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Group actions on stacks and applications to equivariant string topology for stacks

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Abstract

This paper is a continuations of the project initiated in [BGNX]. We construct string operations on the S^1-equivariant homology of the (free) loop space $L\mathfrak{X}$ of an oriented differentiable stack $\mathfrak{X}$ and show that $H^{S^1}_{*+\dim\mathfrak{X}-2}(L\mathfrak{X})$ is a graded Lie algebra. In the particular case where $\mathfrak{X}$ is a 2-dimensional orbifold we give a Goldman-type description for the string bracket. To prove these results, we develop a machinery of (weak) group actions on topological stacks which should be of independent interest. We explicitly construct the quotient stack of a group acting on a stack and show that it is a topological stack. Then use its homotopy type to define equivariant (co)homology for stacks, transfer maps, and so on.

Contents

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Introduction

One of the original motivations for the fundamental work of Chas-Sullivan in String Topology was to study the S^1-equivariant homology of the (free) loop space $LM = \mathrm{Map}(S^1, M)$ of a closed oriented manifold M. In particular, they showed in [CS] that this homology has a natural Lie algebra structure which generalizes the classical Goldman bracket [Go] on (free homotopy classes of) loops on an oriented surface. The main motivation of our paper is to construct a similar Lie algebra structure for 2-dimensional orbifolds. Indeed, we construct a generalization of the Chas-Sullivan *string bracket* for oriented stacks of arbitrary dimension. The following is one of the main results of this paper (see Corollary 9.6 for a more precise statement and Section 8.1 for the definition of the transfer map).

Theorem 0.1 *Let $\mathfrak{X}$ be an oriented Hurewicz stack of dimension d. Let $q : \mathrm{L}\mathfrak{X} \to [S^1\backslash\mathrm{L}\mathfrak{X}]$ be the projection map from the loop stack to its quotient stack by the natural S^1-action. Let $T : H_*^{S^1}(\mathrm{L}\mathfrak{X})[2-d] \to H_*(\mathrm{L}\mathfrak{X})[1-d]$ be the transfer map. Then, for $x, y \in H_*(\mathrm{L}\mathfrak{X})$, the bracket defined by the formula*

$$\{x, y\} := (-1)^{|x|} q\big(T(x) \star T(y)\big)$$

makes the equivariant homology $H_*^{S^1}(\mathrm{L}\mathfrak{X})[2-d]$ into a graded Lie algebra. Furthermore, the transfer map

$$T : H_*^{S^1}(\mathrm{L}\mathfrak{X})[2-d] \to H_*(\mathrm{L}\mathfrak{X})[1-d]$$

is a Lie algebra homomorphism. Here, $H_*(\mathrm{L}\mathfrak{X})[1-d]$ is the Lie algebra structure underlying the **BV***-algebra structure on $H_*(\mathrm{L}\mathfrak{X})$.*

The non-equivariant string topology for manifolds equipped with a G-action (or more generally for differentiable stacks, for instance orbifolds) has been studied by many authors (for example, in [LUX] for finite groups G, in [GrWE, CM] for a Lie group acting trivially on a point, and in our previous work [BGNX] for general oriented differentiable stacks). In [BGNX] we build a general setting allowing us to study string topology for stacks. In particular we define functorial loop stacks $L\mathfrak{X} = Map(S^1, \mathfrak{X})$, which are again topological stacks, and construct functorial S^1-actions on them. Contrary to the case of manifolds, constructing suitable models for mapping stacks is a nontrivial task, as the usual constructions using groupoids are rather complicated and not functorial. That is why we have chosen to work with stacks instead (see [No3]).

In [BGNX], we proved that the appropriately shifted homology of the free loop stack of an oriented stack is a Batalin-Vilkovisky algebra. Thereby, once we have the right tools to deal with the equivariant homology of stacks, it is possible to carry out Chas-Sullivan's original method for constructing the string bracket in the framework of stacks. This was the main motivation for the first part of this paper, in which we study the quotient of a (weak) action of a group G on a (topological or differentiable) stack $\mathfrak{X}$, following the work of [Ro]. This part is of independent interest and is expected to have applications beyond string topology. Our main result in this part is the following (see Section 4.3 and Propositions 4.8, 4.9).

Theorem 0.2 *Let G be a topological (resp., Lie) group acting on a topological (resp., differentiable) stack $\mathfrak{X}$. Then, there is a topological (resp., differentiable) stack $[G\backslash\mathfrak{X}]$ together with a map $\mathfrak{X} \to [G\backslash\mathfrak{X}]$ making $\mathfrak{X}$ into a G-torsor. (Note that we do not need to define $[G\backslash\mathfrak{X}]$ as a 2-stack).*

This result allows us to define the G-equivariant (co)homology of a G-stack $\mathfrak{X}$ as the (co)homology of $[G\backslash\mathfrak{X}]$, in the same way that the G-equivariant (co)homology of a manifold is the (co)homology of the stack $[G\backslash M]$. In particular, we can apply the general machinery of bivariant cohomology for stacks developed in [BGNX], which allows us to construct easily Gysin (or "umkehr") maps for equivariant cohomology (among other applications). In particular, we obtain transfer maps and the long homology exact sequence relating the S^1-equivariant homology of an S^1-stack $\mathfrak{X}$ and its ordinary homology (Sections 8.1, 8.2). This set of tools enable us to perform, in a more or less formal manner, the standard constructions of manifold string topology in the more general setting of oriented stacks.

Plan of the paper

In Section 2 we review some basic results on topological stacks. We recall the notions of a *classifying space* for a topological stack (which enables us to do algebraic topology), a *mapping stack* (which enables us to define functorial loop stacks), and a *bivariant theory* for topological stacks (which allows us to do intersection theory, define Gysin maps, and so on).

Sections 3–6 are devoted to the study of group actions on stacks. In Section 4 we construct the quotient stack $[G\backslash\mathfrak{X}]$ of the action of a topological group G on a topological stack $\mathfrak{X}$ and establish its main properties. We give two explicit constructions for $[G\backslash\mathfrak{X}]$; one in terms of transformation groupoids, and one in terms of torsors. We prove that $[G\backslash\mathfrak{X}]$ is always a topological stack, and that in the differentiable context it is a differentiable stack. In Section 5 we use the results of Section 4 to define the equivariant (co)homology of a G-equivariant stack $\mathfrak{X}$. In Section 6 we focus on the case where G is acting on the mapping stack $\mathrm{Map}(G,\mathfrak{X})$ by left multiplication. In Section 7, we look at the homotopy type of the unparameterized mapping stack $[G\backslash\mathrm{Map}(G,\mathfrak{X})]$. The tools developed in the previous sections are robust enough to allow us to carry forward standard constructions in algebraic topology, such as transfer maps and Gysin spectral sequence, to the stack setting in a more or less straightforward manner. This is discussed is Section 8.

In Section 9 we embark on proving the main result of the paper, namely the existence of a Lie algebra structure on $H_*^{S^1}(\mathrm{L}\mathfrak{X})[2-\dim\mathfrak{X}]$. We illustrate this result by looking at a few examples in Section 9.4. We refer to this Lie algebra structure as the *string algebra* of $\mathfrak{X}$.

Our next goal is to study the case of a 2-dimensional oriented reduced orbifold $\mathfrak{X}$. This is done in Section 11, where we give a Goldman-type description for the Lie bracket of $H_0^{S^1}(\mathrm{L}\mathfrak{X})$. Observe that, as in the case of ordinary surfaces, $H_0^{S^1}(\mathrm{L}\mathfrak{X})$ can be identified with the free module spanned by free homotopy classes of loops on $\mathfrak{X}$ (see Lemma 11.1). In Section 11.2 we give an algorithm for computing the Goldman bracket on a reduced orbifold surface.

The main result that is used to prove that the Goldman-type bracket on $H_0^{S^1}(\mathrm{L}\mathfrak{X})$ coincides with the bracket constructed in Section 9 is the functoriality of the Batalin-Vilkovisky structure on $H_{*+\dim\mathfrak{X}}(\mathrm{L}\mathfrak{X})$ with respect to open embeddings. This is established in Section 10 and is a result which is interesting in its own right.

Further results

In an upcoming paper we will study the Turaev cobracket and the coLie algebra structure on the equivariant homology of the loop stack. We will also investigate the important role played by the ghost loops (the inertia stack).

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1 Notation and conventions

Throughout the notes, by a fibered groupoid we mean a category fibered in groupoids. We often identify a space by the functor it represents and also by the corresponding fibered category. We use the same notation for all.

When dealing with stack, by a fiber product we always mean a 2-fiber product.

We use multiplicative notation for composition of arrows in a groupoid. Namely, given composeable arrows

$$x \xrightarrow{\alpha} y \xrightarrow{\beta} z$$

their composition is denoted $\alpha\beta\colon x \to z$.

If V_* is a graded k-module (or chain complex), we will denote by $V_*[1]$ its suspension, that is the graded k-module given by $(V_*[1])_i := V_{i-1}$.

2 Review of stacks

In this section, we review some basic facts about stacks and fix some notation. For more details on stacks the reader is referred [No1]. For a quick introduction to stacks which is in the spirit of this paper, the reader can consult [No5].

Fix Grothendieck site T with a subcanonical topology (i.e., all representable functors are sheaves). Our favorite Grothendieck sites are Top, the site of all topological spaces (with the open-cover topology), or the site CGTop of compactly generated topological spaces (with the open-cover topology).

A *stack* is a fibered groupoid $\mathfrak{X}$ over the site T satisfying the descent condition ([No5], §1.3). Alternatively, we can use presheaves of groupoids instead of fibered groupoids, however, this is less practical for applications.

Stacks over T form a 2-category St_{T} in which all 2-morphisms are isomorphisms. This is a full subcategory of the 2-category $\mathsf{Fib}_{\mathsf{T}}$ of fibered groupoids over T. An crucial property of the 2-category of fibered groupoids is that it has 2-fiber products. The 2-fiber product is a fiberwise version of the following construction for groupoids.

Let $\mathfrak{X}$, $\mathfrak{Y}$ and $\mathfrak{Z}$ be groupoids and $f\colon \mathfrak{Y} \to \mathfrak{X}$ and $g\colon \mathfrak{Z} \to \mathfrak{X}$ functors. The *2-fiber product* $\mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{Z}$ is the groupoid which is defined as follows:

$$\mathrm{ob}(\mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{Z}) = \{(y, z, \alpha) \mid y \in \mathrm{ob}\mathfrak{Y}, z \in \mathrm{ob}\mathfrak{Z},\ \alpha\colon g(z) \to f(y) \text{ an arrow in } \mathfrak{X}\}$$

$$\mathrm{Mor}_{\mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{Z}}\big((y_1, z_1, \alpha), (y_2, z_2, \beta)\big) = \left\{ \begin{array}{ccc} (u, v) & \mid & u\colon y_1 \to y_2,\ v\colon z_1 \to z_2 \text{ s.t.:} \\ & & \begin{array}{ccc} f(y_1) & \xrightarrow{f(u)} & f(y_2) \\ {\scriptstyle\alpha}\uparrow & \circlearrowleft & \uparrow{\scriptstyle\beta} \\ g(z_1) & \xrightarrow[g(u)]{} & g(z_2) \end{array} \end{array} \right\}$$

The 2-category of stacks is closed under 2-fiber products. Since we wil never use the strict fiber product of groupoids in this paper, we will often refer to the 2-fiber product as fiber product.

To every object T in T we associate a fibered groupoid by applying the Grothendieck construction to te functor it represents. We use the same notation T for this fibered groupoid. This induced a functor from T to the 2-category of stacks over T. This functor is fully faithful thanks to the following lemma. So there is no loss of information in regarding T as a stack over T.

Lemma 2.1 (Yoneda lemma) *Let $\mathfrak{X}$ be a category fibered in groupoids over T, and let T be an object in T. Then, the natural functor*

$$\mathrm{Hom}_{\mathsf{Fib}_{\mathsf{T}}}(T, \mathfrak{X}) \to \mathfrak{X}(T)$$

is an equivalence of groupoids.

A morphism $f \colon \mathfrak{X} \to \mathfrak{Y}$ of fibered groupoids is called an *epimorphism* if for every object T in T, every $y \in \mathfrak{Y}(T)$ can be lifted, up to isomorphism, to some $x \in \mathfrak{X}(T)$, possibly after replacing T with an open cover. For example, in the case where $\mathfrak{X}$ and $\mathfrak{Y}$ are honest topological spaces, $f \colon \mathfrak{X} \to \mathfrak{Y}$ is an epimorphism if and only if it admits local sections.

The inclusion $\mathsf{St}_{\mathsf{T}} \to \mathsf{Fib}_{\mathsf{T}}$ of the 2-category fo stacks in the 2-category of fibered groupoids has a left adjoint which is called the *stackification functor* and is denoted by $\mathfrak{X} \mapsto \mathfrak{X}^+$. There is a natural map $\mathfrak{X} \to \mathfrak{X}^+$ (the unit of adjunction). The naturality means that we have a natural 2-commutative diagram

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "f"] \arrow[d] & \mathfrak{X} \arrow[d] \\ \mathfrak{X}^+ \arrow[r, "f"'] & \mathfrak{Y}^+ \end{tikzcd}$$

(In fact, the stackification functor can be constructed in way that the above diagram is strictly commutative, but we do not really need this here.) The construction of the stackification functor involves taking limits and filtered colimits, hence it commutes with 2-fiber products.

Now let T be a category of topological spaces. A morphism $f \colon \mathfrak{X} \to \mathfrak{Y}$ of stacks over T is called *representable*, if for every morphism $T \to \mathfrak{Y}$ from a topological space T, the fiber product $T \times_{\mathfrak{X}} \mathfrak{Y}$ is equivalent to a topological space.

To any topological groupoid $[R \rightrightarrows X_0]$ in T we can associate its *quotient stack* (see [No1], Definition 3.3, or [No5], §1.5). A stack $\mathfrak{X}$ over T is a **topological stack** if and only if the following two conditions are satisfied [No1], §7):

- The diagonal $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is representable;
- There exists a topological space X (called an *atlas* for $\mathfrak{X}$) and an epimorphism $p \colon X \to \mathfrak{X}$.

The first condition is indeed equivalent to all morphisms $T \to \mathfrak{X}$ from a topological space T to $\mathfrak{X}$ being representable. Given an atlas $X \to \mathfrak{X}$, we obtain a groupoid presentation $[R \rightrightarrows X]$ for $\mathfrak{X}$, where $R = X \times_{\mathfrak{X}} X$ and the source and target maps $s, t \colon R \to X$ are the projection maps.

One can define similarly geometric stacks. For instance, if T is the category of C^∞-manifolds (instead of topological spaces), one obtains the notion of *differentiable stacks*. A manifold has an underlying topological space structure and similarly a differentiable stack has an underlying topological stack structure. Differentiable stacks can be characterized as the topological stacks which can be presented by Lie groupoids, that is, topological groupoids $[X_1 \rightrightarrows X_0]$ such that X_1, X_0 are manifolds, all the structures maps are smooth and, in addition, the source and target maps are subjective submersions.

2.1 Classifying spaces of topological stacks

We recall some basic fact about classifying spaces of topological stacks from [No2].

Let $\mathfrak{X}$ be a topological stack. By a *classifying space* for $\mathfrak{X}$ ([No2], Definition 6.2) we mean a topological space X and a morphism $\varphi \colon X \to \mathfrak{X}$ which is a *universal weak equivalence*. The latter means that, for every map $T \to \mathfrak{X}$ from a topological space T, the base extension $\varphi_T \colon X_T \to T$ is a weak equivalence of topological spaces

Theorem 2.2 ([No2], Theorem 6.3) *Every topological stack $\mathfrak{X}$ admits an atlas $\varphi \colon X \to \mathfrak{X}$ with the following property. For every map $T \to \mathfrak{X}$ from a paracompact topological space T, the base extension $\varphi_T \colon X_T \to T$ is shrinkable map of topological spaces, in the sense that, it admits a a section $s \colon T \to X_t$ and a fiberwise deformation retraction of X_T onto $s(T)$. In particular, $\varphi \colon X \to \mathfrak{X}$ makes X a classifying space for $\mathfrak{X}$.*

The fact that $\varphi \colon X \to \mathfrak{X}$ is universal weak equivalence essentially means that we can identify the homotopy theoretic information in $\mathfrak{X}$ and X via φ.

The classifying space is unique up to a unique (in the weak homotopy category) weak equivalence. In the case where $\mathfrak{X}$ is the quotient stack $[G\backslash M]$ of a group action, it can be shown ([No2], §4.3) that the Borel construction $M \times_G EG$ is a classifying space for $\mathfrak{X}$. Here, EG is the universal G-bundle in the sense of Milnor.

Classifying spaces can be used to define homotopical invariants for topological stack ([No2], §11). For example, to define the relative homology of a pair $\mathfrak{A} \subset \mathfrak{X}$, we choose a classifying space $\varphi \colon X \to \mathfrak{X}$ and define $H_*(\mathfrak{X}, \mathfrak{A}) := H_*(X, \varphi^{-1}\mathfrak{A})$. The fact that φ is a universal weak equivalence guarantees that this is well defined up to a canonical isomorphism. In the case where $\mathfrak{X}$ is the quotient stack $[G\backslash M]$ of a group action and $\mathfrak{A} = [G\backslash A]$ is the quotient of a G-equivariant subset A of M, this gives us the G-equivariant homology of the pair (M, A) (as defined via the Borel construction).

2.2 Bivariant theory for stacks

In [BGNX] we showed that the (singular) (co)homology for topological stacks extends to a (generalized) *bivariant theory* à la Fulton-Mac Pherson [FM]. In fact, in [BGNX], we associate, to any map $f : \mathfrak{X} \to \mathfrak{Y}$ of stacks, graded k-modules $H^\bullet(\mathfrak{X} \stackrel{f}{\to} \mathfrak{Y})$ (called the bivariant homology group of f) such that $H^\bullet(X \stackrel{id}{\to} \mathfrak{X})$ is the singular cohomology of $\mathfrak{X}$ and similarly, the homology groups of $\mathfrak{X}$ are given by $H_n(\mathfrak{X}) \cong H^{-n}(\mathfrak{X} \to pt)$.

This bivariant theory is endowed with three kinds of operations:

- *(composition)* or products generalizing the cup-product;
- *(pushforward)* generalizing the homology pushforward;
- *(pullback)* generalizing the pullback maps in cohomology.

These operations satisfy various compatibilities and allow us to build Poincaré duality and Gysin and transfer homomorphisms easily.

2.3 Mapping stacks

We begin by recall the definition and the main properties of mapping stacks. For more details see [No3].

