_1)x+(w_0-w_1)^2))$ $\mp 5((w_0+w_1-2\lambda_1)x+(w_0-w_1)^2))/(12(x^2+2(w_0+w_1-2\lambda_1)x+(w_0-w_1)^2)^{3/2})$ $\mp((w_0+w_1-2\lambda_1)x+w_0^2-10w_0w_1+w_1^2+8(w_0+w_1)\lambda_1-8\lambda_1^2)$ $/\Big(24(w_0-\lambda_1)(w_1-\lambda_1)\sqrt{x^2+2(w_0+w_1-2\lambda_1)x+(w_0-w_1)^2}\Big)$
3	$x\Big(3x^2+(w_0+w_1-2\lambda_1)x-2(w_0-w_1)^2\Big)$ $\times\Big(x+w_0+w_1-2\lambda_1 \mp \sqrt{x^2+2(w_0+w_1-2\lambda_1)x+(w_0-w_1)^2}\Big)$ $/\Big(4(x^2+2(w_0+w_1-2\lambda_1)x+(w_0-w_1)^2)^3\Big)$

TABLE 4. $S_m^{(\pm)}(x)$ for degree 1 hypersurface in $\mathbb{C}\mathbf{P}^1$ obtained from the GKZ equation (C.3).

is parametrized by a local coordinate z as follows:

$$x(z) = z^2 - \Lambda, \qquad y(z) = z + \frac{1}{2}(w_0+w_1), \qquad \Lambda = \frac{1}{4}(w_0-w_1)^2.$$

The spectral curve $\Sigma_{\mathbb{C}\mathbf{P}^1_{\boldsymbol{w}}}$ has only one simple ramification point at $z=0$ in this local coordinate. Starting from

$$\omega_1^{(0)}(z) = 0, \qquad \omega_2^{(0)}(z_1,z_2) = B(z_1,z_2) = \frac{dz_1 dz_2}{(z_1-z_2)^2},$$

the differentials $\omega_n^{(g)}$ for $(g,n) \neq (0,1),(0,2)$ are defined by the topological recursion (3.9)

$$\omega_{n+1}^{(g)}(z,\boldsymbol{z}_N) = \operatorname*{Res}_{w=0} \frac{\int_{-w}^{w} B(\cdot,z)}{2\,(y(w)-y(-w))\,dx(w)/x(w)} \Bigg(\omega_{n+2}^{(g-1)}(w,-w,\boldsymbol{z}_N) + \sum_{\ell=0}^{g} \sum_{\emptyset = J \subseteq N} \omega_{|J|+1}^{(g-\ell)}(w,\boldsymbol{z}_J)\omega_{|N|-|J|+1}^{(\ell)}(-w,\boldsymbol{z}_{N\backslash J})\Bigg),$$

where $N = \{1,2,\ldots,n\} \supset J = \{i_1,i_2,\ldots,i_j\}$, and $N\backslash J = \{i_{j+1},i_{j+2},\ldots,i_n\}$. Integrating these multi-differentials, one finds the free energies

$$\begin{aligned} &F_1^{(0)}(x) = \int_{z_*}^{z} y(z')\frac{dx(z')}{x(z')}, \qquad F_2^{(0)}(x) = \int_{z_*}^{z}\int_{z_*}^{z}\left(B(z_1',z_2') - \frac{dx(z_1')dx(z_2')}{(x(z_1')-x(z_2'))^2}\right), \\ &F_n^{(g)}(x) = \int_{z_*}^{z}\cdots\int_{z_*}^{z}\omega_n^{(g)}(z_1',\ldots,z_n'), \quad (g,n)\neq(0,1),(0,2), \end{aligned} \tag{C.4}$$

where z_* denotes a reference point.

