ON THE DIFFERENCE EQUATION OF THE POINCARÉ TYPE (Part 1)

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Dedicated to the memory of professor A.O. Gelfond.

Sit ye near some old Cavern's Mouth, and brood
Until ye start, as if the sea nimphs quir'd!
John Keats

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§0. Foreword.

As a participiant of Gelfond's seminar at the Moscow University, I was always excited by his big intellect and the width of his interests, which were not limited only to the Trancendental Number Theory, where he had obtained the most famous results, but spread over many other topics: Function Theory, Analytical Number Theory, Integral Equations, History of Mathematics etc. . He paid a special attention to the Difference Equations, in particular, to the Difference Equations of Poincaré Type. His results (together with I.M. Kubenskaya) on this subject [2] are well-known; he had also inspired the works on this topic by his pupils M.A.Evgrafov [3] and G.A.Frejman [4].

Let the difference equation

$$\sum_{k=0}^{n} a_k(\nu)y(\nu+k)=0, \tag{1}$$

with $n \in \mathbb{N}$, $a_k(\nu) \in \mathbb{C}$ for $k = 0, \dots, n$ and $\nu \in \mathbb{N}-1$ be given. Let further $a_n(\nu) = 1$, where $\nu \in \mathbb{N}-1$, and suppose there exists

$$a_k^{\sim} = \lim_{\nu\to\infty} a_k(\nu). \tag{2}$$

Equation (1) satisfying condition (2) is called a difference equation of Poincaré type. The polynomial

$$T(z) = \sum_{k=0}^{n} a_k^{\sim} z^k \tag{3}$$

is called the characteristic polynomial of the equation (1).

The main result here belongs to O.Perron [1].

Theorem 1 (Perron's Theorem.) *Let $\rho_1, \dots, \rho_s$ are pairwise distinct absolute values of all the roots of the polynomial (3) and let e_σ, $\sigma = 1, \dots, s$, is the sum of multyplicities of these roots with module equal to ρ_σ. So, $\sum\limits_{\sigma=0}^{s} e_\sigma = n$. Let $\rho_k < \rho_i$ for $1 \le i < k \le s$.*

Then the space of solutions of the equation (1) have a basis Γ, which splits into s mutually disjoint classes Ω_σ, $\sigma = 1, \dots, s$, with the following property: if a solution $y(\nu)$ of the equation (1) is a linear combination of the elements of Γ, in which all the elements of Ω_σ, $1 \le \sigma \le q-1$, where $q \in \{1, \dots, s\}$, have coefficients equal to zero, but at least one element in Ω_q has in this linear combination the nonzero coefficient, then

$$\limsup_{\nu \to \infty} |y(\nu)|^{1/\nu} = \rho_q. \tag{4}$$

The equality (4) may be reformulated so:

for any nonzero solution of the equation (1) and any $\varepsilon \in (0, 1)$ there exist a positive constants $\gamma_{0,0}$ and $\gamma_{0,1}$ such that

$$|y(\nu)| \le \gamma_{0,1}(\rho_q + \varepsilon)^\nu$$

for any $\nu \in \mathbb{N} - 1$ and the inequality

$$\gamma_{0,0}(\rho_q(1 - \varepsilon))^\nu \le |y(\nu)| \tag{5}$$

is fulfilled, when ν run over an infinite subset of $\mathbb{N} - 1$; because this infinite subset is undefined, the last assertion is insufficient for some problems of Diophantine Approximations. This gap was partially closed in [6] (Theorem 2 and Theorem 3).

Theorem 2. *Let $\lambda_1, \dots, \lambda_n$ are the zeros of the polynomial (3) and one can meet each root here as many times as the multiplicity of this root is. Let $|\lambda_i| \ge |\lambda_k|$ for $1 \le i \le k \le n$. Let further the equality (2) is fulfilled. Let $a_n(\nu) \ne 0$ for $\nu \in \mathbb{N} - 1$. Then there exists the fundamental system of solutions*

$$y_1(\nu), \dots, y_n(\nu) \tag{6}$$

of the equation (1) such that, if

$$h(V) = \max(|v_1|,\ \dots,|v_n|),\ y(V,\nu) = \sum_{k=1}^{n} v_k y_k(\nu),$$

$$k_1(V) = \inf\{k \in [1,n] \cap \mathbb{N}\colon v_k \neq 0\}, k_2(V) = \sup\{k \in [1,n] \cap \mathbb{N}\colon v_k \neq 0\},$$

where $V \in \mathbb{C}^n \diagdown \{0\}, V = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$, *then the following conditions ar fulfilled:*

(I) for any $\varepsilon \in (0,1)$ *and any* $V \in \mathbb{C}^n \diagdown \{0\}$ *there exists* $\gamma_3(V,\varepsilon) > 0$ *such that*

$$\gamma_3(V,\varepsilon)(|\lambda_{k_1(V)}|(1-\varepsilon))^\nu \leq \sum_{\kappa=0}^{n-1} |y(V,\nu+\kappa)|,$$

where $\nu \in \mathbb{N} - 1$;

(II) for any $\varepsilon \in (0,1)$ *there exist* $\gamma_4(\varepsilon) > 0$ *and* $\gamma_5(\varepsilon) > 0$ *such that*

$$\gamma_4(\varepsilon)h(V)(|\lambda_{k_2(V)}|(1-\varepsilon))^\nu \leq \sum_{\kappa=0}^{n-1} |y(V,\nu+\kappa)|,$$

$$|y(V,\nu)| \leq \gamma_5(\varepsilon)h(V)(|\lambda_{k_1(V)}| + \varepsilon))^\nu,$$

where $\nu \in \mathbb{N} - 1$, $V \in \mathbb{C}^n$.

Theorem 3. *Let are fulfilled all the conditions of the Theorem 2. Let (6) is the declared in the Theorem 2 the fundamental system of solutions of the equation (1). Let* $y_1^{\sim}(\nu),\ \dots,\ y_r^{\sim}(\nu)$ *be some linear independent over* $\mathbb{C}$ *system of solutions of the equation (1) and let* $y_l^{\sim}(\nu) = \sum\limits_{k=1}^{n} \eta_{k,l} y_k(\nu)$, *where* $\eta_{k,l} \in \mathbb{C}, k = 1,\ \dots,\ n;\ l = 1,\ \dots,\ r;\ \nu \in \mathbb{N} - 1$. *Let*

$$\mathfrak{K} = \{k \in [1,n] \cap \mathbb{N}\colon \{l \in [1,r] \cap \mathbb{N}\colon \eta_{k,l} \neq 0\} \neq \emptyset\},$$

and let $k_3 = \inf(\mathfrak{K}), k_4 = \sup(\mathfrak{K})$. *For any* $X \in \mathbb{C}^r, X = \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix}$, *let*

$$y^{\sim}(X,\nu) = x_1 y_1^{\sim}(\nu)\ +\ \dots\ +\ x_r y_r^{\sim}(\nu),\ h(V) = \max(|v_1|,\ \dots,\ |v_n|).$$

Then for any $\varepsilon \in (0,1)$ *there exist* $\gamma_6(\varepsilon) > 0$ *and* $\gamma_7(\varepsilon) > 0$ *such that*

$$\gamma_6(\varepsilon)h(X)(|\lambda_{k_4} h(X)|(1-\varepsilon))^\nu \leq \sum_{\kappa=0}^{n-1} |y^{\sim}(X,\nu+\kappa)|,$$

$$|y^{\sim}(X,\nu)| \le \gamma_7(\varepsilon)h(X)(|\lambda_{k_3}|+\varepsilon))^{\nu},$$

where $\nu \in \mathbb{N}-1$, $X \in \mathbb{C}^r$.

The following Theorem [8] may be considered as development of the above Theorem 2 and Theorem 3.

Theorem 4. *Let* $s \in \mathbb{N}-1, n \in \mathbb{N}$,

$$a_i^{\sim} \in \mathbb{C},\ a_i(\nu) \in \mathbb{C},\ a_n(\nu) = 1,\ a_i(\nu) - a_i^{\sim} = O(1/\nu)$$

for $\nu \in \mathbb{N}$ *and* $0 \le i \le n$. *For* $m \in \mathbb{N}$ *let* V_m *denotes linear over* $\mathbb{C}$ *space of solutions* $y = y(\nu)$ *of the equation*

$$\sum_{k=0}^{n} a_k(\nu)y(\nu+k) = 0, \tag{7}$$

where $\nu \in m+\mathbb{N}-1$. *Let the absolute values of all the roots of the polynomial* $a_0^{\sim}z^0 + \ldots + a_n^{\sim}z^n$ *are among the numbers* $\{\rho_i\colon 1 \le i \le 1+s\}$, *such that* $\rho_{s+1} = 0$ *and* $\rho_j < \rho_i$ *for* $1 \le i < j \le s+1$. *Let* e_i *and* k_i *denote respectively the sum of and the maximum of the multiplicities of those roots, whose absolute value is equal to* ρ_i, $1 \le i \le 1+s$. *For given* $y = y(\nu)$ *in* V_m, *let* $\omega_{n,y}(\nu) = \max(|y(\nu)|, \ldots, |y(\nu+n-1)|)$. *Then there exist* $A > 0$, $m \in \mathbb{N}$, *and the subspaces* $V^{\vee}_{m,1}, \ldots, V^{\vee}_{m,s+1}$ *such that*

$$V_m = V^{\vee}_{m,1} \oplus \ldots, \oplus V^{\vee}_{m,s+1},\ \dim_{\mathbb{C}}(V^{\vee}_{m,i}) = e_i,\ 1 \le i \le s+1.$$

and, moreover, the spaces

$$V^{\wedge}_{m,j} = V^{\vee}_{m,j} \oplus \ldots \oplus V^{\vee}_{m,s+1},\ 1 \le j \le s+1$$

have the folloving property:

if $y \in V^{\wedge}_{m,\theta} \backslash V^{\wedge}_{m,\theta+1}$ *for some* $\theta \in \{1, ..., s\}$, *then*

$$\exp(-A(ln(\nu)+\nu^{1-1/k_\theta}))\beta \le \omega_{n,y}(\nu)(\rho_\theta)^{-\nu} \le \exp(A(ln(\nu)+\nu^{1-1/k_\theta}))\omega_{n,y}(m),$$

where $\beta = \beta(y) > 0$ *depends only from equation and* y *(but does not depend from* ν*); if* $k^* := k_{s+1} > 0$ *and* $y \in V^{\vee}_{m,s+1} (= V^{\wedge}_{m,s+1})$, *then*

$$|y(\nu)| \le (A/\nu)^{\nu/k^*}\omega_{n,y}(m).$$

The main part of Perron's proof of his Theorem is the following

Perron's decomposition Lemma ([1], Hilfsatz 3). *Let*

$$T(z) = T_1(z)T_2(z), \tag{8}$$

where $T(z)$ is characteristic polynomial (3),

$$T_1(z) = \sum_{\alpha=0}^{p} b_\alpha^\sim z^\alpha,\ T_2(z) = \sum_{\beta=0}^{q} u_\beta^\sim z^\beta, \tag{9}$$

$b_p^\sim = u_q^\sim = a_n^\sim = 1$ *and absolute value of each root of $T_1(z)$ is greater than the absolute value of each root of $T_2(z)$.*

Then there exist $m \in \mathbb{N}$, $b_\alpha(\nu) \in \mathbb{C}, \alpha = 0, \dots, p, \nu \in \mathbb{N} + m - 1$,
and $u_\beta(\nu) \in \mathbb{C}$, $\beta = 0, \dots, q$, $\nu \in \mathbb{N} + m - 1$ such that

$$\lim_{\nu\to\infty} b_\alpha(\nu) = b_\alpha^\sim,\ \alpha = 0, \dots, p,\ b_p(\nu) = 1, b_0(\nu) \neq 0, \tag{10}$$

$$\lim_{\nu\to\infty} u_\beta(\nu) = u_\beta^\sim, \beta = 0, \dots, q,\ u_q(\nu) = 1, \tag{11}$$

where $\nu \in \mathbb{N} + m - 1$, and the connected with the equation (1) the equation

$$\sum_{k=0}^{n} a_k(\nu) y(\nu + k) = 0, \tag{12}$$

where $\nu \in \mathbb{N} - 1 + m$, is equivalent to the equation

$$\sum_{\alpha=0}^{p} b_\alpha(\nu) y(\nu + \alpha) = r(\nu),$$

where $\nu \in \mathbb{N} - 1 + m$ and $r(\nu)$ satisfies to the equation

$$\sum_{\beta=0}^{q} u_\beta(\nu) r(\nu + \beta) = 0$$

with $\nu \in \mathbb{N}-1+m$. The following supplement to the Perron's decomposition Lemma was proved in [7].

Theorem 5. *Let the functin $\xi(x)$ is defined on $[0, +\infty)$, let $\xi(x) > 0$ for $x \in [0, +\infty)$, let $\xi(x)$ decreases together with increasing of $x \in [0, +\infty)$ and let $\lim_{x\to\infty} (\xi(x)) = 0$. Let*

$$\xi(x/2) = O(\xi(x)), \tag{13}$$

when $x \to \infty$,

$$\lim_{x\to 0} (\log(\xi(x)))/x = 0. \tag{14}$$

Let further all the conditions of the Perron's decomposition Lemma are fulfilled and together with (2) the more strong condition holds: $a_k(\nu) - a_k^{\sim} = O(\xi(\nu)), k = 0, \ldots, n,$ when $\nu \to \infty$. Then for $b_\alpha(\nu)$, $\alpha = 0, \ldots, p$, and $u_\beta(\nu)$, $\beta = 0, \ldots, q$, from the assertion of the Perron's decomposition Lemma the following conditions are fulfilled (cf. with (10) and (11)):

$$b_\alpha(\nu) - b_\alpha^{\sim} = O(\xi(\nu)),\ \alpha = 0, \ldots, p, b_0(\nu) = 1,\ b_q(\nu) \neq 0,$$

$$u_\beta(\nu) - u_\beta^{\sim} = O(\xi(\nu)),\ \beta = 0, \ldots, q,\ u_0(\nu) = 1,$$

My aim is to present the proofs of the theorems 2, 3, and 4. Here I prove the Theorem 5.

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§1. Difference operators.

Let $m \in \mathbb{N} - 1$ and $\mathfrak{V}_m = \mathbb{C}^{\mathbb{N}-1+m}$ be the algebra over $\mathbb{C}$ of all the maps of $\mathbb{N} - 1 + m$ into $\mathbb{C}$; let $\mathbf{1}_m$ and $\mathbf{0}_m$ are respectively the unity element and the zero element of this algebra. Let $R = \mathfrak{V}_m$, . For each $f \in R$ let $\mathfrak{m}_f$ be the operator on R, which turns each $a \in R$ into fa. Let $\mathfrak{M}_R = \{\mathfrak{m}_f : f \in R\}$. Then $\mathfrak{M}_R$ as the ring consisting of all the $R-$ endomorphisms of the commutative ring R with the unity element is isomorph to R. Let $\mathfrak{M}_R^{\vee}$ is the ring of all the $\mathbb{C}-$endomorphisms of the additive group of the $\mathbb{C}-$algebra R. If $\phi \in \mathfrak{M}_R^{\vee}$ and $\psi \in \mathfrak{M}_R^{\vee}$, then $\phi \circ \psi$ denotes the element in $\mathfrak{M}_R^{\vee}$ such that $(\phi \circ \psi)a = \phi(\psi a)$ for any $a \in R$. On $R = \mathfrak{V}_m$ acts the $\mathbb{C}-$linear operator $\bigtriangledown$, which tansforms each $y = y(\nu) \in \mathfrak{V}_m$ into $\bigtriangledown y$ such that $(\bigtriangledown y)(\nu) = y(\nu + 1)$ for all $\nu \in \mathbb{N} - 1 + m$. Let $R < \bigtriangledown >$ be the subring of the ring $\mathfrak{M}_R^{\vee}$, generated by $\bigtriangledown$ and $\mathfrak{M}_R$. Since $\bigtriangledown \circ \mathfrak{m}_f = \mathfrak{m}_{\bigtriangledown f} \circ \bigtriangledown$, it follows that each $D \in R < \bigtriangledown >$ may be represented in the form $D = \sum\limits_{k=0}^{n(D)} \mathfrak{m}_{\mathfrak{c}_k(D)} \circ \bigtriangledown^k$, with $n(D) \in \mathbb{N} - 1, \bigtriangledown^0 = \mathfrak{m}_{\mathbf{1}_R}$ and $\mathfrak{c}_k(D) \in R$ for $k = 0, \ldots, n(D)$; in particular, $\bigtriangledown^k = \mathfrak{m}_{\mathbf{1}_R} \bigtriangledown^k$ for any $k \in \mathbb{N} - 1$; if we put $\mathfrak{c}_k(D) = 0$ for $k \in n(D) + \mathbb{N}$, then

$$D = \sum_{k=0}^{\infty} \mathfrak{m}_{\mathfrak{c}_k(D)} \circ \bigtriangledown^k. \tag{15}$$

Let $R|_n < \bigtriangledown >= \{D \in R < \bigtriangledown >: (k \in \mathbb{N} + n) \Rightarrow (\mathfrak{c}_k(D) = 0)\}$, where $n \in \mathbb{N} - 1$. Each element $D \in R|_n < \bigtriangledown >$ may be represented in the form

$$D = \sum_{k=0}^{n} \mathfrak{m}_{\mathfrak{c}_k(D)} \circ \bigtriangledown^k, \tag{16}$$