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be stacks over T. We define the stack $\mathrm{Map}(\mathfrak{Y}, \mathfrak{X})$, called the **mapping stack** from $\mathfrak{Y}$ to $\mathfrak{X}$, by the defining its groupoid of section over $T \in \mathsf{T}$ to be $\mathrm{Hom}(\mathfrak{Y} \times T, \mathfrak{X})$,

$$\mathrm{Map}(\mathfrak{Y}, \mathfrak{X})(T) = \mathrm{Hom}(\mathfrak{Y} \times T, \mathfrak{X}),$$

where Hom denotes the groupoid of stack morphisms. This is easily seen to be a stack.

We have a natural equivalence of groupoids

$$\mathrm{Map}(\mathfrak{Y}, \mathfrak{X})(*) \cong \mathrm{Hom}(\mathfrak{Y}, \mathfrak{X}),$$

where $*$ is a point. In particular, the underlying set of the coarse moduli space of $\mathrm{Map}(\mathfrak{Y}, \mathfrak{X})$ is the set of 2-isomorphism classes of morphisms from $\mathfrak{Y}$ to $\mathfrak{X}$.

If $\mathsf{T} = \mathsf{CGTop}$, it follows from the exponential law for mapping spaces that when X and Y are spaces, then $\mathrm{Map}(Y, X)$ is representable by the usual mapping space from Y to X (endowed with the compact-open topology).

The mapping stacks are functorial in both variables.

Lemma 2.3 *The mapping stacks* $\mathrm{Map}(\mathfrak{Y}, \mathfrak{X})$ *are functorial in* $\mathfrak{X}$ *and* $\mathfrak{Y}$*. That is, we have natural functors* $\mathrm{Map}(\mathfrak{Y}, -)\colon \mathsf{St} \to \mathsf{St}$ *and* $\mathrm{Map}(-, \mathfrak{X})\colon \mathsf{St}^{op} \to \mathsf{St}$*. Here,* St *stands for the 2-category of stacks over* T *and* St^{op} *is the opposite category (obtained by inverting the direction of 1-morphisms in* St*).*

The exponential law holds for mapping stacks.

Lemma 2.4 *For stacks $\mathfrak{X}$, $\mathfrak{Y}$ and $\mathfrak{Z}$ we have a natural equivalence of stacks*

$$\mathrm{Map}(\mathfrak{Z}\times\mathfrak{Y},\mathfrak{X}) \cong \mathrm{Map}\big(\mathfrak{Z},\mathrm{Map}(\mathfrak{Y},\mathfrak{X})\big).$$

For the following theorem to be true we need to assume that $\mathsf{T} = \mathsf{CGTop}$.

Theorem 2.5 ([No3], Theorem 4.2) *Let $\mathfrak{X}$ and $\mathfrak{K}$ be topological stacks. Assume that $\mathfrak{K} \cong [K_0/K_1]$, where $[K_1 \rightrightarrows K_0]$ is a topological groupoid with K_0 and K_1 compact. Then, $\mathrm{Map}(\mathfrak{K},\mathfrak{X})$ is a topological stack.*

We define the *free loop stack* of a stack $\mathfrak{X}$ to be $L\mathfrak{X} := \mathrm{Map}(S^1,\mathfrak{X})$. If $\mathfrak{X}$ is a topological stack, then it follows from the above theorem that $L\mathfrak{X}$ is also a topological stack.

Theorem 2.5 does not seem to be true without the compactness condition on the $\mathfrak{K}$. However, it is good to keep in mind the following general fact ([No3], Lemma 4.1).

Proposition 2.6 *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be topological stacks. Then, for every topological space T, every morphism $T \to \mathrm{Map}(\mathfrak{Y},\mathfrak{X})$ is representable. (Equivalently, $\mathrm{Map}(\mathfrak{Y},\mathfrak{X})$ has a representable diagonal.)*

The following result is useful in computing homotopy types of mapping stacks.

Theorem 2.7 ([No3], Corollary 6.5) *Let Y be a paracompact topological space and $\mathfrak{X}$ a topological stack. Let X be a classifying space for $\mathfrak{X}$ with $\varphi\colon X \to \mathfrak{X}$ as in Theorem 2.2. Then, the induced map $\mathrm{Map}(Y,X) \to \mathrm{Map}(Y,\mathfrak{X})$ makes $\mathrm{Map}(Y,X)$ a classifying space for $\mathrm{Map}(Y,\mathfrak{X})$ (in particular, it is a universal weak equivalence).*

Corollary 2.8 ([No3], Corollary 6.6) *Let $\mathfrak{X}$ be a topological stack and X a classifying space for it, with $\varphi\colon X \to \mathfrak{X}$ as in Theorem 2.2. Then, the induced map $L\varphi\colon LX \to \mathrm{L}\mathfrak{X}$ on free loop spaces makes LX a classifying space for $\mathrm{L}\mathfrak{X}$ (in particular, $L\varphi$ is a universal weak equivalence).*

3 Group actions on stacks

3.1 Definition of a group action

In this subsection we recall the definition of a weak group action on a groupoid from [Ro]. This definition is more general than what is needed for our application (§6.5) as in our case the action will be strict (i.e., the transformations α and $\mathfrak{a}$ in Definition 3.1 will be identity transformations).

Let $\mathfrak{X}$ be a fibered groupoid over T and G a group over T. (More generally, we can take $\mathfrak{X}$ to be an object and G a strict monoid object in any fixed 2-category.)

Definition 3.1 ([Ro], 1.3(i)) A left **action** of G on $\mathfrak{X}$ is a triple $(\mu, \alpha, \mathfrak{a})$ where $\mu\colon G \times \mathfrak{X} \to \mathfrak{X}$ is a morphism (of fibered groupoids), and α and $\mathfrak{a}$ are 2-morphisms as in the diagrams

$$\begin{tikzcd}
G \times G \times \mathfrak{X} \arrow[r, "m\times \mathrm{id}_{\mathfrak{X}}"] \arrow[d, "\mathrm{id}_G \times \mu"'] \arrow[dr, phantom, "\Nearrow \alpha"] & G \times \mathfrak{X} \arrow[d, "\mu"] \\
G \times \mathfrak{X} \arrow[r, "\mu"'] & \mathfrak{X}
\end{tikzcd}
\qquad
\begin{tikzcd}
G \times \mathfrak{X} \arrow[r, "\mu"] & \mathfrak{X} \\
\mathfrak{X} \arrow[u, "1\times \mathrm{id}_{\mathfrak{X}}"] \arrow[ur, "\mathrm{id}_{\mathfrak{X}}"'] \arrow[ur, phantom, shift left=3, "\Nwarrow \mathfrak{a}"]
\end{tikzcd}$$

We require the following equalitties:

A1) $(g \cdot \alpha^x_{h,k})\alpha^x_{g,hk} = \alpha^{k\cdot x}_{g,h}\alpha^x_{gh,k}$, for all g, h, k in G and x an object in $\mathfrak{X}$.

A2) $(g \cdot \mathfrak{a}^x)\alpha^x_{g,1} = 1_{g\cdot x} = \mathfrak{a}^{g\cdot x}\alpha^x_{1,g}$, for every g in G and x an object in $\mathfrak{X}$.

The dot in the above formulas is a short for the multiplication μ. Also, $\alpha^x_{g,h}$ stands for the arrow $\alpha(x, g, h)\colon g \cdot (h \cdot x) \to (gh) \cdot x$ in $\mathfrak{X}$ and $\mathfrak{a}^x$ for the arrow $\mathfrak{a}(x)\colon x \to 1 \cdot x$.

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be fibered groupoids over T endowed with an action of G as in definition 3.1.

Definition 3.2 ([Ro], 1.3(ii)) A G-**equivariant** morphism between $(\mathfrak{X}, \mu, \alpha, \mathfrak{a})$ and $(\mathfrak{Y}, \nu, \beta, \mathfrak{b})$ is a morphism $F\colon \mathfrak{X} \to \mathfrak{Y}$ together with a 2-morphism σ as in the diagram

$$\begin{tikzcd}
G \times \mathfrak{X} \arrow[r, "\mu"] \arrow[d, "\mathrm{id}_G \times F"'] \arrow[dr, phantom, "\Swarrow \sigma"] & \mathfrak{X} \arrow[d, "F"] \\
G \times \mathfrak{Y} \arrow[r, "\nu"'] & \mathfrak{Y}
\end{tikzcd}$$

such that

B1) $\sigma^{h\cdot x}_g (g \cdot \sigma^x_h)\beta^{F(x)}_{g,h} = F(\alpha^x_{g,h})\sigma^x_{gh}$, for every g, h in G and x an object in $\mathfrak{X}$.

B2) $F(\mathfrak{a}^x)\sigma^x_1 = \mathfrak{b}^{F(x)}$, for every object x in $\mathfrak{X}$.

Here, σ^x_g stands for the arrow $\sigma(x, g)\colon F(g \cdot x) \to g \cdot F(x)$ in $\mathfrak{Y}$. We often drop σ from the notation and denote such a morphism simply by F.

Definition 3.3 Let (F, σ) and (F', σ') be G-equivariant morphisms from $\mathfrak{X}$ to $\mathfrak{Y}$ as in Definition 3.2. A G**-equivariant 2-morphism** from (F, σ) to (F', σ') is a 2-morphism $\varphi\colon F \Rightarrow F'$ such that

C1) $(\sigma^x_g)(g \cdot \varphi_x) = (\varphi_{g\cdot x})({\sigma'}^x_g)$, for every g in G and x an object in $\mathfrak{X}$.

Here, $\varphi_x\colon F(x) \to F'(x)$ stands for the effect of φ on $x \in \mathrm{ob}\mathfrak{X}$.

3.2 Transformation groupoid of a group action

Suppose now that G is a discrete group and $\mathfrak{X}$ a groupoid (i.e., the base category T is just a point). Given a group action $\mu\colon G \times \mathfrak{X} \to \mathfrak{X}$ as in Definition 3.1, we define the transformation groupoid $[G\backslash\mathfrak{X}]$ as follows. The of objects of $[G\backslash\mathfrak{X}]$ are the same as those of $\mathfrak{X}$,

$$\mathrm{ob}[G\backslash\mathfrak{X}] = \mathrm{ob}\mathfrak{X}.$$

The morphism of $[G\backslash\mathfrak{X}]$ are

$$\mathrm{Mor}[G\backslash\mathfrak{X}] = \{(\gamma, g, x) \mid y \in \mathrm{ob}\mathfrak{X},\ g \in G,\ \gamma \in \mathrm{Mor}\,\mathfrak{X},\ t(\gamma) = g \cdot y\}.$$

We visualize the arrow (γ, g, x) as follows:

$$\begin{tikzcd} & y \\ x \arrow[r, "\gamma"] & g \cdot y \arrow[u, dotted, "g"'] \end{tikzcd}$$

The source and target maps are defined by

$$s(\gamma, g, y) = s(\gamma) = x \text{ and } \ t(\gamma, g, y) = y.$$

The composition of arrows is defined by

$$(\gamma, g, y)(\delta, h, z) = \big(\gamma(g \cdot \delta)\alpha^z_{g,h},\ gh,\ z\big).$$

The identity morphism of an object $x \in \mathrm{ob}[G\backslash\mathfrak{X}] = \mathrm{ob}\mathfrak{X}$ is

$$(\mathfrak{a}^x, 1, x).$$

Picturally, this is

$$\begin{tikzcd} & x \\ x \arrow[r, "\mathfrak{a}^x"] & 1 \cdot x \arrow[u, dotted, "1"'] \end{tikzcd}$$

Finally, the inverse of an arrow (γ, g, y) in $[G\backslash\mathfrak{X}]$ is given by

$$\big(\mathfrak{a}^y(\alpha^y_{g^{-1},g})^{-1}(g^{-1} \cdot \gamma^{-1}),\ g^{-1},\ x\big),$$

where $x = s(\gamma)$.

It follows from the axioms (A1) and (A2) of Definition 3.1 that the above definition makes $[G\backslash\mathfrak{X}]$ into a groupoid. In fact, axioms (A1) and (A2) are equivalent to $[G\backslash\mathfrak{X}]$ being a groupoid.

There is a natural functor $q\colon \mathfrak{X} \to [G\backslash\mathfrak{X}]$ which is the identity on the objects, that is, $q(x) = x$. On arrows it is defined by

$$q(\gamma) = \big(\gamma\mathfrak{a}^y,\ 1,\ y\big),$$

where $y = t(\gamma)$. Pictorially, this is

$$\begin{tikzcd} & & y \\ x \arrow[r, "\gamma"] & y \arrow[r, "\mathfrak{a}^y"] & 1 \cdot y \arrow[u, dotted, "1"'] \end{tikzcd}$$

The functor q is faithful.

Since $q\colon \mathfrak{X} \to [G\backslash\mathfrak{X}]$ is faithful, we can regard $\mathfrak{X}$ as a subcategory of $[G\backslash\mathfrak{X}]$. For this reason, we will often denote $q(\gamma)$ simply by γ, if there is no fear of confusion. We also use the short hand notation g^x for the arrow $(1_{x\cdot g},\, g,\, x)$. This way, we can write $(\gamma, g, x) = \gamma g^x$.

The groupoid $[G\backslash\mathfrak{X}]$ can be defined, alternatively, as the groupoid generated by $\mathfrak{X}$ and the additional arrows g^x subject to certain relations which we will not spell out here. It is important, however, to observe the following commutativity relation $g^x\gamma = (\gamma \cdot g)g^y$, as in the following commutative diagram:

$$\begin{tikzcd} x \arrow[r, "\gamma"] & y \\ g \cdot x \arrow[u, "g^x"] \arrow[r, "g\cdot\gamma"'] & g \cdot y \arrow[u, "g^y"'] \end{tikzcd}$$

Yet another way to define the groupoid $[G\backslash\mathfrak{X}]$ is to define it as the groupoid of trivialized G-torsors P, endowed with a G-equivariant map $\chi\colon P \to \mathfrak{X}$ which satisfies $\chi(g) = g \cdot \chi(1)$, for every $g \in P$. Here, by a trivialized G-torsors P we mean $P = G$ viewed as a G-torsor via left multiplication.

This definition gives a groupoid that is isomorphic to the one defined above. It also explains our rather unnatural looking convention of having the arrow g^x go from $g \cdot x$ to x rather than other way around. If we drop the extra condition $\chi(g) = g \cdot \chi(1)$ in the definition, we get a groupoid which is naturally equivalent (but not isomorphic) to $[G\backslash\mathfrak{X}]$. For more on torsors see §4.2.

Remark 3.4 In the case where the action of G on $\mathfrak{X}$ is strict, an arrow (γ, g, x) in $[G\backslash\mathfrak{X}]$ is uniquely determined by (γ, g), i.e., x is redundant. When $\mathfrak{X}$ is a set, $[G\backslash\mathfrak{X}]$ is equal to the usual transformation groupoid of the action of a group on a set.

Example 3.5 Let $\mathfrak{X}$ be a groupoid with one object, and let H be its group of morphisms. Suppose that we are given a strict action of a group G on $\mathfrak{X}$ (this amounts to an action of G on H by homomorphisms). Then, $[G\backslash\mathfrak{X}]$ is the groupoid with one object whose group of morphisms is $H \rtimes G$. In other words, $[G\backslash BH] = B(H \rtimes G)$.

Given a G-equivariant morphism F as in Definition 3.2, we obtain a functor

$$[F]\colon [G\backslash\mathfrak{X}] \to [G\backslash\mathfrak{Y}]$$

as follows. The effect of $[F]$ on objects is the same as that of F, i.e., $[F](x) := F(x)$. For a morphism (γ, g, y) in $[G\backslash\mathfrak{X}]$ we define

$$[F](\gamma, g, y) := \big(F(\gamma)\sigma_g^y,\ g,\ F(y)\big).$$

It follows from the axioms (B1) and (B2) of Definition 3.2 that $[F]$ is a functor. In fact, axioms (B1) and (B2) are equivalent to $[F]$ being a functor. Furthermore, the diagram

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "F"] \arrow[d, "q_{\mathfrak{X}}"'] & \mathfrak{Y} \arrow[d, "q_{\mathfrak{Y}}"] \\ {[G\backslash\mathfrak{X}]} \arrow[r, "{[F]}"'] & {[G\backslash\mathfrak{Y}]} \end{tikzcd}$$

is 2-cartesian and strictly commutative.

Given a G-equivariant 2-morphism φ as in Definition 3.3, we obtain a natural transformation of functors $[\varphi]\colon [F] \Rightarrow [F']$ whose effect on $x \in \mathrm{ob}\mathfrak{X}$ is defined by $[\varphi](x) := \varphi(x)\colon F(x) \to F'(x)$. It follows from the axiom (C1) of Definition 3.3 that $[\varphi]$ is a natural transformation of functors. In fact, axiom (C1) is equivalent to $[\varphi]$ being a natural transformation of functors.

3.3 The main property of the transformation groupoid

The most important property of the transformation groupoids for us is the fact that the diagram

$$\begin{tikzcd} G \times \mathfrak{X} \arrow[r, "\mu"] \arrow[d, "\mathrm{pr}_2"'] & \mathfrak{X} \arrow[d, "q"] \\ \mathfrak{X} \arrow[r, "q"'] & {[G\backslash\mathfrak{X}]} \end{tikzcd}$$

is 2-cartesian. In other words, the functor

$$(\mathrm{pr}_2, \mu)\colon G \times \mathfrak{X} \to \mathfrak{X} \times_{[G\backslash\mathfrak{X}]} \mathfrak{X}$$

is an equivalence of groupoids. This is an easy verification and we leave it to the reader.

Lemma 3.6 *Let T be a set (viewed as a groupoid with only identity morphisms) and $f\colon T \to [G\backslash\mathfrak{X}]$ a functor. Then, the groupoid $T \times_{[G\backslash\mathfrak{X}]} \mathfrak{X}$ is equivalent to a set. If we denote the set of isomorphism classes of $T \times_{[G\backslash\mathfrak{X}]} \mathfrak{X}$ by P, then the natural left G-action on P (induced from the action of G on the second factor of fiber product) makes P a left G-torsor.*

Proof. This is a simple exercise (e.g., using the above 2-cartesian square). $\square$

4 Quotient stack of a group action

In this section we study the global version of the construction of the transformation groupoid introduced in §3.2 and use it to define the quotient stack of a weak group action on a stack. We fix a Grothendieck site T through this section. The reader may assume that T is the site Top of all topological spaces, or the site CGTop of compactly generated topological spaces.

