ON THE CLASSIFICATION OF VECTOR BUNDLES ON PROJECTIVE CURVES

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ABSTRACT. We propose a new method of classifying vector bundles on projective curves, especially singular ones, according to their "representation type". In particular, we prove that the classification problem of vector bundles, respectively of torsion–free sheaves on projective curves is always either finite, or tame, or wild. We completely classify curves which are of finite, respectively tame, vector bundle type by their dual graph. Moreover, our methods yield a geometric description of all indecomposable vector bundles and torsion–free sheaves for finite and tame curves.

INTRODUCTION

Vector bundles over projective varieties, in particular, over projective curves have been widely studied. Usually, the main emphasis lies in the study of stable bundles and their moduli (cf. [14], [15], [13]). Nevertheless, not too much seems to be known about the overall classification of *all* vector bundles over some variety. The most prominent results here are those of Grothendieck [10] for the projective line and of Atiyah [1] for elliptic curves. On the other hand, the classification of vector bundles on projective curves is closely related to the study of Cohen-Macaulay modules on surface singularities, due to the work of Kahn [12]. Hence, from different points of view, it is important to have some ideas about the complexity of these classification problems.

This article is devoted to the study of vector bundles over projective curves, in particular singular ones, from the point of view of representation theory. Representation theory often distinguishes the following three main cases of the classification problem:

- "finite," when indecomposable objects are completely defined by some "discrete" parameters (this is the case for the projective line);
- "tame," when indecomposable objects form "small" (usually only one-parameter) families (this is the case for elliptic curves);
- "wild," when there are families of indecomposable modules depending of any prescribed number of parameters.

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Note, however, that "wild" means much more: in the wild case the problem usually becomes "versal," i.e., containing other, more or less arbitrary, classification problems (cf. [6, 7]).

Here we prove the "trichotomy" theorem, which claims that a reduced projective curve is either *VB–finite* (when there are only finitely many indecomposable bundles up to isomorphism and shift) or *VB–tame* (when indecomposable bundles form at most one–parameter families) or *VB–wild* (when there are families of indecomposable non–isomorphic bundles depending on arbitrary many parameters).

Namely, the following assertions hold, with C reduced and connected (cf. Theorem 1.6, Proposition 2.7 and Theorem 2.8):

1. *A non–singular projective curve C is:*
   - *VB–finite if and only if it is rational;*
   - *VB–tame if and only if it is elliptic (that is, of genus 1);*
   - *VB–wild in all other cases.*
2. *Let C be a singular projective curve, $C_1, \dots, C_s$ its irreducible components and Δ the* intersection graph *(or the* dual graph*) of C. Then C is:*
   - *VB–finite if and only if all C_i are smooth, rational and Δ is of type A_n (that is, a chain);*
   - *VB–tame if and only if all C_i are smooth, rational and Δ is of type $\widetilde{A}_n$ (that is, a cycle) or C is irreducible, rational with one simple node;*
   - *VB–wild in all other cases.*

In the *wild* case we always construct families of non–isomorphic *semi–stable* vector bundles depending on arbitrary many parameters. But, even worse, we show that the classification of semi–stable vector bundles contains, in some sense, *any* classification problem of representation theory.

Moreover, the above classification extends without problem to torsion–free sheaves. We show that C is VB–finite, respectively VB–tame, respectively VB–wild, if and only if it is TF–finite, respectively TF–tame, respectively TF–wild, where TF denotes the category of torsion–free sheaves on C.

In Theorems 2.10 and 2.11 a full classification of all vector bundles in the finite and tame case for singular curves is described and illustrated in a geometric way.

Our methods are well–known in representation theory. Namely, the techniques of "matrix problems," which are used, for instance, to prove the above "trichotomy" (cf. [6, 7]) or to determine the types of some classes of classification problems. Fortunately, after eliminating the wild cases, we come to a known matrix problem (the so called "Gelfand problem" in the version due to Bondarenko [2]). This gives us the possibility to obtain a complete list of indecomposable vector bundles for the finite and tame cases.

We add two appendices. In the first one we specialise the list of indecomposable objects from [2] to our case, where it is a bit simpler than in general. The second one is devoted to a class of finite dimensional algebras which seems to be closely related to cyclic configurations of projective lines. We think that these relations could be interesting and useful both for algebraic geometry and for representation theory, as was the case, for instance, for "canonical algebras" and weighted projective lines (cf. [8]).

1. VB-type of a curve. Smooth case

Throughout this section we use the following notations:

Notations 1.1. 1. C is an algebraic curve over an algebraically closed field $\mathbf{k}$, which we suppose to be *reduced* and *connected* but usually *singular* and even *reducible.*

2. $\mathcal{O} := \mathcal{O}_C$ denotes the structure sheaf of C and $\mathcal{K}$ denotes the sheaf of *rational functions* on C (its stalk at a point x is the full ring of quotients of $\mathcal{O}_x$).
3. $\mathsf{VB} = \mathsf{VB}(C)$ is the category of (finite dimensional) vector bundles on C or, equivalently, that of locally free (coherent) sheaves on C. (We identify vector bundles with the corresponding locally free sheaves and in our case it is more convenient to deal with sheaves.)
4. Let $\mathcal{M}$ be a sheaf of $\mathcal{O}$-modules. Call the *torsion part* of the sheaf $\mathcal{M}$ and denote by $\mathsf{t}(\mathcal{M})$ the kernel of the natural homomorphism $\mathcal{M} \to \mathcal{K} \otimes_{\mathcal{O}} \mathcal{M}$. The sheaf $\mathcal{M}$ is said to be *torsion free* if $\mathsf{t}(\mathcal{M}) = 0$ and *torsion* if $\mathsf{t}(\mathcal{M}) = \mathcal{M}$. In what follows we always identify a torsion free sheaf $\mathcal{M}$ with its image in $\mathcal{K} \otimes_{\mathcal{O}} \mathcal{M}$.

Obviously, $\mathcal{M}$ is torsion if and only if , for every point $x \in C$ and for every element $t \in \mathcal{M}_x$, there is a non-zero-divisor $a \in \mathcal{O}_x$ such that $at = 0$; $\mathcal{M}$ is torsion free if and only if , for every non-zero $t \in \mathcal{M}_x$ and for every non-zero-divisor $a \in \mathcal{O}_x$, $at \neq 0$. It is also clear that $\mathsf{t}(\mathcal{M})$ is the biggest torsion sub-sheaf of $\mathcal{M}$, while $\mathcal{M}/\mathsf{t}(\mathcal{M})$ is torsion free.

We are going to define the *vector bundle type* (*VB-type*) of a curve, i.e., its type with respect to the classification of vector bundles on it. We take into consideration that such classification involves evident "discrete" parameters, namely, rank and degree. However, if the curve has several irreducible components, these parameters become more complicated.

Definition 1.2. Let C be a projective curve, $C = \cup_{i=1}^{s} C_i$ its decomposition into irreducible components, $\mathcal{B}$ a vector bundle over C and $\mathcal{B}_i$ the restriction of $\mathcal{B}$ onto C_i. The *vector-degree* of $\mathcal{B}$ is defined as the vector $\operatorname{Deg}\mathcal{B} := (d_1, d_2, \dots, d_s)$, where $d_i = \deg \mathcal{B}_i$.

In particular, the mapping Deg defines an epimorphism $\mathrm{Pic}(C) \to \mathbb{Z}^s$. Choose a section of it, $\omega : \mathbb{Z}^s \to \mathrm{Pic}(C)$, and put $\mathcal{O}(\mathbf{d}) := \omega(\mathbf{d})$. Thus, we define $\mathbb{Z}^s$ as a group of "shifts" on the category of coherent sheaves (in particular, on that of vector bundles) by putting $\mathcal{M}(\mathbf{d}) := \mathcal{O}(\mathbf{d}) \otimes_{\mathcal{O}} \mathcal{M}$. Considering "representation types" of categories of sheaves, we should also take into account the action of this big discrete group.

If X is an algebraic variety, there is a natural notion of a family of vector bundles on a curve C with base X. Namely, such a family is just a vector bundle $\mathcal{V}$ on $X \times C$. For our purpose, a non-commutative analogue of this notion is also important.

Definition 1.3. 1. Let Λ be a $\mathbf{k}$-algebra (not necessarily commutative). Denote by $\mathsf{VB}(C, \Lambda)$ the category of coherent flat sheaves of $\mathcal{O} \otimes \Lambda$-modules. The objects of this category are called *families of vector bundles over C with base Λ*.
Given a family $\mathcal{M} \in \mathsf{VB}(C, \Lambda)$ and a finite dimensional[1] Λ-module N, we can construct the tensor product $\mathcal{M}(N) := \mathcal{M} \otimes_\Lambda N$, which is locally free over $\mathcal{O}$, i.e. is a vector bundle over C.

2. A family $\mathcal{M} \in \mathsf{VB}(C, \Lambda)$ is said to be *strict* if the following conditions hold:
   (a) If N is an indecomposable finite dimensional Λ-module, then the sheaf $\mathcal{M}(N)$ is also indecomposable.
   (b) If two finite dimensional Λ-modules N and N' are non-isomorphic, then the sheaves $\mathcal{M}(N)$ and $\mathcal{M}(N')$ are also non-isomorphic.

For any morphism $f : C' \to C$ of curves and any family $\mathcal{M} \in \mathsf{VB}(C, \Lambda)$, the inverse image $f^*(\mathcal{M})$ belongs to $\mathsf{VB}(C', \Lambda)$. It is also quite obvious that if $\mathcal{M} \in \mathsf{VB}(C, \Lambda)$, then also $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{L} \in \mathsf{VB}(C, \Lambda)$ for every invertible sheaf $\mathcal{L}$ on C; in particular, $\mathcal{M}(\mathbf{d}) \in \mathsf{VB}(C, \Lambda)$ for every vector $\mathbf{d} \in \mathbb{Z}^s$. Moreover, if $\mathcal{M}$ is strict, so is $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{L}$ for each invertible sheaf $\mathcal{L}$; in particular, $\mathcal{M}(\mathbf{d})$ is strict for each $\mathbf{d}$. For every finite-dimensional Λ-module N, put $\mathcal{M}(\mathbf{d}, N) := \mathcal{M}(\mathbf{d})(N)$.

If $\Lambda = \mathbf{k}[X]$ for some affine variety X, then an object from $\mathsf{VB}(C, \Lambda)$ is obviously the same as a family of vector bundles on C with base X. However, our construction also produces some "subordinate" families of multiple dimensions. Note that for two different points $p \neq q$ of X the residue fields $\mathbf{k}(p)$ and $\mathbf{k}(q)$ are non-isomorphic as $\mathbf{K}[X]$-modules. Hence, for a strict family $\mathcal{M}$ over X, the fibres over p and q, i.e., the vector bundles $\mathcal{M}(p)$ and $\mathcal{M}(q)$ are also non-isomorphic.

Definitions 1.4. 1. Call a curve C *vector bundle finite* or *VB-finite* if there is a finite set $\mathbf{M}$ of indecomposable vector bundles on C such that every indecomposable vector bundle on C is isomorphic to $\mathcal{B}(\mathbf{d})$ for some $\mathcal{B} \in \mathbf{M}$ and some vector $\mathbf{d} \in \mathbb{Z}^s$. (We shall

[1] "*finite dimensional*" always means finite dimensional as a vector space over $\mathbf{k}$.

see later that indeed $\mathrm{rk}(\mathcal{B}) = 1$ for every indecomposable vector bundle on a VB-finite curve.)

2. Call a curve C *VB-tame* if there is a non-empty set $\mathbf{M} = \{\,\mathcal{M}_i\,\}$ of strict sheaves $\mathcal{M}_i \in \mathsf{VB}(C, \Lambda_i)$ (note that the Λ_i may be different for different i) satisfying the following conditions:
   (a) Each Λ_i is a commutative finitely generated integral smooth $\mathbf{k}$-algebra of Krull dimension 1.
   (b) For each integer r and vector $\mathbf{d}$, the set $\mathbf{M}_{r,\mathbf{d}}$ is finite, where $\mathbf{M}_{r,\mathbf{d}} := \{\,\mathcal{M} \in \mathbf{M} \mid \mathrm{rk}(\mathcal{M}) = r\,, \mathrm{Deg}\,\mathcal{M} = \mathbf{d}\,\}$, where $\mathrm{Deg}\,\mathcal{M}$ is, by definition, $\mathrm{Deg}(\mathcal{M}/\mathbf{m}\mathcal{M})$ for some (and, hence, every) maximal ideal $\mathbf{m} \subset \Lambda_i$.
   (c) For each integer r and vector $\mathbf{d}_0$, all but a finite number of locally free indecomposable sheaves on C of rank r and vector-degree $\mathbf{d}_0$ are isomorphic to those of the form $\mathcal{M}_i(\mathbf{d}, N)$, for some $\mathcal{M}_i \in \mathbf{M}$, $\mathbf{d} \in \mathbb{Z}^s$ and some finite dimensional Λ_i-module N.

   In this case call $\mathbf{M}$ a *parametrising set* for vector bundles over C. Denote $\nu(r)$ the minimal number of sheaves in $\mathbf{M}_{r,\mathbf{d}}$, where $\mathbf{M}$ runs through all such parametrising sets and $\mathbf{d}$ runs through $\mathbb{Z}^s$. Then a VB-tame curve C is said to be:
   - *bounded* if there is an integer m such that $\nu(r) \le m$ for all ranks r;
   - *unbounded* otherwise.
3. Call a curve C *VB-wild* if, for every finitely generated $\mathbf{k}$-algebra Λ, there is a strict sheaf $\mathcal{M} \in \mathsf{VB}(C, \Lambda)$.

   Hence, for wild curves, the classification of vector bundles is at least as complicated as the classification of the representations of *all* finitely generated $\mathbf{k}$-algebras.

Indeed, to prove wildness it is sufficient to check one "typical" algebra as the following result shows.

Proposition 1.5. *A curve C is VB-wild if there is a strict sheaf $\mathcal{M} \in \mathsf{VB}(C, \Gamma)$, where Γ is one of the following algebras:*

- $\mathbf{F} := \mathbf{k}\langle\, z_1, z_2\,\rangle$*, the free algebra in two generators;*
- $\mathbf{k}[z_1, z_2]$*, the polynomial algebra in two generators;*
- $\mathbf{k}[[z_1, z_2]]$*, the power series algebra in two generators.*

Proof. It is well-known (cf. [6]) that if Γ is one of these algebras and Λ is an arbitrary finitely generated algebra, there is a *strict representation* of Γ over Λ, i.e., a Γ-Λ-bimodule V such that:

1. V is finitely generated and free as Λ-module.
2. If N is an indecomposable finite dimensional Λ-module, the Γ-module $V \otimes_\Lambda N$ is also indecomposable.
3. If N, N' are non-isomorphic finite dimensional Λ-modules, the Γ-modules $V \otimes_\Lambda N$ and $V \otimes_\Lambda N'$ are also non-isomorphic.

Therefore, if a sheaf $\mathcal{M} \in \mathsf{VB}(C,\Gamma)$ is strict, so is also $\mathcal{M} \otimes_\Lambda V \in \mathsf{VB}(C,\Lambda)$.

We remind the explicit form of a strict representation V of the free algebra $\mathbf{F}$ over any algebra Λ with generators $a_1, a_2, \dots, a_n$. As Λ-module, $V = (n+2)\Lambda$ while the action of z_1 and z_2 is given by the matrices Z_1 and Z_2, respectively, where Z_1 is a Jordan cell of dimension $n+2$ and

$$Z_2 = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ a_1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & a_2 & 1 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \cdots & a_n & 1 & 0 \end{pmatrix}.$$

□

Note also that this definition of tameness (namely, the condition $\mathbf{M} \neq \emptyset$) implies that "tame" excludes "finite," i.e., we have a real trichotomy.

First of all, consider VB-types of smooth projective curves.

Theorem 1.6. *A smooth projective curve C of genus g is:*

- *VB-finite if $g = 0$, i.e., if $C \simeq \mathbb{P}^1$,*
- *VB-tame if $g = 1$, i.e., if C is an elliptic curve,*
- *VB-wild if $g > 1$.*

Proof. It is known that each vector bundle on $\mathbb{P}^1$ is isomorphic to $\mathcal{O}(n)$ for some n [10]. Hence, $\mathbb{P}^1$ is VB-finite. On the other hand, the classification of vector bundles on elliptic curves [1] implies that all elliptic curves are VB-tame and bounded (indeed, in this case $\nu(r) \le 1$ for each r). So we only have to prove that any curve of genus $g > 1$ is VB-wild, i.e., to construct a strict sheaf $\mathcal{M} \in \mathsf{VB}(C,\mathbf{F})$, where $\mathbf{F} = \mathbf{k}\langle\, z_1, z_2 \,\rangle$. We shall even construct a sheaf $\mathcal{M} \in \mathsf{VB}(C,\mathbf{F})$ such that $\mathcal{M}(N) \simeq \mathcal{M}(N') \otimes_{\mathcal{O}} \mathcal{L}$ for some line bundle $\mathcal{L}$ if and only if $N \simeq N'$ and $\mathcal{L} \simeq \mathcal{O}$. In other words, even the natural action of $\operatorname{Pic} C$ on the set of vector bundles does not simplify their classification.