Clearly,

$$D = \sum_{k=0}^{n} \mathfrak{m}_{\mathfrak{a}_k(D)} \circ (\triangle)^k, \tag{17}$$

where $D \in R|_n < \triangledown >$, $\triangle = \triangledown - \mathfrak{m}_{\mathbf{1}\,R}$, $\triangle^0 = \mathfrak{m}_{\mathbf{1}\,R}$ and $\mathfrak{a}_k(D) \in R$ for $k = 0, \dots, n$; the action of D on $1, \nu, \dots, \nu^n$ shows that $\mathfrak{a}_k(D)$ in (17) and

$$\mathfrak{c}_k(D) = \sum_{\kappa=k}^{n} \binom{\kappa}{k} \mathfrak{a}_\kappa(D)$$

in (16) for $k = 0, \dots, n$ are defined uniquely and are the $\mathbb{C}-$linear maps of the space $R|_n < \triangledown >$ into R. We name by difference polynomials the elements of $R < \triangledown >$. Let

$$\phi D = \sum_{k=0}^{\infty} \mathfrak{m}_{\phi \mathfrak{c}_k(D)} \circ \triangledown^k, \tag{18}$$

where $\phi \in \mathfrak{M}_R^\vee$, $D \in R < \triangledown >$; in particular, $D_1 D = \sum_{k=0}^{\infty} \mathfrak{m}_{D_1 \mathfrak{c}_k(D)} \circ \triangledown^k$, where $D \in R < \triangledown >$, $D_1 \in R < \triangledown >$. Clearly, $D_1 D$ not always coincides with $D_1 \circ D$. The equality (18) determines on $R < \triangledown >$ the structure of the left $\mathfrak{M}_R^\vee$-module. Clearly,

$$\triangledown^s \circ D = (\triangledown^s D) \circ \triangledown^s, \tag{19}$$

for any $D \in R < \triangledown >$ and $s \in \mathbb{N} - 1$. For each $D \in \mathfrak{V}_m < \triangledown >$ the elements $\mathfrak{c}_k(D)$ are defined only for $k \in \mathbb{N} - 1$; let

$$\mathfrak{c}_k(D) = \mathbf{0}_m \tag{20}$$

for $k \in -\mathbb{N}$. Let $D_1 \in \mathfrak{V}_m|_p < \triangledown >$ and $D_2 \in \mathfrak{V}_m|_q < \triangledown >$. In view of (19) and (20),

$$D_2 \circ D_1 = \sum_{s=0}^{q} \mathfrak{m}_{\mathfrak{c}_s(D_2)} \circ \triangledown^s \circ D_1 = \sum_{s=0}^{q} \mathfrak{m}_{\mathfrak{c}_s(D_2)} \circ (\triangledown^s D) \circ \triangledown^s =$$

$$\sum_{s=0}^{q} \sum_{k=0}^{p} \mathfrak{m}_{\mathfrak{c}_s(D_2)} \circ \mathfrak{m}_{\triangledown^s \mathfrak{c}_k(D_1)} \circ \triangledown^{k+s} = \sum_{s=0}^{q} \sum_{k=0}^{p} \mathfrak{m}_{\mathfrak{c}_s(D_2) \triangledown^s \mathfrak{c}_k(D_1)} \circ \triangledown^{k+s} =$$

$$\sum_{\kappa=0}^{p+q} \sum_{s=0}^{q} \mathfrak{m}_{\mathfrak{c}_s(D_2) \triangledown^s \mathfrak{c}_{\kappa-s(D_1)}} \circ \triangledown^{\kappa}.$$

Therefore, for $D = D_2 \circ D_1$, we have

$$\mathfrak{c}_\kappa(D_2 \circ D_1) = \sum_{s=0}^{q} \mathfrak{c}_s(D_2) \bigtriangledown^s \mathfrak{c}_{\kappa-s}(D_1), \tag{21}$$

where $\kappa = 0, \ldots, p+q$. In particular,

$$\mathfrak{c}_\kappa(D_2 \circ D_1) = \mathfrak{c}_q(D_2) \bigtriangledown^q \mathfrak{c}_p(D_1), \tag{22}$$

$$\mathfrak{c}_0(D_2 \circ D_1) = \mathfrak{c}_0(D_2)\mathfrak{c}_0(D_1).$$

If

$$\mathfrak{c}_p(D_1) = \mathbf{1}_m, \mathfrak{c}_q(D_2) = \mathbf{1}_m, \tag{23}$$

then, in view of (22), $\mathfrak{c}_{p+q}(D_2 \circ D_1) = \mathbf{1}_m$. If (23) is fulfilled, then

$$\mathfrak{c}_\kappa(D_2 \circ D_1) = \bigtriangledown^q \mathfrak{c}_{\kappa-q}(D_1) + \sum_{s=0}^{q-1} \mathfrak{c}_s(D_2) \bigtriangledown^s \mathfrak{c}_{\kappa-s}(D_1), \tag{24}$$

where $\kappa \in \mathbb{Z}$. Let

$$a_k = \mathfrak{c}_k(D_2 \circ D_1),\ u_k = \mathfrak{c}_k(D_2),\ b_k = \mathfrak{c}_k(D_1), \tag{25}$$

where $k \in \mathbb{Z}$. Then (24) may be rewritten in the form

$$a_\kappa = \bigtriangledown^q b_{\kappa-q} + \sum_{s=0}^{q-1} u_s \bigtriangledown^s b_{\kappa-s}, \tag{26}$$

where $\kappa \in \mathbb{Z}$. In view of (25),

$$a_\kappa = \sum_{s=0}^{q-1} u_s \bigtriangledown^s b_{\kappa-s}, \tag{27}$$

where $\kappa = 0, \ldots, q-1$. Let K is an arbirary ring, $p \in \mathbb{N}$, $q \in \mathbb{N}$, $n = p+q$; for any $X \in K^n$ let $\pi_1(X)$ be the element in K^p, which have for $i = 1, \ldots, q$ as its $i-$th coordinate the i-coordinate of X and let $\pi_2(X)$ be the element in K^p, which have for $i = 1, \ldots, p$ as its $i-$th coordinate the $q+i$-coordinate of X. Let $B^\wedge$ be the $q \times q$-matrix, in which $\bigtriangledown^{k-1} b_{i-k}$ stands on the intersection of i-th row and k-th columns for $i = 1, \ldots, q$ and $k = 1, \ldots, q$.Clearly $B^\wedge$ is the lower triangle matrix in $\mathrm{Mat}_q(\mathfrak{V}_m)$. Let U be the column with q elements and u_{k-1} for $k = 1, \ldots, q$ is k-th of these elements. Let B be the column with p elements and b_{k-1} for $k = 1, \ldots, p$ is k-th of these elements. Let A be the column with n elements and

a_{k-1} for $k = 1, \dots, n$ is k-th of these elements. We see that equalities (27) may be rewritten in the form

$$\pi_1(A) = B^\wedge U. \tag{28}$$

Taking in (26) $\kappa = q + \mu$ with $\mu = 0, \dots, p-1$ we obtain the equalities

$$a_{q+\mu} = \bigtriangledown^q b_\mu + \sum_{s=0}^{q-1} u_s \bigtriangledown^s b_{q+\mu-s}. \tag{29}$$

Therefore $\bigtriangledown^q b_{i-1} = a_{q+i-1} - \sum\limits_{k=1}^{q} u_{k-1} \bigtriangledown^{k-1} b_{q+i-k}$, where $i = 1, \dots, p$. Let $\mathfrak{B}$ be the $p \times q$-matrix,in which $\bigtriangledown^{k-1} b_{q+i-k}$ stands on the intersection of i-th row and k-th columns for $i = 1, \dots, p$ and $k = 1, \dots, q$. Clearly, (29) may be rewritten in the form

$$\bigtriangledown^q B = \pi_2(A) - \mathfrak{B}U. \tag{30}$$

Let $m \in \mathbb{N}-1$ and let s_m be the map of $\mathfrak{V}_0$ into $\mathfrak{V}_m$, for which $(s_m(y))(\nu) = y(\nu-m)$ for any $\nu \in \mathbb{N} - 1 + m$. Clearly, s_m is the $\mathbb{C}$-isomorphism of $\mathbb{C}$-algebra $\mathfrak{V}_0$ onto $\mathbb{C}$-algebra $\mathfrak{V}_m$. Let $m \in \mathbb{N} - 1$ and let r_m be the map of $\mathfrak{V}_0$ into $\mathfrak{V}_m$, for which $r_m(y)$ is the restriction of $y \in \mathfrak{V}_0$ onto $\mathbb{N} - 1 + m$. Clearly, $s_m \bigtriangledown^m = r_m$ for each $m \in \mathbb{N} - 1$. Let $r_m(D) = \sum\limits_{k=0}^{\infty} \mathfrak{m}_{r_m(\mathfrak{c}_k(D))} \circ \bigtriangledown^k$, where $m \in \mathbb{N} - 1$, $D \in \mathfrak{V}_0 < \bigtriangledown >$; clearly, $r_m(\mathfrak{V}_0 < \bigtriangledown >) = \mathfrak{V}_m < \bigtriangledown >$. Let $D \in (\mathfrak{V}_0)|_n < \bigtriangledown >$, and $a_k = \mathfrak{c}_k(D)$ for $k = 0, \dots, n$. Then $D = \sum\limits_{k=0}^{\infty} \mathfrak{m}_{a_k} \circ \bigtriangledown^k$, $r_m(D) = \sum\limits_{k=0}^{\infty} \mathfrak{m}_{r_m(a_k)} \circ \bigtriangledown^k$, where $m \in \mathbb{N} - 1$, $r_m(D)$ is linear operator on $\mathfrak{V}_m$ and $\operatorname{Ker}(r_m(D))$ for any $m \in \mathbb{N} - 1$ coincides with the space V_m of solutions of the equation (7).

Let the functin $\xi(x)$ is defined on $[0, +\infty)$ and let $m \in \mathbb{N} - 1$; we denote by $\tau_m(\xi)$ the restriction of ξ on $m + \mathbb{N} - 1$. We denote further by $l_{m,\infty}$ the set of all the bounded functions from $\mathfrak{V}_m$, by $l_m^\sim$ - the set all the functions $y = y(\nu) \in \mathfrak{V}_m$, for which there exists $\lim\limits_{\nu\to\infty} y(\nu)$.

If the functin $\xi = \xi(x)$ is defined on $[0, +\infty)$, and $m \in \mathbb{N} - 1$, let

$$l_{m,\xi} = \tau_m(\xi) l_{m,\infty} = \{\tau_m(\xi) y \colon y \in l_{m,\infty}\}.$$

For each $m \in \mathbb{N}-1$, the field $\mathbb{C}$ is embeded into $\mathfrak{V}_m$ in the form $\mathbb{C}\,\mathbf{1}_m$. If S is some nonempty subset of the ring $\mathfrak{V}_m$, let $S < \bigtriangledown >$ (respectively $S|_n < \bigtriangledown >$) be the set of all the $D \in \mathfrak{V}_m < \bigtriangledown >$ (respectively $D \in (\mathfrak{V}_m)|_n < \bigtriangledown >$), for which $\mathfrak{c}_k(D) \in S$ for all the $k \in \mathbb{N} - 1$. Clearly, if S is $\mathbb{C}-$subalgebra of $\mathfrak{V}_m$ and $< \bigtriangledown >$ turns S into S, then $S < \bigtriangledown >$ is $\mathbb{C}-$subalgebra of $\mathfrak{V}_m < \bigtriangledown >$; in particular,$(\mathbb{C}\,\mathbf{1}_m) < \bigtriangledown >$

is $\mathbb{C}-$subalgebra of $\mathfrak{V}_m < \bigtriangledown >$. On $l_m^{\sim}$ there exists the $\mathbb{C}$-linear functional lim, for which $\lim(y) = \lim\limits_{\nu\to\infty} y(\nu)$ for any $y \in l_m^{\sim}$; the functional lim may be uniquely continued to linear map of $l_m^{\sim} < \bigtriangledown >$ onto $(\mathbb{C}1_m) < \bigtriangledown >$ by means of the equality

$$\lim(D) = \sum_{k=0}^{\infty} \mathfrak{m}_{(\lim(\mathfrak{c}_k(D)))\,\mathbf{1}\,_m} \circ \bigtriangledown^k.$$

For each $D \in l_m^{\sim} < \bigtriangledown >$ we name the polynomial

$$T_D(z) = \sum_{k=0}^{\infty} \lim(\mathfrak{c}_k(D))z^k \in \mathbb{C}[z]$$

by charactristic polynomial of the operator D.

§2. Reformulation of the problem.

Reformulation of the Perron's decomposition Lemma. *Let $D \in l_0^{\sim}|_n < \bigtriangledown >$ and $\mathfrak{c}_n(D) = \mathbf{1}\,_0$. Let the polynomial $T_D(z)$ coinsides with the polynomial $T(z)$ in (8). Let, as before, the equality (8) holds wih $T_1(z)$ and $T_2(z)$ defined in (9). Let absolute value of each root of the polynomial $T_1(z)$ is greater than the absolute value of each root of the polynomial $T_2(z)$. Then there exist $m \in \mathbb{N}-1$, $D_1 \in l_m^{\sim}|_p < \bigtriangledown >$ and $D \in l_m^{\sim}|_q < \bigtriangledown >$ such that $\mathfrak{c}_p(D_1) = \mathbf{1}\,_m$, $\mathfrak{c}_q(D_2) = \mathbf{1}\,_m$, $T_{D_1}(z) = T_1(z)$, $T_{D_2}(z) = T_2(z)$ and*

$$r_m(D) = D_2 \circ D_1. \tag{31}$$

We reformulate the Theorem 5 also.

Reformulation of the Theorem 5. *Let $\xi(x)$ satisfies to all the conditions of the Theorem 5. Let all the conditions of the reformulated Perron's decomposition Lemma are fulfilled and together with (2) the more strong condition holds: $D - \lim(D) \in l_{0,\xi} < \bigtriangledown >= \mathfrak{m}_{\tau_0(\xi)} l_{0,\infty} < \bigtriangledown >$ Then for D_1 and D_2 in (31) the following conditions hold:*

$$D_k - \lim(D) \in l_{m,\xi} < \bigtriangledown >= \mathfrak{m}_{\tau_m(\xi)} l_{m,\infty} < \bigtriangledown >,$$

where $k = 1, 2$.

§3. The general plan of the proof of the Theorem 5.

For given $a_k = a_k(\nu)$ in (25) we prove that $u_k = u_k(\nu)$ and $b_k = b_k(\nu)$ in (25) are determined by means some succesive procedure together with increasing of ν. More precisely, if for given $\nu \in \mathbb{N}-1$ are known $b_{i-1}(\nu+s)$ for $i = 0, \dots, p$ and $s = 0, \dots, q-1$ and the matrix $B^\wedge(\nu)$ for this ν is invertible (we remark that $B^\wedge$

is the map of $\mathbb{N}+m-1$ *into* $\mathrm{Mat}_q(\mathbb{C})$ *and therefore* $\mathrm{Mat}_q(\mathbb{C})$ *must contain* $B^\wedge(\nu)$ *for each given* $\nu \in \mathbb{N}+m-1$*), then* $U(\nu)$ *is determined by means of the equality*

$$U(\nu) = (B^\wedge(\nu))^{-1}\pi_1(A(\nu), \tag{32}$$

which follows from (28). But when is known $U(\nu)$*, the value* $b_{i-1}(\nu+q)$ *for* $i = 1, \dots, p$ *is determined by means of the equality*

$$(\bigtriangledown^q B)(\nu) = \pi_2(A(\nu)) - \mathfrak{B}(\nu)U(\nu), \tag{33}$$

which follows from (30). We show that, if for some $\nu = m \in \mathbb{N}-1$ *the values* $b_{i-1}(\nu+s)$ *for* $i = 0, \dots, p$ *and* $s = 0, \dots, q-1$ *are taken sufficiently near to* $b^\sim_{i-1}$ *for* $i = 1, \dots, p$*, then this passage from* $b_{i-1}(\nu+s)$ *for* $i = 1, \dots, p$ *and* $s = 0, \dots, q-1$ *to* $b_{i-1}(\nu+1+s)$ *for* $i = 1, \dots, p$ *and* $s = 1, \dots, q$ *is infinite. Moreover, for specially constructed norm on* $\mathbb{C}^p$ *in this situation acts some kind of the fixed point theorem, which determines the reqired speed of convergence of* $b_{i-1}(\nu)$ *for* $i = 1, \dots, p$ *to coordinates of this fixed point, i.e. to* $b^\sim_{i-1}$*, where* $i = 1, \dots, p$*.*

§4. The polynomials with coefficients in a linear space.

Let x *be a variable and let* L *be a linear space over the field* K*. The expression* $P = P(x) = \sum_{k=0}^{n} x^k a_k$*, where* $n \in \mathbb{N}-1$*,* $a_k \in L$*,* $k = 0, \dots, n$*, I name by polynomial on* x *with coefficients in* L*; all the coefficients I write from the right of degrees of variable* x*; I name* P *by the zero polynomial, if* $a_k = \mathbf{0}_L$*,* $k = 0, \dots, n$*; if* $a_n \neq 0_L$*, then I name the number* n *by degree of the polynomial* P *and I denote the degree of the polynomial* P *by* $\deg(P)$*. If* $\alpha \in K$ *then* $\sum_{k=0}^{n} \alpha^k a_k \in L$*; I name this element in* L *by value of the polynom* P *at the point* $x = \alpha$ *and denote by* $P(\alpha)$*. For given* $P \in L[x]$ *I shall say that the map*

$$f = f_P : \alpha \to P(\alpha) \tag{34}$$

of K *into* L *is defined by* P*; the connected with (34) the map*

$$P \to f_P \tag{35}$$

of the set $L[x]$ *into* L^K *has the following property:*

if $\deg(P) \leq n$ *and the field* K *contains at list* $n+1$ *elements, then the layer over the image of the element* P *by the map (35) consists of single element* P*. The set* $L[x]$ *has the natural structure of a left* $K[x]-$*module; moreover* $L[x]$ *is a graduated* $K[x]-$*module, in which homogeneous cpomponents of nonnegative degree are isomorph to* L *and homogeneous cpomponents of negative degrees consit of the single element* 0*; in particular, there is a natural algorithm of division of* $P \in L[x]$

by $Q \in K[x]$ with the remainder, and therefore the equality $P(\alpha) = 0$ is fulfilled if and only if $P(x) \in (x - \alpha)L[x]$. If $\deg(P) \le n$, then as it was mentioned above, the coefficients of P are uniqely determined by the values of P in the $n + 1$ mutually distinct points $x_0, \ldots, x_n$ and holds the formula of Lagrange:

$$P(x) = \sum_{k=0}^{n} \frac{T_k(x)}{T_k(x_k)} P(x_k),$$

where $T_k(x) \in K[x]$ is defined by means the equality

$$(x - x_k)T_k(x) = \prod_{i=0}^{n} (x - x_i).$$

If $P[x_k] = 0_L$ for $k = 1, \ldots, n$, then $P(x) = ((T_0(x))/T_0(x_k))P(x_k)$. Reasonings similar to those, which done in [5], volume 2, chapter 10, section 4, show that for any $f \in L^K$ and for any sequence $x_1, \ldots, x_n, x$ of mutually distinct elements holds the equality

$$f(x) = \sum_{k=0}^{n} \left(\left(\prod_{\kappa=0}^{k-1} (x - x_\kappa) \right) f_{k-1}(x_1, \ldots, x_k) \right) + \tag{36}$$

$$\left(\prod -i = 0^n (x - x_i) \right) f_n(x_1, \ldots, x_n, x)$$

with

$$f_0(x) = f(x), \tag{37}$$

where $x \in K$,

$$f_1(x_1, x) = (x - x_1)^{-1}(f_0(x) - f_0(x_1)), \tag{38}$$

where $x \in K \diagdown x_1$,

$$f_2(x_1, x_2, x) = (x - x_2)^{-1}(f_0(x_1, x) - f_0(x_1, x_2)), \tag{39}$$

where $x \in K \diagdown \{x_1, x_2\}$,

$$f_k(x_1, \ldots, x_{k-1}, x_k, x) = (x - x_k)^{-1} \times \tag{40}$$

$$(f_1(x_1, \ldots, x_{k-1}, x) - f_1(x_1, \ldots, x_{k-1}, x_k))$$

where $x \in K \diagdown \{x_1, \ldots, x_k\}$.

