Cancellation of Hyperbolic Forms and Topological Four-Manifolds

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This is the second in a series of three papers about cancellation problems (referred to as [I], [II] and [III]). The general questions for this part are:

(i) if M, M', N are quadratic modules with $M \perp N \cong M' \perp N$, is $M \cong M'$?

(ii) if X, Y are topological 4-manifolds with finite fundamental group and $X \sharp (S^2 \times S^2)$ homeomorphic to $Y \sharp (S^2 \times S^2)$, is X homeomorphic to Y?

In part [III], the techniques developed here will be used to study smooth structures on algebraic surfaces with finite fundamental group, extending the results of [**12**]. We also obtain classification theorems for four-manifolds (up to homeomorphism) in some special cases, extending the results of [8] and [10], [11].

We begin by stating some of our algebraic results (proved in §1). The general stable range condition for cancellation of hyperbolic forms over orders (e.g. integral group rings $\mathbf{Z}\pi$, π a finite group) is free hyperbolic rank ≥ 2 [**2**, (3.6), p.238]. This is a special case of the general results on cancellation over noetherian rings due to A. Bak [1], H. Bass [**2**], and L. N. Vaserstein [24]. It is also known that this assumption can be weakened to a local rank condition at all primes (compare [**3**, Thm. 1]).

Let R be a Dedekind domain and F its field of quotients. and recall that a *lattice* over an R-order A is an A-module which is projective as an R-module. Let A and B be orders in separable algebras over F [**6**, 71.1, 75.1], and suppose that there is a surjective ring homomorphism $\epsilon : A \to B$. We obtain an improvement in the stable range, assuming some local information about the lattices. The problem is to show that certain groups of elementary automorphisms act transitively on unimodular elements or hyperbolic planes in quadratic modules, and as in [I] our result gives information about transitivity over A from corresponding information over B. The arguments in §1 are modelled closely on the ones given in [2].

In [I] we introduced the following definition: a finitely generated A-module L has *(A, B)-free rank ≥ 1 at a prime* $\mathfrak{p} \in R$, if there exists an integer r such that $(B^r \oplus L)_\mathfrak{p}$ has free rank ≥ 1 over $A_\mathfrak{p}$. Here $A_\mathfrak{p}$ denotes the localized order $A \otimes R_{(\mathfrak{p})}$. In the extreme case $B = 0$, this is just the condition that $L_\mathfrak{p}$ has a free direct summand. In the other extreme case $A = B$, there is no condition on L.

Similarly, we will say that a quadratic module V has *(A, B)-hyperbolic rank ≥ 1 at a prime* $\mathfrak{p} \in R$ if there exists an integer r such that $(H(B^r) \oplus V)_\mathfrak{p}$ has free hyperbolic rank ≥ 1 over $A_\mathfrak{p}$.

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The other terms used in the statement below are defined precisely in §1 or in [**2**, pp. 80, 87]. Note in particular that a *unitary module* is a (λ, Λ)-quadratic form on a finitely-generated *projective* A-module. A totally isotropic submodule is one on which the quadratic form is identically zero.

Theorem A: *Let V be a (λ, Λ)-quadratic module over a unitary (R, λ) algebra (A, Λ) and put $(M, [h]) = V \perp H(A)$. Suppose that there exists a surjection of orders $\epsilon : A \to B$ such that V has (A, B)-hyperbolic rank ≥ 1 at all but finitely many primes. If $U_2(A)$ acts transitively on the set of unimodular elements in $H(B \oplus B)$ of fixed length, then for any unitary module N, $M \perp N \cong M' \perp N$ implies $M \cong M'$.*

An important special case for the geometric applications is $B = \mathbf{Z}$. We check that for $B = \mathbf{Z}$, the condition on "transitive action" in Theorem A is satisfied (1.21), hence can be omitted from the statement.

The topology of 4-dimensional manifolds can be studied by stabilizing with connected sums of $S^2 \times S^2$ away from the boundary (compare [**26**], [**5**], [**14**]). The connected sum of X with r copies of $S^2 \times S^2$ is denoted $X \sharp r(S^2 \times S^2)$. To recover information about the original manifold we must prove a "cancellation theorem".

Theorem B: *Let X and Y be closed oriented topological 4-manifolds with finite fundamental group. Suppose that the connected sum $X \sharp r(S^2 \times S^2)$ is homeomorphic to $Y \sharp r(S^2 \times S^2)$. If $X = X_0 \sharp (S^2 \times S^2)$, then X is homeomorphic to Y.*

This was proved in [**11**, (1.3)b] for closed manifolds with finite cyclic fundamental group (with certain restrictions on w_2). Note that the assumption that X splits off one $S^2 \times S^2$ cannot be omitted in general. There are, for example, even simply-connected closed 4-manifolds which are stably homeomorphic but not homeomorphic because they have non-isometric intersection forms. Examples of distinct but stably homeomorphic manifolds with the same equivariant intersection form were constructed in [**16**].

The analogous result where $M = M_0 \sharp 2(S^k \times S^k)$ holds for all compact topological $2k$-manifolds ($k \geq 2$) with finite fundamental group, without assumption on the boundary (compare [**8**]). This was proved for topological 4-manifolds in [**11**, (1.3)a]. Essentially the same argument in higher dimensions proves the result for smooth or PL manifolds of dimension $2k \geq 6$.

For special fundamental groups we have a result similar to Theorem B about manifolds with non-empty boundary (see §3).

Theorem B$'$: *Let M and N be compact oriented topological 4-manifolds with finite fundamental group, and let $A = \mathbf{Z}[\pi_1(M)]$. Suppose that the interior connected sum $M \sharp r(S^2 \times S^2)$ is homeomorphic to $M \sharp r(S^2 \times S^2)$ by a homeomorphism inducing the identity on the boundary. If (i) $U_2(A)$ acts transitively on the set of unimodular elements in $H(A \oplus A)$ of a fixed length, (ii) $L_5^s(A) = 0$, and (iii) $M = M_0 \sharp (S^2 \times S^2)$; then the identity map on the boundary extends to a homeomorphism of M with N.*

Again the similar result holds for smooth or PL manifolds in higher dimensions. In the 1-connected case assumptions (i) and (ii) are satisfied (see Lemma 1.21), and the results of [**4**, 5.6] show that assumption (iii) can not be dropped or replaced by assuming that the intersection form of M is isomorphic to that of N.

To obtain the geometric applications, the algebraic results are combined with topological surgery in dimension four due to M. Freedman [**9**], and the technique of C. T. C. Wall [**26**] and S. Cappell and J. Shaneson [5] for constructing diffeomorphisms of 4–manifolds (see §2). The same methods in various geometric contexts give further results which we list below.

Applications:

(i) The conditions in Theorem B$'$ on $\mathbf{Z}[\pi_1(M)]$ are restrictive but do hold for fundamental groups $\pi = \rho \times \sigma$, where ρ has odd order and σ is a cyclic 2–group. It can be applied to classify manifolds with a prescribed boundary and fundamental group of this form (see §3):

Corollary 3.6: *Let M and N be compact oriented topological 4–manifolds with $\pi_1(M) = \rho \times \sigma$. Suppose that the interior connected sum $M \sharp r(S^2 \times S^2)$ is homeomorphic to $M \sharp r(S^2 \times S^2)$ by a homeomorphism inducing the identity on the boundary. If $M = M_0 \sharp (S^2 \times S^2)$, then the identity map on the boundary extends to a homeomorphism of M with N.*

(ii) By applying Theorem B$'$ to the case where the manifold has cyclic fundamental group and lens space boundary, we obtain information about the existence and uniqueness of locally flat simple embeddings of 2–spheres in a 1-connected 4-manifold N. These problems were studied in [**17**] for homology classes of odd divisibility. For the notation, see §4.

Theorem 4.5: *Let N be a closed 1-connected topological 4-manifold.*
i) Let $x \in H_2(N; \mathbf{Z})$ be a homology class of divisibility $d \neq 0$. Then x can be represented by a simple locally flat embedded 2-sphere in N if and only if

$$KS(N) = (1/8)(\sigma(N) - x \cdot x) \pmod 2$$

when x is a characteristic class, and if

$$b_2(N) \geq \max_{0 \leq j < d} |\sigma(N) - 2j(d-j)(1/d^2)x \cdot x|.$$

ii) Any two locally flat simple embeddings of S^2 in N representing the homology class x are ambiently isotopic if $b_2(N) > |\sigma(N)| + 2$ and

$$b_2(N) > \max_{0 \leq j < d} |\sigma(N) - 2j(d-j)(1/d^2)x \cdot x|.$$

(iii) Another geometric problem which has been studied recently [**7**], [**30**], is the classification of pseudo–free actions (i.e. semi–free with isolated fixed points) of finite cyclic groups on 1–connected 4–manifolds. Here we assume that the fixed-point set of the action is *non-empty*: free actions, or equivalently 4-manifolds with finite cyclic fundamental group, will be classified in **[III]**.

In Corollary 4.1 we improve on the results of [**30**, 3.1], [**31**, Thm. A]:

Corollary 4.1: *Let M be a closed, oriented, simply-connected topological 4-manifold. Let G be a finite cyclic group acting locally linearly and pseudo-freely on M, preserving the orientation, with M^G non-empty. Let M_0 denote the complement of a set of disjoint open G-invariant 4-disks around the fixed points, and assume that $M_0/G = W \sharp (S^2 \times S^2)$, where $\partial W = \partial(M_0/G)$. Then the action (M, G) is classified up to equivariant homeomorphism by the local fixed-point data, the signature, type, and Euler characteristic of M and the Kirby-Siebenmann invariant of M_0/G.*

The "local fixed-point data" is the equivalence class of pairs consisting of the tangential G-representations at the fixed points together with, when M is spin and $|G|$ is even, a preferred set of spin structures on the lens spaces bounding M_0/G. If M is spin and $|G|$ is even, then M_0/G has two spin structures whose restrictions to $\partial(M_0/G)$ give the preferred set.

The "type" of M is the parity, even or odd, of its intersection form. We also remark that $KS(M_0/G) = KS(M_0) = KS(M)$ when G has odd order, since connected sum with the Chern manifold changes the $\mathbf{Z}/2$-valued Kirby-Siebenmann invariant.

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§1: Cancellation of Hyperbolic Forms

We adopt the notation and conventions of Bass in [**2**, pp.61-90, 233] for (λ, Λ)-quadratic modules $(M, [h])$ over a unitary (R, λ)-algebra (A, Λ). For our geometric applications it is convenient to introduce also (λ, Λ)-Quadratic modules. By this we mean triples $(M, \langle -, - \rangle, [q])$, where $[q] : M \to A/\Lambda$ is a (λ, Λ)-quadratic form (see [**2**, pp. 80-81]) and $\langle x, y \rangle = q(x+y) - q(x) - q(y)$ is the associated hermitian form. There is a functor from the category of (λ, Λ)-quadratic modules to the category of (λ, Λ)-Quadratic modules, induced by setting $[q](x) = [h(x, x)]$. This functor is an equivalence of categories when M is a projective A-module. However the second notion is the one usually encountered in geometric applications as a quadratic refinement of the intersection form on the kernel of w_2. The quantity $[q](x) = [h(x, x)]$. is referred to as the *length* of x.

A *unitary* module is a non-singular (λ, Λ)-quadratic form on a finitely generated projective A module. Since R is a Dedekind domain, $X = \max(R_0)$ has dimension $d = 1$, where $R_0 \subseteq R$ is the subring generated by all norms $t\bar{t}$ $(t \in R)$. Note that $\lambda\bar{\lambda} = 1$. The form parameter Λ is *ample* at $\mathfrak{m} \in X$ if given a, $b \in A[\mathfrak{m}]$, the semisimple quotient of $A_{\mathfrak{m}}$ there exists $r \in \Lambda[\mathfrak{m}]$ such that

$$(1.1) \qquad A[\mathfrak{m}](a + rb) = A[\mathfrak{m}]a + A[\mathfrak{m}]b.$$

In [2, §2, p.218ff] there is a discussion of this condition. If $R = \mathbf{Z}$ and $\Lambda = \{a - \lambda\bar{a} | a \in A\}$, the minimal form parameter, then Λ is not ample at any prime when $\lambda = 1$ and Λ is not ample at 2 if $\lambda = -1$. Let $\mathfrak{A}_\Lambda \subseteq R_0$ be the ideal such that Λ is ample at all $\mathfrak{m} \notin V(\mathfrak{A}_\Lambda) = \{\mathfrak{p} \in X \mid \mathfrak{A}_\Lambda \subseteq \mathfrak{p}\}$, and d_Λ the dimension of the closed set $V(\mathfrak{A}_\Lambda)$ in X. Note that $d_\Lambda \leq 1$ for all Λ, and $d_\Lambda \leq 0$ when Λ is ample at all but finitely many primes.