For any two points $x \neq y$ of C,

$$\operatorname{Hom}_{\mathcal{O}}(\mathcal{O}(x), \mathcal{O}(y)) \simeq \mathrm{H}^0(C, \mathcal{O}(y-x)) = 0.$$

On the other hand,

$$\operatorname{Ext}^1_{\mathcal{O}}(\mathcal{O}(x), \mathcal{O}(y)) \simeq \mathrm{H}^1(C, \mathcal{O}(y-x)),$$

as $\mathcal{E}xt^1_{\mathcal{O}}(\mathcal{O}(x), \mathcal{O}(y)) = 0$. Using the Riemann–Roch theorem for the divisor $y - x$, we get

$$\dim \mathrm{H}^1(C, \mathcal{O}(y-x)) = g - 1 \ge 1.$$

Now consider the divisor $2x$ for some point x. The Riemann-Roch theorem together with the Clifford theorem [11, Theorem IV.5.4] gives that $\dim \mathrm{H}^0(C, \mathcal{O}(2x)) \le 2$. Hence, all functions from this space are of the form $\lambda + \mu f_0$ for some fixed f_0 and $\lambda, \mu \in \mathbf{k}$. In particular, there is at most one pair (y, z) of points distinct from x such that $2x \sim y + z$.

Choose 5 different points $x_1, \dots, x_5$ in such a way that $2x_i \not\sim x_j + x_k$ for $x_j \ne x_i$, $x_k \ne x_i$, and consider the class of locally free sheaves $\mathcal{A}$ admitting an exact sequence:

$$0 \longrightarrow \mathcal{A}_1 \longrightarrow \mathcal{A} \longrightarrow \mathcal{A}_2 \longrightarrow 0, \tag{1}$$

where

$$\mathcal{A}_1 = r_1\mathcal{O}(x_1) \oplus r_2\mathcal{O}(x_2) \oplus r_3\mathcal{O}(x_3)$$

and

$$\mathcal{A}_2 = r_4\mathcal{O}(x_4) \oplus r_5\mathcal{O}(x_5).$$

Let $\xi \in \mathrm{Ext}_{\mathcal{O}}(\mathcal{A}_2, \mathcal{A}_1)$ be the element corresponding to the sequence (1). As there are no homomorphisms from the sub-sheaf to the factor-sheaf, one can easily check that two elements $\xi, \xi' \in \mathrm{Ext}_{\mathcal{O}}(\mathcal{A}_2, \mathcal{A}_1)$ lead to isomorphic modules $\mathcal{A}$ and $\mathcal{A}'$ if and only if there are automorphisms $\alpha : \mathcal{A}_1 \xrightarrow{\sim} \mathcal{A}_1$ and $\beta : \mathcal{A}_2 \xrightarrow{\sim} \mathcal{A}_2$ such that $\alpha\xi = \xi\beta$ (we mean here the Yoneda multiplication). Choose some non-zero elements $\xi_{ij} \in \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{O}(x_j), \mathcal{O}(x_i))$. Put $\mathcal{S} = \mathcal{O} \otimes \mathbf{F}$, where $\mathbf{F} = \mathbf{k}\langle\, z_1, z_2 \,\rangle$, the free $\mathbf{k}$-algebra in two generators, $\mathcal{S}(x) = \mathcal{S} \otimes_{\mathcal{O}} \mathcal{O}(x)$ for $x \in C$. Then $\mathrm{Ext}^1_{\mathcal{S}}(\mathcal{S}(x), \mathcal{S}(y)) \simeq \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{O}(x), \mathcal{O}(y)) \otimes \mathbf{F}$. Consider the exact sequence of locally free $\mathcal{S}$-modules

$$0 \longrightarrow \mathcal{S}(x_1) \oplus \mathcal{S}(x_2) \oplus \mathcal{S}(x_3) \longrightarrow \mathcal{M} \longrightarrow \mathcal{S}(x_4) \oplus \mathcal{S}(x_5) \longrightarrow 0$$

corresponding to the element of the Ext-space given by the matrix

$$\begin{pmatrix} \xi_{14} & \xi_{15} \\ \xi_{24} & z_1\xi_{25} \\ \xi_{34} & z_2\xi_{35} \end{pmatrix}.$$

If N is any finite dimensional $\mathbf{F}$-module, then the locally free $\mathcal{O}$-module $\mathcal{M}(N)$ corresponds to the element of Ext-space given by the matrix

$$\begin{pmatrix} \xi_{14}I & \xi_{15}I \\ \xi_{24}I & \xi_{25}Z_1 \\ \xi_{34}I & \xi_{35}Z_2 \end{pmatrix}.$$

Here I denotes the identity matrix of size $\dim_{\mathbf{k}} N$, while Z_1 and Z_2 are the matrices describing the action of z_1 and z_2, respectively, on the module N. Then an easy straightforward calculation shows that $\mathcal{M}(N) \simeq \mathcal{M}(N')$ if and only if $N \simeq N'$.

Suppose now that $\mathcal{M}(N) \simeq \mathcal{M}(N') \otimes_{\mathcal{O}} \mathcal{L}$, where $\mathcal{L} = \mathcal{O}(D)$ for some divisor D on C. Then for each $i \in \{\,1, 2, 3\,\}$, there are $j, k \in \{\,1, 2, 3, 4, 5\,\}$ such that

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}(x_i), \mathcal{O}(D + x_j)) = \mathrm{H}^0(C, \mathcal{O}(D + x_j - x_i)) \ne 0$$

and

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}(D+x_i),\mathcal{O}(x_k)) = \mathrm{H}^0(C,\mathcal{O}(-D+x_k-x_i)) \neq 0\,.$$

The first inequality implies that $\deg D \geq 0$, while the second one implies $\deg D \leq 0$. Hence, $\deg D = 0$. But then both $D + x_j - x_i$ and $-D + x_k - x_i$ are equivalent to zero, whence $2x_i \sim x_k + x_j$. The choice of these points implies that $x_j = x_k = x_i$ and $D \sim 0$, i.e., we return to the just considered case. □

Remark 1.7. Certainly, if C is a VB-wild curve, there are families of indecomposable vector bundles on C consisting of pairwise non-isomorphic bundles and depending on any number of parameters. Indeed, any strict sheaf $\mathcal{M} \in \mathsf{VB}(C,\Lambda)$ for $\Lambda = \mathbf{k}[x_1,x_2,\dots,x_n]$ gives rise to such a family consisting of the vector bundles $\mathcal{M}(p)$, where $p \in \mathbb{A}^n$.

Nevertheless, the notion of wildness is much stronger than just the existence of "big families," which is known in this case (cf. [15]). Namely, if a curve C is VB-wild, then the problem of the classification of vector bundles on C is, in some sense, "versal": it contains the classification of the representations of any finitely generated algebra. Note that there are examples of classification problems, when there also are families depending on any number of parameters, but no such versality. Consider, for instance, the classification of square matrices under the conjugation by diagonal ones. The $n\times n$ matrices (a_{ij}) with $a_{i,i+1} = 1$ for $1 \leq i < n$ form a family depending on $n^2 - n + 1$ parameters, but one can easily give in this case a list of "normal forms" of all matrices, so no versality occurs here and the problem is not wild.

Just as above, in what follows we give an explicit construction of strict sheaves from $\mathsf{VB}(C,\mathbf{F})$ (hence, from $\mathsf{VB}(C,\Lambda)$ for any Λ) for any VB-wild curve C. It gives an explicit "representation embedding" from the category of finite dimensional Λ-modules to $\mathsf{VB}(C)$.

Note also that all families which we obtain for wild curves consist of *semi-stable* bundles (cf.[14], [15]). In the proof of Proposition 1.6 it follows from the fact that such a bundle has a filtration whose factors are all of rank 1 and of degree 1 (analogous observations are also valid in the other cases considered below).

2. VB-types of singular curves

Consider now the case of singular curves. We introduce, in addition to Notations 1.1, the following

Notations 2.1. 1. $\nu : \tilde{C} \to C$ denotes the *normalisation* of C.(Note that $\tilde{C}$ can be reducible or, what is equivalent, non-connected.)

2. $S = S(C)$ denotes the set of *singular points* of C and we put $\tilde{S} := \nu^{-1}(S)$.
3. Set $\tilde{\mathcal{O}} = \nu_*(\mathcal{O}_{\tilde{C}})$; we identify $\mathcal{O}$ with its natural image in $\tilde{\mathcal{O}}$.

4. Let $\mathcal{J}$ be the *conductor* of $\mathcal{O}$ in $\tilde{\mathcal{O}}$, i.e., the biggest sheaf of $\tilde{\mathcal{O}}$-ideals contained in $\mathcal{O}$.
5. Set $\mathcal{F} = \mathcal{O}/\mathcal{J}$ and $\tilde{\mathcal{F}} = \tilde{\mathcal{O}}/\mathcal{J}$.
6. For any torsion free sheaf $\mathcal{B}$ of $\mathcal{O}$-modules, put $\tilde{\mathcal{B}} = \tilde{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{B}/\,\mathsf{t}(\tilde{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{B})$ (cf. 1.1) and $\overline{\mathcal{B}} = \mathcal{B}/\mathcal{J}\mathcal{B}$. In particular, $\mathcal{F} = \overline{\mathcal{O}}$ and $\tilde{\mathcal{F}} = \overline{\tilde{\mathcal{O}}}$.
   As $\mathcal{B}$ is torsion free, the canonical map $\mathcal{B} \to \tilde{\mathcal{B}}$ is a monomorphism and we always consider $\mathcal{B}$ as a sub-sheaf in $\tilde{\mathcal{B}}$. Note also that if $\mathcal{B}$ is a vector bundle, then $\tilde{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{B}$ has no torsion part, hence, coincides with $\tilde{\mathcal{B}}$.

Lemma 2.2. *For every torsion free sheaf $\mathcal{B}$ of $\mathcal{O}$-modules $\tilde{\mathcal{B}}$ is naturally isomorphic to the $\tilde{\mathcal{O}}$-sub-sheaf in $\mathcal{K} \otimes_{\mathcal{O}} \mathcal{B}$ generated by $\mathcal{B}$.*
(Remind that $\mathcal{K}$ denotes the sheaf of rational functions on C.)

Proof. By definition of the torsion part, we have an exact sequence

$$0 \longrightarrow \mathsf{t}(\tilde{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{B}) \longrightarrow \tilde{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{B} \longrightarrow \mathcal{K} \otimes_{\mathcal{O}} (\tilde{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{B}) \simeq \mathcal{K} \otimes_{\mathcal{O}} \mathcal{B}.$$

The image of $\tilde{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{B}$ is hence isomorphic to $\tilde{\mathcal{B}}$ and obviously coincides with the $\tilde{\mathcal{O}}$-sub-sheaf of $\mathcal{K} \otimes_{\mathcal{O}} \mathcal{B}$ generated by $\mathcal{B}$. □

Note that $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are sky-scraper sheaves, zero outside S and with finite dimensional fibres. Hence, we may (and will) identify them with the finite dimensional $\mathbf{k}$-algebras $\bigoplus_{x \in S} \mathcal{F}_x$ and $\bigoplus_{x \in S} \tilde{\mathcal{F}}_x$ respectively. Just in the same way we identify the sky-scraper sheaf $\overline{\mathcal{B}}$ with the $\mathcal{F}$-module $\bigoplus_{x \in S} \overline{\mathcal{B}}_x$.

When considering families of torsion free sheaves, we have to impose some conditions, which guarantee that they are "uniformly embedded" into their $\tilde{\mathcal{O}}$-closures. Thus, we give the following definition for such families (in [9] they are called *δ-constant*).

Definition 2.3. Let Λ be a $\mathbf{k}$-algebra (not necessarily commutative). Denote by $\mathsf{TF}(C, \Lambda)$ the category whose objects are coherent sheaves of $\mathcal{O} \otimes \Lambda$-modules $\mathcal{B}$ satisfying the following conditions:
1. $\mathcal{B}$ is torsion free over $\mathcal{O}$.
2. $\tilde{\mathcal{B}}$ is flat over $\tilde{\mathcal{O}} \otimes \Lambda$.
3. $\tilde{\mathcal{B}}/\mathcal{B}$ is flat over Λ.

Such sheaves are called *families of torsion free sheaves* on C with base Λ.

Lemma 2.4. *If $\mathcal{B} \in \mathsf{TF}(C, \Lambda)$, then it is flat over Λ and, for every Λ-module N, the sheaf $\mathcal{B}(N) = \mathcal{B} \otimes_{\Lambda} N$ is also torsion free over $\mathcal{O}$; moreover, the natural homomorphism $\mathcal{B}(N) \to \tilde{\mathcal{B}}(N)$ is an embedding and induces an isomorphism $\widetilde{\mathcal{B}(N)} \simeq \tilde{\mathcal{B}}(N)$.*

Proof. Put $\mathcal{T} = \tilde{\mathcal{B}}/\mathcal{B}$, which is a torsion sheaf over $\mathcal{O}$. Consider the exact sequence $0 \to \mathcal{B} \to \tilde{\mathcal{B}} \to \mathcal{T} \to 0$. As $\tilde{\mathcal{B}}$ and $\mathcal{T}$ are both Λ-flat, so is $\mathcal{B}$. Tensoring by N over Λ, we get again an exact sequence:

$$0 \longrightarrow \mathcal{B}(N) \longrightarrow \tilde{\mathcal{B}}(N) \longrightarrow \mathcal{T}(N) \longrightarrow 0\,.$$

As $\tilde{\mathcal{B}}$ is flat over $\tilde{\mathcal{O}} \otimes \Lambda$ and $(\tilde{\mathcal{B}} \otimes_\Lambda N) \otimes_{\tilde{\mathcal{O}}} \mathcal{X} \simeq \tilde{\mathcal{B}} \otimes_{\tilde{\mathcal{O}}\otimes\Lambda} (N \otimes \mathcal{X})$ for any sheaf of $\tilde{\mathcal{O}}$-modules $\mathcal{X}$, the sheaf $\tilde{\mathcal{B}}(N) = \tilde{\mathcal{B}} \otimes_\Lambda N$ is flat over $\tilde{\mathcal{O}}$, hence, torsion free. Therefore, $\mathcal{B}(N)$ is also torsion free. Moreover, as the image of $\mathcal{B}(N)$ obviously generates $\tilde{\mathcal{B}}(N)$, the latter coincides with $\widetilde{\mathcal{B}(N)}$ in view of Lemma 2.2. □

Using this notion, we are able to define *TF-finite, TF-tame* and *TF-wild* curves just in the same way as we have defined the corresponding VB-types. Nevertheless, it so happens that indeed the TF-type of a curve coincides with its VB-type. We formulate in this section the corresponding results; the remaining part of the article will be devoted to their proofs. First the following holds:

Proposition 2.5. *If a singular curve C is not VB-wild, then:*

1. *All irreducible components of C are rational curves.*
2. *Any singular point $x \in S$ is a simple node (simple double point). In other words, the pre-image $\nu^{-1}(x)$ consists of 2 points and $\tilde{\mathcal{F}}_x \simeq \mathbf{k}^2$.*

Proof: of *1* cf. Section 4, of *2* cf. Section 6. □

If all irreducible components of C are rational and all its singular points are simple nodes, one calls C a *projective configuration.* Such configurations are defined by their *dual graphs.* Recall the corresponding definition.

Definitions 2.6. If C is a projective configuration, its *dual graph* is the graph $\Delta(C)$ whose vertices are the irreducible components of C, the edges are the singular points of C and an edge corresponding to the point p_j is incident to the vertex corresponding to the component C_i if and only if $p_j \in C_i$.

Note that the graph $\Delta(C)$ is non-oriented, but may have loops and multiple edges. A loop appears if a singular point p_j belongs to a unique component C_i (in this case the edge corresponding to p_j is only incident to the vertex corresponding to C_i). As we always suppose C to be connected, the graph $\Delta(C)$ is connected as well.

