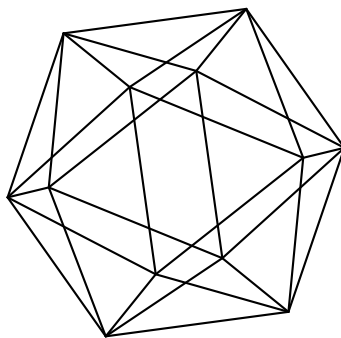# Max-Planck-Institut für Mathematik Bonn

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by

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COISOTROPIC DISPLACEMENT AND SMALL SUBSETS OF A SYMPLECTIC MANIFOLD

JAN SWOBODA AND FABIAN ZILTENER

We prove a coisotropic intersection result and deduce the following:
- **Lower bounds on the displacement energy of a subset of a symplectic manifold, in particular a sharp stable energy-Gromov-width inequality.**
- **A stable non-squeezing result for neighborhoods of products of unit spheres.**
- **Existence of a "badly squeezable" set in $\mathbb{R}^{2n}$ of Hausdorff dimension at most d, for every $n \geq 2$ and $d \geq n$.**
- **Existence of a stably exotic symplectic form on $\mathbb{R}^{2n}$, for every $n \geq 2$.**
- **Non-triviality of a new capacity, which is based on the minimal symplectic area of a regular coisotropic submanifold of dimension d.**

CONTENTS

1. Motivation and main results

1.1. Questions. The theme of this article is the following.

Question 1. *How much symplectic geometry can a small subset of a symplectic manifold carry?*

We approach this question from several points of view, interpreting "small" as "of Hausdorff dimension bounded above by a given number". Our main tool is an intersection result for coisotropic submanifolds (Theorem 1 below). Further results will appear in [**SZ**].

One instance of Question 1 is the following. Let (M, ω) be a symplectic manifold. (For simplicity all manifolds in this paper are assumed to have empty boundary.) For a subset $X \subseteq M$ we denote by $e(X) := e(X, M) := e(X, M, \omega)$ its displacement energy (see (29) below).

Question 2. *What lower bounds on $e(X)$ can be detected by special small subsets of X?*

To make this more precise, let $\mathcal{X}$ be a collection of subsets of M and $f : \mathcal{X} \to [-\infty, \infty]$ a function. We define

$$\widehat{f} : \{\text{subset of } M\} \to [-\infty, \infty], \quad \widehat{f}(X) := \sup \{f(Y) \,|\, Y \in \mathcal{X}, Y \subseteq X\}.$$

Note that the estimate

$$e(X) \geq \widehat{f}(X), \quad \forall X \subseteq M$$

holds, provided that the inequality

$$e(X) \geq f(X), \quad \forall X \in \mathcal{X} \tag{1}$$

is satisfied. Our goal is therefore to find a collection $\mathcal{X}$ containing a lot of small subsets of M and a big function $f : \mathcal{X} \to [-\infty, \infty]$ for which the inequality (1) is satisfied. Our ansatz in this article is to define $\mathcal{X}$ to be the set of all closed regular coisotropic submanifolds $N \subseteq M$ and $f(N)$ to be a refined version of the minimal symplectic area of N (see (4) below). Inequality (1) is then a direct consequence of Theorem 1 below.

Another instance of Question 1 is the following. Let $n \in \mathbb{N} = \{1, 2 \ldots\}$ and $a \in (0, \infty)$. We denote by $B^{2n}(a)$ ($\overline{B}^{2n}(a)$) the open

(closed) ball in $\mathbb{R}^{2n}$ of radius $\sqrt{a/\pi}$, around 0. Furthermore, we denote by $Z^{2n}(a) := B^2(a) \times \mathbb{R}^{2n-2}$ the open symplectic cylinder of area a. (Note that $B^{2n}(a)$ and $Z^{2n}(a)$ both have Gromov-width a.) We abbreviate $B^{2n} := B^{2n}(\pi)$ and $Z^{2n} := Z^{2n}(\pi)$.

Let (M, ω) and (M', ω') be symplectic manifolds. We write $(M, \omega) \hookrightarrow (M', \omega')$ iff there exists a symplectic embedding of M into M'. Let $n \in \mathbb{N}$. We denote by ω_0 the standard symplectic form on $\mathbb{R}^{2n}$. Gromov's non-squeezing result [**Gr**, Corollary, p. 310] states that $(B^{2n}(a), \omega_0) \not\hookrightarrow (Z^{2n}, \omega_0)$ if $a > \pi$. We may ask whether the boundary of the unit ball (or more generally, a finite product of unit spheres) is already too big to be squeezed into the unit cylinder. To make this more precise, let $k \in \mathbb{N}$ and $n_1, \ldots, n_k \in \mathbb{N}$. We denote $n := \sum_{i=1}^{k} n_i$.

Question 3 (Skinny non-squeezing). *If $U \subseteq \mathbb{R}^{2n}$ is any open neighborhood of $\times_{i=1}^{k} S^{2n_i-1}$ then is it true that*

$$(2) \qquad\qquad (U, \omega_0) \not\hookrightarrow (Z^{2n}, \omega_0)?$$

If $n_i = 1$ for some i then an elementary argument shows that there exists a U as above for which $(U, \omega_0) \hookrightarrow (Z^{2n}, \omega_0)$. Hence assume that $n_i \geq 2$, for every i. In this case Corollary 5 below provides a positive answer to a stabilized version of Question 3 (and in particular to the original question).

