NILPOTENT ORBITS, PRIMITIVE IDEALS AND CHARACTERISTIC CLASSES

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R. MACPHERSON

A geometric Perspective in Ring theory

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Comments

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The present preprint version of the book consists of the following elements: Chapter 1, which is essentially congruent to a paper that appeared in Mathematische Annalen volume 278; chapters 2-4, that evolved from earlier versions preprinted at the IHES in Bures-sur-Yvette and at BUGH Wuppertal, chapter 3 being revised in respond to the valuable advices of Klaus Bongartz (BUGH Wuppertal); plus a new chapter five, complemented by some additional introductions and bibliographies.

CONTENTS

§ 5: PRIMITIVE IDEALS and CHARACTERISTIC CLASSES

General introduction

Here we roughly summarize the contents, and comment on the methods and exposition, of this book. For more precise statements, and a more detailed discussion of the contents, we refer to the subsequent separate introductions to chapters 1 resp. 2 - 4 resp. 5.

0.1 Summary

Let G be a complex semisimple Lie group. We study

(1) the geometry of nilpotent orbits in the Lie algebra $\underline{g}$ of G, and

(2) the classification of primitive ideals in the enveloping algebra $U(\underline{g})$.

A "primitive ideal" is a kernel of an irreducible infinitesimal representation (for simplicity, assume with trivial central character in this summary). Our principal object is to gain insight into (1) and (2) simultaneously, and to understand their relation.

Originally, both topics appeared fairly unrelated, and evolved for some time quite independently into highly cultivated research areas, with remarkable theories. For an excellent exposition of (2), see for instance [Ja]. However, both (1) and (2) have been related to irreducible representations of the Weyl group W, by fundamental work of T.A. Springer resp. A. Joseph, with some superficial similarities on one hand indicating some deep relations (as was suggested from the outset by conjectures of Borho resp. Jantzen [B2]), but with intriguing discrepancies on the other hand, which remained a mystery for several years.

As an illustration for the non-expert reader, let $G = SL(n,\mathbb{C})$, where W is the symmetric group of n letters (in fact, of the n eigenvalues of a matrix). Then

(1) a "nilpotent orbit" is a <u>conjugacy class</u> of nilpotent n by n matrices, and is specified by a <u>partition</u> of n (theory of Jordan normal form), whereas

(2) a primitive ideal is specified by a Young standard <u>tableau</u> (theory of Joseph's Goldie rank polynomials [Ja]).

Here both Springer's resp. Joseph's W representations are equivalent to the one specified by the corresponding Young <u>diagram</u> (Frobenius' theory of representations of the symmetric group).

We give a reformulation of both Springer's and Joseph's irreducible representations in a uniform fashion, in terms of <u>characteristic classes</u> of cone bundles on the flag variety X of G.

(1) To a nilpotent orbit, we attach a bunch of cone bundles in the cotangent bundle T^*X as follows: Take the preimage under the (Kostant-Souriau) momentum map $T^*X \longrightarrow \underline{g}$, and decompose its closure into irreducible components.

(2) For a primitive ideal J, we obtain a cone bundle by localizing the left module $U(\underline{g})/J$ as a $\mathcal{D}$-module on X (Beilinson-Bernstein localization), and taking its characteristic variety in T^*X.

Our "characteristic classes" are then given in both cases by the (Fulton-MacPherson) Segre classes of these cone bundles, as lowest degree term of the product with the Chern class of T^*X. The characteristic class of a primitive ideal, when interpreted as a harmonic polynomial

(Borel's picture of the cohomology of X), turns out to be proportional to Joseph's Goldie rank polynomial, while the bunch of characteristic classes attached to a nilpotent orbit identify with the canonical basis of Springer's representation; so Joseph's representation becomes conceptually identified with a "special" [L] one of Springer.

We only recall here [Ja] that this representation is finally open to explicit computation (recently extended to rank ≤ 5 by Borho-Steins)*) in terms of integer matrices, as a consequence of the Kazhdan-Lusztig conjecture, proved by Beilinson-Bernstein and Brylinski-Kashiwara.*)

0.2 Methods

Our treatement is based on three relatively new methods: We use the intersection homology approach to (1) as developed in joint work of the first and third author [BM1], [BM2] (in chapter 1), resp. the $\mathcal{D}$-module approach to (2) as developed in joint work of the first and second author [BB1], [BB3](in chapter 5), and we furthermore introduce here (in chapters 2 - 4) equivariant K-theory on T^*X as a unifying concept, which provides an elegant common frame work for the simultaneous investigation of nilpotent orbits and primitive ideals in terms of characteristic classes, and makes their relations appear quite natural. This new perspective was outlined in [BBM2], and is presented in full detail here. Let us mention that there is some minor overlap with parallel work of Victor Ginsburg [Gi].

*) See the notes added in proof at the end of chapter five.

0.3 Exposition

Since this book is a research report in the first place, we do not make a systematic attempt to be self-contained here; in particular we make free use of [BM1], [BM2] resp. [BB1], [BB3] in the proofs of chapters 1 resp. 5. More explicitly speaking, the work of the first and third author on nilpotent orbits and Weyl group representations, resp. the work of the first and the second author on primitive ideals and their characteristic varieties, are taken for granted, as a kind of basement for the ideas we build up here. It also goes without saying that we always have to build on some (back-)ground, in algebraic groups (chapters 1,4), in topology (chapter 1), in algebraic geometry (chapters 2, 4), in representation theory (chapter 5), or non-commutative ring theory (5.10-12), although we do spend some care on keeping the necessary prerequisites down to a minimum. We give full statements or at least references for material used from other sources; we spend some effort to make our fundamental definitions and the statement of main results easily accessable, and to make the logical dependencies between different chapters explicit, where they exist; the core material of the individual chapters can then be studied independently.

So our presentation does not lack ambitions towards independent readability: To return to our above picture, the reader is invited into the building, or into his favourite chambre, without having to worry too much about those parts of the basement, or buildung ground, or other chambres, he might be unfamiliar with.

In fact, our purpose is not so much presentation of individual items of new research (that do occur at various places), but primarily our unified perspective or understanding of certain known results, which we therefore wish to reprove here in full detail. This includes e.g.

(i) a very concrete construction (due to Joseph-Hotta) of Springer's irreducible W representation in terms of integer matrices (due to Hotta-Lusztig) (4.14),

(ii) the irreducibility of Joseph's W representation (5.13),

(iii) the irreducibility of the associated variety of a primitive ideal (5.14) (alias the relation between nilpotent orbits and primitive ideals suggested in [B2]),

(iv) the equivalence (due to Barbash-Vogan) of Joseph's with Springer's representation (5.14),

(v) Joseph's computation of Goldie ranks of primitive ideals (5.18).

While (i), (iii), (iv) have appeared only in research articles so far, (ii) and (v) have already been central themes in Jantzen's book [Ja] (following Joseph), so let us explain the point of our new exposition here. In [Ja], (ii) depends on (v), which in turn depends on several chapters of hard non-commutative algebra in that book. Our point is to totally avoid this core part of [Ja], and to logically separate (ii) from (v), that is to treat the classification of primitive ideals, and the analysis of their Goldie ranks, as two separate purposes. Another point is to show all of the five above mentioned results (along with others) arise in a closely related way in our geometric approach.

INTRODUCTION to CHAPTER 1

Summary.

A nilpotent orbit $\mathcal{O}_u$ in a complex semisimple Lie algebra gives rise to a collection of cone bundles on the flag variety, by taking the closed components of its preimage under Springer's resolution of singularities. Using the generalization of inverse Chern classes of vector bundles to Segre classes of cone bundles due to Fulton and the third author, we attach to each such cone bundle a characteristic class in the cohomology of the flag variety, which is interpreted as a harmonic polynomial on the Cartan subalgebra. Using the intersection homology approach to the study of nilpotent varieties as in [BM1], [BM2] we show that this collection of polynomials transforms under the action of the Weyl group according to Springer's irreducible representation ρ_u which is usually constructed from $\mathcal{O}_u$ by quite different means.

Introduction.

Consider the set N of all nilpotent complex n by n matrices (n $\geq$2). With respect to the action of the group $G = SL(n,\mathbb{C})$ by conjugation, N decomposes into finitely many conjugacy classes. By the theory of Jordan normal form, these classes (or "nilpotent orbits") are in bijective correspondence to the partitions of n. On the other hand, the classical theory of representations of the symmetric group due to Frobenius and Young classifies the irreducible complex linear representations of S_n by the same set of combinatorial objects, i.e. the partitions of n. Here we realize the symmetric group S_n as the group of permutations of the eigenvalues of the diagonal matrices, to identify it with the Weyl group W of $G = SL(n,\mathbb{C})$. In conclusion, this may be used to set up a bijective correspondence between the set of nilpotent orbits of G in its Lie algebra $\underline{g}$, and the set of classes of irreducible representations of its Weyl group W.

The more recent (1976) theory of T.A. Springer [S1] gives a very elegant, deep geometrical explanation for this correspondence, and simultaneously generalizes it to the case of an arbitrary complex semisimple Lie group G as follows. Given a nilpotent element $u \in N$, we denote $\mathcal{O}_u$ its G orbit (conjugacy class), and X^u the variety of all flags (maximal chains of linear subspaces in $\mathbb{C}^n$) which are preserved by u. Then Springer constructs a linear W action on the cohomology $H^*(X^u)$ of this variety. This action commutes with the obvious action of the isotropy group G_u, and is <u>irreducible</u> on its invariants $H^{2d}(X^u)^{G_u}$ in the highest nonzero cohomology group of X^u (of degree $2d = \dim_{\mathbb{R}} X^u$). The resulting irreducible representation of W will be refered to as <u>Springer's representation</u> ρ_u corresponding to the nilpotent orbit $\mathcal{O}_u$. In our example $G = SL(n,\mathbb{C})$, Springer's correspondence $\mathcal{O}_u \longmapsto \rho_u$ gives an intrinsic description of the bijection described above in completely classical, but more superficial (combinatorial) terms. For an arbitrary semisimple group G, this correspondence turns out to be <u>injective</u>, that is to say different nilpotent orbits $\mathcal{O}_u \neq \mathcal{O}_v$ correspond to non-equivalent representations $\rho_u \not\cong \rho_v$; but it is no longer <u>surjective</u> in general, and Springer's theory explains more precisely why this is so, by relating the "missing" irreducible W representations to non-trivial local systems on some of the nilpotent orbits.

After Springer's original version [S1], this remarkable theory has been further investigated and improved in several respects. We can only mention here some of the many research contributions by various authors. Various alternative constructions of Springer's W action on $H^*(X^u)$ have been given by Springer himself [S2], by Kazhdan-Lusztig [KL], Slodowy [Sl], and Lusztig [Lu]; for a detailed account of why all these very different approaches yield essentially the same W action, we may refer to Hotta [Ho], appendix, and Spaltenstein [Sp], § 2. An explicit calculation of the correspondence $\mathcal{O}_u \longmapsto \rho_u$ was carried out by Shoji (G classical [Sh1], type F_4 [Sh2]), Springer (type G_2 [S1]), resp. Alvis and Lusztig (types E_6, E_7, E_8 [AL]). This latter calculation, and even the complete tabulation of the W action on the full cohomology groups $H^*(X^u)$ (in all degrees) by Beynon and Spaltenstein [BS], had to make use of certain new formulae for the multiplicities in Springer's representations on $H^*(X^u)$ obtained by two of us in [BM1], [BM2].

In these last mentioned papers, a reformulation of Springer's theory of Weyl group representations was given in terms of intersection homology theory. It seems to us that this new approach, which will also provide essential methods of proof for our present paper, offers a more satisfactory, most natural conceptual frame-work to understand Springer's theory. One of its key points is to relate the multiplicities of Springer's W representation on $H^*(X^u)$ to the local Betti numbers of the intersection homology groups of closures of nilpotent orbits. This means that Springer's representation is - up to equivalence - completely described by certain numerical topological invariants of the singularities of the closures of nilpotent orbits. The precise formula [BM1] relates two a priori unknown sets of numbers; however, its structure offers an opportunity for recursive calculation, which eventually yields complete knowledge of both sets of numbers.

As a consequence of [BM1], we may say that we totally know Springer's representations - but only up to equivalence. Let us suggest now an even more ambitious goal: To refine our geometrical analysis to an internal description of the representations themselves, in terms of matrices with respect to a suitable fixed vector-space basis; all items in this description should be defined or interpreted in geometrical terms. We do not know at present how to achieve this for the full W representation on $H^*(X^u)$, but we do know such a description at least for Springer's irreducible representation ρ_u, and the purpose of our present paper is to explain this in some detail.

Our description of the representation ρ_u is in terms of characteristic classes of cone bundles on the flag variety X, which are constructed from the nilpotent orbit $\mathcal{O}_u$ as follows: Take the preimage of $\mathcal{O}_u$ in the cotangent bundle T^*X (which maps onto the nilpotent cone N under the so-called momentum map), and then decompose its closure into irreducible components $K_1,\dots,K_r$. These are cone bundles on X (i.e. locally trivially fibred over X by conical sets of covectors) of codimension $d = \dim X^u$. Using the notion of Segre class $s(K)$ of a cone bundle K in the sense of Fulton and the third author [Fu], which coincides with the inverse Chern class $c(K)^{-1}$ in the special case of a vector-bundle K, we may define our characteristic class $Q(K)$ as the lowest degree homogeneous term in $H^*(X)$ of $c(T^*X)s(K)$. Then $Q(K_1),\dots,Q(K_r)$ are cohomology classes in $H^{2d}(X)$, which we may interpret as well as degree d homogeneous harmonic polynomials on the Cartan subalgebra ("Borel picture" of $H^*(X)$). Using the methods of [BM1], we show that we may pass from Springer's original representation space for ρ_u to the vector-space spanned by these r polynomials by composition of various canonical, W equivariant isomorphisms. The classes $Q(K_1),\dots,Q(K_r)$ turn out to be linearly independent, and hence provide a basis for our representation space. Now the representation ρ_u can be described with respect to this basis in terms of certain integer matrices, which have non-zero entries only for those indices i, j for which K_i intersects K_j in codimension ≤ 1.

More precise formulae for this matrix representation (cf. 1.15 below) were previously found by R. Hotta [Ho2] in a somewhat different geometrical setting and were later used in [Ho] to identify certain W representations constructed by A. Joseph [J1] with Springer's ρ_u.

Hotta's formulae hold for our characteristic class basis without change. Moreover, as we shall explain in detail in a subsequent paper, our new approach may be used to reprove these formulae, independently of Hotta's work, and even in such a way, that the above mentioned identification with Joseph's construction becomes simultaneously apparent.

Introduction to chapters 2-4.

Our object here is to suggest, as a part of our general program outlined in [BBM2], that equivariant K-theory provides a very appropriate frame-work for the simultaneous study of nilpotent orbits in a semisimple Lie algebra, and primitive ideals in the enveloping algebra.

To make this slightly more precise, let us first sketch very briefly some of the crucial techniques from equivariant K-theory which are frequently used throughout these chapters . We investigate here primarily the equivariant K-theory of a semisimple Lie group G , acting on the flag variety $X = G/B$, and on its cotangent bundle T^*X . This means, in other words, that we work in and with the Grothendieck ring $K_G(X)$ (resp. $K_G(T^*X)$) of the category of G-equivariant vector bundles (or equivalently, of coherent sheaves) on X (resp. on T^*X) . On the other hand, let E be a single cotangent space of X at some point fixed by the maximal torus $T \subset G$, so that T acts linearly on the vector space E . Then our key technique, frequently employed in this paper, consists in switching from G-equivariant K-theory on X to T-equivariant K-theory on E . This is performed as follows : Starting from equivariant sheaves on T^*X , we restrict them to the zero section on one hand, and to a single fibre on the other hand, which gives isomorphisms

$$K_G(X) \cong K_G(T^*X) \cong K_T(E) \quad .$$

The point of this manipulation is now that for a linear action of a torus on a vector space, equivariant K-theory can be carried out very conveniently in terms of calculations with formal characters. On the other hand, the link to purely geometrical considerations as in our §1 (in terms of characteristic classes in the cohomology of the flag variety) is made simply by the homomorphisms

$$K_G(X) \to K(X) \to H^*(X) \quad ,$$

where the first arrow forgets the G action, and the second maps a vector bundle to its Chern class. In conclusion, this shows how we can translate statements from a context most convenient for computations (formal characters) into a context most convenient for geometrical interpretation (cohomology of the flag variety), and vice versa.

Let us next summarize very roughly the contents of individual chapters. In chapter 2, we have collected a few generalities about equivariant K-theory, which are basic for the subsequent chapters; this chapter is very short, and is mainly meant to help the reader unfamiliar with this theory to read the other chapters. - In chapter 3, we offer a somewhat systematic treatment of the equivariant K-theory of linear torus actions. We assume that the torus acts with positive weights (as in the case of T acting on E as above), to make sure that formal characters exist. Let us note here that chapter 3 may also be viewed - to some extent - as a systematic study of multigraded modules over multigraded rings. For some readers, it could therefore be of interest independently of any applications to semisimple Lie theory. Specific topics treated in §3 include for example the characterization of Grothendieck's γ-filtration in terms of codimensions of supports (theorem 3.10), or also in terms of order of growth of "Hilbert functions" of graded modules (3.18).

Applications of the general formalism as exposed in chapters 2 and 3 to nilpotent orbits in the Lie algebra $\underline{g}$ = Lie G resp. to primitive ideals in the enveloping algebra $U(\underline{g})$ are elaborated in chapters 4 resp. 5. As the reader will realize, some of the formal results stated and proved in chapter 3, turn into significant items of nilpotent orbit or primitive ideal theory, if one learns to translate them appropriately into these fields of applications. For example, the dimension formula in theorem 3.10 translates into the (well-known) formula expressing the Gelfand-Kirillov dimension of a primitive ideal as a function of the degree of its Goldie rank polynomial (cf. [BBM2], proposition 3). A similar example gives Joseph's well-known formula saying how to compute his Goldie rank polynomials from multiplicities of Verma modules, and hence from the Kazhdan-Lusztig polynomials (see e.g. [Bl], 6.9, or [Ja]), an embryonic, formal version of which is proposition 3.9.(*).

Let us now explain the applications to nilpotent orbits contained in chapter 4. As usual, we consider the G-equivariant map π of the cotangent bundle T^*X into the Lie algebra $\underline{g}$ known as Springer's resolution or the Kostant-Souriau moment map (cf. [BB]I). For the collection of ("orbital") cone bundles $K_1,\dots,K_r$ attached to a nilpotent orbit $\mathcal{O}_u$ in $\underline{g}$ (that is the irreducible components of $\overline{\pi^{-1}\mathcal{O}_u)}$, we have already defined in chapter 1 certain characteristic classes $Q(K_i)$ in $H^{2d}(X)$, where d is the common codimension of these bundles in T^*X. Since π is G equivariant, these cone bundles are even G-equivariant, and so they determine classes in $K_G^d(X)$ (the degree $\geq d$ part of $K_G(X)$ with respect to γ-filtration). Therefore, our notion of characteristic classes $Q(K_i)$ can be <u>refined</u> into that of <u>"equivariant characteristic classes"</u> $Q_G(K_i)$ in the <u>equivariant cohomology group</u> $H_G^{2d}(X)$, which is isomorphic to $K_G^d(X)/K_G^{d+1}(X)$. By means of the Borel picture of the (resp. equivariant) cohomology of the flag variety, an adapted version of which is reviewed right at the beginning of chapter 4, we may interpret both $Q(K_i)$ resp. $Q_G(K_i)$ as polynomials on the Cartan subalgebra, and then the former can be considered just as the <u>harmonic part</u> of the latter. Actually, this is only a fact about general G-stable cone bundles (4.6); but the "orbital cone bundles" $K_1,\dots,K_r$ have the following remarkable property in addition : Their equivariant characteristic classes turn out to be harmonic (4.7), and so to actually coincide with the "purely geometrically" defined characteristic classes of chapter 1.

This fact is established in chapter 4 only as a by-product of a much stronger result : We prove directly, by an argument partially following Joseph [J1], that the classes $Q_G(K_1),\dots,Q_G(K_r)$ transform under a simple reflection in the Weyl group according to a certain formula, which was first given by Hotta for the canonical basis of Springer's representations (see 4.14). In combination with our previous work on the "purely geometrical" level in chapter 1, this provides a new approach to the results of Hotta and Joseph [Ho], [J1], and even to Hotta's original transformation formula [Ho2], if one likes. Thus we reprove in a quite natural fashion the coincidence of Weyl group representations constructed by Joseph resp. Springer, which Hotta had established [Ho] only in a rather indirect manner.

Since Hotta's transformation formulae are fairly sophisticated (see 4.14), one has to accept a fair amount of effort for proving them, as we do here. However, we wonder whether there is a simple elegant argument proving the harmonicity of $Q_G(K_i)$ more directly, without reproving Hotta's results. For the case $G = SL_n$, or more generally for $\mathcal{O}_u$ a "special" orbit (in the sense of Lusztig [L]) , an easy proof is obtained by arguing via the equivariant characteristic classes (of the characteristic cycles) of primitive ideals, for which we, curiously enough, do have an easy direct proof of harmonicity (see chapter 5).

Introduction to chapter 5

The systematic study of primitive ideals of the universal enveloping algebra $U(\underline{g})$ of a complex Lie algebra $\underline{g}$ was initiated by J. Dixmier. Obviously, one of the motivations he had in mind was to support our understanding of irreducible Lie group representations, which was greatly advanced under the influence of Harish-Chandra and I.M. Gelfand and their schools, by contributing an additional, purely algebraic new tool. It was clear from the outset that the primitive ideal, the kernel of an irreducible infinitesimal representation of the corresponding Lie group, could carry only relatively rough information about the representation. But as it turned out soon, that amount of information was already sufficiently sophisticated to be of high interest, and in the sequel, the evolution of theoretical insight into the primitive spectra turned into a dramatic series of research developments, heavily interacting with representation theory and several other fields in mathematics. As a result of these events, non-commutative algebra was dramatically advanced, and was more seriously interrelated in a variety of - often unexpected - ways with various other important developments in mathematics.

To be slightly more specific, one has to distinguish the three cases $\underline{g}$ solvable resp. semisimple, resp. general, which showed remarkably different developmentsin both respects, history and result. Remarkably enough, the original initiative is due to Dixmier in each case, roughly 20, resp. 15, resp. 10 years ago. In the solvable case, a fairly satisfactory theory was achieved by the early seventies, with contributions, due to Conze, Duflo, Rentschler, Vergne, and others: The so-called Dixmier map established a continuous (and conjecturally bi-continuous) bijection of the primitive spectrum Prim $U(\underline{g})$ with the coadjoint orbit space $\underline{g}^*/G$, see [Di] or [BGR] for detailed accounts. The general case, after Dixmier's initial work in the late seventies, was theoretically penetrated by work of Duflo, Moeglin, and Rentschler, which essentially achieved (by the mid eighties) a reduction to the semisimple case, see [MR] and [D2], see also [Re].

There is not (yet?) such an easy way of summarizing the history and main achievements in the semisimple case, to which the present book is intended to make another contribution. So we do not attempt to systematically review the complicated history, nor the present situation, of this subject as a whole. Let us try, however, to sketch some very rough features of it, which may help to put our present contribution into an appropriate perspective. For this purpose, let us roughly split the development of the subject over the past 15 years into three 5 year periods. The first one was a phase of first explorations of the subject, and its link up with the theories of highest weight an Harish-Chandra modules. Duflo's characterization of the primitive ideals [D] established the intimate relation of the subject with work of Verma, Bernstein-Gelfand-Gelfand, Jantzen, and others on highest weight representations, and early papers of Joseph, [BJ], and others exploited the achievements of this link up. On the other hand, the Dixmier-Kirillov orbit method, which had been so succesful in the solvable case, was realized to be inadequate for dealing with the semisimple case for various reasons, in the first place because of the phenomenon of non-trivial Goldie ranks. The Goldie rank of a primitive ideal J in $U(\underline{g})$ is the rank of the matrix ring (over a skew field) obtained from $U(\underline{g})/J$ by appropriate localization (in the sense of Ore). A Dixmier map $\underline{g}^*/G \longrightarrow \mathrm{Prim}\, U(\underline{g})$ was defined for $\underline{g} = \underline{sl}_n$ in [B4] and was proven to be injective in [BJ]. It was clear a priori that it could surject at most onto the "completely prime" (i.e. Goldie rank one) part of $\mathrm{Prim}\, U(\underline{g})$; that it actually <u>does</u> was only recently established by Moeglin [M1].However, this approach fails for simple Lie algebras other then $\underline{sl}_n$, and misses the ideals of Goldie rank > 1, so seemed (and still seems) hopelessly inadequate, although optimistic experts may still hope an appropriate modification to extend the orbit method may ultimately be found. For a summary of this <u>exploration phase</u> of the subject, and the emerging problems, see [B2].

The <u>second</u> 5 year period saw most dramatic transformations of the subject. In the first place, the problem of classification of all primitive ideals was reduced to the problem of computing multiplicities in Jordan-Hölder series of highest weight modules (by work of Jantzen, Joseph, Vogan and others), which in turn were conjecturally interpreted by Kazhdan-Lusztig [KL] in terms of topology (in fact, intersection homology) of Schubert varieties. And this conjecture was soon proved by Beilinson-Bernstein [BeBe], and Brylinski-Kashiwara [BK] by means of the Riemann-Hilbert correspondence to modules of differential operators ($\mathcal{D}$-modules) on the flag variety. Since Kazhdan-Lusztig had simultaneously provided a combinatorial recipe to compute those multiplicities, the set theoretic classification of primitive ideals could thus be considered to be done - at least in principle - as a result of this sequence of events. On the other hand the combinatorics involved (for a major Lie group like E_8, say) is so prohibitively complicated, that in practice the relation thus revealed was sometimes even applied to prove combinatorial statements by means of primitive ideal theory, rather then the converse. So in the second place, there was still the need for a more adequate classification theory of primitive ideals . Such a theory emerged - completely independently of the development mentioned before - from extensive work of A. Joseph culminating in [J3] in his beautiful bijective correspondence between primitive ideals (of a specified central character) and bases of certain irreducible Weyl group representations. Joseph's method consisted in a complicated analysis of Goldie ranks of primitive ideals, using very heavily very sophisticated ring and representation theory. For an excellent exposition of Joseph's beautiful theory (along with much further material from this "second 5 year period"), we refer to [Ja]. For an exposition of the proof of the Kazhdan-Lusztig conjecture and related material, we refer to [Mi].

The present book deals with, and contributes to, the <u>third</u> of those 5 year periods. This one is characterized by the purpose of gaining geometric insight into the classification already achieved. The (smooth) change from period 2 to this period 3 was marked e.g. by such papers as Joseph's [J1], which revealed a relation from Goldie rank polynomials to nilpotent orbits, and Barbasch-Vogan's [BV1], [BV2], which first indicated "experimentally" that Joseph's irreducible Weyl group representation must be deeply related to Springer's, a relation which was theoretically understood first by Hotta-Kashiwara [HK]. A crucial achievement of this "third period" was to establish the simple relation between primitive ideals and nilpotent (coadjoint) orbits that had been suggested in [B2], or in other words the irreducibility of the associated variety of a primitive ideal; this was done first for integral central characters in [BB1], and in general by Joseph [J2]. Simultaneously, the various reformulations of Springer's Weyl group representations, discussed in the previous chapters of this book, and appropriate for our present investigation of primitive ideals, were developed [BM1], [J1], [Ho]. Also, the geometric tool of associated varieties (in $\underline{g}^*$) was refined to that of characteristic varieties (in T^*X) of the $\mathcal{D}$-module corresponding to a primitive ideal, see [BB1], [BB3], and V. Ginsburg [Gi]; cf. also Kashiwara-Tanizaki [KT].

Let us just mention the existence of more recent intriguing work of Barbash-Vogan on "unipotent primitive ideals" in [BV3], and further new work of Joseph [J5], Hotta-Kashiwara [H3], which indicate further deep relations between nilpotent orbits and primitive ideals, which remain to be fully integrated into a unifying theoretical picture in the future. Let us also mention in this context, that a "reastauration attempt" of the Dixmier-Kirillov orbit method was made by Vogan [V2], but turned out [MG] to be not yet fully successful.

The list of more recent developments, which we complement by pointing out new interactions with ring and representation theory in [M2] and [LSS], seems to promise that there might be yet another 5 year period of further evolution of the subject to come. We hope that our present contribution, which aims at putting some essential achievements of "periods 2 and 3" into a unified geometric perspective, might be helpful to prepare for it.

Let us now summarize what we do in the present chapter 5. Roughly speaking , it splits into three parts. In the first part (5.1-5.9) we introduce our notion of a characteristic class attached to a primitive ideal J of $U(\underline{g})$, and identify it as a character polynomial of a related highest weight module. This characteristic class $P(U(\underline{g})/J)$ is a cohomology class in $H^*(X)$ defined as follows: Take the characteristic cycle $\underline{\underline{C}}$ on T^*X of the (Beilinson-Bernstein) localization of the left module $U(\underline{g})/J$, and define $P(U(\underline{g})/J) := Q(\underline{\underline{C}})$ to be the characteristic class of that cycle, in the sense of chapter 1. The identification of $P(U(\underline{g})/J)$ with a character polynomial is a very crucial point, and so we offer two alternative proofs. The first one (in 5.6) is very short now, but depends heavily on the corresponding results about nilpotent orbits proved in chapter 4. The second one (in 5.9) requires the introduction of a G equivariant version of our characteristic class concept (see 5.8), but seems more satisfactory and more natural then.

In the second part of the chapter (5.10-5.14), we reprove the irreducibility of Joseph's Weyl group representations (5.13) and of associated varieties (5.14) in our picture. Let us state here our version of the classification theorem for primitive ideals:

Theorem: Let J be a primitive ideal (with trivial central character) of $U(\underline{g})$. Let $\mathcal{O}$ be the nilpotent orbit which is dense in the associated variety of J. Let $J_1,\dots, J_r$ be the set of all primitive ideals corresponding to $\mathcal{O}$ in this way. Let $P_i := P(U(\underline{g})/J_i)$ for $i = 1,\dots,r$. Then:

a) Those characteristic classes $P_1,\dots,P_r$ are linearly independent.

b) They span a W submodule in $H^{2d}(X)$. (Note $2d = \operatorname{codim} \mathcal{O}$.)

c) This W representation is equivalent to Springer's $\rho_{\mathcal{O}}$.

Moreover, our P_i is proportional to Joseph's Goldie rank polynomial attached to J_i, hence the equivalence of our version of stating the classification theory with Joseph's [J3], [Ja]. The above version summarizes the more detailed statement of theorems in 5.13 and 5.14. The reader interested in more details will also realize that the above version differs from the statements actually given in 5.13,5.14 by the choice of scale factors (the equivalence of both versions following from corollary 5.11), which we arrange in the text in a certain new, more natural ("translation invariant") fashion by definition 5.11. Let us also draw attention to the point that we minimize efforts in non-commutative algebra (which is implemented essentially in 5.10).

In the third and final part of this chapter (5.15-5.18), our purpose is to reprove also Joseph's beautiful results about the computation of Goldie ranks of primitive ideals. In doing this, we draw attention to a crucial factorization of polynomials (5.17) due to Joseph and D. King [Ki], [J1]. Finally,we simultaneously obtain formulas for the behaviour of characteristic cycles and our characteristic classes of primitive ideals under "coherent translation".

For the expert readers, let us briefly comment on our choice of attitude as to allowance for central characters. While [BB1], [BB3] and also these introductions are formulated only for trivial central characters, it is necessary for our purposes in the second and third part of the present chapter, that we allow for arbitrary regular integral central characters. We do not consider non-integral central characters here, because that would require a lot of additional basic preparations in $\mathcal{D}$-modules (which will anyway be provided by Miličić's book [Mi]). Nor do we consider non-regular central characters, because it is clear from [Ja] for the experts, how to extend all results to the "walls". One essential reason for this choice was not to obscure the essential points of our new perspective by a lot of additional technicalities and notational machinery. We hope that this choice might help to invite, and encourage newcomers in the subject, and that the experts will accept this as an excuse.

W. Borho, MPI für Mathematik, Bonn,
December 1987

Note added in Proof (January 18, '88):

We just received a thesis by Anna Melnikov, Weizman Institute (Rehovot), which among other things overlap with computations of Andreas Steins, BUGH (Wuppertal), in cases B_3, B_4, and A_5, and also disproves stimulating recent conjectures by Anthony Joseph and Colette Moeglin.

We also wish to add that the proof of the Kazhdan-Lusztig conjecture, a quantum leap in the evolution of the field, was based on a so-called Riemann-Hilbert correspondence revealed by Kashiwara-Mebkhout.

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§1. A description of Springer's Weyl group representations in terms of characteristic classes of cone bundles.

Our "algebraic varieties" are reduced, and defined over an algebraically closed base field k of characteristic 0. We restrict attention to the case $k = \mathbb{C}$ whenever we find it convenient for topological interpretations. If not otherwise stated, we consider (co) homology etc. with coefficients in k.

1.1. Segre classes of cone bundles [FM], [Fu].

A cone in a vector-space is a union of lines through the origin. Let Y be a non-singular algebraic variety. A cone bundle over Y is an algebraic variety K equipped with 1. an action of the multiplicative group k^* (or $\mathbb{C}_m$), and 2. a morphism $K \to Y$ making K a fibre bundle over Y, which admits a closed embedding into a vector bundle E over Y, respecting both data 1. and 2. (In particular, each fibre of K is embedded into the corresponding fibre of E as a closed cone.)

Each cone bundle K over Y determines a certain "characteristic class" $s(K)$ in the cohomology ring $H^*(Y)$ $(= H^*(Y,k))$, called its Segre class. It can be axiomatically defined by the following two properties :

(1) The Segre class of a vector-bundle E over Y is the inverse of its total Chern class $c(E)$, i.e.

$$s(E) = c(E)^{-1} .$$

(2) The Segre class is compatible with proper pushforwards in the following sense : Given a commutative square of two cone bundles

$$\begin{tikzcd} K' \arrow[r, "\tilde{f}"] \arrow[d] & K \arrow[d] \\ Y' \arrow[r, "f"] & Y \end{tikzcd}$$

with f proper and $\tilde{f}$ proper birational, the functorial ring homomorphism $f_*: H^*(Y') \to H^*(Y)$ maps $s(K')$ to $s(K)$.