It is also convenient to consider $\mathbb{P}^1$ as a projective configuration. As it has only 1 component and no singular points, its dual graph has one vertex and no edges at all. The following result will be proved in Section 6:

Proposition 2.7. *Let C be a projective configuration. Then:*

1. *C is TF-finite (hence, VB-finite) if and only if $\Delta(C)$ is a Dynkin diagram of type A, i.e., a chain. (For instance, this is the case if*

$C = \mathbb{P}^1$.) Moreover, in this case all indecomposable vector bundles on C are of rank 1 and are determined up to isomorphism by their vector-degrees.

2. *C is TF-tame (hence, VB-tame) if and only if $\Delta(C)$ is an extended Dynkin digram of type $\tilde{\mathrm{A}}$, i.e., a cycle. (For instance, this is the case if C is irreducible rational and has only one simple node.) Moreover, in this case it is VB-unbounded (hence, TF-unbounded).*
3. *In all other cases C is VB-wild (hence, TF-wild).*

In the first two cases we call C a projective configuration of type A *(resp., $\tilde{\mathrm{A}}$).*

Altogether, it gives the following theorem completely describing the VB-types of projective curves:

Theorem 2.8. *Let C be a reduced projective curve.*

1. *If C is a projective configuration of type A, then it is TF-finite (hence, VB-finite).*
2. *If C is a smooth elliptic curve, then it is VB-tame, bounded.*
3. *If C is a projective configuration of type $\tilde{\mathrm{A}}$, then it is both TF- and VB-tame, unbounded.*
4. *In all other cases C is VB-wild (hence, TF-wild).*

Remark 2.9. Note some evident corollaries of Theorem 2.8.

1. An irreducible projective curve C is
   (a) VB-finite if and only if $C \simeq \mathbb{P}^1$;
   (b) VB-tame bounded if and only if its is elliptic;
   (c) VB-tame unbounded if and only if it is rational and has only one singular point which is a simple node;
   (d) VB-wild otherwise.
2. If a curve C is VB-finite, its arithmetic genus is always 0, if it is VB-tame, its arithmetic genus is always 1.
3. Any deformation of a VB-finite curve is also VB-finite; any deformation of a VB-tame one is also VB-tame.

We also give a description of all vector bundles and torsion free sheaves on projective configurations with the dual graphs of types A and $\tilde{\mathrm{A}}$. Such a sheaf $\mathcal{B}$ can be given by its "normalisation" $\tilde{\mathcal{B}}$ and the "rule of gluing," which describes the image of the factor $\mathcal{B}/\mathcal{J}\mathcal{B}$ in $\tilde{\mathcal{B}}/\mathcal{J}\tilde{\mathcal{B}}$.

First, let C be a projective configuration of type A, $\{C_1, C_2, \dots, C_s\}$ be its irreducible components. Each of C_i is isomorphic to $\mathbb{P}^1$ and the normalisation $\tilde{C}$ of C can be identified with their disjoint union $\bigsqcup_i C_i$. Let $\{x_1, x_2, \dots, x_{s-1}\}$ be the singular points of C; we suppose that $x_i \in C_i \cap C_{i+1}$ and denote by x_i' (resp., x_i'') the pre-image of x_i on C_i (resp., on C_{i+1}).

Consider any vector of the form $\mathbf{d} = (0, \dots, 0, d_k, d_{k+1}, \dots, d_l, 0, \dots, 0)$, where $0 \le k \le l \le s$ and define the torsion-free sheaf $\mathcal{B} = \mathcal{B}_{\mathbf{d}}$ in the following way:

- $\tilde{\mathcal{B}} = \bigoplus_{i=k}^{l} \mathcal{O}_{C_i}(d_i)$;
- For each $i = k, \dots, m-1$, the image of the stalk $\mathcal{B}_{x_i}$ in $\tilde{\mathcal{B}}_{x_i}/\mathcal{J}\tilde{\mathcal{B}}_{x_i} = \mathbf{k}(x_i') \oplus \mathbf{k}(x_i'')$ is one-dimensional subspace spanned by $(1, 1)$.

Certainly, $\mathcal{B}_{\mathbf{d}} \not\simeq \mathcal{B}_{\mathbf{d}'}$ if $\mathbf{d} \neq \mathbf{d}'$ and the sheaf $\mathcal{B}_{\mathbf{d}}$ is a vector bundle if and only if its support coincides with the whole C, i.e., $k = 1$ and $l = s$.

Theorem 2.10. *If C is a projective configuration of type* A, *then all torsion free sheaves $\mathcal{B}_{\mathbf{d}}$ defined above are indecomposable and pairwise non-isomorphic and every indecomposable torsion free sheaf on C is isomorphic to one of the sheaves $\mathcal{B}_{\mathbf{d}}$.*

In particular, an indecomposable torsion free sheaf on C is completely determined by its vector-degree and any indecomposable vector bundle on C is of rank 1.

Let now C be a projective configuration of type $\tilde{\mathrm{A}}$, $\{ C_1, C_2, \dots, C_s \}$ be its irreducible components. If $s > 1$, each of C_i is again isomorphic to $\mathbb{P}_1$ and $\tilde{C} \simeq \bigsqcup_i C_i$. If $s = 1$, $\tilde{C} \simeq \mathbb{P}^1$. We denote by $\mathcal{O}_i$ the structure sheaf of the *normalisation* of C_i (which coincides with C_i if $s > 1$). Let $x_1, x_2, \dots, x_s$ be the singular points of C. We suppose that $x_i \in C_i \cap C_{i+1}$ (putting $C_{s+1} = C_1$) and denote, if $s > 1$, its pre-image on C_i (resp., on C_{i+1}) by x_i' (resp., x_i''). If $s = 1$, x_1 has two pre-images on $\tilde{C}$ and we also denote them by x_1' and x_1''. Put also $\mathcal{F}_i = \tilde{\mathcal{O}}_{x_i}/\mathcal{J}_{x_i}$ and identify it with $\mathbf{k}(x_i') \oplus \mathbf{k}(x_i'')$.

Call a *band datum* a triple $\mathbf{b} = (\mathbf{d}, m, \lambda)$, where m is a positive integer, λ is a non-zero element of the field $\mathbf{k}$ and $\mathbf{d}$ is a sequence of integers $(n_1, n_2, \dots, n_{sr})$, which is *$s$-aperiodic*, i.e., cannot be obtained by a repetition of a shorter sequence whose length is also a multiple of s. For every band datum $\mathbf{b}$ define the vector bundle $\mathcal{B} = \mathcal{B}_{\mathbf{b}}$ as follows:

- Put $\mathcal{A}^j := m\mathcal{O}_j(n_j)$, where $\mathcal{O}_j := \mathcal{O}_k$ for $j = ls + k$, and $\mathcal{A} := \oplus_{j=1}^{sr} \mathcal{A}^j$.
  Remark that
  $$(\mathcal{A}^j/\mathcal{J}\mathcal{A}^j)_{x_i} \simeq \begin{cases} m\mathbf{k}(x_i') & \text{if } j \equiv i \pmod{s} \\ m\mathbf{k}(x_i'') & \text{if } j \equiv i+1 \pmod{s} \\ 0 & \text{otherwise} \end{cases}$$
  We identify a non-zero factor $(\mathcal{A}^j/\mathcal{J}\mathcal{A}^j)_{x_i}$ with $m\mathbf{k}(x_i')$ or $m\mathbf{k}(x_i'')$ and denote by $\{ \mathbf{e}_{jk}' \mid 1 \le k \le m \}$ or by $\{ \mathbf{e}_{jk}'' \mid 1 \le k \le m \}$ its "standard" basis (consisting of the vectors $(0, \dots, 1, \dots, 0)$).
- Define $\mathcal{B} = \mathcal{B}_{\mathbf{b}}$ as the vector bundle on C such that $\tilde{\mathcal{B}}$ coincides with the sheaf $\mathcal{A}$ defined above and the image of $\mathcal{B}/\mathcal{J}\mathcal{B}$

in $\mathcal{A}/\mathcal{J}\mathcal{A}$ coincides with the subspace spanned by the following vectors:

$$\begin{aligned} \mathbf{e}'_{jk} + \mathbf{e}''_{j+1,k} \qquad & \text{for all } j \neq rs \text{ and all } k\,; \\ \mathbf{e}'_{rs,1} + \lambda \mathbf{e}''_{1,1}\,; \qquad & \\ \mathbf{e}'_{rs,k} + \lambda \mathbf{e}''_{1,k} + \mathbf{e}''_{1,k} \qquad & \text{for } 1 < k \leq m\,. \end{aligned}$$

One easily sees that $\mathcal{B}_{\mathbf{b}} \simeq \mathcal{B}_{\mathbf{b}'}$ if $\mathbf{b}' = (\mathbf{d}', m, \lambda)$, where $\mathbf{d}'$ is obtained from $\mathbf{d}$ by an s-cyclic permutation, i.e., $\mathbf{d}' = (n_{sl+1}, n_{sl+2}, \dots, n_{sl})$ for some l. In this case we say that the whole $\mathbf{b}'$ is obtained from $\mathbf{b}$ by an s-cyclic permutation.

Now call a *string datum* a sequence $\mathbf{s} = (m; d_1, d_2, \dots, d_r)$, where $1 \leq m \leq s$ and d_i are integers (r being any positive integer). Define the torsion free sheaf $\mathcal{B}_{\mathbf{s}}$ on C as follows:

- Put $\mathcal{A}^j := \mathcal{O}_{m+j}(d_j)$ for $1 \leq j \leq r$ and $\mathcal{A} := \oplus_{j=1}^r \mathcal{A}^j$.
  Again we identify here a non-zero factor $(\mathcal{A}^j/\mathcal{J}\mathcal{A}^j)_{x_i}$ with $\mathbf{k}(x'_i)$ or $\mathbf{k}(x''_i)$ and denote, respectively, by $\mathbf{e}'_j$ or by $\mathbf{e}''_j$ its basic vector.
- Define $\mathcal{B} = \mathcal{B}_{\mathbf{s}}$ as the torsion free sheaf on C such that $\tilde{\mathcal{B}}$ coincides with the sheaf $\mathcal{A}$ defined above and the image of $\mathcal{B}/\mathcal{J}\mathcal{B}$ in $\mathcal{A}/\mathcal{J}\mathcal{A}$ coincides with the subspace spanned by the vectors $\mathbf{e}'_j + \mathbf{e}''_{j+1}$ ($1 \leq j < r$).

Note that this sheaf is *never* locally free.

Theorem 2.11. *If C is a projective configuration of type $\tilde{\mathrm{A}}$, then all torsion free sheaves $\mathcal{B}_{\mathbf{b}}$ and $\mathcal{B}_{\mathbf{s}}$, where $\mathbf{b}$ ($\mathbf{s}$) runs through all possible band (resp., string) data, are indecomposable and every indecomposable torsion free sheaf on C is isomorphic to one of the sheaves $\mathcal{B}_{\mathbf{b}}$ or $\mathcal{B}_{\mathbf{s}}$. The only possible isomorphisms between these sheaves are $\mathcal{B}_{\mathbf{b}} \simeq \mathcal{B}_{\mathbf{b}'}$, where $\mathbf{b}'$ is obtained from $\mathbf{b}$ by an s-cyclic permutation.*

In particular, $\mathcal{B}_{\mathbf{b}}$ are all indecomposable vector bundles, while $\mathcal{B}_{\mathbf{s}}$ are all indecomposable torsion free sheaves which are not vector bundles.

Note that in the latter case there are only "discrete" sets of sheaves: there are no non-trivial families of torsion free sheaves, which are not vector bundles.

The proofs of Theorems 2.10 and 2.11 are also given in Section 6.

Remark 2.12. Suppose that C is a projective configuration of type $\tilde{\mathrm{A}}$. One easily checks that in this case the dualizing sheaf ω_C is isomorphic to $\mathcal{O}_C$. Hence, using the Harder–Narasimhan filtration [15], one immediately prove, just as in [3], that all indecomposable vector bundles are $\mathbf{a}$-*semi-stable* in the sense of [15, Définition VII-9] for any "weight" $\mathbf{a} = (a_1, a_2, \dots, a_s)$. Obviously, if $\mathbf{b}_m = (m, \lambda, \mathbf{d})$, there is a filtration in $\mathcal{B}_{\mathbf{b}}$ with all factors isomorphic to $\mathcal{B}_{\mathbf{b}_1}$; hence, the only possibilities for stable vector bundles are $\mathcal{B}_{\mathbf{b}}$ with $\mathbf{b} = (1, \lambda, \mathbf{d})$. A straightforward calculation shows that, if $\mathbf{b}' = (1, \mu, \mathbf{d}')$ with $\mathbf{d}' \neq \mathbf{d}$ or $\mu \neq \lambda$, there is no homomorphism $f : \mathcal{B}_{\mathbf{b}} \to \mathcal{B}_{\mathbf{b}'}$ such that $f(p)$ were inclusion for all points $p \in C$ (even for all $p \in S$). Therefore, all vector bundles $\mathbf{b}_{\mathcal{B}}$ with $\mathbf{b} = (1, \lambda, \mathbf{d})$ are indeed $\mathbf{a}$-stable for arbitrary $\mathbf{a}$. In particular, there is a coarse moduli space for $\mathbf{a}$-stable sheaves of any prescribed rank and vector-degree, which is a disjoint union of several copies of $\operatorname{Pic} C^\circ \simeq \mathbf{k}^\times$ On the other hand, for every vector $\mathbf{d} = (d_1, d_2, \dots, d_{rs})$, put $\mu_{\mathbf{a}}(\mathbf{d}) = \sum_{i=1}^{rs} a_i d_i / r$ (here we put $a_i = a_j$ if $i \equiv j \pmod{s}$). Then the $\mathbf{a}$-semi-stable vector bundles are just the direct sums of $\mathcal{B}_{\mathbf{b}}$, where $\mathbf{b}$ runs through all triples $(m, \lambda, \mathbf{d})$ with a fixed value of $\mu_{\mathbf{a}}(\mathbf{d})$.

Recall that if C is an elliptic curve, then, again, all indecomposable vector bundles on C are semi-stable, while the stable vector bundles on C are just those having coprime rank and degree. All semi-stable vector bundles on C are the direct sums of indecomposable ones with a fixed value of the ratio "degree : rank" (cf. [3]).

We are going to illustrate the above classification by some pictures: Let $t = 3$, C_1, C_2, C_3 the three components of C, $C_i = \mathbb{P}^1$ and x_1, x_2, x_3 the three intersection points as in Figure 1:

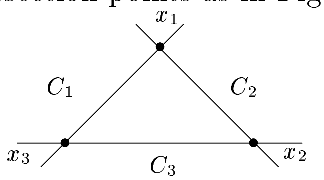


Figure 1

According to the description in Theorem 2.10, a vector bundle $\mathcal{B} = \mathcal{B}_{\mathbf{b}}$ is given by an s-aperiodic band datum $\mathbf{b} = (m, \lambda, \mathbf{d})$, where $\mathbf{d} = (d_1, \dots, d_{sr}) \in \mathbb{Z}^{sr}$, $\operatorname{rk}(\mathcal{B}) = mr$, $m \in \mathbb{N}$, $\lambda \in \mathbf{k}^*$. (In our example $s = 3$.) The sequence $\mathbf{d}$ describes $\nu^*\mathcal{B}$ on $\widetilde{C}$ where $\nu : \widetilde{C} \to C$ is the normalisation.

Since we consider bundles together with a trivialisation, we have fixed a basis of each vector space $\mathcal{B}(x_i)$ and also of $\nu^*\mathcal{B}$ at the two preimages of x_i. Hence, any vector bundle $\mathcal{B}$ on C is completely described by $\nu^*\mathcal{B}$, where $\nu^*\mathcal{B}|_{\tilde{C}_j} \equiv \mathcal{B}|_{C_j}$ is a direct sum of $\mathcal{O}_{C_j}(d_i)$ for certain d_i, and by the glueing of $\mathcal{B}|_{C_j}$ with $\mathcal{B}|_{C_{j+1}}$ at x_j with respect to the given bases. The glueing of $\mathcal{B}|_{C_j}$ with $\mathcal{B}|_{C_{j+1}}$ can be trivialised for $1 \leq j < s$, that is, given by the identity matrix, but the glueing of $\mathcal{B}|_{C_t}$ with $\mathcal{B}|_{C_1}$ at x_1 may, perhaps, not be. Indeed, if $m \geq 1$ and $\mathcal{B}$ of rank mr, this glueing can be described as being the identity on the first $m(r-1)$ basis vectors, and the Jordan cell of rank d with eigenvalue λ on the last m basis vectors (in particular, multiplication with λ if $m = 1$).

