Boundary values of holomorphic functions and some spectral problems for unitary representations

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May 26, 1997

Consider a (reducible) unitary representation of a semisimple Lie group G in a hilbert space H (for instance we can consider an action of the group G in some functional space on some homogeneous space, a tensor product of irreducible unitary representations, a restriction of a irreducible unitary representation to a subgroup etc.). There arises a natural problem: to decompose our representation to irreducible representations or (it is a synonym) to find spectrum of the representation. Problems of this type were widely investigated for last 50 years (since [Kre]) and now many such spectral problems are solved completly or partialy.

In several simplest cases the spectra are continuous and more or less 'uniform'. But sometimes such spectra contains strange discrete increments (or equivalently there exist minimal G-invariant subspaces in the hilbert space H). First examples of discrete increments were observed in [Nai], [Puk], [H-Ch], [Mol1] (1961 - 1966) and now there exist large literature devoted to discrete spectra, see [Boy], [Ism], [Mol3-4], [Str], [Far], [Pat], [F-J1], [F-J2], [Sch], [Kob1-2], [RSW], [Tsu], [How], [Ada], [Li], [BO] etc.

Quite often 'discrete increments' are very singular ('exotic') representations and it is very difficult to construct such unitary representations by some other way. Even Harish-Chandra discrete series now don't seems quite hand objects. Remarkable paper [F-J1] (1980, see also [F-J2]) contained serious generalization of Harish-Chandra discrete series and this paper put origin to systematic search of 'exotic' unitary representations in discrete spectra.

*supported by RFRF (grant 95-01-00814) and Russian program of support of scientific schools

In [Ner1] (1986), [Ols3],[Ols4], [NO], [Ner2] it was observed that very often discrete increments to spectra are related to some functional-theoretical phenomena, namely to so-called 'trace theorems' (i.e theorems about restrictions of discontinuos functions to submanifolds). On various analytical trace theorems see [RS], IX.3,IX.9, [ESh], 2.3, [Beu], [Bar],5.12, [NR], [Rud], [Vla], [VS], [Ner2].

This paper is some kind of addendum to Olshanskii and my papers [NO], [Ner2]. Its purpose is to discuss some open problems related to applications of trace theorems in noncommutative harmonic analysis.

I am grateful to G.I.Olshanskii for collaboration. I also thanks V.F.Molchanov, H.Schlichtkrull, G.Zuckerman, B.Ørsted, M.Flensted-Jensen, T.Kobayshi, A.G.Sergeev, G.Olafson, B.Kostant, A.Dvorsky, R.S.Ismagilov, R.Howe, V.M.Gichev, V.V.Lebedev for discussions, comments and references. These notes are mainly based on my lectures [Ner5] on the school 'Analysis on homogeneous spaces' (Tambov, august 1996) and I thanks organizers of this school.

1 Tensor product of two representations of $SL(2,\mathbb{R})$ of complementary series

1.1. Complementary series. Consider the sircle T^1 with coordinates $z = e^{i\phi}$ (where $|z| = 1, \phi \in [0, 2\pi]$). Let $-1 < s < 1, s \neq 0$. Consider a scalar product

$$< f_1, f_2 >= \int_0^{2\pi} \int_0^{2\pi} \frac{f_1(\phi_1)\overline{f_2(\phi_2)}d\phi_1 d\phi_2}{|\sin(\phi_1 - \phi_2)/2|^{(1+s)/2}} \tag{1}$$

in the space of C^∞-functions on the circle. The functions $e^{in\phi}$ are pairwise orthogonal with respect to this scalar product and

$$< e^{in\phi}, e^{in\phi} >= \frac{\Gamma(n + (1+s)/2)}{\Gamma(n + (1-s)/2)} \tag{2}$$

If $s > 0$ then the integral (1) diverges; neverless the formula (2) has sence and we can consider (2) as the definition of divergent integral (1) . Denote by $\mathcal{H}_s$ the completion of the space $C^\infty(T^1)$ with respect to the scalar product (1). We have

$$f(\phi) = \sum c_n e^{in\phi} \in \mathcal{H}_s \Leftrightarrow \sum n^s |c_n|^2 < \infty$$

i.e. $\mathcal{H}_s$ is a *Sobolev space* on the sircle T^1.

We realize the group $SL(2,\mathbb{R})$ as the group of matrices $g = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}$ satisfying the condition $|a|^2 - |b|^2 = 1$. Define a representation of the group $SL(2,\mathbb{R})$ in the space $\mathcal{H}$ by the formula

$$T_s \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} f(z) = f\left(\frac{az + b}{\overline{b}z + \overline{a}}\right) |\overline{b}z + \overline{a}|^{s-1}$$

The representations T_s are unitary in the hilbert space $\mathcal{H}_s$ (this fact is simple but it is not trivial). These representations are called by *representations of $SL(2,\mathbb{R})$ of complementary series.* Recall that representations T_s and T_{-s} are equivalent.

Consider the tensor product $T_{s_1} \otimes T_{s_2}$ of two representations of the compementary series. The space $\mathcal{H}_{s_1} \otimes \mathcal{H}_{s_2}$ is the space of functions on two-dimensional torus $T^2 : z_1 = e^{i\phi_1}, z_2 = e^{i\phi_2}$ equipped with the scalar product

$$< f_1, f_2 >= \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{f_1(\phi_1, \phi_2)\overline{f_2(\psi_1, \psi_2)} d\phi_1 d\phi_2 d\psi_1 d\psi_2}{|\sin(\phi_1 - \psi_1)/2|^{(1+s_1)/2} |\sin(\phi_2 - \psi_2)/2|^{(1+s_2)/2}}$$

The group $SL_2(\mathbb{R})$ acts in this space by the formula

$$(T_{s_1} \otimes T_{s_2}) \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} f(z_1, z_2) = f\left(\frac{az_1 + b}{\overline{b}z_1 + \overline{a}}, \frac{az_2 + b}{\overline{b}z_2 + \overline{a}}\right) |\overline{b}z_1 + \overline{a}|^{s_1 - 1} |\overline{b}z_2 + \overline{a}|^{s_2 - 1}$$

1.2. Restriction to the diagonal.

Denote be Δ the diagonal $\phi_1 = \phi_2$ of the torus T^2.

Theorem 1.1. *Let $s_1 + s_2 > 1$. Then operator of restriction of function $f \in C^\infty(T^2)$ to the diagonal Δ extends to a bounded operator from the space $\mathcal{H}_{s_1} \otimes \mathcal{H}_{s_2}$ to the space $L^2(\Delta)$.*

We emphasis that functions $f \in \mathcal{H}_{s_1} \otimes \mathcal{H}_{s_2}$ are discontinuous and hence a function $f \in \mathcal{H}_s$ have no value in a individual point of the torus. Neverless the operator R of the restriction of a function f to the diagonal is well defined .

Proof of this fact is very simple. The space $\mathcal{H}_{s_1} \otimes \mathcal{H}_{s_2}$ consist of functions

$$f(\phi_1, \phi_2) = \sum c_{n,m} e^{in\phi_1} e^{im\phi_2}$$

satisfying the condition

$$\sum n^{s_1} m^{s_2} |c_{n,m}|^2 < \infty$$

The restriction of f to the diagonal Δ has the form

$$\sum_k (\sum_{m+n=k} c_{m,n}) e^{ik\phi}$$

and the convergence of the series $\sum c_{m,n}$ easily can be deduced from the Cauchy-Bunyakovsky unequality. $\square$

The restriction operator is a bounded $SL(2,\mathbb{R})$-interwinning operator from $T_{s_1} \otimes T_{s_2}$ to $T_{s_1+s_2-1}$. The representation $T_{s_1} \otimes T_{s_2}$ is unitary and hence $T_{s_1+s_2-1}$ is a subrepresentation in $T_{s_1} \otimes T_{s_2}$.

1.3. Dual picture: disributions supported on the diagonal. Now let $s_1 + s_2 < -1$ (we remind that $T_{-s} = T_s$). It turns out to be that in this case the hilbert space $\mathcal{H}_{s_1} \otimes \mathcal{H}_{s_2}$ contains distributions supported on the diagonal Δ. Obviously the space of distributions supported on the diagonal is a invariant subspace. It is easy to show that the representation of $SL(2,\mathbb{R})$ in this subspace is equivalent to $T_{s_1+s_2+1}$ and we again obtain a embedding of representation of complementary series to the tensor product.