$F_n^{(g)}(x)$
$F_2^{(0)}(x) = -\frac{1}{2}\log\left[(x+\Lambda)/(x(z_*)+\Lambda)\right]$
$F_3^{(0)}(x) = \mp\Lambda/(2(x+\Lambda)^{3/2})$ $F_1^{(1)}(x) = \pm(3x+2\Lambda)/(48(x+\Lambda)^{3/2})$
$F_4^{(0)}(x) = \Lambda(-3x+\Lambda)/(4(x+\Lambda)^3)$ $F_2^{(1)}(x) = (3x^2-2\Lambda^2-6x\Lambda)/(96((x+\Lambda)^3)$
$F_5^{(0)}(x) = \mp\Lambda(2\Lambda^2-21x\Lambda+12x^2)/(8(x+\Lambda)^{9/2})$ $F_3^{(1)}(x) = \pm(4\Lambda^3+18x\Lambda^2-63\Lambda x^2+6x^3)/(192(x+\Lambda)^{9/2})$ $F_1^{(2)}(x) = \mp(186\Lambda x^2+72x\Lambda^2+16\Lambda^3-45x^3)/(15360(x+\Lambda)^{9/2})$

TABLE 5. Free energies for the equivariant $\mathbb{C}\mathbf{P}^1$ model. Here we have two types free energies corresponding to the branches $z = \pm\sqrt{x+\Lambda}$.

m	$F_m^{(\pm)}(x)$
1	$-\frac{1}{4}\log\left[(x+\Lambda)/(x(z_*)+\Lambda)\right]$
2	$\pm(3x-2\Lambda)/\left(48\,(x+\Lambda)^{3/2}\right)$
3	$x(x-4\Lambda)/\left(64\,(x+\Lambda)^3\right)$
4	$\pm\left(375x^3-3654x^2\Lambda+1512x\Lambda^2+16\Lambda^3\right)/\left(46080\,(x+\Lambda)^{9/2}\right)$

TABLE 6. $F_m(x) = F_m^{(\pm)}(x)$ for the equivariant $\mathbb{C}\mathbf{P}^1$ model.

The WKB reconstruction (3.13) of wave function is defined by $F_n^{(g)}(x)$'s as

$$\psi_{\mathbb{C}\mathbf{P}^1_{\boldsymbol{w}}}(x) = \exp\left(\sum_{g=0,n=1}^{\infty}\frac{\hbar^{2g-2+n}}{n!}F_n^{(g)}(x)\right). \tag{C.5}$$

We fix the reference point by $z_* = \infty$ which corresponds to $x(z_*) = \infty$. Here note that $F_1^{(0)}$ and $F_2^{(0)}$ need to be regularized by certain constant shifts so as to depend on z_*. Some explicit computational results of the free energies $F_n^{(g)}(x)$ are listed in Table 5.

The wave function (C.5) is reorganized by

$$\psi_{\mathbb{C}\mathbf{P}^1_{\boldsymbol{w}}}(x) = \exp\left(\sum_{m=0}^{\infty}\hbar^{m-1}F_m(x)\right), \qquad F_m(x) = \sum_{\substack{g\geq 0,\ n\geq 1,\\ 2g+n-1=m}}\frac{1}{n!}F_n^{(g)}(x),$$

where corresponding to two branches $z = \pm\sqrt{x+\Lambda}$ we find two types of free energies $F_m(x) = F_m^{(\pm)}(x)$. Since the leading term $S_0(x)$ of the asymptotic expansion of the J-function obeys

$$x\frac{dS_0(x)}{dx} = y(x),$$

the free energy $F_1^{(0)}(x)$ in (C.4) agrees with $S_0^{(\pm)}(x)$ up to a constant shift. Using the computational results in Table 5, $F_m^{(\pm)}(x)$'s ($m \geq 1$) are computed immediately and summarized in Table 6. Comparing the computational results in Tables 3 and 6, one finds the agreement

$$F_m^{(\pm)}(x) = S_m^{(\pm)}(x),$$

for $m = 1, 2, 3, 4$ up to a constant shift of $F_1^{(\pm)}(x)$.

References

[1] Aganagic, M. and Vafa, C.: *Mirror symmetry, D-branes and counting holomorphic discs.* [hep-th/0012041].

[2] Aganagic, M., Cheng, M. C. N., Dijkgraaf, R., Krefl, D. and Vafa, C.: *Quantum Geometry of Refined Topological Strings.* JHEP **1211**, 019 (2012) [arXiv:1105.0630 [hep-th]].

[3] Aganagic, M., Dijkgraaf, D., Klemm, A., Marino, M. and Vafa, C.: *Topological strings and integrable hierarchies.* Commun. Math. Phys. **261** 451 (2006) [hep-th/0312085].

[4] Aganagic, M., Klemm, A., Marino, M. and Vafa, C.: *The Topological vertex.* Commun. Math. Phys. **254** 425 (2005) [hep-th/0305132].