Generalizing Question 3, we may wonder how much small subsets of a symplectic manifold can be squeezed. This leads to the following definition. We denote the Hausdorff dimension of a metric space (X, d) by $\dim(X)$. Let (M_0, ω_0) be a symplectic manifold of dimension $2n$ and $d \in [0, \infty)$. We define the map

$$(3) \quad \mathrm{emb}_d^{M_0, \omega_0} : \left\{ \text{symplectic manifold } (M, \omega) \,\big|\, \dim M = 2n \right\} \to [0, \infty]$$

as follows. We define

$$\mathcal{X}^{2n} :=$$
$$\left\{ (M, \omega, X) \,\big|\, (M, \omega) : \text{ symplectic manifold, } \dim M = 2n, \, X \subseteq M \right\},$$

$$\mathrm{emb}^{M_0, \omega_0} : \mathcal{X}^{2n} \to [0, \infty]$$
$$\mathrm{emb}^{M_0, \omega_0}(M, \omega, X) :=$$
$$\inf \left\{ a > 0 \,\big|\, \exists U \subseteq M \text{ open: } X \subseteq U, \, (U, \omega|_U) \hookrightarrow \left(M_0, a\omega_0 \right) \right\}.$$

For $d \in [0, \infty)$ we now define the map (3) by

$$\mathrm{emb}_d^{M_0, \omega_0}(M, \omega) :=$$

$$\sup \big\{ \operatorname{emb}^{M_0,\omega_0}(M,\omega,X) \, \big| \, X \subseteq M : \text{compact}, \dim(X) \leq d \big\}.$$

Question 4 (Squeezing small sets)**.** *What is the value of* $\operatorname{emb}_d^{M_0,\omega_0}(M,\omega)$*?*

Theorem 6 below provides a lower bound on this number in the case $M_0 := Z^{2n}$, equipped with the standard form ω_0, and $(M,\omega) := (B^{2n},\omega_0)$. In particular, it shows that for every $n \geq 2$ and $d \geq n$ the number $\operatorname{emb}_d^{Z^{2n},\omega_0}(B^{2n},\omega_0)$ is positive, and therefore $\operatorname{emb}_d^{Z^{2n},\omega_0}$ is an intrinsic symplectic capacity on $(\mathbb{R}^{2n},\omega_0)$ in the sense of [**Schl1**, Definition C.1, p. 224].

Looking at Question 1 from yet another point of view, we may ask the following.

Question 5 (Distinguishing symplectic structures)**.** *How can the (non-)existence of certain subsets be used to distinguish symplectic structures?*

Corollary 7 below is concerned with this question. It says that every coisotropically infinite symplectic structure on $\mathbb{R}^{2n}$ is stably exotic. (For definitions see page 9.) It follows that there exists a stably exotic symplectic form on $\mathbb{R}^{2n}$, if $n \geq 2$.

1.2. Coisotropic Intersections and displacement energy.

Coisotropic intersections. The main results of this article are consequences of the following key result. In order to state it, let (M,ω) be a symplectic manifold. We call it *(geometrically) bounded* iff there exist an almost complex structure J on M and a complete Riemannian metric g such that the following conditions hold:

- The sectional curvature of g is bounded and $\inf_{x \in M} \iota_x^g > 0$, where ι_x^g denotes the injectivity radius of g at the point $x \in M$.
- There exists a constant $C \in (0,\infty)$ such that

$$|\omega(v,w)| \leq C|v|\,|w|, \quad \omega(v,Jv) \geq C^{-1}|v|^2,$$

for all $v,w \in T_xM$ and $x \in M$. Here $|v| := \sqrt{g(v,v)}$.

For examples see Section 1.6.

Let $N \subseteq M$ a coisotropic submanifold. We denote by $A(N)$ its minimal symplectic action (see (32) below). We define the *split minimal symplectic action of N, $A_\times(M,\omega,N)$* as follows. We define a *bounded splitting of (M,ω,N)* to be a tuple $(M_i,\omega_i,N_i)_{i=1,\dots,k}$, where $k \in \mathbb{N}$ and for every $i = 1,\dots,k$, (M_i,ω_i) is a bounded symplectic manifold and $N_i \subseteq M_i$ a coisotropic submanifold, such that there exists

a symplectomorphism φ from $\left(\times_{i=1}^{k} M_i, \oplus_{i=1}^{k} \omega_i \right)$ to (M, ω), satisfying $\varphi \left(\times_{i=1}^{k} N_i \right) = N$. We define

$$(4) \qquad\qquad A_\times(N) = A_\times(M, \omega, N) :=$$

$$\sup \left\{ \min_{i=1,\dots k} A(M_i, \omega_i, N_i) \,\middle|\, (M_i, \omega_i, N_i)_i \text{ bounded splitting of } (M, \omega, N) \right\}.$$

Here our convention is that $\sup \emptyset = 0$.

Remark. If (M, ω) is not bounded then (M, ω, N) does not admit any bounded splitting, and therefore $A_\times(N) = 0$. This follows from the facts that a finite product of bounded symplectic manifolds is bounded, and boundedness is invariant under symplectomorphisms. $\square$

We call a coisotropic submanifold $N \subseteq M$ *regular* iff its isotropy relation (see (30) below) is a closed subset and a submanifold of $N \times N$. Equivalently, the symplectic quotient of N is well-defined.

We denote by $\mathrm{Ham}(M, \omega)$ the group of Hamiltonian diffeomorphisms on M and by $\|\cdot\|_\omega$ the Hofer norm on $\mathrm{Ham}(M, \omega)$. (See Section 2, page 14). The key result of this article is the following.

