Invariant of the hypergeometric group associated to the quantum cohomology of the projective space.

Susumu TANABÉ

ABSTRACT - *We present a simple method to calculate the Stokes matrix for the quantum cohomology of the projective spaces* $\mathbf{CP}^{k-1}$ *in terms of certain hypergeometric group. We present also an algebraic variety whose fibre integrals are solutions to the given hypergeometric equation.*

1. Generalized hypergeometric function

We begin with a short review on the motivation of our problem making reference to the works [5], [9] where one can find precise definitions of the notions below.

At first, we consider a $k-$ dimensional Frobenius manifold F with flat coordinates $(t_1, \cdots, t_k) \in F$ where the coordinate t_i corresponds to coefficients of the basis Δ_i of the quantum cohomology $H^*(\mathbf{CP}^{k-1})$. On $H^*(\mathbf{CP}^{k-1})$ one can define so called quantum multiplication

$$\Delta_\alpha \bullet \Delta_\beta = C^\gamma_{\alpha,\beta}\Delta_\gamma,$$

or

$$\frac{\partial}{\partial t_\alpha} \cdot \frac{\partial}{\partial t_\beta} = C^\gamma_{\alpha,\beta}\frac{\partial}{\partial t_\gamma},$$

on the level of vector fields on F. The Frobenius manifold is furnished with the Frobenius algebra on the tangent space T_tF depending analytically on $t \in F$, $T_tF = (A_t, <\ ,\ >_t)$ where A_t is a commutative associative $\mathbf{C}$ algebra and $<\ ,\ >_t$: $A_t \times A_t \to \mathbf{C}$ a symmetric non-degenerate bilinear form. The bilinear form $<\ ,\ >_t$ defines a metric on F and the Levi-Civita connexion ∇ for this metric can be considered. Dubrovin introduces a deformed flat connexion $\tilde{\nabla}$ on F by the formula $\tilde{\nabla}_u v := \nabla_u v + xu \cdot v$ with $x \in \mathbf{C}$ the deformation parameter. Further he extends $\tilde{\nabla}$ to $F \times \mathbf{C}$. Especially we have $\tilde{\nabla}_{\frac{\partial}{\partial x}} = \frac{\partial}{\partial x} - E(t) - \frac{\mu}{x}$, where $E(t)$ corresponds to the multiplication by the Euler vector field $E(t) = \sum_{1 \leq j \neq 2 \leq k-1}(2-j)t_j\frac{\partial}{\partial t_j} + kt_2\frac{\partial}{\partial t_2}$.

After [5], [9] the quantum cohomology $\vec{u}(x) = (u_1(x), \cdots, u_k(x))$ for the projective space $\mathbf{CP}^{k-1}$ at a semisimple point $(0, t_2, 0, \ldots, 0)$ (i.e. the algebra $(A_t, <\ ,\ >_t)$ is semisimple there) satisfies the following system of differential equation:

$$\partial_x \vec{u}(x) = (k\mathcal{C}_2(t) + \frac{\mu}{x})\vec{u}(x). \tag{1.1}$$

where

$$\mathcal{C}_2(0, t_2, 0, \ldots, 0) = (C^\gamma_{2,\beta})_{1 \leq \beta, \gamma \leq k} = \begin{pmatrix} 0 & 0 & \cdots & 0 & e^{t_2} \\ 1 & 0 & \ddots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

The matrix μ denotes a diagonal matrix with rational entries:

$$\mu = diag\{-\frac{k-1}{2}, -\frac{k-3}{2}, \cdots, \frac{k-3}{2}, \frac{k-1}{2}\}.$$

The last component $u^k(z)$ (after the change of variables $z := kxe^{\frac{t_2}{k}}$) of the above system for the quantum cohomology satisfies a differential equation as follows [9]:

$$[(\vartheta_z)^k - z^k]z^{\frac{-k+1}{2}}u^k(z) = 0, \tag{1.2}$$

with $\vartheta_z = z\frac{\partial}{\partial z}$. After the Fourier-Laplace transformation

$$\tilde{u}(\lambda) = \int e^{\lambda z} z^{\frac{-k+1}{2}} u^k(z) dz,$$

we obtain an equation as follows:

$$[(\vartheta_\lambda + 1)^k - (\frac{\partial}{\partial \lambda})^k]\tilde{u}(\lambda) = 0.$$

Here the notation ϑ_λ stands for $\lambda\frac{\partial}{\partial\lambda}$. After multiplying λ^k from the left, we obtain

$$[\lambda^k(\vartheta_\lambda + 1)^k - \vartheta_\lambda(\vartheta_\lambda - 1)(\vartheta_\lambda - 2)\cdots(\vartheta_\lambda - (k-1))]\tilde{u}(\lambda) = 0.$$

The equation for $\lambda\tilde{u}(\lambda)$, the Fourier-Laplace transform of $\frac{\partial}{\partial z} z^{\frac{-k+1}{2}} u^k(z)$ should be

$$[\lambda^k(\vartheta_\lambda)^k - (\vartheta_\lambda - 1)(\vartheta_\lambda - 2)\cdots(\vartheta_\lambda - k)](\lambda\tilde{u}(\lambda)) = 0. \tag{1.3}$$

It is evident that the Stokes matrix for $\frac{\partial}{\partial z} z^{\frac{-1+k}{2}} u^k(z)$ is identical with that of the original solution $u^k(z)$.