If $(M, [h])$ is any (λ, Λ)-quadratic module over A [2, p.80], then a *transvection* [2, p.91] is a unitary automorphism $\sigma = \sigma_{u,a,v} : M \to M$ given by

$$\sigma(x) = x + u\langle v, x\rangle - v\bar{\lambda}\langle u, x\rangle - u\bar{\lambda}a\langle u, x\rangle, \tag{1.2}$$

where u, $v \in M$ and $a \in A$ satisfy the conditions

$$h(u, u) \in \Lambda, \langle u, v\rangle = 0, h(v, v) \equiv a \pmod{\Lambda}. \tag{1.3}$$

Note that $\langle x, y\rangle = h(x, y) + \lambda\overline{h(y, x)}$ is the associated hermitian form. Transvections for (λ, Λ)-Quadratic modules are defined using the quadratic form $[q]$ in these formulas instead of h. For any submodule $L \subseteq M$,

$$L^\perp = \{x \in M | \langle x, y\rangle = 0 \text{ for all } \ y \in L\}.$$

If $M = M' \perp M''$ is an orthogonal direct sum, with $L' \subseteq M'$ a *totally isotropic* submodule (*i.e.* $h(x, y) = 0 \pmod{\Lambda}$ for all x, $y \in L'$), and $L'' \subseteq M''$, then we define

$$EU(M', L'; L'') = \langle\sigma_{u,a,v} | u \in L' \text{ and } \ v \in L''\rangle. \tag{1.4}$$

We will need the relation (see [2, p.92]): if $\alpha : (M, [h]) \to (M', [h'])$ is an isometry, then

$$\alpha \circ \sigma_{u,a,v} \circ \alpha^{-1} = \sigma'_{\alpha u, a, \alpha v} \tag{1.5}$$

where $\sigma \in U(M, [h])$ and $\sigma' \in U(M', [h'])$.

The hyperbolic rank of a (λ, Λ)-Quadratic module $(M, [h])$ is ≥ 1 if $(M, [h]) = H(A) \perp (M', [h'])$, where $H(P)$ denotes the hyperbolic form on $P \oplus \bar{P}$ [2, p.82] and elements denoted by pairs $x = (p, q)$ with $p \in P$, $q \in \bar{P}$. Here we are using the notation $\bar{P}$ for the dual module P^* regarded as a right A-module in the usual way. Since we will always be working with P containing at least one distinguished A-free direct summand, we will write $P = p_0A \oplus P_1$, $\bar{P} = q_0A \oplus \bar{P}_1$ and denote the element

$$(p, q) = (p_0a + p_1, q_0b + q_1).$$

In [I, §1] we introduced various subgroups of elementary automorphisms of $L \oplus P$, including $E(P)$, $E_\pm(L, P)$, $E_\pm(P_0, L \oplus P_1)$ and their $\mathfrak{O}$-analogues, where $\mathfrak{O}$ is a two-sided ideal in A. These are defined by products of certain elementary automorphisms. For the $\mathfrak{O}$-analogues we assume that the elementary automorphisms are $\equiv id \pmod{\mathfrak{O}}$.

Groups of transvections were used in [**2**, pp.142-143] to describe two important subgroups of $U(H(P))$, namely $H(E(P))$ and $EU(H(P))$. For the $\mathfrak{O}$ version of the first, we take $H(E(P;\mathfrak{O}))$. For the second, define

$$EU(H(P);\mathfrak{O}) = \langle \sigma_{u,a,v} \in EU(H(P)) \,|\, u, v \in P_0, P_1\mathfrak{O}, \ or\ \bar{P} \rangle.$$

An automorphism $\tau \in GL(L \oplus P)$ is *realized* by a transvection $\sigma_{u,a,v}$ if $\sigma_{u,a,v}(x) = \tau(x)$ for all $x \in L \oplus P \subset V \perp H(P)$.

The main result of this section is a unitary analogue of the transitivity result in [**I**,§**1**]. Before stating it, we need two lemmas.

Lemma 1.6: *Let V be a (λ, Λ)-Quadratic module which has (A, B)-hyperbolic rank ≥ 1 at a prime $\mathfrak{p} \in R_0$, for which $A_\mathfrak{p}$is maximal. Then*
(i) V contains a totally isotropic submodule L which has (A, B)-locally free rank ≥ 1 at all but finitely many primes, and $H(L \otimes_R F) \subseteq V \otimes_R F$
(ii) if $P \cong A^r$ and $f : P \to L$ is an A-homomorphism, let $\tau = 1 + f \in E_+(P, L)$ where f is extended by zero on L. Then there are elements $q_i \in \bar{P}$, $v_i \in L (1 \leq i \leq r)$ such that $\sigma = \prod \sigma_{q_i,o,v_i} \in EU(H(P), \bar{P}; V)$ and $\sigma(x) = \tau(x)$ for all $x \in L \oplus P \subseteq V \perp H(P)$.

Proof: (i) Since A_p is maximal, we can write $A_\mathfrak{p} = B' \times C'$ and work over the C' factor V' of $V_\mathfrak{p}$. Then V' has free hyperbolic rank ≥ 1 and for $L_\mathfrak{p}$ we choose a maximal rank totally isotropic C'-free direct summand with $H(L_\mathfrak{p}) \subseteq V_\mathfrak{p}$. Let $L = L_\mathfrak{p} \cap V$ and compare it to a direct sum of copies of the A-lattice $C := \ker\{\epsilon : A \to B\}$. Since $C_\mathfrak{p} \cong C'$ we may choose a direct sum $N = C^r$ with the same R-rank as L and so $N_\mathfrak{p} \cong L_\mathfrak{p}$. Therefore N and L are full lattices on the same F-vector space (F is the quotient field of R), and hence agree at all but finitely many primes. If we further avoid all the primes where A is not maximal, then L has (A, B)-free rank ≥ 1 at the remaining primes.
(ii) Let $\{q_1, ..., q_r\}$ be a basis for $\bar{P}$. Then there exist v_1, ..., $v_r \in L$ such that $f(x) = -\sum \bar{\lambda} v_i \langle q_i,\ x \rangle$ for all $x \in P$. ∎

Corollary 1.7: *Let $(M, [h]) = V \perp H(P)$, where P is a free A-module, and $L \subset V$ is a totally isotropic submodule. Every element of $E_+(P, L; \mathfrak{O})$, can be realized by a product of transvections in $EU(H(P), \bar{P}; V\mathfrak{O})$.*

Proof: This follows directly from the definitions and Lemma 1.6. ∎

Lemma 1.8: (A. Bak [**2**, (3.11),p.241]) *Suppose that (C, Λ) is a semisimple unitary algebra over (R, λ). Assume either that (i) P has free rank ≥ 2, or (ii) Λ is ample in C and $P = C$. Write $x \in H(P)$ as $x = (p_0 a + p_1, q_0 b + q_1)$. Then there is an element $\sigma \in H(E(P)) \cdot EU(H(P))$ such that $\sigma(x) = (p_0 a' + p_1', q_0 b' + q_1')$ and $O(x) = Aa'$. In case (i), $\sigma \in EU(H(P_0), Q; H(P_1))$ where $Q = P_0$ or $\bar{P}_0$, and in case (ii) $\sigma \in EU(H(P_0))$.*

Definition 1.9: *Let $(N,\ [h])$ be a (λ, Λ)-Quadratic module. An element $x \in N$ is $[h]$-unimodular if there exists $y \in N$ such that $\langle x,\ y \rangle = 1$.*

If $(N, [h])$ is non-singular then an element is $[h]$-unimodular if and only if it is unimodular.

Definition 1.10: *Let $\mathfrak{F}$ denote a simple involution invariant factor of $A[\mathfrak{m}]$, for some prime $\mathfrak{m} \in \max(R_0)$. A form parameter Λ is called ample at $(\mathfrak{m}, \mathfrak{F})$ if the projection of Λ to $\mathfrak{F}$ is ample. If Λ is ample at $(\mathfrak{m}, \mathfrak{F})$ for all factors of $C[\mathfrak{m}]$ in a splitting $A[\mathfrak{m}] = B[\mathfrak{m}] \times C[\mathfrak{m}]$ of semisimple rings, then we say that Λ is (A, B)–ample at $\mathfrak{m}$.*

The following is our main result in the quadratic case.

Theorem 1.11: *Let V be a (λ, Λ)-Quadratic module and put $(M, [h]) = V \perp H(P)$ where $P = p_0A \oplus p_1A$ is A-free of rank 2. At all but finitely many primes $\mathfrak{m} \in \max(R_0)$, assume either that V has (A, B)-hyperbolic rank ≥ 1 or that Λ is (A, B)–ample. For any two-sided ideal $\mathfrak{O}$ in A and any $b \in A$, the subgroup of*

$$G_1(\mathfrak{O}) = \langle EU(H(P), Q; V\mathfrak{O}), H(E(P; \mathfrak{O})) \cdot EU(H(P); \mathfrak{O})\rangle$$

fixing $\epsilon_*(p_0 + q_0b)$, where $Q = P\mathfrak{O}$ or $\bar{P}$, acts transitively on the set of $[h]$-unimodular elements $x \in M$ of a fixed length $[b] \in A/\mathfrak{O}$ such that $x \equiv p_0 + q_0b (\mathrm{mod}\,\mathfrak{O})$ and $\epsilon_*(x) = \epsilon_*(p_0 + q_0b)$.

The first part of the proof will be stated separately:

Lemma 1.12: *Let $x = (v; p, q) \in M$ be a $[h]$-unimodular element with $x \equiv p_0 + q_0b(\mathrm{mod}\,\mathfrak{O})$ and $\epsilon_*(x) = \epsilon_*(p_0 + q_0b)$. Then, after applying an element from $G_1(\mathfrak{O})$ we may assume that p is unimodular.*

Proof: (i) Let $\mathfrak{g} = \prod\{\mathfrak{m} | \mathfrak{m} \in \mathcal{S}\}$ where $\mathcal{S}$ is a finite set in X containing all the primes at which A is not maximal, or V does not have (A, B)-hyperbolic rank ≥ 1 (resp. Λ is not (A, B)–ample). Then $A[\mathfrak{g}] = B[\mathfrak{g}] \times \bar{C}$ and we may achieve "$O(x) = Aa$ over $A[\mathfrak{g}]$" using (1.8) and the fact that P is free of rank 2. Note that nothing needs to be done over B or over any simple factor of $A[\mathfrak{g}]$ in which the ideal $\mathfrak{O}$ has zero reduction. This step uses $EU(H(P_0), Q; H(P_1))$ where $Q = P_0$ or $\bar{P}_0$.

At all primes not in $\mathcal{S}$, we may assume (by Lemma 1.6) that V contains a non-zero totally isotropic submodule L which has (A, B)-free rank ≥ 1 (resp. Λ is (A, B)–ample).
(ii) Let $\mathfrak{t} \subseteq R_0$ be an ideal, maximal such that $A\mathfrak{t} \subseteq Aa$, and put $X' = V(\mathfrak{t})$, $X'_\Lambda = V(\mathfrak{t} + \mathfrak{A}_\Lambda)$, $d'_\Lambda = \dim\, X'_\Lambda$. Let $\pi : A \to A' = A/A\mathfrak{t}$ be the natural projection and note that $\dim\, X' = 0$ and $\dim\, X'_\Lambda \leq 0$. As in [2, p.244] we see that $\mathfrak{m} \notin X'$ for all $\mathfrak{m} \in \mathcal{S}$, hence $\mathfrak{t} \neq 0$ and A' is semilocal. We have $O(p_1, q_1 + q_0b) + Aa = A$ and so $O(\pi p_1, \pi(q_1 + q_0b)) + \pi(Aa) = A'$. Over $B' = B/B\mathfrak{t}$ we do nothing. Over the complementary factor C' of A', apply [2, (2.5.2) p.225] to find an element $u \in \pi P_1\mathfrak{O}$ such that u projects to zero over B' and

$$O(\pi p_1 - ub) + O(\pi q_1) + \pi(Aa) = A'.$$

(Note that this already holds over B' by assumption. Choose $z \in P_1\mathfrak{O}$ such that $\pi z = u$ and $\epsilon_*(z) = 0$. Since $\mathfrak{t}$ and $\mathfrak{g}$ are relatively prime, we can choose $z \in (P_1\mathfrak{O}) \cdot \mathfrak{g}$.