Theorem 1 (Coisotropic intersections). *Let (M, ω) be a symplectic manifold, $\emptyset \neq N \subseteq M$ a closed connected regular coisotropic submanifold, and $\varphi : M \to M$ a Hamiltonian diffeomorphism. If*

$$(5) \qquad\qquad \|\varphi\|_\omega < A_\times(N)$$

then

$$(6) \qquad\qquad N \cap \varphi(N) \neq \emptyset.$$

In the case where N is a Lagrangian submanifold the statement of this result is an immediate consequence of the Main Theorem in Y. Chekanov's paper [**Ch**]. Furthermore, Theorem 1 is related to the main result, Theorem 1, in [**Zi**]. Namely, it has weaker hypotheses and a weaker conclusion than that result. Its proof is an adaption of the proof of [**Zi**, Theorem 1]. It is based on a certain Lagrangian embedding of N and on the Main Theorem in [**Ch**].

Displacement energy. The next result provides an answer to Question 2. To formulate it, we define the map

$$A_\times^{\mathrm{coiso}} : \left\{ (M, \omega, X) \,\middle|\, (M, \omega) \text{ symplectic manifold}, X \subseteq M \right\} \to [0, \infty],$$

by taking $A_\times^{\mathrm{coiso}}(X) = A_\times^{\mathrm{coiso}}(M, \omega, X)$ to be the supremum of all numbers $A_\times(N)$, where $N \neq \emptyset$ is a closed regular coisotropic submanifold of M that is contained in X (with the convention that $\sup \emptyset = 0$). Let (M, ω) be a symplectic manifold and $X \subseteq M$ a subset.

Corollary 2 (Displacement energy). *If (M, ω) is bounded then*

$$e(X) \geq A_\times^{\mathrm{coiso}}(X).$$

This is an immediate consequence of Theorem 1. In an example on page 11 below we will compute $A_\times^{\mathrm{coiso}}(X)$ for certain products. As a special case, let ω_1 be an area form on S^2 with total area at least π. Then Corollary 2 and inequality (22) below imply that

$$e\big((S^3)^2, \mathbb{R}^8, \omega_0\big) = \pi,$$

$$e\big((S^1)^2 \times S^2, \mathbb{R}^4 \times S^2, \omega_0 \oplus \omega_1\big) = \pi.$$

To our knowledge, these equalities are new.

Stable sharp energy-Gromov-width inequality. As a consequence of Corollary 2, we obtain the following result. Let (M, ω) be a symplectic manifold. We call it *aspherical* iff $\int_{S^2} u^*\omega = 0$ for every $u \in C^\infty(S^2, M)$. We denote $2n := \dim M$ and by

$$w(M) := w(M, \omega) := \sup\big\{a \,\big|\, (B^{2n}(a), \omega_0) \hookrightarrow (M, \omega)\big\}$$

the Gromov-width of (M, ω). Let (M', ω') be another symplectic manifold.

Corollary 3 (Energy-Gromov-width inequality). *Assume that (M, ω) and (M', ω') are aspherical, (M, ω) is bounded, and M' is closed. Then for every open subset $U \subseteq M$, we have*

$$(7) \qquad e\big(U \times M', M \times M'\big) \geq w(U).$$

This inequality appears to be new. (There are previous results about the case $M' = \{\mathrm{pt}\}$ or with a constant factor on the right hand side of (7), see Section 1.6.)

Remark. Even in the case $M' = \{\mathrm{pt}\}$ the result is sometimes new. As an example, let X be a closed manifold and σ a closed two-form on X such that $\int_{S^2} u^*\sigma = 0$, for every $u \in C^\infty(S^2, X)$. We denote by $\pi : T^*X \to X$ and ω_{can} the canonical projection and two-form on T^*X. We define $(M, \omega) := \big(T^*X, \omega_{\mathrm{can}} + \pi^*\sigma\big)$ and $M' := \{\mathrm{pt}\}$.

Then the hypotheses of Corollary 3 are satisfied, and therefore, applying the corollary, we have $e(U) \geq w(U)$, for every open subset $U \subseteq M$. For X equal e.g. to the sphere S^2 or the two-torus $\mathbb{T}^2$ this result appears to be new. $\square$

The statement of the corollary is sharp in the sense that for every pair of symplectic manifolds (M, ω) and (M', ω') there exists an open subset $U \subseteq M$ for which equality holds in (7). Namely, denoting $2n := \dim M$, and by $B_r^{2n} \subseteq \mathbb{R}^{2n}$ the open ball of radius r around 0,

we may choose $r > 0$ and an embedding $\varphi : (B^{2n}_{3r}, \omega_0) \hookrightarrow (M, \omega)$. We define $U := \varphi(B^{2n}_r)$. The opposite inequality in (7) then follows from an elementary argument using Remarks 37 and 38 below.

1.3. The regular coisotropic capacity. Let $d \in \mathbb{N}$. The minimal coisotropic area gives rise to a map

$$A^d_{\mathrm{coiso}} : \big\{\text{symplectic manifold}\big\} \to [0, \infty].$$

Namely, we define $A^d_{\mathrm{coiso}}(M, \omega)$ to be the supremum of all numbers $A(N)$, where $N \subseteq M$ is a non-empty closed regular coisotropic submanifold of dimension d, satisfying the following condition:

(8) $\forall F$ isotropic leaf of N, $\forall x \in C(S^1, F) : x$ is contractible in M.

Our next result involves the function

(9) $$\mathrm{k} : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \cup \{\infty\},$$

which is defined as follows. Let $(n, d) \in \mathbb{N} \times \mathbb{N}$. We define $\mathrm{k}(n, d)$ to be the infimum of all integers $\sum_{i=1}^{\ell} k_i$, where $\ell \in \mathbb{N}$ and $k_1, \ldots, k_\ell \in \mathbb{N}$ are such that there exist $n_i \in \mathbb{N}$, for $i = 1, \ldots, \ell$, satisfying

(10) $$n_i \geq k_i,$$

(11) $$\sum_i k_i n_i = n, \quad \sum_i k_i(2n_i - k_i) = d.$$

Here our convention is that the infimum of the empty set is ∞. Note that $\mathrm{k}(n, d) = \infty$, if $d < n$ or $d > 2n - 1$. On the other hand,

$$\mathrm{k}(n, d) \leq 2n - d, \quad \text{if } n \leq d \leq 2n - 1.$$

(See inequality (37) in Proposition 8 below). Let $n, n' \in \mathbb{N}$, $d \in \{n, \ldots, 2n\}$, and (M, ω) and (M', ω') be symplectic manifolds of dimensions $2n$ and $2n'$, respectively.