In Figure 2 and Figure 3, a thick line with label d_i corresponding to the component C_j symbolises $\mathcal{O}_{Cj}(d_i)$, a thin line corresponds to a trivial glueing and a dotted line to a non-trivial glueing, described by the matrix A. The marked points symbolise the basis of the fibres over the corresponding intersection points:

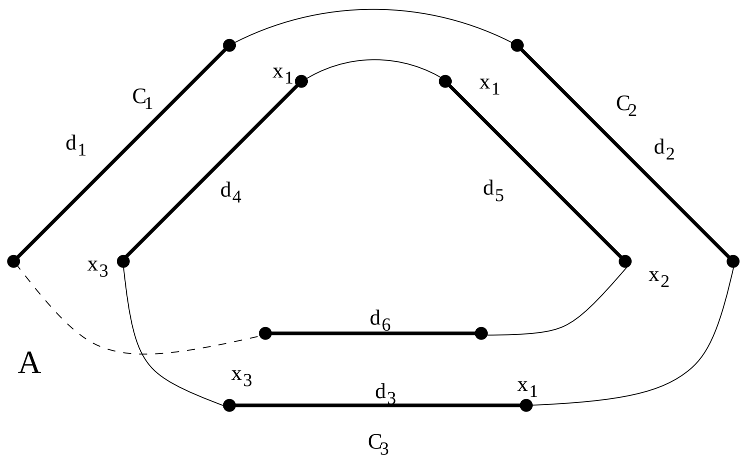


Figure 2
Bundle of rank 2, $\mathbf{b} = (1, \lambda; d_1, \ldots, d_6)$,
A $= (\lambda)$

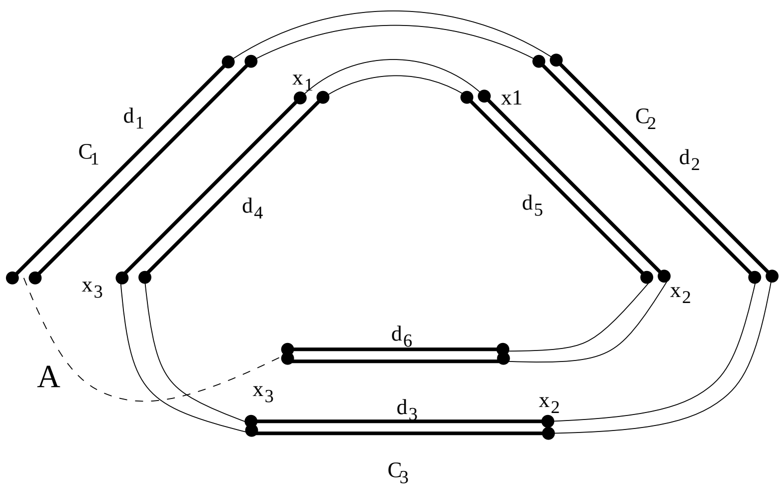
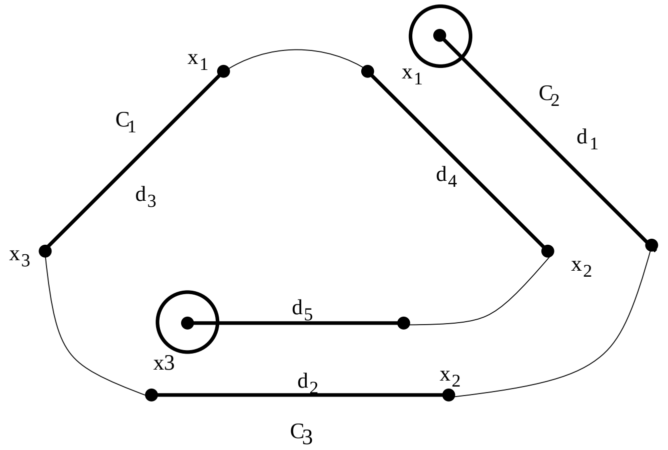


Figure 3
Bundle of rank 4, $\mathbf{b} = (2, \lambda; d_1, \ldots, d_6)$,
A $= \left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right)$

A torsion-free sheaf $\mathcal{M}$ on C which is not locally free is given by a string datum which can be coded as $\mathbf{s} = (m; \mathbf{d})$, $\mathbf{d} = (d_1, \ldots, d_r) \in \mathbb{Z}^r$, $1 \leq m \leq s$.

Here, $\mathbf{d}$ describes $\nu^*\mathcal{M}$ as before, while C_m is the component where the glueing starts. Again we do cyclic glueing, though the cycle does not close but has two free ends corresponding to the basis elements which are not glued (encircled in Fig. 4). Hence all glueings are trivial and we see that (not locally free) torsion free sheaves on C are characterised by discrete data without moduli.



Figure‘ 4
Torsion-free sheaf of rank 1 on C_1, rank 2 on C_2 and C_3, $\mathbf{s} = (2; d_1, \dots, d_5)$.

3. Shifting bimodules

The study of vector bundles on singular curves is closely related to “bimodule problems” considered in [5, 6]. They appear during the description of “gluing” necessary to obtain a sheaf on such a curve from one on its normalisation. This relation will be studied in the next section and here we recall and make precise some of the corresponding definitions and, in addition, modify them to take into account more complicated sets of “discrete parameters.”

In what follows “*category*” usually means a category over an algebraically closed field $\mathbf{k}$. This means that all Hom-spaces are vector spaces over $\mathbf{k}$ and the product of morphisms is $\mathbf{k}$-bilinear. Given two categories A_1 and A_2, an A_1-A_2-*bimodule* is, by definition, a functor $\mathsf{U} : \mathsf{A}_1^\circ \otimes \mathsf{A}_2 \to \mathsf{Vect}$, the category of $\mathbf{k}$-vector spaces. Here $\otimes$ denotes, as usually, $\otimes_{\mathbf{k}}$. If $\mathsf{A}_1 = \mathsf{A}_2 = \mathsf{A}$, call U an A-*bimodule.* Usually we suppose that our categories are additive. Whenever some category C is not additive, we consider its *additive hull*, i.e., the smallest additive category $\mathsf{A} = \operatorname{add}\mathsf{C}$ containing C, and identify C-bimodules with A-bimodules.

Let U be an A-bimodule. Define the category $\mathsf{El}(\mathsf{U})$ of *elements of the bimodule* U as follows. Its object set is

$$\operatorname{Ob}\mathsf{El}(\mathsf{U}) := \bigcup_{A \in \operatorname{Ob}\mathsf{A}} \mathsf{U}(A, A)$$

and morphisms from $u \in \mathsf{U}(A, A)$ to $v \in \mathsf{U}(B, B)$ are morphisms $f \in \mathsf{A}(A, B))$ such that $fu = vf$. Here (and later on) we write fu instead of $\mathsf{U}(1, f)u$ and vf instead of $\mathsf{U}(f, 1)v$. Note that both these elements belong to $\mathsf{U}(A, B)$.

Call a *shift* in a category A any self-equivalence $\mathsf{A} \xrightarrow{\sim} \mathsf{A}$. Define now a *shifting category* to be a triple $(\mathsf{A}, \Sigma, \rho)$, where A is a category, Σ

is a group and ρ is a homomorphism of Σ to the group of shifts in A. As usually, we write $\sigma(a)$ instead of $\rho(\sigma)(a)$, where a is an object or a morphism of A. We call Σ the *group of shifts* in A.

Let U be an A-bimodule and σ be some shift in A. A *shift in* U *compatible with* σ is, by definition, an isomorphism of A-bimodules $\sigma : \mathsf{U} \xrightarrow{\sim} \mathsf{U}^\sigma$, where U^σ denotes the bimodule obtained from U by pulling it back via σ, i.e. $\mathsf{U}^\sigma(a_1, a_2) := \mathsf{U}(\sigma(a_1), \sigma(a_2))$.

Given a shifting category $(\mathsf{A}, \Sigma, \rho)$, define a *shifting* A*-bimodule* as a pair (U, ρ_u), where ρ_u maps each element $\sigma \in \Sigma$ to a shift of U compatible with $\rho(\sigma)$ and $\rho_u(\sigma\tau) = \rho_u(\sigma)\rho_u(\tau)$ for each $\sigma, \tau \in \Sigma$. Again we write $\sigma(u)$ instead of $\rho_u(\sigma)(u)$. Note that in this case the category $\mathsf{El}(\mathsf{U})$ also becomes a shifting category with the same group Σ of shifts.

Let C be another $\mathbf{k}$-category. Then we can consider the tensor product $\mathsf{A} \otimes \mathsf{C}$ and the $\mathsf{A} \otimes \mathsf{C}$-bimodule $\mathsf{U} \otimes \mathsf{C}$. We say that the elements of $(\mathsf{U} \otimes \mathsf{C})(A \otimes P, A \otimes P)$, where $A \in \operatorname{Ob} \mathsf{A}$, $P \in \operatorname{Ob} \mathsf{C}$, are *based on* A. We are mostly interested in the case when $\mathsf{C} := \mathsf{Pr}\,\Lambda$ is the category of finitely generated (right) projective modules over some $\mathbf{k}$-algebra Λ. Then we write U^Λ instead of $\mathsf{U} \otimes \mathsf{C}$ and $\mathsf{El}(\mathsf{U}, \Lambda)$ instead of $\mathsf{El}(\mathsf{U} \otimes \mathsf{C})$. If, moreover, Λ is a commutative domain and $u \in \mathsf{U}^\Lambda(A \otimes P, A \otimes P)$, where $\operatorname{rk}_\Lambda P = r$, we say that u is an *element of rank* r *based on* A.

Note that any functor $\theta : \mathsf{C} \to \mathsf{C}'$ induces the functor $\theta_* : \mathsf{El}(\mathsf{U} \otimes \mathsf{C}) \to \mathsf{El}(\mathsf{U} \otimes \mathsf{C}')$. In particular, given a Λ'-Λ-bimodule N (i.e., left Λ- and right Λ'-), which is finitely generated and projective over Λ', we get the functor $\mathsf{El}(\mathsf{U}, \Lambda) \to \mathsf{El}(\mathsf{U}, \Lambda')$ induced by the tensor product $\_ \otimes_\Lambda N$. The image of an element $u \in \mathsf{El}(\mathsf{U}, \Lambda)$ under this functor will be denoted by $u(N)$.

Denote by vect the category of finite dimensional vector spaces over $\mathbf{k}$. Then $\mathsf{A} \otimes \mathsf{vect} \simeq \mathsf{A}$ for each additive category A and we always identify these categories. Hence, any functor $N : \mathsf{C} \to \mathsf{vect}$ gives rise to the functor $N_* : \mathsf{El}(\mathsf{U} \otimes \mathsf{C}) \to \mathsf{El}(\mathsf{U})$. In particular, if $\mathsf{C} = \mathsf{Pr}\,\Lambda$, such a functor is given by some finite dimensional Λ-module and we shall identify both. For this reason, in the general case, we also call functors $\mathsf{C} \to \mathsf{vect}$ C*-modules* (more precisely, they should be called "finite dimensional modules," but we never deal with other ones). Denote the category of C-modules by C-mod.

Definition 3.1. Let U be a shifting bimodule with the group of shifts Σ. Call an element $u \in \mathsf{El}(\mathsf{U} \otimes \mathsf{C})$ *strict* if it satisfy the following conditions:

1. The element $u(N)$ is indecomposable (in $\mathsf{El}(\mathsf{U})$) for each indecomposable C-module N.
2. For each two C-modules N, N' and for each shift $\sigma \in \Sigma$, the elements $u(N)$ and $\sigma(u)(N')$ are isomorphic (in $\mathsf{El}(U)$) if and only if $\sigma = 1$ and $N \simeq N'$ (as Λ-modules).

Definition 3.2. Let U be a shifting bimodule with the group Σ of shifts. Suppose we have given, for each $\mathbf{k}$-algebra Λ, a full sub-category $\mathsf{El}'(\mathsf{U},\Lambda) \subseteq \mathsf{El}(\mathsf{U},\Lambda)$ satisfying the following conditions:

1. $u(N) \in \mathsf{El}'(\mathsf{U},\Lambda')$ for each element $u \in \mathsf{El}'(\mathsf{U},\Lambda)$ and for each Λ'-Λ-bimodule N which is finitely generated and projective over Λ'.
2. $\sigma(u) \in \mathsf{El}'(\mathsf{U},\Lambda)$ for each shift $\sigma \in \Sigma$ and for each element $u \in \mathsf{El}'(\mathsf{U},\Lambda)$.

Then call the family of sub-categories $\{\, \mathsf{El}'(\mathsf{U},\Lambda) \,\}$ a *correct family* and the elements $u \in \mathsf{El}'(\mathsf{U},\Lambda)$ the *correct elements* (with respect to this correct family).

Definitions 3.3. The *representation type* of a shifting A-bimodule U (with the group Σ of shifts) supplied by a correct family of sub-categories is defined as follows. This bimodule is called to be:

- *correctly finite* if there exists a finite set of indecomposable correct elements $\mathbf{M} \subseteq \mathsf{El}'(\mathsf{U})$ such that each indecomposable correct element is isomorphic to $\sigma(u)$ for some $\sigma \in \Sigma$ and some $u \in \mathbf{M}$.
- *correctly tame* if there exists a set $\mathbf{M}$ consisting of strict elements $u \in \mathsf{El}'(\mathsf{U},\Lambda_u)$ such that:
  1. Each Λ_u is a commutative domain, finitely generated as $\mathbf{k}$-algebra and of Krull dimension 1 (note that it may depend on u).
  2. For each object $A \in \operatorname{Ob}\mathsf{A}$ and for each natural number r, the set $\mathbf{M}_{A,r} = \{u \in M | u$ is an element of rank r based on $A\}$ is finite.
  3. For each object $A \in \operatorname{Ob}\mathsf{A}$, all indecomposable correct elements from $\mathsf{U}(A,A)$, except possibly for a finite number, are isomorphic to $\sigma(u)(N)$ for some element $u \in \mathbf{M}$, some shift $\sigma \in \Sigma$ and some (finite dimensional) Λ_u-module N.

  In this case we call $\mathbf{M}$ a *parametrising set* for correct elements of U.

  Moreover, if U is correctly tame, call it:
  - *bounded* if there exists a parametrising set $\mathbf{M}$ for correct elements of U such that $|\mathbf{M}_{A,r}| \le C$ for some constant C and all possible A and r.
  - *unbounded* if there is no such parametrising set.
- *correctly wild* if there exists a correct strict element $u \in \mathsf{El}'(\mathsf{U},\Lambda)$ for each finitely generated $\mathbf{k}$-algebra Λ.

In the "simplest" case, when *all* elements are considered as correct, we omit the word "correct" at all and speak about *finite*, *tame* (bounded or unbounded) or *wild* shifting bimodule.

Recall once more that to prove U being correctly wild, we only have to find a correct strict element in $\mathsf{El}'(\mathsf{U},\mathbf{F})$, where $\mathbf{F} := \mathbf{k}\langle\, z_1, z_2 \,\rangle$ is a free (non-commutative) $\mathbf{k}$-algebra with 2 generators.

In most cases we deal with so called *bipartite bimodules* [5]. They are defined as follows. If U is an A_1-A_2-bimodule, we can consider it as a bimodule over the direct product $\mathsf{A} := \mathsf{A}_1 \times \mathsf{A}_2$ by putting $\mathsf{U}((a_1, a_2), (b_1, b_2)) := \mathsf{U}(a_1, b_2)$ for $a_i, b_i \in \mathsf{A}_i$. Call this bimodule a *bipartite* A_1-A_2-bimodule. In this case an element of U based on a pair (A_1, A_2), where $A_i \in \operatorname{Ob} \mathsf{A}_i$, is indeed an element of $\mathsf{U}(A_1, A_2)$. All the other above definitions are modified just in the same way. Note that in [6] only bipartite bimodules were considered.

4. Relation between vector bundles and bimodules

We are now going to apply the techniques of the preceding section to the study of torsion free sheaves on a singular curve C. We use Notations 2.1; in particular, $\tilde{C}$ denotes the normalisation of C, $\mathcal{O}$ the structure sheaf of C, $\tilde{\mathcal{O}}$ that of $\tilde{C}$ and $\mathcal{J}$ the conductor of $\mathcal{O}$ in $\tilde{\mathcal{O}}$. To describe the glueing necessary to obtain sheaves of $\mathcal{O}$-modules from those of $\tilde{\mathcal{O}}$-modules the following notions are convenient.