Note that $\sigma_{p_0,0,z} \in EU(H(P); \mathfrak{O})$ by [2, (3.10.2),p.142]. Then

$$\begin{aligned}\sigma(x) &= x + p_0\langle z, x\rangle - z\langle p_0, x\rangle \\ &= (v; p_1 - zb + p_0(a + \langle z, q_1\rangle), q)\end{aligned}$$

Therefore

$$O(p_1 - zb) + O(q_1) + A(a + \langle z, q_1\rangle) + At = A.$$

But $At \subseteq Aa$ and $A(a + \langle z, q_1\rangle) \subseteq O(q_1) + Aa$, so after these changes, we may assume that

$$O(p_1) + O(q_1) + Aa = A. \tag{1.13}$$

(iii) If πV has hyperbolic rank ≥ 1 over C' we can choose an isometry $\alpha : \pi V \cong H(C') \perp W'$ and extend it to an isometry of $\pi V \perp H(\pi P)$ by the identity on $H(\pi P)$. We now apply case (i) of (1.8) to the element $\alpha(\pi(p_1, q_1)) \in H(C') \perp H(\pi P_1)$ over the semisimple ring C', where $A' = B' \times C'$. This uses an element $\sigma' \in EU(H(\pi P_1), \pi Q; H(C'))$ where $Q = P_1\mathfrak{O}$ or $\bar{P}_1$. By (1.5), $\alpha^{-1} \circ \sigma' \circ \alpha \in EU(H(\pi P_1), \pi Q; \pi V)$. Then there exists a lift σ of $\alpha^{-1} \circ \sigma' \circ \alpha$ to $U(M, [h])$ which lies in $EU(H(P_1), Q; V\mathfrak{O})$. If $\pi\Lambda$ is ample, case (ii) of (1.8) applies, and this uses an element of $EU(H(\pi P_1); \mathfrak{O})$. After moving $x = (v; p, q)$ by σ we get

$$A = O(p_1) + At + Aa \subseteq O(p_1) + Aa = O(p).$$

Finally note that after this change p is unimodular and $\epsilon_*(x) = \epsilon_*(p_0 + q_0 b)$, where $\bar{b} \equiv h_P(x,x) \bmod \Lambda$. ∎

The Proof of Theorem 1.11

(i) By Lemma 1.12 we have p unimodular. Since $h(p,p) = 0$, we can split $H(P) = H(pA) \perp H(pA)^{\perp}$. If $H(pA) = pA \oplus \bar{p}A$, where $\bar{p} \in \bar{P}$ then $\sigma_{\bar{p},d,\lambda v} \in EU(H(P), \bar{P}; V\mathfrak{O})$ and

$$\begin{aligned}\sigma_{\bar{p},d,\lambda v}(x) &= x + \bar{p}\langle \lambda v, x\rangle - \lambda v \bar{\lambda}\langle \bar{p}, x\rangle - \bar{p}\bar{\lambda} d\langle \bar{p}, x\rangle \\ &= (0; p, q').\end{aligned}$$

(ii) We now have an element $x = (p,q) \in H(P)$ with p unimodular and $A + \mathfrak{g}A = A$, where $p = p_0 a + p_1$. Recall that V contains a non-zero totally isotropic submodule L which has (A,B)-free rank ≥ 1 at all but finitely many primes. Furthermore, V' contains $H(L')$ by construction. We claim that after applying a suitable transformation in $G_1(\mathfrak{O})$, we can assume that $x = (v; p_0, q)$, with $v \in L$ and a possibly different q.

By [I, (1.6)], there exists an element $\tau \in E_+(P_1, L; \mathfrak{O})$ fixing $\epsilon_*(p_0)$ such that after applying τ, $x = (v; p, q)$ with $z = p_0 a + v$ unimodular and $Aa + O(v) + At = A$. By (1.7), τ can be realized by a transvection in $G_1(\mathfrak{O})$. Now notice that $z \in P_0 \oplus L$ is actually $[h]$-unimodular and $h(z,z) = 0 \pmod \Lambda$. First, z is $[h']$-unimodular in $V' \perp H(P_0')$ since it lies in the non-singular subspace $H(L') \perp H(P_0')$. Therefore $\langle V \perp H(P_0), z\rangle + At = A$. However, $\langle H(P_0), z\rangle = Aa$ and $At \subset Aa$, hence $\langle V \perp H(P_0), z\rangle = A$.

We now refer to [I, (1.8)] for a sequence of elementary automorphisms moving $p_0 a + p_1 + v$ to p_0. To realise $\tau_1 \in E_+(zA, P_1; \mathfrak{O})$ by an isometry, we find a unitary submodule $H(zA) \subseteq V \perp H(P_0)$ and then work inside $H(zA) \perp H(P_1)$. Let $\bar{z} \in H(zA)$ denote a complementary basis element. By [2, (3.10.4), p.143]

$$H(\tau_2|_{zA \oplus P_1}) = \sigma \subseteq EU(H(zA), \bar{z}\bar{A}; P_1\mathfrak{O}) \subseteq EU(H(P); \mathfrak{O}) \cdot EU(H(P_1), P_1\mathfrak{O}; V\mathfrak{O}).$$

The remaining automorphisms can be realized by Corollary 1.7.

After this we have $x = (v, p_0, q)$ and we finish by repeating step (i) above, which does not alter the P–component. The result is $x = (0; p_0, q)$. The proof is completed by applying the following Lemma. ∎

Lemma 1.14: *Let $x = (p_0, q) \in H(P)$ with $\epsilon_*(x) = \epsilon_*(p_0 + q_0 b)$ and $x \equiv p_0 + q_0 b \pmod{\mathfrak{O}}$. Then there is an element $\sigma \in EU(H(P); \mathfrak{O})$, fixing $\epsilon_*(p_0 + q_0 b)$ such that $\sigma(x) = p_0 + q_0 b$. If x is hyperbolic, then we can obtain $\sigma(x) = p_0$.*

Proof: Write $q = q_0 b - q_1 \in q_0 A \oplus \bar{P}_1$. The transvection $\sigma_{q_1, 0, q_0}$ belongs to $EU(H(P); \mathfrak{O})$ by [**2**, (3.10.1), p.142], and

$$\sigma x = x - q_1 \langle q_0, x \rangle - q_0 \bar{\lambda} \langle q_1, x \rangle.$$

Note that $\langle q_1, x \rangle = 0$ since x has no component in P_1 and $\langle q_0, x \rangle = \langle q_0, p_0 \rangle = 1$, so $\sigma x = x + q_1 = (p_0,\ q_0 b)$. We are now finished if x was only unimodular. If x was a hyperbolic element, then $h(\sigma x, \sigma x) = h(p_0,\ q_0 b) = \bar{b} \pmod{\Lambda}$, and so $\bar{b} \in \Lambda$, since x and $\sigma(x)$ are isotropic.

In the hyperbolic plane $p_0 A + q_o A$, the element $X_+(-b) = \left(\begin{smallmatrix} I & -b \\ 0 & I \end{smallmatrix}\right) \in EU(H(P_0))$ transforms $p_0 + q_0 b$ into p_0 (the notation X_+ is from [**2**, p.130]). ∎

In the following definition we suppose that $(M, [h]) = V \perp H(P)$ is a (λ, Λ)-Quadratic module where $P = p_0 A \oplus p_1 A$. We recall the following notation from [**I**]: if N is a submodule of M and $G \subseteq GL(M)$, then $G(N) = \{g \in G \,|\, g(N) = N\}$. If $M = M_1 \oplus M_2$ and $G \subseteq GL(M_1)$, then (by definition) $G(N) = \{g \in G \,|\, (g \oplus 1_{M_2})(N) = N\}$.

Definition 1.15: *Let $N \subseteq (M, [h])$ be a (λ, Λ)-Quadratic submodule, containing $H(P_0)$, and $\mathfrak{O} = \mathrm{Ann}(M/N)$. Let $N_0 = N \cap H(P)$. A subgroup $G_0 \subseteq U(H(P))$ is $(N, H(P_0), \epsilon)$-transitive if*

(i) $G_0(N)$ acts transitively on the images in $N_0/N_0 \cap M\mathfrak{O}$ of the elements $p_0 a + q_0 b$ of a fixed length which are unimodular $(\bmod\, \mathfrak{O})$.

(ii) for each $[b] \in A/\Lambda$, the subgroup of $G_0(N)$ fixing $p_0 + q_0 b \pmod{\mathfrak{O}}$ acts transitively on the images in $H(\epsilon_*(P))$ of the $[h]$–unimodular elements $x \in N \cap H(P)$ of fixed length $[\bar{b}]$ such that $x \equiv p_0 + q_0 b \pmod{\mathfrak{O}}$.

Example 1.16: If $N = M$ so that $\mathfrak{O} = A$, then $G_0 \subseteq U(H(P))$ is $(N, H(P_0), \epsilon)$-transitive if G_0 acts transitively on the images in $H(\epsilon_*(P))$ of the $[h]$–unimodular elements of fixed length.

The unitary transitivity conditions are related to the linear transitivity conditions given in [**I**, Definition 1.9]. In the following statement we let $\sigma_0 \in U(H(P_0))$ be the flip automorphism $\sigma_0(p_0) = q_0, \sigma_0(q_0) = p_0$. The metabolic form on an A-module $L \oplus \bar{L}$ with isotropic summand $0 \oplus \bar{L}$ is denoted $Met(L)$.

Lemma 1.17: *Let $N \subset M = H(P) \perp Met(L)$ be a (λ, Λ)-Quadratic submodule of finite index containing $H(P_0)$, and let $N_1 = N \cap (P \oplus L)$. If $G_0 \subset GL(P)$ is (N_1, p_0, ϵ)-transitive, then the group $\langle H(G_0), \sigma_0, G_1(\mathfrak{O}) \rangle$ is $(N, H(P_0), \epsilon)$-transitive.*

Proof: When $p_0a + q_0b$ is unimodular $(\operatorname{mod} \mathfrak{D})$, then a or b is a unit $(\operatorname{mod} \mathfrak{D})$, and after at most multiplication by -1 and interchanging p_0, q_0 we can get $a \equiv 1 (\operatorname{mod} \mathfrak{D})$. By using a suitable element of $H(G_0)$, we can get $p_0 + q_0b (\operatorname{mod} \mathfrak{D})$.

If $x \in N \cap H(P)$ is $[h]$-unimodular of length $[\bar{b}]$ and $x \equiv p_0 + q_0 b(\bmod \mathfrak{D})$, we apply Lemma 1.12 with $B = 0$ and then an element of $H(G_0)$, by [**I**, Definition 1.9(ii)], to get $\epsilon_*(x) = \epsilon_*(p_0 + q_0b)$. ∎

Lemma 1.18: *Let $N \subseteq (M, [h]) = V \perp H(P)$ be a (λ, Λ)-Quadratic submodule, containing $H(P_0)$, and $\mathfrak{D} = \operatorname{Ann}(M/N)$. Let $N = H(P_0) \perp N'$ and $N'' = N \cap V$.*
(i) Suppose that N has finite index in M and that there exists a subgroup $G_0 \subseteq U(H(P))$ satisfying the condition in Definition 1.15 (i). If $x \in N$ is an $[h]$–unimodular element with length $[\bar{b}]$, then there exist elements $\sigma_1 \in EU(H(P_0), P_0; N')$, $\sigma_2 \in EU(H(P_0), \bar{P}_0; N')$, and $\theta_1 \in G_0(N)$ such that $x' = \sigma_2\theta_1\sigma_1(x)$ has $x' \equiv p_0 + q_0b (\operatorname{mod} \mathfrak{D})$.
(ii) Suppose that there exists a subgroup $G_0 \subseteq U(H(P))$ satisfying the condition in Definition 1.15 (ii). If $x \in N$ is an $[h]$–unimodular element with length $[\bar{b}]$ and $x \equiv p_0 + q_0 b(\bmod \mathfrak{D})$, then there exist elements $\theta_2 \in G_0(N)$ and $\sigma_3 \in EU(H(P), P\mathfrak{D}; N''\mathfrak{D})$, $\sigma_4 \in EU(H(P), \bar{P}; N''\mathfrak{D})$ such that $x' = \sigma_4\theta_2\sigma_3(x)$ has $\epsilon_*(x') = \epsilon_*(p_0 + q_0b)$ and $x' \equiv p_0 + q_0b (\operatorname{mod} \mathfrak{D})$.

