On Exact Asymtotics at Infinity of Solutions to Differential Equations

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On Exact Asymptotics at Infinity of Solutions to Differential Equations

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Abstract

In the paper asymptotic solutions at infinity to differential equations with polynomial coefficients are constructed. The method of constructing of these solutions is based on the multidimensional resurgent analysis and uses the notion of a resurgent function of several independent variables introduced by the authors in the paper [1].

Introduction

The paper is aimed at the construction of asymptotic solutions to differential equations

$$\widehat{P}u \overset{\text{def}}{=} \sum_{|\alpha| \leq M} P_\alpha(x) \left(\frac{\partial}{\partial x} \right)^\alpha u(x) = 0 \tag{1}$$

with polynomial coefficients $P_\alpha(x)$ for large values of the variables $x = (x^1, \ldots, x^n) \in \mathbf{C}^n$. Here $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an integer-valued multiindex and $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

To present the exact definition of an asymptotic solution used below, we introduce the following notions.

Let $f(x)$ be a function (ramifying in general) defined in a conical (that is, $\mathbf{R}_+$-invariant) subset D of the space $\mathbf{C}^n$ for sufficiently large $|x| > R$. We say that this function has the *order* k and the *type* c if the inequality

$$|u(x)| \leq C e^{c|x|^k} \tag{2}$$

is valid in $D \cap \{|x| > R\}$ with some positive constant C.

Let now D_0 be a fixed conical subset in $\mathbb{C}^n$ and k be a positive number. We say that an asymptotic solution to the equation (1) is defined in D_0 if there is a finite covering

$$D_0 = \bigcup_{j=1}^{N} D_j$$

of the set D_0 with the help of conical sets D_j and if in each set D_j a function $u_j(x)$ is given such that:

a) The functions $u_i(x)$ and $u_j(x)$ coincide on the intersection $D_i \cap D_j$ up to functions of order k with arbitrary negative type. This means that for any positive constant c there exists such constant C that the inequality

$$|u_i(x) - u_j(x)| \leq Ce^{-c|x|^k}$$

holds in the intersection $D_i \cap D_j$.

b) The functions $u_j(x)$ are solutions to equation (1) up to functions of an arbitrary negative type, that is, the inequality

$$\left| \widehat{P} u_j(x) \right| \leq Ce^{-c|x|^k}$$

holds for any $c > 0$ with some constant C (possibly dependent on c).

In what follows the tuple of functions $\{u_j(x)\}$ subject to the conditions a) and b) above will be denoted simply by $u(x)$.

To solve the stated problem, we shall use the notion of a resurgent function of several independent variables introduced in [1]. Namely, we say that a function $u_j(x)$ is *a resurgent function* of the variables $(x^1, \ldots, x^n)$ if it is representable in the form

$$u_j(x) = \sum_{S_i \in \Omega_j} \int_{\Gamma_i} e^{-\zeta} U(\zeta, x) d\zeta \tag{3}$$

in each domain D_j introduced above where $U(\zeta, x)$ is an infinitely continuable function of ζ which is homogeneous with respect to (ζ, x) of some order. Each contour Γ_j included in (3) comes around the point $S_i(x)$ as it is shown on Figure 1 and comes to infinity along the direction of the positive part of the real axis. The set $\Omega_j = \{S_1(x), \ldots, S_{N_j}(x)\}$ is a subset of the singularity set of the function $U(\zeta, x)$ which can depend on the domain D_j.

Formula (3) we shall also write down in the form

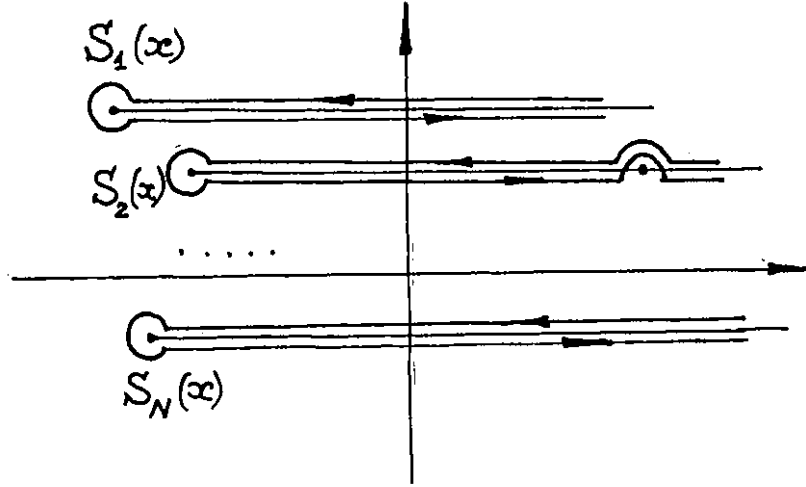$$u(x) = \ell\left[U(\zeta, x).\right] \tag{4}$$

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Figure 1

We emphasize that the set Ω_j included in formula (3), *the support* of the resurgent function (3), in general depends on the domain D_j. This phenomenon is well-known in the theory of differential equations (Stokes phenomenon).

More exactly, the function $U(\zeta, x)$ included into representation (3) must be considered as *an equivalence class* of (ramifying) analytic function with respect to functions holomorphic in the vicinity of the set Ω_j and the integrals in formula (3) are defined up to a function of arbitrary negative type.