1.4. Separation of discrete increment. Let $s_1 = s_2 = s < -1/2$. Consider the operator

$$Jf(\phi_1,\phi_2) = \left|\sin\left(\frac{\phi_1-\phi_2}{2}\right)\right|^{2s} f(\phi_1,\phi_2)$$

Operator J interwines the representation $T_s \otimes T_s$ and representation

$$(T_0\otimes T_0)\begin{pmatrix} a & b \\ \overline{b} & \overline{a}\end{pmatrix} f(z_1,z_2) = f(\frac{az_1+b}{\overline{b}z_1+\overline{a}}, \frac{az_2+b}{\overline{b}z_2+\overline{a}})|\overline{b}z_1+\overline{a}|^{-1}|\overline{b}z_2+\overline{a}|^{-1}$$

The last representation is the standard representation of $SL(2,\mathbb{R})$ in $L^2(T^2)$. The kernel of J consist of distributions supported on the diagonal and the image is dence in L^2. Hence the representation $T_s\otimes T_s$ is equivalent to the direct sum of the representation T_{1+2s} and the representation of $SL(2,\mathbb{R})$ in L^2 on the torus.

Remark. The operator J is unbounded and hence it is nessesary to be careful, see [NO], \$ 7 for details and for the case $s_1 \neq s_2$.

Remark. The existence of embedding $T_{s_1+s_2-1}$ to $T_{s_1}\otimes T_{s_2}$ was obtained by Pukanzky [Puk] in 1961 , analogical rezult for $SL(2,\mathbb{C})$ was obtained by Naimark [Nai] (1961). The explanation of this phenomenon using the restriction operator wasn't observed before [Ner1] (1986).

2 Hilbert spaces defined by positive defined kernels.

2.1. Positive definite kernels. Let Ξ be a set. A function $K(x,y)$ on $\Xi\times\Xi$ is named by a *positive definite kernel* if for each finite collection of points $x_1,\dots,x_n\in\Xi$ we have

$$\det\begin{pmatrix} K(x_1,x_1) & K(x_1,x_2) & \dots & K(x_1,x_n) \\ K(x_2,x_1) & K(x_2,x_2) & \dots & K(x_2,x_n) \\ \vdots & \vdots & \ddots & \vdots \\ K(x_n,x_1) & K(x_n,x_2) & \dots & K(x_n,x_n)\end{pmatrix} \geq 0$$

For each positive definite kernel $K(x,y)$ we associate a Hilbert space H and fixed system of vectors $\Psi_x\in H$ enumerated by $x\in\Xi$ such that

1. The linear span of the system Ψ_x is dence in H

2. For all $x,y\in\Xi$ we have

$$<\Psi_x,\Psi_y>_H = K(x,y) \tag{3}$$

The system Ψ_x is named by *supercompete basis* or *overfilled system.*

Let $h \in H$. We consider a function $f_h(x) = Jh(x)$ on $B_{p,q}$ given by formula

$$JF(x) = f_h(x) = < h, \Psi_x > \tag{4}$$

where $x \in \Xi$.

We will identify the space H and the image of the operator J. In other words *we consider the hilbert space H as a space of functions on Ξ.*

2.2. Hilbert spaces of holomorphic functions. Let Ω be a open domain in $\mathbb{C}^n$. A function $K(z,u)$ on $\Omega \times \Omega$ is called by a *reproducing kernel* on Ω if

1. $K(z,u)$ is a positive definite kernel
2. Function (z,u) is holomorphic with respect to variables z and antiholomorphic with respect to varables u.

Consider associated hilbert space H and let us realize it as a space of functions on Ω by the formula (4) . It is easy to see that functions f_h are holomorphic, i.e. hilbert space H can be considered as a hilbert space of holomorphic functions on Ω.

2.3. Example: Berezin spaces on Cartan domains. Let $p \leq q$. Denote by $B_{p,q}$ the space of all $p \times q$-matrices z satisfying the condition $\|z\| < 1$, i.e $B_{p,q}$ is a *Cartan domain of the first type.*

A pseudounitary group $U(p,q)$ consists of $(p+q) \times (p+q)$-matrices $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfying the condition

$$g \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot g^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5}$$

The group $U(p,q)$ acts on $B_{p,q}$ by the *fractional-linear* transformations

$$z \mapsto z^{[g]} := (a+zc)^{-1}(b+zd) \tag{6}$$

The stabilizer of the point $z = 0$ consists of matrices having the form $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ where $a \in U(p), d \in U(q)$. Hence $B_{p,q}$ is the riemann symmetric space

$$U(p,q)/(U(p) \times U(q))$$

Theorem 2.1. *(see [Ber1], see also [Wal]) Let $s = 0, 1, 2, \ldots, p-1$ or $s > p-1$. Then the kernel*

$$K_s(z,u) = \det{}^{-s}(1 - zu^*)$$

is positive defined on $B_{p,q}$.

Denote by V_s the hilbert space of holomorphic functions on $B_{p,q}$ defined by the kernel $K_s(z,u)$ (for discussion of this space see for instance [Ber1], [NO]). The group $U(p,q)$ acts in V_s by a unitary operators

$$T_s \begin{pmatrix} a & b \\ c & d \end{pmatrix} = f(z^{[g]}) \det{}^{-s}(a+zc) \tag{7}$$

where $z^{[g]}$ is given by the formula (6).

The representations T_s are called by *spherical highest weight representations* of $U(p,q)$.

Remark. If s is integer then T_s is a representation of the group $U(p,q)$ itself. If s is not integer then the function $\det^{-s}$ is multivalued and hence T_s is a representation of the universal covering group of $U(p,q)$ (in other words the representation T_s is a projective representation of the group $U(p,q)$).

Remark. If $s > p-1$ then the space V_s contains all polynomial functions on Cartan domain $B_{p,q}$. If $s = 0, 1, \dots, p-1$ when all functions $f \in V_s$ satisfies to the following system of partial differential equations

$$\left[\det \begin{pmatrix} \frac{\partial}{\partial z_{i_1 j_1}} & \cdots & \frac{\partial}{\partial z_{i_\alpha j_1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial z_{i_1 j_\alpha}} & \cdots & \frac{\partial}{\partial z_{i_\alpha j_\alpha}} \end{pmatrix}\right] f(z) = 0$$

for all $\alpha = 2, 3, \dots, s+1$ and all i_μ, j_ν satisfying unequalities $0 \le i_\mu \le q$, $O \le j_\nu \le p$.

2.4. Cartan domains of second and third types. A Cartan domain C_n of the third type is the set of all $n \times n$ symmetric matrices z with complex coeffitients satisfying the condition $\|z\| < 1$. The group G of biholomorphic transformations (automorphisms) of the domain C_n consists of transformations having form (6) where a matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfies the condition

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^t = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad d = \overline{a}, \quad c = \overline{b} \tag{8}$$

It is easy to see that the group G is isomorphic to a real symplectic group $Sp(2n, \mathbb{R})$ and the domain C_n is a riemann symmetric space

$$C_n = Sp(2n, \mathbb{R})/U(n)$$

A kernel

$$K_s(z, u) := \det{}^{-s}(1 - z\overline{u})$$

is positive definite if and only if

$$\{s = 0, 1/2, \dots, (n-1)/2\} \text{or} \{s > (n-1)/2\}$$

A *Cartan domain* D_n of the second type is a set of all $n \times n$ skew-symmetric matrices z satisfying the condition $\|z\| < 1$. Group G of automorphisms of the domain D_n is a group of transformations (6) where a matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfies the condition

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad d = \overline{a}, \quad c = \overline{b} \tag{9}$$

Remark. Condition (9) implies (5) and also (9) implies (5).

The group G is isomorphic to a real classical group $SO^*(2n)$ and the domain D_n is the symmetric space

$$D_n = SO^*(2n)/U(n)$$

The kernel $K(z,u) = \det^{-s}(1 - zu^*)$ on D_n is positive defined if $s = 0, 1, 2, \dots, n-1$ or $s > n-1$.