Proof:

(i) Write $(p, q) = (p_0a + p_1, q_0b + q_1)$ as above. We begin by working over $N/N \cap M\mathfrak{D}$ to arrange for $p_0a + q_0b (\operatorname{mod} \mathfrak{D})$ to be unimodular. By assumption, $N = H(p_0A) \perp N'$, and N' is a quadratic submodule of finite index in $V \perp H(p_1A)$. There exists some $y \in N$ such that $\langle x, y\rangle = 1$ and so $\langle N', v\rangle + O(p_0a) + O(q_0b) = A$. Choose $w \in N'$ so that $\langle v, w\rangle + Aa + Ab$ contains 1; put $c = \langle v, w\rangle$. From [**I**, (1.3)], there is a $z \in P_0$ such that $O(p_0 + zc) + O(q_0) = A$. Now apply the transvection $\sigma_1 = \sigma_{z,e,w}$ to x, and then the $H(P_0)$-component $p_0a + q_0b$ of x will be unimodular $(\operatorname{mod} \mathfrak{D})$. This isometry lies in $EU(H(P_0), P_0; N')$.

Now there exists $\theta_1 \in G_0(N)$, so that after applying θ_1, $x \equiv (v; p_0, q) (\operatorname{mod} \mathfrak{D})$. By repeating step (i) of the proof of Theorem 1.11, we find an element $\sigma_2 \in EU(H(P_0), \bar{P}_0; N')$ to get $x \equiv p_0 + q_0b (\operatorname{mod} \mathfrak{D})$.

(ii) We now assume that $x \equiv p_0 + q_0b(\bmod \mathfrak{D})$ and try to obtain the condition on $\epsilon_*(x)$. Since P is free of rank 2, it follows as above that we may assume (p, q) is unimodular. More precisely, there exists some $y \in N$ such that $\langle x, y\rangle = 1$ and so $\langle V, v\rangle + O(p) + O(q) = A$. Let $w \in V$ be the V-component of y. Since $x \equiv p_0 + q_0b (\operatorname{mod} \mathfrak{D})$, we may assume that the $w \in V\mathfrak{D}$. Now $\langle v, w\rangle + O(p) + O(q)$ contains 1, and we let $c = \langle v, w\rangle$. From [**I**, (1.1)] with $\mathfrak{a} = O(q)$, there is a $z \in P\mathfrak{D}$ such that $O(p + zc) + O(q) = A$. Now apply the transvection $\sigma_3 = \sigma_{z,e,w}$ to x. This isometry lies in $EU(H(P), P\mathfrak{D}; V\mathfrak{D})$.

Since the subgroup of $G_0(N)$ fixing $p_0 + q_0b (\operatorname{mod} \mathfrak{D})$ acts transitively on the set of unimodular elements of fixed length in $H(\epsilon_*(P))$, we may assume after applying $\theta_2 \in G_0(N)$ that $\epsilon_*(x) = \epsilon_*(v; p_0, q_0b)$, where $\bar{b} \equiv h_P(x, x) \bmod \Lambda$. Finally, apply again step (i) of the proof of Theorem 1.11, we find an element $\sigma_4 \in EU(H(P), \bar{P}; N''\mathfrak{O})$ to get $\epsilon_*(x) = \epsilon_*(p_0, q_0 b)$. ∎

Theorem 1.19: *Let V be a (λ, Λ)-Quadratic module which has (A, B)-hyperbolic rank ≥ 1 at all but finitely many primes, and let $N \subseteq (M, [h]) = V \perp H(P)$ be a (λ, Λ)-Quadratic submodule of finite index, containing $H(P_0)$, and $\mathfrak{O} = Ann(M/N)$. Suppose there exists a subgroup $G_0 \subseteq U(H(P))$ which is $(N, H(P_0), \epsilon)$-transitive.*

Then the subgroup $G(N)$ of

$$G = \langle G_0, EU(H(P), Q; V), H(E(P)) \cdot EU(H(P)) \rangle$$

stabilizing N acts transitively on the set of $[h]$-unimodular elements of a fixed length in N, and the set of hyperbolic pairs and hyperbolic planes in N.

Proof: The same reduction used in [**2**, (3.5),p.237] shows that it is enough to prove that G acts transitively on the set of $[h]$-unimodular elements of a fixed length in N. One can check that G contains all transvections $\sigma_{p_0, a, v}$ with $v \in (p_0)^{\perp} = V \oplus H(P_1) \oplus p_0 A$ (see [**2**, (3.11),p.143] and [**2**, (5.6),p.98]). Now we apply Lemma 1.18 and Theorem 1.11. The isometries used all preserve N. ∎

The special case when $\mathfrak{O} = A$ and $N = M$ will be used later.

Theorem 1.20: *Let V be a (λ, Λ)-Quadratic module which has (A, B)-hyperbolic rank ≥ 1 at all but finitely many primes, and put $(M, [h]) = V \perp H(P)$. Suppose there exists a subgroup $G_0 \subseteq U(H(P))$ such that $\epsilon_*(G_0)$ acts transitively on the set of unimodular elements in $H(\epsilon_*(P))$ of fixed length $\epsilon_*([h](x))$. Then*

$$G = \langle G_0, EU(H(P), Q; V), H(E(P)) \cdot EU(H(P)) \rangle$$

where $Q = P$ or $\bar{P}$, acts transitively on the set of $[h]$-unimodular elements of a fixed length, and the set of hyperbolic pairs and hyperbolic planes in M.

Proof of Theorem A: The argument is the same as for [**2**, (3.6), p.238] using our (1.20).

We conclude this section with a few useful remarks.

Lemma 1.21: *Let $P = p_0\mathbf{Z} \oplus p_1\mathbf{Z}$. For any ideal $\mathfrak{q} \subseteq \mathbf{Z}$, the group $H(SL_2(\mathbf{Z}; \mathfrak{q})) \cdot EU(H(P; \mathfrak{q}))$ acts transitively on unimodular elements $x \in H(P)$ of fixed length $[\bar{b}]$, with $x \equiv p_0 + q_0 b \pmod{\mathfrak{q}}$.*

Proof: Let $x = (p_0 a,\ p_1 c;\ q_0 d,\ q_1 e)$ be a unimodular element in $H(P)$ with $a \equiv 1 \pmod{\mathfrak{q}}$, $d \equiv b \pmod{\mathfrak{q}}$ and $c, e \equiv 0 \pmod{\mathfrak{q}}$. We may assume that $e = 0$ after applying an element of $H(SL_2(\mathbf{Z}; \mathfrak{q}))$, so there exists an integer $r \equiv 0 \pmod{\mathfrak{q}}$ such that $c + rd$ is a unit (mod a). Then

$$X_+\left(\begin{smallmatrix} 0 & -\lambda r \\ r & 0 \end{smallmatrix}\right)(x) = (p_0 a,\ p_1(c + rd);\ q_0 c,\ 0)$$

so that $O(p_0 a) + O(p_1(c + rd)) = \mathbf{Z}$. We may therefore assume in the beginning that for $x = (p_0 a, p_1 c; q_0 d, q_1 e)$, a and c are relatively prime. Using a suitable element of $H(SL_2(\mathbf{Z}; \mathfrak{q}))$ (see [**I**, (1.15)]) we get $x = (p_0,\ 0;\ q_0 b,\ q_1 e)$ and after applying $X_-\left(\begin{smallmatrix} 0 & \lambda d \\ -e & 0 \end{smallmatrix}\right)$ the result is $(p_0,\ 0;\ q_0 b,\ 0)$, where $[\bar{b}]$ is the length of x. ∎

Remark 1.22: *For any surjection of orders $\epsilon : A \to \mathbf{Z}$, the subgroup $G_1 = H(E_2(A)) \cdot EU(H(A \oplus A)) \subseteq U_2(A)$ has the property that $\epsilon_*(G_1) = H(GL_2(\mathbf{Z})) \cdot EU(H(\mathbf{Z} \oplus \mathbf{Z}))$.*

Lemma 1.23: *Let V be a (λ, Λ)-Quadratic module and set $(M, [h]) = V \perp H(P)$ with $P = A$. Let $\sigma_0 \in U(H(P))$ be the isometry interchanging the standard basis elements p_0, q_0. If (A, Λ) is a local unitary ring, then*

$$G = \langle EU(H(P), Q; V), H(GL(P)) \cdot EU(H(P)), 1 \perp \sigma_0 \rangle$$

acts transitively on the set of $[h]$-unimodular elements of a fixed length in M, and the set of hyperbolic pairs and hyperbolic planes in M.

Proof: Again it is enough to prove transitivity on the set of $[h]$-unimodular elements of a fixed length. Let $x = (v; p_0 a, q_0 b)$ be an $[h]$-unimodular element. Then after applying a suitable transvection from $EU(H(P), Q; V)$ we may assume that $(p_0 a, q_0 b) \in H(P)$ is unimodular. Then since A is local, we may apply σ_0 is necessary to assume that a is a unit. Now, after using an element of $H(GL(P))$ we get $x = (v; p_0, q_0 b)$ and we finish by using step (i) of the proof of Theorem 1.11. ∎

Remark 1.24: *For $\lambda = -1$ and any form parameter Λ, $U(H(\mathbf{Z}))$ acts transitively on unimodular elements of fixed length in $H(\mathbf{Z})$.*

§2: The Proof of Theorem B

In this section we apply the algebraic cancellation theorems to prove our main geometric cancellation result for four-manifolds.

Proposition 2.1: *Let X be a closed oriented topological 4–manifold with finite fundamental group, and let $A = \mathbf{Z}[\pi_1(X)]$. There is an A–submodule V of $\pi_2(X)$ which supports a (λ, Λ)-Quadratic refinement of the intersection form on X for $\lambda = 1$ and any form parameter Λ. In addition, V has $(A, \mathbf{Z})$–hyperbolic rank ≥ 1 at all but finitely many primes.*

Proof: We take the submodule

$$V = \ker(\langle w_2, - \rangle : \pi_2(X) \to \mathbf{Z}/2)$$

on which the intersection form S_X has a quadratic refinement $q : V \to A/\{\nu - \bar{\nu}\}$ defined as in [**28**, Chap. 5].

Next we check that V has $(A, \mathbf{Z})$–hyperbolic rank ≥ 1 at all odd primes not dividing the order of $\pi_1(X)$. Since X is a closed manifold, the components of the multi–signature of S_X are all equal. On the other hand, from [10, 2.4] we know that $\pi_2(X)_{(\mathfrak{p})}$ is isomorphic to the localization of $I \oplus I^* \oplus \Lambda^\ell$, where I denotes the augmentation ideal of Λ. It follows that the components of S_X are indefinite at all non–trivial characters of $\pi_1(X)$. Since S_X is unimodular when restricted to $V_\mathfrak{p}$, for $\mathfrak{p}$ as above, we conclude that $V_\mathfrak{p}$ has hyperbolic rank ≥ 1 at each non–trivial character. ∎

We need the following result of Cappell–Shaneson. In the statement a standard basis for the summand $H_2(S^2 \times S^2, \mathbf{Z})$ of $H_2(X \sharp (S^2 \times S^2), \mathbf{Z})$ is denoted $\{p_0, q_0\}$.