We remark that in the paper [1] we considered functions $U(\zeta, x)$ being homogeneous with respect to the standard action of the group $\mathbf{C}_*$, that is

$$U(\lambda\zeta, \lambda x) = \lambda^\sigma U(\zeta, x).$$

Such functions allow to consider only resurgent functions of order 1. As we shall see below, such class of functions is not sufficient while constructing asymptotic solutions to equation (1). Namely, asymptotic solutions to such an equation can have *arbitrary* order depending on degrees of the coefficients $P_\alpha(x)$ in this equation. To overcome this difficulty, we introduce a modification of the notion of resurgent function of several variables considering homogeneous functions $U(x)$ with respect to *more general* action of the group $\mathbf{C}_*$, namely

$$U(\lambda^k\zeta, \lambda x) = \lambda^\sigma U(\zeta, x). \tag{5}$$

for some k. Such generalization allows to include into consideration also functions given by integral (3) with *arbitrary* order k at infinity.

We note that the degree of homogenuity σ of the function $U(x)$ included into representation (3) is not essential. For example, this degree can be changed with the help of integrating by parts in integral (3):

$$\int_{\Gamma_j} e^{-\zeta} U(\zeta, x)d\zeta = \int_{\Gamma_j} e^{-\zeta} \frac{\partial U}{\partial \zeta}(\zeta, x)d\zeta. \tag{6}$$

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From the viewpoint of the functional class described by representation (3) only action (5) of the group $\mathbf{C_*}$ is essential.

Later on, solving equations of the type (1) one often can find no solution $u(x)$ representable in the form of integral (3) with a homogeneous function $U(\zeta, x)$. Hence, it becomes nesessary to consider more wide class of *asymptotically homogeneous* functions for which equality (5) holds up to $O(\lambda^{\sigma'})$ for some $\sigma' < \sigma$.

In order to avoid cumbersome computations, in the first section we consider the theory of asymptotic solutions on the example of the stationary *Schrödinger equation*. This example is rather representative in the sense that all main features of the theory can be illustrated on it.

The second section is aimed at modifications needed for consideration of *general equations* of the type (1). Here we consider also the generalization of the theory to the case when the function $u(x)$ has different orders in different directions of the complex space $\mathbf{C}^n$.

In the third section *examples* of construction of asymptotic solutions to concrete differential equations are presented.

The concluding fourth section is aimed at the illustration of the application of the developed theory to *nonstationary* problems. This illustrarion is carried out on the simplest example of the wave equation.

1   Asymptotic Solutions to Schrödinger Equation

1. The aim of this section is to construct asymptotic solutions to the equation

$$\Delta u(x) + V(x)u = 0 \tag{7}$$

where $V(x)$ is a polynomial of the complex variabes $(x^1, \ldots, x^n) \in \mathbf{C}^n$ of degree $2m$. This equation is essentially the stationary Schrödinger equation

$$-\frac{h^2}{2m}\Delta u + W(x)u = Eu$$

with polynomial potential $W(x)$ for some special choice of units of measure (it allows to eliminate the coefficient $h^2/2m$) for $V(x) = E - W(x)$.

We shall construct asymptotic solutions to equation (7) of the form (3) with the function $U(\zeta, x)$ being asymptotically homogeneous with respect to the action

$$\lambda(\zeta, x) = (\lambda^k \zeta, \lambda x) \tag{8}$$

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of the group $\mathbf{C}_*$.

To illustrate more explicitely the form of asymptotic expansions (3), we consider a function $u(x)$ representable in this form under two simplifying assumptions.

1. The function $U(\zeta, x)$ is a *homogeneous* function with respect to action (8) of the group $\mathbf{C}_*$.

2. The function $U(\zeta, x)$ has *simple singularities*, that is, this function is representable in the form

$$U(\zeta, x) = \frac{a_0}{\zeta - S_j(x)} + \ln(\zeta - S_j(x)) \sum_{l=0}^{\infty} \frac{(\zeta - S_j(x))^l}{l!} a_{l+1}(x) \tag{9}$$

near each its singular point $\zeta = S_j(x)$. We remark that if the function $U(\zeta, x)$ has the expansion of the form (9) with polar singularity of order more than one in a neighbourhood of some singular point $S_j(x)$, then the multiplicity of the pole can be easily diminished to unity with the help of integration by parts in integral (3) (see formula (6)).

Under the above assumptions $S_j(x)$ are evidently homogeneous (in the usual sence) functions of the variables $(x^1, \ldots, x^n)$ of order k. One can show that in this situation integral (3) has the asymptotic expansion of the form

$$u(x) \cong \sum_{j=1}^{N} e^{-S_j(x)} \sum_{l=0}^{\infty} a_l(x) \tag{10}$$

with respect to diminishing powers of homogenuity.

Thus, for functions with simple singularities outside focal points resurgent functions of the considered form are simply expansions of the form (10).

Now let us turn our mind to the consideration of the general case. To derive an equation for the function $U(\zeta, x)$ we substitute integral (3) to equation (7) and use the commutation formula for operator (4) and the differentiation operator[1]

$$\frac{\partial}{\partial x^i} \ell[U(\zeta, x)] = \ell\left[\left(\frac{\partial}{\partial \zeta}\right)^{-1} \frac{\partial U}{\partial x^i}(\zeta, x)\right]. \tag{11}$$

Formula (11) is proved in the paper [1] for $k = 1$ and can be immediately generalized on the case of action (8) of $\mathbf{C}_*$.

As a result of the substitution of (3) into (7) we obtain the following equation for the function $U(\zeta, x)$:

$$\left(\frac{\partial}{\partial \zeta}\right)^{-2} \Delta U(\zeta, x) + V(x)U(\zeta, x) = 0, \tag{12}$$

---

[1]The operator $\partial/\partial\zeta$ is evidently invertible modulo functions holomorphic in the neighbourhood of points $\zeta = S_j(x)$.