Spherical highest weight representations of $Sp(2n,\mathbb{R})$ and $SO^*(2n)$ act in hilbert spaces associated to the kernels K_s by the formula (7).

2.5. Matrix-valued positive definite kernels. Consider a set Ξ and a finite dimensional complex euclidian space $\mathcal{Y}$. Denote by $GL(\mathcal{Y})$ the group of invertible linear operators in $\mathcal{Y}$. We say that a function

$$L : \Xi \times \Xi \to GL(\mathbb{C}, \mathcal{Y})$$

is a *matrix-valued positive definite kernel* if a function

$$\widetilde{L}((z,\xi);(u,\eta)) := < L(z,u)\xi, \eta > ; \quad (z,\xi),(u,\eta) \in \Xi \times \mathcal{Y}$$

is a positive defined kernel on $\Xi \times \mathcal{Y}$.

Let we have a matrix-valued positive defined kernel on Ξ. Then there exists a unique hilbert space H and a map $\Psi : \Xi \times \mathcal{Y} \to H$ such that

a) The map Ψ is linear on each fibre $z \times \mathcal{Y} \subset \Xi \times \mathcal{Y}$.

b) $< \Psi(z,\xi), \Psi(u,\eta) >_H = < L(z,u)\xi, \eta >_{\mathcal{Y}}$

c) The image of map Ψ is dence in H.

Again there exists a canonical map J from the hilbert space H to space of $\mathcal{Y}$-valued functions on Ξ. Let $h \in H$. Then the function $f_h = Jf$ is defined from the equation

$$< f_h, \xi >_{\mathcal{Y}} = < h, \Psi(z,\xi) >_H$$

2.6. Highest weight representations of the groups $G = U(p,q)$, $Sp(2n,\mathbb{R})$, $SO^*(2n)$.

Consider the case $G = U(p,q)$. Consider a irreducible finite-dimensional holomorphic representation $\rho = \rho_1 \otimes \rho_2$ of the group $GL(p,\mathbb{C}) \times GL(q,\mathbb{C})$ or of its universal covering. Let $W = W_1 \otimes W_2$ be the space of the representation ρ. Assume that function

$$K_\rho(z,u) := \rho_1(1 - zu^*) \otimes \rho_2(1 - u^*z)$$

is a matrix-valued positive-definite kernel (see [Ols2] for conditions of positive definiteness, see also [NO]). Consider associated hilbert space H_ρ of $W_1 \otimes W_2$-valued holomorphic functions on $B_{p,q}$. Then *unitary highest weight representation T_ρ* of the group $U(p,q)$ (or of its universal covering) acts in the space H_ρ by the formula

$$T_\rho \begin{pmatrix} a & b \\ c & d \end{pmatrix} F(z) = (\rho_1(a + zc) \otimes (b - dz^{[g]}))F(z^{[g]}) \qquad (10)$$

Let $G = Sp(2n,\mathbb{R})$ or $SO^*(2n)$. Consider a finite dimensional holomorphic representation ρ of group $GL(n,\mathbb{C})$ or of its universal covering group. Assume matrix-valued kernel $K_\rho(z,u) = \det^{-s}(1 - zu^*)$ be positive-definite on the Cartan domain C_n or D_n. Denote by H_ρ associated hilbert space of holomorphic vector-valued functions. Then *unitary highest weight representation* T_ρ of the group G is given by the formula

$$T_\rho \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(z) = \rho(a + zc) F(z^{[g]})$$

2.7. Bibliographical comments.

a) There are also *Cartan domains of 4-th type* $SO(n,2)/(SO(n) \times O(2))$ and two exeptional Cartan domains. There exists associated highest weight representation theory. Cartan domains $SO(n,2)/(O(n) \times O(2))$ are popular object of compex analysis (*tube of future*), see [SV]. These domains often are called by *Lie spheres or Lie balls*, but this term is a result of philological mistake (it is a result of the following sequence of translations: german-chinese-russian-english).

b) Conditions of positive-definiteness of standard scalar-valued kernels on Cartan domains of 1-4 typpes were discovered by Berezin [Ber] (with some additions given by S.G.Gindikin; Berezin and Gindikin didn't consider exeptional domains; later conditions of positive-defineteness in general case (including exeptional) were obtained by Wallach [Wal] and Rossi - Vergne).

c) Conditions of positive-defineteness of standard kernels for $B_{p,q}$ where obtained in [Ols2]. For other domains this problem was discussed by many authors, the final result was obtained in [EHW].

3 Boundary values of holomorphic functions.

3.1. L^1-limits

Let $\Omega \subset \mathbb{C}^n$ be a open domain, let $\partial\Omega$ be its boundary, let $\overline{\Omega}$ be the closure of Ω. We say that Ω is a *regular circle domain* if

a) for all $z \in \Omega$ and $\lambda \in \mathbb{C}$ such that $|\lambda| \leq 1$ we have $\lambda z \in \Omega$

b) for all $z \in \partial\Omega$ and $\lambda \in \mathbb{C}$ such that $|\lambda| < 1$ we have $\lambda z \in \Omega$

Let $K(z,u)$ be a reproducing kernel in Ω satisfying the condition

$$K(e^{i\phi}z, e^{i\phi}u) = K(z,u) \tag{11}$$

Let H be the hilbert space of holomorphic functions associated to this kernel.

Remark. Cartan domains are regular circle domains and Berezin kernels satisfy the condition (11)

Theorem 3.1.*(see [NO]) Let $M \subset \partial\Omega$ be a compact subset. Let μ be a measure supported on M. Let*

a)$K^*(z,u) := \lim_{c\to 1-0} K(cz,cu)$ exist almost sure on $M \times M$ with respect to the measure $\mu \times \mu$.

b)$K^* \in L^1(M \times M, \mu \times \mu)$ and $\lim_{c\to 1-0} K(cz, cu)$ is dominated, i.e. there exists a function $S(z,u) \in L^1(M \times M, \mu \times \mu)$ such that $|K(z,u)| < S(z,u)$ almost sure on $M \times M$.

Then the operator of restriction of a function $f \in H$ on the set M is well-defined operator

$$H \to L^1(M, \mu)$$

This theorem doesn't cover all cases then the restriction operator exists (see below subsections 3.2-3.3) . Hence the following natural problem arises.

Problem. Let $\Omega \subset \mathbb{C}^n$ be a a open domain. Let $K(z,u)$ be a reproducing kernel and let H be the associated hilbert space. Let M be a submanifold in the Shilov boundary of Ω. Find conditions for existence of restriction operator from H to some functional hilbert space on M.

Remark. The case which is interesting for harmonic analysis is the followind case. Let the domain Ω be a Cartan domain G/K and $H = V_s$ be a Berezin space. Consider a subgroup $Q \subset G$ and some Q-orbit M in Shilov boundary of domain Ω. Let the operator of restriction of function $f \in H_s$ to the manifold M exists. Then we obtain a interwinning operator from the Berezin space H_s to some hilbert space $H(M)$ of functions on M. The representation of Q in $H(M)$ is equivalent to a subrepresentation in representation of Q in H. If the group Q is semisimple and orbit M is compact then representation of Q in $H(M)$ has purely discrete spectrum.

3.2. Restrictions of functions of polynomial grouth in a ball. Denote by B_q the unit ball

$$|z_1|^2 + \ldots + |z_q|^2 < 1$$

in $\mathbb{C}^q$. Denote by $H^\infty(B_q)$ Hardy space of bounded holomorphic functions on B_q. Denote by $\langle \cdot, \cdot \rangle$ be standard scalar product in $\mathbb{C}^n$.

Let $\gamma(t)$ be a C^1-curve in the sphere $\partial B^q = S^{2q-1}$.

Theorem 3.2 *(see [NR],[Rud]) Let $\gamma(t)$ satisfies the condition*

$$\forall t : Im\langle \gamma(t), \gamma'(t) \rangle \neq 0 \tag{12}$$

Then for each $f \in H^\infty(B^q)$ the nontangent limit $f(z)$ as $z \to \gamma(t)$ exists almost sure on the curve $\gamma(t)$.