Remark 4.1. All this is almost trivial. There exists in L a Hamel's basis e_i, where i runs through a some set I of indexes; L is embeded as subspace of of all the finite functions in the linear space $L^\vee = K^I$ over K. Each $P \in L^\vee[x]$ with $\deg(P) \le n$ may be considered as system $P = \{P_i(x) \colon i \in I\}$, where $P_i(x) \in K[x], \deg P_i(x) \le n$, where $i \in I$ and for $L = K$ the formula (36) is well known (see, for example, [5], volume 2, chapter 10, section 4). Clearly, $L[x]$ may be regarded also as $K[x] \otimes_K L$.

We show below that, when $f \in L^K$ is defined by means of the polynomial $P \in L[x]$ which is a zero polynomial or have $\deg(P) \le n$, the formula (36) may be spread on the arbitrary sequences

$$x_1, \dots, x_n, x \tag{41}$$

consisting of not necessary mutually distinct elements in K.

Let $k \in \mathbb{N}$, $s \in \mathbb{Z}$. Let K_0 be the ring of all the polynomials over K of countable set of variables

$$u_l, \tag{42}$$

where $l \in \mathbb{N}$ and let $K^\wedge$ be the quotient field of the ring K_0;let $\phi_s(u_1) = u_1^s$ for $s \in \mathbb{N} - 1$; let $\phi_s(u_1, \dots, u_k)$ for $k \ge 2$ be the homogenous form of degree s of the expansion of the fraction $1/\prod\limits_{\kappa=1}^{k}(1 - u_\kappa)$ in the formal multiple power series relatively variables $u_1, \dots, u_k$; so

$$\phi_s(u_1, \dots, u_k) = 0, \tag{43}$$

if $k < 0$;

$$\phi_s(u_1, \dots, u_k) = 1, \tag{44}$$

if $k = 0$;

$$\phi_s(u_1, \dots, u_k) = \sum_{\substack{\alpha_1 \in \mathbb{N}-1, \dots, \alpha_1 \in \mathbb{N}-1 \\ \sum_{\kappa=1}^{k} \alpha_\kappa = s,}} \left(\prod_{\kappa=1}^{k} u_\kappa^{\alpha_\kappa}\right) \tag{45}$$

for $k \in \mathbb{N}$. In view of the equality

$$(u_k - u_{k-1}) \prod_{\kappa=1}^{k}(1 - u_\kappa)^{-1} = ((1 - u_k)^{-1} - (1 - u_{k-1})^{-1}) \prod_{\kappa=1}^{k}(1 - u_\kappa),$$

$$\phi_{s-1}(u_1, u_2)(u_2 - u_1) = \phi_s(u_2) - \phi_s(u_1), \tag{46}$$

$$\phi_{s-1}(u_1, u_2,\, u_3(u_3 - u_2) = \phi_s(u_1, u_3) - \phi_s(u_1, u_2), \tag{47}$$

(48)

$$(u_k - u_{k-1})\phi_{s-1}(u_1,\, \ldots\,,\, u_{k-2},\, u_{k-1},\, u_k) = \sum_{h=0}^{1}(-1)^h \phi_s(u_1,\, \ldots\,,\, u_{k-2},\, u_{k-h}),$$

where $s \in \mathbb{N}$. Let $L_0 = K_0 \otimes_K L,\, n \in \mathbb{N} - 1, k \in \mathbb{N}, a_\kappa \in L, \kappa = 0,\, \ldots\,,\, n$

$$P = P(x) = \sum_{\kappa=0}^{n} x^\kappa a_\kappa. \tag{49}$$

Let

$$\triangledown(P|u_1,\, \ldots\,,\, u_k) = \sum_{\kappa=0}^{n} \phi_{\kappa-k+1}(u_1,\, \ldots\,,\, u_k) \otimes_K a_\kappa. \tag{50}$$

Then for each specialisation $u_1 = \alpha_1 \in K,\, \ldots\,,\, u_k = \alpha_k \in K$ holds the equality $\triangle(P|\alpha_1,\, \ldots\,,\, \alpha_k) = \sum\limits_{\kappa=0}^{n} \phi_{\kappa-k+1}(\alpha_1,\, \ldots\,,\, \alpha_k) \otimes_K a_\kappa$.

Lemma 4.1. *For the sequence (42) we have*

$$\triangle(P|u_1) = P(u_1),\, \triangle(P|u_1, u_2)(u_2 - u_1) = (P|u_2) - (P|u_1), \tag{51}$$

$$(u_k - u_{k-1}) \triangle (P|u_1,\, \ldots\,,\, u_{k-2},\, u_{k-1},\, u_k) = \tag{52}$$

$$\sum_{h=0}^{1}(-1)^h \triangle (P|u_1,\, \ldots\,,\, u_{k-2},\, u_{k-h}).$$

Proof. The assertion of the Lemma follows from (46)-(50). ■

Corollary. *If the sequence (41) consists of mutually distinct elements of the field K and $P \in L[x]$, then*

$$P_0(x) = \triangle(P|x), P_1(x_1,\, x) = \triangle(P|x_1, x),$$

$$P_{n-1}(x_1,\, \ldots\,,\, x_{n-1}, x) = \triangle(P|x_1,\, \ldots\,,\, x_{n-1},\, x),$$

where $n \in \mathbb{N} + 1$.

Proof. The assertion of the Corollary follows from (37)-(40) and Lemma 4.1. ■

Lemma 4.2. For the sequence (42) and $P \in L[x]$ we have

$$P(u_k) = P(u_1) + \sum_{\kappa=2}^{k} \left(\prod_{i=1}^{\kappa-1} (u_k - u_i) \right) \triangle (P|u_1, \ldots, u_\kappa), \tag{53}$$

where $k \in 1 + \mathbb{N}$.

Proof. For $k - 1 = 1$ the assertion of the Lemma follows from (51). Let $k-1 \in 1+\mathbb{N}$ and assertion of the Lemma holds for $k-1-1$. Then for the sequence $\{u_\kappa : \kappa \in [1, k-2] \cap \mathbb{N}\}, u_k$ we have the equality

$$P(u_k) = P(u_1) + \left(\sum_{\kappa=2}^{k-2} \left(\prod_{i=1}^{\kappa-1} (u_k - u_i) \right) \triangle (P|u_1, \ldots, u_\kappa) \right) + \left(\prod_{i=1}^{k-2} (u_k - u_i) \right) \triangle (P|u_1, \ldots, u_{k-2}, u_k). \tag{54}$$

In view of (54) and (52),

$$P(u_k) = P(u_1) + \left(\sum_{\kappa=2}^{k-2} \left(\prod_{i=1}^{\kappa-1} (u_k - u_i) \right) \triangle (P|u_1, \ldots, u_\kappa) \right) + \left(\prod_{i=1}^{k-2} (u_k - u_i) \right) \times$$
$$\left(\triangle(P|u_1, \ldots, u_{k-2}, u_{k-1}) + (u_k - u_{k-1}) \triangle (P|u_1, \ldots, u_{k-2}, u_{k-1}, u_k) \right).$$

The right part of the last equality coincides wih the right part of the equality (53). ■

So, when the function f is represented by means of $P \in L[x]$, the mentioned above spreading of (36) for arbitrary sequence (41) is fulfilled. Let $a^0 = \mathbf{1}_K$, where $a \in K$ (in particular, $a = \mathbf{0}_K$.) Let $n \in \mathbb{N}$. Let $\mathfrak{k}$ be a field contained the field K and and contained elements

$$z_1, \ldots, z_n \tag{55}$$

algebraically independent over K. For the elements (55) we define $\sigma_{k,l}$ for $k \in \mathbb{Z}$ and $l \in [1, n] \cap \mathbb{N}$ and as follows:

$\sigma_{k,l} = 0$, if $k \in -\mathbb{N} \cup (l + \mathbb{N})$, let $\sigma_{0,l} = 1$, and let $\sigma_{k,l} = \sigma_{k,l}(z_1, \ldots, z_l)$ is k-th elementar symmetric function of z_i with $i \in [1, l] \cap \mathbb{N}$, if $k \in [1, l] \cap \mathbb{N}$. Let $n \in \mathbb{N}$ and for $\kappa \in \mathbb{Z}$ let $a_\kappa = (-1)^{n-\kappa} \sigma_{n-\kappa,n}$. Let us consider the polynomial

$$z^n + \sum_{\kappa=0}^{n} a_\kappa z^\kappa. \tag{56}$$

The sequence (55) consists of all the zeros of the polynomial (56). Let

$$A = A(z_1, \dots, z_n) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{pmatrix}. \tag{57}$$

For $i = 1, \dots, n$ and $k = 1, \dots, n$ let

$$c_{i,k} = \phi_{i-k}(z_1, \dots, z_n), \tag{58}$$

$$d_{i,k} = (-1)^{i-k}\sigma_{i-k,i-1};$$

let $C = C(z_1, \dots, z_n)$ and $D = D(z_1, \dots, z_n)$ are $n\times n$-matrices, in which respectively $c_{i,k}$ and $d_{i,k}$ stand on the intersection of i-th row and $k-$th column. Clearly, C and D are a lower triangle matrices, all the diagonal elements of which are equal to 1.

Lemma 4.3. *The following equality holds:*

$$D = C^{-1}. \tag{59}$$

Proof. Applying the Lemma 4.2 to the polinomial $P(z) = z^{i-1}$ and the sequence $z_1, \dots, z_{i-1}, z$, we see that

$$z^{i-1} = z_1^{i-1} + \sum_{k=2}^{i-1}\left(\prod_{\kappa=1}^{k-1}(z - z_\kappa)\right) \triangle\, (z^{i-1}|z_1, \dots, z_k) +$$

$$\prod_{\kappa=1}^{i-1}(z - z_\kappa) \triangle\, (z^{i-1}|z_1, \dots, z_{i-1}, z).$$

Comparing coefficients by $z^{l-1}, l \in [1, n] \cap \mathbb{N}$ we see that assertion of the Lemma 4.3 is true. ∎

Let $L = K^n$, let $i = 1, \dots, n$ and e_i is the $i-$th column of the unit $n \times n$-matrix and let $P_0 = P_0(z) = \sum\limits_{i=1}^{n} z^{i-1}e_i$. Then $P_0(z) \in L(z)$.

Lemma 4.4. *Let* $g_k = \bigtriangledown(P_0|z_1, \dots, z_k)$, *where* $k = 1, \dots, n$; *then* g_k *coincides with* $k-$*th column of matrix* C.

Proof. The assertion of the Lemma follows from (50) ∎

Lemma 4.5. *For the matrix* A *in (57) the following equalities hold :*

$$Ag_1 = z_1 g_1, \tag{60}$$

$$Ag_k = z_k g_k + g_{k-1}, \tag{61}$$

where $k \in \mathbb{N} \cap [2, n]$.

Proof. Since z_1 is the zero of polynomial (56), the equality (60) may be verfied directly. Since the elements (55) are mutually distinct, itfollows that

$$Ag_2 = A(z_2 - z_1)^{-1}(P_0(z_2) - P_0(z_1)) =$$

$$(z_2 - z_1)^{-1}(z_2 P_0(z_2) - z_1 P_0(z_1)) =$$

$$z_2 \bigtriangledown (P_0|z_1,\, z_2) + P_0(z_1) = z_2 g_2 + g_1.$$

Let $k \in [3, n] \cap \mathbb{N}$ and (61) is true for $k-1$. Then

$$Ag_k = A(z_k - z_{k-1})^{-1}(\bigtriangledown(P_0|z_1,\, \ldots\, ,\, z_{k-2},\, z_k) - \bigtriangledown(P_0|z_1,\, \ldots\, ,\, z_{k-2},\, z_{k-1}) =$$

$$(z_k - z_{k-1})^{-1}(z_k \bigtriangledown (P_0|z_1,\, \ldots\, ,\, z_{k-2},\, z_k) - z_{k-1} \bigtriangledown (P_0|z_1,\, \ldots\, ,\, z_{k-2},\, z_{k-1})) =$$

$$z_k \bigtriangledown (P_0|z_1,\, \ldots\, ,\, z_{k-2},\, z_{k-1},\, z_k) + P_0|z_1,\, \ldots\, ,\, z_{k-2},\, z_{k-1}) = z_k g_k + g_{k-1}. \blacksquare$$

Let

$$S = S(z_1,\, \ldots\, ,\, z_n) = \begin{pmatrix} z_1 & 1 & 0 & \ldots & 0 \\ 0 & z_2 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & z_3 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & z_n \end{pmatrix}.$$

Lemma 4.6. *The following equality holds:* $C^{-1}AC = S$.

Proof. The assertion of the Lemma follows from the Lemma 4.5. $\blacksquare$

Lemma 4.7. *The folloing equalities hold:*

$$\sum_{k=0}^{l} (-1)^k \sigma_{k,n} \phi_{l-k}(z_1,\, \ldots\, ,\, z_n) = 0,$$

where $l \in \mathbb{N}$.

Proof. The assertion of the Lemma follows from the equality

$$\left(\sum_{k=0}^{\infty} \phi_k(z_1,\, \ldots\, ,\, z_n) \right) \times \prod_{1}^{n} (1 - z_k) = 1. \blacksquare$$

Corollary. *Let* $m \in [1, n] \cap \mathbb{N}$, P *and* Q *are the lower triangle* $m \times m$*-matrices, in which respectively* $(-1)^{i-k}\sigma_{i-k,n}$ *and* $\phi_{i-k}(z_1,\, \ldots\, ,\, z_n)$, *stand on the intersection of the* i*-th row and* $k-$*th column, where* $i = 1,\, \ldots\, ,\, m$ *and* $k = 1,\, \ldots\, ,\, m$. *Then* $P^{-1} = Q$.

Proof. Let
$$P = (p_{i,j}), Q = (q_{j,k}), R = (r_{i,k}) = PQ,$$
where $\{i, j, k\} \subset \{1, \dots, m\}$. Then
$$r_{i,k} = \sum_{j=1}^{m} (-1)^{i-j} \sigma_{i-j,n} \phi_{j-k}(z_1, \dots, z_n) =$$
$$\sum_{\kappa=0}^{i-1} (-1)^{\kappa} \sigma_{\kappa,n} \phi_{i-k-\kappa}(z_1, \dots, z_n).$$
If $i < k$, then all the summands of the last sum are equal to 0, if $k > i$, then the last sum is equal to 0, according to the Lemma 4.7, and if $i = k$, then last sum have only one nonzero summand $\sigma_{0,n}\phi_0(z_1, \dots, z_n) = 1$. ■

Remark 4.2. . *Since all the elements of the matrices in (57), (58), (59) and in the Corollary of the Lemma 4.7 lie in $K[z_1 \ \dots, z_n]$, it follows that the assertions of the Lemmata 4.3 - 4.7 and of the Corollary of the Lemma 4.7 remain true for each specialisation of the point $(z_1, \dots, z_n)$ into the space K^n. Therefore we may assume that $\mathfrak{k} = K$ and the sequence (55) consists of the elements of the field K. Of course, the objects, which are obtained as result of above specialization, inherit the notations of their prototypes. For example, the element, in which turns $\sigma_{k,n}$ from the Lemma 4.7 by any specialization, we again denote by $\sigma_{k,n}$.*

§5. A several constants.