Theorem 4 (Regular coisotropic capacity). *The following statements hold.*

  (i) *If $d < 2n$ then the restriction of A^d_{coiso} to the class of all aspherical symplectic manifolds of dimension $2n$ is a symplectic capacity (as defined on page 15). For $d = 2n - 1$ this capacity is normalized, i.e., it takes on the value π on B^{2n} and Z^{2n}.*
 (ii) *We have*

(12) $$A^d_{\mathrm{coiso}}(B^{2n}, \omega_0) \geq \tfrac{\pi}{\mathrm{k}(n,d)},$$

(13) $$A^d_{\mathrm{coiso}}(Z^{2n}, \omega_0) \leq \pi.$$

(iii) If (M', ω') is closed and aspherical then

$$(14) \qquad A_{\text{coiso}}^{d+2n'}(M \times M', \omega \oplus \omega') \geq A_{\text{coiso}}^d(M, \omega).$$

For $d \in \{n, \ldots, 2n - 1\}$ we call the restriction of A_{coiso}^d to the class of all aspherical symplectic manifolds of dimension $2n$ the *regular coisotropic capacity*. In the case $d = n$ this is closely related to the Lagrangian capacity introduced by K. Cieliebak and K. Mohnke [**CM**]. Furthermore, in this case, the right hand side in (12) simplifies. Namely, we have

(15)

$$\text{k}(n, n) = \text{K}(n) := \inf \Big\{ \sum_{i=1}^{\ell} k_i \,\Big|\, \ell \in \mathbb{N},\ k_1, \ldots, k_\ell \in \mathbb{N} : n = \sum_i k_i^2 \Big\}.$$

(See equality (35) in Proposition 8 below.) The first few values of K are

$n =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\text{K}(n) =$	1	2	3	2	3	4	5	4	3	4	5	6	5	6	7	4	5

The function K satisfies the upper bound

$$\text{K}(n) < \sqrt{n} + 2^{\frac{3}{2}} \sqrt[4]{n}.$$

(See inequality (36) in Proposition 8 below.)

1.4. Symplectic squeezing of small sets. Our next application of Theorem 1 is the following. Let $k \in \mathbb{N}$ and $n_1, \ldots, n_k \in \mathbb{N}$. We denote $n := \sum_{i=1}^k n_i$. Let (M, ω) be a symplectic manifold.

Corollary 5 (Skinny non-squeezing)**.** *Let $U \subseteq \mathbb{R}^{2n}$ be an open neighborhood of $\times_{i=1}^k S^{2n_i-1}$. Assume that $n_i \geq 2$, for every $i = 1, \ldots, k$, M is closed, and*

$$(16) \qquad \int_{S^2} u^* \omega \in \pi \mathbb{Z}, \quad \forall u \in C^\infty(S^2, M).$$

Then we have

$$(17) \qquad \big(U \times M, \omega_0 \oplus \omega\big) \not\hookrightarrow \big(Z^{2n} \times M, \omega_0 \oplus \omega\big).$$

Taking $M = \{\text{pt}\}$, this result provides a positive answer to Question 3 of Section 1.1 in the case $n_i \geq 2$, for every i. To our knowledge, this fact is new. [1]

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[1] We are not aware of any other written proof. However, K. Cieliebak mentioned to the second author that in the case $M = \{\text{pt}\}$ the non-embedding (17) also follows from a standard "neck stretching" argument similar to the proof of [**CM**, Theorem 1.1].

In order to state our result about Question 4, recall the definition (3) of the map $\mathrm{emb}_d^{M_0,\omega_0}$. We also need the following. We define the map

(18)
$$\mathrm{k}_{\geq} : \mathbb{N} \times [0,\infty) \to \mathbb{R}$$

in the same way as the map k (see (9)), replacing (11) by the conditions

(19)
$$\sum_i k_i n_i \geq n, \quad \sum_i k_i(2n_i - k_i) \leq d.$$

Note that

$$\mathrm{k}_{\geq}(n,d) \leq \mathrm{k}(n,d), \quad \forall (n,d) \in \mathbb{N} \times [0,\infty).$$

We can now formulate the following result.

Theorem 6 (Badly squeezable small sets). *For every $n \in \{2,3,\ldots\}$ and $d \in [n,\infty)$ we have*

$$\mathrm{emb}_d^{Z^{2n},\omega_0}(B^{2n}, \omega_0) \geq \frac{\pi}{\mathrm{k}_{\geq}(n,d)}.$$

The map $\mathrm{k}_{\geq}$ satisfies some explicit upper bounds, see Proposition 8 below. As an example, inequalities (33,35,36) of that proposition imply that for $d \geq n$,

$$\mathrm{k}_{\geq}(n,d) < \sqrt{n} + 2^{\frac{3}{2}} \sqrt[4]{n}.$$

In the proof of Theorem 6 we consider a certain product of Stiefel manifolds. This is a regular coisotropic submanifold N of $\mathbb{R}^{2(n+n')}$ for some $n' \in \mathbb{N}$. We also use the inequality

$$\mathrm{emb}_d^{Z^{2n},\omega_0}(M, \omega) \geq A_{\mathrm{coiso}}^d(M, \omega),$$

see Proposition 21 below. The proof of this inequality is based on Theorem 1. It also relies on an argument in which we glue disks to a given regular coisotropic submanifold N, to make all loops in the isotropic fibers of N contractible.