[5] Aganagic, M. and Yamazaki, M.: *Open BPS Wall Crossing and M-theory.* Nucl. Phys. B **834** 258 (2010) [arXiv:0911.5342 [hep-th]].

[6] Alday, L. F. and Tachikawa, Y.: *Affine $SL(2)$ conformal blocks from 4d gauge theories.* Lett. Math. Phys. **94**, 87 (2010) [arXiv:1005.4469 [hep-th]].

[7] Aoki, T., Iwaki, K. and Takahashi, T. : *Exact WKB analysis of Schrödinger equations with a Stokes curve of loop type.* submitted (2016).

[8] Aoki, T., Kawai, T., Sasaki, S., Shudo, A. and Takei, Y.: *Virtual turning points and bifurcation of Stokes curves.* J. Phys. A, **38**, 3317–3336 (2005).

[9] Aoki, T., Kawai, T. and Takei, Y.: *New turning points in the exact WKB analysis for higher order ordinary differential equations.* Analyse algébrique des perturbations singuliéres. I, Hermann, pp. 69–84, (1994).

[10] Awata, H., Fuji, H., Kanno, H., Manabe, H. and Yamada, Y.: *Localization with a Surface Operator, Irregular Conformal Blocks and Open Topological String.* Adv. Theor. Math. Phys. **16** no.3 725 (2012) . [arXiv:1008.0574 [hep-th]].

[11] Benini, F. and Cremonesi, S.: *Partition Functions of $\mathcal{N} = (2, 2)$ Gauge Theories on S^2 and Vortices.* Commun. Math. Phys. **334**, no. 3, 1483 (2015) [arXiv:1206.2356 [hep-th]].

[12] Benini, F. and Peelaers, W.: *Higgs branch localization in three dimensions.* JHEP **1405**, 030 (2014) [arXiv:1312.6078 [hep-th]].

[13] Benini, F. and Zaffaroni, A. *A topologically twisted index for three-dimensional supersymmetric theories.* JHEP **1507**, 127 (2015) [arXiv:1504.03698 [hep-th]].

[14] Berk, H. L., Nevins, W. M. and Roberts, K. V.: *New Stokes' line in WKB theory.* J. Math. Phys., **23**, 988–1002 (1982).

[15] Bershadsky, M., Cecotti, S., Ooguri, H. and Vafa, C.: *Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes.* Commun. Math. Phys. **165**, 311 (1994) [hep-th/9309140].

[16] Bonelli, G., Sciarappa, A., Tanzini, A. and Vasko, P.: *Vortex partition functions, wall crossing and equivariant Gromov-Witten invariants.* Commun. Math. Phys. **333** no.2, 717 (2015) [arXiv:1307.5997 [hep-th]].

[17] Bonelli, G., Tanzini, A. and Zhao, J.: *Vertices, Vortices and Interacting Surface Operators.* JHEP **1206**, 178 (2012) [arXiv:1102.0184 [hep-th]].

[18] Bouchard, V. and Eynard, B.: *Think globally, compute locally.* JHEP **1302**, 143 (2013) [arXiv:1211.2302 [math-ph]].

[19] Bouchard, V. and Eynard, B. *Reconstructing WKB from topological recursion.* Journal de l'Ecole polytechnique – Mathematiques, **4**, 845–908 (2017) [arXiv:1606.04498 [math-ph]].

[20] Bouchard, V., Hutchinson, J., Loliencar, P., Meiers, M. and Rupert, M.: *A generalized topological recursion for arbitrary ramification.* Annales Henri Poincare **15**, 143 (2014) [arXiv:1208.6035 [math-ph]].

[21] Bouchard, V., Klemm, A., Marino, M. and Pasquetti, S.: *Remodeling the B-model.* Commun. Math. Phys. **287**, 117 (2009) [arXiv:0709.1453 [hep-th]].

[22] Bouchard, V., Klemm, A., Mariño, M. and Pasquetti, S.: *Topological open strings on orbifolds.* Commun. Math. Phys. **296**, 589 (2010) [arXiv:0807.0597 [hep-th]].

[23] Braverman, A.: *Instanton counting via affine Lie algebras. 1. Equivariant J functions of (affine) flag manifolds and Whittaker vectors.* CRM Proc. Lecture Notes **38**, 113–132 (2004) [math/0401409 [math-ag]].