We refer to Fulton-MacPherson [FM], or Fulton [Fu] chapter 4, for the proof of existence and uniqueness of such classes $s(K)$ (even on singular base spaces Y), as well as for more properties and historical back-ground.

1.2 Characteristic class of a subvariety of a vector-bundle.

Let E be a vector-bundle over the non-singular equidimensional variety Y. Then each subvariety V of codimension d in E determines a cohomology class of degree $2d$ on the base space Y, denoted $Q(V) \in H^{2d}(Y)$, and defined as follows. We start here with a geometric definition for the case $k = \mathbb{C}$, and postpone the statement of a more general, formal algebraic definition to the next section (1.3). If m resp. n are the dimension of the fibre resp. base of E, the real dimension of V is $2m+2n-2d$. It therefore defines a canonical homology class $[V]$ in the Borel-Moore (closed support) homology group $H^{c\ell}_{2m+2n-2d}(E)$. Homology with closed supports is sometimes called Borel-Moore homology. It is the homology of the complex of locally finite singular chains. A representative for $[V]$ can be obtained by triangulating V (with infinitely many simplices). Since E is nonsingular, Poincaré duality gives a canonical identification of $H^{c\ell}_{2m+2n-2d}$ with the cohomology group $H^{2d}(E)$. Since the zero-section $\sigma : Y \to E$ is a homotopy equivalence, we have an induced graded ring isomorphism $\sigma^* : H^*(E) \xrightarrow{\sim} H^*(Y)$, which we call "homological intersection with the zero section". Then, by definition, the characteristic class $Q(V)$ is obtained from the canonical class $[V]$ by Poincaré duality followed by homological intersection with the zero section.

For the case of a cone bundle $V = K$, there is the following nice formula for this characteristic class $Q(K)$ in terms of Chern- and Segre classes, due to Fulton-MacPherson [FM], see also [Fu], p.73, 4.1.8.

Proposition : *If the subvariety* $V \subset E$ *is a cone bundle* $K = V$ *over* Y *of codimension* d *in* E*, then*

$$Q(K) = [c(E) \cdot s(K)]^{2d}$$

Here $[\ldots]^i$ means homogeneous part in degree i .

1.3. Characteristic class determined by a sheaf on a bundle.

The following definition, generalizing 1.2, will not be used until chapter 3. It may serve as an algebraic alternative to 1.2, but its main purpose is to link up the results of the present chapter with our work in later chapters. For general background, we refer to [SGA6], [Ma], [Fu], [FL], [BFM].

With notations as in 1.2, we consider now an arbitrary coherent sheaf $\underline{\underline{F}}$ of $\mathcal{O}_E$-modules whose support $\mathrm{supp}(\underline{\underline{F}})$ has codimension d in E , and we define a characteristic class $Q(\underline{\underline{F}})$ determined by $\underline{\underline{F}}$ in $H^{2d}(Y)$ as follows. We consider the Grothendieck ring $K(E)$ of (the category of) all coherent $\mathcal{O}_E$-modules, filtered by the subgroups $K^j(E)$ $(j \in \mathbb{Z})$ generated by coherent $\mathcal{O}_E$-modules of codimension $\geq j$. The ring structure on $K(E)$ comes from its identification with the Grothendieck ring of locally free $\mathcal{O}_E$-modules (cf. 1.4). We note that this filtration coincides with Grothendieck's γ-filtration at least after tensoring with $\mathbb{Q}$, see [FL], p.182, Proposition 5.5; but note here and in the sequel our convention that our coefficients are tacitly extended to k , for convenience. We next apply the Chern character $ch:K(E) \to H^*(E)$, which is a functorial homomorphism of the Grothendieck ring into the cohomology ring [Fu], [FL], [Hi] . Finally, we intersect homologically with the zero-section $\sigma:Y \to E$ (as in 1.2), and take the degree $2d$ homogeneous part. In summary, we define

$$Q(\underline{\underline{F}}) := [\sigma^* ch[\underline{\underline{F}}]]^{2d} \quad . \tag{2}$$

We may sometimes also consider the "total" class attached to $\underline{\underline{F}}$ by

$$Q_{total}(\underline{\underline{F}}) := \sigma^* ch[\underline{\underline{F}}] \quad . \tag{3}$$

Remarks. a) Note that we could reverse the order of σ^* and ch in this definition, since functoriality of the Chern character provides a commutative diagram

$$\begin{array}{ccc} K(E) & \xrightarrow{\sigma^*} & K(Y) \\ {\scriptstyle ch}\downarrow & & \downarrow{\scriptstyle ch} \\ H^*(E) & \xrightarrow{\sigma^*} & H^*(Y) \end{array} .$$

b) Let us point out that $Q_{total}(\underline{\underline{F}})$ has degree $\geq 2d$, or in other words : $Q(\underline{\underline{F}})$ is either the lowest degree term of this class, or else zero. To see this, recall that the ring $K(E)$ is filtered by the Grothendieck subgroups $K^j(E)$ of coherent $\mathcal{O}_E$-modules with support of codimension $\geq j$. Then the Chern character respects this filtration upto a doubling of degrees, that is

$$ch\ K^d(E) \subset \bigoplus_{j\leq 2d} H^j(E) \quad ,$$

which implies our claim.

The Chern character thus induces a homomorphism

$$gr\ ch : gr\ K(E) \to H^*(E)$$

of the <u>associated graded ring</u> $gr\ K(E)$ into cohomology, given by

(4) $\qquad gr_j ch : gr_j K(E) := K^j(E)/K^{j+1}(E) \to H^{2j}(E)$, for all degrees j.

<u>1.4. Comparison of the two definitions for Q .</u>

In order to establish compatibility of the topological definition for $Q(V)$ in 1.2 with the algebraic definition in 1.3, we have to check that in case $k = \mathbb{C}$, the class $Q(\mathcal{O}_{\overline{V}})$ of 1.3 coincides with $Q(V)$ of 1.2. For the comparison, the algebraic analogue of Borel-Moore homology of E is the Grothendieck group $K_o(E)$ of coherent $\mathcal{O}_E$-modules, while the cohomology of E corresponds to the Grothendieck ring of algebraic vector-bundles (or locally free sheaves), denoted $K^o(E)$. Since we are assuming Y (hence E) non-singular, the canonical ("Poincaré duality") map $K^o(E) \to K_o(E)$ is an isomorphism, which justified the notation K(E) for both groups; but for the present specific purpose, the notational distinction will contribute to clarity. We look at the following diagram

$$\begin{tikzcd} K_o(E) \arrow[d, "\tau"] & K^o(E) \arrow[l, "\sim"', "\text{can.}"] \arrow[r, "\sigma^*", "\sim"'] \arrow[d, "ch"] & K^o(Y) \arrow[d, "ch"] \\ H_*^{\mathcal{C}\ell}(E) & H^*(E) \arrow[l, "\sim"', "\text{can.}"] \arrow[r, "\sigma^*", "\sim"'] & H^*(Y) \end{tikzcd}$$

where τ is the "Todd character" defined in [BFM] , which makes the left hand square commutative only upto multiplication by an element in $H_*^{\mathcal{C}\ell}(E)$ which is the image under "can." of the Todd class $\mathcal{T}d(T_E)$ of the tangent bundle of the smooth variety E . However, this multiplication does never affect the top degree term, i.e. in other words, the above diagram induces a commutative one on the associated graded level :

$$\begin{tikzcd} gr_dK_o(E) \arrow[d] & gr_dK^o(E) \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d] & gr_dK^o(Y) \arrow[d] \\ H^{\mathcal{C}\ell}_{2m+2n-2d} & H^{2d}(E) \arrow[l, "\sim"'] \arrow[r, "\sim"] & H^{2d}(Y) \end{tikzcd}$$

Now the desired equality follows from the fact that the canonical class $[\mathcal{O}_{\bar{V}}] \bmod K_o^d(E)$ determined by V in $gr_dK_o(E) = K_o^d(E)/K_o^{d+1}(E)$ maps onto the canonical class $[V]$ determined by V in $H^{\mathcal{C}\ell}_{2m+2n-2d}(E)$.

From the considerations above, we may draw a more general conclusion, which will become significant in chapter 3.. Recall that one may attach to each irreducible component V of the support supp($\underline{\underline{F}}$) of our coherent $\mathcal{O}_E$-module a well-defined integer multiplicity $m_V(\underline{\underline{F}}) \geq 0$. Hence we may define an algebraic cycle (supporting $\underline{\underline{F}}$) as the formal linear combination

$$\underline{\mathrm{supp}}(\underline{\underline{F}}) = \sum_{V} m_V(\underline{\underline{F}})[V] \quad .$$

<u>Proposition</u> : $Q(\underline{\underline{F}}) = \sum_{\operatorname{codim} V=d} m_V(\underline{\underline{F}})Q(V)$.
<u>In particular, the characteristic class</u> $Q(\underline{\underline{F}})$ <u>of a coherent</u> $\mathcal{O}_E$<u>-module</u> $\underline{\underline{F}}$ <u>is completely determined by its supporting cycle.</u>

Let $\underline{\underline{F}} \neq 0$. We claim that

$$[\underline{\underline{F}}] - \sum_{\operatorname{codim} V=d} m_V(\underline{\underline{F}})[\mathcal{O}_V] \in K_o^{d+1}(E)$$

has degree strictly larger than $[\underline{\underline{F}}]$. Then the proposition will be clear from the preceding discussion. Let S denote the (reduced) support of $\underline{\underline{F}}$, and $V_1,\ldots,V_r$ its irreducible components. By an easy dévissage, one may assume that $\underline{\underline{F}}$ is an $\mathcal{O}_S$-module, so we may speak of the class $[\underline{\underline{F}}]$ in $K_o(S)$. In each V_i there exists an open subset $U_i \neq \emptyset$ such that the restriction of $\underline{\underline{F}}$ to U_i is a free module of rank $m_{V_i}(\underline{\underline{F}})$. This means that the element $[\underline{\underline{F}}] - \sum_{1\le i\le r} m_{V_i}(\underline{\underline{F}})[\mathcal{O}_{V_i}]$ of $K_o(S)$ restricts to zero in $K_o(U)$, where $U = U_1 \cup\cdots\cup U_r$. By the localization exact sequence for K_o , this element belongs to the image of $K_o(S\smallsetminus U)$ in $K_o(S)$. Since $S\smallsetminus U$ has codimension $\geq d+1$ in E , this implies our claim.

<u>1.5.</u> <u>Homology of the flag variety</u> [Bo], [Hℓ]

In the present paper, we shall apply the previously defined notions in the special case where $Y = X$ is a flag variety and $E = T^*X$ its cotangent bundle. So let us introduce now some of the fundamental notations and facts for this particular situation. We consider a <u>semisimple</u>, connected, linear <u>algebraic group</u> G defined over k , and fix a <u>Borel subgroup</u> B . Then we may define the <u>flag variety</u> as $X = G/B$, a complete homogeneous space. Let $n = \dim_k X$. We also fix a maximal torus $T \subset B$ and denote by W the <u>Weyl group</u> $W = N_G(T)/T$. For each $w \in W$, the <u>Schubert cell</u> X_w is the locally closed subvariety BwB/B , isomorphic to an affine space of dimension $\ell(w)$, the <u>length</u> of w with respect to the system of simple reflections determined by B . Since the various Schubert cells X_w $(w \in W)$ form a paving of X by affine spaces of even real dimensions $(\dim_{\mathbb{R}} X_w = 2\ell(w))$, they provide a vector-space basis for the homology groups :

$$H_{2j}(X) = \bigoplus_{\ell(w)=j} k[X_w] \qquad \text{for } j = 0,\dots,n \quad .$$

Let us point out that the existence of such an affine paving causes the Chern character to be an isomorphism $K(X) \xrightarrow{\sim} H^*(X)$, so that the two diagrams in 1.4 consist entirely of isomorphisms in the present situation, viz.

$$\begin{tikzcd}
K_o(T^*X) \arrow[d, "\tau"', "\cong"] & K^o(T^*X) \arrow[l, "\sim"'] \arrow[r, "\sigma^*", "\sim"'] \arrow[d, "ch"', "\cong"] & K^o(X) \arrow[d, "ch"', "\cong"] \\
H_*^{c\ell}(T^*X) & H^*(T^*X) \arrow[l, "\sim"'] \arrow[r, "\sigma^*", "\sim"'] & H^*(X)
\end{tikzcd}$$

As another special feature of the flag variety case let us mention here as a side-remark, that the multiplier $\mathcal{T}d(T_{T^*X}) = 1$ is trivial in this particular case, so that even the left hand square of the above diagram is commutative (cf. the remark below).

Let U be the unipotent radical of B . We denote $\underline{g}$,$\underline{b}$,$\underline{t}$, $\underline{u}$ the Lie algebras of G,B,T,U . For convenience, we sometimes identify $\underline{g}$ with its dual $\underline{g}^*$ via the Killing form. We also identify therefore $\underline{u}$ with the cotangent space of $X = G/B$ at the base point $x = \{B\}$. Then T^*X identifies with the associated fiberbundle $G \times^B \underline{u}$ (as usual, [BM1], [BB] I,III). It splits into line bundles with fibre $\underline{u}^\alpha$, $\alpha \in \Phi^+$. Here $\Phi^+ \subset \underline{t}^*$ denotes the system of positive roots relative B , and $\underline{u}^\alpha = \underline{g}^\alpha$ is the root space belonging to $\alpha \in \Phi^+$.

Remark : Let us prove that $\mathcal{T}d(T_{T^*X}) = 1$. It suffices to prove that $\mathcal{T}d(T_X)\mathcal{T}d(T_X^*) = 1$. The filtration of $\underline{g}$ by the B-invariant subspaces : $0 \subset \underline{u} \subset \underline{b} \subset \underline{g}$ induces a filtration of $G \times^B \underline{g}$ by homogeneous sub-bundles. Since the whole bundle $G \times^B \underline{g}$ resp. the bundle subquotient $G \times^B (\underline{b}/\underline{u}$ are trivial vector bundles (the latter since B acts trivially on $\underline{b}/\underline{u}$, the former since the B-action extends to a G-action), we obtain

$$1 = \mathcal{T}d(G \times^B \underline{g}) = \mathcal{T}d(G \times^B \underline{u})\, \mathcal{T}d(G \times^B (\underline{b}/\underline{u}))\mathcal{T}d(G \times^B (\underline{g}/\underline{b})) = \mathcal{T}d(T_X^*)\mathcal{T}d(T_X) ,$$

since $\underline{u}$ resp. $\underline{g}/\underline{b} \cong \underline{u}^*$ identify with the cotangent resp. tangent space of X at B . (This sort of phenomenon for the flag variety was first observed in [Mr].)

1.6 Cohomology of the flag variety.

There is a nice explicit description, due to Borel, of the full ring structure of $H^*(X)$, in terms of polynomial functions on $\underline{t}^*$, which we shall explain in more detail in the adequate context in chapter 3 . Let us state here only basic facts relevant for the present chapter : First, the cohomology of X is trivial in all odd degrees (by the remarks made in 1.5). Second, the cohomology group $H^{2d}(X)$ in some even degree 2d identifies canonically with the vector-space $S^d_{harm}(\underline{t}^*)$ of polynomial functions on $\underline{t}$ which are W-harmonic, and homogeneous of degree d , for all $d = 0,1,\dots,n$. In particular, this provides a linear W-action on $H^{2d}(X)$. Third, the cup-product of cohomology classes is given by multiplying polynomials modulo W-invariants without constant term (cf. 3. or [Bo]).

Convention. From now on, we shall consider the characteristic classes $Q(\mathcal{V})$ resp. $Q(\underline{\underline{F}})$, defined by a subvariety $\mathcal{V} \subset T^*X$ or a coherent $\mathcal{O}_{T^*X}$-module $\underline{\underline{F}}$ as in 1.3 resp. 1.4, as harmonic polynomials on the Cartan subalgebra $\underline{t}$, if this is convenient. If the difference between a characteristic cohomology class on X and the corresponding harmonic polynomial on $\underline{t}$ should matter, then we may refer to the latter as a "characteristic harmonic polynomial".

Example : Let us illustrate the computation of classes $Q(K)$ for cone bundles K in T^*X, using the conventions above and the formula of 1.2. We take the special case where the conical fibre $\underline{k} \subset \underline{u}$ is actually a vector-space, spanned by root-spaces. Let Ψ resp. Ψ' denote the set of positive roots α such that $\underline{u}^\alpha$ is resp. is not contained in $\underline{k}$, so $\underline{k} = \bigoplus_{\alpha\in\Psi} \underline{g}^\alpha$, and $\underline{k}' = \bigoplus_{\alpha\in\Psi'} \underline{g}^\alpha$ is a complement of $\underline{k}$ in $\underline{u}$. Then clearly, the codimension d of K in T^*X is given by $d = \dim \underline{k}' = \# \Psi'$. The total Chern classes are given by :

$$c(T^*X) = \prod_{\alpha\in\Phi^+} (1+\alpha) \quad , \quad c(K) = \prod_{\alpha\in\Psi} (1+\alpha) = s(K)^{-1} \ .$$

Hence we may compute the product (cf. 1.2) :

$$c(T^*X)s(K) = c(T^*X)/c(K) = \prod_{\alpha\in\Psi'} (1+\alpha)$$

which has highest term $\Pi\alpha$, $\alpha \in \Psi'$, of degree $2d$ (if each α is given degree 2). Then the formula in 1.2 says that

$$Q(K) = \prod_{\alpha\in\Psi'} \alpha \ . \tag{*}$$

Strictly speaking, by our convention above, one has always to take only the harmonic parts of these products. However, in the case when e.g. Ψ' is a positive sub-root-system, then the product (*) is already harmonic.

1.7. Orbital cone bundles on the flag variety ([BB] III, Appendix B)

We denote by $\pi : T^*X \to N$ the socalled Springer resolution, which is a G-equivariant proper algebraic map of T^*X onto the cone N of all nilpotent elements in $\underline{g}$. We obtain some particularly nice cone bundles on X by taking the preimage of a "nilpotent-orbit". To be more precise, let $u \in N$ denote a nilpotent element, and $\mathcal{O} = \mathcal{O}_u \subset N$ the G-orbit generated by u under the adjoint action of G on its Lie algebra. Since a nilpotent orbit is obviously stable under multiplication by a nonzero scalar, its closure $\overline{\mathcal{O}}$ is clearly a cone, and so is $\overline{\mathcal{O} \cap \underline{u}}$, by the same reason. Now consider the preimage $\pi^{-1}\mathcal{O}$ of the orbit in T^*X . Interpreting T^*X as the associated fibre bundle $G \times^B \underline{u}$ (notation 1.5), we have obviously

$$\pi^{-1}\mathcal{O} = G \times^B (\mathcal{O} \cap \underline{u}) ,$$

hence the description of its closure by

$$\overline{\pi^{-1}\mathcal{O}} = G \times^B \overline{\mathcal{O} \cap \underline{u}}$$

which exhibits its structure as a cone bundle on $G/B = X$ with fibre $\overline{\mathcal{O} \cap \underline{u}}$. Moreover, if $C_1,\ldots,C_r$ denote the irreducible components of $\overline{\mathcal{O} \cap \underline{u}}$, then their associated fibre bundles

$$K_i = G \times^B C_i \quad , \quad \text{for} \quad i = 1,\ldots,r$$

are the irreducible components of $\overline{\pi^{-1}\mathcal{O}}$. We call these cones C_i resp. cone bundles K_i "<u>orbital</u> for $\mathcal{O}$" ; any cone in $\underline{u}$ resp. cone bundle in T^*X is called "<u>orbital</u>" , if it is orbital for some nilpotent orbit (which is then necessarily uniquely determined). By a result of Spaltenstein [Sp2], all C_i $(1 \leq i \leq r)$ above have the same dimension, and hence all K_i $(1 \leq i \leq r)$ have the same dimension. From Steinberg [St], it follows that the common codimension of these cone bundles K_i in T^*X is given by

$$d = d_u := \dim \pi^{-1}u = \frac{1}{2}\operatorname{codim}_{\mathcal{N}}\mathcal{O} = \operatorname{codim}_{T^*X}K_i \quad .$$

1.8. Another realization of Springer's Weyl group rerpesentation

We are now ready to state the main result of this chapter.

Theorem : Let $K_1,\ldots,K_r$ be the orbital cone bundles for some nilpotent orbit $\mathcal{O}_u$ of codimension $2d_u$ in N (as in 1.7). Let $Q(K_1),\ldots,Q(K_r)$ be their characteristic classes on X, as defined in 1.2. Then :

a) These classes are linearly independent.

b) They span an irreducible Weyl group submodule of $H^{2d_u}(X)$.

c) This irreducible representation is equivalent to Springer's representation ρ_u.

For the last statement, we have to recall that Springer [S1] constructs an irreducible representation of the Weyl group W on $H^{2d_u}(\pi^{-1}u)^{G_u}$, the G_u-invariants of the top cohomology group of the fibre of his resolution, where G_u is the isotropy group in G of the nilpotent element u. We denote this representation by $\rho_u = \rho_{(u,1)}$, following our conventions in [BM] 1,2 and [BB] I,III, which differ from Springer's by a sign-character (cf. 1.13).

Before going into the proof (1.9 - 1.12), let us point out some immediate useful consequences :

Corollary 1 : The set of classes $Q(K)$, where K runs through all orbital cone bundles (for all nilpotent orbits), is linearly independent.

Proof : This follows from a), b), c) of the theorem in combination with the fact that the Springer representations for different nilpotent orbits are pairwise non-equivalent. Q.e.d.

Remark : Note that in a general context, our topological invariant $Q(V)$ defined in 1.2 may often be zero, and hence not of much interest. But for the study of nilpotent orbits, the theorem establishes the usefulness of our concept. Here is another illustration of its use.

Corollary 2 : Let V be the (left) characteristic variety of a primitive ideal in the enveloping algebra $U(\underline{g})$, or of a Harish-Chandra bimodule (cf. [BB] III). Then $Q(V) \neq 0$.

Proof : By loc.cit., V is a union of orbital cone bundles. Let $K_1,\dots K_r$ be those of maximal dimension. Then $Q(V) = Q(K_1)+\dots+Q(K_r) \neq 0$ by corollary 1.

Q.e.d.

Let us also mention that the theorem will allow us to understand and reprove Hotta's results [Ho] in a more natural way, see §3.

1.9 Reformulation of the theorem using intersection homology.

To put the theorem into a more formal language, note that the fundamental classes $[K_1],\dots,[K_r]$ of our orbital cone bundles form a basis of the top homology group of $\overline{\pi^{-1}\mathcal{O}_u}$, and that our characteristic class construction (1.2) provides a group-homomorphism

$$H^{c\ell}_{4n-2d_u}(\overline{\pi^{-1}\mathcal{O}_u}) \xrightarrow{\delta} H^{2d_u}(X) ,$$

sending $[K_i]$ to $Q(K_i)$. Now part a) of the theorem asserts that δ is injective. We shall first reinterpret this map (see the proposition below) in terms of the intersection homology approach to Springer's theory, as developed by two of us in [BM] I, and then work in this alternative frame-work to establish the theorem.

In [BM] I, Springer's theory of Weyl group representations was derived from the Beilinson-Bernstein-Gabber direct sum decomposition theorem [BBD], applied to Springer's resolution $\pi : T^*X \to N$. The direct sum decomposition theorem gives that the sheaf $\underline{\underline{A}}^{\cdot} := R\pi_* \mathbb{C}_{T^*X}$, considered as an object in the bounded derived category $D^b(N)$ decomposes into direct summands $\underline{\underline{A}}^{\cdot}_{(x,\varphi)}$, where x runs over a set of representatives of all nilpotent orbits, φ runs over the irreducible characters of $\pi_1(\mathcal{O}_x)$, and

(*) $$\underline{\underline{A}}^{\cdot}_{(x,\varphi)} := Rj^x_* \underline{\underline{IC}}^{\cdot}(L_\varphi) \otimes V_{(x,\varphi)}$$

in the notion of [B M] I. We recall that $V_{(x,\varphi)}$ is a certain vectorspace, and the formula states that $\underline{\underline{A}}^{\cdot}_{(x,\varphi)}$ is a sum of dim $V_{(x,\varphi)}$ copies of the direct image under the inclusion map $j^x : \mathcal{O}_x \hookrightarrow N$ of the intersection homology sheaf $\underline{\underline{IC}}^{\cdot}(L_\varphi)$ with coefficients in the local system L_φ of monodromy φ. To simplify notation we shall drop φ if $\varphi = 1$ is trivial, so write $\underline{\underline{A}}^{\cdot}_{(x,1)} = \underline{\underline{A}}^{\cdot}_x$ etc. . Now application of the hyper cohomology functor $\mathbb{H}^i$ provides an isomorphism.

$$\beta : H^i(X) \xrightarrow{\sim} \mathbb{H}^i(N, \underline{\underline{A}}^{\cdot}) \quad ,$$

and the direct sum decomposition

$$\mathbb{H}^i(N, \underline{\underline{A}}^{\cdot}) \cong \mathbb{H}^i(N, \bigoplus_{(x,\varphi)} \underline{\underline{A}}^{\cdot}_{(x,\varphi)}) \cong \bigoplus_{(x,\varphi)} \mathbb{H}^i(N, \underline{\underline{A}}^{\cdot}_{(x,\varphi)})$$

provides an inclusion as a direct summand

$$\gamma : \mathbb{H}^i(N, \underline{\underline{A}}^{\cdot}_x) \hookrightarrow \mathbb{H}^i(N, \underline{\underline{A}}^{\cdot}).$$

in each degree i, for each nilpotent orbit $\mathcal{O}_x$.

<u>Proposition</u> : <u>There exists an isomorphism</u> α <u>which makes the following diagram commutative</u> :

$$\begin{tikzcd}
H^{c\ell}_{4n-2d_u}(\overline{\pi^{-1}\mathcal{O}_u}) \arrow[r, "\delta"] \arrow[d, "\cong"', "\alpha"] & H^{2d_u}(X) \arrow[d, "\cong"', "\beta"] \\
\mathbb{H}^{2d_u}(N, \underline{\underline{A}}^{\cdot}_u) \arrow[r, hook, "\mu"] & \mathbb{H}^{2d_u}(N, \underline{\underline{A}}^{\cdot})
\end{tikzcd}$$

Clearly, this implies part a) of the theorem. We shall next explain in 1.10, how also parts b) and c) about the W-action will follow. Then the construction of the isomorphism α will be given in 1.11, 1.12, to complete the proof of the theorem.

1.10 The Weyl group action.

Let us now take into account also Borel's W-action on $H^*(X)$ (1.7) resp. Lusztig's [Lu] W-action on $\underline{\underline{A}}^{\cdot}$, and recall that these make the above isomorphism β W-equivariant (cf. [BM1], section 6, [Sp]). Furthermore, we conclude from the main theorem of [BM1], that the direct summands $\underline{\underline{A}}^{\cdot}_{(x,\varphi)}$ are stable under this action, and that the W-action on each $\underline{\underline{A}}^{\cdot}_{(x,\varphi)}$ is given by a linear representation $\rho_{(x,\varphi)}$ of W on the vector-space $V_{(x,\varphi)}$, which is irreducible and identifies with Springer's representation up to a sign character. Application of the hyper cohomology functor $\mathbb{H}^i$ to formula 1.9 (*) gives

$$\mathbb{H}^i(N,\underline{\underline{A}}^{\cdot}_{(x,\varphi)}) \cong IH^{i-2d_x}(\overline{\mathcal{O}}_x,L_\varphi) \otimes V_{(x,\varphi)} ,$$

which describes the hypercohomology of each direct summand $\underline{\underline{A}}^{\cdot}_{(x,\varphi)}$ as a W-module, which is isotypical of type $\rho_{(x,\varphi)}$, with multiplicity given by the intersection homology group of $\overline{\mathcal{O}}_x$, with coefficients in L_φ, in the appropriate degree (notations as in [BM1]). Taking $x = u$, $\varphi = 1$, $i = 2d_u$ this intersection homology group becomes $IH^o(\overline{\mathcal{O}}_u)$, which is spanned by a single canonical class $[\mathcal{O}_u]$, so that we find

$$\mathbb{H}^{2d_u}(N,\underline{\underline{A}}^{\cdot}_u) \cong IH^o(\overline{\mathcal{O}}_u) \otimes V_u \cong V_u .$$

Or in other words, this says that the map γ in proposition 1.9 maps $\mathbb{H}^{2d_u}(N,\underline{\underline{A}}^{\cdot}_u)$ onto a single copy of Springer's representation ρ_u. This shows that the proposition will also imply parts b) and c) of our theorem.

1.11 Reduction to a crucial lemma.

We now turn to the construction of an isomorphism α as announced in the proposition (1.9). As a preliminary, we shall replace our characteristic class map δ by a similar but more convenient map δ', using the following commutative diagram :

$$\begin{tikzcd} H^{2d_u}(T^*X, T^*X - \overline{\pi^{-1}\Theta_u}) \arrow[r, "\delta'"] \arrow[d, "\varepsilon", "\cong"'] & H^{2d_u}(T^*X) \arrow[d, "\sigma^*", "\cong"'] \\ H_{4n,2d_u}(\overline{\pi^{-1}\Theta_u}) \arrow[r, "\delta"] & H^{2d_u}(X) \end{tikzcd}$$

which will allow us to work entirely in cohomology of T^*X . Here the map δ' is defined just by functoriality of cohomology on pairs of topological spaces, and ε is Lefschetz duality, which is an isomorphism since T^*X is nonsingular. In view of this diagram, proposition 1.9 is equivalent to the following :

<u>Proposition</u> : <u>There is an isomorphism</u> α' <u>making this diagram commutative</u>

$$\begin{tikzcd} H^{2d_u}(T^*X, T^*X - \overline{\pi^{-1}\Theta_u}) \arrow[r, "\delta'"] \arrow[d, "\alpha'", "\cong"'] & H^{2d_u}(T^*X) \arrow[d, "\beta' = \beta\sigma^*", "\cong"'] \\ \mathbb{H}^{2d_u}(N, \underline{\underline{A}}^\cdot_u) \arrow[r, "\gamma"] & \mathbb{H}^{2d_u}(N, \underline{\underline{A}}^\cdot). \end{tikzcd}$$

We prefer to prove this equivalent version. The construction of α' will be accomplished by a discussion of the following diagram, where we denote $\partial\Theta_u = \overline{\Theta}_u \setminus \Theta_u$ the topological boundary of Θ_u , and $d = d_u$, to simplify notation.

$$\begin{tikzcd}
 & H^{2d}(T^*X, T^*X \setminus \overline{\pi^{-1}\Theta_u}) \arrow[dl, "\omega"'] \arrow[dr, "\delta'"] & \\
H^{2d}(T^*X \setminus \pi^{-1}\partial\Theta_u, T^*X \setminus \pi^{-1}\overline{\Theta}_u) \arrow[d, "\upsilon", "\cong"'] & H^{2d}(T^*X, T^*X \setminus \pi^{-1}\overline{\Theta}_u) \arrow[l] \arrow[r] \arrow[u] & H^{2d}(T^*X) \arrow[d, "\beta'", "\cong"'] \\
\mathbb{H}^{2d}(N \setminus \partial\Theta_u, N \setminus \overline{\Theta}_u; \underline{\underline{A}}^{\bullet}) & \mathbb{H}^{2d}(N, N \setminus \overline{\Theta}_u; \underline{\underline{A}}^{\bullet}) \arrow[l] \arrow[r] \arrow[u, "\cong"] & \mathbb{H}^{2d}(N, \underline{\underline{A}}^{\bullet}) \\
\mathbb{H}^{2d}(N \setminus \partial\Theta_u, N \setminus \overline{\Theta}_u; \underline{\underline{A}}^{\bullet}_u) \arrow[u, hook, "\xi"] & \mathbb{H}^{2d}(N, N \setminus \overline{\Theta}_u; \underline{\underline{A}}^{\bullet}_u) \arrow[l, "\eta"] \arrow[r, "\zeta"'] \arrow[u, hook] & \mathbb{H}^{2d}(N, \underline{\underline{A}}^{\bullet}_u) \arrow[u, hook, "\gamma"']
\end{tikzcd}$$

In this diagram, all horizontal maps, and also the top triangles, are induced by inclusions of pairs of topological spaces. The middle row of vertical isomorphisms are examples of the sheaf theoretic isomorphism

$$(*) \qquad \mathbb{H}^i(\pi^{-1}A, \pi^{-1}B; \underline{\underline{S}}^{\bullet}) \xrightarrow{\sim} \mathbb{H}^i(A, B; R\pi_* \underline{\underline{S}}^{\bullet})$$

which holds for any map π of a space with a complex of sheaves $\underline{\underline{S}}^{\bullet}$ to a space with a pair of subspaces $A \supset B$. Finally, the bottom vertical arrows are induced by coefficient inclusions $\underline{\underline{A}}^{\bullet}_u \to \underline{\underline{A}}^{\bullet}$; they are inclusions, since $\underline{\underline{A}}^{\bullet}_u$ is a direct summand of $\underline{\underline{A}}^{\bullet}$. Each of the small squares and triangles in the diagram clearly commutes, so <u>the diagram is commutative</u>.

<u>Lemma</u>. <u>The maps</u> ξ , η , ζ , ω <u>in this diagram are isomorphisms</u>.

Given the lemma, we may define the desired isomorphism α' by tracing around the outside edge of the diagram $(\alpha' = \zeta\eta^{-1}\xi^{-1}\upsilon\omega)$, and then the commutativity of the whole diagram gives $\gamma\alpha' = \beta'\delta'$, which establishes our proposition.