1.5. Stably exotic symplectic forms. To state our last application of Theorem 1, let $n \in \mathbb{N}$. We call a symplectic form ω on $\mathbb{R}^{2n}$ *stably exotic* iff the following holds. Let (X, σ, σ') be a triple consisting of a closed manifold X and symplectic forms σ and σ' on X, with σ aspherical, and let $\varphi : M := X \times \mathbb{R}^{2n} \to X \times \mathbb{R}^{2n}$ be an embedding. Then

(20)
$$\varphi^*(\sigma \oplus \omega_0) \neq \sigma' \oplus \omega.$$

Note that such an ω is exotic in the usual sense, i.e., if $\varphi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is an embedding then $\varphi^* \omega_0 \neq \omega$.

Our result is a sufficient criterion for stable exoticness. Namely, we call a symplectic manifold (M, ω) *(coisotropically) infinite* iff there exists a regular closed coisotropic submanifold $N \subseteq M$ with $A(N) = \infty$. (This means that $\int_{S^2} u^* \omega = 0$, for every $u \in C^\infty(S^2, M)$ such that $u(S^1)$ is contained in some isotropic leaf of N.) We call (M, ω) *strongly (coisotropically) infinite* iff there exists N as above, such that every continuous loop in an isotropic leaf of N is contractible in M.

Corollary 7 (Stably exotic form)**.** *Let $n \in \mathbb{N}$. Then every coisotropically infinite form on $\mathbb{R}^{2n}$ is stably exotic.*

It follows from Corollary 7 and the example on page 11 below that for every $n \geq 2$, there exists a stably exotic symplectic form on $\mathbb{R}^{2n}$.

1.6. Remarks, examples, related work etc.

Remarks. On geometric boundedness: In the article [**Zi**] the second author used the term "geometrically bounded" in a slightly stronger sense.
On Theorem 1: Assume that the hypotheses of Theorem 1 are satisfied and the pair (N, φ) is non-degenerate in the sense of [**Zi**]. Then the number of leafwise fixed points of φ is bounded below by the sum of the $\mathbb{Z}_2$-Betti numbers of N. This follows by adapting the proof of Theorem 1 along the lines of the proof of [**Zi**, Theorem 1]. $\square$
On Corollary 3: The assumption that (M', ω') is aspherical can be weakened as follows. For a symplectic manifold (M, ω) we define

$$(21) \quad A(M, \omega) := \inf \left(\left\{ \int_{S^2} u^* \omega \,\Big|\, u \in C^\infty(S^2, M) \right\} \cap (0, \infty) \right) \in [0, \infty].$$

Assume that (M, ω) is aspherical and bounded, and there exist closed symplectic manifolds (M_i, ω_i), for $i = 1, \ldots, k$, such that $M' = \prod_i M_i$, $\omega' = \oplus_i \omega_i$, and $A(M_i, \omega_i) \geq w(U)$. Then inequality (7) still holds. This follows from an argument using Corollary 2. $\square$
On the regular coisotropic capacity: One can define variants of this capacity by imposing other conditions on the coisotropic submanifold N (e.g., stability or a contact type condition). Note that in order to obtain a capacity c^d satisfying $c^d(Z^{2n}, \omega_0) < \infty$, one can neither completely drop the condition that N is closed nor that it is regular.

Namely, there exists a regular (but not closed) coisotropic submanifold $N \subseteq Z^{2n}$ such that $A(Z^{2n}, \omega_0, N) = \infty$, and there exists a closed (but not regular) coisotropic submanifold $N' \subseteq Z^{2n}$ such that $A(Z^{2n}, \omega_0, N') = \infty$.

As an example, we may choose a coisotropic subspace $W \subseteq \mathbb{R}^{2n}$ of dimension d and define $N := W \cap Z^{2n}$. Furthermore, we may choose N' to be a closed hypersurface in Z^{2n} without any closed characteristic. Such an N' exists by a construction due to V. Ginzburg, see [**Gi**], Example 7.2 p. 158. (Such an N is not regular. We shrink Ginzburg's hypersurface homothetically, so that it fits into Z^{2n}.) □

Examples. Examples for geometric boundedness: (M, ω) is bounded if it is closed (i.e., compact and with empty boundary), a symplectic vector space, convex at infinity (see [**CGK**, Remark 2.3]), or the twisted cotangent bundle $\left(T^*X, \omega_{\mathrm{can}} + \pi^*\sigma \right)$, where X is a closed manifold X, σ is a closed two-form on X, and $\pi : T^*X \to X$ and ω_{can} denote the canonical projection and two-form on T^*X. (For the last example see [**CGK**, Proposition 2.2].) Furthermore, by a straight-forward argument, the product of a two bounded symplectic manifolds is bounded. □

Example for $A_\times^{\mathrm{coiso}}(X)$ and Corollary 2: For $n \in \mathbb{N}$ and $a \in (0, \infty)$ we denote by $S^{2n-1}(a) \subseteq \mathbb{R}^{2n}$ the sphere of radius $\sqrt{a/\pi}$ around 0. Let $k \in \mathbb{N}_0 = \{0, 1, \ldots\}$, $\ell \in \mathbb{N}$, for $i = 1, \ldots, k$ let $n_i \in \mathbb{N}$, and $a_i \in (0, \infty)$, and for $i = 1, \ldots, \ell$ let (M_i, ω_i) be a closed symplectic manifold. We define $n := \sum_i n_i$, and

$$M := \mathbb{R}^{2n} \times \times_{i=1}^{\ell} M_i, \quad \omega := \omega_0 \oplus \oplus_{i=1}^{\ell} \omega_i,$$
$$X := \times_{i=1}^{k} S^{2n_i - 1}(a_i) \times \times_{i=1}^{\ell} M_i.$$