[24] Braverman, A. and Etingof, P.: *Instanton counting via affine Lie algebras II: From Whittaker vectors to the Seiberg-Witten prepotential.* Studies in Lie Theory, 61–78 (2006) [math/0409441 [math-ag]].

[25] Braverman, A., Feigin, B., Finkelberg, M. and Rybnikov, L.: *A Finite analog of the AGT relation I: F inite W-algebras and quasimaps' spaces.* Commun. Math. Phys. **308**, 457 (2011) [arXiv:1008.3655 [math.AG]].

[26] Chiang, T. M., Klemm, A., Yau,S. T. and Zaslow, E.: *Local mirror symmetry: Calculations and interpretations.* Adv. Theor. Math. Phys. **3**, 495 (1999) [hep-th/9903053].

[27] Chriss, N. and Ginzburg, V.: *Representation Theory and Complex Geometry.* Birkhäuser Mathematics, pp.508 (1997) .

[28] Closset, C., Cremonesi, S. and Park, D. S.: *The equivariant A-twist and gauged linear sigma models on the two-sphere.* JHEP **1506**, 076 (2015) [arXiv:1504.06308 [hep-th]].

[29] Coates, T., Corti, A., Iritani, H. and Tseng, H.: *Hodge-theoretic mirror symmetry for toric stacks*, arXiv:1606.07254 [math.AG].

[30] Coates, T. and Givental, A.: *Quantum Riemann-Roch, Lefschetz and Serre.* Ann. Math. **165**, 15-53 (2007) [arXiv:math/0110142 [math.AG]].

[31] Coates, T., Iritani, H. and Jiang, Y.: *The Crepant Transformation Conjecture For Toric Complete Intersections.* arXiv:1410.0024 [math.AG].

[32] Costin, O.: *Asymptotics and Borel Summability.* Monographs and surveys in pure and applied mathematics, vol. **141**, Chapmann and Hall/CRC, 2008.

[33] Delabaere, E., Dillinger, H. and Pham, F.: *Résurgence de Voros et périodes des courbes hyperelliptiques.* Ann. Inst. Fourier (Grenoble) **43** 163–199 (1993).

[34] Dijkgraaf, R., Hollands, L. and Sulkowski, P.: *Quantum Curves and D-Modules.* JHEP **0911**, 047 (2009) [arXiv:0810.4157 [hep-th]].

[35] Dijkgraaf, R., Hollands, L., Sulkowski, P. and Vafa, C.: *Supersymmetric gauge theories, intersecting branes and free fermions.* JHEP **0802**, 106 (2008) [arXiv:0709.4446 [hep-th]].

[36] Dimofte, T., Gukov, S. and Hollands, L.: *Vortex Counting and Lagrangian 3-manifolds.* Lett. Math. Phys. **98** (2011) 225, [arXiv:1006.0977 [hep-th]].

[37] Dorey, N.: *The BPS spectra of two-dimensional supersymmetric gauge theories with twisted mass terms.* JHEP **9811** (1998) 005 [hep-th/9806056].

[38] Doroud, N., Gomis, J., Le Floch, B. and Lee, S.: *Exact Results in $D = 2$ Supersymmetric Gauge Theories.* JHEP **1305**, 093 (2013) [arXiv:1206.2606 [hep-th]].

[39] Dubrovin, B.: *Geometry and analytic theory of Frobenius manifolds.* Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998). Doc. Math. Extra Vol. II, 315–326 (1998).

[40] Dumitrescu, O. and Mulase, M.: *Quantum curves for Hitchin fibrations and the Eynard-Orantin theory.* Lett. Math. Phys. **104**, 635 (2014) [arXiv:1310.6022 [math.AG]].

[41] Dunin-Barkowski, P., Mulase, M., Norbury, P., Popolitov, A. and Shadrin, S.: *Quantum spectral curve for the Gromov-Witten theory of the complex projective line.* [arXiv:1312.5336 [math-ph]].

[42] Eynard, B. and Orantin, N.: *Invariants of algebraic curves and topological expansion.* Commun. Num. Theor. Phys. **1**, 347 (2007) [math-ph/0702045].

[43] Eynard, B. and Orantin, N.: *Computation of Open Gromov-Witten Invariants for Toric Calabi-Yau 3-Folds by Topological Recursion, a Proof of the BKMP Conjecture.* Commun. Math. Phys. **337**, no. 2, 483 (2015) [arXiv:1205.1103 [math-ph]].