Theorem 2.2: [5, 1.5] *Let X be a compact, connected smooth (topological) manifold of dimension four, and suppose $X = X_0 \sharp (S^2 \times S^2)$ for some manifold X_0. Let $\omega \in H_2(X;\Lambda)$ with $w_2(H(\omega)) = 0$ and let $a \in \Lambda = \mathbf{Z}[\pi_1(X)]$ be any element such that $\mu(\omega) \equiv a(\mathrm{mod}\Lambda)$. Then there is a base point preserving diffeomorphism (homeomorphism) ϕ of $X \sharp (S^2 \times S^2)$ with itself which preserves local orientations and induces the identity on $\pi_1(X \sharp (S^2 \times S^2))$, so that $\phi_*(p_0) = p_0$, $\phi_*(q_0) = q_0 + \omega - p_0 a$, and $\phi_*(\xi) = \xi - (\xi \cdot \omega)p_0$ for $\xi \in H_2(X;\Lambda)$.*

In order to prove Theorem B, we need to realize transvections by homeomorphisms of $X \sharp r(S^2 \times S^2)$. For the rest of this section we fix the notation

$$K\pi_2(X) = \ker(\langle w_2, -\rangle : \pi_2(X) \to \mathbf{Z}/2)$$

for the submodule of the intersection form on $H_2(X;\Lambda)$ on which a quadratic refinement is defined. We denote by $H(P_0)$, where $P_0 = p_0\Lambda$, the summand of $H_2(X \sharp (S^2 \times S^2);\Lambda)$ given by $H_2(S^2 \times S^2;\Lambda)$. As further copies of $S^2 \times S^2$ are added to X by connected sum, we denote all these hyperbolic factors of the intersection form by $H(P)$. Note that Theorem 2.2 allows us to realize the transvections $\sigma_{p_0,a,v}$ by self-homeomorphisms of $X \sharp (S^2 \times S^2)$ for any $v \in K\pi_2(X_0)$ with $\langle v, p_0\rangle = 0$, in the case when $X = X_0 \sharp (S^2 \times S^2)$. Cappell and Shaneson use this to realize many isometries (see [5, Thm. A2]), but the conclusions given are not in the exact form we need.

Corollary 2.3: *Suppose that $K\pi_2(X) = V_0 \perp V_1$ with V_0 non-singular under the intersection form S_X. Then for any transvection $\sigma_{p,a,v}$ on $K\pi_2(X) \perp H(P_0)$ with $p \in V_0 \perp P_0$ and $v \in K\pi_2(X)$, the stabilized isometry $\sigma_{p,a,v} \oplus Id_{2(S^2\times S^2)}$ can be realized by a self-homeomorphism of $X \sharp 3(S^2 \times S^2)$.*

Proof: First we consider a unimodular isotropic element $p \in V_0 \perp P_0$. Since $V_0 \perp H(P_0)$ is non-singular, p is automatically a hyperbolic element and thus by Freedman [9] we can resplit $X \sharp (S^2 \times S^2) = X' \sharp (S^2 \times S^2)$ such that p is represented by $S^2 \times *$. Thus $\sigma_{p,a,v} \oplus Id_{S^2\times S^2}$ can be realized by a self-homeomorphism on $(X' \sharp (S^2 \times S^2)) \sharp (S^2 \times S^2)$ for all $v \in K\pi_2(X)$ with $\langle v, p\rangle = 0$.

Next we consider the transvection $\sigma_{p,0,v}$ for an arbitrary $p \in V_0 \perp P_0$, but assume that $v \in K\pi_2(X)$ is isotropic. Then we write $p = \sum p_i$ with $p_i \in V_0 \perp P_0$ unimodular and $\langle v, p_i\rangle = 0$. This uses the fact that $\Lambda = \mathbf{Z}[\pi_1(X)]$ and $P_0 \cong \Lambda$. We obtain: $\sigma_{p,o,v} = \sigma_{v,0,-p} = \sigma_{v,0,-\sum p_i} = \prod \sigma_{p_i,0,v}$. Thus $\sigma_{p,0,v} \oplus Id_{S^2\times S^2}$ is realizable by a self-homeomorphism on $(X \sharp (S^2 \times S^2)) \sharp (S^2 \times S^2)$, since $\sigma_{p_i,0,v} \sharp Id_{S^2\times S^2}$ is realizable.

Finally we realize an arbitrary transvection $\sigma_{p,a,v} \sharp Id_{2(S^2\times S^2)}$, of the form required, by a homeomorphism on $(X \sharp (S^2 \times S^2)) \sharp (S^2 \times S^2)$. We use the fact that v can be expressed as $v = \sum v_i$ with $v_i \in K\pi_2(X) \perp H_2(S^2 \times S^2; A)$ isotropic and $\langle v_i, p\rangle = 0$. Thus $\sigma_{p,0,v} \oplus Id_{2(S^2\times S^2)} = \prod \sigma_{p,0,v_i} \oplus Id_{S^2\times S^2}$ which by the considerations above is realizable. ∎

Corollary 2.4: *Let X_0 be a topological 4-manifold, $V = K\pi_2(X_0)$ and consider an element $\varphi \in EU(H(P), Q; V)$, for $Q = P, \bar{P}$, as an isometry of the intersection form of $X_0 \sharp 2(S^2 \times S^2)$. Then stabilized isometry $\varphi \oplus Id_{2(S^2\times S^2)}$ can be realized by a self-homeomorphism of $X_0 \sharp 4(S^2 \times S^2)$.*

Proof: By definition (1.4) the group $EU(H(P), Q; V)$ is generated by transvections $\sigma_{p,a,v}$ with $p \in P$ or $\bar{P}$ and $v \in V$ fulfilling the conditions of a transvection. It is enough to consider the case $p \in P$. Now Corollary 2.3 applies with the splitting $K\pi_2(X) = V \perp H(A)$ with $H(A)$ the first summand of $H(P)$. This shows that for each $\varphi \in EU(H(P), Q; V)$, the isometry $\varphi \oplus Id_{2(S^2\times S^2)}$ can be realized by a self-homeomorphism on $(X_0 \sharp 2(S^2 \times S^2)) \sharp 2(S^2 \times S^2)$. ∎

Proof of Theorem B: By induction it is enough to consider the case $r = 1$. Let $f : X \sharp (S^2 \times S^2) \to Y \sharp (S^2 \times S^2)$ be a homeomorphism. We will apply Theorem 1.20 and Corollary 2.3 to show that there is a self-homeomorphism g of $X \sharp 3(S^2 \times S^2)$ such that $(f \sharp Id) \cdot g$ induces the identity on the hyperbolic form corresponding to $\sharp 3(S^2 \times S^2)$ in $H_2(X \sharp 3(S^2 \times S^2); A)$. Then it follows that X and Y are s-cobordant **[14]**. By Freedman **[9]** X and Y are homeomorphic.

To begin, we apply Theorem 1.20 together with Lemma 1.21 to

$$V \oplus H(P) \subseteq H_2(X_0 \sharp 2(S^2 \times S^2); A),$$

where $P = A \oplus A$ and $V = K\pi_2(X_0)$. This gives an isometry

$$\varphi \in G = \langle EU(H(P), Q; V), H(E(P)) \cdot EU(H(P))\rangle,$$

where $Q = P$ or $\bar{P}$, such that $f_* \cdot \varphi$ induces the identity on $H_2(2(S^2 \times S^2); A) \subseteq H_2(X_0 \sharp 2(S^2 \times S^2); A)$. We finish the proof by showing that for each $\varphi \in G$, $\varphi \oplus Id$ can be realized by a self-homeomorphism on $X_0 \sharp 4(S^2 \times S^2)$. Note that by definition $G \subseteq \mathrm{Aut}(H_2(X_0 \sharp 2(S^2 \times S^2); A))$.

The elements of $EU(H(P), Q; V)$ are handled by Corollary 2.4. In addition, we have to realize an arbitrary element in $H(E(P)) \cdot EU(H(P))$, stabilized by the identity, by a self-homeomorphism of $(X_0 \sharp 4(S^2 \times S^2))$. This follows again from Corollary 2.3 and the considerations above since this group is generated by transvections $\sigma_{p,a,x}$ with $p \in P_0$ or P_1 **[2**, p.142-143**]**. ∎

§3: Applications to Manifolds with Boundary

In this section we prove Theorem B′, and Corollary 3.6. The other geometric applications, Corollary 4.1 and Theorem 4.5, are postponed to §4. We begin with

Lemma 3.1: *Let M_0 be a closed 4-manifold, and $A = \mathbf{Z}\pi_1(M_0)$. If $L_5^s(A) = 0$ then, for every element σ of $U_2(A)$, there is a self-homeomorphism of $M_0 \,\sharp\, (r+2)(S^2 \times S^2)$ for some $r \geq 0$ inducing $1 \perp \sigma \perp 1$ on $(H_2(\tilde{M}_0), S_{M_0}) \perp H(A \oplus A) \perp H(A^r)$.*

Proof: Since $L_5^s(A) = 0$, for any element σ of $U_2(A)$, there exists an integer $r \geq 0$ such that $\sigma \perp 1$ is in the subgroup $RLU_{(r+2)}$ defined in [5, p.526]. But by [5, Thm. A.2] any element of $RLU_{(r+2)}$ is realizable by a self homeomorphism of $M_0 \,\sharp\, (r+2)(S^2 \times S^2)$ which induces the identity on $\pi_1(M_0)$ and on $H_2(\tilde{M}_0)$.

The Proof of Theorem B′: The proof follows the steps of the proof of Theorem B to obtain geometric cancellation of the $r(S^2 \times S^2)$ factors. The necessary algebraic cancellation is provided by the special case of Theorem 1.20 with $A = B$, using our assumption on $U_2(A)$. The realization of unitary automorphisms by self-homeomorphisms follows from (3.1) and (2.4). ∎

Now we will verify the assumptions of Theorem B′ for special fundamental groups.

Lemma 3.2: *Let P be a free A-module of rank two, and $N = H(p_0A) \oplus p_1\mathfrak{O} \oplus \bar{P}_1 \subset H(P)$, where $\mathfrak{O}$ is an involution-invariant two-sided ideal in A. Suppose that the form parameter Λ is (A, B)-ample at $\mathfrak{m}$, for all but finitely many primes, and that there is a subgroup $G_0 \subseteq U(H(P))$ satisfying condition (1.15)(ii). In addition, let $\Gamma \subseteq GL(P)$ such that the subgroup of Γ, fixing $p_0 \pmod{\mathfrak{O}}$ and $\epsilon_*(p_0)$, acts transitively on unimodular elements $x \in P$ with $x \equiv p_0 \pmod{\mathfrak{O}}$ and $\epsilon_*(x) = \epsilon_*(p_0)$.*

Then the subgroup $G(N)$ of

$$G = \langle G_0, H(\Gamma)) \cdot EU(H(P); \mathfrak{O})\rangle$$

acts transitively on the set of unimodular elements in $x \in H(P)$ of fixed length $[\bar{b}]$, such that $x \equiv p_0 + p_1 b \pmod{\mathfrak{O}}$.

Proof: Let $x = (p, q) \in H(P)$ be a unimodular element with $x \equiv p_0 + p_1 b \pmod{\mathfrak{O}}$, and apply first Lemma 1.18 (ii) and then Lemma 1.12. After this we may assume that p is unimodular, and then use an element of $H(\Gamma)$ to get $x = (p_0, q)$. We complete the proof by using Lemma 1.14. ∎

Remark 3.3: Note that when $\lambda = 1$ and $\mathfrak{F}$ is a field with trivial involution, then Λ is not ample at $(\mathfrak{m}, \mathfrak{F})$ for any odd prime. Our assumption above is therefore very special. It does however apply to finite group rings $A = \mathbf{Z}[\rho \times \sigma]$ and $B = \mathbf{Z}[\mathbf{Z}/2]$, where ρ has odd order and σ is a cyclic 2-group.

Lemma 3.4: *Let $B = \mathbf{Z}[\mathbf{Z}/2]$ and P be a free B-module of rank two. Let $\mathfrak{O} \subset B$ an involution-invariant ideal and $N = H(p_0B) \oplus p_1\mathfrak{O} \oplus \bar{P}_1 \subset H(P)$. Then for the group*

$$G_0 = \langle H(SL_2(B; \mathfrak{O})) \cdot EU(H(P); \mathfrak{O})\rangle,$$

and for each $[b] \in B/\Lambda$, the subgroup of $G_0(N)$ fixing $p_0 + q_0 b (\bmod \mathfrak{O})$ acts transitively on the $[h]$–unimodular elements $x \in N$ of fixed length $[\bar{b}]$ such that $x \equiv p_0 + q_0 b (\bmod \mathfrak{O})$.