1.12 Completion of the proof of theorem 1.8.

It is left to prove the lemma. To prove that ξ is an isomorphism, it suffices to check that for all direct summands $\underline{\underline{A}}^{\cdot}_{(x,\varphi)}$ other than $\underline{\underline{A}}^{\cdot}_{u}$, the group $\mathbb{H}^{2d}(N\smallsetminus\partial\mathcal{O}_u, N\smallsetminus\overline{\mathcal{O}}_u;\underline{\underline{A}}^{\cdot}_{(x,\varphi)})$ vanishes. We may interpret this group as the global sections $\Gamma L_{(u,(x,\varphi))}$ of the local system $L_{(u,(x,\varphi))}$ on $\mathcal{O}_u$ whose fiber at $u \in \mathcal{O}_u$ is given by $\mathbb{H}^{4n-2d}(N,N\smallsetminus\{u\};\underline{\underline{A}}^{\cdot}_{(x,\varphi)})$. Now if $\mathcal{O}_x \neq \mathcal{O}_u$, then this fibre is zero by a dimension count, using the support conditions for intersection homology (axioms (AX2) in [GM2]). If $\mathcal{O}_x = \mathcal{O}_u$, but $\varphi \neq 1$, then the local system has no invariants. It follows that indeed

$$\mathbb{H}^{2d}(N\smallsetminus\partial\mathcal{O}_u,N\smallsetminus\overline{\mathcal{O}}_u;\underline{\underline{A}}^{\cdot}_{(x,\varphi)}) = 0 \quad , \text{ whenever } \quad (x,\varphi) \neq (u,1) \;,$$

and hence ξ is an isomorphism.

To see that η is an isomorphism, look at the case $(x,\varphi) = (u,1)$ of the previous discussion, and interpret the target of η as $\Gamma L = H^{o}(\mathcal{O}_u,L)$ with $L = L_{(u,(u,1))}$, notation as above. Similarly, the source of η may be interpreted as $IH^{o}(\overline{\mathcal{O}}_u,L)$, so we have a commutative diagram of canonical maps

$$\begin{tikzcd}
\mathbb{H}^{2d}(N\smallsetminus\partial\mathcal{O}_u,N\smallsetminus\overline{\mathcal{O}}_u;\underline{\underline{A}}^{\cdot}_u) \arrow[d, equal] & \mathbb{H}^{2d}(N,N\smallsetminus\overline{\mathcal{O}}_u;\underline{\underline{A}}^{\cdot}_u) \arrow[l, "\eta"'] \arrow[d, equal] \\
H^{o}(\mathcal{O}_u,L) & IH^{o}(\mathcal{O}_u,L) \arrow[l]
\end{tikzcd} .$$

Now it is a general fact in intersection homology theory, that the bottom canonical map is an isomorphism : In the language of [GM1], codimension zero cycles are always allowable.

Finally, the map η is an isomorphism since $\underline{\underline{A}}^{\cdot}_u$ is supported on $\overline{\mathcal{O}}_u$. And the map ω is an isomorphism because its source and target are both vector-spaces with basis the irreducible components of $\overline{\pi^{-1}\mathcal{O}_u}$, and ω identifies them via these bases. This completes the proof of the lemma, and hence of the theorem. Q.e.d.

1.13 Comparison with Springer's original construction.

We have seen in theorem 1.8 that our characteristic class map δ : $K \to Q(K)$ embeds $H^{c\ell}_{4n-2d_u}(\overline{\pi^{-1}\mathcal{O}_u})$ into $H^{2d_u}(X)$ as a subspace invariant for Borel's W-action on $H^*(X)$, and we even know already that the W-action on this subspace is equivalent to ρ_u, that is to the Springer representation of W on $H^{2d_u}(\pi^{-1}u)^{G_u}$ (cf. 1.8). It remains to show how this equivalence can be realized by some geometrically defined map between the two representation spaces. This is the purpose of the theorem below, which complements theorem 1.8.

We shall use here the construction of the W action on $H^*(\pi^{-1}u)$ as in [B-M] 1, due to Lusztig [Lu], and which differs from Springer's originally defined representations by a multiplication with the sign character (cf. loc. cit., and also [Ho], Appendix, [Sp], §2 for more details). We shall actually prefer to work with the contragredient representation on the top homology space $H_{2d_u}(\pi^{-1}u)$, dual to $H^{2d_u}(\pi^{-1}u)$. This space, and similarly $H_{4n-2d_u}(\overline{\pi^{-1}\mathcal{O}_u})$, are equipped with canonical bases, given by the (canonical classes $[C]$ resp. $[K]$ of) the irreducible components C resp. K of $\pi^{-1}u$ resp. $\pi^{-1}\mathcal{O}_u$. So the linear maps linking these vector-spaces as explained below can be explicitly described by referring to these bases. Note that G_u acts via the finite group $C(u) := G_u/G_u^o$ of its connected components on (co)homology, since G_u^o acts trivially. Since $C(u)$ acts on $H_{2d_u}(\pi^{-1}u)$ by permuting the canonical basis, we get also a canonical basis of the subspace of G_u-invariants, c corresponding to the $C(u)$-orbits of components C. We even get a canonical projection p of $H_{2d_u}(\pi^{-1}u)$ onto $H_{2d_u}(\pi^{-1}u)^{G_u}$ defined by

$$(1) \qquad p[C] = \sum_{a\in C(u)} [aC] = n_C([C_1]+\cdots+[C_r])$$

where $C_1,\ldots,C_r$ are the <u>different</u> $C(u)$-conjugates of C, and n_C is the number of $a \in C(u)$ fixing C. Starting from K, an irreducible component of $\overline{\pi^{-1}\theta_u}$, we recall from [BB] III, B.2 that

$$K \cap \pi^{-1}u = G_u C = C_1 \cup \cdots \cup C_r$$

is the union of one full $C(u)$-orbit of components of $\pi^{-1}u$ (notation as before). We conclude that the Gysin homomorphism

$$j : H^{c\ell}_{4n-2d_u}(\overline{\pi^{-1}\theta_u}) \longrightarrow H_{2d_u}(\pi^{-1}u)$$

is given by

$$(2) \qquad j[K] = [G_u C] = [C_1] + \cdots + [C_r] \quad .$$

In particular, it is a linear isomorphism onto the G_u-invariants of the target space. We also recall from loc.cit. that the G-saturation GC, that is the union of all $gC (g \in G)$, is an irreducible component of $\pi^{-1}\theta_u$, so its closure is an orbital cone bundle $\overline{GC} = K$, and we have a linear map q in the reverse direction of j by

$$(3) \qquad q[C] = n_C[\overline{GC}] \quad ,$$

where the scale factor n_C is added to make the triangle p,j,q commute. Let us summarize this discussion as follows.

<u>Proposition</u> : <u>We have a commutative diagram of linear maps</u>

$$\begin{tikzcd} H_{2d_u}(\pi^{-1}u) \arrow[d, "p"] \arrow[r, "q"] & H^{c\ell}_{4n-2d_u}(\overline{\pi^{-1}\theta_u}) \arrow[dl, "j", "\sim"'] \arrow[r, hook, "\delta"] & H^{2d_u}(X) \\ H_{2d_u}(\pi^{-1}u)^{G_u} & & \end{tikzcd}$$

described in (1), (2), (3) resp. 1.8 above.

1.14 Theorem : The maps in the above diagram are W-equivariant.

In particular, the Weyl group representation on characteristic classes of orbital cone bundles is equivalent to Springer's representation ρ_u via the the Gysin map j .

Proof : Note that the "inclusion" δ is W-equivariant by definition, and the projection p is W-equivariant since the actions of G_u and W commute [S1], [BM1]. It is therefore sufficient to prove that the linear isomorphism j is W-equivariant.

To do this, chose a tranversal slice A in $\underline{\underline{g}}$ to $\mathcal{O}_u$ at u , and denote $D = A \cap N$ its intersection with the nilpotent cone. (For instance, A may be chosen as an affine subspace of codimension $2d_u = \dim \mathcal{O}_u$ in $\underline{\underline{g}}$ meeting the tangent space of $\mathcal{O}_u$ only in the point u). Consider the following commutative diagram :

$$\begin{tikzcd}
H_{2d_u}(\pi^{-1}u) & H^{c\ell}_{4n-2d_u}(\pi^{-1}O_u) \arrow[l] \arrow[d, "\cong"] & H^{c\ell}_{4n-2d_u}(\overline{\pi^{-1}\mathcal{O}_u}) \arrow[l, "\cong"'] \\
H^{2d_u}(\pi^{-1}D,\pi^{-1}D-\pi^{-1}u) \arrow[u, "\cong"] \arrow[d, "\cong"] & H^{2d_u}(T^*X-\pi^{-1}\partial O_u, T^*X-\pi^{-1}\overline{O}_u) \arrow[l] \arrow[d, "v"', "\cong"] & H^{2d_u}(T^*X, T^*X-\overline{\pi^{-1}\mathcal{O}_u}) \arrow[l, "\omega"'] \arrow[u, "\cong"] \arrow[d, dashed] \\
\mathbb{H}^{2d_u}(D,D\setminus\{u\},\underline{\underline{A}}^{\cdot}) & \mathbb{H}^{2d_u}(N-\partial O_u, N-\overline{\mathcal{O}}_u, \underline{\underline{A}}^{\cdot}) \arrow[l] & {}
\end{tikzcd}$$

Our map j occurs in this diagram as the composition of the two Gysin homomorphisms in the first row. The other horizontal arrows come from inclusions of pairs of topological spaces. The upper row of vertical arrows are Lefschetz duality isomorphisms, while the lower ones are again examples of the isomorphism 1.11 (*). Let us point out that the dotted arrow, analogous to v , would generally not be an isomorphism, because $\overline{\pi^{-1}\theta_u}$ will be strictly smaller than $\pi^{-1}\overline{\theta}_u$ in general and is therefore <u>not</u> a π-preimage. This is the reason why we are inserting the middle column in the diagram.

Now observe that Springer's W representation on $H_{2d_u}(\pi^{-1}u)$ is obtained from Lusztig's W action on $\underline{\underline{A}}^{\cdot}$ by transport of structure up the left column of this diagram. On the other hand, our W action on $H_{4n-2d_u}(\overline{\pi^{-1}\theta_u})$ is induced from Borel's on $H^{2d_u}(X)$, which is again obtained from the same W action on $\underline{\underline{A}}^{\cdot}$ by transport of structure up the right column in the big diagram of 1.11. This shows that j is in fact W equivariant. Q.e.d.

1.15. Hotta's transformation formulas.

We may now describe quite explicitly in terms of integer matrices, how the characteristic classes $Q(K_1),\dots,Q(K_r)$ transform under the action of W (notations as in 1.8).

Theorem (cf. [Ho], Theorem 1 [Ho2]) : For each simple reflection s in W and each $i = 1,\dots,r$, we have either

$$sQ(K_i) = -Q(K_i) \quad ,$$

or else

$$sQ(K_i) = Q(K_i) + \sum_j n^s_{ij}\, Q(K_j)$$

for certain non-negative integer coefficients n^s_{ij}, which are zero unless K_j intersects K_i in codimension 1 .

Moreover, the coefficients n^s_{ij} can be described more precisely in geometrical terms, see Hotta's formula [Ho],1.5, definition 2 (cf. also 4.13 of the present paper).

The fact that our classes $Q(K_1),\dots,Q(K_r)$ satisfy Hotta's transformation formulas may now be viewed as a corollary of theorem 1.14. However, another proof of this theorem (which does not assume Hotta's work),will also follow from chapter 4 of our present paper (see 4.13).

§2. Generalities on equivariant K-theory

For the convenience of those readers not familiar with equivariant K-theory, we have collected here in some detail the general facts needed from this theory as prerequisites for subsequent chapters. In the present chapter, G may be an arbitrary linear algebraic group over k .

2.1. Algebraic notion of fibre bundles [Sr2], [We].

By a G-variety, we mean an algebraic variety Y over k , equipped with an algebraic action of G on Y . A G-morphism $\varphi : Y \to X$ is a map of a G-variety Y into a G-variety X , which is a morphism of algebraic varieties, and respects the G-action. A surjective G-morphism is called a principal G-fibration, if G acts simply transitively on each fibre. The projection of $X \times G$ onto the left factor, with G acting only on the right factor, provides an example, refered to as trivial. A principal fibre bundle with structure group G , base X , and total space Y , is then defined as a principal G-fibration $\varphi : Y \to X$, which is locally trivial, meaning that each point in X has a neighbourhood $\mathcal{U}$ such that $\varphi^{-1}\mathcal{U} \to \mathcal{U}$ is trivial (up to isomorphism). Note that so far, the definitions are only a word by word translation from a topological or analytical context. In the algebraic context, however, there is a subtle point to be clarified here : The "local triviality" in the definition may refer to either the Zariski or the étale topology, that is $\mathcal{U}$ above is a (Zariski-) open affine neighbourhood in the first case, resp. any étale covering of such in the second case. For example, the principal fibration of G by an algebraic subgroup H need not be a bundle in the first, narrow sense (used by Weil [We]), but it is always a bundle in the second, wider sense (introduced by Serre [Sr2]). However, it fortunately turns out, for a lot of groups G , called "special" in [Sr2], that this subtle difference does not matter at all, that is to say local triviality in the weak ("étale") sense implies local triviality in the strong ("Zariski") sense for principal fibrations with this structure group. "Special" groups in this sense include all connected solvable linear groups [Ro], and also $G = GL_n$ [Sr2]. Now let F be any G-variety, and Y a principal G-bundle over X in either sense. Then the associated fibre bundle

$$Y \times^G F \to X$$

with fibre F is defined as usual (see e.g. [Sr2]), and is locally trivial in the corresponding sense. So again, for the "special" groups above, the two notions of local triviality coincide for the associated fibre bundle as well.

For example, taking $F = k^n$ and Y a principal GL_n-bundle on X, we get $Y \times^{GL_n} F$, an (algebraic) vector-bundle over X. Since each vector-bundle is obtained in this fashion (up to isomorphism), we conclude from the preceding remarks that a vectorbundle is locally trivial with respect to Zariski topology, if and only if it is locally trivial with respect to étale topology. - As a consequence of this discussion, we will not have to care anymore about the difference in this paper.

2.2. Equivariant vector-bundles and definition of $K_G(X)$ [SGA6], [A1].

Now we assume that G acts on both the base X and the total space Y of a vector bundle, and that the bundle map $\varphi : Y \to X$ is a G-morphism (2.1). Moreover, we assume that G preserves the linear structure, in the sense that each group element $g \in G$ maps the fibre $Y_x = \varphi^{-1}(x)$ (at $x \in X$) linearly into Y_{gx}. Then Y, equipped with this additional structure, is called a G-equivariant vector-bundle, or just a G-vector-bundle. Morphisms of G-vector-bundles are defined as vector-bundle homomorphisms which are simultaneously G-morphisms.

We denote by $K_G(X)_{\mathbb{Z}}$ the Grothendieck group of the category of G-vector-bundles on X. We write

$$K_G(X) = K_G(X)_{\mathbb{Z}} \times_{\mathbb{Z}} k \quad ,$$

according to our general conventions on coefficients (cf. §1). The formation of direct sums resp. tensor products on G-vector-bundles induces the structure of a commutative <u>ring</u> on $K_G(X)$. The <u>rank</u> of a vector-bundle gives rise to an <u>augmentation</u> homomorphism $\varepsilon : K_G(X) \to k$ of this ring, and the formation of exterior powers of G-vector-bundles defines the so-called λ-<u>operations</u> $\lambda^i : K_G(X) \to K_G(X)$; this equips $K_G(X)$ with the structure of an <u>"augmented λ-ring"</u> in the sense of Grothendieck, as defined and studied from an axiomatic point of view in [SGA6], exposé V, or also in the first chapters of [Kn], [FL]. Our terminology here is completely analogous to Atiyah's, who considers topological vector-bundles equivariant under a finite or compact group G in [A1], §§ 1.6, 2.3.

<u>Remarks</u>. 1) If $G = 1$ is the trivial group, then $K_G(X)$ reduces to the ring $K(X)$ of "ordinary" K-theory (as considered already in §1), see e.g. [Ma], [FL], or [A1].

2) If X is a single point, then a G-vector-bundle on X is just a G-<u>module</u>, that is a finite-dimensional linear representation of G on a k-vector-space. Hence in this case, $K_G(X)$ is nothing else but the <u>representation ring</u> $R(G)$ (cf. [A1]).

So equivariant K-theory is a common generalization of these two important extreme cases.

2.3. Equivariant homogeneous vector-bundles.

as an important non-trivial example, let us consider equivariant vector-bundles on a homogeneous space, say $X = G/B$, with isotropy group B any closed subgroup of G. Starting from an arbitrary B-module F (of finite dimension over k), let us form the associated fibre bundle $Y = G \times^B F$, which is a vector-bundle on X (cf. the remark on local triviality in 2.1). We make it into a G-equivariant vector-bundle, by making G act on G by left multiplication (whereas for the bundle construction, B acted on G by right multiplication).

Proposition : The construction of associated fibre bundles $F \to G \times^B F$ as explained above induces an isomorphism of augmented λ-rings :

$$R(B) \xrightarrow{\sim} K_G(X) \quad .$$

Obviously, the construction induces a λ-ring homomorphism and preserves the augmentation. It is not difficult to see that every G-vector-bundle on X has the form $G \times^B F$ up to isomorphism (cf. e.g. [Sel], p.130), which means that the homomorphism is surjective. An inverse homomorphism will be provided by restriction (to the isotropy group and the base point of X), cf. 2.9. below.

2.4. Functoriality in the group G.

If $\phi : B \to G$ is a morphism of algebraic groups, then our G-variety X becomes also a B-variety, and each G-equivariant vector-bundle on X is also B-equivariant. This provides a functorial homomorphism

$$\phi^* : K_G(X) \to K_B(X) \quad ,$$

which we refer to as restriction from G to B. This is a homomorphism of augmented λ-rings.

If for instance $B = 1$, then this is a canonical homomorphism $K_G(X) \to K(X)$, refered to as the <u>forgetful</u> homomorphism, since it is given by "forgetting" the G-action on a G-vector-bundle. In case $X = \{x\}$ is a single point for example, this forgetful homomorphism $K_G(X) \to K(X) \cong k$ is just the augmentation homomorphism $\varepsilon : R(G) \to k$ (cf. 2.2).

2.5. Functoriality in the space X .

Let X, X' be two G-varieties, and $f : X \to X'$ a G-morphism. Then for each G-vector-bundle E over X' , the pull-back f^*E to a vector-bundle over X is again G-equivariant, and this induces a homomorphism

$$f_G^* : K_G(X') \to K_G(X) \quad ,$$

which also preserves the structures of augmented λ-rings. In the case $G = \{1\}$, this is the so-called <u>Gysin homomorphism</u> $f^* : K(X') \to K(X)$ in ordinary K-theory (cf. [A1], and [Fu], Example 15.1.8, or [SGA6], III.4.1, IV.2.7). The G-equivariant analogue f_G^* <u>induces</u> the ordinary Gysin homomorphism, in the sense that we have a commutative diagram

$$\begin{tikzcd} K_G(X') \arrow[r, "f_G^*"] \arrow[d] & K_G(X) \arrow[d] \\ K(X') \arrow[r, "f^*"] & K(X) \end{tikzcd}$$

of "forgetful" and Gysin homomorphisms.

2.6. The sheaf-theoretic point of view.

It will be convenient for us to work with <u>G-equivariant coherent sheaves</u> of <u>$\mathcal{O}_X$-modules</u> on our (irreducible) G-variety X , as we did in [BB] III.

Let $K_G(X)_o$ denote the Grothendieck group of the category of all such sheaves. (This notational convention follows Fulton [Fu], p.281). Since we can identify a vector-bundle over X with the locally free $\mathcal{O}_X$-module of its germs of sections, and since G-equivariance of the vector-bundle means the same as G-equivariance of the corresponding locally free $\mathcal{O}_X$-module, we obtain a canonical homomorphism $K_G(X) \to K_G(X)_o$. In the cases of interest for our present paper, this turns out to be an isomorphism by the following proposition.

Proposition : If X is a smooth G-variety which (*) admits a G-equivariant ample line bundle , then $K_G(X) \cong K_G(X)_o$ by the canonical homomorphism.

Remarks. 1) Note that the existence of an ample line bundle means exactly that the variety under consideration is quasi-projective. Therefore,, in the case $G = 1$, hypothesis (*) means X quasi-projective, and so the proposition coincides with Borel-Serre's Théorème 2 in [BS] for the non-equivariant case.

2) For example, the hypothesis (*) of this proposition is always satisfied for X projective and G semi-simple. In fact, according to Mumford [Mu], Chapter 1.3, Corollary 1.6, any normal projective G-variety admits a G-equivariant ample line bundle. More generally, it easily follows that Mumford's result extends to any G-variety X for which there exists a G-equivariant affine morphism $X \to Y$ to a normal projective G-variety. Hence the proposition implies as a special case the following criterion, which will suffice for our purposes in this paper :

Corollary : Let E be the total space of a G-vector-bundle over a smooth projective G-variety. Then $K_G(E) \cong K_G(E)_o$, if G is semi-simple.

3) In the sequel, we shall always assume that X satisfies the assumptions of proposition 2.6, and we shall use this proposition to identify

$K_G(X)$ with $K_G(X)_o$.

2.7. Existence of equivariant locally free resolutions.

For the proof of the equivariant Borel-Serre theorem (proposition 2.6 above), one may proceed by imitating the original proof in [BS], pp.105-108. The only step in their argument which requires a new proof in our present, equivariant version is Lemma 10 in loc. cit., which assures that coherent sheaves admit locally free resolutions. So let us just state and prove the "equivariant version" of this lemma here.

Lemma : Every G-equivariant coherent sheaf $\underline{\underline{F}}$ on X is a quotient of a G-equivariant locally free sheaf $\underline{\underline{E}}$ of finite type on X .

Note that repeated application of this lemma will imply the following (with assumptions as in proposition 2.6) :

Proposition : Every G-equivariant coherent sheaf $\underline{\underline{F}}$ on X admits a finite resolution by G-equivariant locally free sheaves of finite type.

Proof of the lemma : Let $\underline{\underline{L}}$ denote a G-equivariant ample line bundle on X , which exists by assumption. Then the tensor product $\underline{\underline{F}} \otimes \underline{\underline{L}}^n$ over $\mathcal{O}_X$ of $\underline{\underline{F}}$ with a sufficiently big number n of copies of $\underline{\underline{L}}$ will be generated by its global sections, cf. [Ha], p.153. Then the natural morphism

$$\Gamma(X, \underline{\underline{F}} \otimes \underline{\underline{L}}^n) \otimes_k \mathcal{O}_X \to \underline{\underline{F}} \otimes \underline{\underline{L}}^n$$

is surjective, and we can even find a finite-dimensional k-subspace $E \subset \Gamma(X, \underline{\underline{F}} \otimes \underline{\underline{L}}^n)$ such that the restriction map $E \otimes_k \mathcal{O}_X \to \underline{\underline{F}} \otimes \underline{\underline{L}}^n$ is still surjective. So we have represented $\underline{\underline{F}} \otimes \underline{\underline{L}}^n$ as a quotient of a locally free sheaf $E \otimes_k \mathcal{O}_X$, and it only remains to observe that this can be done equivariantly. Since $\underline{\underline{F}}$ and $\underline{\underline{L}}$ are G-equivariant, so is $\underline{\underline{F}} \otimes \underline{\underline{L}}^n$. Hence G acts on $\Gamma(X, \underline{\underline{F}} \otimes \underline{\underline{L}}^n)$ algebraically, by locally finite linear endomorphisms. By enlarging the subspace E if necessary, we may assume that E is a G-submodule. Now we have G-actions on E and on $\mathcal{O}_X$, hence on the tensor product $E \times_k \mathcal{O}_X$. It is clear that the above surjective morphism $E \otimes_k \mathcal{O}_X \to \underline{\underline{F}} \otimes \underline{\underline{L}}^n$ is G-equivariant. Tensoring this morphism with the inverse $\underline{\underline{L}}^{-n}$ of the invertible sheaf $\underline{\underline{L}}^n$, we obtain a surjective G-equivariant morphism of the locally free coherent sheaf

$$\underline{\underline{E}} := (E \otimes_k \mathcal{O}_X) \otimes_{\mathcal{O}_X} \underline{\underline{L}}^{-n} = E' \otimes_k \underline{\underline{L}}^{-n} \quad \text{onto} \quad \underline{\underline{F}} \otimes \underline{\underline{L}}^n \otimes \underline{\underline{L}}^{-n} \cong \underline{\underline{F}} .$$

Q.e.d.

Remark : For results of the same type, but with different hypothesis, we refer to Thomason [T1], Corollary 5.2 ("Seshadri's conjecture", see also [Sh]).

2.8. Remarks on Gysin homomorphisms in terms of coherent sheaves.

a) In order to define the ("equivariant" resp. "ordinary") Gysin homomorphisms (cf. 2.5)

$$f_G^* : K_G(X')_o \to K_G(X)_o \quad \text{resp.} \quad f^* : K(X') \to K(X)_o$$

for a (G)-morphism $f : X \to X'$ in working with coherent $\mathcal{O}_X$-modules, one has to assume f to be a "perfect" morphism in the sense of [SGA6], Définition III.4.1, and IV.2.7, see also [Fu], Example 15.1.8. For example, if $f : X \to X'$ is a closed embedding, such that $f_*\mathcal{O}_X$ has a finite resolution by a complex $E_\cdot$ of locally free $\mathcal{O}_{X'}$-modules, then f is perfect, and the Gysin homomorphism f^* (also called "homological intersection with X") is given by the formula

$$f^*[F] = \sum_i (-1)^i \, [\mathrm{Tor}_i^{X'}(\mathcal{O}_X, \underline{\underline{F}})]$$

for any locally free $\mathcal{O}_{X'}$-module. Here $\mathrm{Tor}_i^{X'}(\mathcal{O}_X, \underline{\underline{F}})$ is the i^{th} cohomology group of the complex $E. \otimes f_*(\underline{\underline{F}})$ (cf. loc.cit.). If the morphism f and the complex $E.$ are G-equivariant, then the same formula holds for the equivariant Gysin-homomorphism f_G^*. More generally, if f and $E.$ are only B-equivariant, for a closed subgroup $B \subset G$, then by restricting first the group from G to B, then the space from X' to X, we obtain a composed restriction homomorphism $f_B^* : K_G(X') \to K_B(X)$ satisfying a similar formula.

b) The above assumptions on F are satisfied e.g. for f the inclusion of a complete intersection, or for f a regular embedding (cf. [SGA6], Example III, 4.1.1, or [Fu], chapter 6.2, for a detailed treatment of this case on the level of Chow groups).

2.9. Equivariant K-theory on a vector-bundle : Basic restriction techniques.

A crucial technique frequently applied in this paper, will be the investigation of equivariant K-theory on the total space of a homogeneous vector-bundle, via its restriction to a fibre, or alternatively to the zero-section, and finally to a point. Since the morphisms involved are all regular embeddings, the preceding remarks (2.8) apply to the corresponding restriction homomorphisms (and these are the only three, very special, cases of 2.8 relevant for our present work).

Because of the fundamental significance of this restriction technique for the whole paper, let us explain it in more detail on a general level here. We assume that X is an irreducible G-variety satisfying the assumptions of proposition 2.6.

(1) Restriction to a point. Consider the inclusion map $i : \{x\} \to X$ of a point x in X. Let $B = G_x$ be the isotropy group at x. Then we obtain a restriction homomorphism $i_B^* : K_G(X) \to R(B)$ by the remarks 2.8 a),b). For X a homogeneous G-space, that is $X = G/B$, this turns out (easily) to be an inverse of the homomorphism $R(B) \to K_G(X)$ coming from the formation of associated fibre bundles (see 2.3), and so i_B^* is an isomorphism $K_G(X) \cong R(B)$ in this case.

(2) Restriction to the zero section. Now let E be the total space of a G-vector-bundle over X, and let $\sigma : X \to E$ denote the zero-section. Then the restriction homomorphism

$$\sigma^* : K(E) \to K(X) ,$$

known as "homological intersection with the zero section", has an equivariant analogue

$$\sigma_G^* : K_G(E) \to K_G(X)$$

by 2.8. It is well-known to algebraic geometers, that σ^* is always an isomorphism (see [SGA6], Exposé IX, Proposition 1.6, or on the level of Chow groups [Fu], Theorem 3.3). According to recent work of Thomason, this generalizes to the equivariant situation, that is to say σ_G^* is an isomorphism quite generally [T1], Theorem 4.1. In the cases of interest for our present paper, we shall see more explicit reasons why this map is an isomorphism (see 3.7 and 4.3), independently of [T1].

(3) Restriction to a fibre. With E a G-vector-bundle on X as before, let $j : E_x \to E$ denote the inclusion of the fibre E_x at $x \in X$. Then we obtain a restriction homomorphism

$$j_B^* : K_G(E) \to K_B(E_x) \quad , \quad \text{where } B = G_x \text{ as before.}$$

(4) Combining examples (1), (2) (twice), and (3) into the following diagram of inclusions

$$\begin{tikzcd} E \arrow[r, hookleftarrow, "\alpha"] & X \\ E_x \arrow[u, hook, "j"] \arrow[r, hookleftarrow, "\iota"] & x \arrow[u, hook, "i"'] \end{tikzcd}$$

we obtain a commutative diagram of restriction homomorphisms

$$\begin{tikzcd} K_G(E) \arrow[r, "\sigma_G^*"] \arrow[d, "j_B^*"] & K_G(X) \arrow[d, "i_B^*"] \\ K_B(E_x) \arrow[r, "\iota_B^*"] & R(B) \end{tikzcd}$$

Summarizing our present discussion, we may now state :

Proposition : This is a commutative diagram of isomorphisms of λ-rings.

2.10. Filtrations on $K_G(X)$.

a) Grothendieck's γ-filtration $\ldots \supset K_G^a(X) \supset K_G^{a+1}(X) \supset \ldots$, is defined as follows (cf. [SGA6], V.3.10, or [A2], §12) : Let $I_G = \operatorname{Ker} \varepsilon$ denote the augmentation ideal, and γ^i $(i = 0,1,2,\ldots)$ the operators given by the formula

$$\gamma^i(Z) = \lambda^i(Z+i-1)$$

for all ring elements Z, where ε resp. the λ^i's are defined in 2.2. Then $K_G^a(X)$ is the ideal generated by all monomials

$$\gamma^{i_1}(Z_1)\gamma^{i_2}(Z_2)\ \ldots\ \gamma^{i_r}(Z_r)\quad ,\quad \text{with}\quad i_1+i_2+\cdots+i_r \geq a\ ,$$
$$\text{and}\quad Z_1,\ldots,Z_r \in I_G\ .$$

Then $K_G^o(X) = K_G(X)$, $K_G^1(X) = I_G$, and

$$K_G^a(X)\ K_G^b(X) \subset K_G^{a+b}(X)$$

for all $a,b \in \mathbb{N}$, i.e. $K_G(X)$ becomes a filtered ring. This filtration has the advantage of being obviously functorial (i.e. preserved by the various restriction or Gysin homomorphisms introduced above), since its definition refers only to the structure of $K_G(X)$ as an abstract "augmented λ-ring", which is indeed functorial. - In particular, the various functorial homomorphisms considered induce also homomorphisms on the associated graded rings, for which we shall use a notation generalizing that introduced in 1.4, that is :

$$\mathrm{gr}\ K_G(X) = \bigoplus_{a\geq o} \mathrm{gr}_a K_G(X) = \bigoplus_{a\geq o} K_G^a(X)/K_G^{a+1}(X)\quad .$$

b) Topological filtration. Alternatively, we may filter the ring $K_G(X)$ by co-dimension of supports. More precisely, let $K_G^a(X)'$ denote the Grothendieck group of G-equivariant coherent $\mathcal{O}_X$-modules with support of codimension $\geq a$. Then $\ldots \supset K_G^a(X)' \supset K_G^{a+1}(X)' \supset \ldots$ is another descending filtration of $K_G(X)$ as a ring. Its comparison to the γ-filtration is a delicate problem in general. In the non-equivariant situation (special case $G = 1$), it is known that the γ-filtration coincides with the topological filtration, as we already mentioned in 1.3. In general, the two filtrations will be obviously very different, see the next example.

2.11. Representation rings for example.

The γ-filtration on $R(G)$, for a commutative reductive group G, is given by the powers of the augmentation ideal $I_G = \mathrm{Ker}\,\varepsilon$:

$$R^a(G) := K_G^a(\mathrm{point}) = I_G^a \,, \quad \text{for all } a \in \mathbb{N} \,.$$

(This follows from the fact that all irreducible representations are one-dimensional, cf. [A2], Corollary 12.4.) In general, we have only $I_G^a \subset R^a(G)$ from the definition, but the topology defined by the γ-filtration on $R(G)$ still coincides with the I_G-adic topology. The completed representation ring $\hat{R}(G)$ is defined as the completion of $R(G)$ with respect to this topology.

2.12. Application of equivariant K-theory to $\mathcal{D}$-modules.

In this chapter, Y is a smooth algebraic variety over k, equipped with an action $q : G \times Y \to Y$ of an algebraic group G. It will be convenient to introduce a weaker notion for "equivariant $\mathcal{D}_Y$-modules" than was considered in [BB] III, 2.2. First, the sheaf of algebras $\mathcal{D}_{G\times Y}$ on $G \times Y$, which is isomorphic to the external tensor product $\mathcal{D}_G \boxtimes \mathcal{D}_Y$ (notation [Gd]) contains $\mathcal{O}_G \boxtimes \mathcal{D}_Y$ as a subsheaf of algebras.

Definition : A weakly G-equivariant $\mathcal{D}_Y$-module is a $\mathcal{D}_Y$-module $\mathcal{M}$ equipped with an isomorphism $\alpha : q^*\mathcal{M} \to p^*\mathcal{M}$ of $\mathcal{O}_G \boxtimes \mathcal{D}_Y$-modules, which satisfies a certain cocycle condition which ensures that α induces a group action of G on $\mathcal{M}$ (see [Mu] for more details).

Notice that both $p^*\mathcal{M}$ and $q^*\mathcal{M}$ are $\mathcal{D}_{G\times Y}$-modules, hence are $\mathcal{O}_G \boxtimes \mathcal{D}_Y$-modules.

A good filtration $(\mathcal{M}_n)_{n\in\mathbb{Z}}$ of a weakly G-equivariant $\mathcal{D}_Y$-module $\mathcal{M}$ is said to be G-equivariant if the isomorphism $\alpha : q^*\mathcal{M} \xrightarrow{\sim} p^*\mathcal{M}$ maps $q^*\mathcal{M}_n$ to $p^*\mathcal{M}_n$ (so that each $\mathcal{M}_n$ is a H-equivariant sub $\mathcal{O}_Y$-module of $\mathcal{M}$).