Definitions 4.1. 1. Let Λ be a $\mathbf{k}$-algebra, $\mathcal{A}$ a flat coherent sheaf of $\tilde{\mathcal{O}} \otimes \Lambda$-modules and $\mathcal{M}$ a coherent $\mathcal{F} \otimes \Lambda$-sub-sheaf in $\overline{\mathcal{A}} := \mathcal{A}/\mathcal{J}\mathcal{A}$. Call this sub-sheaf *correct* if it satisfies the following conditions:
(a) $\overline{\mathcal{A}}/\mathcal{M}$ is flat over Λ.
(b) $\mathcal{M}$ is flat over $\mathcal{F} \otimes \Lambda$.
(c) The natural homomorphism $\tilde{\mathcal{F}} \otimes_{\mathcal{F}} \mathcal{M} \to \overline{\mathcal{A}}$ is an isomorphism.
If only condition (a) holds and $\tilde{\mathcal{F}}\mathcal{M} = \overline{\mathcal{A}}$, call $\mathcal{M}$ *semi-correct.*
2. Define the category $\mathsf{C}^s = \mathsf{C}^s(C, \Lambda)$ as follows: objects are pairs $(\mathcal{A}, \mathcal{M})$, where $\mathcal{A}$ is a flat coherent sheaf of $\tilde{\mathcal{O}} \otimes \Lambda$-modules of constant rank on $\tilde{C}$ and $\mathcal{M}$ is a semi-correct submodule in $\overline{\mathcal{A}}$, and a morphism $(\mathcal{A}, \mathcal{M}) \to (\mathcal{A}', \mathcal{M}')$ is a morphisms $f : \mathcal{A} \to \mathcal{A}'$ such that the induced mapping $\overline{f} : \overline{\mathcal{A}} \to \overline{\mathcal{A}}'$ maps $\mathcal{M}$ to $\mathcal{M}'$.
3. Let $\mathsf{C} = \mathsf{C}(C, \Lambda)$ be the full sub-category of $\mathsf{C}^s(C, \Lambda)$ consisting of all pairs $(\mathcal{A}, \mathcal{M})$ with a correct submodule $\mathcal{M}$.
We write $\mathsf{C}^s(C)$ and $\mathsf{C}(C)$ instead of $\mathsf{C}^s(C, \mathbf{k})$ and $\mathsf{C}(C, \mathbf{k})$ correspondingly.
4. Define a functor $F : \mathsf{TF}(C, \Lambda) \to \mathsf{C}^s(C, \Lambda)$ mapping any sheaf $\mathcal{B} \in \mathsf{TF}(C, \Lambda)$ to the pair $(\tilde{\mathcal{B}}, \overline{\mathcal{B}})$ and any morphism $g : \mathcal{B} \to \mathcal{B}'$ to the morphism $\tilde{g} : \tilde{\mathcal{B}} \to \tilde{\mathcal{B}}'$.

One can easily check that $\overline{\mathcal{B}}$ is indeed a semi-correct submodule in $\tilde{\mathcal{B}}/\mathcal{J}\tilde{\mathcal{B}}$ and $\overline{\tilde{g}}(\overline{\mathcal{B}}) \subseteq \overline{\mathcal{B}}'$. Note only that the definitions imply immediately that $\mathcal{J}\tilde{\mathcal{B}} = \mathcal{J}\mathcal{B}$. Moreover, if $\mathcal{B}$ is a flat sheaf, the submodule $\overline{\mathcal{B}}$ is indeed correct. This construction allows to formulate the following result.

Proposition 4.2. *The functor F establishes an equivalence of the categories $\mathsf{TF}(C,\Lambda)$ and $\mathsf{C}^s(C,\Lambda)$. Moreover, the restriction of F to $\mathsf{VB}(C,\Lambda)$ establishes an equivalence of $\mathsf{VB}(C,\Lambda)$ and $\mathsf{C}(C,\Lambda)$.*

Proof. Define the inverse functor G as follows: for any object $\mathcal{P} = (\mathcal{A},\mathcal{M})$ of C^s let $\mathcal{B} = G\mathcal{P}$ be the pre-image of $\mathcal{M} \subseteq \overline{\mathcal{A}}$ in $\mathcal{A}$. It is a coherent sub-sheaf in $\mathcal{A}$ such that $\tilde{\mathcal{O}}\mathcal{B} = \mathcal{A}$. In particular, $\mathcal{B}$ is torsion free and $\tilde{\mathcal{B}} \simeq \mathcal{A}$. As $\mathcal{A}/\mathcal{B} \simeq \overline{\mathcal{A}}/\mathcal{M}$ is flat over Λ, $\mathcal{B} \in \mathsf{TF}(C,\Lambda)$. If $\mathcal{P}' = (\mathcal{A}',\mathcal{M}')$ is another pair and $f : \mathcal{P} \to \mathcal{P}'$ is a morphism from $\mathsf{C}^s(C,\Lambda)$, then, by construction, $f(G\mathcal{A}) \subseteq G\mathcal{A}'$. Therefore, we obtain a functor $G : \mathsf{C}^s \to \mathsf{VB}$, inverse to F.

Let now the pair $\mathcal{P}$ be correct, that is, $\mathcal{M} \simeq \mathcal{B}/\mathcal{J}\mathcal{B}$ is flat over $\mathcal{F}\otimes\Lambda$. As it is also coherent, it is a projective $\mathcal{F}\otimes\Lambda$-module. Fix a point $x \in S$ and put $M = \mathcal{M}_x/\mathfrak{m}\mathcal{M}_x$, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{F}_x$. M is a flat coherent, hence, projective Λ-module. Then $P = \mathcal{O}_x \otimes M$ is a projective $\mathcal{O}_x\otimes\Lambda$-module such that $P/\mathfrak{m}P \simeq \mathcal{B}_x/\mathfrak{m}\mathcal{B}_x$. Hence, there is a homomorphism $f : P \to \mathcal{B}_x$ such that $\operatorname{Im} f + \mathfrak{m}\mathcal{B}_x = \mathcal{B}_x$. As $\mathcal{B}_x$ is finitely generated as $\mathcal{O}_x\otimes\Lambda$-module, then f is an epimorphism. Moreover, as $\overline{\mathcal{A}}_x \simeq \tilde{\mathcal{F}} \otimes_{\mathcal{F}} \mathcal{M}_x$, also $\mathcal{A}_x/\mathfrak{m}\mathcal{A}_x \simeq \tilde{P}/\mathfrak{m}\tilde{P}$, where $\tilde{P} = \tilde{\mathcal{O}}_x \otimes M = \tilde{\mathcal{O}}_x \otimes_{\mathcal{O}_x} P$. As both $\tilde{P}$ and $\mathcal{A}_x$ are finitely generated projective $\tilde{\mathcal{O}}_x\otimes\Lambda$-modules, they are isomorphic and the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_x \otimes M & \xrightarrow{\ f\ } & \mathcal{B}_x \\ \downarrow & & \downarrow \\ \tilde{\mathcal{O}}_x \otimes M & \xrightarrow{\ \sim\ } & \mathcal{A}_x \end{array}$$

shows that f is a monomorphism, hence, isomorphism. Therefore, $\mathcal{B}_x$ is projective (thus flat) $\mathcal{O}_x\otimes\Lambda$-module for every point $x \in S$. As for all other points y, $\mathcal{B}_y = \mathcal{A}_y$ is also flat, the whole sheaf $\mathcal{B}$ is flat over $\mathcal{O}\otimes\Lambda$, that is, belongs to $\mathsf{VB}(C,\Lambda)$. $\square$

For each vector bundle $\tilde{\mathcal{A}}$ of constant rank r on $\tilde{C}$, we can always choose an open affine sub-variety $C' \subset \tilde{C}$ such that $\tilde{S} \subset C'$ and the restriction of $\tilde{\mathcal{A}}$ on C' is trivial: $\tilde{\mathcal{A}}|_{C'} \simeq r\mathcal{O}_{\tilde{C}}|_{C'}$. Using this, we can (and do) always suppose that $\tilde{\mathcal{A}}|_{C'} = r\mathcal{O}_{\tilde{C}}|_{C'}$. Therefore, setting $\mathcal{A} = \nu_*\tilde{\mathcal{A}}$, we get $\mathcal{A}|_{\nu(C')} = r\tilde{\mathcal{O}}|_{\nu(C')}$ and hence $\overline{\mathcal{A}} = r\tilde{\mathcal{F}}$. Such an agreement is compatible with tensor products if we always identify $\tilde{\mathcal{O}}\otimes_{\tilde{\mathcal{O}}}\tilde{\mathcal{O}}$ with $\tilde{\mathcal{O}}$ via the natural isomorphism. A correct sub-sheaf in $\overline{\mathcal{A}}$ is then given by r elements $v_1, v_2, \dots, v_r$ of $r\tilde{\mathcal{F}}$ linearly independent over $\mathcal{F}$, namely, $\mathcal{M} = \sum_{i=1}^r \mathcal{F}v_i$. We often write $(\mathcal{A}, v_1, v_2, \dots, v_r)$ instead of $(\mathcal{A},\mathcal{M})$.

For instance, a line bundle $\mathcal{L}$ over C is given by a line bundle $\tilde{\mathcal{L}}$ over $\tilde{C}$ and an invertible element v of the algebra $\tilde{\mathcal{F}}$. If $\mathcal{B}$ is the torsion free sheaf corresponding to a semi-correct pair $\mathcal{P} = (\mathcal{A},\mathcal{M})$ then their tensor product $\mathcal{B}\otimes_{\mathcal{O}}\mathcal{L}$ is given by the pair $\mathcal{P}^{\mathcal{L}} := (\mathcal{A}\otimes_{\tilde{\mathcal{O}}}\mathcal{L}, v\mathcal{M})$. This is the way how the Picard group $\mathrm{Pic}(C)$ acts on the category $\mathsf{C}^s(C)$. Of course, if $\mathcal{P} \in \mathsf{C}$, also $\mathcal{P}^{\mathcal{L}} \in \mathsf{C}$.

In particular, we get a rule for tensor products of line bundles. Note that the bundles corresponding to the pairs $(\tilde{\mathcal{L}}, v)$ and $(\tilde{\mathcal{L}}', v')$ are isomorphic if and only if $\tilde{\mathcal{L}} \simeq \tilde{\mathcal{L}}'$ and $v' = \theta v$ for some invertible element $\theta \in \mathcal{F}$ (take into account that both $\tilde{\mathcal{L}}|_{C'}$ and $\tilde{\mathcal{L}}'|_{C'}$ coincide with $\nu_* \mathcal{O}_{\tilde{C}}|_{C'}$). Note also that any isomorphism $\tilde{\mathcal{L}} \xrightarrow{\sim} \tilde{\mathcal{L}}'$ is locally constant (as the curve $\tilde{C}$ is projective). Denote by $\mathcal{C}$ the image of all locally constant functions in $\tilde{\mathcal{F}}$. Then the preceding considerations immediately give the following corollary.

Corollary 4.3. $\mathrm{Pic}(C) \simeq \mathrm{Pic}(\tilde{C}) \times (\tilde{\mathcal{F}}^*/\mathcal{C}^*\mathcal{F}^*)$.

If C is irreducible, $\mathcal{C} = \mathbf{k} \subseteq \mathcal{F}$, so the latter factor becomes indeed $\tilde{\mathcal{F}}^*/\mathcal{F}^*$.

The following result is quite obvious.

Proposition 4.4. *If $\tilde{C}$ is VB-wild, so is also C.*

Proof. Indeed, let $\mathcal{A} \in \mathsf{VB}(\tilde{C}, \mathbf{F})$ be a strict sheaf. For any point $x \in \tilde{C}$, $\mathcal{A}_x/\mathcal{J}\mathcal{A}_x$ is a projective $\mathbf{F}$-module, hence, it is free (cf. [4]). It is evident that all these factors are of the same rank r. Fix an $\mathbf{F}$-bases $\mathbf{e}_1^x, \mathbf{e}_2^x, \dots, \mathbf{e}_r^x$ of these factors for all points $x \in \tilde{S}$ and consider the $\mathcal{F} \otimes \mathbf{F}$-sub-sheaf $\mathcal{M} \subset \mathcal{A}/\mathcal{J}\mathcal{A}$ generated by the set

$$\left\{ \sum_{\nu(x)=y} \mathbf{e}_i^x \mid y \in S,\ 1 \le i \le r \right\}.$$

Clearly, this sub-sheaf is correct, hence, defines a family $\mathcal{B} \in \mathsf{VB}(C, \mathbf{F})$. But this family is strict as so is $\mathcal{A}$ and evidently $\widetilde{\mathcal{B}(N)} \simeq \mathcal{A}(N)$ for every N. □

The vector-degree Deg defines a homomorphism $\mathrm{Pic}(C) \to \mathbb{Z}^s$ and it is evident that $\mathrm{Deg}\,\mathcal{B} = \mathrm{Deg}\,\tilde{\mathcal{B}}$. In what follows, we choose a section $\omega : \mathbb{Z}^s \to \mathrm{Pic}(C)$ in such a way that $\omega(\mathbf{e}_i) = \mathcal{O}(p_i)$ for some point $p_i \notin S$, where $\mathbf{e}_i := (0, \dots, 0, 1, 0, \dots 0)$ (1 at the i-th place). Put, as above, $\mathcal{O}(\mathbf{d}) := \omega(\mathbf{d})$ and $\mathcal{A}(\mathbf{d}) := \mathcal{A} \otimes_{\mathcal{O}} \mathcal{O}(\mathbf{d})$ for every sheaf $\mathcal{A}$ of $\mathcal{O}$-modules. Then $\mathsf{VB}(C)$ becomes a shifting category with the group $\Sigma = \mathbb{Z}^s$ of shifts. The same is true for C and C^s: if $\mathcal{P} = (\mathcal{A}, \mathcal{M})$, put $\mathcal{P}(\mathbf{d}) := (\mathcal{A}(\mathbf{d}), \mathcal{M})$. Note that under the restriction imposed above, $\mathcal{A}(\mathbf{d})|_{\nu(C')} = \mathcal{A}|_{\nu(C')}$, hence, $\overline{\mathcal{A}}(\mathbf{d}) = \overline{\mathcal{A}}$. Certainly, the equivalence $\mathsf{VB}(C) \simeq \mathsf{C}$ preserves these shifts.

The category C^s has the advantage that it can be easily reinterpreted with the help of some bimodule category.

Denote by A the category of locally free (coherent) $\tilde{\mathcal{O}}$-sheaves and by B the category of projective (finitely generated) $\mathcal{F}$-modules. Define a bipartite B-A-bimodule U by putting, for $B \in \mathsf{B}$ and $\mathcal{A} \in \mathsf{A}$,

$$\mathsf{U}(B, \mathcal{A}) = \operatorname{Hom}_{\mathcal{F}}(B, \overline{\mathcal{A}}) .$$

This is also a shifting bimodule with the same group $\Sigma = \mathbb{Z}^s$ of shifts. Namely, shifting by $\mathbf{d}$ acts on A as above, acts on B trivially and maps an element $u \in \mathsf{U}(B, \mathcal{A})$, i.e., a homomorphism $B \to \overline{\mathcal{A}}$, to the same element considered as a homomorphism $B \to \overline{\mathcal{A}}(\mathbf{d}) = \overline{\mathcal{A}}$ (we denote it by $u(\mathbf{d})$).

Define two correct families of elements of the bimodule U. The first one is denoted by El_c and called the family of *correct* elements consisting of all elements $u \in \mathsf{U}^{\Lambda}(B, \mathcal{A}) = \operatorname{Hom}_{\mathcal{F}\otimes\Lambda}(B, \overline{\mathcal{A}})$ satisfying the following conditions:

1. $B \simeq F \otimes P$ for some free $\mathcal{F}$-module F and some projective Λ-module P.
2. $\operatorname{Ker} u \subseteq (\operatorname{rad} \mathcal{F}) B$.
3. $\operatorname{Coker} u$ is flat as Λ-module.
4. The induced map $u_{\tilde{\mathcal{F}}} : \tilde{\mathcal{F}} \otimes_{\mathcal{F}} B \to \overline{\mathcal{A}}$ is an isomorphism.

The second family is denoted by El_{sc} and called the family of *semi-correct* elements consisting of all elements satisfying conditions 2 and 3 above and the following one:

4′. $u_{\tilde{\mathcal{F}}}$ is an epimorphism.