Before proceding further, we remind the following theorem that gives connexion between the Stokes matrix of the system (1.1) with the monodromy of the equation (1.3). Let us consider the Fourier-Laplace transform of the system (1.1):

$$(\vartheta_\lambda + id_k)\vec{\tilde{u}}(\lambda) = (k\mathcal{C}_2(t)\partial_\lambda - \mu)\vec{\tilde{u}}(\lambda). \tag{1.1'}$$

It is evident that the system (1.1)' yields the equation (1.3) for its solution component after corresponding modifications.

In a slightly more general setting, let us observe a system with regular singularities:

$$(\Lambda - \lambda \cdot id_k)\partial_\lambda \vec{\tilde{u}}(\lambda) = (id_k + A_1(\lambda))\vec{\tilde{u}}(\lambda) \tag{1.4}$$

with $\Lambda \in GL(k, \mathbf{C})$ whose eigenvalues $(\lambda_1, \cdots, \lambda_k)$ are all distinct, $A_1(\lambda) \in End(\mathbf{C}^k) \otimes \mathcal{O}_{\mathbf{C}}$ with $A_1(0) = diag(\rho_1, \cdots, \rho_k)$ where none of the ρ_j's is an integer.

Theorem 1.1. *([2], [5]) Under the assumption that the eigenvalues of the matrix $A_1(0)$ are distinct, the Stokes matrix S for the solution component of (1.1) expresses the symmetric Gram matrix G of the solution component of (1.1)' as follows:*

$$^tS + S = 2G.$$

As for the definition of the Stokes matrix S for the system (1.4) we refer to [5], [9]. The main theorem of this article is the following:

Theorem 1.2. *The i, j component S_{ij}, $1 \leq i, j \leq k$ of the Stokes matrix to the system (1.1) has the following expression:*

$$S = \begin{cases} (-1)^{i-j}\ {}_kC_{i-j} & i \geq j \\ 0 & i < j \end{cases}$$

This theorem has already been shown by D.Guzzetti [9] by means of a detailed study of braid group actions etc on the set of solutions to (1.2). We present here another approach to understand the structure of the Stokes matrix.

Remark 1. *In this article we observe the convention of the matrix multiplicatin as follows:*

$$A \cdot x = (a_{ij})_{0 \leq i,j \leq k-1}(x_i)_{0 \leq i \leq k-1} = \langle \sum_{i=0}^{k-1} a_{ij}x_i \rangle_{0 \leq j \leq k-1}.$$

The matrix operatoes on the vector from left, in contrast to the convetion used in [5], [9].

On the other hand it has been known since [4] that a collection of coherent sheaves $\mathcal{O}(-i)$ $0 \leq i \leq k-1$ on $\mathbf{CP}^{k-1}$ satisfies the following relation

$$Hom(\mathcal{O}(-i), \mathcal{O}(-j)) = S^{i-j}(\mathbf{C}^k), 0 \leq i, j \leq k-1$$

$$Ext^\ell(\mathcal{O}(-i), \mathcal{O}(-j)) = 0, 0 \leq i, j \leq k-1, \ell > 0$$

These relation entails immediately the equality

$$\chi(\mathcal{O}(-i), \mathcal{O}(-j)) := \sum_{\ell=0} (-1)^\ell Ext^\ell(\mathcal{O}(-i), \mathcal{O}(-j)) = \begin{cases} {}_kC_{i-j} & i \geq j \\ 0 & i < j. \end{cases}$$

The conjugate of the matrix $\chi := (\chi(\mathcal{O}(-i), \mathcal{O}(-j))$ $0 \leq i, j \leq k-1$ with the diagonal matrix $diag(1, -1, \cdots, (-1)^{k-1})$ gives us our Stokes matrix. Therefore the Theorem 1.2 is a confirmation of an hypothesis [6] that the matrix for certain exceptional collection of coherent sheaves on a good Fano variety Y must coincide with the Stokes matrix for the quantum cohomology of Y.

Our strategy to prove theorem 1.2 consists in the study of system (1.1)', instead of (1.1) itself.

Further we consider so called the Kummer covering (naming after N.Katz) of the projective space $\mathbf{CP}^1$ by $\zeta = \lambda^k$ to deduce an hypergeometric equation:

$$[\zeta(\vartheta_\zeta)^k - (\vartheta_\zeta - \frac{1}{k})(\vartheta_\zeta - \frac{2}{k}) \cdots (\vartheta_\zeta - 1)]v(\zeta) = 0, \tag{1.4}$$

for $v(\lambda^k) = \lambda \tilde{u}(\lambda)$. We remind of us here a famous theorem due to A.H.M.Levelt that allows us to express the (global) monodromy group of the solution to (1.4) in quite a simple way. For the hypergeometric equation in general,

$$[\prod_{\ell=1}^{k} (\vartheta_\zeta - \alpha_\ell) - \zeta \prod_{\ell=1}^{k} (\vartheta_\zeta - \beta_\ell)]v(\zeta) = 0, \tag{1.5}$$

we define two vectors $(A_1, \cdots, A_k)$ and $(B_1, \cdots, B_k)$ in the following way:

$$\prod_{\ell=1}^{k} (t - e^{2\pi\alpha_\ell i}) = t^k + A_1 t^{k-1} + A_2 t^{k-2} + \cdots + A_k,$$

$$\prod_{\ell=1}^{k} (t - e^{2\pi\beta_\ell i}) = t^k + B_1 t^{k-1} + B_2 t^{k-2} + \cdots + B_k.$$