Remark. Obviously vector $\gamma'(t)$ is tangent to the sphere S^{2q-1}. Hence we have $Re\langle \gamma(t), \gamma'(t) \rangle = 0$ for all t. Fix $z \in S^{2q-1}$, let $v \in \mathbb{C}^q$be a vector. The equation $\langle z, v \rangle = 0$ defines a subspace of codimension 1 in a tangent space to the sphere S^{2q-1} in a point $z \in \S^{2q-1} \subset \mathbb{C}^n$. We obtain a contact distribution on sphere (see [Rud], 5.4, 10.5). The condition (12) means that $\gamma(t)$ is transversal to this distribution.

Denote by $\mathcal{D}'$ the space of all holomorphic *functions of polynomial grouth* in B^q:

$$f \in \mathcal{D}' \Leftrightarrow \exists N : \sup |f(z)|(1 - |z|^2)^N < \infty$$

Remark. The space $\mathcal{D}'$ is a union $\mathcal{D}' = \cup_{s>0} V_s(B_q)$ of Berezin spaces in balls, see subsection 2.3. The space $V_q(B_q)$ coincides with Hardy space $H^2(B_q)$. The space $V_{q+1}(B_q)$ is Bergman space (i.e intersection of $L^2(B_q)$ and the space of holomorphic functions).

Remark. It is well known (and more or less obvious) that each function $f \in \mathcal{D}'$ has limit on the boundary in the sence of distributions (see [RS],IX.3 for discussion of such type theorems and references).

Denote by R operator of resctriction of function f holomorphic in the ball B_q to the curve $\gamma(t)$. Obviously function Rf is well-defined if the function f is continuous up to the boundary.

Theorem 3.3. *(see [Ner2]) Let $\gamma(t)$ be a C^∞-smooth curve in ∂B^q satisfying the condition (12). Then the operator R of restriction of holomorphic function to $\gamma(t)$ extends to a bounded operator from $\mathcal{D}'$ to the space of distributions on $\gamma(t)$ Moreover distrubution Rf coincides with the limit*

$$\lim_{\epsilon \to +0} f(1-\epsilon)z$$

(where $z \in S^{2q-1}$) in the sence of distributions.

3.3. Restriction of functions of polynomial grouth in polydisk.

Denote by P^n a polydisk $|z_1| < 1, \dots, |z_n| < 1$. Let T^n be the torus $z_1 = e^{i\phi_1}, \dots, z_n = e^{i\phi_n}$. Let $\gamma(t) = (\phi_1(t), \dots, \phi_n(t))$ be a *time-like* C^∞-curve in T^n i.e.

$$\forall t : \phi_1'(t) > 0, \dots, \phi_n'(t) > 0 \tag{13}$$

Denote by $\mathcal{D}'$ the space of holomorhic functions of polynomial grouth in P^n, i.e

$$f \in \mathcal{D}' \Leftrightarrow \exists N : \sup f(z) \prod_{k=1}^{n} (1-|z_k|)^N < \infty$$

This condition is equivalent to the condition

$$f(z_1, \dots, z_n) = \sum c_{k_1,\dots,k_n} z_1^{k_1} \dots z_n^{k_n} \in \mathcal{D}' \Leftrightarrow$$

$$\Leftrightarrow \exists L : \sum \frac{|c_{k_1,\dots,k_n}|}{[(k_1+1)\dots(k_n+1)]^L} < \infty$$

Theorem 3.4. *(see [Ner2]) Operator R of restriction of holomorphic function f on P^n to the curve $\gamma(t)$ extends to a bounded operator from the space $\mathcal{D}'$ to the space of distributions on the curve $\gamma(t)$. Moreover Rf coincides with the radial limit*

$$\lim_{\epsilon \to +0} f((1-\epsilon)\phi_1(t), \dots, (1-\epsilon)\phi_n(t))$$

(where $z \in T^q$) in the sence of distributions.

Example. Consider a bidisk P^2 and the hilbert space H_{s_1,s_2} associated to the reproducing kernel

$$(1 - z_1\overline{u}_1)^{-s_1}(1 - z_2\overline{u}_2)^{-s_2}$$

where $s_1, s_2 > 0$. It is easy to show that $\mathcal{D}' = \cup_{s_1,s_2>0} H_{s_1,s_2}$. If $\gamma(t)$ has a form $(\gamma_1(t), \gamma_2(t)) = (e^{it}, e^{i\sigma})$ where $\sigma = const$, then operator of restriction to $\gamma(t)$ doesn't exist for all s_1, s_2. If $\gamma(t)$ satisfies the condition $\gamma_1'(t) > 0, \gamma_2'(t) < 0$ then the restriction operator exists only in the case $s_1 + s_2 < 1$.

3.4. A conjecture It seems to me that Theorems 3.3-3.4 are partial cases of some general fact. Let $\Omega \subset \mathbb{C}^N$ be a open domain. Let M be a submanifold in the Shilov boundary of Ω. Denote by T_m the tangent space to M in the point $m \in M$. We identify T_m with a linear submanifold in $\mathbb{C}^n$. Denote by $S_m = i \cdot T_m$ the linear submanifold which consists of vectors

$$i \cdot (v - m) + m$$

where $v \in T_m$ (and $i^2 = -1$). We say submanifold M is *time-like* if for each point $m \in M$ there exists a open cone $C_m \subset S_m$ with the vertex m and ϵ-neighbourhood $\mathcal{U}_\epsilon(m)$ of m such that

$$C_m \cap \Omega \supset \mathcal{U}_\epsilon(m) \cap C_m$$

Conjecture. Each holomorphic function of polynomial grouth in Ω has restriction to time-like manifold M in the Shilov boundary in the sense of distributions.

Theorems 3.3-3.4 are partial cases of this conjecture. It is also similiar to the standard facts on limits of functions of polynomial grouth on the *whole* Shilov boundary mentioned above (see [RS], IX.3, [Vla], [VS]). Neverless I couldn't find this fact in the literature.

4 Positive defined kernels on riemann noncompact symmetric spaces

4.1. Spherical kern-representations of the groups $U(p,q)$. Consider a matrix ball $B_{p,q}$, see subsection 2.3. Let $s = 0, 1, 2, \ldots, p-1$ or $s > p-1$. Consider a function

$$L_s(z,u) = |\det(1 - zu^*)|^{-2s}$$

where $z, u \in B_{p,q}$. With notations of subsection 2.3 we have

$$L_s(z,u) = K_s(z,u)\overline{K_s(z,u)}$$

and hence $L_s(z,u)$ is a positive defined kernel on $B_{p,q}$. Denote by H_s the hilbert space associated with the kernel $L_s(z,u)$. Using the formula (4) we identify this space with some space of real analytic functions on $B_{p,q}$.

The group $U(p,q)$ acts in H_s by the unitary operators

$$A_s(g)f(z) = f(z^{[g]})|\det(a + zc)|^{-2s}$$

where $z^{[g]}$ is defined by the formula (6).

Problem. *Decompose the repersentation A_s*

We will name representations T_s by *spherical kern-representations* of the group $U(p,q)$.

Some partial cases of this question were discussed in the end of 70-ies, see [Ber2],[Rep],[Gut]. In fact there was discussed only the case when s is large. In this case the most interesting phenomena don't appear. Then such problems were more or less forgotten. In last several years some this problem attracted interest again (see [NO], [UU], [OØ], [OZ],[Dij]).

I would like to try to explain why this problem is interesting and also to discuss some approaches to this problem.