Proposition 4.5. *For each correct (semi-correct) element* $u \in \mathsf{U}^{\Lambda}(P, \mathcal{A})$, *the image* $\operatorname{Im} u$ *is a correct (semi-correct) submodule in* $\overline{\mathcal{A}}$ *and the mapping* $u \to (\mathcal{A}, \operatorname{Im} u)$ *induces a functor* $F : \mathsf{El}_c(\mathsf{U}, \Lambda) \to \mathsf{C}(C, \Lambda)$ *(correspondingly* $\mathsf{El}_{sc}(\mathsf{U}, \Lambda) \to \mathsf{C}^s(C, \Lambda)$ *) having the following properties:*

1. *It is* dense*, i.e., each object from* C^s *is isomorphic to* Fu *for some object* $\mathsf{El}_{sc}(\mathsf{U}, \Lambda)$ *from* TF.
2. *It is* full*, i.e., all induced maps* $\mathsf{El}(u, u') \to \mathsf{C}^s(Fu, Fu')$ *are surjective.*
3. *It* reflects isomorphisms*, i.e.,* $u \simeq u'$ *if and only if* $Fu \simeq Fu'$.
4. *It* preserves indecomposability*, i.e.,* u *is indecomposable if and only if* Fu *is indecomposable.*
5. *It is* compatible with shifts*, i.e.,* $F(u(\mathbf{d})) \simeq \sigma(F(u))(\mathbf{d})$ *for each* $\sigma \in \mathbb{Z}^n$.

Usually a functor having properties *1*, *3* and *4* is called a *representation equivalence.* We call a functor having properties *1–4* a *full representation equivalence* and a functor having all five properties a *full shifting representation equivalence.* Note that the condition *4* is a formal consequence of the conditions *1* and *3*.

Proof. For each pair $(\mathcal{A}, \mathcal{M})$ from C^s the $\mathcal{F}$-module $U := \mathcal{M}/(\operatorname{rad} \mathcal{F})\mathcal{M}$ is semi-simple, hence, isomorphic to a direct sum of $U_i \otimes P_i$, where U_i are simple $\mathcal{F}$-modules and P_i are projective Λ-modules. Then $U_i \simeq$

$B_i/(\operatorname{rad}\mathcal{F})B_i$, where B_i is a projective $\mathcal{F}$-module, and there is an epimorphism $u : B := \oplus_i B_i \otimes P_i \to \mathcal{M}$ such that $\operatorname{Ker} u \subseteq (\operatorname{rad}\mathcal{F})B$. Obviously, u is a semi-correct (correct if $\mathcal{M}$ is a correct submodule) element of $\mathsf{U}^\Lambda(B,\mathcal{A})$ and $F(u) \simeq (\mathcal{A},\mathcal{M})$. Moreover, as any homomorphism can be lifted to projective covers, the functor F is full, and as $\operatorname{Ker} u \subseteq (\operatorname{rad}\mathcal{F})B$, it reflects isomorphisms. Hence, it is a full representation equivalence. The compatibility with shifts follows immediately from the definition of F. □

Due to condition *2* of the definition of correct elements, if an element $u \in \mathsf{U}(B,\mathcal{A})$ is correct, the number of indecomposable summands in B is not more that $\dim_{\mathbf{k}} \overline{\mathcal{A}}$. Hence, if $\mathcal{A}$ is fixed, there are only finitely many possibilities for B. On the other hand, if $\mathcal{A} \in \mathsf{A}$ is of rank r, then $\operatorname{Deg}\mathcal{A}(\mathbf{e}_i) = \operatorname{Deg}\mathcal{A} + r\mathbf{e}_i$. Hence, any correct element u has a shift lying in $\mathsf{U}(B,\mathcal{A})$ with $0 \le \operatorname{Deg}\mathcal{A} < (r,r,\dots,r)$. Here an inequality for vectors means inequality for all components. Together with Proposition 4.5 it implies the following corollary.

Corollary 4.6. *A curve C is VB-finite, VB-tame or VB-wild if and only if the bimodule $\mathsf{U}(C)$ is correctly finite, correctly tame or correctly wild respectively.*
(The same is true for TF-types of curves and semi-correct types of the corresponding bimodules.)

We accomplish this section with the proof of Proposition 2.5 (*1*):

If a singular curve C is not VB-wild, then all its irreducible components are rational curves.

Proof. Accordingly to Proposition 4.4 and Theorem 1.6, whenever a curve C is not VB-wild, the irreducible components of $\tilde{C}$ are either rational or elliptic curves. We show that the latter case is still impossible.

Let $C_1, C_2, \dots, C_t$ be the irreducible components of $\tilde{C}$, $\mathcal{O}_k := \mathcal{O}_{C_k}$. Suppose there is a component C_1 which is elliptic. As C is singular and connected, there is a point $e \in C$ which lies on $\nu(C_1)$ and such that $\mathcal{O}_e \ne \tilde{\mathcal{O}}_e$. Consider the case when also $e \in \nu(C_2)$ for another component C_2 (other cases are even simpler to handle). Find 4 different points $x_1,\dots,x_4$ on $C_1 \setminus \tilde{S}$ such that $2x_i \not\sim x_j + x_k$ for $x_j \ne x_i, x_k \ne x_i$, and a point $y \in C_2 \setminus \tilde{S}$. Consider the element u from $\mathsf{El}(\mathsf{U},\mathbf{F})$, where $\mathbf{F} = \mathbf{k}\langle\, z_1, z_2 \,\rangle$, defined as follows:

$$\begin{aligned} &u \in \mathsf{U}(B,A)\,, \quad \text{where}\\ &A := \bigoplus_{i=1}^{4}(\tilde{\mathcal{O}}(x_i + iy) \otimes \mathbf{F})\,,\\ &B := 4\mathcal{F} \otimes \mathbf{F}\,. \end{aligned}$$

In this case $u : B \to A$ can be given by a set of 4×4 matrices u_{pk} with entries from $\mathcal{O}_{kp}/\mathcal{J} \otimes \mathbf{F}$, as such a matrix defines a homomorphism $B_p \to A_p$. Here p runs through $S(C)$ and $k = 1, 2, \dots t$. We set all components equal to the identity matrices except of u_{e2} which is

$$\begin{pmatrix} z_1 & z_2 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

It is not difficult to verify that we obtain a strict element, hence C is VB-wild. Moreover, if $\mathcal{M} \in \mathsf{VB}(C, \mathbf{F})$ is the corresponding sheaf, one can verify, just as in the proof of Theorem 1.6, that $\mathcal{M}(N) \simeq \mathcal{M}(N') \otimes_{\mathcal{O}} \mathcal{L}$ for some line bundle $\mathcal{L}$ if and only if $N \simeq N'$ and $\mathcal{L} \simeq \mathcal{O}$. □

5. Rationally composed curves and special bimodules

From now on we suppose that all components of $\tilde{C}$ are rational curves (i.e., isomorphic to $\mathbb{P}^1$). In this case we say that the curve C itself is *rationally composed*. Note that a rationally composed curve is a projective configuration if and only if all its singular points are simple nodes. To find VB-types of rationally composed curves, we introduce the following class of bipartite bimodules.

First of all, consider the category L_0 such that $\operatorname{Ob} \mathsf{L}_0 = \mathbb{Z}$ (the integers) and the set of morphisms is generated by morphisms $x_n : n \to n+1$ and $y_n : n \to n+1$ subject to the relations: $x_{n+1}y_n = y_{n+1}x_n$ for all $n \in \mathbb{Z}$. Let L be its additive hull. It is well-known that L is equivalent to the category of vector bundles over the projective line $\mathbb{P}^1$, the object n corresponding to the sheaf $\mathcal{O}_{\mathbb{P}^1}(n)$. There is a natural shift σ on L mapping n to $n+1$ and we consider L as a shifted category with the group of shifts $\mathbb{Z} \simeq \{\, \sigma^k \mid k \in \mathbb{Z} \,\}$.

Data for a special bimodule consist of:

1. *A finite dimensional commutative algebra* $\mathbf{R}$ *and its sub-algebra* $\mathbf{S}$.
   Let $\mathbf{R} := \prod_{k=1}^{r} \mathbf{R}_i$ and $\mathbf{S} := \prod_{j=1}^{s} \mathbf{S}_j$ be their decompositions into direct product of local algebras. We denote by $\mathbf{J}$ the set of all pairs (k, j) ($1 \le k \le r$, $1 \le j \le s$) such that the projection of $\mathbf{S}_j$ onto $\mathbf{R}_k$ is non-zero.
2. *An equivalence relation* $\sim$ *on the set of local components of* $\mathbf{R}$ *(indeed on the index set* $\{\, 1, 2, \dots, r \,\}$*).*
3. *A mapping* $\kappa : \mathbf{J} \to \mathbb{P}^1$ *(the projective line over* $\mathbf{k}$*).*
   We denote $\kappa(k, j) := (\xi_{kj} : \eta_{kj})$.

We impose the following restrictions on these data:

1. $\mathbf{S}$ *contains no non-zero ideal of* $\mathbf{R}$.
2. *Whenever* $j \ne j'$ *and* $k \sim k'$ *are such that* $(k, j) \in \mathbf{J}$ *and* $(k', j') \in \mathbf{J}$, *then* $\kappa(k', j') \ne \kappa(k, j)$.

3. *For each* $k = 1, 2, \dots, r$ *there exists a* j *such that* $(k, j) \in \mathbf{J}$.
4. *If* $\mathbf{J} = \mathbf{J}' \cup \mathbf{J}''$ *such that* $(k, j) \in \mathbf{J}'$ *implies* $(k', j') \notin \mathbf{J}''$ *for each* j' *and each* $k' \sim k$ *as well as* $(k'', j) \notin \mathbf{J}''$ *for each* k'' *, then either* $\mathbf{J}'$ *or* $\mathbf{J}''$ *is empty.*[2]

We denote such data (somewhat ambiguously) by $[\,\mathbf{R}, \mathbf{S}, \kappa\,]$ and call them "*special data.*"

To special data $[\,\mathbf{R}, \mathbf{S}, \kappa\,]$ we associate a shifting bimodule $\mathbf{U}[\,\mathbf{R}, \mathbf{S}, \kappa\,]$ (called a *special bimodule*) in the following way.

Let $\mathbf{C}$ be the set of equivalence classes of the relation $\sim$, $\mathbf{C} := \{\, c_1, c_2, \dots, c_t \,\}$. For each class c_i put $\mathbf{R}(i) := \prod_{k \in c_i} \mathbf{R}_k$. Consider the category $\mathsf{A} := \mathsf{L}^t$ whose indecomposable objects are in one-to-one correspondence with the pairs (n, i) , where $n \in \mathbb{Z}$, $1 \le i \le t$ and we identify them with such pairs. If some object of A decomposes into a direct sum: $A = \oplus_k (n_k, i_k)$, put $|A| := \oplus_k \mathbf{R}(i_k)$ considered as a projective module over the algebra $\mathbf{R}$. Note that the endomorphism ring of the object (n, i) coincides with $\mathbf{k}$ and the complete set of morphisms in A is generated by the morphisms $x_{ni}, y_{ni} : (n, i) \to (n+1, i)$ originated in the morphisms x_n, y_n of L . The category A is a shifting one with the group of shifts $\Sigma := \mathbb{Z}^t = \langle\, \sigma_i \mid 1 \le i \le t \,\rangle$: the shift σ_i maps (n, i') to $(n + \delta_{ii'}, i')$.

Let now $\mathsf{B} := \mathsf{Pr}\,\mathbf{S} \simeq \prod_{j=1}^{s} \mathsf{Pr}\,\mathbf{S}_j$. Its indecomposable objects are in one-to-one correspondence with the indices $j = 1, \dots, s$ and we identify them. Here the endomorphism ring of the object j is $\mathbf{S}_j$ and there are no morphisms between different indecomposable objects. We consider B as a shifting category with the same groups of shifts Σ but with all shifts acting trivially.

Now define the special (shifted and bipartite) B-A-bimodule $\mathsf{U} := \mathsf{U}[\,\mathbf{G}, \varphi, \xi\,]$ with the same group Σ of shifts in the following way:

$$\begin{aligned}
\mathsf{U}(j, (n, i)) &:= \operatorname{Hom}_{\mathbf{S}}(\mathbf{S}_j, \mathbf{R}(i))\,;\\
x_{ni} f(n) &:= \xi_{ij} f(n+1)\,;\\
y_{ni} f(n) &:= \eta_{ij} f(n+1)\,;\\
\sigma_{i'} f(n) &:= f(n + \delta_{ii'})\,.
\end{aligned}$$

Here $f(n)$ denotes a homomorphism $f \in \operatorname{Hom}_{\mathbf{S}}(\mathbf{S}_j, \mathbf{R}(i))$ considered as an element of $\mathsf{U}(j, (n, i))$ (and we regard such homomorphisms with different indices n as different elements of U). Note that if $(i, j) \notin \mathbf{J}$, we have $\mathsf{U}(j, (n, i)) = 0$.

One can easily check that replacing the pair (ξ_{ij}, η_{ij}) , for some arrow (i, j) , by $(\lambda\xi_{ij}, \lambda\eta_{ij})$, with $\lambda \in \mathbf{k} \setminus \{\, 0 \,\}$, leads to an isomorphic shifting bimodule, so U really depends only on κ as a mapping to

[2]This condition, some sort of connectedness, is not restrictive indeed, but we prefer to impose it to simplify the definitions. In any case, we never need "non-connected" data.

the projective line. Moreover, for each fixed i we can replace all values $\xi(k,j)$ for the arrows with source $k \in c_i$ by $\gamma\xi(k,j)$, where γ is any collineation of $\mathbb{P}^1$ given by an invertible matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$: we only have to change the action of x_{ni} and y_{ni} (for each n) on all spaces $\mathsf{U}(\mathbf{S}_j, \mathbf{R}(i))$ to $ax_{ni}+by_{ni}$ and $cx_{ni}+dy_{ni}$ respectively. This also leads to an isomorphic shifted bimodule.

Let now Λ be a $\mathbf{k}$-algebra. The objects from $\mathsf{A}\otimes\Lambda$ are direct sums of the shape $A = \bigoplus_k A_k \otimes P_k$, where $A_k \in \operatorname{Ob}\mathsf{A}$, $P_k \in \mathsf{Pr}\,\Lambda$. Put $|A| := \bigoplus_k |A_k| \otimes P_k$ and consider it as a projective $\mathbf{R}\otimes\Lambda$-module. On the other hand, objects from $\mathbf{B}\otimes\Lambda$ are indeed projective $\mathbf{S}\otimes\Lambda$-modules. Hence, to each element $u \in \mathsf{U}^\Lambda(B,A)$ we can associate a homomorphism $|u| \in \operatorname{Hom}_{\mathbf{S}\otimes\Lambda}(B, |A|)$.

Call an element $u \in \mathsf{U}^\Lambda(B,A)$ *correct* if it satisfy the following conditions:

1. $B \simeq F\otimes P$ *for some* free S*-module* F *and some* $P \in \mathsf{Pr}\,\Lambda$.
2. $\operatorname{Ker}|u| \subseteq (\operatorname{rad}\mathsf{S})B$.
3. $\operatorname{Coker}|u|$ *is flat as* Λ*-module.*
4. *The induced homomorphism* $|u|_{\mathbf{R}} : \mathbf{R}\otimes_{\mathbf{S}} B \to |A|$ *is an isomorphism.*

If u satisfies the conditions 2 and 3 above and the following one:

$4'$. $|u|_{\mathbf{R}}$ *is an epimorphism,*

we call it *semi-correct.* Note that condition 3 is empty if $\Lambda = \mathbf{k}$ (i.e., for elements of U themselves).

Denote by $\mathsf{El}_c(\mathsf{U},\Lambda)$ and $\mathsf{El}_{sc}(\mathsf{U},\Lambda)$ respectively the full sub-categories of $\mathsf{El}(\mathsf{U},\Lambda)$ consisting of all correct and of all semi-correct elements. Evidently, both families are correct families in the sense defined in Section 3. Therefore, we have for the bimodule U the notion of *correctly* (or *semi-correctly*) *finite*, *tame* (*bounded* or *unbounded*) or *wild.*

Special bimodules naturally arise when we apply the procedure of Section 4 to rationally composed curves. Namely, let C be such a curve, $\tilde{C}$ be its normalisation and $C_1, C_2, \dots, C_t$ be the irreducible components of $\tilde{C}$. Fix isomorphisms $C_k \simeq \mathbb{P}^1$. Then $\mathsf{VB}(\tilde{C}) \simeq \mathsf{L}^t$. Put $\mathbf{R} := \tilde{\mathcal{F}} = \tilde{\mathcal{O}}/\mathcal{J}$ and $\mathbf{S} := \mathcal{F} = \mathcal{O}/\mathcal{J}$. Let $S = S(C) = \{p_1, p_2, \dots, p_s\}$ be the set of singular points of C and $\tilde{S} = \{q_1, q_2, \dots, q_r\}$ the pre-image of S in $\tilde{C}$. Then $\mathbf{S} = \prod_{j=1}^s \mathbf{S}_j$, where $\mathbf{S}_j = \mathcal{F}_{p_j}$, $\mathbf{R} = \prod_{k=1}^r \mathbf{R}_k$, where $\mathbf{R}_k = \tilde{\mathcal{F}}_{q_k}$, and all algebras $\mathbf{S}_j$, $\mathbf{R}_k$ are local. Define the equivalence relation on the indices $1,2,\dots,r$ by putting $k \sim l$ if and only if q_k and $\nu(q_l)$ belong to the same component of $\tilde{C}$. Note that in this case $(k,j) \in \mathbf{J}$ means that $\nu(q_k) = p_j$. In this case put $\kappa(k,j) := q_k$. Now the following result is quite obvious.