4.1 Definition of the quotient stack

Let $\mathfrak{X}$ be a fibered groupoid over T and G a presheaf of groupoids over T viewed as a fibered groupoid). Suppose that we have a right action $\mu\colon G \times \mathfrak{X} \to \mathfrak{X}$ of G on $\mathfrak{X}$ as in Definition 3.1. Repeating the construction of the transformation groupoid as in §3.2, we obtain a a category fibered in groupoids $\lfloor G\backslash\mathfrak{X}\rfloor$. (The reason for not using the square brackets becomes clear shortly.) In terms of section, $\lfloor G\backslash\mathfrak{X}\rfloor$ is determined by the following property:

$$\lfloor G\backslash\mathfrak{X}\rfloor(T) = [G(T)\backslash\mathfrak{X}(T)], \text{ for every } T \in \mathsf{T}.$$

The following lemma is straightforward.

Lemma 4.1 *Notation being as above, if $\mathfrak{X}$ is a prestack and G is a sheaf of groups, then $\lfloor G\backslash\mathfrak{X}\rfloor$ is a prestack.*

If in the above lemma $\mathfrak{X}$ is a stack, it is not necessarily true that $\lfloor G\backslash\mathfrak{X}\rfloor$ is a stack (this is already evident in the case where $\mathfrak{X}$ is a sheaf of sets). Therefore, we make the following definition.

Definition 4.2 Let $\mathfrak{X}$ be a stack and G acting on $\mathfrak{X}$ (Definition 3.1). We define $[G\backslash\mathfrak{X}]$ to be the stackification of the prestack $\lfloor G\backslash\mathfrak{X}\rfloor$.

There is a natural epimorphisms of stacks $q\colon \mathfrak{X} \to [G\backslash\mathfrak{X}]$. This morphism is strictly functorial, in the sense that, for every G-equivariant morphism (Definition 3.2) $F\colon \mathfrak{X} \to \mathfrak{Y}$ of G-stacks, there is a natural induced morphism $[F]\colon [G\backslash\mathfrak{X}] \to [G\backslash\mathfrak{Y}]$ of stack such that the diagram

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "F"] \arrow[d, "q_{\mathfrak{X}}"'] & \mathfrak{Y} \arrow[d, "q_{\mathfrak{Y}}"] \\ {[G\backslash\mathfrak{X}]} \arrow[r, "{[F]}"'] & {[G\backslash\mathfrak{Y}]} \end{tikzcd} \tag{4.1}$$

is 2-cartesian and strictly commutative. This follows from the corresponding statement in the discrete case (see end of §3.2) and the similar properties of the stackification functor §2. Similarly, given a G-equivariant 2-morphism $\varphi\colon F \Rightarrow F'$, we obtain a 2-morphism $[\varphi]\colon [F] \Rightarrow [F']$.

Since the stackification functor commutes with 2-fiber products, we have a 2-cartesian square (see §3.3)

$$\begin{tikzcd} G \times \mathfrak{X} \arrow[r, "\mu"] \arrow[d, "\mathrm{pr}_2"'] & \mathfrak{X} \arrow[d, "q"] \\ \mathfrak{X} \arrow[r, "q"'] & [G\backslash\mathfrak{X}] \end{tikzcd}$$

and the functor

$$(\mathrm{pr}_2, \mu)\colon \mathfrak{X} \times G \to \mathfrak{X} \times_{[G\backslash\mathfrak{X}]} \mathfrak{X}$$

is an equivalence of stacks. Here $\mu\colon G \times \mathfrak{X} \to \mathfrak{X}$ stands for the action of G on $\mathfrak{X}$.

Lemma 4.3 *Let $\mathfrak{X}$ be a stack with a group action as above. Let T be a sheaf of sets (viewed as a fibered groupoid over T) and $f\colon T \to [G\backslash\mathfrak{X}]$ a morphism. Then, the stack $T \times_{[G\backslash\mathfrak{X}]} \mathfrak{X}$ is equivalent to the sheaf of sets P, where P is the sheaf of isomorphism classes of $T \times_{[G\backslash\mathfrak{X}]} \mathfrak{X}$. Furthermore, the natural left G-action on P (induced from the action of G on the second factor of fiber product) makes P a left G-torsor.*

Proof. It follows from Lemma 3.6 that $T \times_{[G\backslash\mathfrak{X}]} \mathfrak{X}$ is (equivalent to) a presheaf of sets, namely P. On the other hand, since stacks are closed under fiber product, $T \times_{[G\backslash\mathfrak{X}]} \mathfrak{X}$ is a stack. Hence, it is (equivalent to) a sheaf of sets. Thus, P is indeed a sheaf. It follows from Lemma 3.6 that P is a left G-torsor. $\square$

4.2 Interpretation in terms of torsors

Let $\mathfrak{X}$ be a stack with an action of a sheaf of groups G. Let T be an object in T. We define the groupoid $\mathfrak{Prin}_{G,\mathfrak{X}}(T)$ as follows.

$$\mathrm{ob}\mathfrak{Prin}_{G,\mathfrak{X}}(T) = \left\{ \begin{array}{ccl} (P,\chi) & | & P \to T \ \text{ left } G\text{-torsor}, \\ & & \chi\colon P \to \mathfrak{X} \ \ G\text{-equivariant map} \end{array} \right\}$$

$$\mathrm{Mor}_{\mathfrak{Prin}_{G,\mathfrak{X}}(T)}\big((P,\chi),(P',\chi')\big) = \left\{ \begin{array}{ccl} (u,\phi) & | & u\colon P \to P' \ \text{map of } G\text{-torsors}, \\ & & \phi\colon \chi \Rightarrow \chi' \circ u \ \ G\text{-equivariant} \end{array} \right\}$$

The groupoid $\mathfrak{Prin}_{G,\mathfrak{X}}(T)$ contains a full subgroupoid $\mathfrak{TrivPrin}_{G,\mathfrak{X}}(T)$ consisting of those pairs (P,χ) such that P admits a section (i.e., is isomorphic to the trivial torsor).

We can enhance the above construction to a fibered groupoid $\mathfrak{Prin}_{G,\mathfrak{X}}$ over T. In fact, $\mathfrak{Prin}_{G,\mathfrak{X}}$ is a stack over T. The stack $\mathfrak{Prin}_{G,\mathfrak{X}}$ contains $\mathfrak{TrivPrin}_{G,\mathfrak{X}}$ as a full subprestack. Furthermore, since every G-torsor is locally trivial, $\mathfrak{Prin}_{G,\mathfrak{X}}$ is (equivalent to) the stackification of $\mathfrak{TrivPrin}_{G,\mathfrak{X}}$.

We define a morphism of prestacks

$$F_{pre}\colon \lfloor G\backslash\mathfrak{X} \rfloor \to \mathfrak{Prin}_{G,\mathfrak{X}}$$

as follows (see §4.1 for the definition of $\lfloor G\backslash\mathfrak{X}\rfloor$). For $T\in\mathsf{T}$, an object $x\in\lfloor G\backslash\mathfrak{X}\rfloor(T)$ is, by definition, the same as an object in $\mathfrak{X}(T)$. This, by Yoneda, gives a map $\mathrm{f}_x\colon T\to\mathfrak{X}$. Define $F(x)$ to be the pair $(G\times T,\chi_x)$, where $G\times T$ is viewed as a trivial G-torsor over T, and $\chi_x:=\mu\circ(\mathrm{id}_G\times f_x)$, as in the diagram

$$G\times T\xrightarrow{\mathrm{id}_G\times f_x}G\times\mathfrak{X}\xrightarrow{\mu}\mathfrak{X}.$$

(Note that producing f_x from x involves making choices, so our functor F_{pre} depends on all these choices.) Symbolically, χ_x can be written as $\chi_x\colon h\mapsto h\cdot x$, where h is an element of G (over T).

The effect of F_{pre} on arrows is defined as follows. Given an arrow (γ,g,y)

$$\begin{tikzcd} & y \\ x \arrow[r, "\gamma"] & g\cdot y \arrow[u, dotted, "g"'] \end{tikzcd}$$

in $\lfloor G\backslash\mathfrak{X}\rfloor(T)$ from x to y, we define $F_{pre}(\gamma,g,y)$ to be the pair (m_g,ϕ), where $m_g\colon G\times T\to G\times T$ is right multiplication by g (on the first factor), and $\phi\colon\chi_x\Rightarrow\chi_y\circ u$ is the composition

$$\chi_x\xRightarrow{\mu\circ(\mathrm{id}_G\times f_\gamma)}\chi_{g\cdot y}\xRightarrow{\alpha^y_{-,g}}\chi_y\circ m_g.$$

It is not hard to see that F is fully faithful and it lands in $\mathfrak{TrivPrin}_{G,\mathfrak{X}}$, hence, after stackification, we obtain an equivalence of stacks

$$F\colon[G\backslash\mathfrak{X}]\xrightarrow{\sim}\mathfrak{Prin}_{G,\mathfrak{X}}.$$

There is an alternative description of F in terms of pullback torsors which is more geometric. For any $x\in[G\backslash\mathfrak{X}](T)$, let $f_x\colon T\to[G\backslash\mathfrak{X}]$ be the morphism obtained from Yoneda, and form the following fiber square

$$\begin{tikzcd} P \arrow[r, "\chi"] \arrow[d, "p_1"'] & \mathfrak{X} \arrow[d, "q"] \\ T \arrow[r, "f_x"'] & {[G\backslash\mathfrak{X}]} \end{tikzcd}$$

Here, P is the sheaf of set obtained from $T\times_{[G\backslash\mathfrak{X}]}\mathfrak{X}$ by contracting each isomorphism class to a point, as in Lemma 4.3. The maps p_1 and χ are obtained from the first and the second projection maps, respectively, by choosing an inverse equivalence to the projection $T\times_{[G\backslash\mathfrak{X}]}\mathfrak{X}\to P$. There is an obvious left action of G on $T\times_{[G\backslash\mathfrak{X}]}\mathfrak{X}$ in which G acts on the second factor (so the projection $\mathrm{pr}_2\colon T\times_{[G\backslash\mathfrak{X}]}\mathfrak{X}\to\mathfrak{X}$ is strictly G-equivariant). This induces a G-action on P such that χ is G-equivariant (not necessarily strictly anymore).

Sending x to the pair (P, χ) gives rise to a morphism of stacks

$$F' \colon [G\backslash\mathfrak{X}] \to \mathfrak{Prin}_{G,\mathfrak{X}}.$$

The effect of F on arrows is defined in the obvious way. The morphism F' is canonically 2-isomorphic to F (hence is an equivalence of stacks). What the functor F' says is that the pair $(\mathfrak{X}, \mathrm{id})$ is a universal pair with $\mathfrak{X} \to [G\backslash\mathfrak{X}]$ a "G-torsor" and $\mathrm{id}\colon \mathfrak{X} \to \mathfrak{X}$ a G-equivariant map.

There is a natural inverse morphism of stacks

$$Q \colon \mathfrak{Prin}_{G,\mathfrak{X}} \xrightarrow{\sim} [G\backslash\mathfrak{X}]$$

for F (or F') which is defined as follows. Let (P, χ) be on object in $\mathfrak{Prin}_{G,\mathfrak{X}}(T)$. The G-equivariant map $\chi\colon P \to \mathfrak{X}$ induced a map $[\chi]\colon [G\backslash P] \to [G\backslash\mathfrak{X}]$ on the quotient stacks (§ 4.1). Since P is a G-torsor, the natural map $[G\backslash P] \to T$ is an equivalence of stacks. *Choose* an inverse $f_P\colon T \to [G\backslash P]$ for it. The composition $[\chi] \circ f_P\colon T \to [G\backslash\mathfrak{X}]$ determines an object in $[G\backslash\mathfrak{X}]$ which we define to be $Q(P, \chi)$. The effect on arrows is defined similarly (for this you do not to make additional choices).

Remark 4.4 As we pointed out above, construction of the morphisms F, F' and Q requires making certain choices. In the case of F and F', the choice involves associating a map $f_x\colon T \to \mathfrak{X}$ to an element $x \in \mathfrak{X}(T)$. The map f_x is unique up to a unique 2-morphism. In the case of Q, the choice involves choosing an inverse for the equivalence of stacks $[G\backslash P] \to T$. Again, such an inverse is unique up to a unique 2-morphism. The conclusion is that the morphisms F, F' and Q are well defined up to a unique 2-morphism. As we pointed out above, F and F' are canonically 2-isomorphic and Q is an inverse equivalence to both.

4.3 Quotients of topological stacks

In this section we assume that our Grothendieck site T is either Top or CGTop. We are particularly interested in the case where the sheaf of groups G indeed comes from a topological group (denoted again by G). We point out that, in this case, the sheaf theoretic notion of a G-torsor used in the previous subsections coincides with the usual one. More precisely, give a a topological space T, a sheaf theoretic G-torsor P over T always comes from a topological space (again denoted by P). The reason for this is that P is locally (on T) of the form $U \times G$, which is indeed a topological space. Gluing these along intersections yields a topological space representing P.

The main result we prove in this subsection is that if G is a topological group and $\mathfrak{X}$ a topological stack, then $[G\backslash\mathfrak{X}]$ is also a topological stack. To prove this we need two lemmas.

Lemma 4.5 *Let $f, g\colon \mathfrak{X} \to \mathfrak{Y}$ be representable morphisms of stacks. Assume further that the diagonal $\Delta\colon \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is representable. Then, $(f, g)\colon \mathfrak{X} \to \mathfrak{Y} \times \mathfrak{Y}$ is representable.*

Proof. We can write (f,g) as a composition of two representable maps $\Delta\colon \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ and $f \times g\colon \mathfrak{X} \times \mathfrak{X} \to \mathfrak{Y} \times \mathfrak{Y}$. $\square$

Lemma 4.6 *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism of stacks and $\mathfrak{Y}' \to \mathfrak{Y}$ an epimorphism of stacks. If the base extension $f'\colon \mathfrak{X}' \to \mathfrak{Y}'$ of f over $\mathfrak{Y}'$ is representable, then so is f itself.*

Proof. This is Lemma 6.3 of [No1]. $\square$

Proposition 4.7 *Let $\mathfrak{X}$ be a topological stack and G a topological group acting on $\mathfrak{X}$. Then the canonical epimorphism $\mathfrak{X} \to [G\backslash\mathfrak{X}]$ is representable.*

Proof. This is Lemma 4.6 applied to the 2-cartesian square

$$\begin{tikzcd}
G \times \mathfrak{X} \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mu"'] & \mathfrak{X} \arrow[d, "q"] \\
\mathfrak{X} \arrow[r, "q"'] & {[G\backslash\mathfrak{X}]}
\end{tikzcd}$$

from § 3.3 since the map $\mathrm{pr}_2\colon G \times \mathfrak{X} \to \mathfrak{X}$ is representable. $\square$

Proposition 4.8 *Let G be a topological group acting on a topological stack $\mathfrak{X}$. Then, the quotient $[G\backslash\mathfrak{X}]$ is also a topological stack.*

Proof. We need to prove two things.

The diagonal $\Delta\colon [G\backslash\mathfrak{X}] \to [G\backslash\mathfrak{X}] \times [G\backslash\mathfrak{X}]$ is representable. To see this, we consider the 2-cartesian diagram

$$\begin{tikzcd}
\mathfrak{X} \times_{[G\backslash\mathfrak{X}]} \mathfrak{X} \arrow[r] \arrow[d, "{(\mathrm{pr}_1,\mathrm{pr}_2)}"'] & {[G\backslash\mathfrak{X}]} \arrow[d, "\Delta"] \\
\mathfrak{X} \times \mathfrak{X} \arrow[r, "q\times q"'] & {[G\backslash\mathfrak{X}] \times [G\backslash\mathfrak{X}]}
\end{tikzcd}$$

Since the map $q \times q$ is an epimorphism, it is enough to prove that the map $(\mathrm{pr}_1, \mathrm{pr}_2)\colon \mathfrak{X} \times_{[G\backslash\mathfrak{X}]} \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is representable. As we saw in § 4.1, this map is equivalent to the map $(\mathrm{pr}_2, \mu)\colon G \times \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$. The map $\mathrm{pr}_2\colon G \times \mathfrak{X} \to \mathfrak{X}$ is clearly representable. On the other hand, $\mu\colon G \times \mathfrak{X} \to \mathfrak{X}$ is equivalent to pr_1 as a map, so μ is also representable. On the other hand, the diagonal $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is representable because $\mathfrak{X}$ is a topological stack. It follows from Lemma 4.5 that $(\mathrm{pr}_1, \mathrm{pr}_2)\colon \mathfrak{X} \times_{[G\backslash\mathfrak{X}]} \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is representable.