We claim that

(22) $$A_\times^{\mathrm{coiso}}(X) \geq a := \min \left(\{a_i\}_i \cup \{A(M_i, \omega_i)\}_i \right),$$

where $A(M_i, \omega_i)$ is defined as in (21). To see this, observe that $N := X$ is a closed regular coisotropic submanifold of $\mathbb{R}^{2n} \times \times_{i=1}^{\ell} M_i$ with $A_\times(N) \geq a$. (This inequality follows from a straight-forward argument involving the splitting $M = \times_i \mathbb{R}^{2n_i} \times \times_i M_i$, and Remark 31 and Proposition 34 below.) The claimed inequality (22) follows.

Combining Corollary 2 with inequality (22) we obtain

(23) $$e(X) \geq a.$$

Assume that $\min_i A(M_i, \omega_i) \geq \min_i a_i$. Then the estimate (23) is sharp. To see this, let $j \in \{1, \ldots, k\}$ be such that $a_j = \min_i a_i$. Remarks 37 and 38 below imply that $e(X) \leq a_j = a$. □

Examples of coisotropically infinite manifolds. Every closed aspherical symplectic manifold and the cotangent bundle of a closed simply-connected manifold are strongly infinite. (In the first example

we may take $N := M$, and in the second example we may take N to be the zero section of the bundle.) Furthermore, by a standard argument, there exists a pair (ω, L), where ω is a symplectic form on $\mathbb{R}^{2n}$ and $L \subseteq \mathbb{R}^{2n}$ is a closed Lagrangian submanifold such that $\int_{\mathbb{D}} u^* \omega = 0$, for every $u \in C^\infty(\mathbb{D}, M)$ satisfying $u(S^1) \subseteq L$. (See e.g. [**ALP**], p. 317.) Such a form ω is strongly infinite. $\square$

Related work. Coisotropic intersections and displacement energy. In [**Zi**] the second author proved a result (Theorem 1) similar to the key result (Theorem 1) of the present article.

Let (M, ω) be a symplectic manifold and $U \subseteq M$ an open subset. In the case $M' = \{\text{pt}\}$ the energy-Gromov-width inequality (7) follows from an elementary argument, whenever one can prove that a certain symplectic capacity c satisfies

$$(24) \qquad\qquad c(B^{2n}, \omega_0) \geq \pi,$$

$$(25) \qquad\qquad c(U, \omega|_U) \leq e(U, M).$$

In the following, we take "e" in inequality (25) to mean variants of the displacement energy. Let $(M, \omega) := (\mathbb{R}^{2n}, \omega_0)$ and c be the **Ekeland-Hofer capacity**. Then inequality (24) was proved by I. Ekeland and H. Hofer in [**EH**, Theorem 1]. (They actually proved equality.) Furthermore, H. Hofer [**Ho1**, Theorem 1.6(i)] proved inequality (25).

Let now c be the π_1-**sensitive Hofer-Zehnder capacity** c_{HZ}°. Then inequality (24) is an easy consequence of the definition. Furthermore, H. Hofer [**Ho2**, Theorem 2] proved inequality (25) for $(\mathbb{R}^{2n}, \omega_0)$. (See also H. Hofer and E. Zehnder [**HZ**, Section 5.5].)

U. Frauenfelder, V. Ginzburg, and F. Schlenk [**FGS**, Corollary 1] proved the inequality (25) (for $c = c_{\mathrm{HZ}}^\circ$) if an exhaustion of (M, ω) admits an action selector. As an example, assume that (M, ω) is aspherical. If it is also closed or convex at infinity, then it admits such an exhaustion. (It even admits an action selector itself. See the examples in [**FGS**, pages 3,4]. See also inequality (2.9), p. 13, and Proposition 3.4 in [**Gi**].)

M. Usher [**Us**, Corollary 1.2] proved that (25) holds if (M, ω) is of type (C) (see [**Us**, p. 3]). Examples of type (C) manifolds are Stein manifolds, closed manifolds, and convex symplectic 4-manifolds. Non-sharp versions of inequality (25) were proved by M. Schwarz [**Schw**, Corollary 5.16] and F. Schlenk [**Schl2**, Theorem 1.1].

In [**LM2**] F. Lalonde and D. McDuff proved that $e(U, M) \geq \frac{1}{2} w(U)$, for *any* symplectic manifold (M, ω) (and any open subset $U \subseteq M$). (This inequality is a non-sharp version of (7) with $M' = \{\text{pt}\}$.) F. Lalonde

and C. Pestieau [**LP**] proved the following stabilized version of this result: Let (M, ω) and (M', ω') be symplectic manifolds, with M' closed, and $U \subseteq M$ an open subset. Then Theorem 1.2 in [**LP**] states that

$$e\big(U \times M', M \times M'\big) \geq \frac{1}{2}w(U).$$

The regular coisotropic capacity. Let $n \in \mathbb{N}$. We denote

$$\mathcal{M} := \big\{(M, \omega) \text{ symplectic manifold} \mid$$
$$\dim M = 2n, \ \pi_1(M) \cong \pi_2(M) \cong \{e\}\big\}.$$

In [**CM**] K. Cieliebak and K. Mohnke defined the *Lagrangian capacity* to be the map

$$c_L : \mathcal{M} \to [0, \infty),$$

$$c_L(M, \omega) := \sup \big\{A(M, \omega, L) \mid L \subseteq M \text{ embedded Lagrangian torus}\big\}.$$