Proof: The group ring B of the cyclic group of order two is the pull-back:

$$\begin{array}{ccc} \mathbf{Z}[\mathbf{Z}/2] & \xrightarrow{\epsilon_+} & \mathbf{Z} \\ {\scriptstyle \epsilon_-}\downarrow & & \downarrow \\ \mathbf{Z} & \longrightarrow & \mathbf{Z}/2 \end{array}$$

If $x \in H(B \oplus B)$ is a unimodular element, we first apply (1.21) to $\epsilon_+(x)$, and obtain the relation $\epsilon_+(x) = \epsilon_+(p_0 + q_0 b)$. This implies $\epsilon_-(x) = \epsilon_-(p_0 + q_0 b) (\bmod 2)$, and uses an element of $H(SL_2(\mathbf{Z}; \mathfrak{q}_+)) \cdot EU(H(P; \mathfrak{q}_+))$, where $\mathfrak{q}_+ = \epsilon_+(\mathfrak{q})$. To lift this element to G_0, notice that $\mathfrak{q}_+ \equiv \mathfrak{q}_- (\bmod 2)$, so we can lift into $H(SL_2(\mathbf{Z}; \mathfrak{q}_-)) \cdot EU(H(P; \mathfrak{q}_-))$ over the $\mathbf{Z}_-$ corner. We now apply (1.21) again, this time to $\epsilon_-(x)$, with the ideal $2\mathfrak{q}_-$ and get an element $\sigma_- \in H(SL_2(\mathbf{Z}; 2\mathfrak{q}_-)) \cdot EU(H(P; 2\mathfrak{q}_-))$ such that $\sigma_-(\epsilon_-(x)) = \epsilon_-(p_0 + q_0 b)$. This element can be lifted over the $\mathbf{Z}_+$ corner to give an element of G_0. ∎

Corollary 3.5: *Let $A = \mathbf{Z}[\rho \times \sigma]$, where ρ has odd order and σ is a cyclic 2-group. Then $U_2(A)$ acts transitively on unimodular elements in $H(A \oplus A)$ of fixed length.*

Proof: We apply Lemma 3.2 and Lemma 3.4 with $\mathfrak{O} = A$ or B respectively. For the group Γ needed in Lemma 3.2 we can take $GL_2(A)$ since A satisfies the Eichler condition. The group $G_0 = \langle H(SL_2(A)) \cdot EU(H(P)) \rangle$ satisfies condition (1.15) by Lemma 3.4. ∎

Corollary 3.6: *Let M and N be compact oriented topological 4–manifolds with $\pi_1(M) = \rho \times \sigma$. Suppose that the interior connected sum $M \sharp r(S^2 \times S^2)$ is homeomorphic to $M \sharp r(S^2 \times S^2)$ by a homeomorphism inducing the identity on the boundary. If $M = M_0 \sharp (S^2 \times S^2)$, then the identity map on the boundary extends to a homeomorphism of M with N.*

We finish this section by considering a very special case of the question (see **[3]**): under what conditions does a quadratic or hermitian form contain a hyperbolic direct summand $H(P)$, for P projective ? For any A-module L, let $rank_A(L) = k$ if k is the largest integer such that $L_{(\mathfrak{p})}$ contains a direct summand $A^k_{(\mathfrak{p})}$, for all primes $\mathfrak{p} \in R$. It is convenient to define the "essential rank" of a hermitian (resp. quadratic) form U by $ess\text{-}rank_A(U) = rank_A(U_0)$ if U_0 is a hermitian (resp. quadratic) submodule of minimum rank such that $U = U_0 \perp H(P)$ for some finitely-generated projective P. Thus the quantity $rank_A(U)-$ $ess\text{-}rank_A(U)$ is twice the (projective) hyperbolic rank of U. If $A = \mathbf{Z}G$ for some finite group G, then L^G denotes the submodule fixed by G. If L has a hermitian (resp. quadratic) module then L^G is a hermitian (resp. quadratic) submodule with values in $A^G \cong Z$.

Theorem 3.7: *Let $(L_1, [h_1])$ be a non-singular hermitian form over $A = \mathbf{Z}G$ where $G = \mathbf{Z}/d$ is a finite cyclic group, and the involution on A maps $g \mapsto g^{-1}$ for all $g \in G$. Suppose that L_1 is a finitely-generated free A-module with $ess\text{-}rank_{A_{(p)}}(L_1, [h_1]) = b > 0$ for all rational primes p. If $d \neq 2, 3$, assume that $b \geq 3$. Given a splitting*

$(L_1,[h_1])^G \cong (N,[k]) \perp H(\mathbf{Z}^r)$, there exists a hermitian submodule $(L,[h])$ such that $(L_1,[h_1]) = (L,[h]) \perp H(A^r)$ and $(L,[h])^G \cong (N,[k])$.

Proof: We first construct the form L and then prove it is isomorphic to L_1. In applying the results concerning "transitivity on unimodular elements" we can mostly work inside the quadratic submodule of L_1, since transvections extend to the whole module. The argument is divided into various steps. Note that L_1 is free over A of rank $b+2r$ with $r>0$, and we assumed that $b \geq 3$ unless $d = 2,3$ (the case $d=1$ is vacuous).

If Γ denotes an involution-invariant maximal order containing A in $\mathbf{Q}G$, then $\Gamma = \bigoplus \Gamma_i$, $i|d$, where $\Gamma_i = \mathbf{Z}[\zeta_i]$. Under our assumptions in (4.5)(i), the induced hermitian modules $\Gamma_i L_1$ are all indefinite and have rank ≥ 3.

(i) We first consider the problem of splitting off $H(\Gamma^r)$ from ΓL_1. For each factor of the maximal order we will show that $\Gamma_i L_1 = J_i \perp H(\Gamma_i^r)$ for some hermitian module of rank b. Then we let $J = \bigoplus\{J_i \,:\, i|d\}$.

When $i=1$ we use the given splitting of $L_1^G = N \perp H(\mathbf{Z}^r)$. If i is divisible by an odd prime, then $\Gamma_i L_1$ admits a quadratic refinement since the trace map is onto. By **[27**, Thm. 11], there exist quadratic modules J_i over Γ_i so that $J_i \perp H(\Gamma_i^{r_i}) = \Gamma_i L_1$, and each J_i has the minimum rank consistent with the multi-signature, and the requirement that $J_i \perp H(\Gamma_i)$ has rank ≥ 3 over Γ_i. We add hyperbolics to some of the J_i as necessary to assume that they all have the same rank b.

If $i=2$, the fact that $b>0$ allows us to write $\Gamma_2 L_1 = J_2 \perp H(\mathbf{Z}^r)$. Finally, if $i = 2^k$ for some integer $k \geq 2$ then the required splitting follows from

Lemma 3.8: *For $i = 2^k \geq 4$, ess-rank$(\Gamma_i L_1) \leq b$.*

(ii) Next we consider the problem of splitting off $H(\hat{A}_p^r)$ from $\hat{\mathbf{Z}}_p \otimes L_1$. If p is odd, this form has a quadratic refinement and hence we can reduce modulo the radical, where the problem is trivial. If $p=2$, we may suppose that d is a power of 2 (otherwise a quadratic refinement exists again), and hence that $\hat{A}_2$ is a local ring. Then any hermitian form splits into an orthogonal direct sum of one and two-dimensional forms. Now we claim that *ess-rank*$(\hat{\mathbf{Z}}_2 \otimes L_1) \leq b$. This follows by induction from the pull-back diagram

$$\begin{array}{ccc} \hat{\mathbf{Z}}_2[\mathbf{Z}/2^k] & \rightarrow & \hat{\mathbf{Z}}_2(\mathbf{Z}/2^{k-1}) \\ \downarrow & & \downarrow \\ \hat{\mathbf{Z}}_2(\zeta_{2^k}) & \rightarrow & \mathsf{F}_2[\mathbf{Z}/2^{k-1}] \end{array}$$

and the fact that *ess-rank*$(\Gamma_i L_1) \leq b$. To lift a splitting over $\hat{A}_2$ given one over the corners of this pull-back square, we must lift enough automorphisms σ of $U = U_0 \perp H(P)$ over $\mathsf{F}_2[\mathbf{Z}/2^{k-1}]$ to act transitively on hyperbolic pairs in U. This problem can be studied over the quadratic submodule of U by the methods of §1. By Lemma 1.8 or Lemma 1.23, we need only lift certain transvections and $H(\mathsf{F}_2[\mathbf{Z}/2^{k-1}]^\times)$, together with the "flip" isometry of a hyperbolic plane which interchanges the standard basis elements. In these cases, the lifting can be done over $\hat{\mathbf{Z}}_2(\mathbf{Z}/2^{k-1})$.

It follows that $\hat{\mathbf{Z}}_2 \otimes L_1$ splits off the required number of hyperbolic summands. Let $\hat{L}_p$ denote an orthogonal complement: $\hat{\mathbf{Z}}_p \otimes L_1 = \hat{L}_p \perp H(\hat{A}_p^r)$.

(iii) We consider the completions $\hat{J}_i = \prod \hat{\mathbf{Z}}_p \otimes J_i$ over the product $\hat{\Gamma}_i$ of the p-adic completions at all primes $p|d$. These are unimodular forms over the rings of integers in local fields, and we will apply the classification theory of such forms (see [**22**]). If p is odd, then the forms are detemined by the rank and determinant. It follows that the forms are *standard*, i.e. either hyperbolic or isometric to $\langle a\rangle \perp \ldots \perp \langle 1\rangle$ [**13**, 7.1,8.2], [**19**, 92:1]. If $p = 2$, the forms $\hat{\mathbf{Z}}_2 \otimes J_i$, $i = 1, 2$ are again standard of the same type [**19**, 93:15, 93:18]. If $p = 2$ and i is not a 2-power, then the trace map $\mathbf{Z}[\zeta_i] \to \mathbf{Z}[\zeta_i + \zeta_i^{-1}]$ is onto and so the global form $\Gamma_i L_1$ admits a quadratic refinement. It follows that the forms are determined by the rank.

We are left with the forms $\hat{\mathbf{Z}}_2 \otimes J_i$ over ramified rings of integers with non-trivial involution. For these, unimodular forms are classified by the rank, determinant, and the norm ideal $n(J_i)$: the ideal in $\hat{\mathbf{Z}}_2[\zeta_i]$ generated by the values $\langle u, u\rangle$ for all elements $u \in \hat{J}_i$ [**13**, 10.4]. It is easy to see that either the form represents a unit, or the form admits a quadratic refinement. In the first case, the norm ideal is the whole ring and the form is $\langle a\rangle \perp \ldots \perp \langle 1\rangle$. In the second case, we can write $\alpha = \langle u, u\rangle$ in terms of the basis $(1, \sqrt{-1}, \zeta^a, \zeta^{-a} \,|\, 1 \leq a < 2^{k-2})$, if $i = 2^k$. Then $\alpha = \bar{\alpha}$ implies that the the coeficient of $\sqrt{-1}$ is zero, and those of ζ^a, ζ^{-a} are equal. Hence α is a non-unit implies that the coefficient of 1 is even and so α is a trace. When a quadratic refinement exists, the rank is even and we may reduce modulo the radical to see that $\textit{ess-rank}(\hat{\mathbf{Z}}_2 \otimes J_i) \leq 2$.

(iv) One consequence of the local classification given in step (iii) is the fact that cancellation of hyperbolics is possible over each factor of $\hat{\Gamma}$. Another is that when the rank $b \geq 3$, then $\hat{J}_i$ splits off a hyperbolic plane $H(\hat{\Gamma}_i)$.

Now we claim that for each $p|d$, the forms $\hat{\mathbf{Z}}_p \otimes J$ and $\hat{L}_p$ are isometric. Indeed, by construction they are stably isometric to $\hat{\Gamma}_p L_1$ and cancellation of hyperbolics is possible over $\hat{\Gamma}_p$. It follows that $\hat{\mathbf{Z}}_p \otimes J \cong \hat{L}_p$ at primes dividing d. Let L over A be the pull-back of the forms $\hat{L}_p$ over $\hat{A}$ and J over Γ. Then $\Gamma_i L = J_i$, for all i and $\hat{Z}_p \otimes L = \hat{L}_p$ for each prime $p|d$.

(v) The final step is to show that $L \perp H(A^r) \cong L_1$. For this it is enough to show that for any isometry α of $\hat{J} \perp H(P)$, $P = \hat{\Gamma}^r$, which is the identity in the $\hat{\Gamma}_1$ component, there exists a *liftable* isometry $\sigma \in U(\hat{J} \perp H(P))$ such that $\sigma\alpha = \beta \perp 1$ for some $\beta \in U(\hat{J})$. Here *liftable* means that σ is the product of isometries which come from the forms over Γ or $\hat{A}$. This proves that $L \perp H(A^r) \cong L_1$. It follows that L is a stably-free A-module and hence (by cancellation of modules) L is a free A-module.