Here we collected some constants, which are used in the further sections. We recommend to omit this section and return here during reading the other sections. Let $n \in \mathbb{N}$, $m \in \mathbb{N}$, $p \in \mathbb{N}, q \in \mathbb{N}$, $\rho > 0$,
$$\varepsilon > 0, \gamma_0(n, \rho) = \max(1, \rho^{n-1}), \gamma_1(n, \rho) = \min(1, \rho^{n-1},)$$

$$\gamma_0^{\sim}(n, \rho) = \sum_{k=0}^{n-1} \rho^k, \gamma_1^{\sim}(n, \rho, \varepsilon) = \gamma_0(n, \varepsilon/\rho)/\rho, \tag{62}$$

$$\gamma_2(n, \rho, \varepsilon) = \max(\varepsilon, 1/\varepsilon)(1 + \rho_0)^2)^{n-1}, \gamma_3(n, m, \rho) = \sum_{\kappa=0}^{m-1} \binom{n}{\kappa} (\rho)^{n-\kappa}, \tag{63}$$

$\gamma_4(n, \rho) = \max(1, \gamma_3(n, n, \rho))$, *for* $n \in \mathbb{N} + 1$ *and* $\gamma_4(1, \rho) = \rho$,

$$\gamma_5(m, n, \rho) = \sum_{k=0}^{m-1} \binom{n+k-1}{k} \rho^k. \tag{64}$$

Clearly, if $0<\rho<1$, *then* $0<\gamma_5(n,1,\rho)=\gamma_0^{\sim}(n,\rho)=(1-\rho^n)/(1-\rho)<1/(1-\rho)$; *moreover, if* $\varepsilon\in[0,\rho]$, *then*

$$1-\varepsilon\gamma_5(n,1,\varepsilon/(2\rho))/(2\rho)>1-(\varepsilon/(2\rho))/(1-\varepsilon/(2\rho))=$$

$$(1-\varepsilon/\rho)/(1-\varepsilon/(2\rho))$$

and therefore

$$0<\gamma_6(n,\rho,\varepsilon)=\frac{\gamma_5(n,1,\varepsilon/(2\rho))/\rho}{1-\varepsilon\gamma_5(n,1,\varepsilon/(2\rho))/\rho}<1/(\rho-\varepsilon). \tag{65}$$

Let $n\in\mathbb{N}$, $\kappa\in\mathbb{N}-1$ *and* $\eta(n,\kappa)$ *be the coefficient by* x^κ *in the formal series*

$$1+\sum_{k=1}^{\infty}((1+x)^n-1)^k,$$

$$\gamma_7(m,n,\rho)=\sum_{\kappa=0}^{m-1}\eta(n,\kappa(\rho)^{n+\kappa}. \tag{66}$$

Let $\mathfrak{M}_\eta$ *be the set of the zeros of the polynomial* $T_\eta,\eta=1,\,2$ *in (9),*

$$\rho_{\eta,0}=\sup\{|\lambda|\colon\lambda\in\mathfrak{M}_\eta\},\rho_{\eta,1}=\inf\{|\lambda|\colon\lambda\in\mathfrak{M}_\eta\},$$

where $\eta=0,1$. *Clearly,* $\rho_{1,1}>\rho_{2,0}$. *Let* $\varepsilon_1\in(0,1/3)$, $\varepsilon_2=\varepsilon_1(\rho_{1,1}-\rho_{2,0})$,

$$\gamma_8=\gamma_2(p;\rho_{1,0},\varepsilon_2/2),\gamma_9=\gamma_2(q;\rho_{2,0},\varepsilon_2/2),\gamma_{10}=\gamma_6(p,\rho_{1,1},\varepsilon_2)$$

$$\gamma_{11}=\gamma_7(q,p,\gamma_{10}),\gamma_{12}=\gamma_4(p,\gamma_6(p,\rho_{1,1},\varepsilon_2)),\gamma_{13}=\gamma_4(p,\rho_{1,0}+\varepsilon_2),$$

$$\gamma_{14}=\gamma_0(1+q,\rho_{1,0}+\varepsilon_2),\gamma_{15}=\min(\gamma_8(\gamma_6(p,\rho_{1,1},\varepsilon_2))^q,\,(\gamma_{12})^q),$$

$$\gamma_{16}=\min(\gamma_8\gamma_{14},\,\gamma_0(q+1,\gamma_{13})),\,\gamma_{17}=2\gamma_8\gamma_{15}\gamma_{16},\,\gamma_{18}=\gamma_4(n,\rho_1,0)+\gamma_8\varepsilon_2/2.$$

Let

$$\varepsilon_3=\frac{\rho_{2,0}}{\rho_{1,1}-\varepsilon_2},\varepsilon_4\in(0,1-\varepsilon_3],\varepsilon_5\in(0,1),\varepsilon_6=1-(1-\varepsilon_5)(\varepsilon_4)^q, \tag{67}$$

$$\gamma_{19}=\min(\gamma_8(\gamma_6(p,\rho_{1,1},\varepsilon_2))^q,\,(\gamma_{12})^q,\gamma_{20}=\min(\gamma_8\gamma_{14},\,\gamma_0(q+1,\gamma_{13})),$$

$$\gamma_{21}=\gamma_8\gamma_{19}\gamma_{20},\gamma_{22}=(1+\rho_{2,0})^{q-1}\gamma_{21},$$

$$\gamma_{23}=(1+q\gamma_{18}\gamma_{11}),\gamma_{24}=q\gamma_{11}\gamma_4(n,\rho_{1,0})(1+q\gamma_{11}\gamma_{18}).$$

§6. Some special norms on $\mathbb{C}^n$.

Let now $K = \mathbb{C}$. According to the remark 4.2, we suppose that sequence (55) consists of the elements of the field $\mathbb{C}$. If $\mathfrak{q}$ is a norm on $\mathbb{C}^n$, the function $q^{\sim}$ on $\mathrm{Mat}\,_n(\mathbb{C})$ *defined by means the equality*

$$\mathfrak{q}^{\sim}(A) = \sup\{\mathfrak{q}(AX) \colon X \in \mathbb{C}^n, \, \mathfrak{q}(X) \leq 1\}$$

is a norm on $\mathrm{Mat}\,_n(\mathbb{C})$ *and they name this norm $\mathfrak{q}^{\sim}$ by the norm associated with the norm $\mathfrak{q}$. We make use of the following norms on the space $\mathbb{C}^n$ and associated with them norms on* $\mathrm{Mat}\,_n(\mathbb{C})$:

$$q_{\infty}(X) = \sup\{|x_i| \colon i = 1, \, \dots \, , \, n, X = (x_i) \in \mathbb{C}^n\},$$

$$q_{\infty}^{\sim}(A) = \sup\{\sum_{k=1}^{n} |a_{i,k}| \colon i = 1, \, \dots \, , \, n, A = (a_{i,k}) \in \mathrm{Mat}\,_n(\mathbb{C})\},$$

$$q_1(X) = \sum_{i=1}^{n} |x_i| \colon i = 1, \, \dots \, , \, n, X = (x_i) \in \mathbb{C}^n,$$

$$q_1^{\sim}(A) = \sup\{\sum_{i=1}^{n} |a_{i,k}| \colon k = 1, \, \dots \, , \, n, A = (a_{i,k}) \in \mathrm{Mat}\,_n(\mathbb{C})\}.$$

Although all the norms on $\mathbb{C}^n$ are equivalent, they are suitable for concrete calculations in a different degree. Below we construct some norms convenient for our purposes.

Let $K = \mathbb{C}$, $\varepsilon > 0$, $n \in \mathbb{N}$. Let $T_{n,\varepsilon}$ be the diagonal $n \times n-$matrix having as its $k-$th diagonal element the number ε^{k-1}. Let further the numbers (55) are contained in $\mathbb{C}$ and

$$\rho_0 = \sup\{|z_i| \colon i = 1, \, \dots \, , \, n\}, \, \rho_1 = \inf\{|z_i| \colon i = 1, \, \dots \, , \, n\}. \tag{68}$$

For the sequence (55) and any $\varepsilon \in (0, 1)$ we define the norm $q_{z_1, \, \dots \, , \, z_n; \varepsilon}$ as follows:

$$q_{z_1, \, \dots \, , \, z_n; \varepsilon}(X) = q_{\infty}((C(z_1, \, \dots \, , \, z_n)T_{n,\varepsilon})^{-1}(X)), \tag{69}$$

where $X \in \mathbb{C}^n$. Clearly, if $B \in$ $\mathrm{Mat}\,_n(\mathbb{C})$, *then*

$$q_{z_1, \, \dots \, , \, z_n; \varepsilon}^{\sim}(B) = q_{\infty}((C(z_1, \, \dots \, , \, z_n)T_{n,\varepsilon})^{-1} FC(z_1, \, \dots \, , \, z_n)T_{n,\varepsilon}). \tag{70}$$

In particular, according to the Lemma 4.5, for the matrix A in (57) holds the equality

$$q_{z_1, \, \dots \, , \, z_n; \varepsilon}^{\sim}(A) = \rho_0 + \varepsilon \min(n - 1, 0). \tag{71}$$

If $\rho_1 > 0$, then, clearly, there exists A^{-1} and the following inequaliy holds:

$$q^{\sim}_{z_1,\,\dots\,,\,z_n;\varepsilon}(A^{-1}) \leq \gamma_1^{\sim}(n, \rho_1, \varepsilon). \tag{72}$$

Let us establish some supplementary connections between the norms q_∞ and $q_{z_1,\,\dots\,,\,z_n;\varepsilon}$ (respectively between the norms $q^{\sim}_\infty$ and $q^{\sim}_{z_1,\,\dots\,,\,z_n;\varepsilon}$).

Lemma 6.1. *The following inequaliy holds:*

$$q^{\sim}_\infty(C(z_1, \dots, z_n)) \leq (1+\rho_0)^{n-1}.$$

Proof. Clearly, $q^{\sim}_\infty(C) = \sup\{\sum_{k=1}^{n} |c_{i,k}| \colon i = 1, \dots, n$. In view of (58), (43) - (45), $|c_{i,k}| \leq \binom{i-1}{k-1}\rho_0)^{i-k}$, where $k = 1, \dots, i$, and $c_{i,k} = 0$ for $i < k$. Therefore

$$\sum_{k=1}^{n} |c_{i,k}| \leq \sum_{k=1}^{i} \binom{i-1}{k-1}\rho_0)^{i-k} = (1+\rho_0)^{i-1}. \blacksquare$$

Lemma 6.2. *The following inequality holds:*

$$q^{\sim}_\infty((C(z_1, \dots, z_n))^{-1}) \leq (1+\rho_0)^{n-1}.$$

Proof. The assertion of the Lemma follows directly from the Lemma 4.6. ■

Lemma 6.3. *The following inequalities hold:*

$$q_{z_1,\,\dots\,,\,z_n;\varepsilon} \leq (\max(1, 1/\varepsilon)(1+\rho_0))^{n-1} q_\infty,$$

$$q_\infty \leq (\max(1, \varepsilon)(1+\rho_0))^{n-1} q_{z_1,\,\dots\,,\,z_n;\varepsilon},$$

$$q^{\sim}_{z_1,\,\dots\,,\,z_n;\varepsilon} \leq \gamma_2(n, \rho, \varepsilon) q^{\sim}_\infty, \tag{73}$$

$$q^{\sim}_\infty \leq \gamma_2(n, \rho, \varepsilon) q^{\sim}_\infty vert z_1, \dots, z_n; \varepsilon, \tag{74}$$

whith $\gamma_2(n, \rho, \varepsilon)$ from (63).

Proof. In view of (69) and (70),

$$q_\infty(X) = q_{z_1,\,\dots\,,\,z_n;\varepsilon}(C(z_1, \dots, z_n) T_{n,\varepsilon} X), \tag{75}$$

$$q^{\sim}_\infty(F) = q_{z_1,\,\dots\,,\,z_n;\varepsilon}((C(z_1, \dots, z_n) T_{n,\varepsilon}) F (C(z_1, \dots, z_n) T_{n,\varepsilon})^{-1}). \tag{76}$$

where $X \in \mathbb{C}^n$ and $F \in \mathrm{Mat}_n(\mathbb{C})$. The assertions of the Lemma follow directly from (69), (70), (75), (76), Lemma 4.6 and Lemma 6.2. ■

We introduce on $\mathbb{C}$ the order relation $\preceq$ as follows: we say that $z_1 \preceq z_2$ if $|z_1| < |z_2|$ or if $0 < |z_1| = |z_2|$ and $\arg(z_1) < \arg(z_2)$, where $\arg(z)$ for any $z \in \mathbb{C}$ is taken in $(-\pi, \pi]$. Let the elements (55) are contained in $\mathbb{C}$ and $z_{k+1} \preceq z_k$, for $k \in [1, n-1] \cap \mathbb{N}$. We suppose that the sequence (55) consists of all the roots of the polynomial (56) and one meets each root in (55) as many times as the multiplicity of this root is. Then the sequence (55) is uniquely defined by means of the matrix A from (57). Therefore the norms $q_{z_1, \dots, z_n;\varepsilon}$ and $q^{\sim}_{z_1, \dots, z_n;\varepsilon}$ further will be denoted respectively by $q_{A,\varepsilon}$ and $q^{\sim}_{A,\varepsilon}$

Lemma 6.4. *Let $\eta(t) = 0$ for $t < 0$ and $\eta(t) = 1$ for $t \geq 0$ The following inequalities hold for P and Q defined (according to the remark 4.2) in the Corollary 1 of the Lemma 4.7 :*

$$q^{\sim}_1(P^{-1}) \leq \gamma_5(m, n, \rho_0), q^{\sim}_{\infty}(P^{-1}) \leq \gamma_5(m, n, \rho_0),$$

$$q^{\sim}_1(P) \leq (1+\rho_0)^n - \eta(n-m)(\rho_0)^n, q^{\sim}_{\infty}(P) \leq (1+\rho_0)^n - \eta(n-m)(\rho_0)^n$$

with ρ_0 from (68) and $\gamma_5(m, n, \rho_0)$ from (64).

Proof. Clearly,

$$\sum_{i=1}^{m} |\phi_{i-k,}(z_1, \dots, z_n)| \leq \sum_{i=k}^{m} \binom{i-k+n-1}{i-k} (\rho_0)^{i-k} =$$

$$\sum_{\kappa=0}^{m-k} \binom{\kappa+n-1}{\kappa} (\rho_0)^{\kappa} \leq \sum_{\kappa=0}^{m-1} \binom{\kappa+n-1}{\kappa} (\rho_0)^{\kappa} = \gamma_5(m, n, \rho_0),$$

$$\sum_{k=1}^{m} |\phi_{i-k,}(z_1, \dots, z_n)| \leq \sum_{k=1}^{i} \binom{i-k+n-1}{i-k} (\rho_0)^{i-k} =$$

$$\sum_{\kappa=0}^{i-1} \binom{\kappa+n-1}{\kappa} (\rho_0)^{\kappa} \leq \sum_{\kappa=0}^{m-1} \binom{\kappa+n-1}{\kappa} (\rho_0)^{\kappa} = \gamma_5(m, n, \rho_0),$$

$$\sum_{i=1}^{m} |\sigma_{i-k,n}| \leq \sum_{i=k}^{m} \binom{n}{i-k} (\rho_0)^{i-k} =$$

$$\sum_{\kappa=0}^{m-k} \binom{n}{\kappa} (\rho_0)^{\kappa} \leq (1+\rho_0)^n - \eta(n-m)(\rho_0)^n,$$

$$\sum_{k=1}^{m} |\sigma_{i-k,n}| \leq \sum_{k=1}^{i} \binom{n}{i-k} (\rho_0)^{i-k} = \sum_{\kappa=0}^{i-1} \binom{n}{\kappa} (\rho_0)^{\kappa} \leq (1+\rho_0)^n - \eta(n-m)(\rho_0)^n. \blacksquare$$

Lemma 6.5. *Let $n \in \mathbb{N}, m \in \mathbb{N}$ Let, as before, the matrix (57) corresponds to the sequence (55), ρ_0 and ρ_1 are defined in (68). Let A_1 be the lower triangle*

$m \times m$-matrix, in which a_{i-k}, stands on the intersection of the i-th row and $k-$th column, where $k = 1, \ldots, m$ and $i = k, \ldots, m$, but all other elements are equal to 0. *Then*

$$q^{\sim}_{\infty}(A) \le \gamma_4(n, \rho_0), q^{\sim}_{\infty}(A_1) \le \gamma_3(n, m, \rho_0), q^{\sim}_{1}(A_1) \le \gamma_3(n, m, \rho_0). \tag{77}$$

If

$$a_0 \ne 0, \tag{78}$$

then

$$q^{\sim}_{\infty}(A^{-1}) \le \gamma_4(n, 1/\rho_1), q^{\sim}_{\infty}((A_1)^{-1}) \le \tag{79}$$

$$\gamma_5(m, n, 1/\rho_1)(\rho_1)^{-n}, q^{\sim}_{1}((A_1)^{-1}) \le \gamma_5(m, n, 1/\rho_1)(\rho_1)^{-n}.$$

Proof. The inequalities (77) are obvious. We show that, if (78) is fulfilled, then (79) hold. Clearly, $A^{-1} = \begin{pmatrix} \frac{-a_1}{a_0} & \frac{-a_2}{a_0} & \ldots & \frac{-1}{a_0} & \frac{-1}{a_n} \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}$. From (78) follows the first of the inequalities (79). Let $G = (a_n)^{-1}A_1, w_k = 1/z_k$, where $k = 1, \ldots, n$. We see that for P all the conditons of the Lemma 6.4 are fulfilled with w_k instead z_k for $k = 1, \ldots, n$. The second of the inequalities (79) follows from this Lemma. ■

Since $\|A\|_{sp} := \rho_0$, $\|A^{-1}\|_{sp} := 1/\rho_1$, it follow that (77) and (78) may be rewritten respectively in the form

$$q^{\sim}_{\infty}(A) \le \gamma_4(n, \|A\|_{sp}), q^{\sim}_{\infty}(A_1) \le \gamma_3(n, m, \|A\|_{sp}), q^{\sim}_{1}(A_1) \le \gamma_3(n, m, \|A\|_{sp}),$$

$$q^{\sim}_{\infty}(A^{-1}) \le \gamma_4(n, \|A^{-1}\|_{sp}), q^{\sim}_{\infty}((A_1)^{-1}) \le \gamma_5(m, n, \|A^{-1}\|_{sp})(\|A^{-1}\|_{sp})^n,$$

$$q^{\sim}_{1}((A_1)^{-1}) \le \gamma_5(m, n, \|A^{-1}\|_{sp})(\|A^{-1}\|_{sp})^n.$$

Lemma 6.6. *Let A be the matrix from (57), let $\varepsilon > 0$, let $B \in \operatorname{Mat}_n \mathbb{C}$ and let*

$$q^{\sim}_{A,\varepsilon/2}(B - A) \le \varepsilon/2. \tag{80}$$

Then

(a) $\|B\|_{sp} \le q^{\sim}_{A,\varepsilon}(B) \le \varepsilon + \|A\|_{sp}$;