The stack $[G\backslash\mathfrak{X}]$ admits an atlas. Let $X \to \mathfrak{X}$ be an atlas for $\mathfrak{X}$. Since $q\colon \mathfrak{X} \to [G\backslash\mathfrak{X}]$ is an epimorphism, the composition $X \to \mathfrak{X} \to [G\backslash\mathfrak{X}]$ is an epimorphism, hence is an atlas for $[G\backslash\mathfrak{X}]$. $\square$

Using Proposition 4.8 we can give an explicit groupoid presentation for $[G\backslash\mathfrak{X}]$ starting from a groupoid presentation $[R \rightrightarrows X]$ for $\mathfrak{X}$. Consider the action map $\mu\colon G \times \mathfrak{X} \to \mathfrak{X}$. It corresponds to a bibundle

$$\begin{tikzcd} & E \arrow[dl, "\mu_1"'] \arrow[dr, "\mu_2"] & \\ G \times X & & X \end{tikzcd}$$

The claim is that $[E \rightrightarrows X]$, with source and target maps $s = \mu_2$ and $t = \mathrm{pr}_2 \circ \mu_1$, is a groupoid presentation for $[G\backslash\mathfrak{X}]$. It is in fact easy to see why this is the case by staring at the 2-cartesian diagram

$$\begin{tikzcd} E \arrow[r] \arrow[rr, bend left, "\mu_2"] \arrow[d, "\mu_1"'] & G \times X \arrow[r, "\mathrm{pr}_2"'] \arrow[d, "\psi"] & X \arrow[d, "p"] \\ G \times X \arrow[r, "\mathrm{id}_G \times p"] \arrow[d, "\mathrm{pr}_2"'] & G \times \mathfrak{X} \arrow[r, "\mu"] \arrow[d, "\mathrm{pr}_2"] & \mathfrak{X} \arrow[d, "q"] \\ X \arrow[r, "p"'] & \mathfrak{X} \arrow[r, "q"'] & {[G\backslash\mathfrak{X}]} \end{tikzcd}$$

Here $\psi\colon G \times X \to G \times \mathfrak{X}$ is the map $(g, x) \mapsto \big(g, g^{-1} \cdot p(x)\big)$. Perhaps it is helful to remind the reader that, in general, the bibundle E associated to a morphism of stacks $f\colon \mathfrak{X} \to \mathfrak{Y}$ given by groupoid presentations $[R \rightrightarrows X]$ and $[S \rightrightarrows Y]$ is defined by the 2-cartesian diagram

$$\begin{tikzcd} E \arrow[rr] \arrow[d] & & Y \arrow[d, "p_Y"] \\ X \arrow[r, "p_X"'] & \mathfrak{X} \arrow[r, "f"'] & \mathfrak{Y} \end{tikzcd}$$

We now work out the composition rule in $[E \rightrightarrows X]$. This relies on the analysis of the axioms of a group action (Definition 3.1) in terms of bibundles. Consider the commutative square in Definition 3.1. The composition

$$G \times G \times \mathfrak{X} \xrightarrow{\mathrm{id}_G \times \mu} G \times \mathfrak{X} \xrightarrow{\mu} \mathfrak{X}$$

is given by the bibundle $E \times_{s,X,t} E$ from $G \times G \times X$ to X as in the diagram

$$\begin{tikzcd}
 & & E \times_{s,X,t} E \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & \\
 & G \times E \arrow[dl, "\mathrm{id}_G \times \mu_1"'] \arrow[dr, "\mathrm{id}_G \times \mu_1"] & & E \arrow[dl, "\mu_1"'] \arrow[dr, "\mu_2"] & \\
G \times G \times X \arrow[rr, dotted, "\mathrm{id}_G \times \mu"'] & & G \times X \arrow[rr, dotted, "\mu"'] & & X
\end{tikzcd}$$

where the maps in the cartesian square are $\pi_1(u,v) = (\mathrm{pr}_1\, \mu_1(v), u)$ and $\pi_2(u,v) = v$. Similarly, the composition

$$G \times G \times \mathfrak{X} \xrightarrow{m \times \mathrm{id}_{\mathfrak{X}}} G \times \mathfrak{X} \xrightarrow{\mu} \mathfrak{X}$$

is given by the bibundle B

$$\begin{tikzcd}
 & & B \arrow[dl, "\pi'_1"'] \arrow[dr, "\pi'_2"] & & \\
 & G \times G \times R \arrow[dl, "\mathrm{id}_{G \times G} \times \tau"'] \arrow[dr, "m \times \sigma"] & & E \arrow[dl, "\mu_1"'] \arrow[dr, "\mu_2"] & \\
G \times G \times X \arrow[rr, dotted, "m \times \mathrm{id}_{\mathfrak{X}}"'] & & G \times X \arrow[rr, dotted, "\mu"'] & & X
\end{tikzcd}$$

where $\sigma, \tau \colon R \to X$ are the source and target maps of groupoid presentation of $\mathfrak{X}$. The 2-isomorphism α in Definition 3.1 corresponds to an isomorphism $E \times_{s,X,t} E \to B$ of bibundles. Composing this with the projection $\pi'_2 \colon B \to E$ gives rise to the desired composition map $E \times_{s,X,t} E \to E$.

The above discussion immediately implies the following.

Proposition 4.9 *Let $\mathfrak{X}$ be a differentiable stacks and G a Lie group acting smoothly on $\mathfrak{X}$. Then $[G\backslash\mathfrak{X}]$ is a differentiable stack.*

5 Equivariant (co)homology of stacks

Let $\mathfrak{X}$ be a topological stack with an action of a topological group G. We saw in Proposition 4.8 that the quotient stack $[G\backslash\mathfrak{X}]$ is again a topological stack. We can apply the definitions in ([No2] §11,12) to define homotopy groups, (co)homology theories, etc., for $[G\backslash\mathfrak{X}]$. The resulting theories are regarded as G-equivariant theories for $\mathfrak{X}$.

For example, let H be the singular homology (with coefficients in any ring). Let $(\mathfrak{X}, \mathfrak{A})$ be a G-equivariant pair, namely, $\mathfrak{X}$ is a topological stack with a G-action, and $\mathfrak{A}$ is a G-invariant substack.

Definition 5.1 We define the *G-equivariant singular homology* of the pair $(\mathfrak{X}, \mathfrak{A})$ to be

$$H_*^G(\mathfrak{X}, \mathfrak{A}) := H_*(Y, B),$$

where $Y \to [G\backslash\mathfrak{X}]$ is a classifying space for $[G\backslash\mathfrak{X}]$ (§ 2.1), and $B \subseteq Y$ is the inverse image of $\mathfrak{A}$ in Y.

More generally, if h is a (co)homology theory for topological spaces that is invariant under weak equivalences, we can define G-equivariant (co)homology $h_G(\mathfrak{X})$ for a G-equivariant stack $\mathfrak{X}$ (or a pair of topological stacks) using the same procedure.

The functoriality of the construction of the quotient stack $[G\backslash\mathfrak{X}]$ implies that a G-equivariant morphism $f\colon \mathfrak{X} \to \mathfrak{G}$ induces an natural morphism $h(f)\colon h_G(\mathfrak{X}) \to h_G(\mathfrak{Y})$ on G-equivariant homology (in the covariant case) or $h(f)\colon h_G(\mathfrak{Y}) \to h_G(\mathfrak{X})$ on G-equivariant cohomology (in the contravariant case). If $f, f'\colon \mathfrak{X} \to \mathfrak{Y}$ are related by a G-equivariant 2-morphism, the induced maps on G-equivariant (co)homology are the same.

5.1 (Co)homology theories that are only homotopy invariant

There are certain (co)homology theories that are only invariant under homotopy equivalences of topological spaces. Among these are certain sheaf cohomology theories or Čech type theories.

As discussed in ([No2] §11.1), such (co)homology theories can be extended to topological stacks that admit a paracompact classifying space $\varphi\colon X \to \mathfrak{X}$ (satisfying the condition of Theorem 2.2).

Proposition 5.2 *Let $\mathfrak{X}$ be a topological stack and and G a topological group acting on it. Let $[R \rightrightarrows X]$ a topological groupoid presentation for $\mathfrak{X}$. Assume that R, X_0 and G are metrizable. Then, the quotient stack $[G\backslash\mathfrak{X}]$ admits a paracompact classifying space (which satisfies the condition of Theorem 2.2).*

Before proving the proposition we need a lemma.

Lemma 5.3 *Let $[R \rightrightarrows X]$ be a topological groupoid such that R is metrizable. Let T be a metrizable topological space and $f\colon E \to T$ a (locally trivial) torsor for $[R \rightrightarrows X]$. Then, E is metrizable.*

Proof. By Smirnov Metrization Theorem, we need to show that E is locally metrizable, Hausdorff and paracompact. By local triviality of E over T, we can find an open cover $\{E_i\}$ of E such that each E_i is homeomorphic to $T_i \times_X R_i$, where T_i is an open subspace of T trivializing E, and R_i a subspace of R. (The map $T_i \to X$ in the fiber product is the composition of the trivializing section $s_i \colon T_i \to E$ with the structure map $E \to X$ of the torsor.) It follows that $E_i = T_i \times_X R_i \subseteq T_i \times R_i$ is metrizable. Furthermore, since T is metrizable (hence paracompact) we may assume that the open cover $\{T_i\}$ is locally finite. Hence, so is the open cover $\{E_i\}$ of E. Since each E_i is metrizable (hence paracompact) it follows that E is paracompact. Finally, to prove that E is Hausdorff, pick two points x and y in E. If $f(x)$ and $f(y)$ are different, then we can separate them in T by open sets U and V, so $f^{-1}(U)$ and $f^{-1}(V)$ separate x and y in E. If $f(x) = f(y)$, then x and y are in some E_i. Since E_i is Hausdorff (because it is metrizable) we can separate x and y. This proves the lemma. □

Now we come to the proof of Proposition 5.2.

Proof. Consider the explicit groupoid presentation $[E \rightrightarrows X]$ for $[G\backslash\mathfrak{X}]$ described in §4.3. Recall that E is a bibundle

$$\begin{tikzcd} & E \arrow[dl, "\mu_1"'] \arrow[dr, "\mu_2"] & \\ G \times X & & X \end{tikzcd}$$

Since $E \to G \times G \times X$ is torsor for $[R \rightrightarrows X]$, and $G \times X$ is metrizable, it follows from Lemma 5.3 that E is metrizable. The proposition follows from Proposition 8.5 of [No2]. □

As a consequence, we see that if G and $\mathfrak{X}$ satisfy any of the conditions in Proposition 5.2, then any (co)homology theory h that is invariant under homotopy equivalences of topological spaces can be defined G-equivariantly for $\mathfrak{X}$. The resulting (co)homology $h_G(\mathfrak{X})$ is functorial in $\mathfrak{X}$ and in invariant under 2-morphisms.

6 Group actions on mapping stacks

6.1 Group actions on mapping stacks

Let $\mathfrak{X}$ be a stack and G a sheaf of groups. There is a natural strict left G-action

$$\mu \colon G \times \mathrm{Map}(G, \mathfrak{X}) \to \mathrm{Map}(G, \mathfrak{X})$$

on the mapping stack $\mathrm{Map}(G, \mathfrak{X})$ induced from the right multiplication of G on G. We spell out how this works. Let T be in T and $g \in G(T)$. We want to define the action of g on the groupoid $\mathrm{Map}(G, \mathfrak{X})(T)$. Let $f \in \mathrm{Map}(G, \mathfrak{X})(T)$ be an object in this groupoid. By definition of the mapping stack, f is a map $f \colon G \times T \to \mathfrak{X}$. We define $g \cdot f$ by the rule $(g \cdot f)(a, t) = f(ag, t)$. More precisely, $g \cdot f$ is the composition $f \circ m_g \colon G \times T \to \mathfrak{X} \in \mathrm{Map}(G, \mathfrak{X})(T)$, where $m_g \colon G \times T \to G \times T$ is the composition

$$G \times T \xrightarrow{(\mathrm{id}_G, g) \times \mathrm{id}_T} (G \times G) \times T \xrightarrow{m \times \mathrm{id}_T} G \times T.$$

Here, $m\colon G\times G\to G$ is the multiplication in G. The action of g on arrows of $\mathrm{Map}(G,\mathfrak{X})(T)$ is defined similarly.

Given a map $\mathfrak{X}\to\mathfrak{Y}$ of stacks, the induced map $\mathrm{Map}(G,\mathfrak{X})\to\mathrm{Map}(G,\mathfrak{Y})$ is strictly G-equivariant.

The case we are interested in is where $G=S^1$ is the circle. We find that the loop stack $\mathrm{L}\mathfrak{X}$ has a natural strict S^1-action.

6.2 Interpretation of $[G\backslash\mathrm{Map}(G,\mathfrak{X})]$ in terms of torsors

We saw in §6.1 that for every stack $\mathfrak{X}$ and every sheaf of groups G, the mapping stack $\mathrm{Map}(G,\mathfrak{X})$ has a natural left G-action. Our goal is the understand the quotient stack $[G\backslash\mathrm{Map}(G,\mathfrak{X})]$ of this action in the spirit of §4.2.

Let $\mathfrak{X}$ be a stack with an action of a sheaf of groups G. Let T be an object in T. We define the groupoid $\mathfrak{Prin}_{G,\mathfrak{X}}(T)$ as follows.

$$\mathrm{ob}\mathfrak{Prin}^{un}_{G,\mathfrak{X}}(T)=\left\{\begin{array}{ccl}(P,\chi) & | & P\to T\ \text{ left } G\text{-torsor,}\\ & & \chi\colon P\to\mathfrak{X}\ \text{ morphism of stacks}\end{array}\right\}$$

$$\mathrm{Mor}_{\mathfrak{Prin}^{un}_{G,\mathfrak{X}}(T)}\big((P,\chi),(P',\chi')\big)=\left\{\begin{array}{ccl}(u,\phi) & | & u\colon P\to P'\ \text{map of } G\text{-torsors,}\\ & & \phi\colon\chi\Rightarrow\chi'\circ u\ \text{ 2-morphism}\end{array}\right\}$$

We can enhance the above construction to a fibered groupoid $\mathfrak{Prin}^{un}_{G,\mathfrak{X}}$ over T. In fact, $\mathfrak{Prin}^{un}_{G,\mathfrak{X}}$ is a stack over T.

The definition of the stack $\mathfrak{Prin}^{un}_{G,\mathfrak{X}}$ is very similar to $\mathfrak{Prin}_{G,\mathfrak{X}}$, except that we have dropped the G-equivariance condition on χ and ϕ. A T-point of $\mathfrak{Prin}^{un}_{G,\mathfrak{X}}$ should be regarded as a 'family of G-torsors in $\mathfrak{X}$ parametrized by T'. In the case when $G=S^1$, $\mathfrak{Prin}^{un}_{S^1,\mathfrak{X}}$ is the stack of unparametrized loops in $\mathfrak{X}$.

There is a natural morphism of stacks

$$p\colon\ \mathrm{Map}(G,\mathfrak{X})\to\mathfrak{Prin}^{un}_{G,\mathfrak{X}}$$

which sends $f\in\mathrm{Map}(G,\mathfrak{X})$, $f\colon G\times T\to\mathfrak{X}$, to the pair $(G\times T,f)$ in $\mathfrak{Prin}^{un}_{G,\mathfrak{X}}(T)$. Here, we are viewing $G\times T$ as a trivial G-torsor over T.

Proposition 6.1 *There is a canonical (up to a unique 2-morphism) equivalence of stacks*

$$\Phi\colon[G\backslash\mathrm{Map}(G,\mathfrak{X})]\xrightarrow{\sim}\mathfrak{Prin}^{un}_{G,\mathfrak{X}}$$

making the diagram

$$\begin{tikzcd}[column sep=small] & \mathrm{Map}(G,\mathfrak{X}) \arrow[dl, "q"'] \arrow[dr, "p"] & \\ {[G\backslash\mathrm{Map}(G,\mathfrak{X})]} \arrow[rr, "\sim", "\Phi"'] & & \mathfrak{Prin}^{un}_{G,\mathfrak{X}}\end{tikzcd}$$

canonically 2-commutative.

Proof. We define the effect of Φ on objects as follows. Let T be an object in T. As we saw in §4.2, a map $T \to [G\backslash \mathrm{Map}(G,\mathfrak{X})]$ is characterized by a pair (P,χ), where $P \to T$ is a left G-torsor and $\chi\colon P \to [G\backslash \mathrm{Map}(G,\mathfrak{X})]$ is a G-equivariant map. Unraveling the definition of G-equivariance, we find the following description of T-points of $[G\backslash \mathrm{Map}(G,\mathfrak{X})]$. A T-point of $[G\backslash \mathrm{Map}(G,\mathfrak{X})]$ is given by triple (P,χ,σ), where $P \to T$ is a left G-torsor, $\chi\colon G\times P \to \mathfrak{X}$ is a morphism, and σ is a 2-morphism as in the diagram

$$\begin{tikzcd} G\times G\times P \arrow[r, "\mathrm{id}_G\times\mu"] \arrow[d, "m\times\mathrm{id}_P"'] \arrow[dr, phantom, "\Leftarrow\sigma"] & G\times P \arrow[d, "\chi"] \\ G\times P \arrow[r, "\chi"'] & \mathfrak{X} \end{tikzcd}$$

where $\mu\colon G\times P \to P$ is the action of G on P and $m\colon G\times G \to G$ is multiplication in G. The following equality is required to be satisfied:

- $\sigma_g^{a,hp}\sigma_h^{ag,p} = \sigma_{gh}^{a,p}$, for every a, g, h in G and p in P.

Here, $\sigma_g^{a,p}$ stands for the arrow $\sigma(a,g,p)\colon \chi(a,gp) \to \chi(ag,p)$.

Observe that, because of the G-equivariance condition above, $\chi\colon G\times P \to \mathfrak{X}$ is uniquely (up to a unique 2-morphism) determined by its restriction to $\{1\}\times P$, that is, by the composition

$$\chi_1\colon P \xrightarrow{(1_G,\mathrm{id}_P)} G\times P \xrightarrow{\ \chi\ } \mathfrak{X}.$$

We define $\Phi(P,\chi,\sigma) := (P,\chi_1)$.

Given two T-point (P,χ,σ) and (P',χ',σ') in $[G\backslash \mathrm{Map}(G,\mathfrak{X})]$, a morphism between them is a pair (u,ϕ), where $u\colon P \to P'$ is a map of G-torsors and $\phi\colon \chi \Rightarrow \chi'\circ u$ is a 2-morphism satisfying

- $\sigma_g^{a,p}\phi^{ag,p} = \phi^{a,gp}{\sigma'}_g^{a,u(p)}$, for every a, g in G and p in P.

Here, $\phi^{a,p}$ stands for the arrow $\phi(a,p)\colon \chi(a,p) \to \chi'\big(a,u(p)\big)$. As in the case of T-points, the G-equivariance implies that an ϕ is uniquely determined by its restriction ϕ_1 to $\{1\}\times P$, which is obtained by precomposing ϕ by $(1_G,\mathrm{id}_P)\colon P \to G\times P$,

$$\begin{tikzcd} P \arrow[r, "(1_G,\mathrm{id}_P)"] & G\times P \arrow[r, bend left, "\chi"] \arrow[r, bend right, "\chi'\circ u"'] \arrow[r, phantom, "\phi\Downarrow"] & \mathfrak{X}. \end{tikzcd}$$

We define $\Phi(u,\phi) := (u,\phi_1)$.

We leave it to the reader to verify the last part of the proposition (2-commutativity of the triangle). $\square$

Example 6.2 Suppose that $\mathfrak{X} = [H\backslash X]$ is the quotient stack of the action of a topological group H on a topological space X. Then, a T-point of $[G\backslash \mathrm{Map}(G,\mathfrak{X})]$ is a sequence

$$Q \to P \to T$$

together with a continuous map $\chi\colon Q \to X$, where Q is an H-torsor over P and P is a G-torsor over T. The map χ is assumed to be H-equivariant. A morphism between such T-points is a commutative diagram

$$\begin{tikzcd} Q \arrow[r] \arrow[d, "u_2"'] & P \arrow[d, "u_1"'] \arrow[dr] & \\ Q' \arrow[r] & P' \arrow[r] & T \end{tikzcd}$$

such that u_1 is G-equivariant, u_2 is H-equivariant, and $\chi = \chi' \circ u_2$.