Definition 1.3. *A linear map $L \in GL(k, \mathbf{C})$ is called pseudo-reflexion if it satisfies the condition $rank(id_k - L) = 1$. A pseudo-reflexion R satisfiying an additional condition $R^2 = id_k$ is called a reflexion.*

Proposition 1.4. *([1], [11]) For the solutions to (1.5), the monodromy action on them at the points $\zeta = 0, \infty, 1$ has the follwing expressions:*

$$h_0 = \begin{pmatrix} 0 & 0 & \cdots & 0 & -A_k \\ 1 & 0 & \ddots & 0 & -A_{k-1} \\ 0 & 1 & \ddots & 0 & -A_{k-2} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -A_1 \end{pmatrix}, \tag{1.6}$$

$$(h_\infty)^{-1} = \begin{pmatrix} 0 & 0 & \cdots & 0 & -B_k \\ 1 & 0 & \ddots & 0 & -B_{k-1} \\ 0 & 1 & \ddots & 0 & -B_{k-2} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -B_1 \end{pmatrix},$$

whereas $h_1 = (h_0 h_\infty)^{-1}$ is a pseudo-reflexion.

It is worthy to notice that the above proposition does not precise for which bases of solution to (1.5) the monodromy is calculated. As a corollary to the Proposition 1.4, however, we see that the monodromy action on the solutions to our equation (1.4) can be written down with respect to a certain basis as follows:

$$h_0 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \ddots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \tag{1.7}$$

$$h_\infty = \begin{pmatrix} {}_kC_1 & 1 & 0 & \cdots & 0 & 0 \\ -{}_kC_2 & 0 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^{k-1}\ {}_kC_{k-2} & 0 & 0 & \ddots & 1 & 0 \\ (-1)^k\ {}_kC_{k-1} & 0 & 0 & \cdots & 0 & 1 \\ -(-1)^k & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

In other words,

$$det(t - h_0) = t^k - 1, det(t - h_\infty) = (t-1)^k. \tag{1.8}$$

Furthermore we have,

$$h_1 = \begin{pmatrix} (-1)^{k-1} & 0 & 0 & \cdots & 0 & 0 \\ (-1)^{k-2}\ {}_kC_{k-1} & 1 & 0 & \ddots & 0 & 0 \\ (-1)^{k-3}\ {}_kC_{k-2} & 0 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ {}_kC_1 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}. \tag{1.9}$$

In the next sectin we will see that the theory of Levelt supplies us with necessary data to calculate further the Stokes matrix of the solutions to (1.1).

2. Invariants of the Hypergeometric group

Let us begin with a detailed description of the generators of the hypergeometric group defined for the solutions to the equation (1.3).

Proposition 2.1. *(cf. [8] I, 8.5) The generators of the hypergeometric group H of the equation(1.3) are expressed in terms of the matrices introduced in the Proposition 1.4 as follows:*

$$M_0 = h_0^k = 1, M_1 = h_1 = (h_0 h_\infty)^{-1}, M_\infty = h_\infty^k, M_{\omega^i} = h_\infty^{-i} h_1 h_\infty^i (i = 1, 2, \cdots, k-1), \tag{2.1}$$

where M_t denotes the monodromy action around the point $t \in \mathbf{CP}^1_\lambda$. These generators naturally satisfy the Riemann-Fuchs relation:

$$M_\infty M_{\omega^{k-1}} M_{\omega^{k-2}} \cdots M_\omega M_1 = id_k. \tag{2.2}$$

proof Let us think of a $k-$leaf covering $\tilde{\mathbf{CP}}^1_\lambda$ of $\mathbf{CP}^1_\zeta$ that corresponds to the Kummer covering $\zeta^k = \lambda$. In lifting up the path around $\zeta = 1$ the first leaf of $\tilde{\mathbf{CP}}^1_\lambda$, the monodromy h_1 is sent to the conjugation with a path around $\lambda = \infty$. That is to say we have $M_\omega = h_\infty^{-1} h_1 h_\infty$. For other leaves the argument is similar. **Q.E.D.**

Let us denote by X^K a $k \times k$ matrix that satisfies the relation

$$\bar{g} X^K \, {}^t g = X^K, \tag{2.3}$$

for every element g of a group $K \subset GL(k, \mathbf{C})$. From the definition, the set of all X^K for a group K reperesents a $\mathbf{C}$ vector space in general. We will call a matrix of this space the quadratic invariant of the group K. In the special case in which we are interested, the following statement holds.

Lemma 2.2. *For the hypergeometric group H generated by the pseudo-reflexions as in (2.1), for every X^H there exists a non zero $k \times k$ matrix $\tilde{X}^H$ such that $X^H = \lambda \tilde{X}^H$ for some $\lambda \in \mathbf{C} \setminus \{0\}$.*

proof The relation

$$h_1 X \, {}^t h_1 = X \tag{2.4}$$

gives rise to equations on x_{0j} and x_{j0}. That is to say, the first row of (2.4) corresponds to

$$(-1)^i \, {}_k C_i x_{00} - (-1)^{k-1} x_{0i} = x_{0i}, \;\; 1 \le i \le k-1,$$

while

$$(-1)^i \, {}_k C_i x_{00} - (-1)^{k-1} x_{i0} = x_{i0}, \;\; 1 \le i \le k-1.$$