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which can be also rewritten in the form

$$\Delta U(\zeta, x) + V(x)\frac{\partial^2}{\partial\zeta^2}U(\zeta, x) = 0. \tag{13}$$

As it can be seen from formulas (9) and (10) above, to construct asymptotic solutions at infinity to equation (7) one has to construct asymptotic solutions to equation (13) *with respect to smoothness*. Such asymptotics are investigated by the authors in a sery of papers on the theory of differential equations on complex manifolds (see, for example, [2, 3]).

We remark that equation (12) is a $\partial/\partial\zeta$-differential one so that for its investigation the $\partial/\partial\zeta$-formalizm can be used (see [3]).

Note, that if the polynomial $V(x)$ is a *homogeneous* one with respect to x of degree $2m$, then equation (13) can be considered as an equation for *homogeneous* functions $U(\zeta, x)$ with respect to action (8) of the group $\mathbf{C}_*$ with $k = m + 1$. Actually, for such choice of k the operators Δ and $V(x)\partial^2/\partial\zeta^2$ diminish the homogenuity power of the function $U(\zeta, x)$ by one and the same amount (namely, by two units). However, in the case when $V(x)$ is not a homogeneous polynomial, equation (13) must be considered as an equation for *asymptotically homogeneous* functions.

The following affirmation is valid.

Theorem 1 *Let $U(\zeta, x)$ be an asymptotically homogeneous solution to equation* (13) *with respect to action* (8) *of the group* $\mathbf{C}_*$. *Then the corresponding resurgent function $u(x)$ given by* (3) *is an asymptotic solution of equation* (7) *of order k up to functions of arbitrary negative type.*

Proof (sketch). The fact that the function $u(x)$ satisfies equation (7) up to functions of arbitrary negative type can be derived from equation (13) with the help of commutation formulas (11).

Later on, the inequalities of the type (2) of the function $u(x)$ given by integral (3) can be obtained by direct estimating of this integral. To do this one has to take into account the fact that the singularity set $\zeta = S(x)$ of solution to (13) is defined by the homogeneous function $S(x)$ of x of the degree k. Below it will be shown that the degree k for an asymptotic solution $U(\zeta, x)$ in action (8) of the group $\mathbf{C}_*$ one must choose to be equal to $m + 1$.

2. Let us investigate now *the resurgent structure* of the solution $u(x)$ to equation (7), that is, the singularity set of solution to equation (13). It is well-known that the set $\zeta = S(x)$ can be the singularity set of a solution to equation (13) only if the function $S(x)$ satisfies the Hamilton–Jacobi equation

$$\left(\frac{\partial S}{\partial x}\right)^2 + V(x) = 0, \tag{14}$$

where $(\partial S/\partial x)^2 = \sum_{j=1}^{n} (\partial S/\partial x^j)^2$.

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This fact can be easily verified if one uses the so-called $\partial/\partial\zeta$-formalism (see [3]). Actually, we remark that equation (12) is a $\partial/\partial\zeta$-differential one (see, for example, [4]). The WKB-asymptotics (by smoothness) of such an equation has the form

$$e^{-S(x)\partial/\partial\zeta}A(\zeta, x), \tag{15}$$

where $A(\zeta, x)$ is asymptotically homogeneous function with respect to action (8) of the group C_*. Substituting function (15) into equation (12) we obtain Hamilton-Jacobi equation (14) as well as the corresponding transport equations.

As it was mentioned above, the function $U(\zeta, x)$ is a homogeneous function for homogeneous polynomial $V(x)$. Hence, the function $S(x)$ is a homogeneous function of the variables x of degree $k = m + 1$. Thus, in this case $S(x)$ determines a *homogeneous* Lagrangian manifold (with respect to the variables x) in the cotangent space T^*C^n.

Let us try to show that in the case when $V(x)$ is an arbitrary polynomial of the degree $2m$, a solution to equation (14) is determined by some *homogeneous* Lagrangian manifold in T^*C^n as well. To do this, we represent the function $V(x)$ as a sum of homogeneous polynomials

$$V(x) = \sum_{j=0}^{2m} V_j(x), \quad V_j(\lambda x) = \lambda^j V_j(x).$$

Let us now to search for solutions of equation (14) in the form of the sum of homogeneous functions

$$S(x) = \sum_{l=0}^{\infty} S_l(x),$$

$S_l(x)$ being homogeneous functions of degree $m - k + 1$. Substituting the latter expression to equation (14) and equating to zero components with different degrees of homogenuity, we obtain the homogeneous Hamilton-Jacobi equation for $S_0(x)$

$$\sum_{j=1}^{n} \left(\frac{\partial S_0}{\partial x^j}\right)^2 + V_{2m}(x) = 0 \tag{16}$$

and the triangle system of equations for functions $S_1(x), S_2(x), \ldots$

$$2\sum_{j=1}^{n} \frac{\partial S_0}{\partial x^j}\frac{\partial S_1}{\partial x^j} + V_{2m-1}(x) = 0,$$

$$2\sum_{j=1}^{n} \frac{\partial S_0}{\partial x^j}\frac{\partial S_2}{\partial x^j} + \sum_{j=1}^{n} \left(\frac{\partial S_1}{\partial x^j}\right)^2 + V_{2m-2}(x) = 0, \tag{17}$$

$\ldots$

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On the diagonal of the latter system stands the transport operator

$$\widehat{\mathcal{P}} = 2 \sum_{j=1}^{n} \frac{\partial S_0}{\partial x^j} \frac{\partial}{\partial x^j}.$$

Equation (16) shows that $S_0(x)$ is a generating function of some homogeneous Lagrangian manifold L_0 in the space $T^*\mathbf{C}^n$ which satisfies Hamilton-Jacobi equation (14) with $V(x)$ substituted by its principal part $V_{2m}(x)$. Equations (17) allow to calculate the functions $S_1(x)$, $S_2(x), \ldots$ as integrals along the Hamiltonian vector field on the manifold L_0 corresponding to Hamiltonian

$$H(x,p) = p^2 + V_{2m}(x).$$

In this sense the Hamiltonian $H(x,p)$ is the principal part of the Hamiltonian

$$\mathcal{H}(x,p) = p^2 + V(x) \tag{18}$$

with respect to asymptotics as $x \to \infty$.