(b) if for some $b_\kappa \in \mathbb{C}$, where $\kappa \in \mathbb{Z}$, $b_n = 1$ and $b_\kappa = 0$ for $\kappa \in (-\mathbb{N}) \cup (n + \mathbb{N})$ the following equality holds:

$$B = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -b_0 & -b_1 & -b_2 & \ldots & -b_{n-1} \end{pmatrix}, \tag{81}$$

B_1 be the lower triangle $m\times m$-matrix, in which b_{i-k}, stands on the intersection of the i-th row and $k-$th column, where $k=1,\dots,m$ and $i=k,\dots,m$, but all other elements are equal to 0, *then*

$$q^{\sim}_{\infty}(B)\le\gamma_4(n,\|A\|_{sp}+\varepsilon),\ q^{\sim}_{\infty}(B_1)\le\gamma_3(n,m,\|A\|_{sp}+\varepsilon),$$

$$q^{\sim}_{1}(B_1)\le\gamma_3(n,m,\|A\|_{sp}+\varepsilon).$$

(c) if (78) is fulfilled and $0<\varepsilon<(\|A^{-1}\|_{sp})^{-1}$, *then*

$$|b_0|\ge\big((\|A^{-1}\|_{sp})^{-1}-\varepsilon\big)^n>0,\tag{82}$$

$$q_{A;\varepsilon_2/2}(B^{-1})\le\gamma_6(n,(\|A^{-1}\|_{sp})^{-1},\varepsilon),\tag{83}$$

$$q^{\sim}_{\infty}(B^{-1})\le\gamma_4(n,\gamma_6(n,(\|A^{-1}\|_{sp})^{-1},\varepsilon)),\tag{84}$$

$$q^{\sim}_{\infty}((B_1)^{-1})\le\gamma_5(m,n,\gamma_6(n,\rho,\varepsilon))\gamma_6(n,\rho,\varepsilon))^n,\tag{85}$$

$$q^{\sim}_{1}((B_1)^{-1})\le\gamma_5(m,n,\gamma_6(n,\rho,\varepsilon))\gamma_6(n,\rho,\varepsilon))^n.$$

Proof. Since $q^{\sim}_{A,\varepsilon/2}(B)\le q^{\sim}_{A,\varepsilon/2}(A)+\varepsilon/2$, the inequality $\|B\|_{sp}\le\mathfrak{q}^{\sim}(B)$ holds for any norm $\mathfrak{q}$ on $\mathbb{C}^n$ and, in fiew of (71),the inequality $q_{A,\varepsilon/2}(A)\le\|A\|_{sp}+\varepsilon/2$ holds, it follows that the assertion (a) of the Lemma is proved. The assertion (b) follows from the assertion (a) and Lemma 6.5. Let us to prove the assertion (c). If (78) is fulfilled, there exisists A^{-1} and, in view of (72),

$$q^{\sim}_{A;\varepsilon/2}(A^{-1})\le\gamma^{\sim}_1(n,\rho_1,\varepsilon/2).\tag{86}$$

Clearly, $B=A(E-A^{-1}(A-B))$. In view of (80) and (86), $q^{\sim}_{A,\varepsilon/2}(A^{-1}(A-B))\le(\gamma^{\sim}_1(n,\rho_1,\varepsilon/2))\varepsilon/2\le\frac{\varepsilon/(2\rho_1))}{1-\varepsilon/(2\rho_1))}<1$ Therefore $(E-A^{-1}(A-B))^{-1}$ exists and

$$q^{\sim}_{A,\varepsilon/2}(E-A^{-1}(A-B))^{-1}\le1/(1-(\gamma_1(n,\rho_1,\varepsilon/2))\varepsilon/2).\tag{87}$$

Consequently, $B^{-1}=(E-A^{-1}(A-B))^{-1}A^{-1}$ exists also; according to the equality $\gamma_5(n,1,\rho)=\gamma^{\sim}_0(n,\rho)$ and in view of (86), (87), and (65),

$$\|B^{-1}\|_{sp}\le q^{\sim}_{A,\varepsilon}(B-1)\le\frac{\gamma^{\sim}_1(n,\rho_1,\varepsilon/2)}{1-(\gamma^{\sim}_1(n,\rho_1,\varepsilon/2))\varepsilon/2}=\tag{88}$$

$$\frac{\gamma_5(n,1\rho_1,\varepsilon/(2\rho_1))/\rho_1}{1-(\varepsilon/2)\gamma_5(n,1\rho_1,\varepsilon/(2\rho_1))/\rho_1}=\gamma_6(n,\rho_1,\varepsilon)<\frac{1}{\rho_1-\varepsilon}.$$

In view of (88) and (81), $|b_0| \geq (\rho_1 - \varepsilon)^n > 0$ and therefore (82) is fulfilled. The inequalities (84) and (85) follow from (88) and (79). ■

Lemma 6.7. *Let $m \in \mathbb{N}$, $n \in \mathbb{N}$. Let A be the matrix of the Lemma 6.6, $b_{i,\kappa} \in \mathbb{C}$, where $i = 1, \dots, m$, $\kappa \in \mathbb{Z}$, $b_{i,n} = 1$ and $b_{i,\kappa} = 0$ for $i = 1, \dots, m$, $\kappa \in (-\mathbb{N}) \cap (n + \mathbb{N})$ and*

$$B_i = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -b_{i,0} & -b_{i,1} & -b_{i,2} & \dots & -b_{i,n-1} \end{pmatrix}.$$

Let $q^{\sim}_{A,\varepsilon/2}(B_i - A) \leq \varepsilon/2$, where $i = 1, \dots, m$, let $B^{\wedge}$ be the lower triangle $m \times m$-matrix, in which $b_{k,i-k}$, stands on the intersection of the i-th row and $k-$th column, where $k = 1, \dots, m$ and $i = k, \dots, m$, but all other elements are equal to 0. Then

$$q^{\sim}_{\infty}(B^{\wedge}) \leq \gamma_3(n, m, \|A\|_{sp} + \varepsilon), q^{\sim}_{1}(B^{\wedge}) \leq \gamma_3(n, m, \|A\|_{sp} + \varepsilon). \tag{89}$$

If (78) is fulfilled and $0 < \varepsilon < (\|A^{-1}\|_{sp})^{-1}$, then $(B^{\wedge})^{-1}$ exists and $q^{\sim}_{\infty}((B^{\wedge})^{-1}) \leq \gamma_7(m, n, \gamma_6(n, (\|A^{-1}\|_{sp})^{-1}, \varepsilon))$,

$$q^{\sim}_{1}((B^{\wedge})^{-1}) \leq \gamma_7(m, n, \gamma_6(n, (\|A^{-1}\|_{sp})^{-1}, \varepsilon)).$$

Proof. In view of the lemma 6.6, $\|B_i\|_{sp} < \varepsilon + \|A\|_{sp}$, where $i = 1, \dots, m$. Consequently, $|b_{i,k}| < \binom{n}{k}(\varepsilon + \|A\|_{sp})^{n-k}$, where $i = 1, \dots, m, k = 0, \dots, n$. Therefore (69) holds. If (78) is fulfilled and $0 < \varepsilon < (\|A^{-1}\|_{sp})^{-1}$, then it follows from the lemma 6.6 that $b_{i,0} \neq 0$ for $i = 1, \dots, m$ and

$$1/|b_{i,0}| \leq \gamma_6(n, (\|A^{-1}\|_{sp})^{-1}, \varepsilon)^n, \tag{90}$$

where $i = 1, \dots, m$. Let D_1 be the diagonal $m \times m$-matrix, in which $b_{i,0}$, stands on the intersection of the i-th row and $i-$th column, where $i = 1, \dots, m$, and let $B^{\wedge}_1 = (D_1)^{-1} B_1$. Then $B^{\wedge}_1$ is the lower triangle $m \times m$-matrix, in which all the diagonal elements are equal to 1 and $B^{\wedge}_1 = E_m - F$, where F^m is the zero $m \times m$-matrix and E_m is the unit $m \times m$-matrix. Let $\mathfrak{D}_m$ and $\mathfrak{R}_m$ are the rings of all the diagonal and respectively the lower triangle $m \times m$-matrices in $\mathrm{Mat}_m(\mathbb{C})$; let N_m be the $m \times m$-matrix having the form $\begin{pmatrix} 0 & 0 \\ E_{m-1} & 0 \end{pmatrix}$, where E_{m-1} is the unit $(m-1) \times (m-1)$-matrix; let $\mathfrak{D}_{m,0} = \mathfrak{D}_m, \mathfrak{D}_{m,l} = \mathfrak{D}_m (N_m)^l = (N_m)^l \mathfrak{D}_m$ where $l \in \mathbb{N}$. Clearly, $\mathfrak{R}_m$ splits into direct sum $\mathfrak{R}_m = \oplus_{l=0}^{\infty} \mathfrak{D}_{m,l} = \oplus_{l=0}^{m-1} \mathfrak{D}_{m,l}$ and is the graduated ring, for which $\mathfrak{D}_{m,l}$ is the additive group of homgeneous elements of degree l. Clearly, $F = \sum\limits_{l=1}^{m-1} F_l$, where $F_l \in \mathfrak{D}_{m,l}$ for $l = 1, \dots, m-1$. According to the Lemma 6.6, $(B_i)^{-1}$ exists and $\|B^{-1}\|_{sp} \le \gamma_6(n, \rho_1, \varepsilon)$. The absolute value of the each element of the matrix F_l is not greater, than $\binom{n}{l}(\gamma_6(n, (\|A^{-1}\|_{sp})^{-1}, \varepsilon))^l$. On the other hand

$$(91) \qquad (B_1^{\wedge})^{-1} = E_m + \sum_{l=0}^{m-1} F^l = E_m + \sum_{k=1}^{m-1} \left(\sum_{l=1}^{m-1} \left(\sum_{\substack{l_1 \in \mathbb{N}, \dots, l_s \in \mathbb{N} \\ \sum\limits_{s=1}^{k} l_s = l}} \prod_{s=1}^{k} F_{l_s} \right) \right).$$

The absolute value of the each element of the matrix

$$\sum_{\substack{l_1 \in \mathbb{N}, \dots, l_s \in \mathbb{N} \\ \sum\limits_{s=1}^{k} l_s = l}} \prod_{s=1}^{k} F_{l_s}$$

is not greater than

$$(\gamma_6(n, (\|A^{-1}\|_{sp})^{-1}, \varepsilon))^l \sum_{\substack{l_1 \in \mathbb{N}, \dots, l_s \in \mathbb{N} \\ \sum\limits_{s=1}^{k} l_s = l}} \prod_{s=1}^{k} \binom{n}{l_s}$$

The last sum is equal to the coefficient by x^l in the polynomial $((1+x)^n - 1)^l$; we denote this coefficient by $\eta^{\sim}(n, k, l)$. Then, in view of (91), (66) for $q = q_1$ and for $q = q_\infty$, we have

$$q^{\sim}((B_1^{\wedge})^{-1}) \le 1 + \sum_{k=1}^{m-1} \sum_{l=1}^{m-1} \eta^{\sim}(n, k, l)(\gamma_6(n, (\|A^{-1}\|_{sp})^{-1}, \varepsilon))^l =$$

$$1 + \sum_{l=1}^{m-1} (\gamma_6(n, (\|A^{-1}\|_{sp})^{-1}, \varepsilon))^l \sum_{k=1}^{m-1} \eta^{\sim}(n, k, l) =$$

$$1 + \sum_{l=1}^{m-1} (\gamma_6(n, (\|A^{-1}\|_{sp})^{-1}, \varepsilon))^l \eta(n, l) =$$

$$\gamma_7(m, n, \gamma_6(n, \|A^{-1}\|_{sp})^{-1}, \varepsilon))(\gamma_6(n, (\|A^{-1}\|_{sp})^{-1}, \varepsilon))^{-n}.$$

In view of (90), $q^{\sim}(D_1) < (\gamma_6(n, (\|A^{-1}\|_{sp})^{-1}, \varepsilon))^n$. Therefore

$$q^{\sim}((B^{\sim})^{-1}) \le q^{\sim}((B_1^{\sim})^{-1}) q^{\sim}((D_1)^{-1}) \le \gamma_7(m, n, \gamma_6(n, (\|A^{-1}\|_{sp})^{-1}, \varepsilon)). \blacksquare$$

§7. Some remarks on contracted maps and fixed point Theorem.

If (Y,ρ) is a bounded metric space (i.e. $d(Y)=\sup\{\rho(x,y)\colon (x,y)\in Y^2\}<+\infty$), then following function $\rho^{\sim}$ is defined on X^2, where $X=Y^Y$: $\rho^{\sim}(A,B)=\sup\{\rho(A(y),B(y))\colon y\in Y)$, where $A\in X=Y^Y$ and $B\in\mathbb{X}$. Clearly, $(X,\rho^{\sim})$ is bounded metric space and $d(X)\le d(Y)$; $(X,\rho^{\sim})$ is complete if (Y,ρ) is complete. In this section we consider only bounded metric spaces. Let $\gamma_0\in(0,1)$. If A is a contacted map of (Y,ρ) into (Y,ρ) with coefficient of contraction equal to γ_0 then, clearly, $\rho^{\sim}(AB,AC)\le\gamma_0\rho(B,C)$ for any $B\in Y^Y$ and $C\in Y^Y$. To avoid the superfluous notations, if $Z\subset Y$, then the restriction of ρ on Z^2 we shall again denote by ρ.

Lemma 7.1. *Let (Y,ρ) is the bounded metric space, $r\in\mathbb{N}$. Let $A\in X=Y^Y$, $\gamma_0\in(0,1)$, A^r is the contracted map (Y,ρ) into (Y,rho) with coefficient of contraction equal to γ_0 and fixed point y^*. Let $y_0\in Y$, $B_n\in X=Y^Y$, where $n\in\mathbb{N}-1$, let $y(n+1)=B_n(y_n)$, where $n\in\mathbb{N}-1$, let $\delta_r(y)=\sup\{\rho(y^*,A^s(y))\colon s=0,\ldots,r-1\}$, where $y\in Y$ and let $\delta_r^{\sim}(B)=\sup\{\rho^{\sim}(A^\kappa,A^{\kappa-1}B)\colon\kappa=1,\ldots,r\}$. Then*

$$\rho(y_n,y*)\le(\gamma_0)^{[n/r]}\delta_r(y_0)+\sum_{\kappa=1}^{n}(\gamma_0)^{[(\kappa-1)/r]}\delta_r^{\sim}(B_{n-\kappa})= \tag{92}$$

$$(\gamma_0)^n\delta_r(y_0)+\sum_{\kappa=1}^{n}(\gamma_0)^{[(n-\kappa)/r]}\delta_r^{\sim}(B_{\kappa-1}).$$

If $\lim\limits_{\nu\to+\infty}\delta_r^{\sim}(B_n)=0$ then $\lim\limits_{n\to+\infty}\rho(y_n,y^*)=0$. Moreover, if the functin $\xi(x)$ is defined on $[0,+\infty)$, $\xi(x)>0$ for $x\in[0,+\infty)$, $\xi(x)$ decreases together with increasing of $x\in[0,+\infty)$, $\lim\limits_{x\to\infty}(\xi(x))=0$, (12), (13) are fulfilled and $\delta_r^{\sim}(B_n)<\xi(n)$ for $n\in\mathbb{N}-1$, then $\rho(y_n,y^*)=O(\xi(n))$.

Proof. Clearly,

$$\rho(y_n,y^*)=\rho(B_{n-1}(y_{n-1}),A(y^*))\le \tag{93}$$

$$\rho(A(y_{n-1}),A(y^*))+\rho(B_{n-1}(y_{n-1}),A(y_{n-1})\le\rho(A(y_{n-1}),A(y^*))+\rho^{\sim}(B_{n-1},A)$$

Iterating of (93) we obtain the inequality

$$\rho(y_n,y^*)\le\rho(A^k(y_{n-k}),A^k(y^*))+\sum_{\kappa=1}^{k}\rho^{\sim}(A^{\kappa-1}B_{n-\kappa},A^\kappa), \tag{94}$$

where $k\in\mathbb{N}$, $n\in k+\mathbb{N}-1$. In fact, (94) coincides with (93) for $k=1$. Let $n\in\mathbb{N}+1$, $k\in[2,n]\cap\mathbb{N}$ and (94) is true for $k-1$. Then

$$\rho(y_n,y^*)\le\rho(A^{k-1}(y_{n-k+1}),A^{k-1}(y^*))+\sum_{\kappa=1}^{k-1}\rho^{\sim}(A^{\kappa-1}B_{n-\kappa},A^\kappa),$$

$$\rho(y_n, y^*) \leq \rho(A^{k-1}(B_{n-k}y_{n-k}), A^{k-1}(Ay^*)) + \sum_{\kappa=1}^{k-1} \rho^{\sim}(A^{\kappa-1}B_{n-\kappa}, A^{\kappa}) \leq$$

$$\rho(A^{k-1}(Ay_{n-k}), A^{k-1}(Ay^*)) + \rho(A^{k-1}(B_{n-k}y_{n-k}), A^{k-1}(Ay_{n-k}))+$$

$$\sum_{\kappa=1}^{k-1} \rho^{\sim}(A^{\kappa-1}B_{n-\kappa}, A^{\kappa}) \leq$$

$$\rho(A^k y_{n-k}), A^k y^*)) + \rho^{\sim}(A^{k-1}B_{n-k}, A^k) + \sum_{\kappa=1}^{k-1} \rho^{\sim}(A^{\kappa-1}B_{n-\kappa}, A^{\kappa}).$$

We take now $k = n = rl + s$, $\kappa = r\lambda + \sigma$, where $s = 0, \dots, r-1$, $l \in \mathbb{N}$, $\sigma = 1, \dots, r$, $\lambda \in \mathbb{N} - 1$ and obtain the inequalities

$$\rho(y_n, y^*) \leq \rho(A^k(y_{n-k}), A^k(y^*)) \leq (\gamma_0)^l \rho(A^s(y_{n-k}), y^*) \leq \tag{95}$$

$$(\gamma_0)^l \sup\{\rho^{\sim}(A^s(y_0), y^*)\colon s = 0, \dots, q-1\} = (\gamma_0)^l \delta_r(y_0),$$

$$\rho^{\sim}(A^{\kappa-1}B_{n-\kappa}, A^{\kappa}) = \tag{96}$$

$$\rho^{\sim}(A^{r\lambda+\sigma-1}B_{n-\kappa}, A^{r\lambda+\sigma}) \leq (\gamma_0)^{\lambda} \rho^{\sim}(A^{\sigma-1}B_{n-\kappa}, A^{\sigma}) \leq$$

$$(\gamma_0)^{\lambda} \sup\{\rho^{\sim}(A^{\sigma-1}B_{n-\kappa}, A^{\sigma})\colon \sigma = 1, \dots, r,\} = (\gamma_0)^{[(\kappa-1)/r]} \delta_r^{\sim}(B_{n-\kappa}).$$

The inequality (92) follows from (95) and (96). Let us rewrite (92) in the form

$$\rho(y_n, y*) \leq (\gamma_0)^n \delta_r(y_0) + \sum_{\kappa=1}^{[n/2]} (\gamma_0)^{[(\kappa-1)/q]} \delta_r^{\sim}(B_{n-\kappa})+ \tag{97}$$

$$\sum_{\kappa=[n/2]+1}^{n} (\gamma_0)^{[(\kappa-1)/q]} \delta_r^{\sim}(B_{n-\kappa}).$$

Other assertions of the Lemma follow from (97). ∎

Let (Y, ρ) is the bounded metric space, $A \in X = Y^Y$. We name A by centripetal map relatively ρ if there exists $\gamma_0 \in (0, 1)$ and fixed point $y^* \in Y$ of the map A such that $\rho(A(y), y^*) \leq \gamma_0 \rho(A(y), y^*)$ for any $y \in Y$. If $\mathfrak{q}$ is an norm on linear space L then we denote by $O_{\mathfrak{q},\delta}(Z^*)$ the open ball with the center Z^* and radius r i.e., $\{Z \in L\colon \mathfrak{q}(Z - Z^*) < \delta\}$.