6.3 A slight generalization

The set of being as in § 6.1, let H be another sheaf of groups, and $F\colon G \to H$ a homomorphism. We have an induced morphism of mapping stacks

$$\mathrm{Map}(H,\mathfrak{X}) \to \mathrm{Map}(G,\mathfrak{X}).$$

The homomorphism Φ also gives rise to an action of G on $\mathrm{Map}(H,\mathfrak{X})$ making the above map a G-equivariant map. Therefore, we have an induced map on the quotient stacks

$$F^*\colon [G\backslash \mathrm{Map}(H,\mathfrak{X})] \to [G\backslash \mathrm{Map}(G,\mathfrak{X})].$$

There is a torsor description for $[G\backslash \mathrm{Map}(H,\mathfrak{X})]$ and F^* as follows. Let $\mathfrak{X}$ be a stack with an action of a sheaf of groups G. Let T be an object in T. We define the stack $\mathfrak{Prin}_{G\to H,\mathfrak{X}}$ by the following rule:

$$\mathrm{ob}\mathfrak{Prin}^{un}_{G\to H,\mathfrak{X}}(T) = \left\{ \begin{array}{ccl} (P,\chi) & | & P \to T \ \text{ left } G\text{-torsor}, \\ & & \chi\colon P_H \to \mathfrak{X} \ \text{ morphism of stacks} \end{array} \right\}$$

$$\mathrm{Mor}_{\mathfrak{Prin}^{un}_{G\to H,\mathfrak{X}}(T)}\big((P,\chi),(P',\chi')\big) = \left\{ \begin{array}{ccl} (u,\phi) & | & u\colon P \to P' \text{ map of } G\text{-torsors}, \\ & & \phi\colon \chi \Rightarrow \chi' \circ u_H \ \text{ 2-morphism} \end{array} \right\}$$

Here, $P_H := H \overset{G}{\times} P$ stands for extension of structure group from G to H, and u_H is the induced map on the extensions. As in Proposition 6.1, it can be shown that there is a canonical (up to a unique 2-morphism) equivalence of stacks

$$[G\backslash \mathrm{Map}(H,\mathfrak{X})] \xrightarrow{\sim} \mathfrak{Prin}^{un}_{G\to H,\mathfrak{X}}.$$

The torsor description of the map $F^*\colon [G\backslash \mathrm{Map}(H,\mathfrak{X})] \to [G\backslash \mathrm{Map}(G,\mathfrak{X})]$ is given by

$$\mathfrak{Prin}^{un}_{G\to H,\mathfrak{X}} \to \mathfrak{Prin}^{un}_{G,\mathfrak{X}},$$

$$(P, \chi) \mapsto (P, \chi \circ f),$$

where $f\colon P \to P_H$ is the maps induced from the map $P \to H \times P$, $x \mapsto (1, x)$.

In the following examples we consider two extreme case of the above construction.

Example 6.3 In the case where H is the trivial group, denoted 1, we have

$$[G\backslash \mathrm{Map}(1, \mathfrak{X})] \cong \mathfrak{Prin}^{un}_{G \to 1, \mathfrak{X}} \cong BG \times \mathfrak{X} \cong [G\backslash \mathfrak{X}],$$

where in the last term G acts trivially on $\mathfrak{X}$. The map F^* coincides with the map

$$[G\backslash c]\colon [G\backslash \mathfrak{X}] \to [G\backslash \mathrm{Map}(G, \mathfrak{X})],$$

where $c\colon \mathfrak{X} \to \mathrm{Map}(G, \mathfrak{X})$ is the morphism parametrizing constant maps $G \to \mathfrak{X}$.

Example 6.4 In the case where $G = 1$ is the trivial group, we have

$$[1\backslash \mathrm{Map}(H, \mathfrak{X})] \cong \mathrm{Map}(H, \mathfrak{X}) \cong \mathfrak{Prin}^{un}_{1 \to H, \mathfrak{X}},$$

and the map $F^*\colon \mathrm{Map}(H, \mathfrak{X}) \to \mathfrak{X}$ is the evaluation map at the identity element of H.

6.4 Existence of a groupoid presentation for $[G\backslash \mathrm{Map}(H, \mathfrak{X})]$

Proposition 6.5 *Let G and H be topological groups and $G \to H$ a homomorphism. Assume that H is compact. Let $\mathfrak{X}$ a topological stack. Then, $[G\backslash \mathrm{Map}(H, \mathfrak{X})]$ is a topological stack.*

Proof. This follows from Theorem 2.5 and Proposition 4.8. $\square$

The above proposition allows us to define G-equivariant (co)homology of the mapping stack $\mathrm{Map}(H, \mathfrak{X})$ using § 5.

Although, strictly speaking, Proposition 6.5 may not be true when H is not compact, there is a slightly weaker version of it which is sufficient for the purpose of defining the G-equivariant (co)homology. We will not need this result in this paper, but we state it as we think it may be useful in other applications.

Proposition 6.6 *Let H be a locally compact topological group and $\mathfrak{X}$ a topological stack. Then, there exists a topological stack $\mathfrak{Y}$ and a morphism of stacks $f\colon \mathfrak{Y} \to [G\backslash \mathrm{Map}(H, \mathfrak{X})]$ with the property that, for every paracompact topological space T, the induced map $f(T)\colon \mathfrak{Y}(T) \to [G\backslash \mathrm{Map}(H, \mathfrak{X})](T)$ on T-points is an equivalence of groupoids.*

Proof. By [No3] Theorem 4.4, $\mathrm{Map}(H, \mathfrak{X})$ is a paratopological stack (see [No3] §2.2 for definition). The proof of Proposition 4.8 can be repeated here to show that $[G\backslash \mathrm{Map}(H, \mathfrak{X})]$ is paratopological. The claim now follows from [No3] Lemma 2.4. $\square$

The above proposition says that, although $[G\backslash\mathrm{Map}(H,\mathfrak{X})]$ may not be a topological stack, from the eye of paracompact topological spaces T it behaves like one. In particular, since most homotopical invariants (such as, homotopy groups, (co)homology, etc.) are defined using paracompact spaces (spheres, simplices, etc.), they make sense for $[G\backslash\mathrm{Map}(H,\mathfrak{X})]$.

6.5 S^1-action on the loop stack

Let $\mathfrak{X}$ be a topological stack. Then the loop stack $\mathrm{L}\mathfrak{X} := \mathrm{Map}(S^1,\mathfrak{X})$ is again a topological stack (Theorem 2.5). By § 6.1, there is a strict S^1-action on $\mathrm{L}\mathfrak{X}$. The quotient stack $[S^1\backslash\mathrm{L}\mathfrak{X}]$ of this action is again a topological stack (Proposition 6.5). The stack $[S^1\backslash\mathrm{L}\mathfrak{X}]$ parametrizes unparametrized loops in $\mathfrak{X}$, in the sense that, for every topological space T, the groupoid $[S^1\backslash\mathrm{L}\mathfrak{X}](T)$ of its T-points is naturally equivalent to the groupoid

$$\mathrm{ob}\mathfrak{Prin}^{un}_{S^1,\mathfrak{X}}(T) = \left\{ \begin{array}{ccl} (P,\chi) & | & P\to T \ \text{ left } S^1\text{-torsor}, \\ & & \chi\colon P\to\mathfrak{X} \ \text{ morphism of stacks} \end{array} \right\}$$

$$\mathrm{Mor}_{\mathfrak{Prin}^{un}_{S^1,\mathfrak{X}}(T)}\big((P,\chi),(P',\chi')\big) = \left\{ \begin{array}{ccl} (u,\phi) & | & u\colon P\to P' \text{ map of } S^1\text{-torsors}, \\ & & \phi\colon \chi\Rightarrow\chi'\circ u \ \text{ 2-morphism} \end{array} \right\}$$

In the case where $\mathfrak{X} = [H\backslash X]$, Example 6.2 (with $G = S^1$) gives a more explicit description of the groupoid of T-points of $[S^1\backslash\mathrm{L}\mathfrak{X}]$.

7 Homotopy type of $[G\backslash\,\mathrm{Map}(G,\mathfrak{X})]$

7.1 Classifying space of $\mathrm{Map}(G,\mathfrak{X})$

Lemma 7.1 *Let $f\colon\mathfrak{X}\to\mathfrak{Y}$ be a representable G-equivariant morphism of topological stacks. Then, the induced map $[f]\colon[G\backslash\mathfrak{X}]\to[G\backslash\mathfrak{Y}]$ is also representable. If f is a universal weak equivalence (§ 2.1), then so is $[f]$.*

Proof. The first statement follows from the 2-cartesian diagram (§4.1)

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "f"] \arrow[d, "q_{\mathfrak{X}}"'] & \mathfrak{Y} \arrow[d, "q_{\mathfrak{Y}}"] \\ {[G\backslash\mathfrak{X}]} \arrow[r, "{[f]}"'] & {[G\backslash\mathfrak{Y}]} \end{tikzcd}$$

and Lemma 6.3 of [No1]. To prove the second part, let $T\to[G\backslash\mathfrak{Y}]$ be a map with T a topological space. Let $P := T\times_{[G\backslash\mathfrak{Y}]}\mathfrak{Y}$ be the corresponding G-torsor on T, with $\chi\colon P\to\mathfrak{Y}$ the second projection map (see § 4.2). We need to show that the projection map

$$F\colon T\times_{[G\backslash\mathfrak{Y}]}[G\backslash\mathfrak{X}]\to T$$

is a weak equivalence. We have

$$T \times_{[G\backslash \mathfrak{Y}]} [G\backslash \mathfrak{X}] \cong G\backslash (P \times_{\mathfrak{Y}} \mathfrak{X}),$$

where the G-action on the right hand side is induced by the one on P. (Note that the action of G on $P \times_{\mathfrak{Y}} \mathfrak{X}$ is free.) Using the above isomorphism, the map F is the same as the map

$$G\backslash (P \times_{\mathfrak{Y}} \mathfrak{X}) \to G\backslash P,$$

induced from the projection $P \times_{\mathfrak{Y}} \mathfrak{X} \to P$ after passing to the quotient of the free G-actions. (here, we have written T as $G\backslash P$ for clarity). The latter is a weak equivalence by assumption, therefore so is the one after passing to the (free) G-quotients. $\square$

Proposition 7.2 *Let $\mathfrak{X}$ be a topological stack and $\varphi\colon X \to \mathfrak{X}$ a classifying space for it as in Theorem 2.2. Let G be a paracompact topological group. Then, there is a natural map*

$$\mathrm{Map}(G, X) \times_G EG \to [G\backslash \mathrm{Map}(G, \mathfrak{X})]$$

making the Borel construction $\mathrm{Map}(G, X) \times_G EG$ *a classifying space for* $[G\backslash \mathrm{Map}(G, \mathfrak{X})]$.

Proof. By Theorem 2.7, the map $\mathrm{Map}(G, X) \to \mathrm{Map}(G, \mathfrak{X})$ is a universal weak equivalence and makes $\mathrm{Map}(G, X)$ a classifying space for $\mathrm{Map}(G, \mathfrak{X})$. By Lemma 7.1, the induced morphism $[G\backslash \mathrm{Map}(G, X)] \to [G\backslash \mathrm{Map}(G, \mathfrak{X})]$ is representable and a universal weak equivalence. We also know that there is a natural map $\mathrm{Map}(G, X) \times_G EG \to [G\backslash \mathrm{Map}(G, X)]$ making the Borel construction $\mathrm{Map}(G, X) \times_G EG$ a classifying space for $[G\backslash \mathrm{Map}(G, X)]$. Composing these two maps give us the desired universal weak equivalence $\mathrm{Map}(G, X) \times_G EG \to [G\backslash \mathrm{Map}(G, \mathfrak{X})]$. $\square$

7.2 Homotopy type of $[S^1\backslash \mathrm{L}\mathfrak{X}]$

Specifying the results of Proposition 7.2 (in the last section) to $G = S^1$ we obtain:

Corollary 7.3 *Let $\mathfrak{X}$ be a topological stack and $\varphi\colon X \to \mathfrak{X}$ a classifying space for it as in Theorem 2.2. Then, there is a natural map*

$$LX \times_{S^1} \mathbb{CP}^\infty \to [S^1\backslash \mathrm{L}\mathfrak{X}]$$

making the Borel construction $LX \times_{S^1} \mathbb{CP}^\infty$ a classifying space for $[S^1\backslash \mathrm{L}\mathfrak{X}]$.

Corollary 7.4 *Let $\mathfrak{X}$ be a topological stack. There is a natural spectral sequence $E^1_{*,*}$ converging to $H^{S^1}_*(\mathrm{L}\mathfrak{X})$ whose first page $E^1_{p,q}$ is isomorphic to $E^1_{p,q} \cong H_{p-q}(\mathrm{L}\mathfrak{X})$ with differential $d^1 : E^1_{p,q} \to E^1_{p,q-1}$ given by the S^1-action operator $D : H_{p-q}(\mathrm{L}\mathfrak{X}) \to H_{p-(q-1)}(\mathrm{L}\mathfrak{X})$ defined below* (8.2).

Proof. Let $\varphi\colon X \to \mathfrak{X}$ be a classifying space for $\mathfrak{X}$ as in Theorem 2.2. By Corollary 2.8 and Corollary 7.3, we are left to the same question with $\mathrm{L}\mathfrak{X}$ replaced by LX. The spectral sequence is now the usual spectral sequence computing S^1-equivariant homology of a S^1-space. $\square$

Remark 7.5 Usually by strings on a manifold M, one means free loops on M, up to reparametrization by (orientation preserving) homeomorphism (or diffeomorphism). Similarly to § 6.1, if $\mathfrak{X}$ is a topological stack, the group $\mathrm{Homeo}^+(S^1)$ of orientation preserving homeomorphism of the circle acts in an natural way on the free loop stack $\mathrm{L}\mathfrak{X}$ (through its natural action on S^1) and we call the quotient stack $[\mathrm{Homeo}^+(S^1)\backslash \mathrm{L}\mathfrak{X}]$, the *stack of strings* of $\mathfrak{X}$. The canonical inclusion $S^1 \hookrightarrow \mathrm{Homeo}^+(S^1)$ induces a map of stacks $[S^1\backslash \mathrm{L}\mathfrak{X}] \to [\mathrm{Homeo}^+(S^1)\backslash \mathrm{L}\mathfrak{X}]$ which is a (weak) homotopy equivalence by Proposition 7.6 below. This justifies the terminology of *string bracket* in Corollary 9.6 and that we only consider $[S^1\backslash \mathrm{L}\mathfrak{X}]$ in Sections 9, 10 and 11.

Proposition 7.6 *Let $\mathfrak{Y}$ be a* $\mathrm{Homeo}^+(S^1)$*-stack. The canonical map* $[S^1\backslash\mathfrak{Y}] \to [\mathrm{Homeo}^+(S^1)\backslash\mathfrak{Y}]$ *is a weak homotopy equivalence and, in particular, induces equivalence in (co)homology.*

Proof. If $\mathfrak{Y}$ is a $\mathrm{Homeo}^+(S^1)$-stack, then it is both a $\mathrm{Homeo}^+(S^1)$-torsor over $[\mathrm{Homeo}^+(S^1)\backslash\mathfrak{Y}]$ and a S^1-torsor over $[S^1\backslash\mathfrak{Y}]$. Since the canonical map $S^1 \hookrightarrow \mathrm{Homeo}^+(S^1)$ is a homotopy equivalence, the result follows from the homotopy long exact sequence [No4, Theorem 5.2]. $\square$

Remark 7.7 The same proof applies to prove that if $\mathfrak{Y}$ is a differentiable stack endowed with an action of the group $\mathrm{Diff}^+(S^1)$ (of orientation preserving diffeomorphism of the circle), then the quotient map $[S^1\backslash\mathfrak{Y}] \to [\mathrm{Diff}^+(S^1)\backslash\mathfrak{Y}]$ is a weak homotopy equivalence.

8 Transfer map and the Gysin sequence for S^1-stacks

8.1 Transfer map for G-stacks

We define natural transfer homomorphisms in (co)homology associated to the projection $q : \mathfrak{Y} \to [S^1\backslash\mathfrak{Y}]$ of an S^1-stack (our main case of interest) and more generally for $\mathfrak{Y} \to [G\backslash\mathfrak{Y}]$ when G is a Lie group. We will use the framework for transfer, i.e., Gysin, maps introduced in [BGNX] and briefly recalled in § 2.2.

Lemma 8.1 *Let G be a compact Lie group. There is a strong orientation class ([BGNX, § 8])*

$$\theta_G \in H^{-\dim(G)}\big(* \to [G\backslash *]\big).$$

In particular, there is a strong orientation class

$$\theta_{S^1} \in H^{-1}(* \to [S^1\backslash *]) \cong \mathbf{k}.$$

PROOF. The canonical map $* \to [G\backslash *]$ factors as $* \cong [G\backslash G] \to [G\backslash *]$. Hence the existence of the class θ_G follows from [BGNX, Proposition 8.32]. In the special case of S^1, this class can be computed easily from the factorization $[S^1\backslash S^1] \hookrightarrow [S^1\backslash \mathbb{R}^2] \to [S^1\backslash *]$ where the first map is the canonical inclusion of S^1 as the unit sphere of $\mathbb{R}^2$ and the last map is a bundle map. Indeed, by definition of bivariant classes [BGNX], we have that $H^i(* \to [S^1\backslash *])$ is isomorphic to $H^{i+2}_{S^1}(\mathbb{R}^2, \mathbb{R}^2 \setminus S^1)$ and the isomorphism $H^{-1}(* \to [S^1\backslash *]) \cong \mathbf{k}$ now follows from the long exact sequence of a pair (in S^1-equivariant cohomology). $\square$

Let G be a compact Lie group and $\mathfrak{Y}$ a G-stack. Since the canonical map $\mathfrak{Y} \to *$ is G-equivariant, we know from Section 4.1 that the diagram

$$\begin{tikzcd} \mathfrak{Y} \arrow[r, "q"] \arrow[d] & {[S^1\backslash \mathfrak{Y}]} \arrow[d, "u"] \\ * \arrow[r, "q"'] & {[S^1\backslash *]} \end{tikzcd} \tag{8.1}$$

is 2-cartesian. Thus, Lemma 8.1 and [BGNX, Section 9.1, 9.2] provide us with canonical Gysin maps as in the following definition.