Thus, the resurgent structure of the function $U(\zeta, x)$, that is, its singularity set is described by the equation

$$\zeta = S(x)$$

for the asymptotically homogeneous function $S(x)$ of degree $k = m+1$ which was determined above with the help of the system (16), (17).

3. Let us investigate now the process of *computation* of asymptotic expansions themselves. To do this, one must investigate singular parts of the function $U(\zeta, x)$ near its singular point $\zeta = S(x)$. In this investigation we shall consider more narrow class of solutions to equation (13) than the class of general resurgent functions.

Namely, let

$$f_1(\xi), \; f_2(\xi), \ldots \tag{19}$$

be some sequence of functions $f_j(\xi)$ of one complex variable ξ regular in the deleted neighbourhood of the origin (these functions are, certainly, singular at the origin itself). We suppose (19) to be a Ludwig's sequence ([5]) This means that functions (19) satisfy the relation

$$f_j'(\xi) = f_{j-1}(\xi), \quad j = 1, 2, \ldots .$$

Definition 1 A function $U(\zeta, x)$ will be called *a resurgent function with simple singularities with respect to the Ludwig's sequence* (19) if in the vicinity of any its singular point $\zeta = S(x)$ this function admits the asymptotic representation of the form

$$U(\zeta, x) \cong \sum_{j=0}^{\infty} a_j(x) f_j(\zeta - S(x)). \tag{20}$$

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Remark 1 A function $U(\zeta, x)$ is a function with simple singularities *in usual sense* if it is a function with simple singularities with respect to the Ludwig's sequence

$$\frac{1}{\xi}, \quad \ln \xi, \quad \xi(\ln \xi - 1), \quad \ldots .$$

However, sometimes it is useful to consider functions with simple singularities with respect to other Ludwig's sequences. For example, one of the often used Ludwig's sequences is

$$\frac{\xi^{\sigma+j}}{\Gamma(\sigma + j + 1)}, \quad j = 0, 1, \ldots$$

for some noninteger number σ. We shall use below only the above two Ludwig's sequences.

Now, substituting expansion (20) to equation (13) and equating to zero coefficients of $f_j(\zeta - S(x))$ one can easily verify the following assertion.

Theorem 2 *Expansion (20) is an asymptotic expansion of solution to equation (13) iff the function $S(x)$ is a solution of Hamilton Jacobi equation (14) and the functions $a_j(x)$ are solutions to the transport equations*

$$\left[\sum_{j=1}^{n} \frac{\partial \mathcal{H}}{\partial p_j} \left(x, \frac{\partial S}{\partial x} \right) \frac{\partial}{\partial x^j} + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 S}{\partial x^i \partial x^j} \frac{\partial^2 \mathcal{H}}{\partial p_i \partial p_j} \left(x, \frac{\partial S}{\partial x} \right) \right] a_j(x) = F_j, \qquad (21)$$

where $F_j = F_j[a_0, \ldots, a_{j-1}]$ is an expression containing the functions $a_0, \ldots, a_{j-1}$ and their derivatives, F_0 being equal to zero.

One can show that equations (21) similar to Hamilton-Jacobi equation (14) can be solved with the help of expansions with respect to homogeneous functions of the variables x.

We remark that Theorem 2 provides one with the *method of solving* equation (13). Namely, if a solution $S(x)$ of Hamilton-Jacobi equation (14) and solutions $a_j(x)$ of system of transport equations (21) are already found, then the presum of series (20) is the solution of equation (13). We shall use this fact while considering examples.

2   General Equations with Polynomial Coefficients

This section is aimed at the description of the changes which must be made in the above theory to adapt it to the investigation of *general* equations of the form (1). Theorem 1 formulated in the previous section can be generalized to the case of equation (1) almost directly. The corresponding affirmation is as follows.

Theorem 3 *Let $U(\zeta, x)$ be a solution to the equation*

$$\sum_{|\alpha| \leq M} P_\alpha(x) \left(\frac{\partial}{\partial \zeta} \right)^{-|\alpha|} \left(\frac{\partial}{\partial x} \right)^\alpha U(\zeta, x) = 0 \tag{22}$$

which is asymptotically homogeneous with respect to action (8) of the group $\mathbf{C}_*$. Then the corresponding resurgent function $u(x)$ given by integral (3) is an asymptotic solution to equation (1) of order k up to functions of arbitrary negative type.

To modify the considerations of the previous section one has to describe the method of the appropriate choice of k such that equation (22) has an asymptotically homogeneous solution.

To do this, we use the method of solving equation (22) with the help of homogeneous Lagrangian manifolds. To begin with, we rewrite equation (22) in the form

$$\sum_{|\alpha| \leq M} P_\alpha(x) \left(\frac{\partial}{\partial \zeta} \right)^{M-|\alpha|} \left(\frac{\partial}{\partial x} \right)^\alpha U(\zeta, x) = 0, \tag{23}$$

excluding negative powers of the derivative $\partial/\partial\zeta$. The Hamilton-Jacobi equation corresponding to equation (23) has the form

$$\sum_{|\alpha| \leq M} P_\alpha(x) \left(\frac{\partial S}{\partial x} \right)^\alpha = 0, \tag{24}$$

where the notation

$$\left(\frac{\partial S}{\partial x} \right)^\alpha = \left(\frac{\partial S}{\partial x^1} \right)^{\alpha_1} \cdots \left(\frac{\partial S}{\partial x^n} \right)^{\alpha_n}$$

is used. If $U(\zeta, x)$ is an asymptotically homogeneous function with respect to the action (8) of the group $\mathbf{C}_*$, then the function $S(x)$ is an asymptotically homogeneous function of degree k. Then degrees of homogenuity of terms included in equation (24) are equal to

$$\sigma_\alpha + (k-1)|\alpha| \tag{25}$$

correspondingly for any multiindex α where σ_α is the homogenuity degree of the polynomial $P_\alpha(x)$. The principal terms of equation (24) with respect to homogenuity are those terms for which the expression (25) has maximal value. The choice of principal terms with respect to homogenuity for fixed value of k can be illustrated graphically as follows.