To esablish this, we remark again that by Lemma 1.8 or Lemma 1.23, σ can be assumed to be a suitable product of transvections and isometries of the hyperbolic plane In particular σ has determinant ± 1, and elements with determinant -1 are only needed when $b = 1$ or 2 since $J_i \otimes \hat{\mathbf{Z}}_p$ splits off a hyperbolic plane whenever its rank $b \geq 3$. Under our assumptions, $b \leq 2$ only occurs if $d = 2, 3$. A second remark is that an isometry is liftable if it is congruent modulo some power of d to a liftable isometry. Indeed, since $d^s\hat{\Gamma} \subseteq \hat{A}$ for some integer s, a matrix for an isometry of the form $(1 + d^s\hat{\tau})$ with respect to a free basis of $\hat{J} \perp H(P)$ can be chosen with entries in $\hat{A}$, giving an isometry $(1 + d^s\tau)$ of $\hat{L}$.

Write $\sigma = \oplus\sigma_i$, where σ_i is the component of σ over $\hat{\Gamma}_i$. To lift the component of σ_2, we need to lift the transvections given in Lemma 1.8, together with those of the form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$, $a \in \hat{\mathbf{Z}}_p^\times$ when $b = 1, 2$. Suppose first that $b \geq 3$. The transvections can be lifted over Γ_2 modulo some high power of d. We then compose α with an isometry over $\hat{A}$, and may assume that σ_i is the identity for $i = 1, 2$. Now if $b = 1, 2$ and $d = 2$ the completion at $p = 2$ is the only prime to consider. For $p = 2$, we use the global flip over $J_2 \perp H(\mathbf{Z})$ and are left with an element of the form $\gamma = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ with $a \in \hat{\mathbf{Z}}_2^\times$. However, the units $(1, a) \in \hat{\Gamma}_1 \oplus \hat{\Gamma}_2$ can be lifted (modulo squares) to units in $\hat{\mathbf{Z}}_2[\mathbf{Z}/2]$. Thus we again get σ_i is the identity for $i = 1, 2$ and we are done if $d = 2$. For $d = 3$ and $b = 1, 2$ the argument is similar: we lift units $(1, a) \in \hat{\Gamma}_1 \oplus \hat{\Gamma}_3$. To complete the case $d = 3$ we need only lift transvections.

To lift the components of σ over Γ_i, where $i \neq 1, 2$, we use [**23**, 5.12]: the group $SU(J_i \perp H(\Gamma_i^r))$ is dense in $SU(\hat{J}_i \perp H(\hat{\Gamma}_i^r))$. Since each σ_i is now a product of transvections (which have determinant 1), this can be done using isometries over Γ and $\hat{A}$. ∎

The Proof of Lemma 3.8: Under our assumptions, $U = \Gamma_i L_1$ rank $(b + 2r)$ over $R = \mathbf{Z}[\zeta_{2^k}]$, with $b \geq 3$. The ring R has non-trivial involution since $k \geq 2$. Let $S = \mathbf{Q}[\zeta_{2^k}]$ and $\hat{R} = \hat{\mathbf{Z}} \otimes R$, $\hat{S} = \hat{\mathbf{Z}} \otimes S$. Here $\hat{\mathbf{Z}}$ denotes the product of all the p-adic completions of the integers. We will prove the Lemma by using the pull-back square [**29**]:

$$\begin{array}{ccc} R & \to & S \\ \downarrow & & \downarrow \\ \hat{R} & \to & \hat{S} \end{array}.$$

Then U is the pull-back of forms $\hat{U}$ over $\hat{R}$ and U_S over S via an isometry $\alpha : (\hat{U} \otimes S) \cong (U_S \otimes \hat{\mathbf{Z}})$. From the local classification, $\textit{ess-rank}(\hat{U}) \leq b$. Since a form over the global field S represents zero if and only if it represents zero at all places [**22**, 10.1.1, 6.6.5], we conclude that $\textit{ess-rank}(U_S) \leq b$ as well. Fix some splittings $\hat{U} = \hat{U}_0 \perp H(\hat{R}^r)$ and $U_S = U_S' \perp H(S^r)$ consider the induced hyperbolic summands over $\hat{S}$. Since $b \geq 3$, we can find a product σ of transvections by Lemma 1.8 such that $\sigma \circ \alpha = \beta \perp 1$ preserves a hyperbolic summand $H(\hat{S}^r)$. It follows that $\textit{ess-rank}(U) \leq b$. ∎

Remark 3.9: With only minor modifications, the argument just given would prove the analogous statement to Theorem 3.7 for forms on many non-free modules. For example if $L_1 = \mathbf{Z}^t \oplus P$ with P free, the same conclusion holds. The only additional facts needed are in [**19**, 102.9, 102.10], to show that the genus equals the class for the form $\Gamma_1 L_1$.

§4: Group Actions on Four-Manifolds

We now give some direct applications of cancellation methods to group actions on topological 4-manifolds.

Corollary 4.1: *Let M be a closed, oriented, simply-connected topological 4-manifold. Let G be a finite cyclic group acting locally linearly and pseudo-freely on M, p-reserving the orientation, with M^G non-empty. Let M_0 denote the complement of* *a set of disjoint open G-invariant 4-disks around the fixed points, and assume that $M_0/G = W \sharp (S^2 \times S^2)$, where $\partial W = \partial(M_0/G)$. Then the action (M, G) is classified up to equivariant homeomorphism by the local fixed-point data, the signature, type, and Euler characteristic of M and the Kirby–Siebenmann invariant of M_0/G.*

The definition of "local fixed–point data" was given in the Introduction.

The Proof of Corollary 4.1: Let $G = \mathbf{Z}_n$ be a finite cyclic group action acting semi-freely and locally linearly on a 1-connected 4-manifold M. We also assume that the action is orientation-preserving and has a non-empty fixed point set consisting of isolated fixed points. Then we can consider the free action of G on $M_0 = M - \bigcup U_i$, where the U_i are small open G–invariant neighbourhoods of the fixed points. Let us define $X = (M - \bigcup U_i)/G$ and $\partial_i X = \partial U_i/G$. If G acts on another 4-manifold M' with the same fixed point data, then we choose a homeomorphism of ∂X to $\partial X'$. The actions are equivalent if one can extend this to a homeomorphism from X to X'.

We note that since the action is not free, M is spin if and only if X is spin. To see this, note that M spin implies that $w_2(X)$ comes from $H^2(G, \mathbf{Z}/2)$, which maps isomorphically onto $H^2(\partial_i X, \mathbf{Z}/2)$. But $\partial_i X$ is spin. The converse is clear.

If n is even and M is spin, choose a spin structure on X and consider its restriction to $\partial_i X$. Since there are precisely two spin structures on X, the spin structure on ∂X is unique up to a simultaneous change on $\partial_i X$ for each i. We call the equivalence class of spin-structures on ∂X the *spin fixed point data.* We say that the local fixed point data in two different actions are *equivalent* if there is a homeomorphism of ∂X to $\partial X'$ preserving the spin fixed point data. If n is odd, there is a unique spin structure on X and ∂X, so we do not need to remember it.

Now suppose that M and M' have the same signature, type, Euler characteristic and equivalent local fixed point data. Then X and X' also have the same signature and Euler characteristic. Since X is spin if and only if M is spin, the w_2-types of X and X' are the same. If in addition they have the same KS-invariant, then it follows from a result of **[14]** that the actions are stably G-homeomorphic. More precisely, it is shown in **[14]** that a homeomorphism between ∂X and $\partial X'$ (which preserves the spin structure if X and X' are spin), extends to a homeomorphism between $X \sharp r(S^2 \times S^2)$ and $X' \sharp r(S^2 \times S^2)$ for some integer r. Here the connected sum is away from the boundary. Since we assume in the statement of Corollary 4.1 that $X = W \sharp S^2 \times S^2$, we are done by Theorem B$'$. ∎

We finish with another application, this time to the existence and uniqueness of locally-flat topological embeddings of 2-spheres in simply-connected 4-manifolds. To prepare for this we need the following sharpening of Theorem B$'$ (in a special situation). Let X_1 and X_2 be compact oriented 4-manifolds with same boundary Y. If Y is non-empty, we will assume that $\pi_1(Y) \to \pi_1(X_i)$ is surjective for $i = 1,2$. We call a map $\alpha: \pi_1(X_1) \to \pi_1(X_2)$ compatible if it commutes with map induced by the inclusions from Y to X_i. Similarly a map $\beta: H^2(X_1) \to H^2(X_2)$ is called compatible if it commutes with the restrictions to $H^2(Y)$ and if the induced map $\beta^*: H^2(X_1, Y)/Tors \to H^2(X_2, Y)/Tors$ is an isometry (here and in the following all homologies are with coefficients in $\mathbf{Z}$). The "w_2-type" of a manifold X with cyclic fundamental group is (I) if $w_2(\tilde{X}) \neq 0$, (II) if $w_2(X) = 0$, and (III) if $w_2(X) \neq 0$ but $w_2(\tilde{X}) = 0$.

Proposition 4.2: *Let X_1 and X_2 be compact oriented 4-manifolds with same w_2-type, KS-invariant, same boundary Y and same fundamental group $G = \mathbf{Z}/d$. Suppose that Y is empty or $\pi_1(Y) \to \pi_1(X_i)$ is surjective. Then for any pair of compatible isomorphisms α and β there is a stable homeomorphism from $X_2 \sharp r(S^2 \times S^2)$ to $X_1 \sharp r(S^2 \times S^2)$ rel. Y inducing α and $\beta \oplus id$.*

Proof: Abbreviate $H_2(X_1)/Tors$ by H. Consider the following three fibrations $B(I)$, $B(II)$ and $B(III)$ over $BTop$, which are the normal 1-types of X_i with w_2-type (I), (II) and (III) respectively. The total space is in all cases $K(G,1) \times K(H,2) \times BTopSpin$. The map to $BTop$ is given by a vector bundle $E_1 \times E_2 \times \gamma$, where in case (I): E_1 is trivial and E_2 a complex line bundle with $w_2(E_2) = w_2(X_1) \in Hom(H, \mathbf{Z}/2)$; in case (II): E_1 and E_2 are trivial; and in case (III): E_1 is a complex line bundle with $u_1^*(w_1(E_1)) = w_1(X_1)$ and E_2 is trivial. Here u_1 is a classifying map of the universal covering of X_1. Let g_1 be a map from X_1 to $K(H,2)$ inducing Id on $H_2/Tors$. Then $u_1 \times g_1 : X_1 \to K(G,1) \times K(H,2)$ is a 2-equivalence and the difference of the tangent bundle of X_1 with the pullback of $E_1 \times E_2$ admits a spin-structure. Choose a spin-structure to get a lift $\bar{\nu}_1 : X_1 \to B$ of the normal Gauss map which is a 2-equivalence, i.e. a normal 1-smoothing [14].

Since α is compatible we can choose $u_2 : X_2 \to K(G,1)$ such that $(u_2)_*\alpha = (u_1)_*$. Consider g_1 as an element of $H^2(X_1; H)$ and let $g_2 : X_2 \to K(H,2)$ represent $\beta((g_1)$. As before u_2 and g_2 define a normal 1-smoothing $\bar{\nu}_2$ of X_2 in B. Since α and β are compatible we have $u_1|Y \sim u_2|Y$ and $g_1|Y \sim g_2|Y$. Finally choose the spin-structures so that they agree on Y (which is possible since if Y is not empty, $H^1(X_i; \mathbf{Z}/2) \to H^1(Y; \mathbf{Z}/2)$ is surjective). Thus our normal 1-smoothings agree on Y.