Thus we obtained $2(k-1)$ linearly independent equations. Further by concrete calculus one can easily see that

$$M_{\omega^\ell} = id_k + T_\ell,$$

where

$$T_\ell = \begin{pmatrix} t_0^{(\ell)}\tau_0 & t_1^{(\ell)}\tau_0 & \cdots & t_\ell^{(\ell)}\tau_0 & 0 & \cdots & 0 \\ t_0^{(\ell)}\tau_1 & t_1^{(\ell)}\tau_1 & \cdots & t_\ell^{(\ell)}\tau_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ t_0^{(\ell)}\tau_{k-1} & t_1^{(\ell)}\tau_{k-1} & \cdots & t_\ell^{(\ell)}\tau_{k-1} & 0 & \cdots & 0 \end{pmatrix},$$

with $(k-\ell)-$ zero columns from the right. The remaining columns are generated from T_1 after simple linear recurrent relations by an inductive way. The relation $M_\omega X \, {}^t M_\omega = X$ gives rise to new equations

$$(1 + t_1^{(1)}\tau_1)^2 x_{11} + \text{linear functions in } (x_{0i}, x_{i0}) = x_{11},$$

with $(1 + t_1^{(1)}\tau_1) = -1 + ({}_k C_1)^2 \neq 1$ and

$$(1 + t_1^{(1)}\tau_1) x_{1i} + \text{linear functions in } (x_{0i}, x_{i0}, x_{11}) = x_{1i},$$

$$(1 + t_1^{(1)}\tau_1) x_{i1} + \text{linear functions in } (x_{0i}, x_{i0}) = x_{i1}.$$

Thus we get $2k-3$ new linearly independent equations. In general for (ℓ, ℓ) term, we get from the relation $M_{\omega^\ell} X \, {}^t M_{\omega^\ell} = X, 1 \le \ell \le k-1$

$$(1 + t_\ell^{(\ell)}\tau_\ell)^2 x_{\ell\ell} + \text{linear functions in } (x_{\nu i}, x_{i\nu}, 0 \le \nu \le \ell - 1) = x_{\ell\ell},$$

with $1 + t_\ell^{(\ell)}\tau_\ell = -1 + ({}_k C_\ell)^2 \neq 1$. For $x_{i\ell}$

$$(1 + t_\ell^{(\ell)}\tau_\ell) x_{i\ell} + \text{linear functions in } (x_{\nu i}, x_{i\nu}, 0 \le \nu \le \ell - 1, x_{\ell\ell}) = x_{i\ell}.$$

In this way we get a set of $2(k-1)+\sum_{\ell=1}^{k-1}\big(2(k-\ell)-1\big)=k^2-1$ independent linear equations with respect to the elements of X. **Q.E.D.**

From the above Lemma 2.2, we can calculate the quadratic invariant X^H after the following relations,

$$\bar{h_0} X^H\, {}^t h_0 = X^H, \bar{h_\infty} X^H\, {}^t h_\infty = X^H. \tag{2.5}$$

From [1] we know that the inverse to $X^{H_0}=X^H$ for $H_0=\{h_0,h_\infty\}$, if it exists, must be a Toeplitz matrix i.e.:

$$(X^{H_0})^{-1}=\begin{pmatrix} x_0 & x_1 & x_2 & \cdots & x_{k-1}\\ x_{-1} & x_0 & x_1 & \cdots & x_{k-2}\\ x_{-2} & x_{-1} & x_0 & \cdots & x_{k-3}\\ \vdots & \ddots & \ddots & \vdots & \vdots\\ x_{-(k-1)} & x_{-(k-2)} & x_{-(k-3)} & \cdots & x_0 \end{pmatrix}.$$

Making use of this circumstances, it is possible to show that the system of equations that arises from the relations

$${}^t h_\infty (X^{H_0})^{-1}\ \bar{h}_\infty = (X^{H_0})^{-1}, {}^t h_0 (X^{H_0})^{-1}\ \bar{h}_0 = (X^{H_0})^{-1},$$

for $(X^{H_0})^{-1}$ consists of $2(k-1)$ equations.

$$x_{k-1-i}=x_{-i-1}, \tag*{(2.6)$'$}$$

$$(-1)^{k+1}x_{k-1-i}+(-1)^k\ {}_kC_{k-1}x_{k-2-i}+\cdots+\ {}_kC_3x_{2-i}-\ {}_kC_2x_{1-i}+kx_{-i}=x_{-1-i} \tag*{(2.6)$''$}$$

This calculates the matrix X^H for the case $k-$ odd.

As for the case $k-$ even, our matrix X^H has the following form

$$X^H=\begin{pmatrix} 0 & y_1 & y_2 & \cdots & y_{k-1}\\ y_{-1} & 0 & y_1 & \cdots & y_{k-2}\\ y_{-2} & y_{-1} & 0 & \cdots & y_{k-3}\\ \vdots & \ddots & \ddots & \vdots & \vdots\\ y_{-(k-1)} & y_{-(k-2)} & y_{-(k-3)} & \cdots & 0 \end{pmatrix},$$

where $y_{-(k-1)},\cdots y_{k-1}$ satisfy $2(k-1)$ equations for some constant y_0,

$$y_i+y_{-i}=0,\ \ y_i-y_{-i}=\ 2(-1)^i\ {}_kC_iy_0\ \ 1\le i\le k-1, \tag*{(2.6)$'''$}$$

which are derived from (2.5). Thus the matrix X^H for the case $k-$even is obtained.