Lemma 7.2. *Let $\mathfrak{q}$ be a norm on the linear space L over $\mathbb{C}$ (or $\mathbb{R}$) and $L^{\sim}$ be te space of the all linear continuous operatos on L. Let further $\mathfrak{q}^{\sim}$ be the norm on $L^{\sim}$ associated with the norm q. Let $Z^* \in L$ and D is a bounded open neighborhood of the zero element in L. Let further $\Phi(Z)$ a differentiable map in the sence of Frech'et of the set $Z^* + D$ in L and let the Frechét derivative $\left(\frac{\partial}{\partial Z}\Phi\right)(Z)$ is continuous in*

$Z^* + D$. *Let* $A = \left(\frac{\partial}{\partial Z}\Phi\right)(Z^*)$, *let* $\mathfrak{q}^{\sim}(A) < 1$ *and let* $\varepsilon \in (0, 1 - \mathfrak{q}(A))$. *Then there exist* $\delta \in (0,1)$ *such that for any* $Z^{\vee}$ *and* $Z^{\wedge}$ *in* $Z^* + \delta D$ *the following inequality holds:* $\mathfrak{q}(\phi(Z^{\vee}) - \phi(Z^{\wedge})) \leq (\mathfrak{q}^{\sim}(A) + \varepsilon)\mathfrak{q}(Z^{\vee} - Z^{\wedge})$.

Proof. Since D is open, it follows that there exists $\delta_1 > 0$, such that $O_{\mathfrak{q},\delta_1}(Z^*) \in D$. Since $\left(\frac{\partial}{\partial Z}\Phi\right)(Z)$ is continuous, it follows that for any $\varepsilon > 0$ there exists $\delta_2 \in (0, \delta_1)$ such that

$$\mathfrak{q}^{\sim}\left(\left(\frac{\partial}{\partial Z}\Phi\right)(Z) - A\right) < \varepsilon$$

for $Z \in O_{\mathfrak{q},\delta_2}(Z^*)$. Then $\mathfrak{q}^{\sim}\left(\left(\frac{\partial}{\partial Z}\Phi\right)(Z)\right) < \mathfrak{q}^{\sim}(A) + \varepsilon$ for $Z \in O_{\mathfrak{q},\delta_2}(Z^*)$. Let

$$Z^{\vee} \in O_{\mathfrak{q},\delta_2}(Z^*), Z^{\wedge} \in O_{\mathfrak{q},\delta_2}(Z^*).$$

Since $O_{\mathfrak{q},\delta_2}(Z^*)$ is convex,it folows that $\phi(t) = \Phi((1-t)Z^{\vee} + tZ^{\wedge})$, is defined $[0,1]$ map of the segment $[0,1]$ in L and, moreover, this map has a continuous derivative $\left(\frac{\partial}{\partial t}\phi\right)(t) = \left(\frac{\partial}{\partial Z}\Phi\right)((1-t)Z^{\vee} + tZ^{\wedge})(Z^{\wedge} - Z^{\vee})$. Therefore

$$\mathfrak{q}(\Phi(Z^{\wedge}) - \Phi(Z^{\vee})) = \mathfrak{q}(\phi(1) - \phi(0)) = \mathfrak{q}\left(\int_0^1 \left(\frac{\partial}{\partial t}\phi\right)(t)dt\right) =$$

$$\mathfrak{q}\left(\left(\int_0^1 \left(\left(\frac{\partial}{\partial Z}\Phi\right)((1-t)Z^{\vee} + tZ^{\wedge})\right)dt\right)(Z^{\wedge} - Z^{\vee})\right) \leq$$

$$\mathfrak{q}^{\sim}\left(\int_0^1 \left(\frac{\partial}{\partial Z}\Phi\right)((1-t)Z^{\vee} + tZ^{\wedge}))dt\right)\mathfrak{q}(Z^{\wedge} - Z^{\vee}) \leq$$

$$\left(\int_0^1 \left(\mathfrak{q}^{\sim}\left(\frac{\partial}{\partial Z}\Phi\right)((1-t)Z^{\vee} + tZ^{\wedge})\right)dt\right)\mathfrak{q}(Z^{\wedge} - Z^{\vee})$$

$$\leq (\mathfrak{q}^{\sim}(A) + \varepsilon)\mathfrak{q}(Z^{\wedge} - Z^{\vee}).$$

Since D is bounded, it follows that there exist $\delta > 0$ such that $Z^* + \delta D \subset O_{\mathfrak{q},\delta_2}(Z^*)$. ■

Lemma 7.3. *Let* L *be a linear normed space over* $\mathbb{C}$ *(or* $\mathbb{R}$*) with norm* $\mathfrak{q}$. *Let further* $\mathfrak{q}^{\sim}$ *be the norm on* $L^{\sim}$ *associated with the norm* q. *Let* $Z^* \in L, D$ *is bounded open neighborhood of the zero element in* L. *Let further* $\Phi(Z)$ *is a map of the set* $Z^* + D$ *in* $\mathbb{C}^n$, *which is differentiable in the sence of Frechét at the point* Z^* *and let* $A = (\frac{\partial}{\partial Z}\Phi)(Z^*)$. *Let there exists* $\gamma_0 \in (0,1)$ *such that* $\mathfrak{q}(\Phi(Z) - Z^*) \leq \gamma_0\mathfrak{q}(Z - Z^*)$ *for any* $Z \in D$. *Then* $\mathfrak{q}^{\sim}(A) < 1$.

Proof. Since D is open, it follows that there exists $\delta_1 > 0$ such that $O_{\mathfrak{q},\delta_1}(Z^*) \in D$. Since $\Phi(Z)$ is differentiable at Z^* in the sence of Frechét, it follows that $\mathfrak{q}(\Phi(Z^*+$

$H) - P(Z^*) - AH) = \omega(H)\mathfrak{q}(H)$, where $H \in O_{\mathfrak{q},\delta_1}(Z^*)$ and $\omega(H)$ is a function defined on $O_{\mathfrak{q},\delta_1}(Z^*)$ such that $\lim\limits_{H\to 0} \omega(H) = \omega(0_L) = 0$. Therefore for $\varepsilon \in (0, 1-\gamma_0)$ there exists $\delta_2 \in (0, \delta_1)$ such that $0 \le \omega(H) < \varepsilon$. for any $H \in O_{\mathfrak{q},\delta_1}(Z^*)$. Then

$$\mathfrak{q}(AH) \le \mathfrak{q}(\Phi(Z^* + H) - P(Z^*) - AH) + \mathfrak{q}(\Phi(Z^* + H) - P(Z^*)) \le$$

$$(\gamma_0 + \omega(H))\mathfrak{q}(H) \le (\gamma_0 + \varepsilon)\mathfrak{q}(H)$$

for any $H \in O_{\mathfrak{q},\delta_1}(Z^*)$. Therefore $\mathfrak{q}^{\sim}(A) \le \varepsilon + \gamma_0 < 1$. ■

Lemma 7.4. *Let $\mathfrak{q}$ is a norm on the linear space L over $\mathbb{C}$ (or $\mathbb{R}$) and $L^{\sim}$ be te space of the all linear continuous operatos on L. Let further $\mathfrak{q}^{\sim}$ be the norm on $L^{\sim}$ associated with the norm q. Let $Z^* \in L$, D is bounded open neighborhood of the zero element in L Let further $\Phi(Z)$ a differentiable map in the sence of Frechét of the set $Z^* + D$ in $Z^* + D$, let Z^* is the fixed point of Φ and let the Frechét derivative $\left(\frac{\partial}{\partial Z}\Phi\right)(Z)$ is continuous in $Z^* + D$. Let Φ is centripetal. Then there exists $r \in \mathbb{N}$ such that Φ is contracted map of set $\Phi^r(Z^* + D)$ into set $\Phi^r(Z^* + D)$.*

Proof. Let $A = (\frac{\partial}{\partial Z}\Phi)(Z^*)$. According to conditions of the Lemma, there exists $\gamma_0 \in (0, 1)$ and such that $\mathfrak{q}(\Phi(Z) - \Phi(Z^*)) \le \gamma_0\mathfrak{q}(Z - Z^*))$ for any $Z \in Z^* + D$. According to the Lemma 6.3, $\mathfrak{q}^{\sim}(A) < 1$. Let $\varepsilon \in (0, 1 - \mathfrak{q}(A))$. According to the Lemma 6.2, there exist $\delta \in (0, 1)$ such that for any $Z^{\vee}$ and $Z^{\wedge}$ in $Z^* + \delta D$ the following inequality holds: $\mathfrak{q}(\phi(Z^{\vee}) - \phi(Z^{\wedge})) \le (\mathfrak{q}^{\sim}(A) + \varepsilon)\mathfrak{q}(Z^{\vee} - Z^{\wedge})$. Since $0 < \mathfrak{q}^{\sim}(A) + \varepsilon < 1$ there exists $r \in \mathbb{N}$ such that $(\mathfrak{q}^{\sim}(A) + \varepsilon)^r < \delta$. Clearly Φ maps of set $\Phi^r(Z^* + D)$ into set $\Phi^{r+1}(Z^* + D) \subset \Phi^r(Z^* + D)$ and for any $Z^{\vee}$ and $Z^{\wedge}$ in $Z^* + \delta D$ the following inequality holds:

$$\mathfrak{q}(\Phi(\Phi^r(Z^{\vee})) - \Phi(\Phi^r(Z^{\wedge}))) \le (\mathfrak{q}^{\sim}(A) + \varepsilon)\mathfrak{q}(\Phi^r(Z^{\vee}) - \Phi^r(Z^{\wedge})). \blacksquare$$

§8. End of the proof of the Theorem 5.

We can now enter upon the realization of the plan marked in the section 3. For any matrix X we denote by $X^{\mathfrak{t}}$ the matrix transposed to the matrix X. We introduce on $\mathbb{C}^q$ the norm $\mathfrak{q}_3$ as follows:

with each column $f \in \mathbb{C}^p$ we connect the matrix $\vartheta(f) \in \mathrm{Mat}_p(\mathbb{C})$, such that all the elements of its first $p - 1$ rows are equal to 0 and last row i equal to $f^{\mathfrak{t}}$; let

$$B_1^{\sim} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -b_0^{\sim} & -b_1^{\sim} & -b_2^{\sim} & \dots & -b_{p-1}^{\sim} \end{pmatrix},$$

$q_3^{\sim} = q^{\sim}_{B_1^{\sim};\varepsilon_2/2}$, $\mathfrak{q}(f) = q_3^{\sim}(\vartheta(f))$.

If $\mathfrak{q}(f) \leq \varepsilon_2/2$, then acording to the Lemma 6.6, the matrix $B_1^{\sim} - \vartheta(f)$ is invertible and, in view of (88),

$$q_3^{\sim}((B_1^{\sim} - \vartheta(f))^{-1}) \leq \gamma_6(n, \rho_{1,1}, \varepsilon_2) < \frac{1}{\rho_{1,1} - \varepsilon_2}. \tag{98}$$

We introduce on the space $A_{p,q}(\mathbb{C})$ of all the $p \times q$ matrices the norm $\mathfrak{q}^*$ as follows: if $F \in A_{p,q}(\mathbb{C})$, then let $F_{*,k-1}$, $k = 1, \ldots, q$ denotes the k-th column of the matrix F and let $\mathfrak{q}^*(F) = \sup\{\mathfrak{q}(F_{*,k-1}) : k = 1, \ldots, q\}$. For any $F \in A_{p,q}(\mathbb{C})$, $i = 1, \ldots, p$, $k = 1, \ldots, q$ let $F_{i-1,k-1}$ denotes the number, which stands in i-th row and k-th column of the matrix F, $F_{p,k-1} = 1$ for $k = 1, \ldots, q$, and let further $F_{i-1,k-1} = 0$ for $k = 1, \ldots, q$, $i \in \mathbb{Z} \diagdown [0, p]$.

Remark 8.1. Actually $F_{i-1,k-1}$ play further the role of $b_{i-1}(\nu+k-1) - b_{i-1}^{\sim}$ for $i = 1, \ldots, p$, $k = 1, \ldots, q$, where $b_{i-1}(\nu + k - 1)$ and $b_{i-1}^{\sim}$ are considered at the end of the section 3.

Let $\mathfrak{O}(\varepsilon) = \{F \in A_{p,q}\mathbb{C} : \mathfrak{q}^*(F) < \varepsilon, \}$, $\mathfrak{O}_0 = \mathfrak{O}(\varepsilon_2/2)$. Let $B_1^{**}(F, k-1) = B_1^{\sim} - \vartheta(F_{*,k-1})$, where $F \in A_{q,p}\mathbb{C}$. If $F \in \mathfrak{O}_0$, and $k = 1, \ldots, q$, then according to Lemma 6.6, the matrix $B_1^{**}(F, k-1)$ is invertible and

$$q_3^{\sim}((B_1^{**}(F, k-1))^{-1}) \leq \gamma_6(p, \rho_{1,1}, \varepsilon_2) < \frac{1}{\rho_{1,1} - \varepsilon_2}.$$

For any $F \in A_{q,p}\mathbb{C}$, let $B_2(F)$ denotes the lower triangle $q \times q$-matrix, in which for $i = 1, \ldots, q$, $k = 1, \ldots, q$, the number $b_{i-k}^{\sim} + F_{i-k,k-1}$ stands on the intersection of the i-th row and k-th column and $B_3(F)$ denotes the $p \times q$-matrix, in which for $i = 1, \ldots, p$, $k = 1, \ldots, q$, the number $b_{q+i-k}^{\sim} + F_{q+i-k,k-1}$ stands on the intersection of the i-th row and k-th column, $B_3^{\sim} = B_3(0)$.

Remark 8.2. Actually $B_2(F)$ will play further the role of the matrix $B^{\wedge}(\nu)$ in (32) and $B_3(F)$ will play further the role of the matrix $\mathfrak{B}(\nu)$ in (33).

Applying the Lemma 6.7 to the case, when the role of n plays p, the role of m plays q, the role of A plays $B_1^{\sim}$, the role of $B^{\wedge}$ plays $B_2(F)$ with $F \in \mathfrak{O}_0$, we see that the following inequalities holds

$$q_\infty^{\sim}(B_2(F)) \leq \gamma_3(p, q, \rho_{1,0} + \varepsilon_2), q_1^{\sim}(B_2(F)) \leq \gamma_3(p, q, \rho_{1,0} + \varepsilon_2);$$

moreover, $(B_2(F))^{-1}$ exists and

$$q_\infty^{\sim}((B_2(F))^{-1}) \leq \gamma_{11}, q_1^{\sim}((B_2(F))^{-1}) \leq \gamma_{11}, \tag{99}$$

where γ_{11} is defined in the section 5. Let $A^{\vee}$ be the column in $\mathbb{C}^n$, in which the i-th element coincides with $a_{i-1}^{\sim}$ for $i = 1, \ldots, n$. Let $B^{\vee}$ be the column in $\mathbb{C}^p$, in which the i-th element coincides with $b_{i-1}^{\sim}$ for $i = 1, \ldots, p$. Let $U^{\vee}$ be the column in $\mathbb{C}^q$, in which the i-th element coincides with $u^{\sim}_{i-1}$ for $i = 1, \dots, q$. For any column $H \in \mathbb{C}^n$, the i-th element of H will be denoted by $H_{i-1,*}$ for $i = 1, \dots, n$. Let

$$U^*(F,H) = (B_2(F))^{-1}\pi_1(A^\vee + H), \tag{100}$$

$$\Phi_H(F) = \pi_2(A^\vee + H) - B_3(F)U^*(F,H) - B_1^\vee = \tag{101}$$

$$\pi_2(A^\vee + H) - B_3(F)(B_2(F))^{-1}\pi_1(A^\vee + H) - B_1^\vee$$

for any column $H \in \mathbb{C}^n$ and any $F \in \mathfrak{O}_0$. Let further $u^*_{i-1}(F,H)$ for $i = 1, \dots, q$ be the i-th element of the column $U^*(F,H)$, $u^*_q(F,H) = 1$ and $u^*_{i-1}(F,H) = 0$ for any column $H \in \mathbb{C}^n$, any $F \in \mathfrak{O}_0$ and any $i \in \mathbb{Z}\diagdown[0,q]$.