Lemma 4.3: *For $B = B(I), B(II)$ or $B(III)$ the map given by the signature, the KS invariant and the fundamental class injects $\Omega_4(B)$ into $\mathbf{Z} \oplus \mathbf{Z}/2 \oplus H_4(K(H,2), \mathbf{Z})$.*

Before we prove this Lemma we finish the proof of Proposition 4.2. Since the signature and the fundamental class are determined by the intersection form and β^* is an isometry, $X_1, \bar{\nu}_1$ and $X_2, \bar{\nu}_2$ are B-bordant rel. boundary. Thus there is a stable homeomorphism rel. boundary from X_1 to X_2 commuting with the maps to $K(H,2)$ [14]. ∎

The Proof of Lemma 4.3: We apply the Atiyah-Hirzebruch spectral sequence with E^2-term $H_i(M(E_1 \times E_2); \Omega_j^{TopSpin})$. The d_2-differential from $H_i(M(E_1 \times E_2); \Omega_1)$ to $H_{i-2}(M(E_1 \times E_2); \Omega_2)$ is the dual of $Sq^2 + w_2(E_1 \times E_2)$ and from $H_i(M(E_1 \times E_2); \Omega_0)$ to $H_{i-2}(M(E_1 \times E_2); \Omega_1)$ is the composition of the reduction from $\mathbf{Z} = \Omega_0$ to $\mathbf{Z}/2$ and the dual of $Sq^2 + w_2(E_1 \times E_2)$. A simple calculation shows that the E^∞-term has $\mathbf{Z} = \Omega_4^{TopSpin}$ on the spot $(0,4)$, a subgroup of $H_4(K(H,2)$ on the spot $(4,0)$ and , in case of type (I) or (III) $\mathbf{Z}/2$ on the spot $(2,2)$. Thus the proof is finished if one has in case (I) or (III) a manifold with a normal B-structure, signature 0 and KS non-zero. Such manifolds are given by the difference of the closed E_8 manifold and the Enriques surface in case (III) or $\mathbf{C}P^2 \sharp 9\overline{\mathbf{C}P}^2$ in case I. ∎

Corollary 4.4: *Suppose that X_1 and X_2 satisfy the assumptions of Proposition 4.2, and addition assume that $X_1 = X_1' \sharp S^2 \times S^2$. Then any pair of compatible isomorphisms α and β can be realized by a homeomorphism between X_1 and X_2 rel. boundary.*

Proof: Note that $H \cong \epsilon_*(\pi_2(X_i))$. Then the statement follows by cancellation as in the proof of Theorem B from Theorem 1.20 and Corollary 2.4. ∎

The method of cancellation can be used effectively to study the existence and classification of locally flat 2-spheres representing a given homology class $x \in H_2(N;\mathbf{Z})$, where N is a closed 1-connected topological 4-manifold. Then $x = dy$ with y primitive and d is called the *divisibility* of x. Such embeddings are called *simple* if the fundamental group of the complement is abelian (and hence isomorphic to $G = \mathbf{Z}/d$). Denote $y \cdot y$ by m, and let $b_2(N)$ and $\sigma(N)$ denote the rank and signature of the intersection form on $H_2(N, \mathbf{Z})$.

Following the original idea of V. Rochlin [21] (compare [17]), these embedding problems will be studied via an associated semi-free cyclic group action: if $f: S^2 \to N$ is an embedding representing a homology class of divisibility d, then there is a d-fold branched cylic covering (M, G) over N, branched along $f(S^2)$.

Theorem 4.5: *Let N be a closed 1-connected topological 4-manifold.*
i) Let $x \in H_2(N;\mathbf{Z})$ be a homology class of divisibility $d \neq 0$. Then x can be represented by a simple locally flat embedded 2-sphere in N if and only if

$$KS(N) = (1/8)(\sigma(N) - x \cdot x) \pmod 2$$

when x is a characteristic class, and if

$$b_2(N) \geq \max_{0 \leq j < d} |\sigma(N) - 2j(d-j)(1/d^2)x \cdot x|.$$

ii) Any two locally flat simple embeddings of S^2 in N representing the homology class x are ambiently isotopic if $b_2(N) > |\sigma(N)| + 2$ and

$$b_2(N) > \max_{0 \leq j < d} |\sigma(N) - 2j(d-j)(1/d^2)x \cdot x|.$$

If d is odd this result was asserted in [17] (under a weaker assumption for part (ii)), but their argument has a gap which we do not see how to overcome. It occurs on [17, p.410] where they claim that: "$[H_2(N), \lambda, x] \oplus H(\mathbf{Z}^r)$ splits off a copy of $H(\mathbf{Z}^{r+1})$" for large enough r. This appears to contradict the example contained in [17, Remark 4.5] which is geometrically realizable by part (i). Their proof of part (i) also appears to be incomplete. In particular, the results of [1] apply to transitivity on *unimodular* elements of the same length, not to "primitive" (i.e. indivisible) elements as claimed in [17, p.407]. In addition, the proof of [17, 4.3] is incorrect. The strong approximation theorem does not say that "$\hat{\sigma} \otimes_{\hat{\Lambda}}$ can be lifted to an isometry $\sigma_\Gamma \in SAut([P, h, z] \otimes_\Lambda$

$\oplus H(\Gamma^r))$", but only provides such liftings modulo a power of d. In particular, an approximate lifting may not preserve the $H(\Gamma^r)$ summand.

Proof: We assume $d > 1$, and leave the necessary changes for the case $d = 1$ to the reader.

i) We will use the connection mentioned above between simple locally flat 2-spheres and semi-free locally linear actions of $G = \mathbf{Z}/d$ with fixed point set S^2 on 1-connected topological 4-manifolds. The correspondence is given by the ramified covering along S^2 and in the other direction by the embedding of the fixed point set S^2 into the orbit space. Since 1-connected 4-manifolds are classified by the intersection form and the KS-invariant it is enough to construct for $x \in H_2(N)$ a locally linear G-action on a 1-connected manifold M as above with corresponding KS-invariant such that the following pointed hermitian forms are isomorphic: $(H_2(M/G), [M^G]) \cong (H_2(N), x)$. To concentrate attention on the differences between our approach and that in [**17**], we refer the reader to that source for background, motivation and some facts about the geometric set-up.

It is easy to construct a G-action on a manifold M with the right KS-invariant such that for some r the pointed hermitian modules are isomorphic: $(H_2(M/G), [M^G]) \cong (H_2(N) \oplus H(\mathbf{Z}^r), x \oplus 0)$ [**17**, Thm. 2.1]. Now $H_2(M)$ is a stably free A-module [**17**, Theorem 3.4], thus we assume that it is free. Suppose that there is an hermitian $A = \mathbf{Z}G$-module L and a class $\alpha \in L^G$ such that $(L \perp H(A^r), \alpha \oplus 0) \cong (H_2(M), [M^G])$. Then one can cancel the $H(A^r)$ geometrically to realize $(H_2(N), x)$. This was carried out for free G-actions in [**10**] and the same proof works here (compare [**17**, Prop. 4.1]).

Lemma 4.6: *There is a hermitian module L such that $L \perp H(A^r) \cong H_2(M)$ and $L^G \cong H_2(N)$.*

It follows that there is a primitive class $z' \in L^G$ such that z' has same norm as $z = [M^G] \in H_2(M)^G$. Then $z' \oplus 0$ and z are primitive elements of the same length.

Lemma 4.7: *There exists an isometry ρ on $L_1 \perp H(A^3)$ such that $\rho(z' \oplus 0) = z$*

Thus the pointed hermitian forms $L \perp H(A^{r+3}, z' \oplus 0)$ and $(H_2(M) \perp H(A^3), z \oplus 0)$ are isomorphic finishing the proof of (i).

ii) By our assumptions we have $b_2(N) - |\sigma(N)| \geq 4$. Thus $N \cong N' \sharp S^2 \times S^2$ for some manifold N'. Choose a class z in $H_2(N')$ of the same type, divisibility and norm as x. By [**25**] the group of isometries acts transitively on elements of same type, divisibility and norm and thus we can assume that $x = z \oplus 0$. Now the uniqueness statement is a consequence of part (i), Proposition 4.2 and the fact that homeomorphisms acting trivially on homology are isotopic to the identity ([**15**, Thm. 1] and [**20**]).

By part (i), we know that there is a simple embedding $f_1 : S^2 \to N'$ representing z. Let $f_2 : S^2 \to N$ be any simple embedding representing $x = z \oplus 0$. Let $X_i = N - \nu_i$, where ν_i denotes a small tubular neighborhood of $f_i(S^2)$ in N, and the boundaries $\partial X_1 = \partial X_2$. We first check that the the w_2-type is the same for X_1 and X_2. Let $x = dy$ where y is a primitive class with $y \cdot y = m$, and assume that d is even. Then X_1 is spin if and only if X_2 is spin, since the map $H^2(N, \mathbf{Z}/2) \to H^2(X_i, \mathbf{Z}/2)$ is an injection. If $\tilde{X}_1$ is non-spin, then its normal 1-type is the same as $X_1 \sharp CP^2 \sharp \overline{CP}^2$ and so the intersection form on $H_2(X_1, \mathbf{Z})$ is odd. But $H_2(X_1, \mathbf{Z}) = \{v \in H_2(N) \,|\, v \cdot x = 0\}$, and so the intersection form on X_2 is also odd. Therefore both have the same w_2-type. In the final case, X_1 is non-spin, but $\tilde{X}_1$ is spin, so $w_2(X_1)$ is the pull-back of some class $\bar{w}_2 \in H^2(\pi_1(X_1), \mathbf{Z}/2)$. But then $\langle w_2, v\rangle = 0$ and X_1 has an even intersection form, similarly for X_2.

The existence of a compatible isomorphism β follows from the fact that we have and isomorphism between (part of) the cohomology exact sequence of the pair $(X_i, \partial X_i)$:

$$\begin{array}{ccccc} H^2(X_i, \partial X_i) & \to & H^2(X_i) & \to & H^3(\partial X_i) \\ \downarrow & & \downarrow & & \downarrow \\ \{v \in H_2(N) \,|\, v \cdot x = 0\} & \to & H_2(N)/\langle x\rangle & \to & H_2(N)/\langle x, \{v \in H_2(N) \,|\, v \cdot x = 0\}\rangle \end{array}$$

Let β be the map induced by the identity on $H_2(N)$. By Corollary 4.4 there is an ambiant homeomorphism from $(N, f_1(S^2)$ to $(N, f_2(S^2)$ inducing the identity on H_2. ∎

The Proof of Lemma 4.6: We will use the fact that the quantity $\sigma_j = (\sigma(N) - 2j(d-j)(1/d^2)x \cdot x)$ is just the formula derived by Rochlin [**21**] for the signature of the eigenspace of $L_1 \otimes \mathbf{C}$ on which G operates as $\exp(2\pi i j/d)$ (compare [**17**]). It follows from the inequality $b_2(N) \geq \max|\sigma_j|$ that $b_2(N) \geq 3$ unless $d = 1, 2$ or 3. The signature of $\Gamma_k L_1$ is the sum of the σ_j over all j such that $k = d/(d, j)$. When $i = 1$, we use the given splitting $(H_2(M/G), [M^G]) \cong (H_2(N) \oplus H(\mathbf{Z}^r), x \oplus 0)$. The result now follows from Theorem 3.7. ∎

Proof of Lemma 4.7: The argument uses the same basic strategy as the proof of Theorem 3.7 but is much easier. We use the pull-back square

$$\begin{array}{ccc} A & \to & \mathbf{Z} \\ \downarrow & & \downarrow \\ I^* & \to & \mathbf{Z}/d \end{array}$$

and consider first the images of the elements $z, z' \oplus 0$ in $\Gamma_1 L_1$ (*i.e.* under the augmentation map projection $A \to \mathbf{Z}$). It is enough to find an isometry ρ_1 with $\rho_1(2z' \oplus 0) = 2z$, so we may consider the problem of transitivity of the unitary group on such elements in the quadratic submodule. By stabilizing the form by $H(P)$, $P = \mathbf{Z}^3$, we can find a product of transvections

$$\rho_1 \in \langle EU(H(P), Q; L_1), EU(H(P)), H(E(P))\rangle$$

which has the required property over $\Gamma_1 L_1$. The images of $z, z' \oplus 0$ are zero in I^* and hence it is enough to lift the reduction of $\rho_1 \pmod d$ over I^*. Since the map $I^* \to \mathbf{Z}/d$ is surjective, this element can be lifted to a product of transvections in $L_1 \perp H(A^3)$. ∎

The arguments in the proof of part (ii) of Theorem 4.5 also give a statement analogous to [**17**, 2.5]:

Theorem 4.8: *Let N be a closed 1-connected topological 4-manifold, such that $N = N_0 \,\sharp\, (S^2 \times S^2)$. Any two locally flat, simple, embeddings $S^2 \hookrightarrow N_0$, representing the same integral homology class, are isotopic in N.*

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