Let us consider the plane with coordinates $(\sigma, |\alpha|)$ and mark on this plane points $(\sigma, |\alpha|)$ corresponding to terms of the form $P'_\sigma(x)(\partial/\partial x)^\alpha$ with homogeneous polynomial $P'_\sigma(x)$ included to equation (1) (see Figure 2).
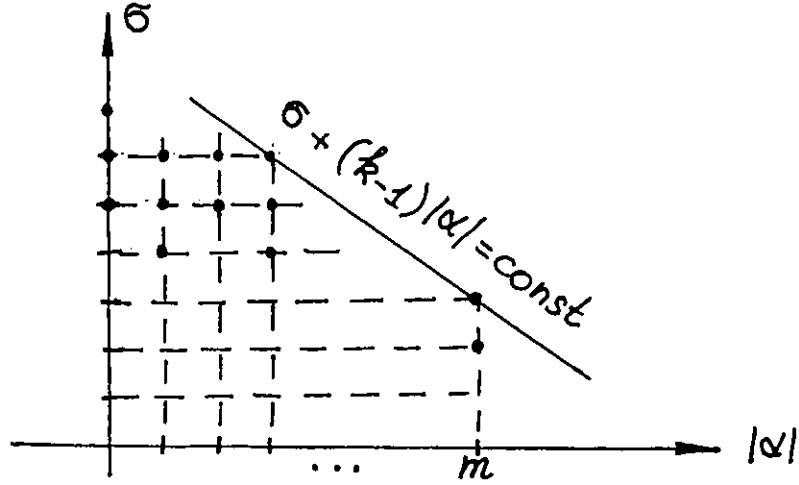
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Figure 2

Consider straight lines $\sigma + (k-1)\,|\alpha| = \text{const}$. Evidently, the points lying on these lines correspond to terms in equation (1) with one and the same homogenuity degree. Therefore, points corresponding to leading (with respect to homogenuity) terms of (1) lie on the line of support with inclination $(k-1)$ to the minimal polygon containing all marked points (Newton's polygon, see Figure 2).

Now, similar to the previous section, we search the solution $S(x)$ to equation (24) in the form

$$S = \sum_{j=0}^{\infty} S_j(x),$$

where the functions $S_j(x)$ are homogeneous functions of degree $k-j$. This leads us to the system of equations first of which is the Hamilton-Jacobi equation

$$H\left(x, \frac{\partial S_0}{\partial x}\right) = 0,$$

and the rest form a triangle system with the transport operator

$$\widehat{P} = \sum_{j=1}^{n} H_{p_j}\left(x, \frac{\partial S_0}{\partial x}\right) \frac{\partial}{\partial x^j} + \frac{1}{2} \sum_{i,j=1}^{n} H_{p_i p_j}\left(x, \frac{\partial S_0}{\partial x}\right) \frac{\partial^2 S_0}{\partial x^i \partial x^j} \tag{26}$$

on the diagonal.

Certainly, the condition of solvability of this system is that the Hamiltonian vector field included in operator (26) does not vanish on zeroes of the function $S(x)$, that is that the equation is (in usual terminology of the theory of differential equations) the equation of the principle type.

We emphasize that in fact the resurgent structure of the solution (that is, the singularity set $\zeta = S(x)$) is described with the help of the *contact* structure in the space (ζ, x, p), where $p = (p_1, \ldots, p_n)$ are dual variables to x, with the structure form
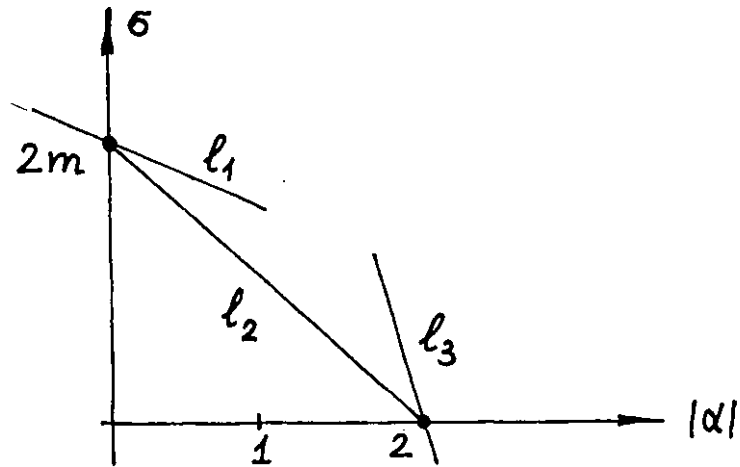
$$d\zeta - p\,dx$$

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Figure 3

rather than with the help of the symplectic structure. However, we can project the Legendre manifold defined by the equation $\zeta = S(x)$ to the symplectic space with coordinates (x, p), deriving ζ from the equation of this manifold. The inverse procedure is, in general, ambiguous: the corresponding Legendre manifold is defined up to adding of an arbitrary constant to the action $S(x)$. In our case, however, this constant is fixed by the homogenuity degree of the function $S(x)$ and, hence, symplectic and contact structures are in our case are isomorphic to each other.