4.2. Limit as $s \to \infty$. We want to show that the limit of kern-representations as $s \to +\infty$ is the canonical representation of $U(p,q)$ in the space L^2 on riemann symmetric space $U(p,q)/(U(p) \times U(q))$

Concider the system of vectors $\Psi_z \in H_s$ (see subsection 2.2). Let χ be a distribution in $B_{p,q}$ with a compact support. Consider the vector $\Theta(\chi) \in H_s$ defined by the equality

$$\Theta(\chi) = \int_{B_{p,q}} |\det{}^s(1 - zz^*)|\chi(z)\Psi_z dz d\overline{z}$$

Consider a scalar product $\{\cdot,\cdot\}_s$ in the spase of distributions on $B_{p,q}$ with compact support given by the formula

$$\{\chi_1, \chi_2\}_s := \langle \Theta(\chi_1), \Theta(\chi_2) \rangle =$$

$$= \int_{B_{p,q}} \int_{B_{p,q}} \left| \frac{\det(1 - zz^*)\det(1 - uu^*)}{\det^2(1 - zu^*)} \right|^s \chi_1(z)\chi_2(u) dz d\overline{z} du d\overline{u}$$

We can identify the space H_s with the completion of the space of distributions with respect to scalar produt $\{\cdot,\cdot\}_s$. The group $U(p,q)$ acts in this space of distributions by the formula

$$B_s(g)f(z) = f(z^{[g]})$$

(the formula doesn't depend of s, neverless the scalar product and spectrum of representation depend on s essentially)

It appeared that for some normalizing multiplier $\omega(s)$ for all continuous functions $\phi_1(z), \phi_2(z)$ on $B_{p,q}$ with compact support we have

$$\lim_{s \to +\infty} \omega(s)\{\phi_1, \phi_2\}_s = \int_{B_{p,q}} \phi_1(z)\phi_2(z) dz d\overline{z}$$

4.3. Another formulation of the problem. Consider a spherical highest weight representation T_s of the group $U(p,q)$ (see subsection 2.3). Denote by V_s the space of representation T_s. Denote by T_s^* the representation contragradient to T_s. Consider the tensor product $T_s \otimes T_s^*$. This representation acts in the space $V_s \otimes V_s$ of holomorphic functions on $B_{p,q} \times B_{p,q}$ by the operators

$$(T_s \otimes T_s^*)(g) = f(z_1, z_2) = f(z_1^{[g]}, z_2^{[\overline{g}]})(a + z_1 c)^{-s}(\overline{a} + z_2 \overline{c})^{-s}$$

The $U(p,q)$-invariant scalar product in the space of holomorphic functions on $B_{p,q} \times B_{p,q}$ is defined by a reproducing kernel

$$M(z_1, z_2; u_1, u_2) = \det{}^{-s}(1 - z_1 u_1^*)\det{}^{-s}(1 - z_2 u_2^*)$$

Consider a operator

$$I : V_s \otimes V_s \to H_s$$

defined by the formula

$$If(z) = f(z, \overline{z})$$

Obviously I is a unitary operator interwining representations

$$T_s \otimes T_s^* \leftrightarrow A_s$$

Hence we can formulate our problem in the form:

Problem. *Decompose the tensor product* $T_s \otimes T_s^*$.

4.4. Orbits of the group $U(p,q)$ on the Shilov boundary.

Denote by $M_{p,q}$ Shilov boundary of $B_{p,q}$. Elements of $M_{p,q}$ are matrices z satisfying the condition

$$z \cdot z^* = 1$$

In the other words z is a matrix of a isometric embedding $\mathbb{C}^p \to \mathbb{C}^q$. Hence $M_{p,q}$ is a *complex Stiefel manifold.*

Shilov boundary of $B_{p,q} \times B_{p,q}$ is $M_{p,q} \times M_{p,q}$. The group $U(p,q)$ has $(p+1)$ orbits on $M_{p,q} \times M_{p,q}$. The unique invariant of a orbit is number

$$\alpha = rk(z - \overline{u}) \tag{14}$$

i.e if $(z_1, u_1), (z_2, u_2) \in M_{p,q} \times M_{p,q}$ and $rk(z_1 - \overline{u}_1) = rk(z_2 - \overline{u}_2)$ when points $(z_)$, $u_1), (z_2, u_2)$ are elements of one orbit.

We denote by Ξ_α the orbit corresponding to a given invariant α.

Remark. Orbit Ξ_0 is compact, orbit Ξ_p is open (in Shilov boundary). For all α the closure of Ξ_α is $\cup_{\sigma \le \alpha} \Xi_\sigma$.

4.5. Restriction of holomorphic functions to $U(p,q)$-orbits in the Shilov boundary. Fix a orbit Ξ_α of $U(p,q)$ in the Shilov boundary of $B_{p,q} \times B_{p,q}$. It can happens (and it realy happens) that for small s function $f \in H_s = V_s \otimes V_s$ has well defined restriction to the orbit Ξ_α. In this case the restriction operator is a interwinning operator from $V_s \times V_s$ to some hilbert space of functions on Ξ_α.

It can also happened (and it realy happens) that for small s all first partial derivatives of function $f \in H_s = V_s \otimes V_s$ have well defined restriction to the orbit Ξ_α etc.(see discussion of this phenomenon in [NO],section 7).

Fix s. For each $\alpha = 0, 1, \dots, p-1$ consder the maximal number τ_α such that all partial derivatives of order $\le \tau_\alpha$ of functions $f \in V_s \otimes V_s$ have well-defined restrictions to Ξ_α. (these numbers aren't known, but Theorem 3.1 give a possibility to estimate them; I don't know are such estimates strict or not). If restrictions of functions $f \in H_s = V_s \otimes V_s$ to Ξ_α don't exist we suppose $\tau_\alpha = -1$.

Remark. For large s restriction operators don't exist, i.e. we have $\tau_\alpha = -1$ for all $\alpha < p-1$.

Fix $\alpha = 0, 1, \dots p-1$. Consider some $i_\alpha = 1, \cdots, \tau_\alpha$. Denote by $Q[\alpha, i_\alpha]$ the space of functions $f \in V_s \otimes V_s$ such that all partial derivatives of f of orders $\le i$ equals zero on Ξ_α. We obtain a filtration

$$\begin{gathered} 0 \subset Q[p-1, \tau_{p-1}] \subset Q[p-1, \tau_{p-1}-1] \subset \dots \subset Q[p-1, 0] \subset \\ \subset Q[p-2, \tau_{p-2}] \subset \dots \subset Q[p-2, 1] \subset Q[p-2, 0] \subset \dots \\ \dots \subset Q[0, \tau_0] \subset \dots Q[0, 1] \subset Q[0, 0] \subset V_s \otimes V_s \end{gathered} \tag{15}$$

Remark. For large s this filtration is trivial, neverless for small s it is quite long.

Consider representations of $U(p,q)$ in subquotients of this filtration. Obviously $A_s = T_s \otimes T_s^*$ is equivalent to the direct sum of the subquotients of the filtration (15).

Remark. The representations of $U(p,q)$ in the subquotients have simple interpretations. For instance $(V_s \otimes V_s)/Q[0,0]$ is a subspace in the space of functions on the orbit Ξ_0. The space $Q[0,1]/Q[0,0]$ is a subspace in space of sections of normal bundle to the orbit Ξ_0 . The space $Q[0,1]/Q[0,2]$ is a subspace in space of sections of symmetric square of normal bundle etc., see discussion in [NO],$ 7.

It is natural to hope that spectrum in each subquotient is more or less 'uniform', i.e. orbit structure of Shilov boundary give separation of quite complicated spectrum of A_s to the different types (compaire with [GG]-project)

4.6. Large s. If s is large enough then the restriction operators don't exist. In this case the representation A_s is equivalent to standart representation of the group $U(p,q)$ in L^2 on riemann symmetric space $U(p,q)/(U(p) \times U(q))$, see [Ber2], [Rep], [Gut], [OØ]. Sufficient (not nessessary) condition for this is $s > p+q-1$ (i.e. T_s is a element of Harish-Chandra discrete seriees).

4.6. Restriction to the compact orbit. The part of spectrum which corresponds to the compact orbit Ξ_0 is purely discrete and it consists of quite exotic representations of $U(p,q)$. This part of spectrum is not empty for small s (for $s < (q-2p-1)/4$), see in [NO] description of analogical situation for compact orbit of $O(p,q)$ on the Shilov boundary of $B_{p,q}$.

In particular it gives a relatively simple way for constructing singular unitary representations of the groups $U(p,q)$, see [NO].