Remark 8.3. Actually $A^\vee + H$ will play further the role of the column $A(\nu)$ in (32) and (33), that $U^*(F,H)$ in (101) will play the role of $U(\nu)$ in (32), and $\Phi_H(F) + B^\vee$ will play the role of $(\triangledown^q B)(\nu)$ in (33).

Clearly,

$$U^*(0,0) = U^\vee, \Phi_0(0) = 0. \tag{102}$$

In view of (45),

$$\Phi_H(F) - \Phi_0(F) = \pi_2(H) - B_3(F)(B_2(F))^{-1}\pi_1(H). \tag{103}$$

Let $y = y(\kappa), \kappa \in [0,n] \cap \mathbb{Z}$ be an arbitrary sequence. Let

$$Y_{p,y}(\kappa) = \begin{pmatrix} y(\kappa) \\ \vdots \\ y(\kappa + p - 1) \end{pmatrix},$$

where $\kappa \in [0, q+1] \cap \mathbb{Z}$. Let E_p be the unit $p \times p$-matrix. Let

$$B_1^*(F,H,k-1) = B_1^{**}(F,k-1), \tag{104}$$

where $k = 1, \dots, q$, and $B_1^*(F,H,q) = B_1^{\sim} - \vartheta(\Phi_H(F))$, where $H \in \mathbb{C}^n$ and $F \in \mathfrak{O}_0$. Making use of these notations we have

$$\sum_{\beta=0}^{q}(u^*_\beta(F,H))(Y_{p,y}(\beta+1) - B_1^*(F,H,\beta)Y_{p,y}(\beta)) = \tag{105}$$

$$Y_{p,y}(q+1) + \sum_{\beta=0}^{q-1}(u^*_{\beta-1}(F,H)E_q - u^*_\beta(F,H)B_1^*(F,H,\beta))Y_{p,y}(\beta)).$$

Let

$$R^*_{\beta-1}(F,H) = u^*_{\beta-1}(F,H)E_q - u^*_\beta(F,H)B^*_1(F,H,\beta), \tag{106}$$

where $\beta \in \mathbb{Z}$. Then

$$R^*_{q+1}(F,H) = E_q. \tag{107}$$

Iterating (106) for $\beta = 0, , \ldots , q$. we have

$$u^*_\beta(F,H)E_q = -R^*_{\beta-1}(F,H)(B^*_1(F,H,\beta))^{-1} + u^*_{\beta-1}(F,H)(B^*_1(F,H,\beta))^{-1} =$$

$$-\sum_{\kappa=0}^{\gamma} R^*_{\beta-\kappa-1}(F,H) \prod_{\mu=\beta-\kappa}^{\beta} (B^*_1(F,H,\mu))^{-1}+$$

$$u^*_{\beta-\gamma-1}(F,H) \prod_{\mu=\beta-\gamma}^{\beta} (B^*_1(F,H,\mu))^{-1},$$

where $\beta = 0, , \ldots , q$, and $\gamma = 0, , \ldots , \beta$; in particular, for $\gamma = \beta$ we have

$$u^*_\beta(F,H)E_q = -\sum_{\kappa=0}^{\beta} R^*_{\beta-\kappa-1}(F,H) \prod_{\mu=\beta-\kappa}^{\beta} (B^*_1(F,H,\mu))^{-1}.$$

If $\beta = q-1$, then

$$u^*_{q-1}(F,H)E_q = \tag{108}$$

$$-\sum_{\kappa=0}^{q-1} R^*_{q-1-\kappa-1}(F,H) \left(\prod_{\mu=q-1-\kappa}^{q-1} (B^*_1(F,H,\mu))^{-1} \right) (B^*_1(F,H,q))^{-1}.$$

It follows from (106) for $\beta = q$ and from (108) that

$$B^*_1(F,H,q) = -R^*_{q-1}(F,H) + u^*_{q-1}(F,H) = \tag{109}$$

$$-R^*_{q-1}(F,H) - \sum_{\kappa=0}^{q-1} R^*_{q-1-\kappa-1}(F,H) \prod_{\mu=q-1-\kappa}^{q-1} (B^*_1(F,H,\mu))^{-1} =$$

$$-R^*_{q-1}(F,H) - \sum_{\kappa=1}^{q} R^*_{q-\kappa-1}(F,H) \prod_{\mu=q-\kappa}^{q-1} (B^*_1(F,H,\mu))^{-1} =$$

$$-\sum_{\kappa=0}^{q} R^*_{q-\kappa-1}(F,H)\Pi_{q-\kappa,q}(F),$$

where $\Pi_{\omega,\omega}(F) = E_q$ for $\omega \in [0,q]\cap\mathbb{Z}$ and $\Pi_{\alpha,\omega}(F) = \prod_{\mu=\alpha}^{\omega-1}(B_1^{**}(F,\mu))^{-1}$ for $\omega \in [1,q]\cap\mathbb{Z}$, $\alpha \in [0,\omega]\cap\mathbb{Z}$.

Let L be a linear space over $\mathbb{C}$; on $L^{\mathbb{Z}}$ acts a $\mathbb{C}$-linear operator $\bigtriangledown$, for which $(\bigtriangledown X)(\nu) = X(\nu+1)$, where $X \in L^{\mathbb{Z}}$, $\nu \in \mathbb{Z}$. Let $\mathfrak{M}^\vee(L)$ be the set of all the $\mathbb{C}$-linear maps of the space L into L; then the equality $(AX)(\nu) = A(\nu)X(\nu)$, where $A \in (\mathfrak{M}^\vee(L))^{\mathbb{Z}}$, $X \in L^{\mathbb{Z}}$, $\nu \in \mathbb{Z}$ determines the natural embeding of the space $(\mathfrak{M}^\vee(L))^{\mathbb{Z}}$ into the space $\mathfrak{M}^\vee(L^{\mathbb{Z}})$ of all the $\mathbb{C}$-linear maps of the space $L^{\mathbb{Z}}$ into $L^{\mathbb{Z}}$. The identity operator on $L^{\mathbb{Z}}$ will be denoted by I and the operator, for wich $(\bigtriangledown A)(\nu) = A(\nu+1)$ for $A \in (\mathfrak{M}^\vee(L))^{\mathbb{Z}}$, $\nu \in \mathbb{Z}$, will be (also) denoted by $\bigtriangledown$. Clearly, for the elements (55) the following equality holds

$$\prod_{k=1}^{l}(I - z_kA) = I + \sum_{k=1}^{l}(-1)^k\sigma_{k,l}(z_1, \ldots, z_l)A^k, \tag{110}$$

where $A \in \mathfrak{M}^\vee(L^{\mathbb{Z}})$ and $l \in \mathbb{N}$. Let z_i for $i = 1, \ldots, p$ are zeros of the polynomial $T_1(z)$ and z_i for $i = p+1, \ldots, n$ are zeros of the polynomial $T_2(z)$ and each zero of the polynomial $T(z)$ is counted according its multiplicity. Clearly, $b_k^\sim = (-1)^{p-k}\sigma_{p-k,p}(z_1, \ldots, z_p)$ for $k = 0, \ldots, p$, and also $u_k^\sim = (-1)^{q-k}\sigma_{q-k,q}(z_p + 1, \ldots, z_{p+k})$ for $k = 0, \ldots, q$. We define $b_k^\sim = 0$ for $k \in \mathbb{Z}\diagdown[0,p]$ and also $u_k^\sim = 0$ for $k \in \mathbb{Z}\diagdown[0,q]$. Let $L = \mathrm{Mat}_p(\mathbb{C})$,

$$x(\kappa) = \Delta^{(0)}(F,H,\kappa) = \Delta(F,H,\kappa) = B_1^*(F,H,\kappa) - B_1^\sim \tag{111}$$

for $\kappa \in [0,q]\cap\mathbb{Z}$, but $x(\kappa) = \Delta^{(0)}(F,H,\kappa) = \Delta^{(0)}(F,H,\kappa) = 0$ for $\kappa\mathbb{Z}\diagdown[0,q]$. In view of (102) and (98), if $\mathfrak{q}^*(F) \le \varepsilon_2/2$ then there exists $(B_1^{**}(F,\kappa-1))^{-1}$ for $\kappa \in [1,q]\cap\mathbb{Z}$, and $q_3^\sim((B_1^{**}(F,\kappa-1))^{-1}) < \frac{1}{\rho_{1,1}-\varepsilon_2}$, where $\varepsilon_2 \in (0,\rho_{1,1})$; let $D \in L^{\mathbb{Z}}$ and $D^\vee \in \mathfrak{M}(L)^{\mathbb{Z}}$ is defined as follows:

$$D(\kappa) = (B_1^{**}(F,\kappa))^{-1} \tag{112}$$

for $\kappa \in [0,q-1]\cap\mathbb{Z}$ and $D(\kappa) = 0$ for $\kappa \in \mathbb{Z}\diagdown[0,q-1]$. Let further $D^\vee(\kappa)$ for $\kappa \in \mathbb{Z}$ be the operator, which turns any matrix $V \in L$ into $VD(\kappa)$. Let

$$\Delta^{(i)}(F,H,\kappa) = \Delta^{(i-1)}(F,H,\kappa) - z_{i+p}\Delta^{(i-1)}(F,H,\kappa-1)D(\kappa-1), \tag{113}$$

where $i \in [1,q]\cap\mathbb{Z}$, $\kappa \in \mathbb{Z}$. Taking in (110) $A = \bigtriangledown^{-1}D^\vee$, we obtain the equality

$$\Delta^{(i)}(F,H,\kappa) = \Delta(F,H,\kappa) + \tag{114}$$

$$\sum_{k=1}^{i}(-1)^k\sigma_{k,i}(z_{p+1},\,\ldots\,,\,z_{p+i})\Delta(F,H,\kappa-k)\prod_{\mu=\kappa-k}^{\kappa-1}D(\mu)=$$

$$\Delta(F,H,\kappa)+\sum_{k=1}^{i}(-1)^k\sigma_{k,i}(z_{p+1},\,\ldots\,,\,_{p+i})\Delta(F,H,\kappa-k)\prod_{\mu=\kappa-k}^{\kappa-1}(B_1^{**}(F,\mu))^{-1}=$$

$$\sum_{k=0}^{i}(-1)^k\sigma_{k,i}(z_{p+1},\,\ldots\,,\,z_{q+i})\Delta(F,H,\kappa-k)\Pi_{\kappa-k,\kappa},$$

where $i\in[1,q]\cap\mathbb{Z}$, $\kappa\in[i,q]\cap\mathbb{Z}$. Clearly, (114) holds also for $i=0$. In view of (114),

$$\Delta^{(q)}(F,H,q)=\sum_{k=0}^{q}u_k^{\sim}\Delta(F,H,q-k)\Pi_{q-k,q}= \tag{115}$$

$$\sum_{\kappa=0}^{q}u_{q-\kappa}^{\sim}\Delta(F,H,k)\Pi_{k,q}.$$

Let in (106) $\Omega^*_{\beta-1}(F,H)=R^*_{\beta-1}(F,H)-R^*_{\beta-1}(0,0)$, where $\beta\in[1,q]\cap\mathbb{Z}$. Then, in view of (109) and (102)

$$B_1^*(F,H,q)=-\left(\sum_{\kappa=0}^{q}R^{\sim}_{q-\kappa-1}(0,0)\Pi_{q-\kappa,q}(F)\right)- \tag{116}$$

$$\sum_{\kappa=0}^{q}\Omega^*_{q-\kappa-1}(F,H)\Pi_{q-\kappa,q}(F)=-\left(\sum_{\kappa=0}^{q}u^{\sim}_{q-\kappa-1}\Pi_{q-\kappa,q}(F)\right)+$$

$$\left(\sum_{\kappa=0}^{q}u^{\sim}_{q-\kappa}B_1^{\sim}\Pi_{q-\kappa,q}(F)\right)-\sum_{\kappa=0}^{q}\Omega^*_{q-\kappa-1}(F,H)\Pi_{q-\kappa,q}(F)=$$

$$B_1^*(F,H,q)=-\left(\sum_{\kappa=1}^{q}u^{\sim}_{q-\kappa}B_1^{**}(F,q-\kappa)\Pi_{q-\kappa,q}(F)\right)+$$

$$\left(\sum_{\kappa=0}^{q}u^{\sim}_{q-\kappa}B_1^{\sim}\Pi_{q-\kappa,q}(F)\right)-\sum_{\kappa=0}^{q}\Omega^*_{q-\kappa-1}(F,H)\Pi_{q-\kappa,q}(F).$$

In view of (116),

$$0=-\left(\sum_{\kappa=0}^{q}u^{\sim}_{q-\kappa}\Delta(F,H,q-\kappa)\Pi_{q-\kappa,q}(F)\right)- \tag{117}$$

$$\sum_{\kappa=0}^{q}\Omega^*_{q-\kappa-1}(F,H)\Pi_{q-\kappa,q}(F).$$

We add (117) with (115) and obtain the equality

$$\Delta^{(q)}(F,H,q)=-\sum_{\kappa=0}^{q}\Omega^*_{q-\kappa-1}(F,H)\Pi_{q-\kappa,q}(F).$$

In view of (113) and (112),

$$\Delta^{(q)}(F,H,q)=\Delta^{(q-k)}(F,H,q)-\sum_{j=1+q-k}^{k}z_{q+1-j+p}\Delta^{(q-j)}(F,H,q-1)D(q-1)=$$

$$\Delta^{(q-k)}(F,H,q-k)-\sum_{j=q+1-k}^{q}z_{j+p}\Delta^{(j-1)}(F,H,q-1)(B_1^{**}(F,q-1))^{-1},$$

where $k\in[0,q]\cap\mathbb{Z}$. Therefore

$$\Delta^{(i)}(F,H,q)=\Delta^{(q)}(F,H,q)+ \tag{118}$$

$$\sum_{j=i+1}^{q}z_{j+p}\Delta^{(j-1)}(F,H,q-1)(B_1^{**}(F,q-1))^{-1},$$

where $i\in[0,q]\cap\mathbb{Z}$. In view of (111), (104) and (114),

$$\Delta^{(i)}(F,H,\kappa)=\Delta^{(i)}(F,0,\kappa)= \tag{119}$$

$$\sum_{k=0}^{i}(-1)^k\sigma_{k,i}(z_{q+1},\,\ldots\,,\,z_{q+i})\vartheta(F_{*,\kappa-k})\Pi_{\kappa-k,\kappa},$$

where $i\in[1,q-1]\cap\mathbb{Z}$, $\kappa\in[i,q-1]\cap\mathbb{Z}$.

For any $F\in\mathfrak{O}_0$, $H\in\mathbb{C}^q$ we denote by $\Psi_H(F)$ the matrix in $A_{p,q}(\mathbb{C})$, in which the last column coincides with $\Phi_H(F)$ and on the first $q-1$ columns stands the matrix obtained by the remowing of the first column in F.

Remark 8.4. The passage $F\to\Psi_H(F)$ corresponds to the described in the section 3 passage from $b_{i-1}(\nu+s)$ for $i=1,\,\ldots\,,p$ and $s=0,\,\ldots\,,q-1$ to $b_{i-1}(\nu+1+s)$ for $i=1,\,\ldots\,,p$ and $s=1,\,\ldots\,,q$.