Definition 8.2 Let $\mathfrak{Y}$ be a G-stack, with G a compact Lie group. The homology *transfer* map $T^G : H_*^G(\mathfrak{Y}) \to H_{*+\dim(G)}(\mathfrak{Y})$ associated to $\mathfrak{Y}$ is the Gysin map

$$T^G := \theta_G^! = x \mapsto u^*\big(\theta_G)\big) \cdot x, \quad \text{for } x \in H_*(\mathfrak{Y}) = H^{-*}(\mathfrak{Y} \to pt).$$

The cohomology transfer map $T_G : H^*(\mathfrak{Y}) \to H_G^{*-\dim(G)}(\mathfrak{Y})$ is similarly defined to be the Gysin map

$$T_G := \theta_{G!} = x \mapsto (-1)^i q_*\big(x \cdot u^*(\theta_G)\big), \quad \text{for } x \in H^i(\mathfrak{Y}) = H^i(\mathfrak{Y} \xrightarrow{id} \mathfrak{Y}).$$

If $G = S^1$, we denote the transfer map T^{S^1} simply by $T : H_*^{S^1}(\mathfrak{Y}) \to H_{*+1}(\mathfrak{Y})$ and call it the *transgression* map. In other words,

$$T = \theta_{S^1}^! := x \mapsto u^*\big(\theta_{S^1})\big) \cdot x, \quad \text{for } x \in H_*(\mathfrak{Y}) = H^{-*}(\mathfrak{Y} \to pt).$$

Proposition 8.3 *The transfer map is natural, that is, if $f : \mathfrak{Z} \to \mathfrak{Y}$ is a G-equivariant morphism of topological stacks, then*

$$f_* \circ T^G = T^G \circ [G\backslash f]_* : H_*^G(\mathfrak{Z}) \to H_{*+\dim(G)}(\mathfrak{Y}).$$

Similarly,

$$[G\backslash f]^* \circ T_G = T_G \circ f^* : H_G^*(\mathfrak{Y}) \to H^{*-\dim(G)}(\mathfrak{Z}).$$

PROOF. This is an easy application of the naturality properties of Gysin maps [BGNX, Section 9.2] applied to the cartesian square (4.1) associated to a G-equivariant morphism of topological stacks. $\square$

8.2 Gysin sequence for S^1-stacks

We now establish the Gysin sequence associated to an S^1-stack $\mathfrak{Y}$.

Proposition 8.4 *Let $\mathfrak{Y}$ be an S^1-stack. There is a (natural with respect to S^1-equivariant maps of stacks) long exact sequence in homology*

$$\cdots \to H_i(\mathfrak{Y}) \xrightarrow{q_*} H_i^{S^1}(\mathfrak{Y}) \xrightarrow{\cap c} H_{i-2}^{S^1}(\mathfrak{Y}) \xrightarrow{T} H_{i-1}(\mathfrak{Y}) \xrightarrow{q_*} H_{i-1}^{S^1}(\mathfrak{Y}) \to \dots,$$

where $q : \mathfrak{Y} \to [S^1\backslash\mathfrak{Y}]$ is the quotient map, T is the transgression map (Definition 8.2), and c is the fundamental class of the S^1-bundle $\mathfrak{Y} \to [S^1\backslash\mathfrak{Y}]$ (that is, the Euler class of the associated oriented disk bundle over $[S^1\backslash\mathfrak{Y}]$).

Proof. The map $q : \mathfrak{Y} \to [S^1\backslash\mathfrak{Y}]$ makes $\mathfrak{Y}$ into an S^1-torsor over $[S^1\backslash\mathfrak{Y}]$ (by § 4.2). This map is representable by Proposition 4.7.

Let $Z \to [S^1\backslash\mathfrak{Y}]$ be a classifying space for $[S^1\backslash\mathfrak{Y}]$ as in Theorem 2.2, and let $Y \to \mathfrak{Y}$ be the classifying space of $\mathfrak{Y}$ obtained by pullback along q. Then, $Y \to Z$ is a principal S^1-bundle (Lemma 4.3) and the long exact sequence in the proposition is the Gysin sequence

$$\cdots \to H_i(Y) \to H_i(Z) \xrightarrow{\cap c} H_{i-2}(Z) \xrightarrow{\widetilde{T}} H_{i-1}(Y) \to H_{i-1}(Z) \to \dots$$

of this S^1-principal bundle under the isomorphisms $H_i(Z) \cong H_i([S^1\backslash\mathfrak{Y}]) \cong H_i^{S^1}(\mathfrak{Y})$ and $H_j(Y) \cong H_j(\mathfrak{Y})$. Here, c is the Euler class of the associated disk bundle (that is, the mapping cylinder of $Y \to Z$).

By [BGNX, § 9], the definition of cup product by bivariant classes [BGNX, § 7.4], and the discussion in Lemma 8.1, the transgression map of Definition 8.2 is induced (under the above isomorphisms) by the connecting homormophism in the long exact sequence of the pair (E, Y) where E is the disk bundle associated to the S^1-principal bundle $Y \to Z$. Hence, the transgression map T is identified with the map $H_{i-1}(Z) \xrightarrow{\widetilde{T}} H_i(Y)$ in the Gysin sequence. □

Let $D : H_i(\mathfrak{Y}) \to H_{i+1}(\mathfrak{Y})$ be the operator, called *the S^1-action operator*, defined as the composition

$$D\colon H_i(\mathfrak{Y}) \xrightarrow{[S^1]\times -} H_{i+1}(S^1 \times \mathfrak{Y}) \xrightarrow{\mu_*} H_{i+1}(\mathfrak{Y}), \tag{8.2}$$

where $[S^1] \in H_1(S^1)$ is the fundamental class and $\mu : S^1 \times \mathfrak{Y} \to \mathfrak{Y}$ is the S^1-action.

Lemma 8.5 *The operator D is equal to the composition*

$$H_i(\mathfrak{Y}) \xrightarrow{q_*} H_i^{S^1}(\mathfrak{Y}) \xrightarrow{T} H_{i+1}(\mathfrak{Y}).$$

In particular, $D \circ D = 0$ and D is natural with respect to S^1-equivariant maps of stacks.

PROOF. By Proposition 8.3 and 8.4, we only need to prove that $D = T \circ q_*$. From § 3.3, we get a tower of 2-cartesian diagrams

$$\begin{tikzcd}
S^1 \times \mathfrak{Y} \arrow[r, "\mu"] \arrow[d, "\mathrm{pr}_2"'] & \mathfrak{Y} \arrow[d, "q"] \\
\mathfrak{Y} \arrow[r, "q"'] \arrow[d] & {[S^1\backslash \mathfrak{Y}]} \arrow[d] \\
* \arrow[r] & {[S^1\backslash *]}
\end{tikzcd}$$

The result follows from naturality of Gysin maps, see [BGNX, Section 9.2]. □

9 Equivariant String Topology for free loop stacks

In this section we look at natural algebraic operations on strings of a stack $\mathfrak{X}$, that is, on the quotient stack $[S^1\backslash \mathrm{L}\mathfrak{X}]$.

9.1 Batalin-Vilkovisky algebras

We first quickly recall the definition of a **BV**-algebra and its underlying Gerstenhaber algebra structure.

A **Batalin-Vilkovisky algebra** (**BV-algebra** for short) is a graded commutative associative algebra with a degree 1 operator D such that $D^2 = 0$ and the following identity is satisfied:

$$\begin{aligned} D(abc) - D(ab)c - (-1)^{|a|}aD(bc) - (-1)^{(|a|+1)|b|}bD(ac) + \\ + D(a)bc + (-1)^{|a|}aD(b)c + (-1)^{|a|+|b|}abD(c) = 0. \end{aligned} \quad (9.1)$$

In other words, D is a second-order differential operator. Note that we do *not* assume **BV**-algebras to be unital.

Let $(A, \cdot, D)$ be a **BV**-algebra. We can define a degree 1 binary operator $\{-;-\}$ by the following formula:

$$\{a; b\} \quad = \quad (-1)^{|a|}D(a \cdot b) - (-1)^{|a|}D(a) \cdot b - a \cdot D(b). \quad (9.2)$$

The BV-identity (9.1) and commutativity of the product imply that $\{;\}$ is a derivation in each variable (and anti-symmetric with respect to the degree shifted down by 1). Further, the relation $D^2 = 0$ implies the (graded) Jacobi identity for $\{;\}$. In other words, $(A, \cdot, \{-;-\})$ is a *Gerstenhaber* algebra, that is a commutative graded algebra equipped with a bracket $\{-;-\}$ that makes $A[-1]$ a graded Lie algebra satisfying a graded Leibniz rule [Ger].

Indeed, it is standard (see [Get]) that a graded commutative algebra $(A, \cdot)$ equipped with a degree 1 operator D such that $D^2 = 0$ is a **BV**-algebra if and only if the operator $\{-;-\}$ defined by the formula (9.2) is a derivation of the second variable, that is

$$\{a; bc\} = \{a; b\} \cdot c + (-1)^{|b|(|a|+1)} b \cdot \{a; c\}. \tag{9.3}$$

The following Lemma was essentially first noticed by Chas-Sullivan [CS].

Lemma 9.1 *Let $(B_*, \star, \Delta)$ be a BV-algebra and H_* a graded module related to it by an "S^1-Gysin exact sequence," that is, sitting in a long exact sequence*

$$\cdots \to B_i \xrightarrow{q} H_i \xrightarrow{c} H_{i-2} \xrightarrow{T} B_{i-1} \xrightarrow{q} H_{i-1} \to \dots$$

such that $\Delta = T \circ q$. Then, we have the following.

1. *The composition*

   $$\{-,-\} \colon H_{i-2} \otimes H_{j-2} \xrightarrow{T \otimes T} B_{i-1} \otimes B_{j-1} \xrightarrow{\star} B_{i+j-2} \xrightarrow{q} H_{i+j-2}$$

   makes the shifted module $H_*[2]$ into a Lie algebra.

2. *The induced map $T : H_*[2] \to B_*[1]$ is a Lie algebra morphism. Here, $B_*[1]$ is equipped with the Lie algebra structure underlying its BV-algebra structure.*

Note that, since T is an operator of odd degree, following the Koszul-Quillen sign convention, the bracket in statement (1) is given by

$$\{x, y\} := (-1)^{|x|} q\big(T(x) \star T(y)\big).$$

PROOF. The proof of statement (1) is the same as the proof of Theorem 6.1 in [CS].

The Lie bracket $\{-,-\}_\Delta$ on the (shifted) modules $B_*[1]$ is defined by the degree 1 operator (from $B_* \otimes B_* \to B_*$)

$$\{a, b\}_\Delta := (-1)^{|a|} \Delta(a \star b) - (-1)^{|a|} \Delta(a) \star b - a \star \Delta(b).$$

We denote the shift operator $(x \mapsto (-1)^{|x|} x)$ by $s : B_* \to B_*[1]$. The Lie bracket on $B_*[1]$ is, by definition, the transport along s of the degree 1 operator above. Now, for $x, y \in H_*$, since $T \circ q = \Delta$ and $\Delta \circ T = T \circ (q \circ T) = 0$, we deduce from the above formula for $\{-,-\}_\Delta$ that

$$\begin{aligned} T\big(\{x, y\}\big) &= (-1)^{|x|} \Delta\big(T(x) \star T(y)\big) \\ &= -\{T(x), T(y)\}_\Delta \\ &= s^{-1}\{s(T(x)), s(T(y))\}_\Delta . \end{aligned}$$

This proves that $s \circ T$ is a Lie algebra map. $\square$

Remark 9.2 We will apply Lemma 9.1 in the context of string topology operations (following [CS]). However, this Lemma also applies when B_* is the Hochschild cohomology of any Frobenius algebra and H_* is its negative cyclic cohomology (for instance see [Tr, ATZ, Me]).

Example 9.3 Lemma 9.1 also applies in the following situation. Let C_* be the graded module $C_* = \bigoplus_{n\geq 0}(A[-1])^{\otimes n}$, where A is a unital associative (possibly differential graded) algebra. In other words, as a $\mathbb{Z}$-graded module, $C_k = CHoch_{-k}(A)$, where $(CHoch_*(A), b)$ is the standard Hochschild chain complex [Lo]. Let $B : C_* \to C_{*-1}$ be the usual Connes operator, which makes (C_*, D) into a chain complex. Since the Connes operator D is a derivation for the shuffle product (see [Lo]), the shuffle product makes $sh : C_* \otimes C_* \to C_*$ into a differential graded commutative algebra, and its homology $B_* := H_*(C_*, B)$ into a graded commutative algebra. Since A is not necessarily commutative, the standard Hochschild differential $b : C_* \to C_{*+1}$ is not necessarily a derivation with respect to the shuffle product, but it is a second order differential-operator. Thus (C_*, B, b) is a differential graded **BV**-algebra and, consequently, (B_*, sh, b) is a **BV**-algebra.

Let $NC_k := \prod_{i\geq 0} C_{k+2i}$. It is immediate to check that $(NC_k, B, (-1)^k b)$ is a bicomplex (which can be thought as an analogue of the standard cyclic chain complex where the role of b and B have been exchanged). Let TC_* be the associated total complex of NC_*, and let $H_* = H_*(TC_*)$ be its homology. The inclusion $q : C_k \hookrightarrow \prod_{i\geq 0} C_{k+2i} = TC_k$ is an injective chain map and, further, its cokernel is $TC_*[2]$. Let $T : TC_k \to C_{k+1}$ be the composition

$$T : TC_k = \prod_{i\geq 0} C_{k+2i} \overset{projection}{\twoheadrightarrow} C_k \overset{b}{\to} C_{k+1}.$$

One check easily that T is a chain map and the connecting homomorphism of the short exact sequence $0 \to C_* \to TC_* \to TC_*[2] \to 0$.

It follows that H_*, B_* satisfy the assumption of Lemma 9.1. Thus, $H_*[2]$ inherit a natural Lie algebra structure.

9.2 Short review of string topology operations for stacks

Let $\mathfrak{X}$ be a Hurewicz oriented stack. We recall (see [No1, BGNX]) that $\mathfrak{X}$ being Hurewicz means that $\mathfrak{X}$ can be presented by a topological groupoid whose source and target maps are local Hurewicz fibrations. In particular, every *differentiable* stack is Hurewicz. We proved in [BGNX], that the homology $H_*(\mathrm{L}\mathfrak{X})$ carries an natural structure of ($\dim(\mathfrak{X})$-dimensional) Homological Conformal Field Theory. In particular, restricting this structure to genus 0-operations, one obtains the following.

Theorem 9.4 ([BGNX], Theorem 13.2) *Let $\mathfrak{X}$ be an oriented Hurewicz stack. Then, the shifted homology $(H_{i+\dim\mathfrak{X}}(\mathrm{L}\mathfrak{X}), \star, D)$ is a BV-algebra, where D is the operator (8.2) induced by the S^1-action on $\mathrm{L}\mathfrak{X}$ and $\star : H_i(\mathrm{L}\mathfrak{X}) \otimes H_j(\mathrm{L}\mathfrak{X}) \to H_{i+j-\dim\mathfrak{X}}(\mathrm{L}\mathfrak{X})$ is the loop product.*

Note that, in general, the multiplication $\star$ may *not* be unital for stacks.

9.3 The Lie algebra structure on the S^1-equivariant homology of the free loop stack

Proposition 9.5 *Let $\mathfrak{Y}$ be an S^1-stack, and assume that the operator* (8.2) *D extends to a BV-algebra structure $(H_{i+d}(\mathfrak{Y}), \star, D)$ on the (shifted) homology of $\mathfrak{Y}$. Then, we have the following.*

- *The composition*

$$\{-,-\}\colon H^{S^1}_{i+d-2}(\mathfrak{Y}) \otimes H^{S^1}_{j+d-2}(\mathfrak{Y}) \xrightarrow{T\otimes T} H_{i+d-1}(\mathfrak{Y}) \otimes H_{j+d-1}(\mathfrak{Y}) \xrightarrow{\star} H_{i+j+d-2}(\mathfrak{Y}) \xrightarrow{q_*} H^{S^1}_{i+j+d-2}(\mathfrak{Y})$$

  makes the shifted equivariant homology $H^{S^1}_{*+d-2}(\mathfrak{Y})$ into a Lie algebra.

- *The induced map $T : H^{S^1}_*(\mathfrak{Y})[2] \to H_*(\mathfrak{Y})[1]$ is a Lie algebra morphism. Here, $B_*[1]$ is equipped with the Lie algebra structure underlying its BV-algebra structure.*

Recall the sign convention for bracket and similarly for higher brackets in statement (2).

Proof. By Proposition 8.4 and Lemma 8.5, the shifted equivariant homology $H^{S^1}_{*+d}(\mathfrak{Y})$ and shifted homology $H_{*+d}(\mathfrak{Y})$ satisfy the assumption of (the purely algebraic) Lemma 9.1. $\square$

Let $\mathfrak{X}$ be a topological stack. Then, the free loop stack $\mathrm{L}\mathfrak{X} = \mathrm{Map}(S^1, \mathfrak{X})$ is a topological stack (Theorem 2.5) with a (strict) S^1-action (see Section 6.1). Further, if $\mathfrak{X}$ is a Hurewicz (for instance differentiable) oriented stack, by Theorem 9.4, its (shifted down by $\dim \mathfrak{X}$) homology carries a structure of a **BV**-algebra. Hence, we can apply Proposition 9.5 to $\mathfrak{Y} = \mathrm{L}\mathfrak{X}$.

Corollary 9.6 *Let $\mathfrak{X}$ be an oriented differentiable (or more generally Hurewicz) stack of dimension d.*

- *For $x, y \in H_*(\mathrm{L}\mathfrak{X})$, the formula*

$$\{x, y\} := (-1)^{|x|} q\big(T(x) \star T(y)\big)$$

  makes the equivariant homology $H^{S^1}_*(\mathrm{L}\mathfrak{X})[2 - \dim \mathfrak{X}]$ into a graded Lie algebra. Here, T is the transgression map (Definition 8.2) and $q : \mathrm{L}\mathfrak{X} \to [S^1 \backslash \mathrm{L}\mathfrak{X}]$ the canonical projection.

- *The transgression $T : H^{S^1}_*(\mathrm{L}\mathfrak{X})[2 - \dim \mathfrak{X}] \to H_*(\mathrm{L}\mathfrak{X})[1 - \dim \mathfrak{X}]$ is a Lie algebra homomorphism. Here, $H_*(\mathrm{L}\mathfrak{X})[1 - \dim \mathfrak{X}]$ is the Lie algebra structure underlying the* **BV***-algebra structure of Theorem 9.4.*

The bracket $\{-,-\}$ defined by Corollary 9.6 is called the **string bracket**.

9.4 Some Examples

Example 9.7 (Oriented manifolds) Let M be an oriented closed manifold. Then by [BGNX, Proposition 17.1], the **BV**-algebra structure of $H_*(LM)$ given by Theorem 9.4 agrees with Chas-Sullivan construction (and other constructions as well). Since the S^1-action on LM agrees with the stacky one ([BGNX, Example 5.8]), it follows immediately that the Lie algebra structure given by Corollary 9.6 agrees with Chas-Sullivan [CS] ones for oriented closed manifolds. Note that Corollary 9.6 also applies to open oriented manifolds.