[44] Fang, B.: *Central charges of T-dual branes for toric varieties.* arXiv:1611.05153.

[45] Fang, B., Liu, C. C. M., Treumann, D. and Zaslow, E.: *T-duality and homological mirror symmetry of toric varieties.* Adv. Math. **229**, 1873–1911 (2012) [arXiv:0811.1228 [math.AG]].

[46] Fang, B., Liu,C. C. M. and Zong, Z.: *All Genus Open-Closed Mirror Symmetry for Affine Toric Calabi-Yau 3-Orbifolds.* Proc. Symp. Pure Math. **93**, 1 (2015) [arXiv:1310.4818 [math.AG]].

[47] Fang, B., Liu, C. C. M. and Zong, Z.: *On the Remodeling Conjecture for Toric Calabi-Yau 3-Orbifolds.* [arXiv:1604.07123 [math.AG]].

[48] Fujitsuka, M., Honda, M. and Yoshida, Y.: *Higgs branch localization of 3d $\mathcal{N} = 2$ theories.* PTEP **2014**, no. 12, 123B02 (2014) [arXiv:1312.3627 [hep-th]].

[49] Gaiotto, D., Moore, G. W. and Neitzke, A.: *Wall-crossing in coupled 2d-4d systems.* JHEP**12**, 082 (2012) [arXiv:1103.2598 [hep-th]].

[50] Gaiotto, D., Moore, G. W. and Neitzke, A.: *Spectral Networks.* Ann. Henri Poincaré, **14**, 1643–1731 (2013) [arXiv:1204.4824 [hep-th]].

[51] Galkin, S., Golyshev, V. and Iritani, H.: *Gamma classes and quantum cohomology of Fano manifolds: Gamma conjectures.* Duke Math. J. **165**, no. 11, 2005–2077 (2016) [arXiv:1404.6407 [math.AG]].

[52] Galkin, S. and Iritani, H. : *Gamma conjecture via mirror symmetry.* [arXiv:1508.00719 [math.AG]].

[53] Givental, A.: *Homological geometry I. Projective hypersurfaces.* Selecta Math. (N.S.), 325–345 (1995).

[54] Givental, A.: *Equivariant Gromov - Witten Invariants.* Internat. Math. Res. Notices, 613–663 (1996) [arXiv:alg-geom/9603021].

[55] Gukov, S.: *Three-dimensional quantum gravity, Chern-Simons theory, and the A polynomial.* Commun. Math. Phys. **255**, 577 (2005) [hep-th/0306165].

[56] Gukov, S. and Sulkowski, P., *A-polynomial, B-model, and Quantization.* JHEP **1202** (2012) 070 [arXiv:1108.0002 [hep-th]].

[57] Gukov, S. and Witten, E.: *Gauge Theory, Ramification, And The Geometric Langlands Program.* [hep-th/0612073].

[58] Guzzetti, D.: *Stokes Matrices and Monodromy of the Quantum Cohomology of Projective Spaces.* Commun. Math. Phys. **207**, 341–383 (1999) [arXiv:math/9904099 [math.AG]].

[59] Harvey, R. and Lawson, H. B.: *Calibrated Geometries.* Acta Math. **148**, 47 (1982).

[60] Hikami, K.: *Generalized Volume Conjecture and the A-Polynomials: The Neumann-Zagier Potential Function as a Classical Limit of Quantum Invariant.* J. Geom. Phys. **57**, 1895 (2007) [math/0604094 [math.QA]].

[61] Honda, N., Kawai, T. and Takei, Y.: *Virtual Turning Points.* Springer Briefs in Mathematical Physics, **4**. Springer, Tokyo, pp 126 (2015).

[62] Honda, D. and Okuda, T.: *Exact results for boundaries and domain walls in 2d supersymmetric theories.* JHEP **1509**, 140 (2015) [arXiv:1308.2217 [hep-th]].

[63] Hori, K., Iqbal, A. and Vafa, C.: *D-branes and mirror symmetry.* [hep-th/0005247].