4.7. Spherical kern-representations of other classical groups All classical riemann noncompact symmetric spaces G/K (up to centrum of G) can be realized as (real, complex or quaternionic) matrix balls (see [Ner3]). Namely the space G/K is the space of matrices z over the field $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ (see below) satisfying the additional condition (see below) such that $\|z\| < 1$.

Now we are enumerate symmetric spaces G/K, fields and additional conditions.

1^*. $U(p,q)/(U(p) \times U(q))$ is the space of $p \times q$ matrices over $\mathbb{C}$.

2^* . $Sp(2n, \mathbb{R})/U(n)$ is the space of symmetric $n \times n$-matrices over $\mathbb{C}$.

3^*. $SO^*(2n)/U(n)$ is the space of skew symmetric $n \times n$-matrices over $\mathbb{C}$.

4^*. $O(p,q)/(O(p) \times O(q))$ is the space of $p \times q$-matrices over $\mathbb{R}$.

5^*. $GL(n, \mathbb{R})/O(n)$ is the space of symmetric $n \times n$-matrices over $\mathbb{R}$.

6^*. $O(n, \mathbb{C})/O(n)$ is the space of skew-symmetric $n \times n$-matrices over $\mathbb{R}$.

7^*. $GL(n, \mathbb{C})/U(n)$ is the space of hermitian $n \times n$-matrices over $\mathbb{C}$.

8^*. $Sp(p,q)/(Sp(p) \times Sp(q))$ is the space of $p \times q$-matrices over $\mathbb{H}$.

9^*. $GL(n, \mathbb{H})/Sp(p,q)$ is the space of hermitian $n \times n$-matrices over $\mathbb{H}$.

10^*. $Sp(2n, \mathbb{C})/Sp(n)$ is the space of skew hermitian (i.e. $z = -z^*$) $n \times n$-matrices over $\mathbb{H}$.

In all cases enumerated above the group G acts on the matrix ball G/K by fractional-linear transformations

$$z \mapsto z^{[g]} := (a + zc)^{-1}(b + zd) \tag{16}$$

Now let us consider positive definite kernels on G/K having a form

$$L_s(z,u) = |\det(1 - zu^*)|^{-2s}$$

(conditions for positive-definiteness are different for different spaces). Consider the hilbert space defined by the positive defined kernel $L_s(z,u)$. We identify this space with some space of real analytic functions on G/K by the same way as in 2.1. The group G acts in H_s by the unitary operators

$$A_s(g)f(z) = f(z^{[g]})|\det(a + zc)|^{-2s}$$

We say A_s is a *spherical kern-representation of* G.

4.8. Nonspherical kern-representations.

Consider a symmetric spaces having the form $2^*, 3^*, 5^*, 6^*, 7^*, 9^*, 10^*$. Consider a finite dimensional irreducible representation ρ of the group $GL(n, \mathbb{K})$ or of its universal covering. Assume that the function

$$L(z,u) = \rho(1 - zu^*)$$

is a matrix valued positive definite kernel. Then we consider the associated hilbert space H_ρ of real-analytic functions $G/K \to \mathcal{Y}$ and the unitary kern-representation of G in H_ρ given by the formula

$$T_\rho(g)f(z) = \rho(a + zc)f(z^{[g]}) \tag{17}$$

Consider the cases $1^*, 4^*, 9^*$. Consider a finite-dimensional irreducible representation $\rho = \rho_1 \otimes \rho_2$ of the group $GL(p, \mathbb{K}) \times GL(q, \mathbb{K})$ or of its universal covering. Assume that the function

$$L_\rho(z, u) := \rho_1(1 - zu^*) \otimes \rho_2(1 - u^* z)$$

is a positive defined matrix-valued kernel on G/K. Then the group G acts in the associated space of real-analytic functions on G/K by the formula

$$T_\rho(g)f(z) = \big(\rho_1(a + zc) \otimes \rho_2(d - cz^{[g]})\big) f(z^{[g]})$$

Remark. Our arguments from subsections 4.3-4.4 are valid for general kern-representations.

4.9. Another description of kern-representations.(see [OØ]) For $G = Sp(2n, \mathbb{R})$,$U(p, q)$, $SO^*(2n)$ a kern representation is a tensor product of irreducible highest weight representation of G and irreducible lowest weight representation of G.

In other cases a kern representation of G is a restriction of a highest weight representation of the group G^* to the **symmetric** subgroup G:

4^*.	$G = O(p, q)$	$G^* = U(p, q)$	5^*.	$G = GL(n, \mathbb{R})$	$G^* = Sp(2n, \mathbb{R})$
6^*.	$G = O(n, \mathbb{C})$	$G^* = SO^*(2n)$	7^*	$G = GL(n, \mathbb{C})$	$G^* = U(n, n)$
8^*	$G = Sp(p, q)$	$G^* = U(2p, 2q)$	9^*	$G = GL(n, \mathbb{H})$	$G^* = SO^*(2n)$
10^*	$G = Sp(2n, \mathbb{C})$	$G^* = Sp(4n, \mathbb{R})$			

Remark. The cases $1^* - 3^*$ can be described by the same way. We have $G^* = G \times G$ and the embedding $G \to G^*$ given by the formula $g \mapsto (g, g^\theta)$ where θ is a outer automorphism of G.

Remark. There are some additional possibilities related to highest weight representations of $O(p, 2)$ and two exeptional groups (see [OØ]).

4.10. Remark: action of Olshanskii semigroup. For each matrix ball G/K consider the set of invertible matrices $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that the fractional-linear mapping (16) maps the matrix ball to itself. Obviously Γ is a semigroup and the group G is the group of invertible elements of Γ. The formula (17) defines representation of semigroup Γ. This representation is irreducible and all irreducible representations of Γ can be obtaned by this way(see[Ols1]). See [Ols1], [Ner3],[Ner4],appendix A, for the explicit description of semigroups Γ. Moreover kern-representations extends to representations of some categories(see[Ner3],[Ner4],appendix A). I don't know any applications of these phenomena to harmonic analysis of kern-representations of groups.

4.11. Bibliographical comments.

a) Let T be a highest weight representation of $G = Sp(2n, \mathbb{R})$, $SO^*(2n)$, $U(p, q), \ldots$ and S be a lowest weight representation of G. Assume that T, S be elements of Harish-Chandra discrete series. Then $T \otimes S$ is equivalent to a representation of G induced from irreducible representation of maximal compact subgroup $K \subset G$ (see [Ber2],[Rep],[Gut]).

b) Spherical kern-representations for large s are equivalent to canonical representation of G in $L^2(G/K)$,see [OØ].

c) Discrete spectra associated to the compact orbit in Shilov boundary were investigated in [NO] for the case $G = O(p,q)$. Analogical results are valid for $G = U(p,q)$, $Sp(p,q)$ ([Ols2], [NO], 7.12.). One method of separation of discrete spectrum is discussed in [Ner1], [NO],7.1-7.8. I think that the restriction operator to the compact orbit doesn't exist for $G \neq U(p,q), Sp(p,q), O(p,q)$.

d) Spectra for spherical kern-representations of $U(2,2)$ were obtained in [OZ2] . For $1 < s < 3/2$ the spectrum consist of two different pieces. One of pieces coincides with the spectrum of $L^2(U(2,2)/(U(2) \times U(2))$. Another piece is a integral of notrivial representations. It is natural to think (it is not proved) that this piece of spectrum is associated to noncompact $U(2,2)$-orbit in the Shilov boundary of $B_{2,2} \times B_{2,2}$. If it is so it is the unique known case when spectrum associated to noncompact orbit is observed. It is natural to think that such spectra exist in various spectral problems (not only for kern-representations).

e)Spectrum of spherical kern-representations of $SU(1,1)$ was obtained in [VGG] and in the case of $U(p,1)$ it was obtained in [Dij] (construction of discrete increments was observed in [Ner1]).

f) Plancherel formula for tensor product of highest weight and lowest weight (single-valued) representations of $SL(2,\mathbb{R})$ is obtained in [Mol5]. In [Ber2] there was announced theorem which is almost equivalent to the Plancherel formula for spherical kern-representations in the case then *s is large* and symmetric space is hermitian.

g) Nonspherical kern-representations have discrete spectrum which is not associated to compact orbit in the Shilov boundary. Some possibilities to observe it are contained in the following two sections. A way to observe Harish-Chandra discrete series increments using trace theorems is proposed in [Ner2].

h) Let ρ be the same as in 2.10. Let $\rho_s(\gamma) = \det^{-s}(\gamma)\rho(\gamma)$. Then limit of T_{ρ_s} as $s \to \infty$ is the representation of G induced from finite dimensional representation of K (i.e the representation in sections of vector bundle on G/K). The theory of such representations is more or less equivalent to Harish-Chandra theory of $L^2(G)$.

e) *Deformation quantization,* see [Ber2]. As we have seen in 4.2 $H_s = V_s \otimes V_s$. Hence we can identify the space H_s and space of Hilbert-Schmidt operators $V_s \to V_s$. Multiplication of Hilbert-Schmidt operators induces some associative operation in H_s. This operation can be concidered as so-called "deformation quantization", see [Ber3].