We remember here a classical theorem on the pseudo-reflexions.

Theorem 2.3. *(cf. Bourbaki Groupe et Algèbre de Lie Chapitre V, §6, Exercise 3) Let E be a vector space with basis $(e_1,\cdots,e_d)$, and their dual basis $(f_1,\cdots,f_d)\in E^*$. Let us set $a_{ij}=f_i(e_j)$. The pseudo-reflexion s_i with respec to the basis f_i is defined as*

$$s_i(e_j)=e_j-f_i(e_j)e_i=e_j-a_{ij}e_i.$$

Set

$$V=\begin{pmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{d1}\\ 0 & a_{22} & a_{32} & \cdots & a_{d2}\\ 0 & 0 & a_{33} & \cdots & a_{d3}\\ \vdots & \ddots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & a_{dd}\end{pmatrix}, U=\begin{pmatrix} 0 & 0 & 0 & \cdots & 0\\ a_{12} & 0 & 0 & \cdots & 0\\ a_{13} & a_{23} & 0 & \cdots & 0\\ \vdots & \ddots & \ddots & \vdots & \vdots\\ a_{1d} & a_{2d} & \cdots & a_{d-1,d} & 0\end{pmatrix} \tag{2.7}$$

Under these notations, the composition of all possible reflexions $s_ds_{d-1}\cdots s_1$ (a Coxeter element) with respect to the basis $(e_1,\cdots,e_d)$ is expressed as follows:

$$s_ds_{d-1}\cdots s_1=(id_d-V)(id_d+U)^{-1}. \tag{2.8}$$

proof For $1 \leq i, k \leq d$ we define

$$y_i = s_{i-1} \cdots s_1(e_i).$$

It is possible to see that

$$e_i = y_i + \sum_{k<i\leq d} a_{ki} y_k, s_d \cdots s_1(e_i) = y_i - \sum_{i\leq k\leq d} a_{ki} y_k.$$

The statement follows immediately from these relations. **Q.E.D.**

To establish a relationship between the invariant X^H and the Gram matrix necessary for calculus of the Stokes matrix, we investigate a situation where the generators of the hypergeomeric group have special forms. Namely consider a hypergeometric group Γ of rank k generated by pseudo-reflexions $R_0, \cdots R_{k-1}$ where

$$R_j = id_k - Q_j, \tag{2.9}$$

with

$$Q_j = \begin{pmatrix} 0 & \cdots & 0 & t_{j0} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & t_{j1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & t_{j2} & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & t_{j,k-1} & 0 & \cdots & 0 \end{pmatrix}, \ 0 \leq j \leq k-1, \tag{2.10}$$

all zero components except for the $j-$th column. Let us define the Gram matrix G associated to the above collection of pseudo-reflexions:

$$G = \begin{pmatrix} t_{00} & t_{10} & \cdots & t_{k-1,0} \\ t_{01} & t_{11} & \cdots & t_{k-1,1} \\ t_{02} & t_{12} & \cdots & t_{k-1,2} \\ \vdots & \vdots & \cdots & \vdots \\ t_{0,k-1} & t_{1,k-1} & \cdots & t_{k-1,k-1} \end{pmatrix}. \tag{2.11}$$

We shall treat the cases where G is either symmetric or anti-symmetric. Let us introduce an upper triangle matrix S satisfying

$$G = S +^t S \quad (resp. \ G = S -^t S),$$

for a symmetric (anti-symmetric) matrix G. In the anti-symmetric case, we shall use a convention so that the diagonal part of S is a scalar multiplication on the unit matrix. It is easy to see that for the symmetric (resp. anti-symemtric) G the diagonal element $t_{jj} = 2$ (resp. $t_{jj} = 0$).

Proposition 2.4. *For an hypergeometric group Γ defined over $\mathbf{R}$, the following statements hold.*

1) Suppose that the space of quadratic invariant matrices X^Γ is 1 dimensional. Then X^Γ coincides with the Gram matrix G (2.11) up to scalar multiplication.

2)The composition of all generators $R_0, \cdots, R_{k-1}$ gives us the Seifert form:

$$R_{k-1} \cdots R_0 = \mp{}^t S \cdot S^{-1}, \tag{2.12}$$

where to the minus sign corresponds symmetric G and to the plus sign anti-symmetric G.

proof 1) It is enough to prove that the Gram matrix is a quadratic invariant. We calculate

$$R_j G^t R_j = (id_k - Q_j) G (id_k - \ {}^t Q_j).$$

It is esy to compute

$$Q_j G = (t_{aj} t_{jb})_{0\leq a,b\leq k-1}, G \ {}^t Q_j = (t_{ja} t_{jb})_{0\leq a,b\leq k-1},$$
$$Q_j G^t Q_j = t_{jj} G \ {}^t Q.$$

It yields the following equality,

$$G^t Q_j + Q_j G - Q_j G^t Q_j = (t_{jb}((1 - t_{jj}) t_{ja} + t_{aj}))_{0\leq a,b\leq k-1},$$

that vanishes for G symmetric with $t_{jj} = 2$ and for G anti-symmetric with $t_{jj} = 0$.