Proposition 5.1. *In the situation above* $[\mathbf{R},\mathbf{S},\kappa]$ *are special data, the special bimodule* $\mathsf{U}[\mathbf{R},\mathbf{S},\kappa]$ *is isomorphic to the bimodule* $\mathsf{U}(C)$

corresponding to the curve C via Proposition 4.5 and the notions of correct and semi-correct elements for these bimodules coincide.

6. Representation type of special bimodules

Now we are going to describe the representation types of special bimodules and hence the VB-types of rationally composed curves.

Step 6.1. *If the algebra $\mathbf{R}$ is non-semi-simple, then the shifting bimodule $\mathbf{U} = \mathbf{U}[\mathbf{R}, \mathbf{S}, \kappa]$ is correctly (hence also semi-correctly) wild.*

For a rationally composed curve C it means that, whenever it is not VB-wild, then for every singular point x, $\tilde{\mathcal{F}}_x \simeq \mathbf{k}^m$, where $m = \#(\nu^{-1}(x))$. In other words, all branches passing through x have different tangents.

Proof. Note first that there are no morphisms in A from (i, n) to (i, n') if $n > n'$. Hence, verifying the second condition of the definition of a strict element from Section 3 we may always suppose that $\sigma = 1$.

Let $\mathbf{F} := \mathbf{k}\langle z_1, z_2 \rangle$ be a free $\mathbf{k}$-algebra with 2 generators, $B := \mathbf{S}^2 \otimes \mathbf{F}$ and

$$A := \bigoplus_{i=1}^{t} ((i, n) \oplus (i, n+1)) \otimes \mathbf{F}$$

for some (arbitrary) $n \in \mathbb{Z}$. Then

$$\begin{aligned} \mathsf{U}^{\mathbf{F}}(B, A) &\simeq \bigoplus_{(i,j)\in \mathbf{J}} U_{ij} \otimes \mathbf{F}, \quad \text{where} \\ U_{ij} &= \operatorname{Hom}_{\mathbf{S}}(\mathbf{S}_j^2, \mathbf{R}_i)(n) \oplus \operatorname{Hom}_{\mathbf{S}}(\mathbf{S}_j^2, \mathbf{R}_i)(n+1). \end{aligned}$$

Put $\mathbf{R}_{ij} := \mathbf{R}_i \mathbf{S}_j$. It is a direct summand of $\mathbf{R}_i$ and, evidently, $\operatorname{Hom}_{\mathbf{S}}(\mathbf{S}_j, \mathbf{R}_i) \simeq \mathbf{R}_{ij}$. We identify the elements of $U_{ij} \otimes \mathbf{F}$ with the 2×2 matrices having the entries from $\mathbf{R}_{ij} \otimes \mathbf{F}$,

$$u_{ij} := \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where the first row corresponds to the (n)-component and the second one to the $(n+1)$-component.

Choose some index i_0 such that $\mathbf{R}_{i_0}$ is non-semi-simple and some minimal ideal $I \subseteq \operatorname{rad} \mathbf{R}_{i_0 j_0}$ for some j_0. As $\mathbf{k}$ is algebraically closed, $I = \mathbf{k} \cdot \alpha$ for some element α. As $\mathbf{S} \not\supset I$, $\mathbf{S} \cap I = \{0\}$. For simplicity, change κ such that $\kappa(i_0, j_0) = (1:0)$.

Consider the element $u \in \mathsf{U}^{\mathbf{F}}(B, A)$ whose components are the following:

$$\begin{aligned} u_{i_0 j_0} &= \begin{pmatrix} 1 & \alpha \\ \alpha z_1 & 1 + \alpha z_2 \end{pmatrix}, \\ u_{ij} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for } (i,j) \neq (i_0, j_0). \end{aligned}$$

It is obviously correct as $|u|$ coincides modulo radical with the natural embedding $\mathbf{S}^2 \otimes \mathbf{F} \to \mathbf{R}^2 \otimes \mathbf{F}$. We prove that it is strict.

Let N be an m-dimensional $\mathbf{F}$-module, Z_k ($k = 1, 2$) be the matrices describing the action of z_k on the vector space N. Then $u(N)$ can be identified with the family of $2m \times 2m$ matrices $u_{ij}(Z_1, Z_2)$ with entries in $\mathbf{R}_{ij}$, where:

$$\begin{aligned} u_{i_0 j_0}(Z_1, Z_2) &= \begin{pmatrix} E & \alpha E \\ \alpha Z_1 & 1 + \alpha Z_2 \end{pmatrix}, \\ u_{ij}(Z_1, Z_2) &= \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} \quad \text{for } (i, j) \neq (i_0, j_0), \end{aligned}$$

(E is the identity matrix). If $u(N') := \{\, u_{ij}(Z'_1, Z'_2) \,\}$ is the element corresponding to another $\mathbf{F}$-module N', any morphism $f : u(N) \to u(N')$ is given by a family of matrices $\{\, f_{ij}, g_j \mid (i, j) \in \mathbf{J},\ 1 \leq j \leq t \,\}$,

$$\begin{aligned} f_{ij} &= \begin{pmatrix} a_i & 0 \\ \xi_{ij} b_i + \eta_{ij} c_i & d_i \end{pmatrix}, \\ g_j &= \begin{pmatrix} s_{1j} & s_{2j} \\ s_{3j} & s_{4j} \end{pmatrix}, \end{aligned}$$

where a_i, b_i, c_i, d_i are $m \times m$ matrices over $\mathbf{k}$ and s_{kj} ($k = 1, 2, 3, 4$) are $m \times m$ matrices over $\mathbf{S}_j$ satisfying the relations:

$$f_{ij} u_{ij}(Z_1, Z_2) = u_{ij}(Z'_1, Z'_2) g_j \text{ for all } (i, j) \in \mathbf{J}. \tag{2}$$

In particular, for $(i, j) = (i_0, j_0)$ these relations give for the first row:

$$\begin{aligned} a_{i_0} &= s_{1j_0} + \alpha s_{3j_0}, \\ \alpha a_{i_0} &= s_{2j_0} + \alpha s_{4j_0}, \end{aligned}$$

whence $a_{i_0} = s_{1j_0}$, $s_{2j_0} = 0 = \alpha s_{3j_0}$ and $\alpha a_{i_0} = \alpha s_{4j_0}$ as $\mathbf{S} \cap \mathbf{R}\alpha = \{\, 0 \,\}$. Now the relations (2) for $(i, j) = (i_0, j_0)$ and the second row give

$$\begin{aligned} b_{i_0} + \alpha d_{i_0} Z_1 &= \alpha Z'_1 s_{1j_0} + s_{3j_0}, \\ \alpha b_{i_0} + d_{i_0} + \alpha d_{i_0} Z_2 &= s_{4j_0} + \alpha Z'_2 s_{4j_0}. \end{aligned}$$

This gives $b_{i_0} = s_{3j_0}$, whence both of them are zero as $\alpha s_{3j_0} = 0$; $d_{i_0} = s_{4j_0}$, whence both of them equal a_{i_0}; at last $d_{i_0} Z_1 = Z'_1 s_{1j_0}$ and $d_{i_0} Z_1 = Z'_1 s_{4j_0}$. Thus

$$f_{i_0, j_0} = g_{j_0} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \quad \text{where} \quad a = a_{1i_0}$$

and $a Z_k = Z'_k a$ ($k = 1, 2$), i.e., $a \in \mathrm{Hom}_{\mathbf{F}}(N, N')$.

The relations (2) for other values of (i, j) easily imply that

$$f_{ij} = g_j = \begin{pmatrix} a & 0 \\ h_{ij} & a \end{pmatrix}, \quad \text{where} \quad h_{ij} = \xi_{ij} b_i + \eta_{ij} c_i.$$

We have only to take into account the "connectedness" condition *3* from the definition of special data. Now it is evident that the functor $u \mapsto u(N)$ maps indecomposable to indecomposable and non-isomorphic to non-isomorphic. Therefore, U is correctly wild. $\square$

For the rest of the section suppose $\mathbf{R}$ (and hence $\mathbf{S}$) to be semi-simple. In this case condition *2* from the definition of correct or semi-correct elements means that $|u|$ is a monomorphism. For each $i = 1, \dots, t$, let m_i be the number of such j that $(k,j) \in \mathbf{J}$ for some $k \in c_i$. On the other hand, for each $j = 1, \dots, s$, let l_j be the number of such k that $(k,j) \in \mathbf{J}$. Note that we always have $l_j > 1$. Otherwise, there were a unique index k such that $(k,j) \in \mathbf{J}$ with $\dim \mathbf{R}_k = 1$. But then $\mathbf{S}_j \subseteq \mathbf{R}_k$, hence, $\mathbf{S}_j = \mathbf{R}_k$, which contradicts condition 1 of the definition of special data. On the other hand, as $\mathbf{S}_j \mathbf{S}_{j'} = 0$ for $j \neq j'$, $(k,j) \in \mathbf{J}$ implies $(k,j') \notin \mathbf{J}$ for $j \neq j'$.

Step 6.2. *If $l_{j_0} > 2$ for some j_0, the shifting bimodule $\mathbf{U}$ is correctly (hence also semi-correctly) wild.*

For a rationally composed curve C it means that, whenever C is not wild, $\#(\nu^{-1}(x)) = 2$ for every singular point x. In other words, x is a simple node, so it accomplishes the proof of Proposition 2.5.

Proof. Suppose there are 3 indices k_1, k_2, k_3 such that $(k_q, j_0) \in \mathbf{J}$ for $q = 1, 2, 3$. Put $B := \mathbf{S}^4 \otimes \mathbf{F}$ and

$$A := \left(\bigoplus_{i=1}^{t} \oplus_{m=0}^{3} (i, n+m) \right) \otimes \mathbf{F}$$

for some (arbitrary) $n \in \mathbb{Z}$. Note that now $\mathrm{Hom}_{\mathbf{S}}(\mathbf{S}_j, \mathbf{R}_k) \simeq \mathbf{k}$ for $(k,j) \in \mathbf{S}$. Hence, the elements from $\mathsf{U}^{\mathbf{F}}(B, A)$ can be regarded as sets of 4×4 matrices $\{ u_{kj} \mid (k,j) \in \mathbf{J} \}$, u_{kj} having entries in $\mathbf{R}_k \otimes \mathbf{F}$.

Now take u such that all its components are identity matrices except of the next two ones:

$$u_{k_1 j_0} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$u_{k_2 j_0} = \begin{pmatrix} 1 & 1 & z_1 & z_2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

Just as in Step 6.1, some easy straightforward calculation, which we omit, shows that u is indeed a strict element. $\square$

Suppose now that $l_j \le 2$ (hence $l_j = 2$) for each j. Then the pair $\mathbf{S} \subset \mathbf{R}$, together with the equivalence relation $\sim$, can be completely described by its *diagram* $\Delta = \Delta[\mathbf{R}, \mathbf{S}, \sim]$. The vertices of Δ are just the indices $i = 1, \dots, t$, its edges are the indices $j = 1, \dots, s$ and an edge j is incident to a vertex i if and only if $(k, j) \in \mathbf{J}$ for some $k \in c_i$. In case when j is incident to a unique i (then, of course, $\dim \mathbf{R}_i = 2$), consider j as a *loop* at the vertex i.

Of course, given any graph Δ, non-oriented but possibly with loops and (or) multiple edges, we can restore some pair $\mathbf{S} \subset \mathbf{R}$ and an equivalence relation $\sim$ (with semi-simple $\mathbf{R}$ and all $l_j = 2$) such that Δ is just their diagram. So, to obtain a special data of this kind, we need only a graph and a function κ. Therefore we call such data and the corresponding bimodule *graphical.*

As we have already noted, graphical data correspond to *projective configurations* (cf. Section 2). Moreover, if a graphical bimodule U with the graph Δ corresponds to a projective configuration C, then Δ coincides with the dual graph of C. So the next Step accomplishes the proof of Proposition 2.7 and hence of Theorem 2.8. Moreover, the description of all indecomposable elements in Cases *1* and *2* below also gives a description of indecomposable sheaves on the corresponding curves. One can easily see that this description coincides with Theorems 2.10 and 2.11.

Step 6.3. *Graphical data with the diagram Δ are:*

1. *Finite (hence also semi-correctly and correctly finite) if Δ is a Dynkin diagram of type A, i.e., a chain.*
2. *Tame (hence also semi-correctly and correctly tame) if Δ is an extended Dynkin diagram of type $\tilde{\mathrm{A}}$, i.e., a cycle (possibly, one vertex with one loop). Moreover, in this case they are correctly (hence, semi-correctly and "absolutely") unbounded.*
3. *Correctly (hence also semi-correctly and "absolutely") wild otherwise.*

Note that in the first two cases $m_i \le 2$ for each $i = 1, 2, \dots, r$ (in particular, each equivalence class c_i consists of at most two elements). Hence, the graph Δ determines the bimodule U up to isomorphism (as any pair of points of a projective line can be moved to any other pair by a collineation).

Proof. Case *1* is very simple. Put $\mathbf{R}_{kj} := \mathbf{R}_k \mathbf{S}_j$ for $(k, j) \in \mathbf{J}$. All these spaces are one-dimensional, hence, we may identify them with $\mathbf{k}$. Moreover, $s = t - 1$ and we can arrange the indices in such a way that the edge i is incident to the vertices i and $i + 1$ for each $i = 1, \dots, t-1$. Then one checks immediately that the indecomposable elements of $\mathsf{El}_{sc}(\mathsf{U})$ are in one-to-one correspondence with the finite sequences ν of integers of the form:

$$\nu = (m, r; \delta_0, \delta_1; n_1, n_2, \dots, n_r),$$

where $1 \le m \le t$, $0 \le r \le t - m + 1$ and both δ_0 and δ_1 are either 0 or 1, while n_i is arbitrary; moreover, if $m = 1$, then $\delta_0 = 1$, if $r = t - m + 1$, then $\delta_1 = 1$, and if $r = 0$, then $\delta_0 = \delta_1 = 0$. Namely, the element $u = u(\nu)$ corresponding to such a sequence lies in $\mathsf{U}(B, A)$, where

$$\begin{aligned} A &:= \oplus_{i=1}^{r} (m + i - 1, n_i), \\ B &:= \oplus_{i=\delta_0}^{r+1-\delta_1} \mathbf{S}_{m+i-1} \end{aligned} \tag{3}$$

and all components of u are equal to 1. Certainly, such an element equals $\sigma_m^{n_0} \sigma_{m+1}^{n_1} \dots \sigma_{m+r-1}^{n_{m+r-1}} u'$, where u' corresponds to the sequence ν' with the same values of m, r, δ_0, δ_1 but with all $n_i' = 0$. So the bimodule U is finite. Note that the element $u(\nu)$ is semi-correct if and only if either $\delta_0 = 0$ or $m = 1$ and, moreover, either $\delta_1 = 0$ or $r = t - m$. This element is correct if and only if $m = 1$ and $r = t$.

Case *2*. Here $s = t$ and we can arrange the indices in such a way that the edge i is incident to the vertices i and $i + 1$ (we define the vertex $t + k$ to be the same as the vertex k). This case fits into the framework of "bunches of chains" (cf. [2] or Appendix A).

Namely, in our case the underlying index set $\mathbf{I}$ is just the set of pairs $\{ (i, i), (i, i + 1) \mid i = 1, 2, \dots, t \}$. We set, for each pair $(i, j) \in \mathbf{I}$, $E_{ij} = \mathbb{Z}$ (with natural ordering) and $F_{ij} = \{ o_{ij} \}$ (a single element). To distinguish the elements from different E_{ij} we write them in the form $n(i, j)$ with $n \in \mathbb{Z}$, $(i, j) \in \mathbf{I}$. The equivalence relation on the union of all E_{ij} and F_{ij} is given by the rule: $n(i, j) \sim n(i, j')$ and $o_{ij} \sim o_{i'j}$ for all possible values of i, i', j, j'. Then one can verify that the bimodule corresponding to this bunch of chains coincides with the graphical bimodule corresponding to the graph Δ. Hence, we can use the results of [2] (cf. also Appendix A) to obtain a complete list of indecomposable elements. Taking into account the shape of this bunch, we can rearrange strings and bands defined in Appendix A as follows.