(See also [**CHLS**], Sec. 2.4, p. 11.) The authors proved that

(26)
$$c_L(B^{2n}, \omega_0) = \frac{\pi}{n}.$$

The capacity c_L is bounded above by the regular coisotropic capacity A^n_{coiso}, since every Lagrangian submanifold is regular. Let $d \in \{n, \ldots, 2n - 1\}$ be an integer. If $d = n$ then assume that $n \geq 4$. Then we have

(27)
$$c_L(B^{2n}, \omega_0) < A^d_{\text{coiso}}(B^{2n}, \omega_0).$$

To see this, observe that $n > \text{k}(n, d)$. (In the case $d = n$ this follows by taking $\ell := n - 3$, $k_1 := 2$, $k_2, \ldots, k_{n-3} := 1$, and in the case $d > n$ from inequality (37) below.) Combining this with the inequality (12) of Theorem 4(ii) and the equality (26), inequality (27) follows.

Squeezing small sets. As mentioned on page 8, an argument by K. Cieliebak and K. Mohnke as in [**CM**] yields the statement of Corollary 5 in the case $M = \{\text{pt}\}$.

Exotic symplectic structures. In [**Gr**] M. Gromov proved that there does not exist a closed exact ω_0-Lagrangian submanifold of $\mathbb{R}^{2n}$. It was folklore that this property of $(\mathbb{R}^{2n}, \omega_0)$ implies the existence of an exotic symplectic structure on $\mathbb{R}^{2n}$, and a proof of this appeared in the paper [**Vi**] by C. Viterbo. Another reference is [**ALP**, p. 317].

Organization of the article. In Section 2 we collect some definitions that are used throughout this article. We also prove some useful properties of the functions K, k, and $k_\geq$, including upper bounds (Proposition 8). Section 3 is devoted to the proofs of the results of Section 1. Since Theorem 4 is used in the proof of Corollary 3, we prove it before the corollary. Appendix A contains some basic facts from (pre-)symplectic geometry, topology, and manifold theory, which are used in the proofs of the main results.

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2. Background and a further result

In this section some standard symplectic geometry is recalled which is used in this article. For more details, see the book by D. McDuff and D. A. Salamon [**MS**]. We also prove upper estimates on the functions k and $k_\geq$.

Let (M, ω) be a symplectic manifold. We define the group $\mathrm{Ham}(M, \omega)$ of *Hamiltonian diffeomorphisms of M*, as follows. We define $\mathcal{H}(M, \omega)$ to be the set of all functions $H \in C^\infty\big([0, 1] \times M, \mathbb{R}\big)$ whose Hamiltonian time-t flow $\varphi_H^t : M \to M$ exists and is surjective, for every $t \in [0, 1]$. We define

$$\mathrm{Ham}(M, \omega) := \big\{ \varphi_H^1 \,\big|\, H \in \mathcal{H}(M, \omega) \big\}.$$

This is a subgroup of the group of diffeomorphisms of M. (See for example [**SZ**].) It contains the group $\mathrm{Ham}_c(M, \omega)$ of Hamiltonian diffeomorphisms generated by a compactly supported time-dependent function. By definition, the Hofer norm on the space of functions is the map

$$\| \cdot \| : C^\infty([0, 1] \times M, \mathbb{R}) \to [0, \infty],$$

$$\|H\| := \int_0^1 \big(\sup_M H^t - \inf_M H^t \big) dt,$$

where $H^t(x) := H(t, x)$. (It follows from Lemma 45 below that this norm is well-defined.) We define the Hofer norm on $\mathrm{Ham}(M, \omega)$ to be the map

$$\| \cdot \|_\omega : \mathrm{Ham}(M, \omega) \to [0, \infty],$$

$$(28) \qquad \|\varphi\|_\omega := \inf \big\{ \|H\| \,\big|\, H \in \mathcal{H}(M, \omega) : \varphi_H^1 = \varphi \big\}.$$

We define the *displacement energy* of a subset $X \subseteq M$ to be

(29)
$$e(X, M) := e(X, M, \omega) := \inf \left\{ \|H\| \,\middle|\, H \in \mathcal{H}(M, \omega) \,\middle|\, \varphi_H^1(X) \cap X = \emptyset \right\}.$$

Let (M, ω) be a symplectic manifold. The *Gromov-width* of (M, ω) is defined to be the supremum of all numbers $a \geq 0$ such that there exists an embedding $\varphi : B(a) \to M$ satisfying $\varphi^* \omega = \omega_0$.

Let (M, ω) be a symplectic manifold and $N \subseteq M$ a submanifold. Then N is called *coisotropic* iff for every $x \in N$ the subspace

$$T_x N^\omega = \left\{ v \in T_x M \,\middle|\, \omega(v, w) = 0, \ \forall w \in T_x N \right\}$$

of $T_x M$ is contained in $T_x N$. As an example, every hypersurface in M is coisotropic.

Let $N \subseteq M$ be a coisotropic submanifold. We define the *isotropy relation* to be the set

(30) $\quad R^{N,\omega} := \left\{ (x(0), x(1)) \,\middle|\, x \in C^\infty([0,1], N) : \right.$

$$\left. \dot{x}(t) \in (T_{x(t)} N)^\omega, \ \forall t \in [0,1], \ x(i) = x_i, \ \forall i = 0, 1 \right\}.$$

This is an equivalence relation on N. For a point $x_0 \in N$ we call the $R^{N,\omega}$-equivalence class of x_0 the *isotropic leaf* through x_0. We denote this subset of N by $N_{x_0}^\omega$. Furthermore, we denote

$$N_\omega := \left\{ \text{isotropic leaf of } N \right\}.$$

We call N *regular* if $R^{N,\omega}$ is a closed subset and a submanifold of $N \times N$. This holds if and only if there exists a manifold structure on the set N_ω such that the canonical projection $\pi_N : N \to N_\omega$ is a submersion, cf. [**Zi**, Lemma 15]. If N is closed then by C. Ehresmann's theorem this implies that π_N is a smooth (locally trivial) fiber bundle.