2) It is possible to apply directly our situation to that of Theorem 2.3. In the symmetric case, $t_{ii}=2$ and

$$V=\begin{pmatrix} 2 & t_{10} & t_{20} & \cdots & t_{k-1,0} \\ 0 & 2 & t_{21} & \cdots & t_{k-1,1} \\ 0 & 0 & 2 & \cdots & t_{k-1,2} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 2 \end{pmatrix}, U=\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ t_{10} & 0 & 0 & \cdots & 0 \\ t_{20} & t_{21} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ t_{k-1,0} & t_{k-1,1} & \cdots & t_{k-1,k-2} & 0 \end{pmatrix},$$

in accordance with the notation (2.7). The formula (2.8) means (2.12) with minus sign. In the anti-symmetric case $t_{ii}=0, 0\leq i\leq k-1$ and (2.7) yields (2.11) with plus sign. **Q.E.D.**

Corollary 2.5. *We can determine the Stokes matrix S by the following relation*

$$S=(id_k-R_{k-1}\cdots R_0)^{-1}G, \tag{2.13}$$

with the aid of the Gram matrix and pseudo-reflexions.

In some sense, a converse to the Proposition 2.4 holds. To show this, we remember a proposition from [12].

Definition 2.6. *The fundamental set $(u_0(\lambda),\cdots u_{k-1}(\lambda))$ of the system (1.4) is a set of solutions satisfying the following asymptotic expansion:*

$$u_j(\lambda)=(\lambda-\lambda_j)^{\rho_j}\sum_{r=0}^{\infty}g_r^{(j)}(\lambda-\lambda_j)^r,$$

where $(\lambda_0,\cdots\lambda_{k-1})$ are eigenvalues of the matrix Λ. The exponenets ρ_j are diagonal elements of the matrix $A_1(0)$.

Proposition 2.7. *([12]) Every generator of an hypergeometric group Γ over $\mathbf{R}$ defined for the system of type (1.4) (without logarithmic solution) is a product of pseudo-reflexions of the follwing form expressed with respect to its fundamental set:*

$$M_j=id_k-\begin{pmatrix} 0 & \cdots & 0 & s_{j0} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & s_{j1} & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & s_{j,k-1} & 0 & \cdots & 0 \end{pmatrix}. \tag{2.14}$$

where $s_{jj}=2$ or 0.

We get the following corollary to the above proposition 2.7.

Corollary 2.8. *Assume that the hypergeometric group Γ is generated by pseudo-reflexions $T_0,\cdots,T_{k-1}$ such that $rank(T_i-id_k)=1$ for $0\leq i\leq k-1$. Then it is possible to choose a suitable set of pseudo-reflexions generators R_j like (2.9), (2.10), up to constant multiplication on Q_j, so that they determine the quadratic invariant Gram matrix like (2.11).*

proof The proposition 2.7 implies that every generator T_i is a product of pseudo-reflexions M_j with s_{jk} possibly different from t_{jk}. From the condition on the quadratic invariant X^{Γ} and the proposition 2.4, s_{jk} must coincide with t_{jk}. That is to say Γ must be generated by $M_0,\cdots,M_{k-1}$ with $\frac{s_{ja}}{t_{ja}}=\frac{s_{jb}}{t_{jb}}$ for all $a,b,j\in\{0,\cdots,k-1\}$. This means that Γ has as its generators the pseudo-reflexions $R_0,\cdots,R_{k-1}$ of (2.12) up to constant multiplication on Q_j. **Q.E.D.**

proof of Theorem 1.2 First we remark that solutions to (1.3) have no logarithmic asymptotic behaviour around any of their singular points except for the infinity.

In the case with k odd for X^{H_0}, there exists $a\neq 0$ such that the vector $\vec{v_0}:={}^t(1+(-1)^{k-1},-k,\ {}_kC_2,\cdots,(-1)^{k-2}\ {}_kC_{k-2},(-1)^{k-1}\ {}_kC_{k-1})\in\mathbf{R}^k$ satisfies the relation:

$$X^{H_0}\vec{v_0}={}^t(a,0,0,\cdots,0).$$

Actually this fact can be proven almost without calculation in the follwing way. First we introduce a series of vetors

$$\vec{w_\ell} = (x_{-\ell}, x_{-\ell+1}, \cdots, x_{k-1-\ell}), \quad \ell = 0, 1, \cdots, k-1.$$

Then the equation $(2.6)''$ can be rewritten in terms of $\vec{w_\ell}$:

$$\vec{w_\ell} \cdot \vec{v_0} = \sum_{i=0}^{k-1} (-1)^i \, {}_kC_i \cdot x_{i-\ell} = 0 \text{ for } 1 \leq \ell \leq \mathrm{k}-1.$$

On the other hand, the vector $\vec{w_0}$ is linearly independent of the vectors $\vec{w_1}, \ldots \vec{w_{k-1}}$ by virtue of the construction of the matrix X. Therefore $\vec{w_0} \cdot \vec{v_0} \neq 0$ as $\vec{v_0} \neq 0$. This means the existence of the non zero constant a as above.