A weakly G-equivariant $\mathcal{D}_Y$-module $\mathcal{M}$, which is coherent (as a $\mathcal{D}_Y$-module), always admits a G-equivariant good filtration. Indeed, let $(\mathcal{M}_n)_{n\in\mathbb{Z}}$ be any good filtration of $\mathcal{M}$; then the intersection $\mathcal{M}'_n$ of $\mathcal{M}_n$ and of $p_*q^*\mathcal{M} = p_*p^*\mathcal{M}$, is $\mathcal{O}_Y$-coherent and $(\mathcal{M}'_n)_{n\in\mathbb{Z}}$ is a G-equivariant good filtration of $\mathcal{M}$.

Now for $(\mathcal{M}_n)$ a G-equivariant good filtration of $\mathcal{M}$, the associated graded module $\mathrm{gr}(\mathcal{M}) = \bigoplus_n (\mathcal{M}_n/\mathcal{M}_{n-1})$ is a G-equivariant coherent $\mathcal{O}_{T^*Y}$-module, and therefore determines a class $[\mathrm{gr}\,\mathcal{M}]$ in $K_G(T^*Y)$. The lemma below asserts that this class does not depend on the choice of $(\mathcal{M}_n)$. Hence a weakly G-equivariant $\mathcal{D}_Y$-module $\mathcal{M}$ determines a well-defined class in $K_G(T^*Y) \cong K_G(Y)$, which we call the class of $\mathcal{M}$ in $K_G(Y)$.

Lemma : Let $(\mathcal{M}_n)_{n\in\mathbb{Z}}$ and $(\mathcal{M}'_n)_{n\in\mathbb{Z}}$ be two G-equivariant good filtrations of $\mathcal{M}$. Then the corresponding associated graded modules $\mathrm{gr}\,\mathcal{M}$ resp. $\mathrm{gr}'\mathcal{M}$ determine the same class in $K_G(T^*Y)$.

Proof : By changing the numbering of the second filtration (if necessary), we may assume without loss of generality that $\mathcal{M}_n \subset \mathcal{M}'_n$ for all n. It is known that for a suitable integer $d \geq 0$, we have $\mathcal{M}'_n \subset \mathcal{M}_{n+d}$ for all n. Our proof proceeds by induction on d. Assume first $d = 1$. Then we have, for all $n \in \mathbb{Z}$, exact sequences of G-equivariant coherent $\mathcal{O}_Y$-modules

$$0 \to \mathcal{M}'_n/\mathcal{M}_n \to \mathcal{M}_{n+1}/\mathcal{M}_n \to \mathcal{M}_{n+1}/\mathcal{M}'_n \to 0$$

$$0 \to \mathcal{M}_{n+1}/\mathcal{M}'_n \to \mathcal{M}'_{n+1}/\mathcal{M}'_n \to \mathcal{M}'_{n+1}/\mathcal{M}_{n+1} \to 0,$$

Hence we get exact sequences of G-equivariant coherent graded $\mathcal{O}_{T^*Y}$-modules,

$$0 \to A \to \mathrm{gr}\, M \to B \to 0$$

$$0 \to B \to \mathrm{gr}' M \to A \to 0 \quad ,$$

where we put

$$A := \bigoplus_n M'_n/M_n \quad , \quad \text{and} \quad B := \bigoplus_n M_{n+1}/M'_n \quad .$$

So in the group $K_G(T^*Y)$, we get the equality

$$[\mathrm{gr}\, M] = [A] + [B] = [\mathrm{gr}' M] \quad .$$

Now the induction step : Assume for some $d > 1$ that $M_n \subset M'_n \subset M_{n+d}$ for all n . We consider the further filtration of M given by $M''_n := M_n + M'_{n-1}$. It is again G-equivariant. It is also a <u>good</u> filtration, because for n big enough, we have

$$M''_n = M_n + M'_{n-1} = \mathcal{D}_Y(1)\, M_{n-1} + \mathcal{D}_Y(1) M'_{n-2} = \mathcal{D}_Y(1) M''_{n-1} \quad .$$

We observe that this new filtration satisfies $M_n \subset M''_n \subset M_{n+d-1}$. Hence $[\mathrm{gr}\, M] = [\mathrm{gr}'' M]$ by the induction hypothesis. On the other hand, $M'_{n-1} \subset M''_n \subset M'_n$ for all n . So we know by the $d = 1$ case of the proof that also $[\mathrm{gr}\, M'] = [\mathrm{gr}\, M'']$. This completes the proof of the lemma. Q.e.d.

§3. Equivariant K-theory of torus actions and formal characters.

In this paragraph, we consider a torus T, that is a commutative connected reductive group over k, and a linear action of T on a vector space E of finite dimension r over k. We shall assume that all weights of T in E are positive with respect to some partial ordering. For example, T might be the group of homotheties of E. In the applications in subsequent chapters, T will be the maximal torus in a semisimple group, E will be the nilradical of a Borel subalgebra, and the weights of T in E will be the set of positive roots.

3.1. The completed representation ring of a torus.

The characters $\chi : T \to \mathbb{C}_m$ of our torus T form a free abelian group $X^*(T)$, and the representation ring of T is isomorphic to the group ring of this character group, that is

$$R(T)_{\mathbb{Z}} \cong \mathbb{Z}[X^*(T)] \quad , \quad \text{and}$$

$$R(T) \cong k[X^*(T)] \cong \mathcal{O}(T) \, .$$

The differential of $\chi \in X^*(T)$ is a linear form $d\chi : \underline{t} \to k$ on the Lie algebra $\underline{t}$ of T, called an integral weight. Let $\Lambda = \Lambda(T)$ denote the lattice in $\underline{t}^*$ of all integral weights. For each $\lambda \in \Lambda$, we define the formal power series

$$e^{\lambda} := \sum_{i \geq 0} \frac{1}{i!} \lambda^i \quad ,$$

considered as an element in $\hat{S}(\underline{t}^*)$, the completion of the ring $S(\underline{t}^*)$ of polynomial functions on $\underline{t}$ (with respect to the $\underline{t}^*S(\underline{t}^*)$-adic topology.) We define a homomorphism $\varphi: R(T) \longrightarrow \hat{S}(t^*)$ by putting

$$\varphi(\chi) := e^{d\chi}$$

for all $\chi \in X^*(T)$, and we observe (cf. e.g. [AH], proposition 4.3) that this extends to an isomorphism (notation 2.11)

$$\varphi : \hat{R}(T) \xrightarrow{\sim} \hat{S}(\underline{t}^*) \quad .$$

This isomorphism will allow us to interpret the elements of the completed representation ring $\hat{R}(T)$ as formal power series functions on the Lie algebra. Note that this is an isomorphism of filtered rings. Let us make this slightly more explicit. Given a power series $P \in \hat{S}(\underline{t}^*)$, we shall use the notation $[P]^i$ for its degree i homogeneous term, that is the unique homogeneous polynomial of degree i on $\underline{t}$ such that

$$P = \sum_{i \geq 0} [P]^i \quad .$$

Recalling that the γ-filtration on $\hat{R}(T)$ is given by the I_T-adic filtration, where I_T is the augmentation ideal (2.11), we get the following

Lemma : If an element $Q \in \hat{R}(T)$ corresponds to the power series $P = \varphi(Q) \in \hat{S}(\underline{t}^*)$, then its degree with respect to the γ-filtration is equal to the smallest number a such that $[P]^a \neq 0$.

Remark. We shall then refer to $[P]^a$ as the lowest order term of the series, and denote it also

$$\text{gr } P := [P]^a \quad .$$

3.2. Formal characters of T-modules

Let M be a T-module, that is to say a k-vector-space equipped with a linear algebraic action of T . For each character $\chi \in X^*(T)$ with differential $d\chi = \lambda \in \Lambda$, we denote by M_χ or M_λ the corresponding "weight space"

$$M_\chi = M_\lambda = \{v \in M \mid tv = \chi(t)v, t \in T\} \quad .$$

Since T is a reductive group, $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ is a direct sum of weight spaces. If the <u>weight multiplicities</u> $\dim M_\chi$ are all <u>finite</u>, then we call M <u>admissible</u> (or we say that M admits a formal character), and we <u>define</u> the <u>formal character</u> of M as the formal sum

$$\mathrm{ch}(M) := \sum_\chi (\dim M_\chi)\,[\chi].$$

We note that this definition coincides with the one used e.g. in [Di], 7.5, and [Ja], 4.5.

Let us call $\mathrm{ch}(M)$ <u>bounded</u> (by χ_0), if its nonzero coefficients occur only at characters $\chi \leq \chi_0$, for some $\chi_0 \in X(T)$. Then the multiplication in $R(T)$ extends to a multiplication of bounded formal characters, defined formally by

$$\Big(\sum_{\xi\leq\xi_0} a_\xi\,[\xi]\Big)\Big(\sum_{\zeta\leq\zeta_0} b_\zeta\,[\zeta]\Big) = \sum_{\chi\leq\chi_0} c_\chi\,[\chi],$$

where

$$c_\chi := \sum_{\xi\zeta=\chi} a_\xi b_\zeta \quad \text{for all } \chi \in X(T),$$

and $\chi_0 = \xi_0\zeta_0$. This makes the group of all bounded formal characters into an extension ring of $R(T)$, which is denoted $R_{-\infty}(T)$. This ring may be described as a power series field $R_{-\infty}(T) \cong k[[x_1,\ldots,x_\ell]]$, where $x_1,\ldots,x_\ell$ are the <u>negative</u> fundamental weights . (In particular, $R_{-\infty}(T)$ should not be confused with the completed representation ring $\hat{R}(T)$.) Note that $R_{-\infty}(T)$ contains the <u>field of fractions</u> of $R(T)$.

3.3. Example.

We consider here a finite dimensional T-module E whose weights are all positive with respect to some partial ordering. Then the symmetric algebra $M = S(E)$ admits a formal character, which is bounded, and (with respect to multiplication in $R_{-\infty}(T)$) given by the formula

$$(1) \qquad ch(S(E)) = \Delta(E)^{-1},$$

where we define

$$(2) \qquad \Delta(E) := \prod_{\lambda} (1-e^{\lambda})^{\dim E_{\lambda}},$$

the product being extended over all weights of E. In fact, for the special case where $E = E_{\lambda}$ is onedimensional, one gets immediately the geometric series

$$ch(S(E_{\lambda})) = 1 + e^{\lambda} + e^{2\lambda} + \ldots = (1-e^{\lambda})^{-1}$$

and then the general case follows by repeated application of the formula

$$ch(S(E \oplus F)) = ch(S(E) \otimes S(F)) = ch(S(E))\cdot ch(S(F)) ,$$

which holds for any E,F satisfying the above positivity assumption.

As a corollary, we note that

$$(3) \qquad ch(\mathcal{O}(E)) = \Delta(E^{*})^{-1} = \prod_{\lambda}(1-e^{-\lambda})^{-\dim E_{\lambda}},$$

product over all weights λ of E.
This is because $\mathcal{O}(E) \cong S(E^{*})$, and the weights of the contragredient T-module E^{*} have opposite signs, that is $\dim E^{*}_{\lambda} = \dim E_{-\lambda}$ for all $\lambda \in \Lambda$.

3.4. T-equivariant modules with highest weight.

Let M be a finitely generated $S(E^*)$-module, which is equipped with a linear, locally finite T-action, such that

$$t(sm) = (ts)(tm) \quad \text{for all} \quad t \in T \, , \, m \in M \, , \, s \in S(E^*) \quad .$$

For short, we say that M is a T-equivariant finitely generated $S(E^*)$-module. It is easy to see that such M admits a formal character. Let us be slightly more specific.

Consider a cyclic $S(E^*)$-submodule $M' = S(E^*)v$ generated by some weight vector $0 \neq v \in M$, of weight λ, say. Then $M' = kv + M''$, where $M'' = E^*M'$ has only weights strictly smaller than λ, because of our positivity assumption on the weights of E. Therefore λ is called the highest weight of M', and v resp. M' is called a cyclic highest weight vector resp. (sub-)module of highest weight λ. Now let I be the ideal of elements in $S(E^*)$ annihilating v. Then obviously I is a T-submodule, and

$$(*) \qquad ch(M') = e^{\lambda} \, ch(S/I) \quad .$$

Here and in the sequel, we sometimes write $S = S(E^*)$, for short.

Lemma : a) A finitely generated T-equivariant $S(E^*)$-module M admits a composition series

$$0 = M_o \subset M_1 \subset M_2 \subset \cdots\cdots \subset M_\ell = M$$

of T-equivariant $S(E^*)$-submodules M_i, such that the composition factors M_i/M_{i-1} are cyclic highest weight modules.

b) Then M admits the formal character

$$ch(M) = \sum_{i=1}^{\ell} e^{\lambda_i} ch(S/I_i) \quad ,$$

where λ_i is the highest weight of M_i/M_{i-1}, and I_i the annihilator of the corresponding cyclic highest weight vector.

c) The composition series can be chosen in such a way, that $I_1,\ldots,I_\ell$ are prime ideals.

Proof : Chose $M_1 = M'$ as in the preceding discussion, then repeat the same discussion for M/M_1 etc. Since M is noetherian, this process will terminate after a finite number of steps, proving a). Now b) follows from (*). Assuming $M \neq 0$, let P be a minimal associated prime ideal of the $S(E^*)$-module M. Since T is connected and acts by automorphisms on M, it stabilizes P. By equivariance, T stabilizes also the submodule of M annihilated by P, which is therefore a T-equivariant submodule $N \neq 0$. By choosing the weight vector $0 \neq v \in M$ in the discussion preceding the lemma even in N, we can achieve there $I = P$ prime. Repetition of this procedure proves part c) of the lemma.

Q.e.d.

3.5. Projective and free cyclic highest weight modules.

Proposition. Each projective finitely generated T-equivariant $S(E^*)$-module M admits a finite composition series with all composition factors free cyclic highest weight modules.

Here the terms "free" resp. "projective" refer to the $S(E^*)$-module structure. The proposition follows by repeated application of the following lemma.

Lemma : A maximal weight space M_λ generates a free highest weight submodule $N = S(E^*)M_\lambda \cong S(E^*) \otimes_k M_\lambda$, and the quotient M/N is again projective.

Proof : Let $\underline{m} = E^* S(E^*)$ denote the maximal ideal corresponding to the zero-point $0 \in E$. Consider the canonical linear map of $\tilde{N} := S(E^*) \otimes M_\lambda$ into M , which maps $\tilde{N}$ onto N . The weight space M_λ injects into $M/\underline{m}M$, because the weights in $\underline{m}M$ are all smaller than λ. Hence the map from $S(E^*) \otimes_k M_\lambda$ to M induces an injection on the modules tensored with $S(E^*)/\underline{m}S(E^*)$. Because both modules are projective over $S(E^*)$, it follows that $N = S(E^*) \otimes_k M_\lambda$ is a submodule of M, and that the quotient is locally free at $\underline{m}$. Hence M/N is locally free in a neighbourhood $U \subset E$ of the zero-point. But then the T-saturation is $TU = E$ (since we assume $\sum_{\lambda > 0} E_\lambda = E$), and since M/N is T-equivariant, it is locally free everywhere on E. This means that M/N is projective. Q.e.d.

Corollary: The formal character of M is given by

$$ch(M) = \Delta(E^*)^{-1}(e^{\lambda_1}+\dots+e^{\lambda_\ell}) \quad ,$$

where $\lambda_1,\dots,\lambda_\ell$ are the highest weights of the free cyclic composition factors, mentioned in the proposition.

In fact, for a free cyclic highest weight module N_i of highest weight λ_i we clearly have

$$ch(N_i) = e^{\lambda_i} ch(S) = e^{\lambda_i} \Delta(E^*)^{-1} \quad ,$$

by combining 3.4(*) and 3.3(1).

Remark : The proposition states that, in the frame-work of lemma 3.4, the case "M projective" means that the prime ideals $I_1,\dots,I_\ell$ there are all zero; hence the corollary is also a special case of 3.4b).

3.6. Formal characters of equivariant coherent sheaves.

Now let $\mathrm{Mod}_T^{\mathrm{coh.}}(E)$ denote the category of T-equivariant coherent sheaves of $\mathcal{O}_E$-modules, and $\mathrm{Mod}_T^{\mathrm{f.g.}}(S(E^*))$ the category of T-equivariant finitely generated $S(E^*)$-modules. Then the functor $\underline{\underline{F}} \mapsto \Gamma(E,\underline{\underline{F}}) = M$ establishes an equivalence of these categories, because E is affine. Moreover, a locally free $\underline{\underline{F}}$ corresponds to a projective M, and since each $\underline{\underline{F}}$ admits a finite locally free resolution in $\mathrm{Mod}_T^{\mathrm{coh.}}(E)$ by 2.7), we have simultaneously that each M admits a projective resolution in $\mathrm{Mod}_T^{\mathrm{f.g.}}(S(E^*))$. In particular, the (isomorphic) Grothendieck K-groups of these categories are generated (over $\mathbb{Z}$) by locally free resp. projective objects, and so it follows from the preceding proposition 3.5 that they are even generated by the free cyclic highest weight modules. -By abuse of language, the formal character of $M = \Gamma(E,\underline{\underline{F}})$ is also called the formal character of $\underline{\underline{F}}$, notation $\mathrm{ch}(\underline{\underline{F}}) := \mathrm{ch}(M)$. Now it follows from the preceding discussion and from corollary 3.5 that $\Delta(E^*)\mathrm{ch}(\underline{\underline{F}})$ is an integer linear combination of exponentials.

Corollary : For an arbitrary T-equivariant coherent $\mathcal{O}_E$-module $\underline{\underline{F}}$ (resp. f.g. $S(E^*)$-module $M = \Gamma(E,\underline{\underline{F}})$) the formal character is of the form

$$\mathrm{ch}(\underline{\underline{F}}) = \mathrm{ch}(M) = \Delta(E^*)^{-1} \sum_{\lambda\in\Lambda} a_\lambda e^\lambda \quad ,$$

where $a_\lambda = a_\lambda(\underline{\underline{F}})$ are integer coefficients, only finitely many of them being nonzero.

Obviously, these integers a_λ are uniquely determined by $\underline{\underline{F}}$ resp. M. Following traditional terminologies (cf. [Ja]), we refer to a_λ as the (integer) multiplicity of the free cyclic highest weight module of highest weight λ in the module M, notation

$$a_\lambda(\underline{\underline{F}}) = a_\lambda(M) = (M:\mathrm{gr}\, M(\lambda)) \quad .$$

In the sequel, we consider $ch(\underline{\underline{F}})$ as an element of the fraction field $\mathrm{Fract}(R(T))$ of the domain $R(T)$.

3.7. Restriction to the zero point.

Since our T-action on E is linear, it fixes the zero point, and so the inclusion $\iota : \{0\} \to E$ gives rise to a restriction homomorphism $\iota_T^* : K_T(E) \to R(T)$.

Proposition : a) ι_T^* is an isomorphism of λ-rings $K_T(E) \xrightarrow{\sim} R(T)$.

b) This isomorphism can be computed within $\mathrm{Fract}(R(T))$ by the following formula :

$$\iota_T^*[\underline{\underline{F}}] = \Delta(E^*)ch(\underline{\underline{F}})$$

which holds for any T-equivariant coherent sheaf $\underline{\underline{F}}$ on E.

Comments. 1) We note that a) may be viewed as a very special case of a theorem of Thomason [T1], Theorem 4.1 on arbitrary equivariant vector-bundles E, cf. 2.9(2). But it is convenient to prove the whole proposition here more directly below.

2) The proposition holds even with coefficients in $\mathbb{Z}$.

Proof : a) Since ι_T^* preserves the λ-ring-structure by functoriality, it suffices to prove that it is bijective. In fact, let us exhibit the inverse map. Starting from an arbitrary finite-dimensional T-module F, we form the locally free coherent $\mathcal{O}_E$-module

$$\underline{\underline{F}} = \mathcal{O}_E \otimes_k F \,, \tag{1}$$

and make T act diagonally on it. Then $\psi : [F] \to [\underline{\underline{F}}]$ defines a homomorphism of $R(T)$ into $K_T(E)$. It is clear that $\underline{\underline{F}}$ has fibre F at the zero-point, so

(2) $$\iota_T^*[\underline{\underline{F}}] = [F] \quad .$$

In particular, ψ is <u>injective</u>. We may even conclude from (2) that ψ is an inverse of ι_T^* , provided that we know ψ is <u>surjective</u>.

To see this, it suffices to show that <u>the sheaves of the particularly nice form</u> (1) <u>are sufficient to generate the whole Grothendieck group</u> $K_T(E)$. Observe that each locally free $\underline{\underline{F}}$ of the form (1) gives rise to a <u>projective</u> module $M := \Gamma(E,\underline{\underline{F}})$ of the form $M \cong S(E^*) \otimes_k F$, (under the equivalence discussed in 3.6), <u>and conversely</u>. But the free cyclic highest weight modules are of this form, and we have observed in 3.6, that even these suffice to generate the full Grothendieck group under consideration. A fortiori, the sheaves of the form (1) generate $K_T(E)$ as a group.

b) Since both sides of the formula claimed in b) are additive in $[\underline{\underline{F}}]$, it suffices to verify the formula only on a nice set of generators of $K_T(E)$, for example just for those sheaves $\underline{\underline{F}}$ of the form (1). Assuming $\underline{\underline{F}}$ of this form, we get

$$M := \Gamma(E,\underline{\underline{F}}) = \Gamma(E,\mathcal{O}_E) \otimes_k F = S(E^*) \otimes_k F \ ,$$

$$ch(\underline{\underline{F}}) = ch(M) = ch(S(E^*) \otimes_k F) = ch(S(E^*))ch(F) \ ,$$

and so by 3.3

$$\Delta(E^*)ch(\underline{\underline{F}}) = ch(F) \quad .$$

On the other hand, as pointed out in (2), we have

$$ch(F) = [F] = \iota^*[\underline{\underline{F}}] \quad ,$$

which completes the proof of our proposition. Q.e.d.

3.8. Computation of γ-degree.

Proposition 3.7a) says in particular, that the isomorphism $\iota_T^* : K_T(E) \xrightarrow{\sim} R(T)$ preserves degrees with respect to the γ-filtrations, and then formula b) of the proposition tells us, how we may compute the γ-degree of $\underline{\underline{F}}$, as an element in $K_T(E)$, from the formal character $ch(\underline{\underline{F}})$, considered as a formal power series on $\underline{t}$ (cf. 3.1) :

Corollary : For any T-equivariant coherent sheaf $\underline{\underline{F}}$ on E , the γ-degree of $[\underline{\underline{F}}]$ in $K_T(E)$ is equal to the degree of the lowest order term of the power series $\Delta(E^*)ch(\underline{\underline{F}})$ in $\hat{S}(\underline{t}^*)$.

Using our notation for lowest order term as introduced in 3.1, we may write this statement as a formula :

$$\gamma\text{-deg}[\underline{\underline{F}}] = \deg \, gr(\Delta(E^*)ch(\underline{\underline{F}})) \ .$$

We shall determine the right hand side more explicitly below (3.10). Before doing this, let us first look more systematically at the lowest order term of $\Delta(E^*)ch(\underline{\underline{F}})$.

3.9. Character polynomials.

Let $\underline{\underline{F}}$ be a T-equivariant coherent sheaf on E , and let $a = \gamma\text{-deg}[\underline{\underline{F}}]$ denote its γ-degree. By the filtered isomorphisms $K_T(E) \xrightarrow{\sim} R(T)$ (3.7) and $\hat{R}(T) \xrightarrow{\sim} \hat{S}(\underline{t}^*)$ (3.1), we may identify the associated graded ring of $K_T(E)$,

$$gr\ K_T(E) \xrightarrow{\sim} S(\underline{t}^*)$$

with the graded ring of polynomial functions on $\underline{t}$, by identifying for each degree j the vector spaces

$$K_T^j(E)/K_T^{j+1}(E) \tilde{\to} \hat{S}_j(\underline{t}^*)/\hat{S}_{j+1}(\underline{t}^*) \cong S^j(\underline{t}^*) \quad .$$

In this manner, the class $[\underline{\underline{F}}]$ modulo $K_T^{a+1}(E)$ determined by the sheaf $\underline{\underline{F}}$ is identified with a homogeneous polynomial of degree a on $\underline{t}$.

Definition : We call this polynomial the character polynomial of $\underline{\underline{F}}$, and denote it by $q_{\underline{\underline{F}}}^T$. For a module $M = \Gamma(E,\underline{\underline{F}})$, we use the similar notation and terminology, calling $q_M^T = q_{\underline{\underline{F}}}^T$ the character polynomial of M . Normally, we drop the superscript T , if the torus of reference is clear enough from the context.

The term "character polynomial" refers to the fact that this polynomial is computable from the formal character by means of formula 3.7b) as the lowest degree term of the series $\Delta(E^*)ch(\underline{\underline{F}})$:

Corollary : $q_{\underline{\underline{F}}} = q_M = gr\ \Delta(E^*)ch(\underline{\underline{F}})$.

Let us illustrate the use of this formula by a few immediate applications.

Proposition : $q_M = \frac{1}{a!} \sum_{\lambda\in\Lambda} a_\lambda(M)\lambda^a$, (*)

where the integer $a_\lambda(M)$ is the "multiplicity" of the free cyclic highest weight module of highest weight λ in M (notation 3.6). Moreover, the number $a = \gamma\text{-deg}[M]$ may be computed from these integers $a_\lambda(M)$ as the smallest positive integer a for which the righthand side of (*) becomes a nonzero polynomial.

Proof : This is now an immediate consequence of 3.5, by writing out the sum of exponentials given there as a power series, and taking the lowest order term.

Q.e.d.

Example : Let us compute the character polynomial q_M for $M = k$ the trivial, one-dimensional module. In this case, $ch(M) = 1$, and so q_M is the lowest degree term of

$$\Delta(E^*)ch(M) = \Delta(E^*) = \prod_{i=1}^{r} (1-e^{-\lambda_i}) =$$

$$\prod_{i=1}^{r} (1-1-\lambda_i - \frac{1}{2}\lambda_i^2 - \dots) = \prod_{i=1}^{r} (\lambda_i) + \dots$$

where $\lambda_1,\dots,\lambda_r$ are the weights of T in E , and dots are terms of higher degree. Hence q_M is equal to the product of all weights of T in E^* , and in particular $\deg q_M = r := \dim E$ is the vector-space dimension of E in this case. - Similarly, one obtains for an <u>arbitrary</u> M of <u>finite vector-space dimension</u> the formula

$$q_M = \dim M \prod_{i=1}^{r} (\lambda_i) \quad ,$$

$$\deg q_M = \dim E =: r \quad .$$

3.10. <u>Degree of character polynomial equals codimension of support.</u>

<u>Theorem</u> : <u>Let</u> $\underline{\underline{F}}$ <u>be a</u> T-<u>equivariant coherent sheaf on</u> E . <u>Then</u>

(1) $\qquad \deg q_{\underline{\underline{F}}} = \operatorname{codim}_E \operatorname{supp} \underline{\underline{F}} \quad .$

<u>Comments</u> : Note that, by definition of the character polynomial $q_{\underline{\underline{F}}}$, its degree is also the γ-degree of $[\underline{\underline{F}}]$. - Let us also restate the theorem in terms of the corresponding T-equivariant module $M = \Gamma(E,\underline{\underline{F}})$:

(2) $\qquad \deg q_M = r-d(M) \quad ,$

where $r := \dim E$, and $d(M)$ denotes Krull- (or Gelfand-Kirillov-) dimension. We refer to Joseph [J1], 2.4 (ii) for a similar result; the precise relationship is explained later in our present paper (4.8).

Our proof of this theorem, given in 3.12-3.13, proceeds by induction on d(M). To make the induction argument work, however, we first have to sharpen the theorem by the following technical complement.

3.11. Positivity property of character polynomials

Let us call a polynomial q on $\underline{t}^*$ positive, if it takes only positive values on all regular dominant integral weights. In more detail, let $\omega_1,\dots,\omega_\ell$ denote the fundamental weights corresponding to our choice of partial ordering on $\underline{t}^*$, and $\rho = \omega_1 + \dots + \omega_\ell$. Then $\Omega := \mathbb{N}\omega_1 + \dots + \mathbb{N}\omega_\ell$ resp. $\rho + \Omega$ denote the semigroups of (resp. regular) dominant integral weights, and "q positive" means

$$q(\lambda) > 0, \quad \text{for all } \lambda \in \rho + \Omega.$$

In particular, $q(\rho) > 0$.

Complement of Theorem 3.10: The polynomial q_F is positive.

Our induction argument, which will prove the theorem and its complement simultaneously, is based on the following key lemma.

3.12. Division by a non-zero-divisor.

Lemma: Let $M \neq 0$ be a finitely generated T-equivariant $S(E^*)$-module, and let $0 \neq f \in S(E^*)$ be a weight vector of weight $-\mu \neq 0$, which acts as a nonzero-divisor on M. Then we have

(1) $$q_{M/fM} = \mu q_M$$

and

(2) $$d(M/fM) = d(M)-1.$$

Proof: In fact, the assumptions on f imply that multiplication by f maps each weight space M_λ injectively into $M_{\lambda-\mu}$, so that

$$ch(fM) = ch(M)e^{-\mu},$$

hence

$$ch(M/fM) = ch(M)-ch(fM) = (1-e^{-\mu})ch(M).$$

Since $1-e^{-\mu} = \mu - \frac{1}{2}\mu^2 + \dots$, it follows that the lowest order term of $\Delta(E^*)ch(M/fM)$ is obtained from that of $\Delta(E^*)ch(M)$ by multiplication with μ. This proves (1). The last equation (2) is a well known property of Krull dimension. Q.e.d.

As a consequence of (1), we can conclude that q_M <u>will be positive if</u> $q_{M/fM}$ <u>is</u>.

<u>3.13.</u> <u>Proof of theorem 3.10 and 3.11.</u>

Let us proceed by induction on $d(M)$. The case $d(M) = 0$ is settled by example 3.9. So assume $d(M) > 0$. We choose prime ideals $I_1,\dots,I_\ell$ as in 3.4c), so that by 3.4b):

$$ch(M) = e^{\lambda_1}ch(S/I_1) + \dots + e^{\lambda_\ell}ch(S/I_\ell).$$

By 3.9, q_M is the lowest order term of the series

$$ch(M)\Delta(E^*) = \sum_i e^{\lambda_i}ch(S/I_i)\Delta(E^*). \tag{*}$$

Let $J \subset \{1,\dots,\ell\}$ denote the set of indices i for which $\deg q_{S/I_i}$ assumes its minimum value, m say. Observing that multiplication by an exponential series e^{λ_i} does not affect the lowest order term of a power series, we see that the power series (*) has its degree m term equal to $\sum_{i\in J} q_{S/I_i}$, and has only zero terms in smaller degree. So we may conclude from (*) that $\deg q_M = m$, and

$$q_M = \sum_{i\in J} q_{S/I_i}, \tag{**}$$

unless this sum vanishes. But suppose for a moment we knew already that theorem 3.10 and its complement hold for the modules $M = S/I_i$ involved in the sum. Then the polynomials q_{S/I_i} are all <u>positive</u>, and so their sum (**) obviously cannot vanish, but must be positive as well.

Hence

$$\deg q_M = m := \min_{1 \leq i \leq \ell} \deg q_{S/I_i}.$$

On the other hand we have

$$r\text{-}d(M) = \min_{1 \leq i \leq \ell} (r\text{-}d(S/I_i)).$$

Therefore, we conclude that theorem 3.10 and 3.11 hold for M, provided that we can prove that they hold for all modules of the form S/I, where I is a T-stable prime ideal in $S = S(E^*)$, and $d(S/I) \leq d(M)$.

In other words, by the preceding argument we have reduced the proof to the particular case $M = S/I$, with I a T-stable prime ideal. To prove the theorem for this case, pick some weight vector $0 \neq f \in M_{-\mu}$ of strictly negative weight $-\mu$; this is possible: in fact, we can take any weight vector in S/I except the scalars. Then $q_M = \mu^{-1} q_{M/fM}$ by the lemma (3.12), and $d(M/fM) < d(M)$, so q_M will be positive, since $q_{M/fM}$ is by induction hypothesis. Moreover, since

$$\deg q_{M/fM} = r\text{-}d(M/fM)$$

by induction, the lemma gives also

$$\deg q_M = (\deg q_{M/fM}) -1 = r\text{-}d(M/fM)-1 = r\text{-}d(M).$$

This completes our inductive proof of theorem 3.10 and 3.11. Q.e.d.

3.14. Determination of character polynomials by supports.

If $V \subset E$ is a T-stable closed subvariety, then the structure sheaf $\underline{\underline{F}} = \mathcal{O}(V)$ is T-equivariant, and so has a character polynomial, which we also denote $q_V = q_{\mathcal{O}(V)}$, for short. If $\underline{\underline{F}}$ is an arbitrary T-equivariant coherent sheaf on E , then its support in E is clearly a T-stable closed subvariety. Recall our notation for the supporting cycle of $\underline{\underline{F}}$: This is the formal linear combination

$$(*) \qquad \operatorname{supp}(\underline{\underline{F}}) = \sum_V m_V(\underline{\underline{F}})[V] \quad ,$$

extended over the irreducible components V of the support (notation 1.4).

Proposition : Let $\underline{\underline{F}}$ be a T-equivariant coherent sheaf on E , let d denote the codimension of its support in E , and let (*) be its supporting cycle. Then

$$q_{\underline{\underline{F}}} = \sum_{\operatorname{codim} V = d} m_V(\underline{\underline{F}}) q_V \quad .$$

In particular, the character polynomial of $\underline{\underline{F}}$ is completely determined by its supporting cycle.

Proof : We let $M = \Gamma(E,\underline{\underline{F}})$ and chose prime ideals $I_1,\ldots,I_\ell$ as in lemma 3.4. To introduce our new notation, we let V_j denote the support of S/I_j in E ; then $q_{S/I_j} = q_{V_j}$. We already knew that

$$q_{\underline{\underline{F}}} = \sum_j{}' \, q_{S/I_j} = \sum_j{}' \, q_{V_j} \quad ,$$

where Σ' is summation over those j for which q_{S/I_j} has minimal degree (cf. 3...). But from theorem 3.10, we now know in addition that

$$\deg p_{\underline{\underline{F}}} = d \quad , \quad \deg q_{V_j} = d_j$$

for $j = 1,\dots,\ell$, where d_j denotes the codimension of V_j in E . We conclude that

$$(**) \qquad q_{\underline{\underline{F}}} = \sum_{d_j=d} q_{V_j} \quad .$$

On the other hand, it is clear that the V_j of codimension $d_j = d$ are the irreducible components of the support of $\underline{\underline{F}}$ of minimal codimension. Now (**) gives the proposition. Q.e.d.