Example 9.8 (Classifying stack of compact Lie groups) Let G be a compact Lie group; its associated classifying stack $[G\backslash *]$ is oriented (see [BGNX]) of dimension $-\dim G$. Hence its S^1-equivariant homology $H_*^{S^1}(\mathrm{L}[G\backslash *])$ has a degree $2+\dim G$ Lie bracket.

Proposition 9.9 *If k is of characteristic zero and G is either connected or finite the Lie algebra $H_*(L[G\backslash *], k)$ is abelian.*

Proof. By [BGNX, Theorem 17.23], if G is connected, the hidden loop product (which coincides with the loop product by [BGNX, Lemma 17.14]) vanishes. Thus, the string bracket vanishes as well.

If G is finite, then $H_*(\mathrm{L}[G\backslash *], k)$ is concentrated in degree 0 and it follows that the transfer map $T : H_*^{S^1}(\mathrm{L}[G\backslash *], k) \to H_{*+1}(\mathrm{L}[G\backslash *], k)$ vanishes, hence so does the string bracket. $\square$

If G is a finite group with order coprime with the characteristic of k, then the same proof shows that $H_*(\mathrm{L}[G\backslash *], k)$ is abelian.

However, if the characteristic of k divides the order of G, then, in view of the results of [SF] on the non triviality of the Gerstenhaber bracket in Hochschild cohomology of the group algebra $k[G]$ of G and the close relationship between the Gerstenhaber product and loop bracket [FT, GTZ], one may expect that the string Lie algebra of $[G\backslash *]$ is no longer abelian in this case.

Example 9.10 (A non-nilpotent example) Let $(\mathbb{Z}/2\mathbb{Z})^{n+1}$ acts on the euclidean sphere

$$S^{2n+1} = \{|z_0|^2 + \cdots + |z_n|^2 = 1, z_i \in \mathbb{C}\}$$

as the group generated by the reflections across the hyperplanes $z_i = 0$ $(0 \leq i \leq n)$. Let $\mathfrak{X} = [S^{2n+1} \times S^{2n+1}/(\mathbb{Z}/2\mathbb{Z})^{n+1}]$ be the induced quotient stack (where $(\mathbb{Z}/2\mathbb{Z})^{n+1}$ acts diagonally), which is an oriented orbifold (in the sense of [BGNX]). Recall that there is an isomorphism of coalgebras $H_*^{S^1}(*) \cong H_*(BS^1) \cong k[u]$ where $|u| = 2$.

Proposition 9.11 *Let k be a field of characteristic different from* 2.

- *There is an isomorphism of (graded) Lie algebras*

$$H_*^{S^1}(L\mathfrak{X}, k) \;\cong\; H_*^{S^1}(L(S^{2n+1} \times S^{2n+1}), k) \otimes_k k[(\mathbb{Z}/2\mathbb{Z})^{n+1}].$$

- *As a $k[(\mathbb{Z}/2\mathbb{Z})^{n+1}]$-module and $k[u]$-comodule (where $|u|=2$), $H_*^{S^1}(L\mathfrak{T},k)$ is free and spanned by the basis elements*

$$(e_{i,j})_{(i,j)\in\mathbb{N}^2\setminus\{(0,0)\}}, \qquad (f_{i,j})_{(i,j)\in\mathbb{N}^2}$$

where $|e_{i,j}| = 2n(i+j)$ and $|f_{k,l}| = 2n(i+j+2)+1$.

- *The string bracket is $k[(\mathbb{Z}/2\mathbb{Z})^{n+1}][u]$-linear and satisfies the formula*

$$\begin{aligned}
[f_{i,j}, e_{k,l}] &= \binom{i+k}{i}\binom{j+l}{j}\frac{il-jk}{(i+k)(j+l)} f_{i+k-1,j+l-1}\\
[e_{i,j}, e_{k,l}] &= \binom{i+k}{i}\binom{j+l}{j}\frac{jk-il}{(i+k)(j+l)} e_{i+k-1,j+l-1}\\
[e_{i,j}, e_{k,l}] &= 0.
\end{aligned}$$

Since $[e_{1,1}, e_{i,j}] = (i-j)e_{i,j}$, it follows that $H_*^{S^1}(\mathrm{L}\mathfrak{T},k)$ is not nilpotent as a Lie algebra.

Proof. The explicit computations follows from the first one and the standard computations of equivariant homology of loop spheres, see [FTV, BV].

The first point follows from [BGNX, § 17]. By [BGNX, Proposition 5.9], the free loop stack $\mathrm{L}\mathfrak{T}$ is presented by the transformation topological groupoid

$$LT := [\coprod_{g\in R} \mathcal{P}_g \rtimes (\mathbb{Z}/2\mathbb{Z})^{n+1} \rightrightarrows \coprod_{g\in R} \mathcal{P}_g]$$

where $\mathcal{P}_g$ is the space of continuous maps

$$\mathcal{P}_g := \{f : \mathbb{R} \to S^{2n+1} \times S^{2n+1},\ f(t) = f(t+1)\cdot g \text{ for all } t\}.$$

The $(\mathbb{Z}/2\mathbb{Z})^{n+1}$ action on $\mathcal{P}_g$ is pointwise. The S^1 or rather $[\mathbb{Z}\backslash\mathbb{R}]$-action on $\mathrm{L}\mathfrak{T}$ is presented by the topological groupoid morphism

$$\big(\mathbb{Z}\times\mathbb{R}\big) \times \big(\coprod_{g\in R} \mathcal{P}_g \rtimes (\mathbb{Z}/2\mathbb{Z})^{n+1}\big) \xrightarrow{\theta} \big(\coprod_{g\in R} \mathcal{P}_g \rtimes (\mathbb{Z}/2\mathbb{Z})^{n+1}\big)$$

defined, for any $(n,x)\in\mathbb{R}\times\mathbb{Z}$, $h\in(\mathbb{Z}/2\mathbb{Z})^{n+1}$ and $f\in\mathcal{P}_g$, by

$$\theta(x,n,f,h) = \big((t\mapsto f(t+x)), g^n h\big).$$

The map is compatible with the group structure of the stack $[\mathbb{Z}\backslash\mathbb{R}]$ and thus is a groupoid morphism representing the S^1-action on $\mathrm{L}\mathfrak{X}$ as in Section 6.

Since $(\mathbb{Z}/2\mathbb{Z})^{n+1}$ is a subgroup of the connected Lie group $SO(2n+2)$, which acts diagonally on $S^{2n+1}\times S^{2n+1}$, for all $g\in(\mathbb{Z}/2\mathbb{Z})^{n+1}$ there is a continuous path $\rho : [0,1]\to SO(2n+2)$ connecting g to the identity (that is $\rho(0)=g$, $\rho(1)=1$). This allows to define a map

$$\Upsilon : \coprod_{g\in(\mathbb{Z}/2\mathbb{Z})^{n+1}} \mathcal{P}_g \to \coprod_{g\in(\mathbb{Z}/2\mathbb{Z})^{n+1}} \mathrm{L}\big(S^{2n+1}\times S^{2n+1}\big)$$

given, for any path $f \in P_g$, by the loop

$$\Upsilon_g(f)(t) = \begin{cases} f(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ f(0).\rho(2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

It is a general fact that Υ is a $(\mathbb{Z}/2\mathbb{Z})^{n+1}$-equivariant homotopy equivalence (see [LUX, § 6] for details), where $(\mathbb{Z}/2\mathbb{Z})^{n+1}$ acts pointwise; note that this action is trivial in homology with coefficients in a field of characteristic coprime with 2 since the (naive) quotient map $S^{2n+1} \times S^{2n+1} \to (\mathbb{Z}/2\mathbb{Z})^{n+1} \backslash (S^{2n+1} \times S^{2n+1}) \cong S^{2n+1} \times S^{2n+1}$ is invertible in homology with coefficient coprime with 2. It follows that Υ induces an isomorphism between $H_*(\mathrm{L}\mathfrak{T})$ and $H_*\big(\mathrm{L}(S^{2n+1} \times S^{2n+1}), k\big) \otimes_k k[(\mathbb{Z}/2\mathbb{Z})^{n+1}]$ and, by Corollary 7.4, similarly an isomorphism of $k[u]$-comodules

$$H_*^{S^1}(\mathrm{L}\mathfrak{T}, k) \;\cong\; H_*^{S^1}(\mathrm{L}(S^{2n+1} \times S^{2n+1}), k) \otimes_k k[(\mathbb{Z}/2\mathbb{Z})^{n+1}]. \qquad (9.4)$$

The proof that the above isomorphism is multiplicative with respect to the loop multiplication is similar to the proof of [BGNX, Proposition 17.10]. Further, by naturality of the Gysin sequence (Proposition 8.4), the Gysin sequence of the S^1-stack $\mathrm{L}\mathfrak{T}$ is identified with the Gysin sequence of the S^1-stack $\coprod_{g \in (\mathbb{Z}/2\mathbb{Z})^{n+1}} [(\mathbb{Z}/2\mathbb{Z})^{n+1} \backslash \mathrm{L}(S^{2n+1} \times S^{2n+1})]$ (where the $[\mathbb{Z} \backslash \mathbb{R}]$-action is induced by the map Υ). By definition of the Lie algebra structure, it follows that the isomorphism (9.4) is an isomorphism of Lie algebras (after the appropriate degree shifting by $\dim \mathfrak{X} - 2$). □

10 Functoriality of the Batalin-Vilkovisky structure and string bracket with respect to open embeddings

In this section we show that the string bracket is functorial with respect to open embeddings (Proposition 10.3). This will be used later on in describing the string bracket for 2-dimensional orbifolds.

Lemma 10.1 *Let $\mathfrak{X}$ be a topological stack whose coarse moduli space $|\mathfrak{X}|$ is paracompact. Let $\mathfrak{F}$ be a metrizable vector bundle over $\mathfrak{X}$, and let $\mathfrak{U} \subset \mathfrak{F}$ be an open substack of the total space of $\mathfrak{F}$ through which the zero section $s\colon \mathfrak{X} \to \mathfrak{F}$ factors. Then, the map $s\colon \mathfrak{X} \to \mathfrak{U}$ admits a tubular neighborhood ([BGNX], Definition 8.5). That is, there is a vector bundle $\mathfrak{N}$ over $\mathfrak{X}$ and a factorization*

$$\mathfrak{X} \overset{i}{\hookrightarrow} \mathfrak{N} \overset{j}{\hookrightarrow} \mathfrak{U}$$

for s, where i is the zero section of $\mathfrak{N}$ and j is an open embedding.

Proof. We show that there is a function $f\colon \mathfrak{X} \to \mathbb{R}^{>0}$ such that the map $\Phi\colon \mathfrak{F} \to \mathfrak{F}$ defined by fiberwise multiplication by f identifies the open unit ball bundle $\mathfrak{D} \subset \mathfrak{F}$ with an open substack $\mathfrak{V} \subseteq \mathfrak{U}$. It would then follows that $\mathfrak{V}$ is isomorphic, as a stack over $\mathfrak{X}$, with $\mathfrak{D}$, which is in turn isomorphic to the total space of $\mathfrak{F}$. Thus, taking $\mathfrak{N} := \mathfrak{F}$ gives the desired factorization.

Since we have partition of unity on $|\mathfrak{X}|$, construction of $f\colon \mathfrak{X} \to \mathbb{R}^{>0}$ can be done locally on $|\mathfrak{X}|$, so we are allowed to pass to open substacks of $\mathfrak{X}$. Thus, we may assume that $\mathfrak{X}$ admits a chart $\pi\colon X \to \mathfrak{X}$ such that after base extending along π, the resulting bundle F over X and the open set $U \subseteq F$ corresponding to $\mathfrak{U}$ have the property that U contains an ε-ball bundle of F for some $\varepsilon > 0$. So it is enough to take $f\colon \mathfrak{X} \to \mathbb{R}^{>0}$ to be the constant function ε. $\square$

Lemma 10.2 *Consider the 2-cartesian diagram of topological stacks*

$$\begin{tikzcd}
\mathfrak{X} \arrow[r, "\theta_f", "f"'] \arrow[d] & \mathfrak{Y} \arrow[d] \\
\mathfrak{X}' \arrow[r, "\theta_{f'}", "f'"'] & \mathfrak{Y}'
\end{tikzcd}$$

in which the vertical arrows are open embeddings. If f' is bounded proper (respectively, normally nonsingular), then so is f (see [BGNX], Definitions 6.1 and 8.15). Suppose, in addition, that f and f' are strongly proper (see [BGNX], Definitions 6.2), and let θ_f and $\theta_{f'}$ be the corresponding strong orientation classes ([BGNX], Proposition 8.25). Then θ_f is the independent pullback of θ'_f in the sense of bivariant theory ([BGNX], 7.2).

Proof. Being bounded proper is invariant under arbitrary base extensions. Suppose that f' is normally nonsingular, and let

$$\begin{tikzcd}
\mathfrak{N}' \arrow[r, hook, "i'"] & \mathfrak{E}' \arrow[d, "p'"] \\
\mathfrak{X}' \arrow[u, "s'"] \arrow[r, "f'"'] & \mathfrak{Y}'
\end{tikzcd}$$

be a normally nonsingular diagram for it. Base extending the diagram along the open embedding $\mathfrak{Y} \to \mathfrak{Y}'$, we obtain a diagram

$$\begin{tikzcd}
\mathfrak{U} \arrow[r, hook, "i"] & \mathfrak{E} \arrow[d, "p"] \\
\mathfrak{X} \arrow[u, "s"] \arrow[r, "f"'] & \mathfrak{Y}
\end{tikzcd}$$

where $\mathfrak{U} = (p' \circ i')^{-1}(\mathfrak{Y})$ is an open substack of the vector bundle $\mathfrak{F} := \mathfrak{N}'|_{\mathfrak{X}}$ over $\mathfrak{X}$ which contains the zero section $s\colon \mathfrak{X} \to \mathfrak{F}$. This diagram is not quite a normally nonsingular diagram, as $\mathfrak{U}$ is not a vector bundle over $\mathfrak{X}$, but by Lemma 10.1 the map $s\colon \mathfrak{X} \to \mathfrak{U}$ admits a tubular neighborhood $\mathfrak{N}$. Replacing $\mathfrak{U}$ by his tubular neighborhood $\mathfrak{N}$ we obtain the desired normally nonsingular diagram for f.

The statement about θ_f being the independent pullback of θ'_f follows from the definition of independent pullback ([BGNX], 7.2) and excision. $\square$

Proposition 10.3 *Let $\mathfrak{X}$ be an oriented Hurewicz stack of dimension d, and $\mathfrak{U} \subseteq \mathfrak{X}$ an open substack. Then, $\mathfrak{U}$ inherits a natural orientation from $\mathfrak{X}$, and the induced map $H_{*+d}(L\mathfrak{U}) \to H_{*+d}(L\mathfrak{X})$ is a morphism of BV-algebras. Therefore, the induced map $H_*^{S^1}(L\mathfrak{U})[2-d] \to H_*^{S^1}(L\mathfrak{X})[2-d]$ is a morphism of graded Lie algebras.*

The Proposition applies, in particular, to an embedding of oriented manifolds (see Example 9.7).

Proof. To prove that $\mathfrak{U}$ inherits an orientation, we have to show that $\Delta_{\mathfrak{U}}\colon \mathfrak{U} \to \mathfrak{U} \times \mathfrak{U}$ is strongly proper ([BGNX], Definition 6.2), normally nonsingular, and that the class $\theta_{\mathfrak{U}} \in H^d(\Delta_{\mathfrak{U}})$ obtained by pulling back the strong orientation class $\theta_{\mathfrak{X}} \in H^d(\Delta_{\mathfrak{X}})$ via independent pullback, as in the 2-cartesian diagram

$$\begin{tikzcd} \mathfrak{U} \arrow[r, "\theta_{\mathfrak{U}}", "\Delta_{\mathfrak{U}}"'] \arrow[d] & \mathfrak{U} \times \mathfrak{U} \arrow[d] \\ \mathfrak{X} \arrow[r, "\theta_{\mathfrak{X}}", "\Delta_{\mathfrak{X}}"'] & \mathfrak{X} \times \mathfrak{X} \end{tikzcd}$$

is a strong orientation ([BGNX], Definition 8.21). These all follow from Lemma 10.2, except the fact that $\Delta_{\mathfrak{U}}$ is strongly proper (the lemma only says that it is bounded proper). The fact that $\Delta_{\mathfrak{U}}$ is strongly proper follows from the observation made in [BGNX], Example 6.3.2.

Let us now prove that the string product is preserved under the map $H_{*+d}(L\mathfrak{U}) \to H_{*+d}(L\mathfrak{X})$. By the construction of the string product ([BGNX], 10.1), proving this boils down to showing that in the 2-cartesian diagram

$$\begin{tikzcd} \mathrm{Map}(8, \mathfrak{U}) \arrow[r, "\omega"] \arrow[d, "g"'] & L\mathfrak{U} \times L\mathfrak{U} \arrow[d, "f"] \\ \mathrm{Map}(8, \mathfrak{X}) \arrow[r, "\theta"'] & L\mathfrak{X} \times L\mathfrak{X} \end{tikzcd}$$

the Gysin maps $\theta^!$ and $\omega^!$ are compatible in the sense that

$$g_*(\omega^!(c)) = \theta^!(f_*(c)), \quad \text{for every } c \in H_*(L\mathfrak{U} \times L\mathfrak{U}). \tag{10.1}$$

Here, the bivariant class θ is the one obtained via independent pullback from the strong orientation class $\theta_{\mathfrak{X}} \in H^d(\Delta_{\mathfrak{X}})$, as in the diagram

$$\begin{tikzcd}
\mathrm{Map}(8, \mathfrak{X}) \arrow[r, "\theta"] \arrow[d] & L\mathfrak{X} \times L\mathfrak{X} \arrow[d] \\
\mathfrak{X} \arrow[r, "\theta_{\mathfrak{X}}", "\Delta_{\mathfrak{X}}"'] & \mathfrak{X} \times \mathfrak{X}
\end{tikzcd}$$

(Similarly, the class ω is obtained from the strong orientation class $\theta_{\mathfrak{U}} \in H^d(\Delta_{\mathfrak{U}})$.)