Let us illustrate the procedure of choosing the principal terms on the example of operator (7), considered in the previous section. For such an operator the Newton's polygon drawn on Figure 3 is defined by two points. It is evident that the homogeneous Hamiltonian corresponding to values of k less than $m + 1$ is equal to $V(x)$ (the line of support l_1 on Figure 3) and, hence, the Hamilton-Jacobi equation degenerates to the equation

$$V(x) = 0,$$

which does not contain the function $S_0(x)$. Evidently, the latter equation has no solution and, hence, the equation (7) has no asymptotic solutions of order $k < m + 1$. Later on, the case $k = m + 1$ (the line of support l_2 on Figure 3) was considered in details in the previous section. Finally, the choice $k > m + 1$ (the line of support l_3 on Figure 3) leads to the Hamilton-Jacobi equation

$$\left(\frac{\partial S_0}{\partial x^1}\right)^2 + \left(\frac{\partial S_0}{\partial x^2}\right)^2 = 0,$$

which has solutions $(x^1 \pm ix^2)^k$ of the degree k. The corresponding Hamiltonian vector field (for the function $(x^1 + ix^2)^k$) equals to

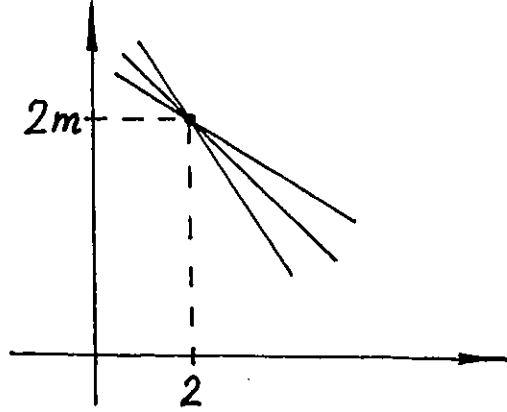$$2k(x^1 + ix^2)^{k-1}\left(\frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2}\right)$$

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Figure 4

and vanishes identically on zeroes of the function $S_0(x)$. So, the only appropriate choice of k is $k = m + 1$.

To conclude this section we remark that for constructing of asymptotic solutions to certain differential equations the described method needs further modification connected with considering *weighted* homogenuity not only with respect to ζ, but also with respect to x. Let us illustrate this effect on the simple example.

Consider the equation

$$\frac{\partial^2 u}{(\partial x^1)^2} + (x^1)^{2m} \frac{\partial^2 u}{(\partial x^2)^2} = 0. \tag{27}$$

The 'Newton's polygon' for such an equation drawn on Figure 4 consists of the two points posited one under another. Hence, for any choice of k the homogeneous Hamilton-Jacobi equation has the form

$$(x^1)^{2m} \left(\frac{\partial S}{\partial x^2} \right)^2 = 0.$$

Evidently, this equation is not an equation of principal type for every k (the corresponding Hamiltonian vector field vanishes identically on zeroes of the latter equation).

To be able to construct an asymptotic solution to equation (27), we can consider more general action of the group C_*

$$\lambda \left(\zeta, x^1, x^2 \right) = \left(\lambda^{k_0} \zeta, \lambda^{k_1} x^1, \lambda^{k_2} x^2 \right) \tag{28}$$

compared with those in (8).For such action of C_* the principal part $S_0(x)$ must be a weighted-homogeneous function of degree k_0 with weights (k_1, k_2):

$$S_0 \left(\lambda^{k_1} x^1, \lambda^{k_2} x^2 \right) = \lambda^{k_0} S_0(x^1, x^2).$$

Evidently, the notion of homogenuity with respect to action (28) depends only on the ratio $(k_0 : k_1 : k_2)$ and one can normalize the choice of these numbers putting $k_0 = 1$.

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Let us try now to choose the numbers k_1 and k_2 in such a way that for this choice both terms of the 'correct' Hamilton-Jacobi equation

$$\left(\frac{\partial S}{\partial x^1}\right)^2 + \left(x^1\right)^{2m} \left(\frac{\partial S}{\partial x^2}\right)^2 = 0$$

are of one and the same homogeneous degree. This condition can be satisfyed by the choice

$$k_1 = 1, \quad k_2 = m + 1.$$

The rest of the procedure of constructing of asymptotic solutions to equation (27) is of no difference with those described above. We remark only that for the present example the order of an asymptotic solution is equal to 1 along the axis x^1 and is equal to $m + 1$ along the axis x^2. Thus, this order depends on the direction.

3 Examples

The aim of this section is to illustrate the work of the described method on two elementary examples.

1. The Helmhölz equation. Let us compute the form of asymptotic solutions to the equation

$$\Delta u + k^2 u = 0 \tag{29}$$

for large values of $|x|$; here $x = (x^1, x^2) \in \mathbf{C}^2$. As above, we search for the solution in the form

$$u = e^{-S(x)} a(x).$$

We remark that the corresponding Hamilton-Jacobi equation has the form

$$\left(\frac{\partial S}{\partial x^1}\right)^2 + \left(\frac{\partial S}{\partial x^2}\right)^2 + k^2 = 0. \tag{30}$$

Evidently, $S = S(x^1, x^2)$ must be a homogeneous function of degree 1. To solve equation (30) we introduce the polar coordinates

$$x^1 = r \cos \varphi, \quad x^2 = r \sin \varphi.$$

Then equation (30) can be rewritten in the form

$$\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial S}{\partial \varphi}\right) = 0.$$

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Taking into account that the function S is a homogeneous function of the first degree, we have

$$S = irs(\varphi),$$

where the function $s(\varphi)$ satisfies the equation

$$s^2(\varphi) + \left(\frac{ds(\varphi)}{d\varphi}\right)^2 = k^2.$$

Solutions of the latter equation are

$$s(\varphi) = \pm k$$

or

$$s(\varphi) = \pm k \sin(\varphi - \varphi_0).$$

The expressions of the found solutions to Hamilton-Jacobi equation (30) in Cartesian coordinates (x^1, x^2) are

$$S_1^\pm = \pm k \sqrt{(x^1)^2 + (x^2)^2} \tag{31}$$

and

$$S_2 = k\left(ax^1 + bx^2\right) \tag{32}$$

where a and b are subject to the relation $a^2 + b^2 = 1$.