<u>Corollary</u> : <u>Let</u> $0 \to M_1 \to M \to M_2 \to 0$ <u>be a short exact sequence of</u> T-<u>equivariant</u> <u>finitely generated</u> $S(E^*)$-<u>modules. If</u> $d(M_1) = d(M_2)$, <u>then</u>

$$q_M = q_{M_1} + q_{M_2} \quad ,$$

<u>if</u> $d(M_i) < d(M_j)$ $(i,j = 1,2)$, <u>then</u>

$$q_M = q_{M_j} \quad .$$

<u>3.15</u>. <u>The theory of Hilbert-Samuel polynomials as a special case</u> [AC], [AM].

Let us now look more carefully at the particular case of a onedimensional torus, that is $T = \mathbb{G}_m = k^*$ is the multiplicative group. Then the representation ring $R(T)$ is the group ring of an infinite cyclic group $X^*(T) \cong \mathbb{Z}$, or in other words $R(T)$ identifies with the ring $k[t,t^{-1}]$ of Laurent polynomials in one variable t (corresponding to the character $a \longmapsto a^{-1}$ of $\mathbb{G}_m$). The augmentation ideal I_T is then the principal ideal generated by $t-1$, and so the I_T -adic degree of a Laurent polynomial $f(t)$

is given by the order of vanishing of $f(t)$ at $t = 1$. More generally, this extends to infinite Laurent series as well. Recalling 3.1, let us summarize :

Lemma : The completed representation ring of the multiplicative group is the ring of formal power series in the variable $t-1$,

$$\hat{R}(\mathbb{G}_m) = k[[t-1]] \quad ,$$

the γ-degree of a power series being its order of vanishing at $t = 1$.

A T-action on a module M is now just a $\mathbb{Z}$-grading $M = \bigoplus_{i \in \mathbb{Z}} M_i$, and M will admit a formal character iff the homogeneous subspaces M_i are finite-dimensional, the formal character being known as the Poincaré series

$$ch(M) = \sum_i (\dim M_i) \cdot t^i$$

in this case. - Now let $d_1, \ldots, d_r \in \mathbb{Z}$ denote the weights of our linear action of $T = \mathbb{G}_m$ in the r-dimensional vectorspace E . By our assumption, they are all positive. Recalling 3.3, we find that now

$$\Delta(E) = (1-t^{-d_1}) \ldots (1-t^{-d_r}) \quad ,$$

so

$$ch(S(E^*)) = \Delta(E^*)^{-1} = (1-t^{d_1})^{-1} \ldots (1-t^{d_r})^{-1} \quad .$$

Now let $\underline{\underline{F}}$ be a T-equivariant coherent $\mathcal{O}_E$-module, and $M = \Gamma(E, \underline{\underline{F}})$ the corresponding graded $S(E^*)$-module. Then our result in proposition 3.7 and theorem 3.10, specialized to this case, yields in particular the following facts about Poincaré series.

Corollary : The Poincaré series of the $\mathbb{Z}$-graded module M (as above) takes the form of a rational function

$$\mathrm{ch}(M) = \frac{P(t)}{(1-t^{d_1})\dots(1-t^{d_r})} ,$$

<u>where</u> $P(t)$ <u>is a Laurent polynomial, whose order of vanishing at</u> $t = 1$ <u>equals</u> $r-d(M)$, <u>the codimension of the support of</u> M <u>in</u> E .

For example, let T be the group of homotheties on E . Then $d_1 = d_2 = \cdots = d_r = 1$, so the Poincaré series equals

$$\mathrm{ch}(M) = \frac{P(t)}{(1-t)^r} = (1-t)^{-d(M)}\, Q(t) ,$$

where $Q(t)$ is a polynomial with $Q(1) \neq 0$.

<u>Remark</u> : These are essentially classical results of Hilbert and Samuel in commutative algebra, for expositions of which we refer to e.g; Bourbaki, [AC] VIII, §4, n°3, Théorème 2, and n°4, Théorème 3, or also [AM], theorems 11.1 and 11.14, or [Sr], under the headlines of "degree of the Hilbert-Samuel functions", resp. "Hilbert functions" and "dimension theory".

3.16. <u>Restriction to one-parameter subgroups.</u>

Let $\omega_1,\dots,\omega_\ell$ be free generators of the lattice Λ of integral weights, called the "fundamental weights". Their choice corresponds to the choice of the partial ordering of weights. Then $\Omega := \mathbb{N}\omega_1 + \dots + \mathbb{N}\omega_\ell$ is the semi-lattice of "positive integral weights", so our positivity assumption on the T-module E means that its weights belong to Ω , and furthermore for any T-equivariant $S(E^*)$-module M , there exists some $\lambda \in \Lambda$ such that

$$\mathrm{ch}(M) = \sum_{-\mu\in\lambda+\Omega} (\dim M_\mu) e^\mu \qquad (*)$$

for each T-equivariant $S(E^*)$-module.

We note that the exponentials $t_1 := e^{-\omega_1}, \ldots, t_\ell := e^{-\omega_\ell}$ are algebraically independent. So we may consider the formal character as a formal Laurent power series in $t_1,\ldots,t_\ell$, as mentioned in 3.2, that is

$$ch(M)=F(t_1,\ldots,t_\ell) = \sum_{m_1,\ldots,m_\ell \geq -k} a_{m_1,\ldots,m_\ell} t_1^{m_1} t_2^{m_2} \ldots t_\ell^{m_\ell}$$

with $m_1,\ldots, m_\ell$ integers bounded below by some $-k$.

Now we consider a <u>one parameter subgroup</u> $\phi: \mathbb{G}_m \longrightarrow T$. We may describe ϕ in terms of the <u>perfect pairing</u> $X_*(T) \times X^*(T) \longrightarrow \mathbb{Z}$ of the (rank ℓ free abelian) groups of all one parameter subgroups resp. characters of T (see e.g. [S2], 2.5.12) as follows: For a character $\chi \in X^*(T)$, we define $\langle\chi,\phi\rangle$ as the integer n describing the restricted character $\phi^*(\chi)$ of $\mathbb{G}_m$ as $a \longmapsto a^n$ ($a \in \mathbb{G}_m$). We denote $\chi_1,\ldots,\chi_\ell$ the "fundamental characters" of T (that is $[\chi_i] = e^{\omega_i} = t_i^{-1}$). Then ϕ is uniquely determined by the ℓ integers

$$n_i := \langle\chi_i,\phi\rangle \qquad (i = 1,\ldots,\ell).$$

<u>Definition</u>: ϕ is called <u>positive</u>, if all $n_1,\ldots,n_\ell$ are <u>positive</u>.

<u>Lemma</u>: <u>The $\mathbb{G}_m$-module obtained from M by restriction to a positive one parameter subgroup ϕ admits a formal character $\phi^*ch(M)$. If $ch(M) = F(t_1,\ldots,t_\ell)$ as above, then</u>

(*) $$\phi^* ch(M) = F(t^{n_1},\ldots,t^{n_\ell}).$$

Here $t = e^{-\omega}$ denotes the character $a \longmapsto a^{-1}$ of $\mathbb{G}_m$.

The lemma is obvious for a one dimensional module $M = k_\chi$ of character $ch(M) = [\chi] = t_1^{m_1} \ldots t_\ell^{m_\ell}$, since then

$$\phi^* ch(M) = t^{\langle\chi,\phi\rangle} = t^{n_1m_1+\ldots+n_\ell m_\ell} = (t_1^{n_1})^{m_1} \ldots (t_\ell^{n_\ell})^{m_\ell},$$

and it extends linearly to arbitrary modules M of finite dimension, i.e. to <u>polynomials</u> F. For the extension to arbitrary bounded formal characters $ch(M)$, i.e. to Laurent <u>power series</u> F, we need the positivity assumption on ϕ. This will guarantee the existence and boundedness of the formal character of M, considered as a $\mathbb{G}_m$-module by restriction, and then formula (*) follows as in the finite dimensional case.

<u>Proposition</u>: Let ϕ be a positive one parameter subgroup of T.
Let $\phi^*: K_T(E) \longrightarrow K_{\mathbb{G}_m}(E)$ denote restriction. Let $[M] \in K_T(E)$ denote the class determined by some T-equivariant $S(E^*)$-module M, and d its γ-degree. Then

a) $\gamma\text{-deg}\ \phi^*[M] \geq d = \gamma\text{-deg}\ [M]$.

b) $\gamma\text{-deg}\ \phi^*[M] = d$ if and only if

$$q_M^T(y) \neq 0, \text{ at } y := d\phi(1) \in \underline{t}.$$

c) <u>If such is the case, then the character polynomial of the "restricted" module is obtained by restriction of the character polynomial, i.e.</u> $q_M^{\mathbb{G}_m} = \phi^* q_M^T$. <u>If we write</u> $q_M^T = f(\omega_1, \ldots, \omega_\ell)$ <u>as a polynomial in the fundamental weights, then</u>

$$\phi^* q_M^T = f(n_1, \ldots, n_\ell)\ \omega^d$$

<u>with</u> $n_i = \langle x_i, \phi\rangle$ <u>as above</u>.

<u>Proof</u>: Refering to 3.7 and the lemma above, we may describe the restriction map ϕ^* explicitely by means of the following commutative diagram,

$$\begin{array}{ccccc}
K_T(E) & \xrightarrow[\sim]{\iota_T^*} & R(T) & \xrightarrow{\sim} & k[t_1, t_1^{-1}, \ldots, t_\ell, t_\ell^{-1}] \\
\downarrow{\scriptstyle \phi^*} & & \downarrow{\scriptstyle \phi^*} & & \downarrow{\scriptstyle \phi^*} \\
K_{\mathbb{G}_m}(E) & \xrightarrow[\iota_{\mathbb{G}_m}^*]{\sim} & R(\mathbb{G}_m) & \xrightarrow{\sim} & k[t, t^{-1}]
\end{array}$$

where the vertical arrows will all be denoted ϕ^*, but the last one is explicitely described as a specialization map of Laurent polynomials as follows:

$$\phi^*P(t_1,\dots,t_\ell) = P(t^{n_1},\dots,t^{n_\ell}).$$

Hence if [M] is represented by the multi-variable Laurent polynomial $P(t_1,\dots,t_\ell)$, then $\phi^*[M]$ is represented by the one-variable Laurent polynomial $P(t^{n_1},\dots,t^{n_\ell})$ =: $P'(t)$. Now recalling $t_i = e^{-\omega_i}$, and $t = e^{-\omega}$, we may develop these polynomials into power series in the fundamental weights $\omega_1,\dots,\omega_\ell$ resp. ω.

If we write

$$P(t_1,\dots,t_\ell) = \sum_{j\geq d} q_j\ (\omega_1,\dots,\omega_\ell),$$

with q_j a homogeneous polynomial of degree j in $\omega_1,\dots,\omega_\ell$, and $q_d \neq 0$, then the character polynomial of M with respect to T is

(1) $$q_M^T = q_d(\omega_1,\dots,\omega_\ell),$$

whereas that of M with respect to ϕ (i.e. $q_M^{\mathbf{G}_m}$) is the lowest term of the power series

(2) $$P'(T) = \sum_j c_j\ \omega^j.$$

Now

$$P'(t) = P(t^{n_1},\dots,t^{n_\ell}) = P(e^{-n_1\omega},\dots,e^{-n_\ell\omega}) =$$

(3) $$= \sum_{j\geq d} q_j\ (n_1\omega,\dots,n_\ell\omega) = \sum_{j\geq d} q_j\ (n_1,\dots,n_\ell)\ \omega^j,$$

and the statements of the proposition are then obvious by comparing coefficients in (3) and (2), in particular

$$q_M^{\mathbf{G}_m} = q_d(n_1,\dots,n_\ell)\ \ \omega^d,$$

if this is non-zero. Q.e.d.

<u>Corollary</u>: <u>For a given T-equivariant module</u> M, <u>restriction preserves γ-degree, that is equality holds in a) above, for "almost all" positive one parameter subgroups</u> ϕ, <u>in the sense that the exceptional set is contained in a Zariski closed subset of</u> $X_*(T) \otimes_{\mathbb{Z}} k$.

Recall that $X_*(T)$ denotes the group (isomorphic to $\mathbb{Z}^\ell$) of all one parameter subgroups ϕ of T.

Remarks: 1) Equality holds in a) above even for all positive one parameter subgroups ϕ by theorem 3.10. In fact, theorem 3.10 says that the γ-degree of M equals the codimension of its support, and hence is completely independent of the torus action under consideration, provided, that this action has only positive weights on E, which is guaranteed in the present situation for our one parameter subgroup ϕ since it is assumed positive.

2) The reason for stating the proposition in the weaker form above which we have easily proved directly (independent of 3.10), is that it provides an alternative method of proof for theorem 3.10: In fact, the corollary above reduces the proof of the theorem to the special case $T = \mathbb{G}_m$ of a one-dimensional torus. It seems to us that even this case is not fully covered by the existing literature (cf. e.g. [Bo], [AM], [Sr], [Sm], theorem 5.5). Let us give therefore another full proof in 3.18, which is essentially based on a lemma about real power series (3.17), and which might have some independent interest.

3.17. A lemma on the growth of coefficients of a power series.

Let $(H_j)_{j\in \mathbb{Z}}$ be a sequence of non-negative real numbers, with a minimal positive term H_{j_o} (that is $H_j = 0$ for $j < j_o$) . Let us define its order of growth as the infimum of all real numbers $\gamma \geq 0$ such that $\sum_{j<n} H_j = nO(n^\gamma)$ as n tends to infinity. (For example, the order of growth of a polynomial function is its degree). Now we assume that the Laurent power series

$$H(t) = \sum_j H_j t^j \qquad (1)$$

is a rational function of the form

$$H(t) = P(t)/\prod_{i=1}^{r} (1-t^{m_i}) , \qquad (2)$$

where $P(t)$ is a Laurent polynomial, and $r, m_1, \ldots, m_r$ are positive integers.

Lemma: The order of growth of (H_j) is $r-d-1$ where d is the order of vanishing of $P(t)$ at $t = 1$.

Proof : We first note that with $H(t)$ also $(1+t+t^2+\ldots+t^{s-1})H(t)$ has non-negative coefficients and satisfies all assumptions of the lemma (for all $s\geq 0$); moreover, since all its coefficients are the sums of s successive coefficients $H_{j-s+1} + \ldots + H_j$ of $H(t)$, it is obvious that the order of growth of the coefficients is not changed by multiplication with such a polynomial $(1+t+t^2+\ldots+t^{s-1})$. By applying this observation r times, we conclude that the power series

$$U(t) = H(t) \prod_{i=1}^{r} (1+t+t^2+ \ldots t^{m_i-1})$$

has also non-negative coefficients of the the same growth as $H(t)$. On the other hand, equation (2) gives now

$$U(t) = P(t) / (1-t)^r.$$

Consequently, we have reduced the proof of the lemma to the special case where all degrees are one, $m_1 = m_2 = \ldots = m_r = 1$. In this case, the proof of the lemma is easy by use of the binomial series

$$(1-t)^{-r} = \sum_{i=0}^{\infty} \binom{-r}{i} (-t)^i = \sum_{i=0}^{\infty} \binom{r+i-1}{r-1} t_i,$$

see [Bo]. Q.e.d.

3.18. An alternative proof of theorem 3.10.

As mentioned before, in view of Corollary 3.16, it suffices to prove the theorem for the special case $T = \mathbb{C}_m$, acting on E with positive weights $m_1,\ldots,m_r$. Let $\underline{\underline{F}}$ and $M = \Gamma(E,\underline{\underline{F}})$ as in 3.10, and consider the formal character or Poincaré series of M, denoted

$$ch(M) = \sum_j H_j t^j = H(t) \quad ,$$

$(H_j = \dim M_{-j})$. We know from 3.7, 3.15 (independently of theorem 3.10) that this series is a rational function of the form considered above, in 3.17 (2). Hence by lemma 3.17, we know the order of growth of (H_j) to be $r-d-1$, where $r = \dim E$, and $d = \gamma\text{-}\deg[M]$ (use lemma 3.15). Now the proof is completed by the following

Lemma : The order of growth of (H_j) is equal to $\dim \operatorname{supp} \underline{\underline{F}}-1$.

This lemma is easily proven via an analogue, for orders of growth, of lemma 3.11.

Q.e.d.

3.19. Character polynomials of subalgebras.

The following generalization of the notion of character polynomials will be useful in chapter 4. Let $Y \subset E$ be an irreducible closed T-equivariant subvariety. We have defined the character polynomial $q_Y (= q_B = q_B^T)$, for $B = \mathcal{O}(Y)$, as the lowest degree term of $\Delta(E^*)ch(B)$, considered as a power series on $\underline{t}$. Similarly, we define for any T-stable subalgebra $A \subset B$ the character polynomial q_A $(= q_A^{T,B})$ to be the lowest degree term of $\Delta(E^*)ch(A)$. This is a homogeneous polynomial of degree $\deg q_A \geq \deg q_B$. This notation is extended also to finitely generated T-equivariant A-modules. - Now let A be finitely generated, say $A = \mathcal{O}(Z)$ the coordinate ring of the affine T-variety Z. Then we also write $q_Z = q_A$. Let $\varphi : Y \to Z$ denote the T-equivariant dominant map corresponding to the inclusion $A \subset B$.

Lemma : If $\varphi : Y \to Z$ is birational, then $q_Z = q_Y$.

Proof : There exists a dense, open, T-stable subset $Z' \subset Z$, such that the restriction of φ to $Y' = \varphi^{-1}Z'$ is an isomorphism $Y' \xrightarrow{\sim} Z'$. Let $\partial Z' = Z-Z'$, $\partial Y' = Y-Y'$ be their complements, and let I resp. J be the ideal in A resp. B of functions vanishing on $\partial Z'$ resp. $\partial Y'$.

Obviously, all these sets and ideals are T-stable, so that we may argue in terms of formal characters. Since $A_I = B_I$, the difference between A and B can be calculated over I :

$$\mathrm{ch}(B)-\mathrm{ch}(A) = \mathrm{ch}(B/A) = \mathrm{ch}((B/IB)/(A/A \cap IB))$$

$$= \mathrm{ch}(B/IB)-\mathrm{ch}(A/A \cap IB)$$

We may assume $0 \neq I$ (the other case being easily settled separately). Then by lemma 3.11, the lowest degree term of $\Delta(E^*)\mathrm{ch}(B/IB)$ has degree strictly bigger than $\Delta(E^*)\mathrm{ch}(B)$. Since $\mathrm{ch}(A/A \cap IB) \leq \mathrm{ch}(B/IB)$ coefficient wise, we conclude that also $\Delta(E^*)\mathrm{ch}(A/A \cap IB)$ cannot contribute to terms in degree $\leq \deg p_B$. (Here we use the characterization of γ-degrees by order of growth, 3.18). It follows then on the left hand side of the equation

$$\Delta(E^*)\mathrm{ch}(B)-\Delta(E^*)\mathrm{ch}(A) = \Delta(E^*)\mathrm{ch}(B/IB)-\Delta(E^*)\mathrm{ch}(A/A \cap IB)$$

the lowest degree terms must cancel each other, that is to say $q_B = q_A$.

Q.e.d.

<u>Proposition</u> : <u>If</u> $\varphi : Y \to Z$ <u>is dominant, and</u> $\dim Y = \dim Z$, <u>then</u> $q_Y = mq_Z$ <u>for some integer</u> $m > 0$.

More precisely, m is the generic degree of the map φ, or also $m = [L:K]$, where K,L denote the fields of rational functions on Z resp. Y.

<u>Proof</u> : The equality of dimensions implies that K and L have the same transcendence degree, so that $m = [L:K]$ is <u>finite</u>. Let $\tilde{A}$ denote the integral closure of $A = \mathcal{O}(Z)$ in $B = \mathcal{O}(Y)$. Then we have $q_B = q_{\tilde{A}}$ by the lemma.

We may chose m weight vectors $v_1,\dots,v_m \in \widetilde{A}$, which are independent over $K = \mathrm{Fract}\, A$, so that $M := Av_1+\dots+Av_m$ is a free A-submodule of rank m in $\widetilde{A}$. Then we have

$$\mathrm{ch}(\widetilde{A}) = \mathrm{ch}(A)(e^{\mu_1}+\dots+e^{\mu_m}) + \mathrm{ch}(\widetilde{A}/M) \quad , \tag{*}$$

where μ_i is the weight of v_i. Now $\widetilde{A}/M$ is a finitely generated (!) torsion module over A. Therefore, it follows by an obvious generalization of lemma 3.11, that $q_{\widetilde{A}/M}$ has strictly larger degree than $q_{\widetilde{A}}$. Hence it follows from (*), that up to terms of higher degree :

$$\Delta(E^*)\mathrm{ch}(A) \equiv \Delta(E^*)\mathrm{ch}(A)(e^{\mu_1}+\dots+e^{\mu_m}) \equiv m\Delta(E^*)\mathrm{ch}(A) \quad (\text{mod higher degree terms}) .$$

It follows that $q_{\widetilde{A}} = mq_A$, which completes the proof. Q.e.d.

§4. Equivariant characteristic classes of orbital cone bundles.

In this chapter, G is a connected semisimple algebraic group over k, and T a maximal torus in G. We use the notations introduced in 1.5. So in particular, U is a maximal unipotent subgroup normalized by T, and $B = TU = UT$ is a fixed Borel subgroup; $\underline{g},\underline{b},\underline{t},\underline{u}$ are the Lie algebras of G,B,T,U etc.

4.1. Borel picture of the cohomology of a flag variety [Bo], [H1].

The purpose of the theorem below is to recall, and simultaneously to rephrase in an equivariant K-theory language convenient for our present paper, the Borel picture of the cohomology ring $H^*(X)$ of the flag variety $X = G/B$. This picture describes $H^*(X)$ as a quotient of the representation ring $R(T)$ by a certain ideal defined as follows. Note that the Weyl group W of G relative to T acts on $R(T)$, stabilizing the augmentation ideal I_T (notation 2.11); so let $I_T^W = I_T \cap R(T)^W$ denote the subspace of W-invariants. Then the ideal mentioned above is the one generated by I_T^W.

Theorem (Borel picture) : We have a commutative diagram of canonical ring homomorphisms

$$\begin{tikzcd}
K_G(X) \arrow[r, "\cong"] \arrow[d] & R(B) \arrow[r, "\cong"] \arrow[d] & R(T) \arrow[dl] \\
K(X) \arrow[r, "\cong"] \arrow[d, "ch"', "\cong"] & R(T)/I_T^W R(T) \arrow[dl, "\cong"] & \\
H^*(X) & &
\end{tikzcd}$$

Here the map from $K_G(X)$ onto $K(X)$ forgets the G action (2.1), and the isomorphism of $K_G(X)$ onto $R(B)$ is given by restriction to the base point (2.4(1)). The link from $K(X)$ to $H^*(X)$ is of course made by the Chern character ch (an isomorphism, cf. 1.5), and that from $R(B)$ to $R(T)$ is given by restriction from B to T (2.1). Observing that U acts trivially on each finite dimensional irreducible (hence one-dimensional) B-module, we conclude that this restriction homomorphism $R(B) \to R(T)$ is also an isomorphism. - For further information and more classical formulations of the Borel picture, we refer to e.g. Borel [Bo], or Hiller [Hl] III, theorem 4.1.

4.2. Description in terms of harmonic polynomials on a Cartan subalgebra.

The isomorphism φ of the completed representation ring $\hat{R}(T)$ with the formal power series ring $\hat{S}(\underline{t}^*)$ (cf. 3.1) is canonical, hence W-equivariant; it maps the ideal generated by I_T^W onto the ideal generated by

$$J_T^W := S(\underline{t}^*)^W \cap \underline{t}^* S(\underline{t}^*) \quad ,$$

that is the W-invariant polynomial functions on $\underline{t}$ vanishing at 0 . Therefore φ induces a diagram of isomorphisms

$$\begin{array}{ccccc} \hat{R}(T) & \longrightarrow & \hat{R}(T)/\hat{R}(T)I_T^W & \xleftarrow{\cong} & R(T)/R(T)I_T^W \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \hat{S}(\underline{t}^*) & \longrightarrow & \hat{S}(\underline{t}^*)/\hat{S}(\underline{t}^*)J_T^W & \xleftarrow{\cong} & S(\underline{t}^*)/S(\underline{t}^*)J_T^W \end{array}$$

Furthermore, we recall that the residue class map of $S(\underline{t}^*)/S(\underline{t}^*)J_T^W$ admits a natural section, given by the graded subspace $S(\underline{t}^*)^{\mathcal{H}} \subset S(\underline{t}^*)$ of all W <u>harmonic polynomials</u>. In conclusion, Borel's description of $H^*(X)$ as formulated in 4.1 also says that

$$H^*(X) \cong R(T)/R(T)I_T^W \cong S(\underline{t}^*)/S(\underline{t}^*)J_T^W \cong S(\underline{t}^*)^{\mathcal{H}}$$

canonically. These isomorphisms clearly respect gradings and W-actions. Thus we reobtain the following formulation of the Borel picture, which is perhaps more familiar to some readers :

Corollary (loc. cit.) : *For all degrees* $d \geq 0$, *we have canonically*

$$H^{2d}(X) \cong S^d(\underline{t}^*)^{h} \quad .$$

Here the right-hand side denotes the space of harmonic polynomials which are homogeneous of degree d . If we add that the cohomology of X vanishes in all odd degrees, then the above statement completes the *description of* $H^*(X)$ *as a graded ring with* W *action*.

Remark : Let us recall our convention, already made in 1.6, that whenever convenient, we identify a cohomology class on X with a harmonic polynomial on $\underline{t}$, by means of the above isomorphism.

4.3. Equivariant K-theory on T^*X .

We shall now return to the study of certain conical subvarieties in - resp. sheaves on - the cotangent bundle T^*X of the flag variety X , which has been our main object in §1 already. We now observe that T^*X is G-equivariant as a vector-bundle over X , and that the subvarieties and sheaves in question are also G-equivariant. This will allow us to study them by means of calculations in the ring $K_G(T^*X)$. This study will amount in a certain sense to "lifting" the geometrical investigations of our §1 to the more refined level of *equivariant* K-theory in the present chapter, and will finally lead to improvements of the results in §1.

The purpose of the following proposition is to provide a fairly explicit description of the ring $K_G(T^*X)$, as well as a method for performing actual calculations. We recall that the cotangent space of X at the base point $x = \{B\}$ is identified with the "co-isotropy space" at x, that is to say $(T^*X)_x = \underline{b}^{\perp} = \underline{u}$, and then T^*X is identified with the associated fibre-bundle $T^*X = G \times^B \underline{u}$ (notations 1.5). Consider the following diagram of inclusion maps :

$$\begin{tikzcd} T^*X & \underline{u} \arrow[l, "j"'] \\ X \arrow[u, hook, "\sigma"] & \{x\} \arrow[l, "i"] \arrow[u, hook, "\iota"'] \end{tikzcd}$$

Here σ resp. j denote the inclusion of X resp. $\underline{u}$ into T^*X as the zero-section resp. the fibre at x.

<u>Proposition</u> : <u>We have a commutative diagram of canonical isomorphisms of</u> λ<u>-rings</u> :

$$\begin{tikzcd} K_G(T^*X) \arrow[r, "j_B^*"] \arrow[d, "\sigma_G^*"] & K_B(\underline{u}) \arrow[r] \arrow[d, "\iota_B^*"] & K_T(\underline{u}) \arrow[d, "\iota_T^*"] \\ K_G(X) \arrow[r, "i_B^*"] & R(B) \arrow[r, "\rho"] & R(T) \end{tikzcd}$$

<u>Here all maps are given by restriction to a smaller group and/or to a subvariety</u>.

<u>Proof</u> : All these restriction homomorphisms are clearly λ-ring homomorphisms, so it suffices to check bijectivity. For the two horizontal arrows j_B^*, i_B^*, this is a consequence of the G-homogeneity of X (cf. 2.4(1)). Also, the horizontal arrow ρ is an isomorphism, as observed already in 4.1, and the vertical arrow ι_T^* is an isomorphism by proposition 3.7. Identifying $R(B)$

with $R(T)$, we may repeat the proof of proposition 3.7 almost word by word to construct an inverse map of ι_B^* from $R(T) = R(B)$ to $K_B(\underline{u})$. Now the bijectivity of the remaining two arrows follows by commutativity of the diagram.

Q.e.d.

Remark. Alternatively, one may invoke Thomason's general result [T1], Theorem 4.1 (cf. 2.9(2))to conclude directly that all three vertical arrows $\iota_T^*, \iota_B^*, \iota_G^*$ are isomorphisms. However, for the convenience of the reader, we are avoiding this here.

4.4. Restriction to a fibre of T^*X .

Let us now explain in more detail the use of the proposition above for computational purposes. Let us compute for instance the class in $K_G(T^*X)$ determined by some G-equivariant coherent sheaf $\underline{\underline{F}}$ on T^*X . To identify this class $[\underline{\underline{F}}]$ as an element in $R(T)$ by means of the proposition, we proceed as follows : We restrict $\underline{\underline{F}}$ to a sheaf $j^*\underline{\underline{F}}$ on the fibre $\underline{u}$ (which is affine!) , then take its module of global sections $M = \Gamma(\underline{u}, j^*\underline{\underline{F}})$ and the formal character $ch(M)$ in $\hat{R}(T)$ (3.2). Then the desired element in $R(T)$ is given by the explicit formula for the map ι_T^* given in 3.7, that is to say :

Corollary 1 :

$$\iota_T^* \, j_B^*[\underline{\underline{F}}] = \Delta(\underline{u}^*) \text{ ch } \Gamma(\underline{u}, j^*\underline{\underline{F}}) \; .$$

Here $\Delta(\underline{u}^*)$ is the product $\Pi(1-e^{-\alpha})$, expanded over all positive roots α (relative $\underline{b}$ and $\underline{t}$).

Corollary 2 : Let $\underline{\underline{F}}$ denote a G-equivariant coherent sheaf on T^*X Then γ-degree of its class $[\underline{\underline{F}}]$ in $K_G(T^*X)$ equals the codimension of the

<u>support of $\underline{\underline{F}}$ in</u> T^*X .

<u>Proof</u> : This follows from theorem 3.10 by means of proposition 4.3. Q.e.d.

4.5. <u>Definition of equivariant characteristic classes.</u>

Let $\underline{\underline{F}}$ be a G-<u>equivariant</u> coherent sheaf on T^*X , whose support has codimension d in T^*X . Then its <u>equivariant characteristic class</u> $Q_G(\underline{\underline{F}})$ is a homogeneous polynomial of degree d on $\underline{t}$, which is defined as follows. First of all, $\underline{\underline{F}}$ determines a class $[\underline{\underline{F}}]$ in $K_G(T^*X)$. By "homological intersection with the zero section", it determines a class $\sigma_G^*[\underline{\underline{F}}]$ in $K_G(X)$, which we may consider as a power series on $\underline{t}$, by means of the identifications

$$K_G(X) \cong R(T) \subset \hat{R}(T) \cong \hat{S}(\underline{t}^*)$$

explained in 4.3 resp. 3.1. Then we <u>define</u> $Q_G(\underline{\underline{F}})$ as the lowest degree term of this power series :

$$Q_G(\underline{\underline{F}}) := \mathrm{gr}\ i_B^*\sigma_G^*[\underline{\underline{F}}] \quad .$$

<u>Proposition</u> : <u>The polynomial</u> $Q_G(\underline{\underline{F}})$ <u>may be computed from a formal character by means of the formula</u>

$$Q_G(\underline{\underline{F}}) = \mathrm{gr}\ \Delta(\underline{u}^*)\mathrm{ch}\ \Gamma(\underline{u}, j^*\underline{\underline{F}}) \quad .$$

<u>In particular, it is homogeneous of degree</u> d .

<u>Proof</u> : The first statement follows from corollary 4.4.1 - in view of the commutativity of the diagram in proposition 4.3. The second statement follows from corollary 4.4.2, so is a consequence of theorem 3.10. Q.e.d.

Remark. Using the terminology and notations of character polynomials as defined in 3.9, we may also state the proposition this way : As a polynomial on $\underline{t}$, the equivariant characteristic class of $\underline{\underline{F}}$ coincides with the character polynomial of its restriction to a fibre of T^*X , that is to say

$$Q_G(\underline{\underline{F}}) = q^T_{j^*\underline{\underline{F}}} \quad .$$

4.6. Comparison to the characteristic classes defined in §1.

Retaining the notations of 4.5, let us recall that we already defined a characteristic class $Q(\underline{\underline{F}})$ in chapter 1, as an element in $H^{2d}(X) \cong S^d(\underline{t}^*)^{\natural}$ (see 1.3 and 4.2). Let us denote $p \mapsto p^{\natural}$ the projection of a polynomial p on $\underline{t}$ onto its uniquely determined harmonic part $p^{\natural}$.