[64] Hori, K., Katz, S., Klemm, A., Pandharipande, R., Thomas, R., Vafa, C., Vakil, R. and Zaslow, E.: *Mirror symmetry.* Clay Mathematics Monographs. Volume 1. American Mathematical Society, pp 929 (2003).
[65] Hori, K. and Romo, M.: *Exact Results In Two-Dimensional* $(2,2)$ *Supersymmetric Gauge Theories With Boundary.* [arXiv:1308.2438 [hep-th]].
[66] Hori, K. and Vafa, C.: *Mirror symmetry.* [hep-th/0002222].
[67] Iqbal, A. and Kashani-Poor, A. K.: *The Vertex on a strip.* Adv. Theor. Math. Phys. **10**, no. 3, 317 (2006) [hep-th/0410174].
[68] Iqbal, A., Nekrasov, N., Okounkov, A. and Vafa, C.: *Quantum foam and topological strings.* JHEP **0804**, 011 (2008) [hep-th/0312022].
[69] Iritani, H.: *Quantum D-modules and equivariant Floer theory for free loop spaces* Math. Z. 252 (3) 577–622 (2006) [arXiv:math/0410487 [math.DG]].
[70] Iritani, H.: *An integral structure in quantum cohomology and mirror symmetry for toric orbifolds.* Adv. Math. **222** (3), 1016–1079 (2009) [arXiv:0903.1463 [math.AG]].
[71] Iritani, H.: *A mirror construction for the big equivariant quantum cohomology of toric manifolds.* Math. Ann., **368** (1), 279–316 (2017) [arXiv:1503.02919 [math.AG]].
[72] Iwaki, K. and Nakanishi, T.: *Exact WKB analysis and cluster algebras.* J. Phys. A: Math. Theor. **47**, 474009 (2014) [arXiv:1401.7094 [math.CA]].
[73] Iwaki, K. and Takahashi, A.: *Stokes Matrices for the Quantum Cohomologies of Orbifold Projective Lines.* J. Math. Phys. **A54**, 101701 (2013) [arXiv:1305.5775 [math.AG]].
[74] Kanno, H. and Tachikawa, Y.: *Instanton counting with a surface operator and the chain-saw quiver.* JHEP **1106**, 119 (2011) [arXiv:1105.0357 [hep-th]].
[75] Katz, S. H., Klemm, A. and Vafa, C.: *Geometric engineering of quantum field theories.* Nucl. Phys. B **497**, 173 (1997) 173 [hep-th/9609239].
[76] Katz, S., Mayr, P. and Vafa, C.: *Mirror symmetry and exact solution of 4D* $N = 2$ *gauge theories: 1.* Adv. Theor. Math. Phys. **1**, 53 (1998) [hep-th/9706110].
[77] Katzarkov, L., Kontsevich, M. and Pantev, T., : *Hodge theoretic aspects of mirror symmetry*, in From Hodge theory to integrability and TQFT tt*-geometry, Proc. Sympos. Pure Math., vol. **78**, Amer. Math. Soc., Providence, RI, pp. 87–174 (2008) [arXiv:0806.0107 [math.AG]].
[78] Kawai, T. and Takei, Y.: *Algebraic Analysis of Singular Perturbation Theory.* Translations of Mathematical Monographs **227**, AMS, pp 129 (2005) (Japanese ver. 1998).
[79] Koike, T. and Schäfke, R.: *On the Borel summability of WKB solutions of Schrödinger equations with rational potentials and its application.* in preparation; also Talk given by Koike, T. in the RIMS workshop "Exact WKB analysis — Borel summability of WKB solutions" September, 2010.
[80] Kozcaz, C., Pasquetti, S., Passerini, F. and Wyllard, N.: *Affine* $sl(N)$ *conformal blocks from* $\mathcal{N} = 2$ $SU(N)$ *gauge theories.* JHEP **1101**, 045 (2011) [arXiv:1008.1412 [hep-th]].
[81] Kronheimer, P. B. and Mrowka, T. S.: *Gauge Theory for Embedded Surfaces: I.* Topology **32** (4), 773–826 (1993).
[82] Kronheimer, P. B. and Mrowka, T. S.: *Knot homology groups from instantons.* J. Topol. **4** (4) 835–918 (2011) [arXiv:0806.1053 [math.GT]].
[83] Lerche, W. and Mayr, P.: *On* $N = 1$ *mirror symmetry for open type II strings.* [hep-th/0111113].
[84] Mariño, M.: *Open string amplitudes and large order behavior in topological string theory.* JHEP **0803**, 060 (2008) [hep-th/0612127].
[85] Mariño, M.: *Chern-Simons Theory, Matrix Models, and Topological Strings.* Oxford University Press pp.197 (2015).
[86] Mulase, M. and Sulkowski P.: *Spectral curves and the Schrödinger equations for the Eynard-Orantin recursion.* Adv. Theor. Math. Phys. **19**, 955 (2015) [arXiv:1210.3006 [math-ph]].
[87] Nawata, S.: *Givental J-functions, Quantum integrable systems, AGT relation with surface operator.* Adv. Theor. Math. Phys. **19**, 1277 (2015) [arXiv:1408.4132 [hep-th]].
[88] Ooguri, H. and Vafa, C.: *Knot invariants and topological strings.* Nucl. Phys. B **577** 419 (2000) doi:10.1016/S0550-3213(00)00118-8 [hep-th/9912123].
[89] Peelaers, W.: *Higgs branch localization of* $\mathcal{N} = 1$ *theories on* $S^3 \times S^1$. JHEP **1408**, 060 (2014) [arXiv:1403.2711 [hep-th]].
[90] Saito, K. : *Period Mapping Associated to a Primitive Form*, Publ. RIMS, Kyoto University **19**, 1231–1264 (1983).
[91] Saito, K. and Takahashi, A. : *From primitive forms to Frobenius manifolds*, in From Hodge theory to integrability and TQFT tt^*-geometry, Proc. Sympos. Pure Math., vol. **78**, Amer. Math. Soc., Providence, RI, pp. 31–48 (2008) [Preprint RIMS-1623].
[92] Sanda, F. and Shamoto, Y.: *An analogue of Dubrovin's conjecture.* [arXiv:1705.05989 [math.AG]].
[93] Shadchin, S.: *On F-term contribution to effective action.* JHEP **0708**, 052 (2007) [hep-th/0611278].
[94] Sugishita, S. and Terashima, S.: *Exact Results in Supersymmetric Field Theories on Manifolds with Boundaries.* JHEP **1311**, 021 (2013) [arXiv:1308.1973 [hep-th]].
[95] Takei, Y.: *WKB analysis and Stokes geometry of differential equations.* RIMS preprint 1848, March (2016).