To prove the compatibility relation (10.1), we note that, by what we just showed in the first part of the proof, the bivariant class ω is the independent pullback of θ. Hence, the relation (10.1) follows from the 'Naturality' of Gysin maps ([BGNX], 9.2). $\square$

11 An example: Goldman bracket for 2-dimensional orbifolds

By Corollary 9.6, when $\mathfrak{X}$ is an oriented 2-dimensional Hurewicz stack, the equivariant homology $H_*^{S^1}(\mathrm{L}\mathfrak{X})$ is a graded Lie algebra. When $\mathfrak{X} = X$ is an honest surface, it is well-known that the degree 0 part $H_0^{S^1}(LX)$ is freely generated by the homotopy classes of free loops on X, and the Lie bracket is the so-called Goldman bracket. The above relationship between equivariant homology and free loops holds for general stacks as well (see Lemma 11.1 below). In this section, we describe the case were $\mathfrak{X}$ is a reduced (or effective) oriented 2-dimensional orbifold and we call the induced bracket on $H_0^{S^1}(\mathrm{L}\mathfrak{X})$ of an oriented 2-dimensional orbifold the *Goldman bracket* of the orbifold.

11.1 Goldman bracket for oriented 2-dimensional orbifolds

The functoriality lemma proved in the previous section allows us to explicitly write down the Goldman bracket for a reduced oriented 2-dimensional orbifold $\mathfrak{X}$. The idea is that the inclusion $\mathfrak{U} \hookrightarrow \mathfrak{X}$ of the complement of the orbifold locus of $\mathfrak{X}$ induces a surjection on fundamental groups $\pi_1(\mathfrak{U}) \to \pi_1(\mathfrak{X})$. Therefore, thanks to the functoriality (Proposition 10.3) and the following lemma, to compute the bracket of free loops in $\mathfrak{X}$, we can first lift them to free loops in $\mathfrak{U}$, compute the bracket there, and then project back down to $\mathfrak{X}$.

Lemma 11.1 *Let $\mathfrak{X}$ be a topological stack. Then, we have a natural isomorphism*

$$H_0^{S^1}(L\mathfrak{X}) \cong k[C],$$

where C is the set of free homotopy classes of loops on $\mathfrak{X}$, and k is the coefficients of the homology.

PROOF. The result is standard when $\mathfrak{X} = X$ is an honest topological space. We reduce the general stacks to this case as follows. By definition, $H_0^{S^1}(L\mathfrak{X}) = H_0[S^1\backslash L\mathfrak{X}]$. By Corollary 2.8, $H_0[S^1\backslash L\mathfrak{X}] \cong H_0[S^1\backslash LX] = k[C']$, where $\varphi\colon X \to \mathfrak{X}$ is a classifying space for $\mathfrak{X}$ and C' is the set of free homotopy classes of loops on X. Since φ induces a bijection $C' \xrightarrow{\sim} C$, the result follows. $\square$

Now, let $\mathfrak{X}$ be a 2-dimensional reduced orbifold, and let $\mathfrak{U} \subseteq \mathfrak{X}$ be the complement of the orbifold locus. We recall that a reduced 2-dimensional orbifold is a surface together with a discrete set of orbifold points. Each such orbifold point x has an isotropy group which is a cyclic group of the form $\mathbb{Z}/n\mathbb{Z}$ and the complement of the orbifold locus is a surface with a discrete set of punctures.

A simple application of van Kampen shows that $\pi_1(\mathfrak{U}) \to \pi_1(\mathfrak{X})$ is surjective. Indeed, $\pi_1(\mathfrak{X})$ is obtained by quotienting $\pi_1(\mathfrak{U})$ out by the relations $a^n = 1$, one relation for each orbifold point. Here, $a \in \pi_1(\mathfrak{U})$ is a simple loop going around the given orbifold point, and n is the order of the orbifold point. Combining Lemma 11.1 and Proposition 10.3, we thus obtain the following.

Lemma 11.2 *Let $\mathfrak{X}$ be a 2-dimensional reduced orbifold, and let $\mathfrak{U} \subseteq \mathfrak{X}$ be the complement of the orbifold locus. The natural map $H_0^{S^1}(\mathrm{L}\mathfrak{U}) \to H_0^{S^1}(\mathrm{L}\mathfrak{X})$ is a surjective map of Lie algebras.*

The Lemma allows us to compute the string bracket of an orbifold by lifting free loops to loops on the complement of the orbifold locus and compute the string bracket here using the usual intersection theory of curves in an honest surface.

11.2 Explicit description for disk with orbifold points

As a trivial example, consider a disk with an orbifold point with isotropy group $\mathbb{Z}/n\mathbb{Z}$. Then, the Goldman Lie algebra is the free k-module spanned by the set $\mathbb{Z}/n\mathbb{Z}$. It is an abelian Lie algebra since every two loops around the orbifold point can be homotopically deformed so that they do not intersect. We describe below the general case of disks with orbifold points. We start with the case of two orbifold points.

The disk with two orbifold points

Consider a disk $\mathfrak{D}$ with two orbifold points x, y. Let $a \in \mathbb{Z}/n\mathbb{Z}$ and $b \in \mathbb{Z}/m\mathbb{Z}$ be the generators of the isotropy groups at the points x and y, respectively. By van Kampen, the fundamental group of $\mathfrak{D}$ at a chosen based point is isomorphic to the free product of the isotropy groups of the orbifold points

$$\pi_1(\mathfrak{D}) \cong \mathbb{Z}/n\mathbb{Z} * \mathbb{Z}/m\mathbb{Z}$$

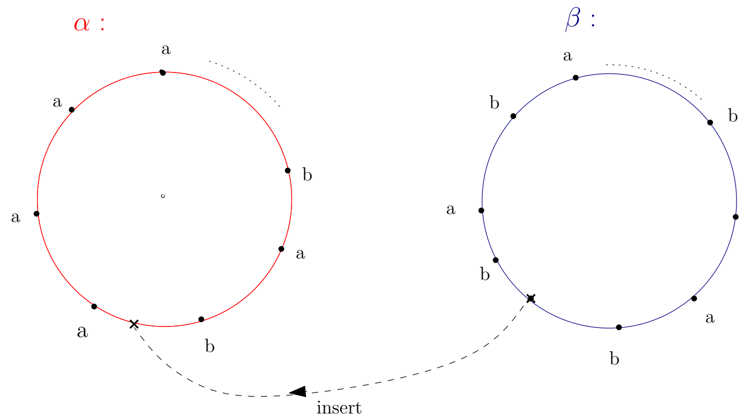

Figure 1: A presentation of α and β with an admissible insertion

so that every free loop is given by a (cyclic) word in the generators a and b, as in Figure (1). Since a and b have finite order, we do not need to consider negative powers of a or b to present a loop.

We now describe the Goldman bracket. Let α, β be two free loops presented by cyclic words as in Figure (1), that is as circle with finitely many points labeled by either a or b; the intervals between these points are colored red for α and blue for β.

We define the bracket $\{\alpha, \beta\}$ as follows.

1. Determine all *admissible pairs*, that is the pairs consisting of a red and a blue interval such that

   (a) the end points in the blue interval have the same labels;

   (b) the end points in the red interval have different labels.

2. For each admissible pair, cut both circles in the middle of the chosen intervals and insert the the blue circle into the red one by joining the cut intervals and preserving the cyclic ordering.

3. Assign the sign $+$ to the new circle obtained at step 2 if

   - the red interval is ab (in the cylic ordering) and the blue interval is aa,
   - or if the red interval is ba (in the cylic ordering) and the blue interval is bb

   Otherwise, assign the opposite sign.

4. Sum all the new circles obtained by choosing all admissible pairs of red and blue intervals satisfying the condition of step 1, using the signs given by step 3. This sum is denoted $\{\alpha, \beta\}$.

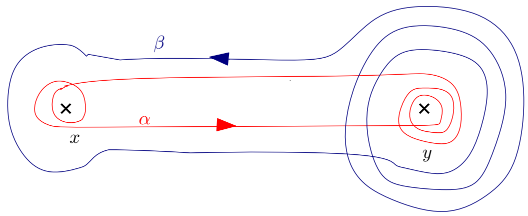

Figure 2: A red loop α sitting inside a blue loop β

In the above procedure, a loop given by a single generator is counted as a loop with a single interval which has the same end points. It is straightforward to check that its bracket with any other loop is trivial.

Proposition 11.3 *The bracket $\{\alpha, \beta\}$ given by the above procedure is the Goldman bracket of α and β in $H_0^{S^1}(\mathrm{L}\mathfrak{D})$.*

Proof. By Lemma 11.2, α and β can, respectively, be presented by a red and a blue loop, as in Figure (2). These loops lie entirely in the surface $\mathfrak{D}\setminus\{x, y\}$, and their Goldman bracket is simply their usual loop bracket computed in $\mathfrak{D}\setminus\{x, y\}$.

Note that, up to homotopy, we may assume that the red loop is sitting inside the blue loop in the neighborhood of x and y and the intersection between the loops α, β are transverse. Further, we may also assume that, when going from the neighborhood of a point to the other, the blue loop and the red loop does not intersect along the way; see Figure (2).

With these conventions, the blue loop and the red loop intersect (necessarily transversely) only when the blue loop is making turns around one of the orbifold points x and y, and the red loop is traveling from one orbifold point to the other; see Figure (2). Now we apply the usual formula for computing the Goldman bracket on the honest surface $\mathfrak{D} \setminus \{x, y\}$, which yields the result in Step 4 after quotienting out the relations $a^n = 1$ and $b^m = 1$. $\square$

Example 11.4 Suppose $a^2 = 1$ and $b^4 = 1$. Let $\alpha := a^2 b$ and $\beta := ab^2$. Note that $\alpha = a^2 b = b$, so

$$\{\alpha, \beta\} = \{b, b^2 a\} = 0,$$

because we can choose a very small representative for the loop b which does not intersect (a given representative of) the loop $b^2 a$.

On the other hand, if we follow our algorithm above, we get

$$\{\alpha, \beta\} = bab^2a^2 - b^2aba^2.$$

It is not obvious that this is equal to zero. Using the relation $a^2 = 1$, we can reduce it to

$$\{\alpha, \beta\} = bab^2 - b^2ab.$$

The latter is zero because these are free loops so we are allowed to perform cyclic permutation on the words.

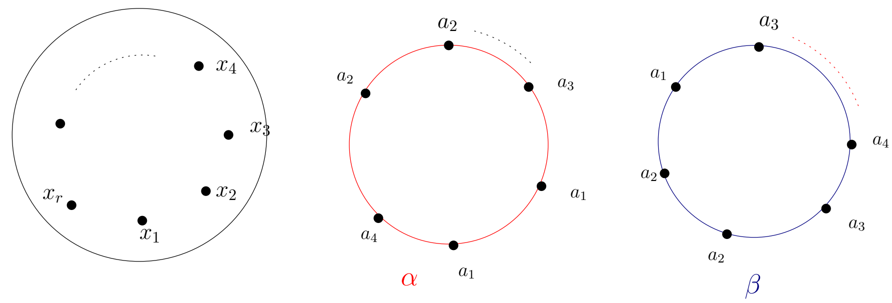

Figure 3: A Disk with r orbifold points and the presentation of a red loop α and a blue loop β

Example 11.5 Suppose $a^3 = 1$ and $b^4 = 1$. Let $\alpha := a^2b$ and $\beta := ab^2$, as in the previous example. Using our algorithm above, we get

$$\{\alpha, \beta\} = bab^2a^2 - b^2aba^2.$$

It is easy to see that this is indeed non-zero.

The disk with finitely many orbifold points

We now describe below the general case of a disk $\mathfrak{D}$ with finitely many orbifold points $\{x_i\}_{i=1\ldots r}$. We write n_i for the order of the orbifold point x_i, i.e., the isotropy group of $\mathfrak{D}$ at the point x_i is $\mathbb{Z}/n_i\mathbb{Z}$. Up to an orbifold isomorphism we may assume the points x_i are cyclically ordered. By van Kampen theorem, the fundamental group of $\mathfrak{D}$ at a chosen based point is isomorphic to the free product of the isotropy groups of the orbifold points

$$\pi_1(\mathfrak{D}) \cong \mathbb{Z}/n_1\mathbb{Z} * \cdots * \mathbb{Z}/n_r\mathbb{Z}$$

so that every free loop is given by a finite product of generators of the isotropy groups.

The above procedure can be generalized to a disk with r many orbifold points (denoted $x_1, \ldots, x_r$). Let a_i be the generator of the inertia group $\mathbb{Z}/n_i\mathbb{Z}$ at the point x_i. As in the case of two orbifold points, we can present two free loops α, β by (cyclic) words on the generators a_i, as in Figure (3). Again, the intervals in α are colored red and the ones in β are colored blue. The bracket $\{\alpha, \beta\}$ is given by a similar cut and insert procedure as above. The only difference is in Steps 1 and 3 where we determine which red and blue intervals are to be cut and what sign to assign after inserting the blue loop into the red loop.

We define the bracket $\{\alpha, \beta\}$ as follows.

(i) Determine which pairs consisting of a red interval a_ia_j and a blue interval a_ka_l are *admissible* (for the cut and insert process 2). Here, a pair is said to be admissible if it satisfies the following conditions:

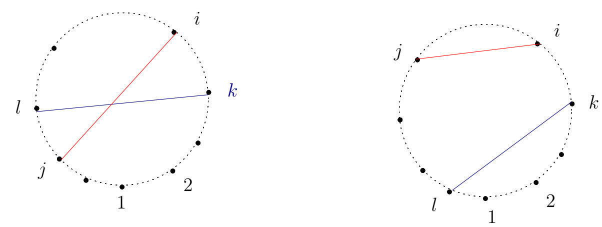

Figure 4: The left pair is admissible while the right pair is not admissible

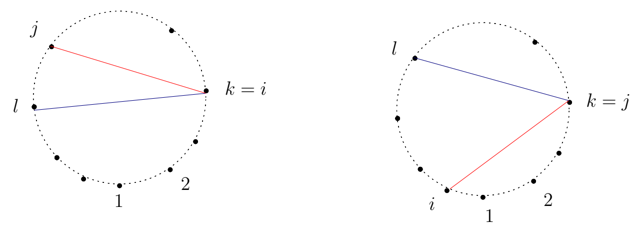

Figure 5: The left triangle is an admissible pair while the right triangle is not

- We have that $i \neq j$, the (unoriented) intervals are distinct, and the (colored in red) segment $[ij]$ intersects the (possibly degenerated and colored in blue) segment $[kl]$ (in the cyclic ordering see Figure (4)).
- If either $k = l$ or all the points are distinct, then there is no further condition. Otherwise, the red segment $[ij]$ and the blue segment $[kl]$ form a non-degenerate triangle (inscribed in the unit circle) and there are two possible cases:
  - Case 1: the intervals intersect in k, the starting point of the blue interval. In this case, the pair is admissible if the red segment is above the blue segment in the cyclic ordering, see Figures (5) (and is not admissible otherwise).
  - Case 2: the intervals intersect in l, the end point of the blue interval. In this case, the pair is admissible if the red segment is below the blue segment in the cyclic ordering (and is not admissible otherwise).

(ii) For every admissible pair (as given by Step (i)), cut the blue circle in the middle of the segment $a_i a_j$ and the red circle in the middle of the segment $a_k a_l$. Then insert the blue circle into the red one by joining the cut intervals and preserving the cyclic ordering.

(iii) To determine what sign to assign to each new loop obtained in step (ii), think of the segments $[kl]$ and $[ij]$ as oriented lines in $\mathbb{R}^2$ (with the orientation given by the one of the disk $\mathfrak{D}$); in the case where $k = l$, use the tangent line to the circle at $k = l$, with the anti-clockwise orientation. Then, the sign rule is the same as the sign rule for oriented lines in $\mathbb{R}^2$.

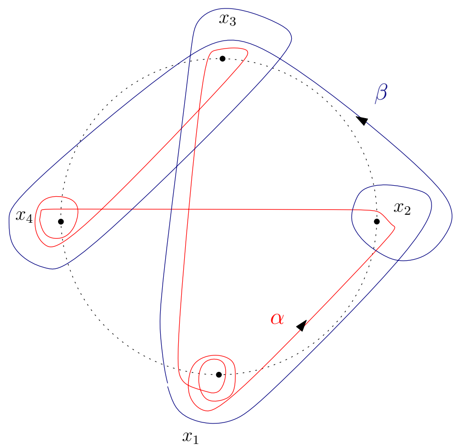

Figure 6: A presentation a the red loop α inside a blue loop β

(iv) Sum up all the new circles obtained by taking all admissible pairs of red and blue intervals (satisfying the condition of step (i)), with the signs given by step (iii). This sum is denoted $\{\alpha, \beta\}$.

Proposition 11.6 *The bracket $\{\alpha, \beta\}$ given by the above procedure is the Goldman bracket of α and β in $H_0^{S^1}(\mathrm{L}\mathfrak{D})$.*

PROOF. The proof is similar to the case of two orbifold points. By Lemma 11.2, α and β can, respectively, be presented by a red and a blue loop, as in Figure (6). These loops lie entirely in the surface $\mathfrak{D} \setminus \{x_1, \ldots, x_r\}$, and their Goldman bracket is simply their usual loop bracket computed in $\mathfrak{D} \setminus \{x_1, \ldots, x_r\}$. Note that, up to homotopy, we may assume that the red loop is sitting inside the blue loop in the neighborhood of any orbifold point, and that α and β intersect transversally. Further, we may also assume that, when going from the neighborhood of one orbifold point to another, say from x_k to x_l, the blue loop intersects the red loop only when this red loop is going from a neighborhood of x_i to a neighborhood of a distinct x_j in a way that $a_i a_j$ and $a_k a_l$ form an admissible pair; see Figure (6). Now, we apply the usual formula for computing the Goldman bracket on the honest surface $\mathfrak{D} \setminus \{x_1, \ldots, x_r\}$ which yields the result in Step (iv) after quotienting out the relations $a_i^{n_i} = 1$, $i = 1 \ldots r$. $\square$

Case of a reduced 2-dimensional orbifold with finitely many orbifold points

Now, let $\mathfrak{X}$ be a reduced orbifold with finitely many orbifold points $x_1, \ldots, x_r$. Up to an orbifold isomorphism, we may assume that $\mathfrak{X}$ is the connected sum of a surface with no orbifold points and a genus 0 surface with r orbifold points, denoted $\mathfrak{D}$.

Then a free loop in $\mathfrak{X}$ can be presented by a sequence of loops lying alternatively in $\mathfrak{D}$ and $\mathfrak{X} \setminus \mathfrak{D}$ (the first loops being given by cyclic words on the generators of the isotropy groups of the orbifold points and are referred to as the purely orbifold segment of a loop). It follows that the Goldman bracket $\{\alpha, \beta\}$ can be computed by using the standard intersection procedure for all possible sequences of loops in $\mathfrak{X} \setminus \mathfrak{D}$ and using the cut and insert process described above for disks with orbifold points for all orbifolds segments.

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