Proposition : The characteristic class $Q(\underline{\underline{F}})$ may be computed from the equivariant one $Q_G(\underline{\underline{F}})$ by taking the harmonic part,

$$Q(\underline{\underline{F}}) = Q_G(\underline{\underline{F}})^{\natural} \quad .$$

Proof : From the "Borel picture" (theorem 4.1, 4.2) we obtain the following commutative diagram by passing to the associated graded level (terminology and notations as in 1.3, 1.4) :

$$\begin{array}{ccccc}
gr_d K_G(X) & \xrightarrow{\sim} & gr_d R(T) & \xrightarrow{\sim} & S^d(\underline{t}^*) \\
\downarrow & & \downarrow & & \downarrow \natural \\
gr_d K(X) & \xrightarrow{\sim} & gr_d R(T)/R(T)I_T^W & \xrightarrow{\sim} & S^d(\underline{T}^*)^{\natural}
\end{array}$$

Here the left vertical arrow is given by forgetting the G-action, while the right one is taking harmonic parts. So the diagram says that forgetting the

G-action in graded K-groups corresponds to taking harmonic parts of the corresponding polynomials on $\underline{t}$. Now note that $Q(\underline{\underline{F}})$ was <u>defined</u> as an element in

$$gr_d K(X) \cong H^{2d}(X) \cong S^d(\underline{t}^*)^W ,$$

and $Q_G(\underline{\underline{F}})$ is an element in

$$gr\ K_G(X) \cong gr\ R(T) \cong S(\underline{t}^*)$$

by definition. The only delicate point is that both $Q_G(\underline{\underline{F}})$ and $Q(\underline{\underline{F}})$ should occur in the same degree, that is in degree d. But this was made sure by theorem 3.10 (cf. 4.4.2). Now it is clear that $Q(\underline{\underline{F}})$ is obtained from $Q_G(\underline{\underline{F}})$ by forgetting the G-action, and so the proposition follows from the above commutative diagram. Q.e.d.

<u>Complementary remarks</u>.

1) If $K \subset T^*X$ is a G-stable closed subvariety, then we define its equivariant characteristic class by

$$Q_G(K) := Q_G(\mathcal{O}_K) .$$

Since we may define $Q(K)$ analogously as $Q(\mathcal{O}_K)$ (see 1.4), we see that also

$$Q(K) = Q_G(K)^W .$$

2) Let us recall that it was a delicate question in §1, whether or not the characteristic class $Q(V)$ of a given subvariety $V \subset T^*X$ vanishes. So let us point out here for the case of a G-stable subvariety $V \neq \emptyset$ that $Q_G(V)$ is always nonzero by definition, and that the delicate question is, whether or

not the corresponding polynomial has <u>nonzero harmonic part</u>. In our case of main interest, that is for V an orbital cone bundle, it actually turns out that even the equivariant class itself is already harmonic, so that $Q(V) = Q_G(V)$ in this case (so in particular $Q(V) \neq 0$) .

3) Let us point out that the <u>equivariant characteristic class</u> $Q_G(\underline{\underline{F}})$ may be described as a class in the <u>equivariant cohomology group</u> $H_G^{2d}(X)$. We use the the standard definition of equivariant cohomology, that is $H_G^*(X) = H^*(X \times^G EG)$ (see [Bo2], chapter 4). It is also easy to define a purely algebraic equivariant de Rham cohomology, by viewing $X \times^G EG$ as a simplicial algebraic variety over k , and taking the hypercohomology of the de Rham complex (which is a simplicial complex of sheaves) (see [D1] or [Fr] for general notions about simplicial varieties and simplicial sheaves). In our case, where $X = G/B$, we have $H_G^{2d}(X) \xrightarrow{\sim} S^d(\underline{t}^*)$ (and $H_G^k(X) = 0$ for k odd). Also, the restriction to the zero-section induces an isomorphism $H_G^*(T^*X) \xrightarrow{\sim} H_G^*(X)$. There is a general equivariant Chern character $ch_G = ch : K_G(-) \to H_G^{even}(-)$ where $H_G^{even}(-)$ is the <u>product</u> of all $H_G^{2i}(-)$. For the flag variety X , we have a commutative diagram.

$$\begin{tikzcd}
K_G(X) \arrow[r] \arrow[d, "\wr"] & H_G^{even}(X) \arrow[d, "\wr"] \\
R(T) \arrow[r] & \hat{S}(\underline{t}^*)
\end{tikzcd}$$

where the lower line is defined in 3.1. It follows that ch maps $K_G(X)_d$ to $\prod_{i \geq d} H_G^{2i}(X)$. So we may give an alternative geometric definition of $Q_G(\underline{\underline{F}})$ as follows. Take $\underline{\underline{F}}$ as in 4.5, then $Q_G(\underline{\underline{F}})$ is the component of $ch([\underline{\underline{F}}])$ in $H_G^{2d}(X)$. Remembering the isomorphism $H_G^{2d}(X) \xrightarrow{\sim} S^d(\underline{t}^*)$ we get the same class as in 4.5. The commutative diagram used in the above proof of Proposition 4.6 may be completed as follows

$$\begin{array}{ccccc}
gr_d K_G(X) & \xrightarrow{\sim} & H_G^{2d}(X) & \xrightarrow{\sim} & S^d(\underline{t}^*) \\
\downarrow & & \downarrow & & \downarrow \\
gr_d K(X) & \xrightarrow{\sim} & H^{2d}(X) & \xrightarrow{\sim} & S^d(\underline{t}^*)
\end{array}$$

We will not try to give a purely homological definition of $Q_G(V)$, in the style of 1.3, because of the lack of references concerning equivariant Borel-Moore homology.

4) The cotangent bundle T_X^* may be viewed as a G-equivariant vector bundle on X. Since the equivariant Chern classes $c_i(T_X^*) \in H_G^{2i}(X)$ are defined, we may, by the usual procedure [Hi], §10.1, introduce the Todd class $Td(T_X^*)$. To compute it, we freely identify $H_G^{even}(X)$ with $\hat{S}(\underline{t}^*)$, as in the previous remark. We have $\sum_{i=1}^{n} c_i(T_X^*)x^i = \prod_{\alpha>o} (1+\alpha x)$, hence $Td(T_X^*) = \prod_{\alpha>o} \left(\frac{\alpha}{1-e^{-\alpha}}\right) = \frac{c_n(T_X^*)}{\Delta(\underline{u}^*)}$ (this is an equality in the fraction field of $\hat{S}(\underline{t}^*)$).

Now let $\underline{\underline{F}}$ be a G-equivariant coherent sheaf on T^*X, let $M = \Gamma(\underline{u}, j^*\underline{\underline{F}})$. Then the formula for $ch(M)$ in 4.4 may be rewritten as :

$$ch(M) = \frac{Td(T_X^*).Q_G(\underline{\underline{F}})}{c_n(T_X^*)}$$

4.7. Equivariant characteristic classes of orbital cone bundles.

We are now ready to state some of the key results of the present chapter.

Theorem 1 : Let K be an orbital cone bundle in T^*X (terminology 1.7). Then its equivariant characteristic class $Q_G(K)$ is a harmonic polynomial.

In other words, we have $Q_G(K) = Q(K)$ in this case.

Theorem 2 : Let $K_1,\dots,K_r$ be the orbital cone bundles for some nilpotent orbit $\mathcal{O}_u$ (notations as in theorem 1.8). Then the classes $Q_G(K_1),\dots,Q_G(K_r)$ span a Weyl group submodule.

Theorem 3 : The corresponding representation of W is isomorphic to Springer's irreducible representation ρ_u (notation as in theorem 1.8).

We note that theorems 2 and 3 above would be immediate consequences of theorem 1.8, if we would assume theorem 1. However, our proof proceeds in a different logical order : Logically, the proof of theorem 2 comes first; it will be completed in 4.13. But let us show here first how to derive theorems 1 and 3 from theorem 2.

Theorem 2 implies theorem 3: Let M resp. N denote the linear span of $Q_G(K_1),\dots,Q_G(K_r)$ resp. of $Q(K_1),\dots,Q(K_r)$. By proposition 4.6, M is mapped onto N by the projection onto harmonic parts, which is a W module homomorphism. Its restriction to M must be a linear isomorphism by 1.8a). So assuming that M is a W submodule, we conclude that it carries the same W representation as N , which is ρ_u by theorem 1.8c). Q.e.d.

Theorems 2 and 3 imply theorem 1 : Suppose not all $Q_G(K_i)$ were harmonic. With notations as before, this means $M \neq N$, and so M,N are two different copies of the irreducible representation ρ_u in $S^{d_u}(\underline{t})$ (notation 1.8). However, it is known that Springer's representation ρ_u occurs only once in (the lowest possible) degree d_u , see [BM1], Corollaire 4. Hence we must have $M = N$. Q.e.d.

The proof of theorem 2 will consist in verifying directly the following much more precise statement :

Theorem 2' : Even the equivariant characteristic classes $Q_G(K_1),\dots,Q_G(K_r)$ satisfy the Hotta transformation formulas (cf. 1.15).

This is our version of a result essentially due to Joseph [J1], 3.1, and reinterpreted by Hotta [Ho], 3.4. In addition to proving theorem 2, this stronger theorem 2' will simultaneously prove theorem 1.15 (see 4.14 below), and hence reprove also the main results of Hotta's work [Ho], [Ho2]. The proof will be given in sections 4.8 to 4.13. The crucial final part of the proof (cf. 4.13) is essentially Joseph's so that we could refer partially to [J1]. However, for convenience of the reader, we prefer to give here an essentially self-contained, full proof.

4.8. Comparison with Joseph's notion of "characteristic polynomial"

The purpose of this section is to link up terminology and notations of Joseph [J1] with our present language. The main point to be made is that our notion of "equivariant characteristic classes" essentially coincides with Joseph's notion of "characteristic polynomial", providing a more conceptual geometric reinterpretation for it. Let us give a few more explanations, which should help the reader to verify the coincidence.

We shall identify here polynomials on $\underline{t}^*$ with polynomials on $\underline{t}$ by means of the Killing form, that is by the isomorphism $\underline{t} \cong \underline{t}^*$, $t \mapsto t^*$, characterized by $\mu(t) = (\mu, t^*)$. Let $C \subset \underline{u}$ be a B-stable closed cone, and let $K = G \times^B C \subset T^*X$ denote the corresponding G-stable closed cone-bundle.

Proposition : Considered as a polynomial on $\underline{t} = \underline{t}^*$, our equivariant characteristic class $Q_G(K)$ (= q_C^T by 4.5) coincides with Joseph's "characteristic polynomial" p_C, as defined in [J1], 2.4.

1) Let us first recall Joseph's definition (loc.cit). Let M denote the T-equivariant $S(\underline{u}^*)$-module $M = \mathcal{O}(C)$. Let $\nu \in \underline{t}^*$ denote an integral weight, which is assumed dominant and regular, that is $(\alpha,\nu) > 0$ for all positive roots α, which will eventually be considered a variable. Now Joseph considers the "Poincaré series"

$$R_M(t,\nu) := \sum_{\mu\in\Lambda} (\dim M_\mu)t^{(\mu,\nu)}$$

(notation as in [J1]), which is a formal power series in one variable t, and which depends on ν as a parameter. Next he studies the leading coefficient of the Hilbert-Samuel polynomial of $R_M(t,\nu)$, considered as a function of ν, denoted $r_M(\nu)/d(M)!$, and he finds that the function

$$p_M(\nu) := r_M(\nu)\cdot \prod_{\alpha>o} \alpha(\nu) \qquad (1)$$

is given by a polynomial p_M on $\underline{t}^*$, homogeneous of degree $d := \operatorname{codim}_{\underline{u}} C = n-d(M)$ This polynomial is denoted $p_C := p_M$, and is called the "characteristic polynomial of C" in loc.cit..

2) Next let us rewrite Joseph's procedure in our present language. We consider the restrictions of the T-action on our T-equivariant module $M = \mathcal{O}(C) = \Gamma(C,\mathcal{O}_K)$ to the various one parameter subgroups $\psi : \mathbb{G}_m \to T$. On a weight-vector $v \in M$ of weight μ, such a subgroup acts by

$$\psi(t)v = t^{(\mu,\nu)}v \qquad (t \in k^*) ,$$

where ν is the integral weight corresponding to ψ (that is $\nu = (d\psi)(1)^*$, cf. 3.16). This shows that in our notation,

$$R_M(t,\nu) = \psi^*(\mathrm{ch}(M)) \quad ,$$

that is Joseph's one variable Poincaré series is just the restriction of the formal character of M to a one parameter subgroup. As discussed in detail in 3.16, this one variable series is obtained from the multi-variable power series $ch(M) = \sum_\mu (\dim M_\mu)e^\mu$ by "specializing" the variables $t_i = e^{\omega_i}$ $(i = 1,\ldots,\ell)$ all to a single variable t via $t_i \mapsto t^{(\omega_i,\nu)}$. This specialization may also be interpreted as an "evaluation" of $ch(M)$, considered as a function on $\underline{t} = \underline{t}^*$: In fact, evaluation on $\tau\nu$, for a variable scalar $\tau \in k$ gives

$$t_i(\tau\nu) = e^{\omega_i}(\tau\nu) = \sum_m \frac{1}{m!}\,\omega_i^m(\tau\nu) = \sum_m \frac{1}{m!}(\omega_i,\tau\nu)^m$$
$$= e^{(\omega_i,\tau\nu)} = e^{\tau(\omega_i,\nu)} = t^{(\omega_i,\nu)} ,$$

putting $e^\tau = t$. So we consider

$$\psi^*(ch(M)) = ch(M)(\tau\nu) = R_M(t,\nu) \tag{3}$$

as the (formal) evaluation of $ch(M)$ on the Lie algebra of our one parameter subgroup. Similarly, we get

$$\psi^*(\Delta(\underline{u}^*)ch(M)) = \prod_{\alpha>o}(1-t^{(\alpha,\nu)})R_M(t,\nu) \tag{4}$$

which we denote by $P_M(t,\nu)$. We know from §3, that this is a Laurent polynomial $P_M(t,\nu) \in k[t,t^{-1}]$; in fact, it is the class determined by the restriction of M (with respect to ψ) in the equivariant K-group

$$K_{\psi(\mathbb{G}_m)}(\underline{u}) \cong R(\mathbb{G}_m) \cong k[t,t^{-1}] .$$

Now let us assume again that ν is regular (all $(\alpha,\nu) > 0$, for all $\alpha > 0$). Then we know that the order of vanishing of $P_M(t,\nu)$ at $t = 1$ is $d = \operatorname{codim}_{\underline{u}} C$ (cf. 3.18), so let us write

$$P_M(t,\nu) = (t-1)^d \tilde{P}_M(t,\nu) \quad ,$$

where now $\tilde{P}_M(t,\nu)$ is a Laurent polynomial such that $\tilde{P}_M(1,\nu) \neq 0$. From the resulting expression

$$R_M(t,\nu) = \tilde{P}_M(t,\nu) \frac{(t-1)^d}{\prod_{\alpha>o} (1-t^{(\alpha,\nu)})} \quad , \quad (\tilde{P}_M(1,\nu) \neq 0) \qquad (5)$$

for Joseph's Poincaré series, one may now deduce some information about its Hilbert-Samuel function (by the methods used in [AC] VIII, §4, cf. also [Sm], or [J1], or our 3.17) : It has degree $n-d = \dim C$ (cf. 3.18), and leading coefficient $\tilde{P}_M(1,\nu)/(n-d)!$. Comparing this with Joseph's notation (see 1), we get

$$\tilde{P}_M(1,\nu) = p_M(\nu) = p_C(\nu) \ .$$

Combining this with (5) and (4), we obtain that

$$\psi^*(\Delta(\underline{u}^*)ch(M)) = p_C(\nu)(t-1)^d + \cdots = p_C(\nu).\tau^d + \cdots$$

up to higher terms in $t-1$ resp. in τ , so the homogeneous term of degree d must map to

$$\psi^*([\Delta(\underline{u}^*)ch(M)]^d) = p_C(\nu)\tau^d \quad . \qquad (6)$$

On the other hand, ψ^* means evaluation at $\tau\nu$ by (3) above, and $[\Delta(\underline{u}^*)ch(M)]^d$ is our "character polynomial" q_M^T by corollary 3.9 (which has degree d by theorem 3.10). Hence equation (6) reads as follows : $q_M^T(\tau\nu) = p_C(\nu)\tau^d$. But $q_M^T(\tau\nu) = q_M^T(\nu)\tau^d$ by homogeneity, and so we conclude that our polynomial q_M^T coincides with Joseph's p_C . - Finally, we have also $q_M^T = Q_G(K)$, as was already explained in 4.5.

4.9. Generalization to the case of sheaves.

Let us extend the considerations of the previous section to the case of an arbitrary G-equivariant conical coherent sheaf $\underline{\underline{E}} \neq 0$ on T^*X. Let $\underline{\underline{F}} = j^*\underline{\underline{E}}$ denote the corresponding B-equivariant coherent sheaf on the fibre $\underline{u}^{\cdot}$ of T^*X, obtained by restriction. As before, we let M denote the T-equivariant $S(\underline{u}^*)$-module $M = \Gamma(\underline{u},\underline{\underline{E}}) = \Gamma(\underline{u},\underline{\underline{F}})$. Generalizing proposition 4.8, we may now state :

Proposition 1 : Considered as polynomials on $\underline{t} = \underline{t}^*$, our equivariant characteristic class $Q_G(\underline{\underline{E}})$ $(= q^T_{\underline{\underline{F}}})$, and Joseph's "characteristic polynomial" $p_{S(M)}$, as defined in [J1], 5.5, coincide.

Let us denote the supporting cycles for $\underline{\underline{E}}$ resp. $\underline{\underline{F}}$ (cf. 1.4) by

$$\underline{\text{supp}}(\underline{\underline{E}}) = \sum_i m_i(\underline{\underline{E}})[\mathcal{U}_i] \qquad \text{resp.}$$

$$\underline{\text{supp}}(\underline{\underline{F}}) = \sum_i m_i(\underline{\underline{F}})[\mathcal{V}_i] \quad .$$

Then the irreducible components $\mathcal{U}_i$ resp. $\mathcal{V}_i$ are clearly G-resp. B-stable closed cone bundles resp. cones, and by chosing the correct numbering, each $\mathcal{U}_i$ will be a homogeneous cone bundle with fibre $\mathcal{V}_i$, that is $\mathcal{U}_i = G \times^B \mathcal{V}_i$ for all i . Moreover, the multiplicities and codimensions will coincide :

$$m_i(\underline{\underline{E}}) = m_i(\underline{\underline{F}}) \quad \text{for all } i ,$$

and
$$\operatorname{codim}_{T^*X} \operatorname{supp}(\underline{\underline{E}}) = \operatorname{codim}_{\underline{u}^{\cdot}} \operatorname{supp}(\underline{\underline{F}}) =: d \quad .$$

Proposition 2 : With notations as above,

a) $Q_G(\underline{\underline{E}}) = \sum_{\operatorname{codim} \mathcal{U}_i = d} m_i(\underline{\underline{E}}) Q_G(\mathcal{U}_i)$

b) $q^T_{\underline{\underline{F}}} = \sum_{\operatorname{codim} \mathcal{U}_i = d} m_i(\underline{\underline{F}}) q^T_{\mathcal{V}_i}$.

Proof of proposition 2 : As explained in 4.5, we have $Q_G(\underline{\underline{E}}) = q^T_{\underline{\underline{F}}}$, and $Q_G(U_i) = q^T_{V_i}$, that is statements a) and b) coincide term by term. In the Grothendieck group $K_T(\underline{u})$, we have up to higher degree terms :

$$[\underline{\underline{F}}] = \sum_i m_i(\underline{\underline{F}})[\mathcal{O}_{V_i}] \quad .$$

By theorem 3.10, we know (since $\underline{\underline{E}}$ and hence $\underline{\underline{F}}$ and M are $\neq 0$) that the γ-degree of $[\underline{\underline{F}}]$ is d , and that of $[V_i]$ is codim V_i , so in $gr_d K_T(\underline{u})$ we get

$$[\underline{\underline{F}}] \equiv \sum_{\text{codim } V_i=d} m_i(\underline{\underline{F}})[\mathcal{O}_{V_i}] \bmod K_T^{d+1}(\underline{u}) \quad .$$

By definition of our character polynomials, this means exactly

$$q^T_{\underline{\underline{F}}} = \sum_{\text{codim } V_i=d} m_i(\underline{\underline{F}}) q^T_{V_i} \qquad \text{Q.e.d.}$$

Corollary : The equivariant characteristic class of $\underline{\underline{E}}$ is completely determined by its (G-equivariant!) characteristic cycle.

So far, we have established an equivariant version of proposition 1.4. Let us now make the link to Joseph's work, i.e. let us prove proposition 1 above. Joseph's definition of a character polynomial p_M , as recalled in 4.8, generalizes in an obvious way to our present module $M = \Gamma(\underline{u},\underline{\underline{E}}) = \Gamma(\underline{u},\underline{\underline{F}})$, and then we get the coincidence $Q_G(\underline{\underline{E}}) = q^T_{\underline{\underline{F}}} = p_M$ by the same proof as before. Note, however, that Joseph in loc.cit. does not explicitly use this obvious extension of his notation. Instead, he introduces the new notation $p_{S(M)}$ (see loc.cit., 5.5). In our terminology, his $S(M)$ is the top-dimensional part of the supporting cycle of M (that is of $\underline{\underline{F}}$) :

$$S(M) = \sum_{\text{codim } V_i=d} m_i(\underline{\underline{F}})[V_i] \quad .$$

Then he defines

$$p_{S(M)} := \sum_{\operatorname{codim} V_i = d} m_i(\underline{\underline{F}}) p_{V_i} \quad .$$

But now it suffices to observe that $p_{V_i} = q^T_{V_i}$ by 4.8, and so $p_{S(M)} = p_M$ by proposition 2b) above. This completes the proof of proposition 1. Q.e.d.

Remark : From now on, we shall normally no longer care much about notational distinctions between Joseph's characteristic polynomials p and our character polynomials q .

4.10. Equivariance under a Levi subgroup.

Now let G' be a connected reductive closed subgroup of G containing T . We denote $W' \subset W$ its Weyl group, and U' any maximal unipotent subgroup (e.g. $U' = G' \cap U$). Let us now assume that our coherent sheaf $\underline{\underline{F}}$ on $\underline{u}$, viewed as a coherent sheaf on $\underline{g}$, is not only T- , but even G'-equivariant. Then $M = \Gamma(\underline{u}, \underline{\underline{F}})$ is a direct sum of finite-dimensional simple G'-modules, and so its formal character is of the form

$$\operatorname{ch}(M) = \sum_{\lambda \in \Lambda} \operatorname{mtp}(L'(\lambda), M) \operatorname{ch}(L'(\lambda)) \quad , \qquad (1)$$

where $L'(\lambda)$ denotes the simple G'-module of highest weight λ . The λ-weight-space of $L'(\lambda)$ equals the U'-invariants,

$$L'(\lambda)_\lambda = L'(\lambda)^{U'} \quad ,$$

and since it is one-dimensional, we may also write

$$\operatorname{mtp}(L'(\lambda), M) = \dim M_\lambda^{U'} \qquad (2)$$

for the multiplicity of $L'(\lambda)$ in M .

Lemma : $ch(M) = \Delta(u'^*)^{-1} e^{-\rho'} \left(\sum_{w\in W'} \det(w)w)e^{\rho'} ch(M^{U'}\right)$. Here $\underline{u}'$ is the Lie algebra of U' , ρ' is half the sum of roots occurring in $\underline{u}'$, and $M^{U'}$ is the space of U'-invariants in M .

Proof : By Weyl's character formula ([Hu], 24.3),

$$ch(L'(\lambda)) = \left(\sum_{w\in W'} \det(w)e^{w(\lambda+\rho')}\right)/\left(\sum_{w\in W'} \det(w)e^{w\rho'}\right) \tag{3}$$

By the denominator formula (cf. loc.cit.), we have

$$\sum_{w\in W'} \det(w)e^{w\rho'} = \prod_{\alpha>o} (e^{\alpha/2}-e^{-\alpha/2}) = \Delta(\underline{u}'^*)e^{\rho'} \tag{4}$$

(notation 3.3). Inserting (3), (4), (2) into (1) above, we obtain

$$ch(M) = \sum_{\lambda}(\dim M_{\lambda}^{U'})\left(\sum_{w\in W'} \det(w)e^{w(\lambda+\rho')}\right)e^{-\rho'}\Delta(\underline{u}'^*)^{-1} \quad .$$

Further computation gives then

$$\begin{aligned} e^{\rho'}\Delta(\underline{u}'^*)ch(M) &= \sum_{w\in W'} \sum_{\lambda}(\dim M_{\lambda}^{U'})\det(w)w(e^{\lambda+\rho'}) \\ &= \sum_{w\in W'} \det(w)w \sum_{\lambda}(\dim M_{\lambda}^{U'})e^{\lambda}e^{\rho'} \\ &= \left(\sum_{w\in W'} \det(w)w)e^{\rho'}ch(M^{U'}\right) \quad , \end{aligned}$$

which is the formula claimed in the lemma. Q.e.d.

Proposition : <u>If $\underline{\underline{F}}$ is G'-equivariant, then its character polynomial $q_{\underline{\underline{F}}} = q_M$ is W'-anti-invariant</u> (i.e. $sq_M = -q_M$ <u>for each simple reflection</u> s <u>in</u> W').

Proof : Since q_M is the lowest degree term of $\Delta(\underline{u}^*)ch(M)$, and since the lowest degree term of $\Delta(\underline{u}^*)$ equals $\prod_{\alpha>o} \alpha$, which is clearly anti-invariant, it is enough to show that $ch(M)$ itself is W'-invariant. This follows from the formula in the above lemma, or more directly from (1) and the W'-invariance of $ch(L'(\lambda))$. Q.e.d.

4.11. Multiple cross section of a unipotent action.

With assumptions and notations as in 4.10, let us now study further the case $\underline{\underline{F}} = \mathcal{O}_V$, where $V \subset \underline{u}$ is a G'-stable closed irreducible subvariety. The ring of U'-invariants $A := \mathcal{O}(V)^{U'}$ is known to be finitely generated by a theorem of Hadžiev ([Ha], cf. also [Kr]) , and so we can define an affine variety $V/U' :=$ Spec A , called the quotient of V by U' . It is equipped with the canonical morphism $\pi : V \to V/U'$ provided by the inclusion $A \subset \mathcal{O}(V)$. Generically, this morphism is a "good quotient map" for the group action, that is an (affine) fibration with a single U'-orbit as a fibre, see e.g. [BGR], Satz 16.6; more precisely, for a suitable weight vector $0 \neq a \in A$, the localization $\mathcal{O}(V)_a$ is a polynomial ring over A_a , and

$$\mathcal{O}(V)_a = A_a \otimes \mathcal{O}[\mathrm{V}]$$

with V an affine homogeneous U-space, U acting via its action on the second factor (cf. loc.cit.). In particular, $\dim V/U' = d(A) = \dim V - \dim \mathrm{V}$, that is the GK-dimension of the U'-invariants is given by the dimension of the variety, minus the "generic orbit dimension".

Let us now make the assumption, for simplicity, that this latter dimension is as big as possible, that is that V contains a free U'-orbit (hence a generic subset of them, cf. loc.cit., or [BK]). Following [J1], 2.6, and using our notation ψ,ν from 4.8 for a one parameter group, we consider the Poincaré series

$$R_A(t,\nu) := \sum_{\mu\in\Lambda} (\dim A_\mu) t^{(\mu,\nu)} = \psi^*(ch(A))$$

and <u>define</u> a function $r_A(\nu)$ and a polynomial on $\underline{t}^*(=\underline{t})$

$$p_A(\nu) := r_A(\nu) \prod_{\alpha>o} \alpha(\nu)$$

in a manner completely analogous to the procedure of 4.8, which defined $R_M(t,\nu) = \psi^*(ch(M))$, $r_M(\nu)$, and $p_M(\nu)$ for $M = \mathcal{O}(V)$. We shall also use the notations $p_A = p_{V/U'}$, and $p_M = p_V$. We note that these polynomials coincide with $q_A = q_{V/U'}$, as defined in 3.19.

<u>Lemma A</u> : <u>If</u> U' <u>acts generically freely on the</u> G'-<u>stable subvariety</u> $V \subset \underline{u}$ <u>as above, then</u> :

$$p_V = \prod_\beta{}' \left(\frac{1}{\beta}\right) \sum_{w\in W'} w\, p_{V/U'} ,$$

<u>where the product extends over all roots</u> β <u>occurring in</u> U' .

<u>Proof</u> : By restriction to a one parameter subgroup, lemma 4.10 gives :

$$R_M(t,\nu) = \prod_\beta{}' \frac{1}{(1-t^{-(\beta,\nu)})} e^{-\rho'} \sum_{w\in W'} \det(w) w\cdot e^{\rho'} R_A(t,\nu) .$$

$$= \prod_\beta{}' (1-t^{-(\beta,\nu)})^{-1} \sum_{w\in W'} \det(w) w\, R_A(t,\nu) + \cdots$$

where ... are power series of higher lowest degree term, which may be neglected in computing r_V and $r_{V/U'}$. Now by taking lowest order terms, as in the special case considered in [J1], 2.6, we get

$$r_M(\nu) = \prod_\beta{}' (\beta,\nu)^{-1} \sum_{w\in W'} \det(w) w\, r_A(\nu) ,$$

and so multiplication by the product of all positive roots, which is W' anti-invariant, gives the lemma. Q.e.d.

Now we consider a T-stable "multiple cross section" C for the U'-action on V. By this, we mean a closed (T-stable) irreducible subvariety $C \subset V$ of dimension $\dim C = \dim V/U' = \dim V - \dim U'$, such that $\overline{U'C} = V$.

By the last assumption, the restriction of functions on V to C embeds $A = \mathcal{O}(V)^{U'}$ into $\mathcal{O}(C)$ as a subring, and by the first assumption, $d(A) = d(\mathcal{O}(C))$, so that the restriction of the quotient map $V \to V/U'$ gives a generically finite map $C \to V/U'$. We denote $d(C, V/U')$ the degree of this map, or in other words the degree of the field extension

$$d(C, V/U') = [\mathrm{Fract}(\mathcal{O}(C)) : \mathrm{Fract}\, A] .$$

Note that this is the number of times that a generic orbit in V meets C.

Lemma B :
$$p_C = d(C, V/U') p_{V/U'} .$$

A proof of this lemma was provided by 3.19.

4.12. For example SL_2-equivariance.

Let us specialize the previous discussions to the case where G' is of type A_1, and $U' = U_{-\alpha}$ for a simple root α of G. (Here we denote U_β the root subgroup of G, for any β). If $C \subset \underline{u}$ is any closed irreducible subvariety which is stable under $B_\alpha = TU_\alpha$, but not under $G_\alpha = G'$, then its G_α-saturation $V := G_\alpha C$ will be a closed (!) G_α-subvariety (use [St2], p.68, lemma 2) satisfying our assumptions in 4.11, provided that $G_\alpha C \subset \underline{u}$, and we obtain as special cases of 4.10, 4.11:

Lemma (cf. [J1], 2.5-2.7) *) : Let C be an irreducible closed B_α-stable subvariety of $\underline{u}$, and $G_\alpha C \subset \underline{u}$. Then

$$-\frac{1}{\alpha}(s_\alpha+1)p_C = z\cdot p_{G_\alpha C} \quad ,$$

where $z = 0$ if C is G_α-stable, and $z = d(C, V/U_{-\alpha}) > 0$ otherwise.

4.13. Completing the proof of theorem 4.7.2.

We are now ready to complete the proof of our main results annonced in 4.7. The argument below is due to Joseph [J1], 3.1, and is repeated for convenience of the reader in our present frame-work.

Let $s \in W$ be a simple reflection, α the corresponding simple root and $P_s = B \cup sBs^{-1}$ the corresponding minimal parabolic subgroup of G. For all orbital cone bundles $K_i = G \times^B C_i$, we wish to compute $sQ_G(K_i)$ as a linear combination of $Q_G(K_j)$'s, where $i,j = 1,\dots,r$. By 4.6 resp. 4.8, we may equivalently work in terms of the cones C_j and their character polynomials $q_T(C_j)$ resp. characteristic polynomials p_{C_j}. If C_i should be P_s-stable, then $sp_{C_i} = -p_{C_i}$ by 4.12, and we are done.

So let us assume from now on that C_i is not P_s-stable. Let $\underline{u}_s$ denote the hyperplane in $\underline{u}$ orthogonal to the root space for $-\alpha$. This hyperplane is also the nilradical of the Lie algebra of P_s. In particular, $\underline{u}_s$ is P_s-stable. We conclude that our cone C_i is not contained in this hyperplane, because otherwise also $P_sC_i \supsetneq C_i$ would be contained in $\underline{u}_s \subset \underline{u}$, and in $GC_i = \mathcal{O}_u$, contradicting the fact that C_i is a maximal irreducible

*) In comparing with [J1], note that there are sign errors in propositions 2.6, and 2.7, and in lemma 2.9 of [J1], which happen to cancel each other in the calculation in [J1], 3.1.

subset of $\overline{\mathcal{O}_u \cap \underline{u}}$. Therefore $\underline{u}_s$ intersects C_i in codimension one; more precisely, by lemma 3.11, the corresponding character (istic) polynomials are related by

$$p_M = \alpha p_{C_i} \quad , \tag{1}$$

where the module M describes the intersection (including multiplicities),

$$M := \Gamma(C_i \cap \underline{u}_s, \mathcal{O}_{C_i}) = \mathcal{O}(C_i)/x_{-\alpha}\mathcal{O}(C_i) \quad ,$$

where $x_{-\alpha}$ is a root vector for $-\alpha$. Let us denote its supporting cycle as in 1.4, i.e.

$$\underline{\underline{\mathrm{supp}}}(M) = \sum_{V} m_V(M)\ [V] \quad . \tag{2}$$

Here the irreducible components V of $C_i \cap \underline{u}_s = \mathrm{supp}(M)$ are all of codimension one in C_i , by Krull's Hauptidealsatz. Now we claim that for each such component V , the P_s-saturation $P_s V$ is either V , or else one of the other C_j's , for some $j = j(s,V) \neq i$. In fact, since each V is B-stable, it follows that it is either even P_s-stable, or else has dimension

$$\dim P_s V = 1 + \dim V = \dim C_i \quad .$$

But since $P_s V$ is irreducible and contained in $\overline{\mathcal{O}_u \cap \underline{u}}$ (in fact even in $\underline{u}_s$), which is equidimensional, it follows that then $P_s V$ must be one of the other (P_s-stable!) irreducible components of $\overline{\mathcal{O}_u \cap \underline{u}}$, that is $P_s V = C_j$ for some $j = j(s,V) \neq i$. Now we apply 4.12 to conclude that

$$-\frac{1}{\alpha}(1+s)\ p_V = z p_{C_j} \tag{3}$$

for $j = j(s,V)$ and some integer

$$z = z(s,V) = d(V,C_j/U_{-\alpha}). \tag{4}$$

Note that (3) holds for all V , if we just put $z(s,V) = 0$ in case $P_sV = V$ (use 4.11).