Remark 2 If one considers the restriction of the solution $u(x^1, x^2)$ to the equation (29) on the real space $\mathbf{R}^2$ then action (31) corresponds to the divergent or convergent (dependent on the sign) spherical wave and the action (32) to the plane wave propagating in the direction of the vector (a, b).

Let us compute now the amplitude functions of the corresponding waves, that is, functions $a_l(x)$ included into expansion (10). The transport equation for the amplitude function a_0 reads

$$\left[2\frac{\partial S}{\partial x^1}\frac{\partial}{\partial x^1} + 2\frac{\partial S}{\partial x^2}\frac{\partial}{\partial x^2} + \Delta S\right] a_0 = 0,$$

and for the amplitude functions $a_1, a_2, \ldots$ the transport equation is

$$\left[2\frac{\partial S}{\partial x^1}\frac{\partial}{\partial x^1} + 2\frac{\partial S}{\partial x^2}\frac{\partial}{\partial x^2} + \Delta S\right] a_j = F_j\left[a_0, \ldots, a_{j-1}\right], \tag{33}$$

where $j = 1, 2, \ldots$. To begin with, consider the action $S = S_1^\pm$. Then in polar coordinates the transport equation reads

$$\frac{\partial a_0}{\partial r} + \frac{1}{2r}a_0 = 0.$$

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The latter equation gives $a_0 = r^{-1/2}a(\varphi)$ with an arbitrary function $a(\varphi)$. Thus, the leading term of the asymptotic solution has the form

$$u \cong \frac{e^{\pm ikr}}{\sqrt{r}}a(\varphi).$$

The rest terms $a_1, a_2, \ldots$ of the asymptotic solution can be determined from equation (33) *uniquely* due to homogenuity of functions a_j.

The recurrent formulas for the functions $a_1, a_2, \ldots$ for $j \geq 1$ are

$$a_j = -\left[\frac{(j + 1/2)^2}{j} + \frac{1}{j}\frac{d^2}{d\varphi^2}\right]a_{j-1}.$$

Remark 3 Thus, we have obtained that a solution of equation (29), at least up to functions decreasing exponentially at infinity, determines uniquely by the function $a(\varphi)$. This fact is well-known in the radiophysics where the function $a(\varphi)$ is called *a diagram* of the wave field u (certainly, similar to Remark 2 we must restrict all functions to the real space $\mathbf{R}^2$).

For action (32) with the help of similar computations one can obtain that an asymptotic solution $u(x^1, x^2)$ with the action S_2 has the form $u(x^1, x^2) = e^{ik(ax^1 + bx^2)}a_0$, that is, in this case $a_0 = const$, $a_j = 0$ for $j \geq 1$.

2. The stationary Schrödinger equation for the harmonic oscillator. Consider equation (7) in the two-dimensional complex space $\mathbf{C}^2$ with the function $V(x)$ equal to $(x^1)^2 + (x^2)^2$:

$$\Delta u + \left[(x^1)^2 + (x^2)^2\right]u = 0. \tag{34}$$

Let us compute the form of asymptotic solutions at infinity to this equation.

Similar to the considerations in the previous example, we obtain the Hamilton-Jacobi equation

$$\left(\frac{\partial S}{\partial x^1}\right)^2 + \left(\frac{\partial S}{\partial x^2}\right)^2 + (x^1)^2 + (x^2)^2 = 0. \tag{35}$$

In this example, unlike the previous one, the action $S(x)$ being a solution of equation (35) must have the homogenuity degree 2 with respect to variables (x^1, x^2). Now, solving equation (35) with the help of the substitution $S = ir^2 s(\varphi)$ one comes to the equation for $s(\varphi)$ of the form

$$\left(\frac{ds}{d\varphi}\right)^2 + 4s^2 = 1.$$

This equation has two possible types of solution:

$$s(\varphi) = \pm\frac{1}{2}, \quad S_1^{\pm} = \pm i\frac{(x^1)^2 + (x^2)^2}{2}, \tag{36}$$

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$$s(\varphi) = \frac{1}{2}\sin 2(\varphi - \varphi_0), \quad S_2 = r^2 \sin(\varphi - \varphi_0)\cos(\varphi - \varphi_0). \tag{37}$$

The function S_2 can be evidently transformed to the function $S_1^{\pm}$ by a linear change of variables (with complex coefficients). Therefore we present the computation of the amplitude function only for function (36). For this action the leading transport equation reads

$$\left(2x^1 \frac{\partial}{\partial x^1} + 2x^2 \frac{\partial}{\partial x^2} + 2\right) a_0 = 0,$$

that is, in the polar coordinates

$$\left(r \frac{\partial}{\partial r} + 1\right) a_0 = 0.$$

Thus, we obtain $a_0 = r^{-1} a(\varphi)$ with an arbitrary function $a(\varphi)$. The leading term of the corresponding asymptotic solution has the form

$$u(x^1, x^2) = \frac{e^{\pm i r^2/2}}{r} a(\varphi).$$

Similar to the previous example, the rest of the terms of the asymptotic solution are uniquely determined by $a(\varphi)$.