String representations are very much like the representations of the previous case. They correspond to *string data*, i.e., sequences of integers:

$$\nu = (m, r; \delta_0, \delta_1; d_0, d_1, \dots, d_{r-1}),$$

where $1 \le m \le t$, $r \ge 0$ and both δ_0 and δ_1 is either 0 or 1, while d_i are arbitrary. The corresponding elements $u(\nu)$ lies in $\mathsf{U}(A, B)$, where A and B are defined by the same formulae (3) if we put $(nt + j, k) := (j, k)$ and $\mathbf{S}_{nt+j} := \mathbf{S}_j$ for each n. The non-zero components of u are only those belonging to $\mathsf{U}(\mathbf{S}_{m+i-1}, (m + i - 1, d_i))$, except for $i = 1$ if $\delta_0 = 1$, and $\mathsf{U}(\mathbf{S}_{m+i}, (m + i - 1, d_i))$, except for $i = r$ if $\delta_1 = 1$. All these components are equal to 1. A string element is semi-correct if and only if $r > 0$ and $\delta_0 = \delta_1 = 0$. It is never correct.

Band representations correspond to *band data*, i.e., triples $\mathbf{b} = (\mathbf{d}, m, \lambda)$, where d is a positive integer, $\lambda \in \mathbf{k}^*$ and $\mathbf{d}$ is a sequence of integers $(d_1, d_2, \dots, d_{tr})$ which is t-*aperiodic*, i.e., is not a multiple self-concatenation of a shorter sequence whose length is also divisible by t.

The corresponding element $u_{\mathbf{b}} = u(\mathbf{d}, m, \lambda)$ lies in $\mathsf{U}(A, B)$ for

$$A := \oplus_{i=1}^{tr} m(i, d_i)\,,$$
$$B := \oplus_{i=1}^{tr} m\mathbf{S}_i\,,$$

Its non-zero components are those belonging to $\mathsf{U}(\mathbf{S}_i, (i, d_i))$ and $\mathsf{U}(\mathbf{S}_{i+1}, (i, d_i))$, which are given by unit matrices of dimension m, and the component belonging to $\mathsf{U}(\mathbf{S}_1, (tr, d_{tr}))$, which is given by the Jordan cell of dimension m with the eigenvalue λ. All band elements are correct.

All string elements are pairwise non-isomorphic. The isomorphic band elements are those corresponding to the triples $(\mathbf{d}, m, \lambda)$ and $(\mathbf{d}', m, \lambda)$, where $\mathbf{d}'$ is a t-*cyclic* permutation of $\mathbf{d}$, i.e., $\mathbf{d}' = (d_{tl+1}, d_{tl+2}, \dots, d_{tl})$.

As string and band elements exhaust all indecomposable elements of U, this bimodule is tame. Moreover, the band elements $u(\mathbf{d}, 1, T) \in \mathsf{El}(\mathsf{U}, \mathbf{k}[T, T^{-1}])$ form a parametrising family of elements of U. But their number grows with r. For instance, consider the band elements corresponding to sequences $\mathbf{d}_{r,l}$ of length rt having all components 0 except the first and the $(tl+1)$-st ones, which are equal to 1. If $l \le [\frac{r}{2}]$ these band elements are pairwise non-isomorphic even up to shift and there are $[\frac{r}{2}]$ of them. Hence, U is correctly unbounded.

Case *3*. Suppose that Δ is neither a chain nor a cycle. Then it contains a vertex i_0 incident either to at least three edges or to a loop and to at least one more edge. We consider the former situation (the latter can be treated in the same way). Let j_k $(k = 1, 2, 3)$ be three edges incident to the vertex i_0. Denote the second end of the edge j_k by i_k (some of i_k may be equal). We put again $B := \mathbf{S} \otimes \mathbf{F}$,

$$A := \left(\bigoplus_{i=1}^{t} \oplus_{k=0}^{3} (i, n+k) \right) \otimes \mathbf{F}$$

for some (arbitrary) n and take u, all whose components are identity matrices except of the next three ones:

$$u_{i_0j_1} = u_{i_0j_2} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$u_{i_0j_3} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & z_1 \\ 1 & 1 & 0 & z_2 \end{pmatrix}.$$

Again some easy straightforward calculation, which we omit, shows that u is a strict element. □

Appendix A. Bunches of chains

Here we recall some definitions and results related to the *bunches of chains* considered by Bondarenko in [2]. We rearrange the definitions to make them more convenient for our purpose and consider only the case of *chains* (not semi-chains) as we need only this one and it is technically much easier.

Definition A.1. A *bunch of chains* $\mathbf{C} = \{\,\mathbf{I}, E_i, F_i, \sim\}$ is defined by the following data:

1. A set $\mathbf{I}$ of indexes.
2. Two chains (i.e., linear ordered sets) E_i and F_i given for each $i \in \mathbf{I}$.
   Put $\mathbf{E} := \bigcup_{i\in\mathbf{I}} E_i$, $\mathbf{F} := \bigcup_{i\in\mathbf{I}} F_i$ and $|\mathbf{C}| := \mathbf{E} \cup \mathbf{F}$.
3. An equivalence relation $\sim$ on $|\mathbf{C}|$ such that each equivalence class consists of at most 2 elements.

We also write $a - b$ if $a \in E_i$, $b \in F_i$ or vice versa (with the same index i). Moreover, we consider the ordering on $|\mathbf{C}|$, which is just the union of all orderings on E_i and F_i (i.e., $a < b$ means that a, b belong to the same chain E_i or F_i and $a < b$ in this chain).

If a bunch of chains $\mathbf{C} = \{\,\mathbf{I}, E_i, F_i, \sim\}$ is given, define the corresponding category $\mathsf{A} = \mathsf{A}(\mathbf{C})$ and the corresponding A-bimodule $\mathsf{U} = \mathsf{U}(\mathbf{C})$ as follows:

- The objects of A are the equivalence classes of $|\mathbf{C}|$ with respect to $\sim$.
- If x, y are two such equivalence classes, a basis of the morphism space $\mathsf{A}(x, y)$ consists of elements p_{ab} with $b \in x$, $a \in y$, $b < a$ and, if $x = y$, the identity morphism 1_x.
- The multiplication is given by the rule: $p_{ab}p_{bc} = p_{ac}$ if $c < b < a$, while all other possible products are zeros.

- A basis of $\mathsf{U}(x,y)$ consists of elements u_{ab} with $a \in y \cap \mathbf{E}$, $b \in x \cap \mathbf{F}$.
- The action of A on U is given by the rule: $p_{ca}u_{ab} = u_{cb}$ if $a < c$; $u_{ab}p_{bd} = u_{ad}$ if $d < b$, while all other possible products are zeros.

The category of representations of the bunch $\mathbf{C}$ is then defined as the category $\mathsf{El}(\mathsf{U})$ of the elements of this bimodule. One can easily verify that this definition gives just the same representations as the definition from [2]. Note that in [2] a more general situation was investigated, but we need only this case, which is essentially simpler than the general one. The following result is the specialisation of the description of the representations given in [2] to our case, though it can be obtained directly using the same recursive procedure. First define some combinatorial objects called "strings" and "bands."

Definitions A.2. Let $\mathbf{C} = \{\, \mathbf{I}, E_i, F_i, \sim \}$ be a bunch of chains.

1. A $\mathbf{C}$-*word* is a sequence $w = a_0r_1a_1r_2a_2 \dots r_ma_m$, where $a_k \in |\mathbf{C}|$ and each r_k is either $\sim$ or $-$, such that, for all possible values of k:
   (a) $a_{k-1}r_ka_k$ in $|\mathbf{C}|$.
   (b) $a_k \neq a_{k+1}$ and $r_k \neq r_{k+1}$.
   Possibly $m = 0$, i.e., $w = a$ for some $a \in \mathbf{C}$.
2. If $a_m = a_0$, $r_1 = \sim$ and $r_m = -$ call the word w a $\mathbf{C}$-*cycle.* *Note that in this case m is always even.*
3. Call a $\mathbf{C}$-word *full* if, whenever a_0 is not a unique element in his equivalence class, then $r_1 = \sim$ and whenever a_m is not a unique element in his equivalence class, then $r_m = \sim$.
4. Call a $\mathbf{C}$-cycle $w = a_0r_1a_1r_2a_2 \dots r_ma_m$ *aperiodic* if the sequence $a_0r_1a_1r_2a_2 \dots r_m$ cannot be written as a multiple self-concatenation $vv\dots v$ of a shorter sequence v.
5. We say that an equivalence class x *occurs* in a word w if w contains a sub-word a in case $x = \{\, a \,\}$ is a singleton, or either a sub-word $a \sim b$ or a sub-word $b \sim a$ in case $x = \{\, a, b \,\}$ with $a \neq b$. In the former case we say that this occurrence corresponds to the occurrence of a, while in the latter case we say that it corresponds to both the occurrence of a and to the occurrence of b. Denote by $\nu(x,w)$ the number of occurrences of x in w.

Definition A.3. For a $\mathbf{C}$-word $w = a_0r_1a_1r_2a_2 \dots r_ma_m$ call its $\sim$-*sub-word* any sub-word of the form $v = a \sim b$ as well as that of the form $v = a$, where $a \in \mathbf{C}$ is unique in its equivalence class. In the latter case put $|v| = \{\, a \,\}$, while in the former case put $|v| = \{\, a, b \,\}$. Denote by $[w]$ the collection of all $\sim$-sub-words of w.

Note that if w is a cycle it contains no entries $a \in \mathbf{C}$ such that a is unique in its equivalence class.

Definition A.4. For any full **C**-word $w = a_0 r_1 a_1 r_2 a_2 \dots r_m a_m$ define the corresponding *string representation* $u = u_s(w)$ of the bunch **C** as follows.

1. $u \in \mathsf{U}(A, A)$, where $A = \bigoplus_{v \in [w]} |v|$.
2. Suppose there is a sub-word $v_1 - v_2$ in w with $v_i \in [w]$. Let a be the right end of the word v_1 and b be the left end of the word v_2. Then $\mathsf{U}(A, A)$ has a direct summand $\mathsf{U}(|v_1|, |v_2|) \oplus \mathsf{U}(|v_2|, |v_1|)$ and we define the corresponding components of u to be $(0, u_{ab})$ if $a \in \mathbf{E}$ and $(u_{ba}, 0)$ if $a \in \mathbf{F}$.
3. All other components of u are defined to be zero.

Definition A.5. For any triple (w, d, λ), where w is an aperiodic **C**-cycle, d is a positive integer and $\lambda \in \mathbf{k}^* = \mathbf{k} \setminus \{0\}$, define the corresponding *band representation* $u = u_b(w, d, \lambda)$ of the bunch **C** as follows.

1. $u \in \mathsf{U}(A, A)$, where $A = \bigoplus_{v \in [w]} d|v|$.
2. Suppose there is a sub-word $v_1 - v_2$ in w with $v_i \in [w]$. Let a be the right end of the word v_1 and b be the left end of the word v_2. Then $\mathsf{U}(A, A)$ has a direct summand
$$\mathsf{U}(d|v_1|, d|v_2|) \oplus \mathsf{U}(d|v_2|, d|v_1|) \simeq \\ \mathrm{Mat}(d \times d, \mathsf{U}(|v_1|, |v_2|) \oplus \mathrm{Mat}(d \times d, \mathsf{U}(|v_2|, |v_1|)))$$
and we define the corresponding components of u to be $(0, u_{ab} I)$ if $a \in \mathbf{E}$ and $(u_{ba} I, 0)$ if $a \in \mathbf{F}$, where I denotes the identity matrix.
3. Let now v_1 be the last and v_2 be the first $\sim$-sub-word in w (they may coincide), a be the right end of the word v_1 and b be the left end of the word v_2. Then $\mathsf{U}(A, A)$ has a direct summand
$$\mathsf{U}(d|v_1|, d|v_2|) \oplus \mathsf{U}(d|v_2|, d|v_1|) \simeq \\ \mathrm{Mat}(d \times d, \mathsf{U}(|v_1|, |v_2|) \oplus \mathrm{Mat}(d \times d, \mathsf{U}(|v_2|, |v_1|)))$$
and we define the corresponding components of u to be $(0, u_{ab} J)$ if $a \in \mathbf{E}$ and $(u_{ba} J, 0)$ if $a \in \mathbf{F}$, where J denotes the Jordan cell of dimension d with the eigenvalue λ.
4. All other components of u are defined to be zero.

Theorem A.6. 1. *All representations $u_s(w)$ and $u_b(w, d, \lambda)$ defined above are indecomposable and each indecomposable representation of* **C** *is isomorphic to one of these representations.*

2. *The only possible isomorphisms between these representations are the following:*
   (a) $u_s(w) \simeq u_s(w')$ *if* $w = a_0 r_1 a_1 \dots r_m a_m$ *and* $w' = a_m r_m a_{m-1} \dots r_1 a_0$, *the reversed word.*
   (b) $u_b(w, d, \lambda) \simeq u_b(w', d, \lambda')$ *if* $w = a_0 r_1 a_1 \dots r_m a_m$, $w' = a_{2k} r_{2k+1} a_{2k+2} \dots r_{2k} a_{2k}$ *is a cyclic permutation of* w, *and* $\lambda' = \lambda$ *for* k *even, while for* k *odd* $\lambda' = \lambda^{-1}$.

(c) $u_b(w,d,\lambda) \simeq u_b(w',d,\lambda')$ *if* $w = a_0 r_1 a_1 \dots r_m a_m$, $w' = a_{2k+1} r_{2k+1} a_{2k} \dots r_{2k+2} a_{2k+1}$ *is a cyclic permutation of the reversed word, and* $\lambda' = \lambda$ *for* k *odd, while for* k *even* $\lambda' = \lambda^{-1}$.

Corollary A.7. *For any bunch of chains* $\mathbf{C}$ *the bimodule* $\mathsf{U}(\mathbf{C})$ *is tame (finite if there are no* $\mathbf{C}$*-bands at all). Moreover, a parametrising set for its elements consists of all band representations* $u_b(w,1,T) \in \mathsf{El}(\mathsf{U},\mathbf{k}[T,T^{-1})$.

Appendix B. Relation to finite dimensional algebras

There exists an amazing correspondence between rationally composed curves and some finite dimensional algebras. Suppose C is a rationally composed curve such that the algebra $\tilde{\mathcal{F}}$ is semi-simple. Then C can be completely described by its normalisation $\tilde{C}$, the set S of its singular points (just a finite set), its pre-image $\tilde{S} \subset \tilde{C}$ and the projection $\nu : \tilde{S} \to S$. Define the corresponding algebra $\mathbf{A} := \mathbf{A}(C)$ by its diagram $\Gamma := \Gamma(C)$ (quiver) and relations as follows.

Let $\tilde{C} = \cup_{i=1}^{t} C_i$, where $C_i \simeq \mathbb{P}^1$, $S = \{\, p_1, p_2, \dots, p_s \,\}$ and $\nu^{-1}(p_j) \cap C_i = \{\, q_{ijk} \mid 1 \le k \le m_{ij} \,\}$ for each j . Then Γ has $2t+s$ vertices $\{\, a_i, b_i, c_j \mid 1 \le i \le t,\ 1 \le j \le s \,\}$. There are two arrows $x_i, y_i : a_i \to b_i$ and m_{ij} arrows $z_{ijk} : c_j \to a_i$. The defining relations for these arrows are:

$$\xi_{ijk} x_i z_{ijk} = \eta_{ijk} y_i z_{ijk}, \text{ where } q_{ijk} = (\xi_{ijk} : \eta_{ijk})\,.$$

Then the following theorem holds.

Theorem B.1. *The algebra* $\mathbf{A}(C)$ *is tame (wild) if and only if the curve* C *is VB-tame (VB-wild).*

The proof is quite easy and straightforward, so we only sketch it. For each $i = 1, 2, \dots, t$ the sub-algebra generated by x_i, y_i is a Kronecker algebra, for which all representations are known. So we can reduce all these arrows and then get a bimodule problem for the remainder. The observation is that this bimodule problem "almost coincides" with that corresponding to the curve C, as in Section 5. At least, it is not simpler, which implies the "only if" part of the theorem. But if C is not VB-wild, the points q_{ijk} can always be chosen as $(1:0)$ or $(0:1)$. In this case the resulting algebra $\mathbf{A}$ is special biserial, hence, tame [16].

We hope that there should be another explanation of this fact, which does not involve explicit calculations, but at the moment we cannot even predict of which kind such explanation could be.

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