Now we have

$$p_{C_i} = \alpha^{-1}\ p_M = \alpha^{-1}\sum_V m_V(M)\ p_V$$

by (1), (2), and 4.9, and so we compute, using (3) :

$$(s-1)p_{C_i} = -\alpha^{-1}(s+1)p_M = -\alpha^{-1}\sum_V m_V(M)(s+1)p_V$$

$$= \sum_V m_V(M)z(s,V)p_{C_{j(s,V)}} \quad .$$

Q.e.d.

4.14. Reproving Hotta's transformation formula.

The proof yields the following more precise formula. A subset of T^*X is called s-vertical (cf. Hotta's terminology [Ho]), if it is a union of projective lines of type s , i.e. projecting onto a conjugate of the line P_s/B in $X = G/B$.

Corollary : If K_i is s-vertical, then $sQ_G(K_i) = -Q_G(K_i)$, and otherwise

$$sQ_G(K_i) = Q_G(K_i) + \sum_j n^s_{ij}\ Q_G(K_j) \quad , \tag{4}$$

where $n^s_{ij} = 0$ unless K_j is s-vertical and meets K_i in codimension 1, in which case

$$n^s_{ij} = \sum_W z_W\, m_W \geq 0 \quad ,$$

Here the summation is over all irreducible components W of $K_i \cap K_j$ (necessarily of codimension one), m_W is the intersection multiplicity of K_i and K_j at W, and z_W is the non-negative integer $z(s, W \cap \underline{u})$ defined in 4.13.

Remark : By applying the canonical map $K_G(X) \to K(X) \cong H^*(X)$, which forgets the G-action, to (4), we reobtain Hotta's formula in the version of our theorem 1.15.

4.15. On explicit computations of our characteristic classes.

Let us conclude this chapter with a few remarks and examples concerning the explicit computation of our characteristic classes $Q(K_i)$ in $H^*(X)$ introduced in chapter 1. The first point to be made here is that it is more convenient to perform the computation on the level of equivariant K-theory, using the fact that the cone bundle under consideration is a G-equivariant one. Of course, to know $Q(K_i)$, or to know $Q_G(K_i)$, amounts to the same, in view of 4.6 and 4.7 (theorem 1). However, actual computations tend to be much more pleasant in $K_G(X) \cong R(T)$, a unique factorization domain, than in $H^*(X)$, which has lots of nilpotent elements.

The second point is that the interpretation of $Q_G(K_i)$ as a character polynomial $q_T(C_i)$ by 4.5 is helpful for calculations. (Here C_i denotes the fibre of K_i.) Let us give an example.

Proposition : Suppose C_i is a complete intersection of codimension d in $\underline{u}$, defined by a regular sequence of equations $f_1, \ldots, f_d \in \mathcal{O}(C_i)$ which are T-~~semi~~invariant, of weights $-\mu_1, \ldots, -\mu_d$. Then

$$Q_G(K_i) = p_{C_i} = \mu_1 \mu_2 \cdots \mu_d \quad .$$

Proof : This follows by repeated application of lemma 3.12. Q.e.d.

For example, if $f_1,\dots,f_d$ are root vectors $x_{-\alpha_1},\dots,x_{-\alpha_d}$, then $Q_G(K_i) = \alpha_1\alpha_2\dots\alpha_d$, and we recover the example treated in 1.6. In this example, all equations are linear, and the cone bundle K_i is actually a vector-bundle, so all computations are very easy. Let us therefore conclude with a less trivial case, where K_i is not a vector-bundle, and which is also covered by the proposition above.

4.16. Example.

Take $G = SL_n$. For $n \leq 3$, all orbital cone bundles are vector bundles, and the computation of their characteristic classes are covered by 1.6. For $n = 4$, there occurs the following orbital cone C, which is not linear, but quadratic : It consists of all block triangular matrices of rank ≤ 1 of the form

$$\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}, \quad \text{where} \quad A = \begin{bmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{bmatrix}.$$

This cone C is a complete intersection given by the three equations

$$a_{12} = 0\ ,\ a_{34} = 0\ ,\ a_{13}a_{24}-a_{23}a_{14} = 0\ ;$$

these equations are semi-invariant under the group T of diagonal matrices in SL_n ; their weights are

$$\alpha_1\ ,\ \alpha_3\ ,\ \alpha_1+2\alpha_2+\alpha_3 = \alpha_{14}+\alpha_{23}$$

respectively. Here we denote α_{ij} the root with root vector the matrix unit e_{ij}, and $\alpha_i := \alpha_{i,i+1}$ the simple roots $(1 \leq i < j \leq n)$.

Now we conclude from 4.15 that the character polynomial of C is given by

$$p_C = \alpha_1\alpha_3(\alpha_1+2\alpha_2+\alpha_3) \quad .$$

This example may be generalized as follows. Let $n = p+q$ with $q \geq p \geq 2$. Let $C(r,p,q)$ denote the set of all block triangular matrices of the form

$$\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} \quad ; \quad \text{where} \quad A = \begin{bmatrix} a_{1,p+1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{p,p+1} & \cdots & a_{p,n} \end{bmatrix} \qquad (*)$$

is a p by q matrix of rank $\leq r$. Then $C(0,p,q) \subset C(1,p,q) \subset \ldots \subset C(p,p,q)$ is a chain of orbital cones, and these are all the orbital cones which are contained in the vector space of all matrices of this form (*). Now we take the particular case $p = q$, and $r = q-1$: The cone $C(p-1,p,p)$ is again a complete intersection, given by the linear equations $a_{ij} = 0$ $(1 \leq i,j \leq p$, or $p \leq i,j \leq n)$, and the single non-linear equation $\det A = 0$, of weight

$$\begin{aligned}\mu &= \alpha_1+2\alpha_2+3\alpha_3+\cdots+p\alpha_p+(p-1)\alpha_{p+1}+\cdots+\alpha_n \\ &= p\alpha_p + \sum_{i=1}^{p-1} i(\alpha_i+\alpha_{n-i}) \quad .\end{aligned}$$

So again by 4.14, we conclude that

$$p_{C(p-1,p,p)} = (\alpha_{1,n}+\alpha_{2,n-1}+\cdots+\alpha_{p,p+1}) \prod_{1\leq i<j\leq p} \alpha_{ij} \prod_{p\leq i<j\leq n} \alpha_{ij} \quad .$$

For other values r,p,q , the cone $C(r,p,q)$ need no longer be a complete intersection, and the computation of characteristic classes becomes more delicate.

4.17. Remark. A possible geometric generalization of proposition 4.15

We conjecture (along with W. Fulton) the following generalization of the Whitney sum formula for Chern classes of vector bundles ([Fu], p.51) to Segre classes of cone bundles. Suppose we have a diagram of cone bundles

$$K \hookrightarrow E \xrightarrow{\oplus f_i} \bigoplus_{i=1}^{m} \ell_i$$

where the vector bundle E is mapped by a fibre preserving algebraic but not necessarily linear map to a direct sum of line bundles, and the local complete intersection cone bundle K of codimension m in E is defined scheme theoretically by the vanishing of $\oplus f_i$. Then

$$s(E) = s(K)\, s(\oplus \ell_i) ,$$

or equivalently, the Segre class of K may be computed from the Chern classes of $E, \ell_1, \ldots, \ell_m$, as

$$s(K) = c(\ell_1) \ldots c(\ell_m)/c(E) .$$

Notice that a special case of this is implied by proposition 4.15. This illustrates the power of the methods of equivariant K-theory in algebraic geometry.

§5. Characteristic classes and primitive ideals

In this chapter, G is again a semisimple group with Lie algebra $\underline{g}$, and we use the notations introduced in 1.5. As in chapter 3 and 4, we denote by Λ the lattice in $\underline{t}^*$ of integral weights (3.1) of our maximal torus $T \subset G$, by $\Omega \subset \Lambda$ the "dominant integral weights" (3.16) with respect to the ordering fixed by our choice of a Borel subgroup $B \supset T$, and by $\rho \in \Omega$ half the sum of weights in $\underline{b}$. We furthermore denote by $U(\underline{g})$ the enveloping algebra of $\underline{g}$, that is the ring of differential operators on G invariant under rigth translations. Our purpose is to study $\underline{g}$-modules, that is to say $U(\underline{g})$ modules. In particular, we are interested in the annihilators in $U(\underline{g})$ of simple $\underline{g}$-modules, called primitive ideals. We denote by $L(\lambda)$ the simple $\underline{g}$-module of highest weight λ (which is defined as the unique simple quotient of the universal (or Verma-)module $M(\lambda) = U(\underline{g}) \otimes_{U(\underline{b})} k_\lambda$, where k_λ is a one-dimensional $\underline{b}$-module of weight λ). Then the center of $U(\underline{g})$ is a polynomial ring in dim T variables (Harish-Chandra, Chevalley), acts by a character on $L(\lambda)$ which is denoted χ_λ; we note that by Harish-Chandra's theorem, $\chi_\lambda = \chi_\mu$ if and only if $\mu = w.\lambda$, for some Weyl group element $w \in W$, where the "shifted Weyl group action" $w.\lambda := w(\lambda+\rho)-\rho$ occurs. Finally, it will be convenient to identify $\underline{g}^*$ with $\underline{g}$, and $\underline{t}^*$ with $\underline{t}$ by means of the Killing form. We apply analogous notations to the group $G \times G$, so for instance $(\lambda,\mu) = \underline{t}^* \times \underline{t}^*$ defines a central character $\chi_{(\lambda,\mu)}$ of $U(\underline{g} \times \underline{g})$ etc..

5.1 Characteristic class attached to a $\underline{g}$-module

Let M be a finitely generated $\underline{g}$-module, with central character χ_λ, given by the dominant integral weight λ. Let $\mathcal{M} = \mathcal{D}_X^\lambda \otimes_{U(\underline{g})} M$ denote its localization as a coherent $\mathcal{D}_X^\lambda$-module on X [BeBe], where $\mathcal{D}_X^\lambda$ is the sheaf of differential operators on X with coefficients in the line bundle $L(\lambda)$ given by λ. Chose a good filtration on $\mathcal{M}$, and consider the associated graded sheaf $\mathrm{gr}\mathcal{M}$ as a coherent sheaf on T^*X. Then we define the <u>characteristic variety</u> of M by

$$Ch\,(M) := Ch\,(\mathcal{M}) = \mathrm{supp}\ \mathrm{gr}\ \mathcal{M},$$

the <u>characteristic cycle</u> of M by (notation 1.4)

$$\underline{\underline{Ch}}\,(M) := \underline{\underline{Ch}}\,(\mathcal{M}) := \underline{\underline{\mathrm{supp}}}\,(\mathrm{gr}\ \mathcal{M}),$$

and the <u>characteristic class</u> associated to M by (notation 1.3)

$$P(M) := Q(\mathrm{gr}\ \mathcal{M}).$$

This is a cohomology class in $H^{2d}(X)$, where d is the codimension of $Ch(M)$ in T^*X; if convenient, we also consider P(M) as a harmonic polynomial, homogeneous of degree d on $\underline{t}$ (convention 1.6). Let us point out that the above notions are <u>well-defined</u>, i.e. independent of the choice of a good filtration. For P(M), this is proved the same way as for $\underline{\underline{Ch}}(M)$ (or a special case, G = 1, of Lemma 2.12). In fact, even more is true:

<u>Proposition</u> [illegible]

[illegible]

Proposition: The characteristic class of a module is entirely determined by its characteristic cycle. More precisely, if $V_1, \dots, V_r$ are the irreducible components of $Ch(M)$ of mimimal codimension (d), with multiplicities $m_1, \dots, m_r$ in $\underline{\underline{Ch}}(M)$, then (notation 1.4)

$$P(M) = m_1\, Q(V_1) + \dots + m_r\, Q(V_r).$$

This is true by proposition 1.4.

5.2 Translation invariance

Let λ, μ be dominant integral weights. Let $E(\mu-\lambda)$ denote the finite dimensional simple $\underline{g}$-module with extremal weight $\mu-\lambda$. Let M be a $\underline{g}$-module with central character χ_λ. Then $T_\lambda^\mu M$ is defined as the direct summand of central character χ_μ of $M \otimes E(\mu-\lambda)$. The "translation functor" T_λ^μ is then an equivalence of the categories of finitely generated $\underline{g}$-modules with central character χ_λ resp. χ_μ ([BeGe], [Ja]).

Lemma: $\underline{\underline{Ch}}(T_\lambda^\mu M) = \underline{\underline{Ch}}(M)$, and $P(T_\lambda^\mu M) = P(M)$.

Proof: The second statement follows from the first one by proposition 5.1. Let $\mathcal{M}$ resp. $T_\lambda^\mu \mathcal{M}$ denote the $\mathcal{D}_X^\lambda$- resp. $\mathcal{D}_X^\mu$-module corresponding to M resp. $T_\lambda^\mu M$. Then the functor $\mathcal{T}_\lambda^\mu$ thus defined is equivalent to the "geometry, translation functor" $\mathcal{M} \longmapsto \mathcal{O}(\mu-\lambda) \otimes_{\mathcal{O}_X} \mathcal{M}$, where $\mathcal{O}(\mu-\lambda)$ denotes the invertible sheaf corresponding to the line bundle $\mathcal{L}(\mu-\lambda)$. Since $\mathcal{O}(\mu-\lambda)$ is locally isomorphic to $\mathcal{O}_X$, it follows that this functor does not change the characteristic cycle. Q.e.d.

Remark: By the lemma, all of our results stated in [BB3] for the trivial central character (case $\lambda = 0$) only, extend to the case of an arbitrary central character χ_λ with $\lambda \in \Omega$ without change.

5.3 Characteristic variety of a Harish-Chandra bimodule

By a Harish-Chandra bimodule , we mean a $(\underline{g} \times \underline{g}, K)$-module with finite K-multiplicities, where K is the diagonal copy of G in $G \times G$. (Here we use the terminology of [BB3]; so the module is K-equivariant in the strong sense that the differential of the K-action coincides with the action of the diagonal subalgebra $\underline{k}$ of $\underline{g} \times \underline{g}$.) For any finitely generated Harish-Chandra bimodule H with integral central character $\chi_{(\lambda,\mu)}$ given by a pair (λ,μ) of dominant integral weights we define modules L^l resp. L^r of central character χ_λ resp. χ_μ by

$$L^l := M(\mu)^V \otimes_{\mathfrak{U}(\underline{g}^r)} H \quad \text{resp.} \quad L^r := M(\lambda)^V \otimes_{\mathfrak{U}(\underline{g}^l)} H,$$

where e.g. $\underline{g}^r = 0 \times \underline{g}$ denotes the right copy of $\underline{g}$ in $\underline{g} \times \underline{g}$, and $M(\mu)^V$ is the universal (Verma-)module of highest weight μ, considered as a right $\underline{g}$-module via the principal anti-automorphism of $\mathfrak{U}(\underline{g})$. Then $H \longmapsto L^l$ resp. $H \longmapsto L^r$ are equivalences of the category of Harish-Chandra bimodules with central character $\chi_{(\lambda,\mu)}$ with the category of all finitely generated $(\underline{g},B)$-modules with central character χ_λ resp. χ_μ.
For a geometric interpretation (and proof) of this well-known result of Bernstein-Gelfand [BeGe], Joseph [J4], and Enright [E], see [BB3], cf. also 5.9 below. In particular, this establishes bijections of the simple objcets (up to isomorphism); we shall de-

note by $H_w^{(\lambda,\mu)}$ for any $w \in W$ the simple Harish-Chandra bimodule corresponding to $L^l = L(w.\lambda)$ resp. to $L^r = L(w^{-1}.\mu)$ (f. [BB3], 3.4). We denote H^l the $\underline{g} = \underline{g}^l$-module obtained from H by forgetting the $\underline{g}^r$-action.

<u>Theorem</u> ([BB3], 5.8, [Gi]): <u>Let</u> H <u>be any finitely generated Harish-Chandra module of central character</u> $\chi_{(\lambda,\mu)}$, <u>and let</u> L^r <u>be the corresponding</u> $(\underline{g},B)$-<u>module of central character</u> χ_μ <u>as above. Then</u>

$$Ch(H^l) = G \times^B V(L^r).$$

Here $V(L^r)$ denotes the associated variety of L^r, that is the support of $\operatorname{gr} L^r$ in $\underline{g}^* = \underline{g}$ with respect to some good filtration. Of course we obtain an analogous result $Ch(H^r) = G \times^B V(L^l)$ by interchanging left and right.

<u>Corollary</u>: $Ch(H^l)$ <u>is a union of orbital cone bundles</u> (terminology 1.7).

In fact, it is enough to show that $V(L^r)$ is a union of orbital cones $C_1,\dots,C_r$, because then $Ch(H^l)$ is the union of the orbital cone bundles $K_i = G \times^B C_i$ by the theorem. Let us briefly recall the reason: The localization of L^r on X is a holonomic $\mathcal{D}$-module and its characteristic variety is a union of closures of conormal bundles of Schubert cells, which project onto orbital cones under Springer's map $\pi: T^*X \longrightarrow \underline{g}$. On the other hand, π maps $Ch(L^r)$ onto $V(L^r)$ by [BB3], 1.9.

5.4 Homogeneous Harish-Chandra bimodules

We denote $d(M) = \dim V(M)$ the Gelfand Kirillov dimension of a $\underline{g}$-module M. for a Harish-Chandra bimodule H as in 5.3 it is obvious, that $d(H^l) = d(H) = d(H^r)$. We call H (left) homogeneous, if $d(M) = d(H)$ for each left submodule $0 \neq M \subset H$. It is easy to see that the following are examples:

a) Each simple Harish-Chandra bimodule $H = H_w^{(\lambda,\mu)}$ is homogeneous (cf. lemma 5.10).

b) For each primitive ideal J, $H = U(\underline{g})/J$ is a homogeneous Harish-Chandra bimodule. The $\underline{g} \times \underline{g}$ action is defined by

$$(x,y)u := xu - uy \quad \text{for} \quad x,y \in \underline{g},\ u \in H,$$

see [BB3], § 3.

Corollary 1 (notation 5.3): *If H is homogeneous, then $Ch(H^l)$ is a union of orbital cone bundles $K_1,\ldots,K_r$, all of the same dimension*

$$\dim Ch(H^l) = \dim X + d(L^r) = \dim X + \tfrac{1}{2}\, d(H).$$

Proof: By theorem 5.3, it suffices to show that $V(L^r)$ is equidimensional of dimension $d(L^r) = \frac{1}{2}\, d(H)$. This follows from the following lemma by a general theorem of Gabber-Kashiwara [Le]. Q.e.d.

Lemma: L^r *is homogeneous of dimension* $d(L^r) = \frac{1}{2}\, d(h)$.

Proof: The equivalence of categories, sending H to L^r, described in 5.3, induces an isomorphism of the lattices of submodules. Let

$L' \neq 0$ be a submodule of L^r, and let $H' \subset H$ be the corresponding bisubmodule $H' \neq 0$ in H. Then $V(H') = \pi(Ch(H'^1)) = \pi(G \times^B V(L')) = GV(L') = GC_1 \cup \ldots \cup GC_r$ for some orbital cones $C_1, \ldots, C_r$ by 5.3. But $\dim GC_i = 2 \dim C_i$ for each orbital cone C_i by well-known results of Steinberg and Spaltenstein (see 1.7). Hence $d(H') = \dim V(H'^1) = \dim GV(L') = \max_i \dim GC_i = 2 \max \dim C_i = 2\, d(L')$. Since H is homogeneous of dimension $d(H)$, it follows now that $d(L') = \frac{1}{2} d(H') = \frac{1}{2} d(H)$. Q.e.d.

From the above proof, we make the following

<u>Observation</u>: <u>Each irreducible component</u> K_i of $Ch(H^1)$ <u>maps</u> (under π) <u>onto an irreducible component of</u> $V(H^1)$.

In fact, $\pi(K_i) = \pi(G \times^B C_i) = GC_i$ has dimension $\dim GC_i = 2 \dim C_i$, while $\dim K_i = \dim X + \dim C_i$. Since the last dimension is independent of i by the corollary above, the first one is also, hence $\dim \pi(K_i) = \dim V(H^1) = d(H)$, so $\pi(K_i)$ must be an irreducible component of $V(H^1)$.

Note that in particular this says that $V(H^1)$ is also equidimensional, which is of course a direct consequence of the equidimensionality theorem of Gabber-Kashiwara quoted above.

<u>Corollary 2</u>: <u>If</u> $M \subset M'$ <u>are left submodules</u> $\neq 0$ <u>of a homogeneous Harish-Chandra bimodule</u> H, <u>then</u>

a) $Ch(M)$ <u>is equidimensional of dimension</u> $\dim Ch(H^1)$, <u>and</u>

b) $d(M'/M) < d(H)$ <u>implies</u> $\underline{Ch}(M) = \underline{\underline{Ch}}(M')$.

Proof:

a) Let H' be the bimodule generated by M. Then H' is finitely generated as a left module, hence is a finite sum of homomorphic images M_j of M. It follows that $Ch(H'^l) = \bigcup_j Ch(M_j) \subset Ch(M)$, hence $Ch(H'^l) = Ch(M)$. Now a) follows from Corollary 1.

b) If M'/M would contribute to the characteristic cycle of M', then $Ch(M'/M)$ would contain an irreducible component of $Ch(M')$, hence $V(M'/M) = \pi(Ch(M'/M)$ would contain an irreducible component of $V(M')$ by the above observation, hence $d(M'/M) = d(M') = d(H)$, contradicting the assumption. Q.e.d.

5.5 Characteristic cycle and class of a Harish-Chandra bimodule

With notations as in 5.3, we have the even stronger result:

Theorem: (cf. [BB3], 5.9): $\underline{\underline{Ch}}(H^l) = G \times^B \underline{V}(L^r)$.

In more detail, this means that if the associated cycle of L^r is

$$\underline{V}(L^r) = m_1[C_1] + \ldots + m_r\,[C_r],$$

with $C_1,\ldots,C_r$ the different irreducible components of $V(L^r)$, then the characteristic cycle of H^l is

$$\underline{\underline{Ch}}(H^l) = m_1[K_1] + \ldots + m_r\,[K_r]$$

where $K_i = G \times^B C_i$, and the positive integers m_i are as in (2) $(i = 1,\ldots,r)$.

Corollary 1: If H is homogeneous, then $P(H^l) = m_1Q(K_1)+\ldots+m_rQ(K_r)$, and $\deg P(H^l) = \dim X - \frac{1}{2}d(H)$.

This follows from 5.1 and Corollary 5.4.1.

Corollary 2: If H is homogeneous, then $Ch(H^l)$ is entirely determined by $P(H^l)$ (and conversely).
In fact, the polynomials $Q(K_1),\ldots,(K_r)$ are linearly independent by corollary 1.8.1. Hence $P(H^l)$ determines the multiplicities $m_1,\ldots,m_r$ uniquely.
Corollary 3: If $H \neq 0$, then $P(H^l) \neq 0$.

5.6 Identification with a character polynomial

Recall that we defined in 3.9 a character polynomial q_M for any finitely generated T-equivariant $S(\underline{u})$-module M. Now we consider a finitely generated $(\underline{g},B)$-module L as in 5.3, and we define its character polynomial p_L or $p(L)$ by

$$p_L := q_{grL}, \tag{1}$$

where grL denotes the associated graded module with respect to some T-equivariant good filtration of L. It is obvious that such a filtration exists, and that grL is a T-equivariant finitely generated $S(\underline{u})$-module, so q_{grL} is defined. Since the formal characters of L and grL are the same, Corollary 3.9 gives

$$p_L = gr\,\Delta(\underline{u}^*)\ ch(L). \tag{2}$$

We take this formula as an alternative definition of p_L, in terms of the formal character of L, which exhibits the independence of the choice of a filtration.

Theorem (notations 5.3): $P(H^l) = p(L^r)$ (3)

Proof: Let $C_1,\dots,C_r$ denote the irreducible components of $V(L^r)$ of minimal codimension d, and let $m_1,\dots,m_r$ denote their multiplicities in $\underline{\underline{V}}(L^r)$, so that by (1) and proposition 4.9.2:

$$p(L^r) = q_{grL} = \sum_i m_i\, q_{C_i}. \tag{4}$$

By theorem 5.4, $K_i := G \times^B C_i$, for $i = 1,\dots r$, are the irreducible components of $\mathcal{Ch}(H^l)$ of minimal codimension d in T^*X, so

$$P(H^l) = \sum_i m_i\, Q(K_i) \tag{5}$$

by 5.1 and theorem 5.5. Now we have by 4.6, resp. theorem 4.7.1, resp. 4.5

$$Q(K_i) = Q_G(K_i)^{\natural} = Q_G(K_i) = q_{C_i}$$

for each $i = 1,\dots,r$. So the sums in (4) and (5) are equal term by term, and (3) follows. Q.e.d.

Remark: This proof of the theorem is based on the harmonicity of the character polynomials q_{C_i} (theorem 4.7.1). We shall give in 5.9 an alternative proof, based more directly on the harmonicity of the character polynomial $p(L^r)$, which seems more satisfactory.

Corollary: For a simple Harish-Chandra bimodule, we have (notation 5.3)

$$P(H_w^{(\lambda,\mu),l}) = p(L(w^{-1}.\mu)),$$

$$P(H_w^{(\lambda,\mu),r}) = p(L(w.\lambda)).$$

5.7 Harmonicity of character polynomials

Let L be any finitely generated $(\underline{g},B)$-module of central character χ_μ. Let us recall from the theory of highest weight modules [Di], [Ja] that within an appropriate Grothendieck group, L is expressible as an integer linear combination of Verma-modules $M(\nu)$ of highest weight $\nu = w.\mu$, $w \in W$, say

$$[L] = \sum_{w\in W} a_w(L)\,[M(w.\mu)], \tag{1}$$

where the integer coefficients $a_w(L)$ are uniquely determined. We list some properties of the character polynomial $p(L)$, which are well-known, but briefly reproved here for convenience of the reader.

Proposition:

a) $p(L)$ is homogeneous of degree $a := \dim \underline{u} - d(L)$.

b) $$p(L) = \frac{1}{a!} \sum_{w\in W} a_w(L)\,(w.\mu)^a \tag{2}$$

c) a is also characterized as the smallest integer ≥ 0 that makes the right hand side of (2) nonzero.

d) Let $\underline{a}(L)$ denote the group ring element $\sum_{w\in W} a_w(L)w$.
Then $\underline{a}(L)\,\frac{1}{j!}\,(\mu+\rho)^j$ is zero for $j<a$, and is $p(L)$ for $j = a$.

e) $P(L)$ is harmonic.

Proof:

a) By theorem 3.10, the degree of the character polynomial equals to the codimension of the support of grL in $\underline{u}^*$, hence a) follows from $d(L) = \dim V(L) = d(grL)$.

b),c) Since the formal character of a Verma-module is obviously given by $ch(M(\nu)) = \Delta(\underline{u}^*)^{-1}e^{\nu}$ [Di], equation (1) gives

$$\Delta(\underline{u}^*)ch(L) = \sum_{w\in W} a_w(L)e^{w.\mu}. \tag{3}$$

Writing out the exponentials as power series, we obtain

$$[\Delta(\underline{u}^*)ch(L)]^j = \sum_{w\in W} a_w(L) \frac{1}{j!} (w.\mu)^j$$

for the homogeneous term of degree j of (3), for all $j \geq 0$. Now b) and c) follow from a).

d) follows from a),b),c) by binomial development of $(w.\mu)^j = (w(\mu+\rho)-\rho)^j$, using induction on j.

e) Let Q be a constant coefficient differential operator on $\underline{t}$ which is W invariant, say Q homogenous of degree $d > 0$. We have to prove $Qp(L) = 0$. This is clear for $\deg Q > \deg p(L)$. So let $d \leq a$. By Leibniz' rule, we have $Q\lambda^a = c.Q(\lambda)\lambda^{a-d}$ for some scalar c independent on $\lambda \in \underline{t}^*$, where we consider Q as a polynomial function on $\underline{t}^*$. Using the formula for $p(L)$ given in d), we obtain

$$\begin{aligned} Qp(L) = Q\,\underline{a}\,(L)\,\frac{1}{a!}\,(\mu+\rho)^a &= \underline{a}\,(L)\,Q\,\frac{1}{a!}\,(\mu+\rho)^a \\ &= \frac{c}{a!}\,Q(\mu+\rho)\,\underline{a}\,(L)\,(\mu+\rho)^{a-d} = 0, \end{aligned}$$

where the second equation comes from W invariance of Q, and the last comes from d), since a-d ist strictly smaller than a.

Q.e.d.

5.8 Equivariant characteristic class for a Harish-Chandra bimodule

Let H be a finitely generated Harish-Chandra bimodule with central character $\chi_{(\lambda,\mu)}$ as in 5.3. Then H has a localization $\underline{\underline{H}}$ on the flag variety $Z = X \times X$ of the group $G \times G$. By definition (cf. [BeBe], [BB3]), $\underline{\underline{H}}$ is the coherent $(\underline{\underline{\mathcal{D}}}_Z^{(\lambda,\mu)}, K)$-module

$$\underline{\underline{H}} := \underline{\underline{\mathcal{D}}}_Z^{(\lambda,\mu)} \otimes_{U(\underline{g}\times\underline{g})} H,$$

where $\underline{\underline{\mathcal{D}}}_Z^{(\lambda,\mu)}$ denotes the sheaf of rings of differential operators with coefficients in the line bundle $\underline{\underline{L}}(\lambda,\mu)$ on Z given by (λ,μ). Now let $p^1: Z = X \times X \longrightarrow X$ denote projection onto the left copy of X. Then the direct image sheaf $p^1_* \underline{\underline{H}}$ is a $\underline{\underline{\mathcal{D}}}_X^\lambda$-module, which still carries a G action, G acting via K. Note, however, that while $\underline{\underline{H}}$ is K-equivariant in the strong sense of [BB3], 2.2, the module $p^1_*\underline{\underline{H}}$ is only "weakly G-equivariant" in the sense of 2.12. The following is easy to see (cf. [BB3], 5.10a) resp. 5.5a)):

Lemma:

a) $\underline{\underline{H}}$ admits a K-equivariant good filtration $(\underline{\underline{H}}_n)_{n\in\mathbb{Z}}$.

b) This induces a G-equivariant good filtration $(p^1_*\underline{\underline{H}}_n)_{n\in\mathbb{Z}}$ on p^1_*H.

With respect to such a filtration, the associated graded module $\mathrm{gr}\, p^1_*\underline{\underline{H}}$ is a G-equivariant $\mathcal{O}_{T^*X}$-module, and hence defines a class in $K_G(T^*X)$, which is independent of the choice of filtration by lemma 2.12. Under the composition of the maps $K_G(T^*X) \longrightarrow K_G(X) \longrightarrow R(T) \subset \hat{S}(\underline{t}^*)$, this class determines a power series on $\underline{t}$, of lowest degree homogeneous term $Q_G(\mathrm{gr}\,p^1_*\underline{\underline{H}})$ (see definition 4.5).

Now we define

$$P_G(H) := Q_G(\mathrm{gr}\, p_*^1 \underline{\underline{H}}).$$

Proposition: $P(H^1)$ is the harmonic part of $P_G(H)$.

Proof: One has to observe that $p_*^1 \underline{\underline{H}}$ is isomorphic to the localization of H^1 on X (cf. [BB3], proposition 5.4). Then

$$P(H^1) = Q(\mathrm{gr}\; p_*^1\; \underline{\underline{H}}) = Q_G(\mathrm{gr}\; p_*^1 \underline{\underline{H}})^{\natural} = P_G(H)^{\natural}$$

follows from proposition 4.6. Q.e.d.

5.9 Alternative proof of identification with character polynomials (5.6).

With notations as in 5.8, let us compute $P_G(H)$, using the functor $H \longmapsto L^r$ described in 5.3. Let us first recall the geometric interpretation of this functor (cf. [BB3], 3.6): The inclusion of X into $Z = X \times X$ as the right copy $X^r = \{B\} \times X$ is denoted i_r. The restriction $\underline{\underline{L}}^r := i_r^* \underline{\underline{H}}$ of the $\underline{\underline{\mathcal{D}}}_Z^{(\lambda,\mu)}$-module $\underline{\underline{H}}$ to $X^r = X$ is a B-equivariant $\underline{\underline{\mathcal{D}}}_X^{\mu}$-module, which is canonically isomorphic to the localization of L^r on X, and the functor $i_r^*\colon \underline{\underline{H}} \longmapsto \underline{\underline{L}}^r$ establishes an equivalence of the category of coherent $(\mathcal{D}_Z^{(\lambda,\mu)}, K)$-modules with the category of coherent $(\underline{\underline{\mathcal{D}}}_X^{\mu}, B)$-modules. We note that $L^r \cong \Gamma(X, \underline{\underline{L}}^r)$ (cf. loc.cit.).

Lemma (cf. [BB3], 5.10): A K-equivariant good filtration on $\underline{\underline{\mathcal{H}}}$ induces a B-equivariant good filtration on $i_r^*\underline{\underline{\mathcal{H}}} = \underline{\underline{\mathcal{L}}}^r$. With respect to such filtrations, we have a B-equivariant isomorphism of graded B-equivariant $\mathcal{O}_{\underline{u}}$-modules

$$j^*(\mathrm{gr}\ p_*^1\ \underline{\underline{\mathcal{H}}}) \cong \mathrm{gr}\ \Gamma(X, i_r^*\underline{\underline{\mathcal{H}}}) \cong \mathrm{gr}\ L^r.$$

Here $j: \underline{u} \hookrightarrow T^*X$ denotes the inclusion of the cotangent space at the base point; also we notationally do not distinguish between the $\mathcal{O}(\underline{u})$-module $\mathrm{gr}\ L^r$ and the corresponding sheaf of $\mathcal{O}_{\underline{u}}$-modules, since $\underline{u}$ is affine.

Theorem (notation 5.3): $P_G(H) = p(L^r)$.

Proof: By definition, $P_G(H) = Q_G(\mathrm{gr}\ p_*^1\underline{\underline{\mathcal{H}}})$, with notations as in 5.9. By proposition 4.5, $Q_G(\mathrm{gr}\ p_*^1\underline{\underline{\mathcal{H}}})$ is the character polynomial of $j^*(\mathrm{gr}\ p_*^1\underline{\underline{\mathcal{H}}})$. By the lemma, this coincides with the character polynomial of $\mathrm{gr}\ L^r$, hence with $p(L^r)$ by definition 5.6(1). Q.e.d.