4   Concluding Remarks

The aim of this section is to illustrsate on the simplest example how the above technique can be applied to the investigation of asymptotic solutions to nonstationary equations. For simplicity we carry out our considerations for the Cauchy problem for the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta u,$$

$$u\big|_{t=0} = u_0(x), \quad \frac{\partial u}{\partial t}\bigg|_{t=0} = u_1(x) \tag{38}$$

in the complex space $\mathbf{C}^{n+1}$ with coordinates $(t, x) = (t, x^1, \dots, x^n)$. For simplicity we suppose that the initial functions $u_0(x)$ and $u_1(x)$ are resurgent functions of order k with one and the same support (in particular, it is supposed that the singularity sets of the corresponding functions $U_0(\zeta, x)$ and $U_1(\zeta, x)$ coincide to each other). Suppose also that the functions U_j, $j = 0, 1$ are homogeneous with respect to action (8) of the group $\mathbf{C}_*$:

$$U_j(\lambda^k \zeta, \lambda x) = \lambda^{\sigma_j} U_j(\zeta, x)$$

for some powers σ_j, $j = 0, 1$.

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A remarkable property of representation (3) of a resurgent function is that this representation commutes with the restriction operators. This means that for any submanifold X the following relation takes place

$$i_X^* u = \ell \left(i_X^* U \right),$$

where the operator ℓ is defined with relations (3) and (4) and i_X^* is the restriction on X. Hence, denoting by $U(\zeta, x, t)$ the function corresponding to an unknown solution of (38) due to representation (3), one comes to the relations

$$
\begin{aligned}
U(\zeta, x, t)|_{t=0} &= U_0(\zeta, x), \\
\frac{\partial U}{\partial t}(\zeta, x, t)\Big|_{t=0} &= U_1(\zeta, x).
\end{aligned}
\tag{39}
$$

In the latter formula we used the fact that the homogenuity degree of the function $U(\zeta, x)$ can be changed arbitrarily with the help of integration by parts (see formula (6) above). Therefore, without loss of generality one can assume that the homogenuity degrees σ_0 and σ_1 of the functions U_0 and U_1 respectively satisfy the relation $\sigma_0 = \sigma_1 + 1$.

Theorem 3 together with formulas (39) allow one to derive the Cauchy problem for the function $U(\zeta, x, t)$:

$$
\frac{\partial^2 U(\zeta, x, t)}{\partial t^2} = \Delta_x U(\zeta, x, t),
$$
$$
U|_{t=0} = U_0(\zeta, x), \quad \frac{\partial U}{\partial t}\Big|_{t=0} = U_1(\zeta, x)
\tag{40}
$$

which is the family of Cauchy problems with variables (x, t) and the parameter ζ. The fact that ζ is included in this problem only as a parameter is purely accidental and is due to the particular form of initial problem (38).

The existence of an analytic (in general, ramifying) solution to problem (40) can be proved with the help of the elementary solution (see, for example, [6, 7]). The singularities of the constructed solution lye on the characteristic conoid of the singularity set of the initial data $U_0(\zeta, x)$, $U_1(\zeta, x)$.

From the viewpoint of the theory developed in this paper it is important that the solution $U(\zeta, x, t)$ of problem (40) is a homogeneous function of degree σ_0:

$$U(\lambda^k \zeta, \lambda x, \lambda t) = \lambda^{\sigma_0} U(\zeta, x, t).$$

The easy proof of this fact is left to the reader.

Let us compute the form of asymptotic solution $u(x, t)$ for concrete initial data of the form

$$u_0(x) = e^{-S(x)} a(x), \quad u_1(x) = 0,$$

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where $S(x)$ is a homogeneous function of degree k and $a(x)$ is a homogeneous function of degree σ with respect to x. In this case one has

$$U_0(x) = \frac{a(x)}{\zeta - S(x)}, \quad U_1(\zeta, x) = 0,$$

the homogeneous degree σ_0 of the function $U(\zeta, x)$ being equal to $\sigma - k$.

As it can be easily seen, the solution $U(\zeta, x, t)$ of problem (40) will have the form

$$U(\zeta, x, t) = \sum_{j=1}^{2} \left[\frac{a_0^{(j)}(x,t)}{\zeta - S_j(x,t)} + \ln(\zeta - S_j(x,t)) \sum_{l=0}^{\infty} \frac{(\zeta - S_j(x,t))^l}{l!} a_{l+1}^{(j)}(x,t) \right], \qquad (41)$$

where $S_j(x,t)$ are two solutions of the Cauchy problem for the Hamilton-Jacobi equation

$$\left(\frac{\partial S_j}{\partial t} \right)^2 = \left(\frac{\partial S_j}{\partial x} \right)^2,$$

$$S_j|_{t=0} = S(x),$$

corresponding to two components of the characteristic conoid of the singularity set $\zeta = S(x)$ of the function $U_0(\zeta, x)$ and $a_l^{(j)}$ are solutions of the corresponding transport equations. Certainly, representation (41) is valid in a neighbourhood of those poins where both components of the characteristic conoid are regular manifolds. In the vicinity of the rest points the form of singular parts of the function $U(\zeta, x, t)$ can be investigated with the help of the elementary solution mentioned above.

Thus, in a neighbourhood of all nonfocal points (x, t) the asymptotic solutions $u(x, t)$ to problem (40) will have the form

$$u(x, t) = \sum_{j=1}^{2} \left[e^{-S_j(x,t)} \sum_{l=0}^{\infty} a_l^{(j)}(x,t) \right],$$

the functions $S_j(x, t)$ being homogeneous functions of degree k with respect to the variables (x, t).

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