[96] Ueda, K.: *Stokes Matrices for the Quantum Cohomologies of Grassmannians.* International Mathematics Research Notices, **34**, 2075–2086 (2005) [arXiv:math/0503355 [math.AG]].
[97] Ueda, K.: *Stokes Matrix for the Quantum Cohomology of Cubic Surfaces.* [arXiv:math.AG/0505350].
[98] Ueda, K. and Yoshida, Y. *Equivariant A-twisted GLSM and Gromov–Witten invariants of CY 3-folds in Grassmannians.* arXiv:1602.02487 [hep-th].
[99] Voros, A.: *The return of the quartic oscillator. The complex WKB method.* Ann. Inst. Henri Poincaré **39**, 211–338 (1983).
[100] Witten, E.: *Phases of $N = 2$ theories in two-dimensions.* Nucl. Phys. B **403**, 159 (1993) AMS/IP Stud. Adv. Math. **1**, 143 (1996) [hep-th/9301042].
[101] Yoshida, Y.: *Localization of Vortex Partition Functions in $\mathcal{N} = (2,2)$ Super Yang-Mills theory.* [arXiv:1101.0872 [hep-th]].
[102] Yoshida, Y.: *Factorization of 4d $\mathcal{N} = 1$ superconformal index.* [arXiv:1403.0891 [hep-th]].
[103] Zhou, J.: *Local Mirror Symmetry for the Topological Vertex.* [arXiv:0911.2343 [math.AG]].

Faculty of Education, Kagawa University, Takamatsu 760-8522, Japan, AND QGM, Department of Mathematics, Aarhus University, DK-8000 Aarhus C, Denmark
E-mail address: `fuji@ed.kagawa-u.ac.jp`

Graduate School of Mathematics, Nagoya University, Nagoya 464-8602, Japan
E-mail address: `iwaki@math.nagoya-u.ac.jp`

Faculty of Physics, University of Warsaw, ul. Pasteura 5, 02-093 Warsaw, Poland, AND Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany
E-mail address: `masahidemanabe@gmail.com`

Faculty of Education, Kagawa University, Takamatsu 760-8522, Japan
E-mail address: `satakeikuo@ed.kagawa-u.ac.jp`