We denote below the distance determined by a norm on linar space in the same way as this norm. Let

$$\rho_0(F)=\sup\{q^{\sim}_{B_1;\varepsilon_2/2}(\varepsilon_4)^{1-i}\Delta^{(i-1)}(F,0,k-1)\colon i=1,\,\ldots\,,q,\;k=1,\,\ldots\,,q\},$$

where ε_4 is defined in (67) and $Y \subset \mathfrak{O}_0$. Clearly, $\rho^{\sim}(F)$ is continuous function on $\mathfrak{O}_0$. Let $Y = \{F \in \mathfrak{O}_0 \colon \rho_0(F) < \varepsilon_2/2.\}$ I view of (102), $0 \in Y$ and Y is nonempty open subset in $\mathfrak{O}_0$. We study the map $\psi_H(F)$ on the open set Y for $H \in \mathbb{C}^p$. Let

$$0 < \gamma \leq \frac{\varepsilon_2}{2}, F \in Y, \rho_0(F) < \gamma, q_3^{\sim}(\Delta^{(\mu)}(F, 0, \beta) < \gamma(\varepsilon_4)^{\mu}, \tag{120}$$

where $\mu = 0, \ldots, q-1$ and $\beta = \mu, \ldots, q-1$. Let

$$\theta^{(i)}(\gamma) = \gamma(\varepsilon_4)^i, \tag{121}$$

where $i = 0, \ldots, q$. Clearly,

$$\theta^{(i)}(\gamma) = \theta^{(q)}(\gamma) + (1 - \varepsilon_4) \sum_{\mu=i}^{q-1} \theta^{(\mu)}(\gamma). \tag{122}$$

In view of (120), (121) and (98),

$$q_3^{\sim}(\Delta^{(\mu)}(F, 0, \beta)) \leq \theta^{(\mu)}(\gamma), \tag{123}$$

where $\mu = 0, \ldots, q-1$ and $\beta = \mu, \ldots, q-1$,

$$q_3^{\sim}(B_1^{**}(F, 0, \beta)) \leq \frac{1}{\rho_{1,1} - \varepsilon_2}, \tag{124}$$

where $\beta = 0 \ldots, q-1$. In view of (118), (119), (122), (123), (124), (67),

$$q_3^{\sim}(\Delta^{(i)}(F, H, q)) \leq q_3^{\sim}(\Delta^{(q)}(F, H, q)) + \tag{125}$$

$$q_3^{\sim}((B_1^{**}(F, q-1))^{-1}) \sum_{j=i+1}^{q} |z_{j+p}| q_3^{\sim}(\Delta^{(j-1)}(F, 0, q-1)) <$$

$$q_3^{\sim}(\Delta^{(q)}(F, H, q)) + (1 - \varepsilon_4) \sum_{j=i+1}^{q} \theta^{(j-1)}(\gamma) = q_3^{\sim}(\Delta^{(q)}(F, H, q)) + \theta^{(i)}(\gamma) - \theta^{(q)}(\gamma),$$

where $i \in [0, q] \cap \mathbb{Z}$. Let $b_{\alpha}^{\sim} = 0$, $u_{\beta}^{\sim} = 0$ for $\alpha \in \mathbb{Z} \diagdown [0, p]$ and $\beta \in \mathbb{Z} \diagdown [0, q]$. In view of (3), (8) and (9), $a_{\tau}^{\sim} = \sum_{\alpha=0}^{\tau} b_{\alpha}^{\sim} u_{\tau-\alpha}^{\sim}$, where $\tau \in \mathbb{Z}$. Let $b_{k-1}^{*}(F, H, i))$ be the product of -1 and the value, which stands in the matrix $B_1^{*}(F, H, i)$ in the last row and the $k-th$ column, where $i = 0, \ldots, q$ and $k = 1, \ldots, p$. Then

$$\Delta_y^{*}(F, H, i) = Y_{p,y}(i+1) - B_1^{*}(F, H, i) Y_{p,y}(i) = \tag{126}$$

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \sum\limits_{\alpha=0}^{p} b^*_\alpha(F,H,i)y(i+\alpha), \end{pmatrix},$$

where $i = 0, \ldots, q$. In view of (105), (106) and (107), we have

$$\sum_{\beta=0}^{q+1}(u^*_\beta(F,H))\Delta^*_y(F,H,\beta) = \sum_{\beta=0}^{q} R_{\beta-1}Y_{p,y}(\beta).$$

It follows from the definition of the matrices $B^*_1(F,H,\beta)$ for $\beta = 0, \ldots, q$, from the dfinition of the matrices $B_2(F), B_3(F)$ and from the definition of the numbers $b^*_k(F,H,i)$ for $i = 0, \ldots, q$ and $k = 1, \ldots, p$ that $B_2(F)$ is the lower triangle $q\times q$-matrix, in which for $i = 1, \ldots, q$ and $k = 1, \ldots, q$ the number $b^*_{i-k}(F,H,k-1) = b^\sim_{i-k} + F_{k-1,i-k}$, stands on the intersection of the i-th row and k-th column, $B_3(F)$ is the $p \times q$-matrix, in which for $i = 1, \ldots, p$, $k = 1, \ldots, q$, the number $b^*_{q+i-k}(F,H,k-1) = b^\sim_{q+i-k} + F_{q+i-k,k-1}$ stands on the intersection of the i-th row and k-th column. In view of (100),

$$a^\sim_{i-1} + H_{i-1,*} = \sum_{k=1}^{q} u^*_{k-1}(F,H)b^*_{i-k}(F,H,k-1), \tag{127}$$

where $i = 1, \ldots, q$. In view of (101), for $i = 1, \ldots, p$ the $i - th$ element of the column $\pi_2(A^\vee + H) - B_3(F)(B_2(F))^{-1}\pi_1(A^\vee + H)$ coincides with $b^*_{i-1}(F,H,q)$. Therefore

$$a^\sim_{q+i-1} + H_{i-1,*} = b^*_{i-1}(F,H,q) + \sum_{k=1}^{q} u^*_{k-1}(F,H)b^*_{q+i-k}(F,H,k-1),$$

where $i = 1, \ldots, p$. Consequently, in view of (126) - (127),

$$\sum_{\beta=0}^{q} u^*_\beta(F,H)\sum_{\alpha=0}^{p} b^*_\alpha(F,H,\beta)y(\beta+\alpha) =$$

$$\sum_{\tau=0}^{n}\left(\sum_{\beta=0}^{q} u^*_\beta(F,H)b^*_{\tau-\beta}(F,H,\beta)\right)y(\tau) = \sum_{\tau=0}^{n} y(\tau)(a^\sim_\tau + H^\sim_{\tau,*})$$

and therefore

$$\sum_{\beta=0}^{q+1} R_{\beta-1}(F,H)Y_{p,y}(\beta) = \sum_{\beta=0}^{q}(u^*_\beta(F,H))\Delta^*_y(F,H,\beta) = \tag{128}$$

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \sum\limits_{\tau=0}^{n} y(\tau)(a_\tau^\sim + H_{\tau,*}^\sim) \end{pmatrix},$$

$$\sum_{\beta=0}^{q+1} R_{\beta-1}(0,0)Y_{p,y}(\beta) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \sum\limits_{\tau=0}^{n} y(\tau)a_\tau^\sim \end{pmatrix}. \tag{129}$$

Clearly, $H_{n,*}^\sim = 0$ and, in view of (107), $\Omega_{q+1}^*(F,H) = 0$. After the substraction of the equality (129) from the equality (128) we have

$$\sum_{\beta=0}^{q} \Omega_{\beta-1}(F,H)Y_{p,y}(\beta) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \sum\limits_{\tau=0}^{n-1} y(\tau)H_{\tau,*}^\sim \end{pmatrix}. \tag{130}$$

Let $Z(F,H.0) = E_p$,

$$Z(F,H.\mu) = \prod_{\kappa=1}^{\mu} (B_1^*(F,H,\mu-\kappa))^{-1}, \tag{131}$$

where $\mu \in [0,q+1] \cap \mathbb{Z}$. Clearly,

$$\Pi_{\mu,q}(F) = Z(F,H.\mu)\Pi_{0,q}(F), \tag{132}$$

where $\mu \in [0,q] \cap \mathbb{Z}$. Therefore (128) may be rewritten in the form

$$\Delta^{(q)}(F,H,q) = -\sum_{\mu=0}^{q} \Omega_{\mu-1}^*(F,H)Z(F,H.\mu)\Pi_{0,q}(F). \tag{133}$$

Let $y_k^*(F,H,k-1) = 1$ for $k = 1, \dots, p$, and let $y_k^*(F,H,\mu) = 0$ for $k = 1, \dots, p$ and $\mu \in [0,p-1] \cap \mathbb{Z} \setminus \{k-1\}$. Let further

$$y_k^*(F,H,\mu) = -\sum_{\alpha=0}^{p-1} b_\alpha^*(F,H,\mu-p)Zy_\alpha^*(F,H,\mu),$$

for $k = 1, \ldots, p$ and $\mu \in [p, n] \cap \mathbb{Z}$. Let

$$Y_k^*(F, H, \mu) = Y_{p, y_k^*(F,H,\mu)}) = \begin{pmatrix} y_k^*(F, H, \mu) \\ \vdots \\ y_k^*(F, H, \mu + p - 1) \end{pmatrix}, \tag{134}$$

for $k = 1, \ldots, p$ and $\mu \in [0, q+1] \cap \mathbb{Z}$. Clearly,

$$Z(F, H.\mu) = (Y_k^*(1, H, \mu)), \ldots, Y_p^*(F, H, \mu)). \tag{135}$$

Let

$$\alpha_k(F, H) = \sum_{\tau=0}^{n-1} H^{\sim}_{\tau,*} y_k^*(F, H, \tau), \tag{136}$$

where $k = 1, \ldots, p$. In view of (131), (134), (135) , (130) and (136),

$$\sum_{\beta=0}^{q} \Omega_{\beta-1}(F, H) Z(F, H, \beta) = \begin{pmatrix} 0 & \ldots & 0 \\ 0 & \ldots & 0 \\ . & \ldots & . \\ 0 & \ldots & 0 \\ \alpha_1(F, H) & \ldots & \alpha_p(F, H) \end{pmatrix}. \tag{137}$$

In view of (123),

$$q_3^{\sim}(\Delta(F, H, \beta)) = q_3^{\sim}(\Delta(F, 0, \beta)) < \varepsilon_2/2, \tag{138}$$

where $\beta = 0, \ldots, q-1$. In view of (84), (83), (74),

$$q_\infty^{\sim}(\Pi_{0,q}) \leq \gamma_{19} \tag{139}$$

In view of (136) and (137),

$$q_\infty^{\sim}(\sum_{\beta=0}^{q} \Omega_{\beta-1}(F, H) Z(F, H, \beta) = \tag{140}$$

$$\sum_{\tau=0}^{n-1} |H^{\sim}_{\tau,*}| \sum_{k=1}^{p} |y_k^*(F, H, \tau)|.$$

In view of (134) and (135),

$$\sum_{k=1}^{p} |y_k^*(F, H, \tau)| \leq \sup\{q_\infty^{\sim}(Z(F, H, \beta)) \colon \beta \in [0, q] \cap \mathbb{Z}.\}$$

In view of (138),

$$\|B_1^*(F,H,\beta)\|_{sp} \le q_3^{\sim}(B_1^*(F,H,\beta)) \le \rho_{1,0} + \varepsilon_2, \tag{141}$$

where $\beta = 0, \ldots, q-1$. In view of (77),

$$q_\infty(B_1^*(F,H,\beta)) \le \gamma_{13},$$

where $\beta = 0, \ldots, q-1$,

$$\sup\{q_\infty^{\sim}(Z(F,H,\beta))\colon \beta \in [0,q] \cap \mathbb{Z}\} \le \gamma_0(q+1, \gamma_{13}). \tag{142}$$

In view of (141) and (74),

$$\sup\{q_\infty^{\sim}(Z(F,H,\beta))\colon \beta \in [0,q] \cap \mathbb{Z}\} \le \gamma_8\gamma_{14}. \tag{143}$$

In view of (142) and (143),

$$\sup\{q_\infty^{\sim}(Z(F,H,\beta))\colon \beta \in [0,q] \cap \mathbb{Z}.\} \le \gamma_{20}. \tag{144}$$

In view of to (73), (133), (139), (140), (142),

$$q_3^{\sim}(\Delta^{(q)}(F,H,q)) \le \gamma_8 q_\infty(\Delta^{(q)}(F,H,q)) \le \gamma_8\gamma_{19}\gamma_{20} q_\infty(H) = \gamma_{21} q_\infty(H). \tag{145}$$

Let now

$$q_\infty(H) \le \gamma\varepsilon_5(\varepsilon_4)^q/\gamma_{21}. \tag{146}$$

It follows then from (144), (145) that

$$q_3^{\sim}(\Delta^{(q)}(F,H,q)) \le \gamma\varepsilon_5(\varepsilon_4)^q. \tag{147}$$

In view of (125), (147), (123) and (122),

$$q_3^{\sim}(\Delta^{(i)}(F,H,q)) < \gamma\varepsilon_5(\varepsilon_4)^q) + \theta^{(i)}(\gamma) - \theta^{(q)}(\gamma) = \gamma(\varepsilon_4)^i(1-\varepsilon_5((\varepsilon_4)^{q-i}) \le \gamma(\varepsilon_4)^i\varepsilon_6 = \theta^{(i)}(\gamma)\varepsilon_6 < \theta^{(i)}(\gamma). \tag{148}$$

Taking in (146) $\gamma = \varepsilon_2/2$, we see that, if (146) is fulfilled, then Ψ_H maps the open set Y into Y. Taking in (146) $\gamma = \rho^{\sim}(F)$, we see that, if (146) is fulfilled, then $\rho^{\sim}(\Psi_0^q(F)) \le \varepsilon_6\rho^{\sim}(F)$ for any $F \in Y$, moreover $\Psi_0(0) = 0$. In view of (132), (139),

$$\sup\{q_\infty^{\sim}(\Pi_{\mu,q}(F)\colon \mu \in [0,q] \cap \mathbb{Z}\} \le \gamma_{19}\gamma_{20}.$$

Let $q^*_\infty(F) = \sup\{q_\infty(F_{*,k-1})\colon k = 1, \dots, q\}$, where $F \in A_{p,q}(\mathbb{C})$. In view of (114),

$$q_3^\sim \Delta^{(i)}(F, H, \kappa) \le (1 + \rho_{2,0})^i \gamma_8 \gamma_{19} \gamma_{20} q^*_\infty(F)$$

for $i \in [0, q-1] \cap \mathbb{Z}$. Therefore $\rho^\sim(F) \le \gamma_{22} q^*_\infty(F)$ for any $F \in Y$. Clearly,

$$q^*_\infty(F) \le \gamma_8 \rho^\sim(F)$$

for any $F \in Y$. So,

$$q^*_\infty(\Psi_0^{qs}(F)) \le \gamma_8 \rho^\sim(\Psi_0^{qs}(F)) \le \gamma_8(\varepsilon_6)^s \rho^\sim(F, 0) \le \gamma_8 \gamma_{22} (\varepsilon_6)^s q_\infty(F),$$

where $F \in Y$, $s \in \mathbb{N}$. Let is chosen $s \in \mathbb{N}$ such that $\gamma_8\gamma_{22}(\varepsilon_6)^s < 1$. Then Ψ_0^{qs} is centripetal relaively q^*_∞. Since q^*_∞ is a norm on $F \in A_{p,q}(\mathbb{C})$, it follows that we may apply the Lemma 6.4. Therefore there exists $m \in \mathbb{N}$ such that Ψ_0^{qsm} is contracted map of the metric space (Y, q^*_∞). In view of (101), $\Phi_0(F^\vee) - \Phi_0(F^\wedge) = W_1 + W_2$, where

$$W_1 = (B_3(F^\wedge) - B_3(F^\vee))(B_2(F)^\wedge)^{-1}\pi_1(A^\vee),$$
$$W_2 = B_3(F^\vee)(B_2(F)^\vee)^{-1}(B_2(F)^\vee - B_2(F)^\wedge)(B_2(F)^\wedge)^{-1}\pi_1(A^\vee).$$

Therefore

$$q_\infty(\Phi_0(F^\vee) - \Phi_0(F^\wedge)) \le q_\infty(W_1) + q_\infty(W_2). \tag{149}$$

In view of (77), if $F \in \mathfrak{O}_0$, then the absolute value of the each element of the the matrix $B_3(F)^\vee$ is not greater than $\gamma_4(n, \rho_1, 0) + \gamma_8\varepsilon_2/2 = \gamma_{18}$. The absolute value of the each element of the matrix $B_2(F^\wedge) - B_2(F^\vee)$ and the matrix $B_3(F^\wedge) - B_3(F^\vee)$ is not greater than $q^*_\infty(F^\wedge - F^\vee)$. According to the inequality $q_\infty(\pi_1(A^\vee)) \le \gamma_4(n, \rho_1, 0)$, and, in view of (103), (100), (77), (101) and (149),

$$q_\infty(B_3(F)(B_2(F))^{-1}H) \le q\gamma_{18}\gamma_{11} q_\infty(H), \tag{150,)}$$

$$q_\infty(\Phi_H(F) - \Phi_0(F)) \le \gamma_{23} q_\infty(H) \tag{151}$$

for any $F \in O_0$; moreover

$$q^*_\infty(\Psi_0(F^\wedge) - \Psi_0(F^\vee)) \le \gamma_{24} q^*_\infty(F^\wedge - F^\vee), \tag{152}$$

where

$$\gamma_{24} = q\gamma_{11}\gamma_4(n, \rho_{1,0})(1 + q\gamma_{11}\gamma_{18}),\ \{F^\wedge, F^\vee\} \subset Y.$$

In view of (151) and (152),

$$q_\infty((\Phi_0(F))^{k-1}\Phi_H(F) - (\Phi_0(F))^k) \le \gamma_{23}(\gamma_{24})^k q_\infty(H),$$

where $k = 1, \dots, qsm$ and $F \in \mathfrak{O}_0$ and therefore

$$q^*_\infty((\psi_0(F))^{k-1}\psi_H(F) - (\psi_0(F))^k) \le \gamma_{23}\gamma_0(qsm, \gamma_{24})q_\infty(H), \tag{153}$$

where $k = 1, \dots, qsm$ and $F \in \mathfrak{O}_0$.

For the bounded metric space (Y, q^*_∞) are fulfilled all the conditions of the Lemma 7.1, where Ψ_0 plays the role of the map A, the map $\psi_{H(\nu)}$ with $H(\nu) = A(\nu) - A^\vee$ plays the role of the map B_n, the number qsm plays the role of the number r and $0 \in Y$ plays the role of the fixed point Z^*. Let

$$\lim_{\nu\to\infty} q_\infty(H_\nu) = 0, \tag{154}$$

then there exists $m \in \mathbb{N}$ such that $q_\infty(H_\nu) \le \gamma\varepsilon_5(\varepsilon_4)^q/\gamma_{21}$ for $\nu \in m+\mathbb{N}-1$. Then Ψ_{H_ν} maps Y in Y, if $\nu \in m+\mathbb{N}-1$. Let $F_m \in Y$, $F_{\nu+1} = \Psi_{H_\nu}(F_\nu)$ for all the $\nu \in m+\mathbb{N}-1$. In view of (154),

$$\lim_{\nu\to\infty} \delta_{qsm}(\Psi_{H_\nu}) = 0;$$

in view of the Lemma 7.1, $\lim\limits_{\nu\to\infty} q^*_\infty(F_\nu) = 0$. In view of (100), (99) and (101),

$$\lim_{\nu\to\infty} q_\infty(U^*(F_\nu, H_\nu) - U^\vee) = 0, \ \lim_{\nu\to\infty} q_\infty\Psi_{H_\nu}(F_\nu) = 0.$$

So, we obtain the proof of the Perron's decomposition Lemma. If $q_\infty(H_\nu) = O(\xi(\nu))$, then, in view of (152) and (153),

$$\delta_{qsm}(\Psi_{H_\nu}) = O(\xi(\nu)).$$

In view of the Lemma 7.1, (100) and (99),

$$q^*_\infty(F_\nu) = O(\xi(\nu)), q_\infty(U^*(F_\nu, H_\nu) - U^\vee) = O(\xi(\nu))$$

and

$$q_\infty(\Psi_{H_\nu}(F_\nu)) = O(\xi(\nu)). \blacksquare.$$

Theorem 5 is proved.

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