This relation with the corollary 2.8 gives immediately the expression below for the pseudo-reflexions

$$R_j = id_k - \begin{pmatrix} 0 & \cdots & 0 & (-1)^{j+k-1} \, {}_kC_j \cdot r & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & -(-1)^{k-1} k \cdot r & 0 & \cdots & 0 \\ 0 & \cdots & 0 & (1+(-1)^{k-1}) \cdot r & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -k \cdot r & 0 & \cdots & 0 \\ 0 & \cdots & 0 & {}_kC_2 \cdot r & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & (-1)^{k-1-j} \, {}_kC_{k-j-1} \cdot r & 0 & \cdots & 0 \end{pmatrix}, \quad 0 \leq j \leq k-1, \tag{2.15}$$

whose Gram matrix is equal to

$$\begin{array}{ll} G_{ij} = (-1)^{i-j+k-1} \, {}_kC_{i-j} \cdot r & i > j \\ G_{ii} = (1+(-1)^{k-1}) \cdot r & i = j \\ G_{ij} = (-1)^{i-j} \, {}_kC_{j-i} \cdot r & j > i \end{array}$$

with some constant r. As for the case $k-$ even, the equations $(2.6)'''$ and the Corollary 2.8 gives us the expression (2.15) for the pseudo-reflexion generators.

Taking into account the theorem 1.1 for the symmetric Gram matrix, we obtain the desired statement for the case $k-$ odd, as it is required $G_{ii} = 2 = 2r$.

For the case $k-$even, we remember a statement on the Stokes matrix from [2] which claims that if the matrix μ of (1.1) has integer eigenvalues, the equality $det(S + {}^tS) = 0$ must hold. The Corllary 2.5 gives us the relation

$$S = (id_k + (id_k - V)(id_k + U)^{-1})^{-1} G = (id_k + U) G^{-1} G = id_k + U,$$

with

$$U_{ij} = (-1)^{i-j+k-1} \, {}_kC_{i-j} \cdot r \quad i > j.$$

We shall choose the constant $r = 1$ so that $S + {}^tS = 2id_k + U + {}^tU$ posesses an eigenvector $(1, -1, \cdots, 1, -1)$ with zero eigenvalue. **Q.E.D.**

Remark 2. The Gram matrix (2.13) that has been calculated for the fundamental set (Definition 2.6) of the equation (1.3) gives directly a suitable Stokes matrix we expected. For other Fano varieties, however, the Gram matrix calculated with respect to the fundamental set does not necessarily give a desirable form, as it is seen from the case of odd dimensional quadrics. This situation makes us to be careful in the choice of the base of solutions for which we calculate the Gram matrix.

3. Geometric interpretation of the hypergeometric equation

In this section we show that the equation (1.4) arises from the differential operator that annihilates the fibre integral associated to the family of variety defined as a complete intersection

$$X_s := \{(x_0, \cdots x_k) \in \mathbf{C}^{k+1}; f_1(x) + s = 0, f_2(x) + 1 = 0\}. \tag{3.1}$$

where

$$f_1(x) = x_0x_1\cdots x_k, f_2(x) = x_0 + x_1 + \cdots x_k.$$

This result has been already announced by [7], [8] and [3]. Our main theorem of this section is the following

Theorem 3.1. *Let us assume that* $\Re(f_1(x)+s)|_\Gamma < 0$, $\Re(f_2(x)+s)|_\Gamma < 0$, *out of a compact set for a Leray coboundary cycle* $\Gamma \in H^{k+1}(\mathbf{C}^{k+1}\setminus X_s)$ *avoiding the hypersurfacess* $f_1(x)+s=0$ *and* $f_2(x)+1=0$. *For such a cycle we consider the following residue integral:*

$$I^{(v_1,v_2)}_{x^{\mathbf{i}},\Gamma}(s) = \int_\Gamma x^{\mathbf{i}}(f_1(x)+s)^{-v_1}(f_2(x)+1)^{-v_2}dx, \tag{3.2}$$

for the monomial $x^{\mathbf{i}} := x^{i_0}\cdots x^{i_{k-1}}$. *Then the integral* $I^{(1,1)}_{1,\Gamma}(s)$ *satisfies the following hypergeometric differential equation*

$$[\vartheta_s^k - k^k s(\vartheta_s + \frac{1}{k})(\vartheta_s + \frac{2}{k})\cdots(\vartheta_s + \frac{k}{k})]I^{(1,1)}_{1,\Gamma}(s) = 0 \tag{3.3}$$

which has unique holomorphic solution at $s = 0$,

$$I_0(s) = \sum_{m\geq 0}\frac{(km)!}{(m!)^k}s^m. \tag{3.4}$$

We shall put $\zeta = \frac{k^k}{s}$, to get (1.4) from (3.3). Our calculus is essentially based on the Cayley trick method developed in [13].

Proof of Theorem 3.1 Let us consider the Mellin transform of the fibre integral (3.2)

$$M^{(v_1,v_2)}_{\mathbf{i},\Gamma}(z) := \int_\Pi s^z I^{(v_1,v_2)}_{x^{\mathbf{i}}}(s)\frac{ds}{s} \tag{3.5}$$

For the Mellin transform (3.5), we have the following

$$M^{(v_1,v_2)}_{\mathbf{i},\Gamma}(z) = g(z)\prod_{\ell=0}^{k-1}\Gamma(z+i_\ell+1-v_2)\Gamma(-\sum_{\ell=0}^{k-1}(i_\ell+1)-kz+v_1+kv_2)\Gamma(-z+v_2)\Gamma(z), \tag{3.5'}$$

with $g(z)$ a rational function in $e^{\pi i z}$. The formula (3.5)' shall be proven below. In substituting $\mathbf{i}=0, v_1=v_2=1$, we see that

$$I^{(1,1)}_{1,\Gamma}(s) = \int_{\check{\Pi}} s^{-z}g(z)\frac{\Gamma(z)^k}{\Gamma(kz)}dz,$$

where $\check{\Pi}$ denotes the path $(-i\infty, +i\infty)$ avoidoing the poles of $\Gamma(z) = 0,-1,-2,\cdots$. From this integral representation, the equation (3.3) immediately follows in taking account the fact that the factor $g(z)$ plays no role in establishment of the differential equation. **Q.E.D.**