As a corollary, we obtain the following alternative proof of theorem 5.6:

$$P(H^1) = P_G(H)^\natural = p(L^r)^\natural = p(L^r)$$

by propositon 5.8, the theorem above, and proposition 5.7 e).
Q.e.d.

Remark: In all preceding considerations, we may obviously interchange left and right sides to obtain analogous results. For instance, we define the "right" equivariant characteristic class of

H by $P_G(H^r) := Q_G\ (\mathrm{gr}\ p^r_* \underline{\underline{H}})$, where $p^r\colon X \times X \longrightarrow X$ denotes projection onto the right copy, in complete analogy to the "left" one, $P_G(H^l) = P_G(H)$ in 5.8, and then we obtain

$$P_G(H^r) = p(L^l) = P(H^r)$$

in analogy to the theorem and corollary above. In the sequel, we have the statement of right analogues to the reader, and consider only <u>left</u> characteristic cycles and classes $P(H^l)$; we shall eventually even give up to write the "l", in order to avoid too clumsy notation in our formulas.

5.10 Some non-commutative algebra

The <u>Goldie rank</u> of a module M, denoted $\mathrm{rk}\ M$, is the maximal number r such that M contains a direct sum of r submodules $\neq 0$. A module $\neq 0$ is called <u>uniform</u>, if any two submodules $\neq 0$ have intersection $\neq 0$, i.e. if $\mathrm{rk}\ M = 1$. If M is noetherian, then clearly $\mathrm{rk}\ M < \infty$, and M contains a direct sum of $\mathrm{rk}\ M$ uniform submodules; moreover, as a matter of fact from general ring theory, any direct sum within M of uniform submodules (necessarily of $\leq \mathrm{rk}\ M$ terms) can be extended to a direct sum within M of $\mathrm{rk}\ M$ uniform terms (see e.g. [Go], Theorem 1.07).

<u>Proposition</u>: <u>Let H be a simple Harish-Chandra bimodule. Let U be a uniform submodule of H^l. Then</u>

a) H^l <u>contains a direct sum M of $\mathrm{rk}\ H^l$ copies of U.</u>

b) $d(H^l/M) < d(H)$.

Corollary: a) $\underline{\underline{Ch}}(H^l) = \text{rk } H^l \ \underline{\underline{Ch}}(U)$

b) $\underline{\underline{Ch}}(U)$ is independent of the choice of U.

Note that corollary a) is immediate from the proposition by corollary 5.4.2, and b) follows from a) on dividing by the positive integer rk H^l.

Lemma:

a) H^l is homogeneous of *GK*-dimension $d(H) = d(A)$, where $A = U(\underline{g})/\text{Ann } H^l$.

b) Every proper homomorphic image of U has smaller *GK*-dimension.

Proof of the proposition: Here we write H as a left-right $U(\underline{g})$-bimodule. Let $0 \neq E \subset H$ be a finite dimensional K-submodule; then $H = U(\underline{g})EU(\underline{g}) = U(\underline{g})E$, so H^l is finitely generated (hence noetherian). By simplicity of H, $UU(\underline{g}) = H$, hence $UF = H$ for some finite dimensional subspace $F \subset U(\underline{g})$ by noetherianness of H^l. For each $f \in F$, Uf is a homomorphic image of U. If $Uf \neq 0$, then $d(Uf) = d(H) = d(U)$ by homogeneity (lemma a)), so $Uf \cong U$ by lemma b). We have proved that $H = UF$ is a finite sum of copies of U. Let $M = Uf_1+\ldots+Uf_r$ be a maximal sub-sum ($f_1,\ldots,f_r \in F$) such that the sum is direct. It remains to prove that $d(H/M) < d(H)$, and that $r = \text{rk } H^l$. Suppose $d(H/M) = d(H)$. Then since H^l is a finite sum of Uf's ($f \in F$), $d(H/M)$ is the maximum of $d(\overline{U\bar{f}})$, where $\overline{}$ denotes the quotient map $H \longrightarrow H/M$. Hence $d(H/M) = d(H)$ implies $d(\overline{Uf}) = d(H)$ for some $0 \neq f \in F$, so Uf maps isomorphically onto $\overline{Uf}$ by lemma b), so $Uf \cap M = 0$, contradicting the maximality of M. Hence $d(H/M) < d(H)$ is proved. Clearly $r \leq \text{rk } H^l$. If $r < \text{rk } H^l$, then by the remark preceding the proposition, there exists a submodule $0 \neq U' \subset H^1$ such that $U' \cap M = 0$. Since this implies $d(H/M) \geq d(U') = d(H)$ by homogeneity of H^1, this contradicts $d(H/M) < d(H)$. Hence we must have $r = \operatorname{rk} H^1$. Q.e.d.

Proof of the lemma:

a) If $N \subset H^1$ has $d(N) < d(H)$, then $d(NU(\underline{g})) \leq d(N) < d(H)$, so $NU(\underline{G})$ would be a proper bisubmodule of H, hence $N = 0$ by simplicity of H. The inequality $d(H) \leq d(A)$ is trivial, and the opposite inequality follows e.g. from the fact that A^1 embeds into a finite direct sum of copies of H^1 (cf.e.g. [BB3], 4.10).

b) Let $\overline{} : U \longrightarrow \overline{U}$ be a homomorphism with kernel $N \neq 0$. Let $u \in U$, and $L = \{a \in A \mid a\bar{u} = 0\}$. We claim that this is an essential left ideal of A, i.e. that every left ideal $L' \neq 0$ meets L non-trivially. In fact, suppose $L'u \cap L = 0$. Then $L' \longrightarrow L'\bar{u}$ is injective, so $L' \longrightarrow L'u$ is injective, and $L' \cap N = 0$, contradicting uniformicity of U. Now it follows from [BGR], 2.7 that L contains a nonzerodivisor s of A, since A is a prime noetherian ring. Now we conclude that $d(A\bar{u}) = d(A/L) \leq d(A/As) \leq d(A) - 1$ by the argument given in [BK2], 3.4 or by [B3], 1.3. By a) this proves $d(\overline{U}) < d(A) = d(U)$. Q.e.d.

5.11 Definition of the polynomials P_w

Lemma: *Let* U *be a uniform left submodule of* $H_w^{(o,o)}$ *for some* $w \in W$. *Then for every pair* $\lambda, \mu \in \Omega$, *the module* $T_0^\lambda U$ (notation 5.2) *is isomorphic to a uniform submodule of* $H_w^{(\lambda,\mu)}$.

Proof: We have $T_0^\lambda H_w^{(o,o),l} = (T_{(o,o)}^{(\lambda,o)} H_w^{(o,o)})^l = H_w^{(\lambda,o),l}$, so $U' := T_0^\lambda U$ is a left submodule of $H_w^{(\lambda,o)}$; since T_0^λ is an equivalence of categories, it preserves the lattice of submodules up to isomorphism, so U uniform implies U' uniform.

Now we have $H_w^{(\lambda,\mu)} = T_{(\lambda,o)}^{(\lambda,\mu)} H_w^{(\lambda,o)}$ is a direct summand of $H_w^{(\lambda,o)} \otimes E(0,\mu)$ (notation 5.2); let $p: H_w^{(\lambda,o)} \otimes E(0,\mu) \longrightarrow H_w^{(\lambda,\mu)}$ be the projection map. As a left module, $H_w^{(\lambda,o)} \otimes E(0,\mu)$ is a finite direct sum of copies of $H_w^{(\lambda,o)}$; hence a finite sum of copies of U' (by the first part of the argument in 5.10). So $p(U'') \neq 0$ for at least one of these copies, U'' say. But then $d(p(U'')) = d(H_w^{(\lambda,\mu)}) = d(H_w^{(\lambda,o)}) = d(U')$ by homogeneity (lemma 5.10a)), hence $p(U'') \cong U'' \cong U'$ by lemma 5.10 b). Q.e.d.

Theorem: *Let* $w \in W$. *Take any two pairs* (λ,μ), (λ',μ') *of dominant integral weights. Let* U *resp.* U' *be uniform left submodules of* $H_w^{(\lambda,\mu)}$ *resp.* $H_w^{(\lambda',\mu')}$. *Then* $\underline{\underline{Ch}}(U) = \underline{\underline{CH}}(U')$, *and* $P(U) = P(U')$.

Proof: The characteristic cycle $\underline{\underline{Ch}}(U)$ is independent of the choice of U by corollary 5.10.b). Making a choice as in the lemma, we see that this cycle is independent of μ, and also of λ by 5.2. The second claim follows by 5.1. Q.e.d.

Definition: For each $w \in W$, we denote $P_w := P(U)$ the polynomial uniquely determined by the theorem.

Corollary: $P(H_w^{(\lambda,\mu)}) = \mathrm{rk}\, H_w^{(\lambda,\mu),l}\, P_w$.

This follows from corollary 5.10 a) by 5.1. In particular, we note that P_w is a homogeneous harmonic polynomial of degree $a = a(w)$.

5.12. Relation to primitive ideals

Let J be a primitive ideal of $U(\underline{g})$, and $A := U(\underline{g})/J$. We note that the left resp. right modules have the same Goldie rank, denoted $\mathrm{rk}\, A$, also called the Goldie rank of J. This number has the following well-known alternative interpretation: Let S denote the set of nonzero-divisors of A; by Goldie's theorem, A admits a ring of fractions $S^{-1}A$, which is simple artinian, and hence isomorphic to the ring of n by n matrices over some skew field (Wedderburn-Artin theorem); then $n = \mathrm{rk}\, A$.

Now assume J of integral central character χ_λ. We recall that

$$J = I(w.\lambda) := \mathrm{Ann}\, L(w.\lambda) \qquad (1)$$

for some $w \in W$ (Duflo's theorem [D]), and that a simple Harish-Chandra module $H_w^{(\lambda,\mu)}$ has left resp. right annihilator

$$\mathrm{Ann}\, H_w^{(\lambda,\mu),l} = I(w.\lambda) \quad \text{resp.} \quad \mathrm{Ann}\, H_w^{(\lambda,\mu),r} = I(w^{-1}.\mu). \qquad (2)$$

(Note that we are still considering left-right $U(\underline{g})$-modules.) The last result (due to Joseph [J5],is easily deduced from the equivalences stated in 5.3 (cf. also [Ja], 7.9).

<u>Theorem</u>: <u>Let</u> $A := U(\underline{g})/I(w.\lambda)$, <u>where</u> $w \in W, \lambda \in \Omega$. <u>Let</u> U <u>be a uniform left ideal of</u> A. <u>Then</u>

a) $P(A) = rkA\ P(U)$

b) $P(U) = P_w$ (definiton 5.11).

<u>Corollary 1</u>: $P_w = P_y$ <u>for</u> $w,y \in W$ <u>in the same left cell</u>.

Here two Weyl group elements w,y are said to be in the same <u>left cell</u>, if $I(w.\lambda) = I(y.\lambda)$ for one (and hence for all) $\lambda \in \Omega$. Note that the corollary follows from the theorem because $P(A)/rkA$ depends only on the ideal $I(w.\lambda)$.

<u>Notation</u>: The corollary justifies the notation $P_J = P_w$ if w is any element in W such that $J = I(w.\lambda)$.

<u>Proof</u>:

a) Considered as a Harish-Chandra bimodule, A has finite length, and hence a socle $H \neq 0$, which obviously must be simple by primality of A (as a ring), so

$$\text{socle}(A) = H \cong H_v^{(\lambda,\lambda)} \tag{3}$$

for some $v \in W$. Also $d(A/H) < d(A)$ by [BK2], Satz 3.4, and hence $P(A) = P(H)$ by corollary 5.4.2 and 5.1. Now let $U' := U \cap H$; then also $d(U/U') \leq d(A/H) < d(A)$, so that we again conclude $P(U) = P(U')$ by loc.cit.. We also must have $rk\ A = rk\ H^l$, because otherwise $rk\ H^l < rk\ A$, and then we could find a uniform submodule of A^l injecting into A/H

by the remark preceding proposition 5.10, which would contradict $d(A/H) < d(A)$ (using homogeneity of A^1). Applying corollary 5.10a) (plus 5.1) to H and U' (which is of course uniform), we may now conlude

$$P(A) = P(H) = \mathrm{rk}\, H^1\, P(U') = \mathrm{rk}\, A\, P(U).$$

b) Note that (3) gives us $P(U) = P(U') = P_v$, but only for a particular element v in the left cell of w, so this is not enough to prove b). Instead, we proceed as follows. We embed A^1 into a finite direct sum of copies of $H_w^{(\lambda,o)}$ (using (2) and [BB3], lemma 4.10). Projecting onto a suitable one of these copies, we get a homomorphism of U' into $H_w^{(\lambda,o),1}$ with nonzero image. Since $H_w^{(\lambda,o),1}$ is homogeneous of dimension $d(A)$ (use (2) and lemma 5.10a)) we have $d(pU') = d(A)$, so $pU' \cong U'$ by lemma 5.10b). We have proved that $H_w^{(\lambda,o),1}$ contains a copy of U', so (by definition 5.11)

$$P(U) = P(U') = P_w. \qquad \text{Q.e.d.}$$

<u>Remark 1</u>: We have also seen in the proof that:

<u>Corollary 2</u>: <u>There exists</u> $v \in W$ <u>such that</u> $I(w.\lambda) = I(v.\lambda)$,

$$P(U(\underline{g})/I(w.\lambda) = P(H_v^{(\lambda,\lambda)}), \tag{4}$$

<u>and</u>

$$\mathrm{rk}\, U(\underline{g})/I(w.\lambda) = \mathrm{rk}\, H_v^{(\lambda,\lambda),1}. \tag{5}$$

Remark 2: The choice of v above in the left cell of w is uniquely determined by (3). We further note that $v^2 = 1$ (Duflo [D], Proposition 9). Since $H_v^{(\lambda,\lambda)} = T_{(o,o)}^{(\lambda,\lambda)} H_v^{(o,o)}$ is the simple socle of $U(\underline{g})/I(w.\lambda) = T_{(o,o)}^{(\lambda,\lambda)} (U(\underline{g})/I(w.0))$, it can be concluded that v is uniquely determined by w, i.e. independent of λ (cf. [Ja], 7.11). Logically, we shall not have to use this fact here, because we anyway know by corollary 1 that P_v is independent of λ.

Remark 3: We note that in general (4) does not hold with w in place of v. Although we do have (even for arbitrary $\mu \in \Omega$)

$$Ch(U(\underline{g})/I(w.\lambda)) = Ch\ (H_w^{(\lambda,\mu)})\ (= G\ x^B\ V\ (L(w^{-1}.\mu))),$$

the corresponding equation for characteristic cycles (or equality for characteristic classes, Cor. 5.5.2) is only true up to a proportionality factor, as is seen by combining theorems 5.11 and 5.12. In fact, the factor is the ratio of two Goldie-ranks and is a function of λ,μ, and w in the left cell. We shall analyze the "Goldie rank functions" in the last sections of this chapter (5.15-5.18). But before doing this, let us first make some remarkable applications of theorems 5.11 resp. 5.12.

5.13 Irreducibility of Joseph's Weyl group representations

Theorem 1: Each polynomial P_w $(w \in W)$ generates an irreducible W submodule of $S(\underline{t})$.

We shall see (in 5.17) that our polynomial P_w is proportional to $\tilde{p}_w$ in Joseph's notation [J3], [J1], so this theorem 1 and also theorem 2 below are just reformulations of theorems of Joseph [J3]. Our point to make here is that theorem 1 is an easy consequence of the result (5.11) underlying our definition of P_w. The argument is the same as in [Ja], 14.10, but we make it explicit here for convenience of the reader:

<u>Proof</u>: We consider the group ring element $\underline{a}_w := \underline{a}\,(L(w^{-1}.\mu)) \in k[W]$, and the degree $a = a(w)$ as defined in 5.7. Then we have

$$\underline{a}_w \frac{1}{a!}\, \mu^a = p(L(w^{-1}.(\mu-\rho)) = P(H_w^{(\lambda,\mu-\rho)}) \in \mathbb{N}P_w \subset kP_w$$

by 5.7d), theorem 5.6, and corollary 5.11, for all $\mu \in \Omega + \rho$. Using that $\Omega + \rho$ is Zariski dense in $\underline{t}^* = \underline{t}$, and that the powers μ^a $(\mu \in \underline{t})$ span $S^a(\underline{t})$, we conclude that the linear operator $\underline{a}_w$ projects all of $S^a(\underline{t})$ onto the line kP_w. (In particular, P_w is an eigenvector.) On the other hand, $S^a(\underline{t})$ splits into a direct sum of irreducible W submodules, each of which is projected into itself by $\underline{a}_w$. So $\underline{a}_w$ must kill all but one irreducible summand, E say. Then $P_w \in \underline{a}_w E \in E$, so P_w generates E. Q.e.d.

<u>Remark</u>: The argument shows simultaneously that E has multiplicity one in $S^a(\underline{t})$ resp. zero in $S^j(\underline{t})$ for $j < a$.

Although we do not use it in the sequel, let us restate here - for the sake of completeness - also Joseph's result about the <u>classification of primitive ideals</u> [J3].

<u>Theorem</u>: <u>If</u> $J_1,\ldots,J_r$ <u>are all primitive ideals of a given central character</u> χ_λ $(\lambda \in \Omega)$ <u>corresponding to a given irreducible</u> W <u>mo-</u>

dule E by theorem 1, then $P_{J_1},\dots,P_{J_r}$ (notation 5.12) form a basis of E.

This is now derived from Vogan's characterization of the order relation of primitive ideals [V1], as exposed e.g. in chapters 7 and 14 of Jantzen's book [Ja]

5.14 Irreducibility of associated varieties of primitive ideals

Theorem 1 [BB1]: Each primitive ideal J with integral central character has an irreducible associated variety $V(U(\underline{g})/J)$.

Since this variety is obviously G invariant, and contained in the cone N of nilpotent elements, it is then necessarily the closure of a single nilpotent orbit, say $\mathcal{O}_X$.

Proof [BBM2], [Gi]: Let $J = I(w.\lambda)$ with $w \in W$, $\lambda \in \Omega$. Then the (left) characteristic class $P(U(\underline{g})/J)$ is proportional to P_w by theorem 5.12, hence generates an irreducible W module by theorem 5.14. On the other hand, we have

$$P(U(\underline{g})/J) = m_1\, Q(K_1)+\dots+m_r\, Q(K_r), \qquad (*)$$

as in 5.5.1, where K_i is an orbital cone bundle of codimension $a = a(w)$ (=deg P_w) in T^*X, and $0 \neq m_i \in \mathbb{N}$ is also the multiplicity of K_i in the characteristic cycle $\underline{\underline{CH}}(U(\underline{g})/J)$ (by 5.5.2), for each $i = 1,\dots,r$. By theorem 1.8, each $Q(K_i)$ generates an irreducible W submodule of $S^a(\underline{t})$ equivalent to Springer's representation ρ_{u_i}, if $\mathcal{O}_{u_i}$ is the nilpotent orbit determined by K_i (notation 1.8), i.e. if $\pi(K_i) = \overline{\mathcal{O}}_{u_i}$. Suppose $\mathcal{O}_{u_i} \neq \mathcal{O}_{u_1}$ for some i. Then the corresponding Springer representations ρ_{u_i}, ρ_{u_1} would be inequivalent by Springer's theory (cf. [BM1]), so they both occur in the cyclic W module generated by(*), contradicting the irreducibility of this module (5.16). Hence the nilpotent orbits $\mathcal{O}_{u_i}$ are all the same $\mathcal{O}_X$. Now by [BB3], 1.9,

$$V(U(\underline{g})/J) = \pi\, Ch(U(\underline{g})/J)^1 = \pi(\bigcup_i K_i) = \overline{\mathcal{O}}_X .$$

Q.e.d.

Note that we have proved simultaneously:

Theorem 2 (Barbasch-Vogan)[BV1],[BV2]): Joseph's Weyl group representation generated by $P(U(\underline{g})/J)$ is equivalent to Springer's Weyl group representation ρ_X corresponding to the dense nilpotent orbit in $V(U(\underline{g})/J)$.

Remark Theorems 1 resp. 2 had first been verfified by the first two authors [BB1] resp. Barbasch-Vogan [BV1], [BV2] using case by case arguments; conceptual proofs were then given by Joseph [J2], Kashiwara-Tanizaki [KT], the first two authors [BB3], resp. by Hotta-Kashiwara [HK]. The argument given in the present subsection appeared independently in Ginsburg's [Gi] and in [BBM2].

5.15 Evaluation of character polynomials

In the notation of 5.7, we define a $W \times W$ matrix of integers by

$$a(w,y) := a_y(L(w.\lambda)) \qquad \text{for all } w,y \in W.$$

By a well known "translation principle" (due to Jantzen), these integers are independent of $\lambda \in \Omega$, as is the degree

$$a := a(w) := \dim \underline{u} - d(L(w.\lambda))$$

in 5.7.

Remark: Let us mention that in the notation of Kazhdan-Lusztig [KL],

$$a(w,y) = (-1)^{l(w)-l(y)} P_{y,w}(1), \tag{1}$$

where the Kazhdan-Lusztig polynomials $P_{y,w}$ are defined by a purely combinatorial recursion formula. We recall that (1) was conjectured by Kazhdan-Lusztig, and proved by Beilinson-Bernstein [BeBe] and Brylinski-Kashiwara [BKa]. As a consequence, these numbers can be effectively calculated on a computer.

Definition: For each $w \in W$, we define polynomial functions p_w resp. $\tilde{p}_w$ on $\underline{t} \times \underline{t}^*$ resp. $\underline{t}$ by

$$p_w(\xi,\eta) = \sum_{y \in W} a(w,y) \langle y(\xi+\rho), \eta+\rho \rangle^a \tag{2}$$

resp.

$$\tilde{p}_w(\xi) = p_w(\xi,0); \tag{3}$$

in this context, we shall identify $\underline{t}$ with $\underline{t}^*$ by the usual scalar product. Then our $p_w, \tilde{p}_w$ coincide with those considered in [J1], 5.1, up to a shift by ρ.

Lemma: For $\xi,\eta \in \underline{t} = \underline{t}^*$, and each $w \in W$ we have

$$p_w(\xi,\eta) = p_{w^{-1}}(\eta,\xi).$$

Proof: This follows from (2) by W invariance and symmetry of the scalar product, using the following property of the coefficients $a(w,y)$:

$$a(w,y) = a(w^{-1},y^{-1}) \qquad \text{for all } w,y \in W.$$

This property follows e.g. by composing the equivalence functors $L^l \longmapsto H \longmapsto L^r$ (notation 5.3), sending $L(w.\lambda)$ to $L(w^{-1}.\lambda)$, and $M(w.\lambda)$ to $M(w^{-1}.\lambda)$, hence identifying $a(w,y)$ with $a(w^{-1},y^{-1})$. Q.e.d.

Proposition: For all $w \in W$, $\lambda,\mu \in \Omega$, $\nu \in \underline{t}^* = \underline{t}$, we have

$$P(H_w^{(\lambda,\mu)})(\nu+\rho) = p(L(w^{-1}.\mu))\ (\nu+\rho) = \frac{1}{a!}\ p_w(\nu,\mu).$$

Proof: The first equation is corollary 5.6, the second is proposition 5.7d), in combination with the lemma. Q.e.d.

<u>Corollary</u>: $p_w(\lambda,\mu)$ <u>is a positive integer for all</u> $\lambda,\mu \in \Omega$.

<u>Proof</u>: They are integers by definition. They are positive, because the character polynomial $p(L(w^{-1}.\mu))$ takes only strictly positive values on the set $\Omega+\rho$ of regular dominant integral weights, by 3.11. Q.e.d.

<u>5.16 Computation of Goldie-ranks</u>

From corollary 5.11 we obtain that for all $\lambda,\mu \in \Omega$

$$\frac{P(H_w^{(\lambda,\mu)})}{\text{rk } H_w^{(\lambda,\mu)}} = P_w = \frac{P(H_w^{(o,o)})}{\text{rk } H_w^{(o,o)}}$$

(From now on, let rk denote always <u>left</u> Goldie rank, to slightly simplify notation.) By proposition 5.15, we get from this equality of polynomials an equality of values, so

$$p_w(\nu,\mu)/\text{rk } H_w^{(\lambda,\mu)} = p_w(\nu,o)/\text{rk } H_w^{(o,o)} \tag{1}$$

for all $\nu \in \underline{t}^*$. Using corollary 5.15, we get at least for all $\nu \in \Omega$

$$\text{rk } H_w^{(\lambda,\mu)} = \text{rk } H_w^{(o.o)} \frac{p_w(\nu,\mu)}{p_w(\nu,o)} . \tag{2}$$

Taking $\nu = 0$, this gives:

<u>Theorem: The left Goldie-rank of a simple Harish-Chandra bimodule is given by</u>

$$\mathrm{rk}\ H_w^{(\lambda,\mu)} = c_w \cdot \tilde{p}_{w^{-1}}(\mu) \qquad (3)$$

<u>for all</u> $\lambda,\mu \in \Omega$, $w \in W$, <u>where</u> c_w <u>is a positive rational constant</u>.

In fact, $c_w = \mathrm{rk}\ H_w^{(o,o)}/p_w(o,o)$. Note that the polynomial thus describing the left Goldie ranks by (3) for given w is uniquely determined by (3), since Ω is Zariski dense in $\underline{t}^*$.

<u>Remark</u>. It follows now from corollary 5.12.2, that also the Goldie ranks of primitive ideals are given by polynomial functions. More precisely,

$$\mathrm{rk}\ U(\underline{g})/I(w.\lambda) = c_v \cdot \tilde{p}_{v^{-1}}(\mu),$$

where $v(= v^{-1})$ is chosen as in 5.12.2. This is a famous result of Joseph [J3], of which we shall obtain a more complete version in 5.18 below.

<u>5.17 Joseph-King factorization of polynomials</u> p_w

From 5.16 (1) we get more generally

$$\mathrm{rk}\ H_w^{(\lambda,\mu)} = c_w(\nu_o) \cdot p_w(\nu_o,\mu), \qquad (1)$$

for any $\nu_o \in \Omega$, where $c_w(\nu_o)$ is a positive constant

(= rk $H_w^{(o,o)}/p_w(\nu_o,o)$), independent of λ,μ. Now we use 5.16 (1) for a second choice of μ, say $\mu' \in \Omega$, to get

$$p_w(\nu,\mu) \;/\; \mathrm{rk}\, H_w^{(\lambda,\mu)} = p_w(\nu,\mu') \;/\; \mathrm{rk}\, H_w^{(\lambda,\mu')} \qquad (2)$$

for all $\nu,\mu,\mu' \in \Omega$. Going with (1) into (2) twice, and cancelling the nonzero factor $c_w(\nu_o)$ on both sides, we obtain

$$p_w(\nu,\mu) \;/\; p_w(\nu_o,\mu) = p_w(\nu,\mu') \;/\; p_w(\nu_o,\mu')$$

for all $\nu,\nu_o,\mu,\mu' \in \Omega$. Using Zariski density of Ω in $\underline{t}^*$, we derive the following remarkable polynomial identity:

<u>Proposition</u>: $p_w(\xi,\eta)\, p_w(\xi',\eta') = p_w(\xi,\eta')\, p_w(\xi',\eta)$.

Taking the special case $\xi' = \eta' = 0$, we obtain the following factorization

<u>Corollary</u> 1 (Joseph-King, cf. [J1], 5.1 [Ki]):

$$p_w(\lambda,\mu) = \frac{p_w(\lambda,o)\cdot\, p_w(o,\mu)}{p_w(o,o)} = c\cdot \tilde{p}_w(\lambda)\, \tilde{p}_{w^{-1}}(\mu).$$

This factorization, up to the constant factor $c = p_w(o,o)^{-1}$, was given in [J1], 5.1; a direct proof by combinatorics of the Kazhdan-Lusztig polynomials seems to be not known. - Now we may rewrite proposition 5.15 as follows:

<u>Corollary</u> 2: $P(H_w^{(\lambda,\mu)})\,(\nu+\rho) = p(L(w^{-1}.\mu))\,(\nu+\rho) = \frac{c}{a!}\, p_{w^{-1}}(\mu)\, \tilde{p}_w(\nu)$.

<u>Corollary</u> 3: <u>The polynomial</u> P_w (5.11) <u>coincides with</u> p_w, <u>upto a positive integer constant factor, and a shift by</u> ρ; <u>more precisely</u>:

$$\tilde{p}_w(\nu) = a!\; \tilde{p}_w(o)\; P_w(\nu+\rho).$$

This formula follows from corollary 2, using corollary 5.11, and theorem 5.16.

<u>Remark</u>: At this point, it becomes clear that our formulation of Joseph's irreducibility theorem (5.13) is equivalent to Joseph's.

<u>Corollary</u> 4: $P_w(\rho) = \frac{1}{a!}$.

<u>Corollary</u> 5 (Joseph): <u>For</u> $w,y \in W$ <u>in the same left cell</u>, $\tilde{p}_w$ <u>is proportional to</u> $\tilde{p}_y$.

In fact, $P_w = P_y$ by corollary 5.11.1, and of course $a(w) = a(y)$, so

$$\tilde{p}_w = \frac{\tilde{p}_w(o)}{\tilde{p}_y(o)} \cdot \tilde{p}_y$$

by corollary 3.

<u>Remark</u>: The converse of this statement is also true cf.[Ja], but we do not reprove this here.

5.18 Goldie-ranks of primitive ideals

We may now use corollary 5.17.3 to reformulate theorem 5.16 as follows:

Proposition: $\operatorname{rk} H_w^{(\lambda,\mu)} / \operatorname{rk} H_w^{(o,o)} = a!\, P_{w^{-1}}(\mu+\rho)$

for all $\lambda,\mu \in \Omega$, $w \in W$.

The significance of this alternative formulation is that $P_{w^{-1}}$ depends only on the left cell of w^{-1} (by cor. 5.12.1), so only on the primitive ideal $J := I(w^{-1},\lambda)$, which is the right annihilator of $H_w^{(\lambda,\mu)}$. Since the same is true for

$$a = a(w) = a(w^{-1}) = \dim X - \tfrac{1}{2}\, d\, (U(\underline{g})/J),$$

let us point out the

Corollary 1: The ratio of left Goldie-ranks of two simple Harish-Chandra bimodules as in the proposition is a function of the right annihilators.

Let us mention that this ratio is also the ratio of the corresponding left characteristic classes (by corollary 5.11), i.e.

$$P(H_w^{(\lambda,\mu)}) / P(H_w^{(o,o)}) = a!\, P_{w^{-1}}(\mu+\rho). \tag{1}$$

<u>Theorem</u> (Joseph [J3]): <u>For all</u> $w \in W$ <u>there is a positive rational constant</u> c_w <u>such that</u>

$$\operatorname{rk} U(\underline{g})/I(w.\lambda) = c_w \, \tilde{p}_w(\lambda),$$

<u>for all</u> $\lambda \in \Omega$.

<u>Proof</u>: We take $v \in W$ in the left cell of w as in 5.12, remarks 1, 2, so $H_v^{(\lambda,\lambda)}$ is isomorphic to the socle of $U(\underline{g})/I(w.\lambda)$, and $v^2 = 1$. The right annihilators of these modules are $I(v^{-1}.\lambda)=I(w.\lambda)$ (by 5.12(2)), so

$$P_w = P_{v^{-1}} = P_v \tag{2}$$

by corollary 5.12.1 and $v^{-1} = v$. Hence 5.12(5) gives us

$$\begin{aligned} &\operatorname{rk} U(\underline{g})/I(w.\lambda)/\operatorname{rk} U(\underline{g})/I(w.o) = \operatorname{rk} H_v^{(\lambda,\lambda)} / \operatorname{rk} H_v^{(o,o)} \\ &= a!\, P_{v^{-1}} (\lambda+\rho) = a!\, P_w(\lambda+\rho), \end{aligned} \tag{3}$$

using $a(v) = a(w) = a$, and the above proposition. Now the theorem follows by using again cor. 5.17.3. Q.e.d.

<u>Corollaries</u>:

$$P(U(\underline{g}) / I(w.\lambda)) = c_w p_w(\lambda)\, P_w, \tag{4}$$

$$\underline{\underline{Ch}}\, (U(g) / I(w.\lambda)) = c'_w \tilde{p}_w(\lambda)\, [G \times^B \underline{\underline{V}}(L(w^{-1}.0))] \tag{5}$$

$$= c''_w\, [G \times^B \underline{\underline{V}}\, (L(w^{-1}.\lambda))] \tag{6}$$

<u>with positive rational factors</u> c_w, c'_w, c''_w independent of λ.

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CORRIGENDA:

p.6, footnote: "of the introduction to chapter 5"

p.125, l.6: after "given by λ " add "(note that our notation is slightly different from loc. cit.)"

p.127,l.-5: delete "which are locally $\underline{b}$-finite"

p.128, bottom: add

"Remark: The $\underline{\underline{D}}_X^{(\lambda,\mu)}$-module $\underline{\underline{H}}_w^{(\lambda,\mu)}$, with global sections $H_w^{(\lambda,\mu)}$ may be described simply as the middle extension of $\underline{H}_{Z_w}^{d_w}(\underline{\underline{L}}_{Z\setminus(\partial Z_w)}^{(\lambda,\mu)})$, from $Z\setminus(\partial Z_w)$ to Z, where d_w is the codimension of the K-orbit Z_w in Z (see BB 3, §2.7 for the case $\lambda=\mu=0$)."

p.129, after l.1: Insert "All the Harish-Chandra bimodules we consider admit integral central character, as in § 5.3."

p.129, l.6: "the following is true:"

p.135, l.5: "Verma module of highest weight ν"

p.135,l.-3: "second equality"

p.136, l.8: "with coefficients in the K-equivariant line bundle"

p.142, l.9: replace "is a" by "; it is a"

p.144, l.-8: after "obviously must be simple" continue "since for H_1, H_2 two sub-bimodules of A with $H_1\cap H_2=0$, the H_i are two-sided ideals of A such that $H_1H_2=0$, hence one of them is 0 because A is prime. So "

p.145, l.-4: Replace "There exists" by "For any $w\in W$ as in the theorem, there exists"