5 Dual pairs

5.1. Harmonic representation of $Sp(2n,\mathbb{R})$. We remind a definition of *harmonic representation* W_{2n} (= Weil representation = Segal-Shale-Weil representation = Friedrichs-Segal-Berezin-Shale-Weil representation = oscilator representation) of the group $Sp(2n,\mathbb{R})$ (see [KV],[Ner4] for discussion of this representation).

Consider a hilbert space F_n of holomorpric (entire) functions on $\mathbb{C}^n$ associated to the reproducing kernel $\exp(\sum z_j \overline{u}_j)$ (it is so-called *boson Fock space*). Let $\begin{pmatrix} \Phi & \Psi \\ \overline{\Psi} & \overline{\Phi} \end{pmatrix} \in Sp(2n,\mathbb{R})$ (see the realization of $Sp(2n,\mathbb{R})$ described in subsection 2.4) The harmonic representation is a unitary projective representation of the group $Sp(2n,\mathbb{R})$ in the space F_n defined by the *Berezin formula* (see [Ner4]):

$$W\begin{pmatrix} \Phi & \Psi \\ \overline{\Psi} & \overline{\Phi} \end{pmatrix} f(z) =$$

$$= \int_{\mathbb{C}^N} \exp\{(z \quad \overline{u}) \begin{pmatrix} \overline{\Psi}\Phi^{-1} & \Phi^{t-1} \\ \Phi^{-1} & -\Phi^{-1}\Psi \end{pmatrix} \begin{pmatrix} z \\ \overline{u} \end{pmatrix} f(z) \exp(-|z|^2) dz d\overline{z}$$

5.2. Noncompact dual pairs. Consider the following subgroups in the symplectic group *(noncompact Howe dual pairs)*:

$$\begin{gathered} Sp(2k(p+q),\mathbb{R}) \supset Sp(2k,\mathbb{R}) \times O(p,q) \\ Sp(2(k+l)(p+q),\mathbb{R}) \supset U(k,l) \times U(p,q) \\ Sp(4k(p+q),\mathbb{R}) \supset SO^*(2k) \times Sp(p,q) \end{gathered}$$

Remark. Let us describe the first embedding. Consider a space $\mathbb{R}^{2n}$ equipped with a nondegenerate skew-symmetric bilinear form. Consider a space $\mathbb{R}^{p+q}$ equipped with symmetric bilinear form with inertia indices p, q. The tensor product of symmetric and skew-symmetric forms is a skew-symmetric form in the space $\mathbb{R}^{2n} \otimes \mathbb{R}^{p+q}$ and we obtain required embedding of groups. □

Let us restrict the harmonic representation W_{2N} of large symplectic group to these subgroups and then let us consider restriction to subgroups

$$O(p,q), Sp(2k,\mathbb{R}), U(k,l), U(p,q), SO^*(2k), Sp(p,q)$$

It was proved in [How], [Ada], [Li] that the spectra of these restrictions have discrete increments (and moreover discrete spectra are exetremely rich). This construction is one of standard way to obtain singular unitary representations of the groups $U(p,q), O(p,q), Sp(p,q)$.

Proposition 3.1. *(see [NO]) Each representation of* $G = O(p,q)$, $Sp(2k,\mathbb{R})$, $U(k,l)$, $U(p,q)$, $SO^*(2k)$, $Sp(p,q)$ *which occurs in spectrum of a dual pair discretly (resp. weakly) occurs in spectrum of some kern-representation discretly (resp. weakly).*

Proof. This proposition is more or less obvious. Concider for instance the case $Sp(2k,\mathbb{R}) \times O(p,q)$. The restriction of $W_{2k(p+q)}$ to the subgroup

$$Sp(2k,\mathbb{R}) \subset Sp(2k,\mathbb{R}) \times O(p,q) \subset Sp(2k(p+q),\mathbb{R})$$

is equivalent to the representation

$$W_{2k}^{\otimes p} \otimes (W_{2k}^{*})^{\otimes q} \tag{18}$$

First tensor factor is a direct sum of highest weight representations and second tensor factor is a direct sum of lowest weight representations. Hence (18) is a direct sum of kern-representations. Consider the following subgroups in $Sp(2k(p+q),\mathbb{R})$:

$$\begin{array}{ccccccc} Sp(2k(p+q),\mathbb{R}) & \supset & O(p,q) & \times & Sp(2k,\mathbb{R}) & \supset & O(p,q) \\ \| & & \cap & & \cup & & \cap \\ Sp(2k(p+q),\mathbb{R}) & \supset & U(p,q) & \times & U(k) & \supset & U(p,q) \end{array}$$

The restriction of $W_{2k(p+q)}$ to $U(p,q)$ is a direct sum of highest weight representations and hence the restriction of $W_{2k(p+q)}$ to $O(p,q)$ is a direct sum of kern-representations.

5.3. Restriction to orbits.

Consider a Cartan domain $C_N = Sp(2N,\mathbb{R})/U(N)$. Consider a reproducing kernel

$$K(z,u) = \det{}^{-1/2}(1 - zu^*)$$

on C_N. Denote by H the associated hilbert space. The group $Sp(2N,\mathbb{R})$ acts in H by the unitary operators

$$W_{2N}^{+}(g) = f(z^{[g]})\det{}^{-1/2}(\Phi + z\overline{\Psi})$$

The representation W_{2N}^{+} is one of two irreducible components of the representation W_{2N} (another irreducible component can be realized in some space of holomorphic vector-valued functions). Again we have question about restrictions of holomorphic functions to $O(p,q) \times Sp(2k,\mathbb{R})$-orbits in the Shilov boundary of C_N. The orbit structure of Shilov boundary in this case is very complicated. In any case there exists a orbit

$$\mathcal{F} = Sp(2k,\mathbb{R})/U(n) \times O(p,q)/Q$$

where Q is stabilizer of maximal isotropic subspace in pseudoeuclidean space $\mathbb{R}^{p+q}$. I can show that operator of restriction to orbit $\mathcal{F}$ exists and this observation give a way to observe a part of discrete spectra for dual pair. It is interesting to calculate this part of spectrum.

Another question which seems interesting to me: is it possible to obtain by such way some hand realizations of some Harish-Chandra discrete series representations?

6 Space L^2 on Stiefel manifolds

.

6.1. Stiefel manifolds. We name by *Stiefel manifolds* the following 10 types of homogeneous spaces G/Q:

1°.	$O(p,q)/O(p-t,q-s)$	2°.	$U(p,q)/U(p-t,q-s)$
3°.	$Sp(p,q)/Sp(p-t,q-s)$	4°.	$Sp(2n,\mathbb{R})/Sp(2(n-t),\mathbb{R})$
5°.	$Sp(2n,\mathbb{C})/Sp(2(n-t),\mathbb{C})$	6°.	$O(n,\mathbb{C})/O(n-t,\mathbb{C})$
7°.	$SO^*(2n)/SO^*(2(n-t))$		

$8^\circ - 10^\circ$. The spaces of all linear embeddings

$$\mathbb{R}^{n-t} \to \mathbb{R}^n \quad \mathbb{C}^{n-t} \to \mathbb{C}^n \quad \mathbb{H}^{n-t} \to \mathbb{H}^n$$

In the last 3 cases the group G is $GL(n,\mathbb{R})$,$GL(n,\mathbb{C})$,$GL(n,\mathbb{H})$ respectively and Q is the group of matrices having the form

$$\begin{pmatrix} 1_s & * \\ 0 & * \end{pmatrix} \tag{19}$$

Remark. The Stiefel manifold $Sp(2n,\mathbb{R})/Sp(2(n-t),\mathbb{R})$ is the space of isometric (form preserving) embeddings of the space $\mathbb{R}^{2t}$ equipped with a non-degenerate skew symmetric bilinear form to the space $\mathbb{R}^{2n}$ equipped with non-degenerated skew symmetric bilinear form. Other Stiefel manifolds $1^\circ - 7^\circ$ have the analogical description.