Proof of (3.5)' In making use of the Cayley trick, we transform the integral (3.5) into the following form

$$M^{(v_1,v_2)}_{\mathbf{i},\Gamma}(z) = \int_{\Pi\times\Gamma\times\mathbf{R}^2_+} x^{\mathbf{i}}e^{y_1(f_1(x)+s)+y_2(f_2(x)+1)}y_1^{v_1}y_2^{v_2}s^z\frac{dx}{x^{\mathbf{1}}}\frac{dy}{y^{\mathbf{1}}}\frac{ds}{s^{\mathbf{1}}}. \tag{3.6}$$

Here we introduce new variables $T_0,\cdots T_{k+2}$,

$$T_i = y_1x_i,\ \ 0\leq i\leq k-1, T_k = y_1s, T_{k+1} = y_2x_0x_1\cdots x_{k-1}, T_{k+2} = y_2 \tag{3.7}$$

in such a way that the phase function of the right hand side of (3.6) becomes

$$y_1(f_1(x)+s) + y_2(f_2(x)+1) = T_0 + T_1 + \cdots + T_{k+2}.$$

If we set

$$Log\ T :=^t (log\ T_0,\cdots, log\ T_{k+2})$$
$$\Xi :=^t (x_0,\cdots,x_{k-1},s,y_1,y_2)$$
$$Log\ \Xi :=^t (\log\ x_0,\cdots,\log\ x_{k-1},\log\ s,,\log\ y_1, log\ y_2).$$

Then the above relationship (3.7) can be written down as

$$Log\, T = \mathsf{L} \cdot Log\, \Xi, \tag{3.8}$$

where

$$\mathsf{L} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 \end{bmatrix}.$$

This yields immediately

$$Log\, \Xi = \mathsf{L}^{-1} \cdot Log\, T,$$

with

$$\mathsf{L}^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & -1 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 & 0 & 0 \\ 1 & 1 & 1 & \cdots & 1 & -k & -1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & \cdots & -1 & k & 1 & 0 \end{bmatrix}.$$

If we set

$$(i_0, \cdots i_{k-1}, z, v_1, v_2) \cdot \mathsf{L}^{-1} = \big(\mathcal{L}_0(\mathbf{i}, z, v_1, v_2), \cdots, \mathcal{L}_{k+2}(\mathbf{i}, z, v_1, v_2)\big). \tag{3.9}$$

then we can see that

$$M^{(v_1,v_2)}_{\mathbf{i},\Gamma}(z) = \int_{\Pi \times \Gamma \times \mathbf{R}^2_-} x^{\mathbf{i}} e^{T_0 + \cdots + T_{k+2}} y_1^{v_1} y_2^{v_2} s^z \frac{dx}{x^{\mathbf{1}}} \frac{dy}{y^{\mathbf{1}}} \frac{ds}{s^{\mathbf{1}}}$$

$$= \int_{T_0+\cdots+T_{k+2}\neq 0} e^{T_0+\cdots+T_{k+2}} \prod_{0\le i\le k+2} T_i^{\mathcal{L}_{k+2}(\mathbf{i},z,v_1,v_2)} \bigwedge_{0\le i\le k+2} \frac{dT_i}{T_i}.$$

In view of the choice of the cycle Γ, we can apply the formula to calculate Γ function to our situation:

$$\int_C e^{-T} T^{\sigma} \frac{dT}{T} = (1 - e^{2\pi i \sigma}) \Gamma(\sigma),$$

for the unique nontrivial cycle C turning around $T = 0$ that begins and returns to $\Re T \to +\infty$. The direct calculation of (3.9) shows that

$$\mathcal{L}_\ell(\mathbf{i}, z, v_1, v_2) = z + i_\ell + 1 - v_2, 0 \le \ell \le k-1,$$

$$\mathcal{L}_k(\mathbf{i}, z, v_1, v_2) = -\sum_{\ell=0}^{k-1} (i_\ell + 1) + v_1 + kv_2), \mathcal{L}_{k+1}(\mathbf{i}, z, v_1, v_2) = -z + v_2, \mathcal{L}_{k+2}(\mathbf{i}, z, v_1, v_2) = z.$$

This shows the formula (3.5)'. **Q.E.D.**

Remark 3. In combining Theorems 1.2, 3.1, we can state that we found out a deformation of an algebraic variety $X_\lambda = \{(\frac{\lambda}{k})^k(x_0x_1\cdots x_k)+1=0, x_0+x_1+\cdots+x_k=1=0\}$. such that its variation gives rise to the equation (1.3). It means that we establish a connexion between an exceptional collection of $\mathbf{CP}^{k-1}$ and a set of vanishing cycles for its mirror counter part X_λ. Thus our theorems give an affirmative answer to the hypothesis stating the existence of such relationship between two mirror symmetric varieties (so called Bondal-Kontsevich hypothesis) in a special case. See [8] and [10] in this respect for the detail.

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Independent University of Moscow
Bol'shoj Vlasijevskij Pereulok 11,
MOSCOW, 121002,
Russia
E-mails: tanabe@mccme.ru,