6.2. Additional symmetries.

Consider the case $G/Q = Sp(2n,\mathbb{R})/Sp(2(n-t),\mathbb{R})$. Then the group $Sp(2t,\mathbb{R})$ acts by the obvious way on the space of symplectic-isometric embeddings $\mathbb{R}^{2t} \to \mathbb{R}^{2n}$ (sinse it acts on the space $\mathbb{R}^{2t}$) . Hence the manifold $Sp(2n,\mathbb{R})/Sp(2(n-t),\mathbb{R})$ is a $Sp(2t,\mathbb{R}) \times Sp(2n,\mathbb{R})$-homogeneous space:

$$Sp(2n,\mathbb{R})/Sp(2(n-t),\mathbb{R}) = (Sp(2t,\mathbb{R})\times Sp(2n,\mathbb{R}))/(Sp(2t,\mathbb{R})\times Sp(2(n-t),\mathbb{R}))$$

Analogical additional group of symmetries exists in all cases $1^\circ - 10^\circ$. These additional symmetries are useful since the spaces $L^2(G/Q)$ have G-spectrum of infinite multiplicity.

6.3. Spectra of L^2 on Stiefel manifolds. A few is known about spectral decomposition of $L^2(G/Q)$. Neverless it is known that this problem is interesting. See [Sch] and [Kob1] for Flensted-Jensen type constructions of discrete spectra in L^2 on

$$O(p,q)/O(p-r,q) \quad U(p,q)/U(p-r,q) \quad Sp(p,q)/Sp(p-r,q)$$

For the case $r=1$ the Plancerel formula is obtained in [ØZ1]. Some consrtructions for discrete increments to spectra of

$$L^2(U(p,q)/(U(p-t,q-t)\times U(p)\times U(q))) \subset L^2(U(p,q)/(U(p)\times U(q)))$$

are contained in [RSW].

Remark. The cases G/Q where $G = GL(n,\mathbb{R})$,$GL(n,\mathbb{C})$,$GL(n,\mathbb{H})$ are very simple. We can consider two-step unitary induction from the subgroup Q (see (19)) to parabolic and from parabolic to the group $GL(n)$ itself.

Proposition 6.1. *Each representation of G which is contained in spectra $L^2(G/Q)$ discretly (resp. weakly) is contained in specrtrum of some kern-representation of G discretly (resp. weakly).*

Proof. We use arguments from [How],[NO]. Concider for instance the case $G = Sp(2n,\mathbb{R})$. The representation

$$W_{2n} \otimes W_{2n}^*$$

of $Sp(2n,\mathbb{R})$ is equivalent to representation of $Sp(2n,\mathbb{R})$ in $L^2(\mathbb{R}^{2n})$. Hence the representation

$$(W_{2n} \otimes W_{2n}^*)^{\otimes 2k} = W_{2n}^{\otimes 2k} \otimes (W_{2n}^*)^{\otimes 2k}$$

is equivalent to representation of $Sp(2n,\mathbb{R})$ in L^2 on the space $Mat_{2k,2n}$ of all $2k \times 2n$-matrices. A generic orbit of $Sp(2n,\mathbb{R})$ in $Mat_{2k,2n}$ is a Stiefel manifold $Sp(2n,\mathbb{R})/Sp(2(n-k),\mathbb{R})$.

Remark. For other groups G Proposition 6.1 can be proved by the same arguments. The basic observation is

$$W_{2n}\Big|_{GL(n,\mathbb{R})} \simeq L^2(\mathbb{R}^{\ltimes})$$

(see real model of harmonic representation in [KV]).

6.3. Some pseudoriemann symmetric spaces. By the obvious way we have

$$\begin{array}{rcl} L^2\big(O(p,q)/(O(p)\times O(p-r,q))\big) & \subset & L^2\big(O(p,q)/O(p-r,q)\big) \\ L^2\big(U(p,q)/(U(r)\times U(p-r,q))\big) & \subset & L^2\big(U(p,q)/U(p-r,q)\big) \\ L^2\big(Sp(p,q)/(Sp(r)\times Sp(p-r,q))\big) & \subset & L^2\big(Sp(p,q)/Sp(p-r,q)\big) \end{array}$$

and spectra of these spaces are contained in spectra of Stifel manifolds. I don't know such embeddings of spectra for other pseudoriemannian symmetric spaces.

7 Miscellaneous

7.1 Molchanov representations In 1970 Molchanov [Mol2] investigated the most degenerate representations of the groups $O(p,q)$. It is representations induced from maximal-dimensional parabolic subgroup and unitary subquotients of these induced representations. Molchanov representations have very simple $K = O(p)\times O(q)$-spectra (spheric harmonics with multiplicities ≤ 1). By this reason Molchanov representations are very simple and hand objects. On another hand these representations are very singular in different sences: they have small rank in the sence of Howe (on How rank see [Li]) and their matrix coeffitients decreasing very slowly (on decreasing of matrix coeffitients see [CHH]).

Recently these representations attract interest again, see [HT]. The most degenerate Molchanov representations now are popular objects, see [BK], [Kos], [KØ] (they are minimal representations of $O(p, q)$ in the sence of A.Joseph, see in [BK] discussion of minimal representations for $O(p, q)$ and other series of simple groups; the most strange case is 'semi-exeptional' group $O(4, 4)$, see [Kos].

I only want to mention that Molchanov representations have very interesting spectral properties. Restrictions of Molchanov representations to the subgroup $O(p, q-1)$ were discussed in [Mol] and [KØ]. The problem is solved for continuous series of Molchanov representations (in this case we have continuous spectrum with discrete increments, see [Mol2]) and for the most degenerate representation (in this case spectrum is discrete for restriction to $O(p, q-r)$, see [KØ]).

Questions. It seems to me that very interesting problem is decomposition of tensor product of two Molchanov representations. Another interesting question is 'hunting' on discrete spectra in tensor products of several Molchanov representations (paper [Ner2] contains false remark on this subject).

Remark. It is possible to show that spectra of tensor products of Molchanov representations contains spectra of dual pairs. I don't know relations between spectra of kern-representations and spectra of tensor products of Molchanov representations.

7.2. Representations unitary with respect to idefinite scalar product. There exist folk-lor (originated by Ismagilov and Molchanov) on harmonic analysis of representations unitary in indefinite hermitian scalar products (partially it is published, see [Ism], [Sul]). It is known that these representations often (more often than in definite unitary case) contains discret increments. It was observed in [Ner1] that these discrete increments usualy can be explained by restriction theorems. Here there are very much nonsolved problems (for instance let us consider negative s in our section 4). Neverless the main (well-known) question here in this moment is the following:

Question. Let a representation ρ of a semisimple group G unitary in indefinite metrics is given. When indefinite scalar product defines by some natural way a topology in space of functions?

The question is very dangerous and sophisticated (this is also well-known), neverless in some cases answer is positive (see [Ism], [Sul], []).

7.3. Harish-Chandra discrete series increments. Tools which are proposed in [NO] don't give possibility to search Harish-Chandra discrete series increnents in unitary representations. Neverless posibilities to expose Harich-Chandra discrete series increments with restriction theorems exist (this is the subject of [Ner2]).

7.4. On totaly discrete spectra. In [Ols4], [Kob2] there were discovered a large classes of restriction problems which have totaly discrete spectra (relations between these works are not clear to me, probably they have a open nondense intersection). It seems to me that it is interesting to understand are these spectra related to some functional-theoretical phenomena or not?

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