We define the *action (or area) spectrum* and the *minimal action* of N as

(31) $$S(N) = S(M, \omega, N) :=$$

$$\left\{ \int_{\mathbb{D}} u^* \omega \,\middle|\, u \in C^\infty(\mathbb{D}, M) : \exists F \in N_\omega : u(S^1) \subseteq F \right\},$$

(32) $\quad A(N) = A(M, \omega, N) := \inf \left(S(N) \cap (0, \infty) \right) \in [0, \infty].$

By a *symplectic capacity* we mean a pair $(\mathcal{C}, c)$, where $\mathcal{C}$ is a subclass of the class of all symplectic manifolds of dimension $2n$, for some $n \in \mathbb{N}$, and $c : \mathcal{C} \to [0, \infty]$ is a map, such that the following conditions are satisfied:

(**Monotonicity**) $c(M, \omega) \leq c(M', \omega')$, if $(M, \omega) \hookrightarrow (M', \omega')$ (i.e., M symplectically embeds into M'),

(**Conformality**) $c(M, a\omega) = |a| c(M, \omega)$, for every $a \in \mathbb{R} \setminus \{0\}$,

(**Nontriviality**) $0 < c(B^{2n})$ and $c(Z^{2n}) < \infty$.

The next result summarizes some properties of the functions K, k, $k_{\geq}$ (see (15,9,18)). In particular, it provides upper bounds on these functions.

Proposition 8. *Let $n, n', k \in \mathbb{N}$ and $d, d' \in [0, \infty)$. Then the following (in-)equalities hold:*

$$(33) \qquad\qquad k_{\geq}(n, d) \geq k_{\geq}(n, d'), \quad \text{if } d \leq d',$$

$$(34) \qquad\qquad k(n, d) \geq k_{\geq}(n, d),$$

$$(35) \qquad\qquad k(n, n) = k_{\geq}(n, n) = K(n),$$

$$(36) \qquad\qquad K(n) < \sqrt{n} + 2^{\frac{3}{2}} \sqrt[4]{n},$$

$$(37) \qquad\qquad k(n, d) \leq 2n - d, \quad \text{if } n \leq d \leq 2n - 1$$

$$(38) \quad k_{\geq}(n, d) < \sqrt{2n - d} + 3, \quad \text{if } n \geq 9,\ n + 6\sqrt{n} - 9 \leq d \leq 2n,$$

$$(39) \qquad\qquad k_{\geq}(n + n', d + d') \leq k_{\geq}(n, d) + k_{\geq}(n', d'),$$

$$(40) \qquad k(n, 2n - k^2) \leq k, \quad \text{if } k \text{ divides } n \text{ and } k^2 \leq n.$$

For the proof of Proposition 8, we need the following.

Remark 9. For every $m \in \mathbb{N}$ and $a > 0$ we have

$$\max \Big\{ \sum_{i=1}^{m} x_i \,\Big|\, x_i \in \mathbb{R},\ \forall i = 1, \ldots, m,\ \sum_i x_i^2 = a \Big\} = \sqrt{ma}.$$

(The maximum is attained at the point $\sqrt{\frac{a}{m}}(1, \ldots, 1)$.) $\square$

Proof of Proposition 8. **Inequalities (33,34)** are direct consequences of the definitions.

We show that the **equalities (35)** hold: We claim that

$$(41) \qquad\qquad k_{\geq}(n, n) \geq K(n).$$

To see this, let $\ell \in \mathbb{N}_0$ and $k_1, \ldots, k_\ell$ be as in the definition of $k_{\geq}(n, d)$ with $d = n$. Inequality (41) is a consequence of the next claim.

Claim 1. *We have*

$$
\text{(42)} \qquad \sum_{i=1}^{\ell} k_i^2 = n.
$$

Proof of Claim 1. We choose integers $n_1, \ldots, n_\ell$ such that the inequalities (10,19) are satisfied. Subtracting the first from the second inequality in (19), we obtain $\sum_i k_i(n_i - k_i) \leq d - n = 0$. Using the inequalities (10), it follows that $n_i = k_i$, for every $i = 1, \ldots, \ell$. Combining this with (19), the equality (42) follows. This proves Claim 1. $\qquad \square$

We complete the proof of (35): In view of (41) and the inequality (34) it suffices to show that

$$
\text{(43)} \qquad \mathrm{k}(n, n) \leq \mathrm{K}(n).
$$

To see this, let $\ell \in \mathbb{N}_0$ and $k_1, \ldots, k_\ell \in \mathbb{N}$ be as in the definition of $\mathrm{K}(n)$. This means that $\sum_{i=1}^{\ell} k_i^2 = n$. We define $n_i := k_i$, for $i = 1, \ldots, \ell$. Then the conditions (10,11) in the definition of $\mathrm{k}(n, n)$ are satisfied with $d = n$. Inequality (43) follows. This proves (35).

To prove **inequality (36)**, let $n \in \mathbb{N}$. We define $\ell := 5$ and k_1 to be the biggest integer $\leq \sqrt{n}$. By the Four Squares Theorem there exist integers $k_2, \ldots, k_5 \in \mathbb{N}_0$ such that $\sum_{i=2}^{5} k_i^2 = n - k_1^2$. (See for example Theorem 2.10 in the book [**EW**].) Since $n = \sum_{i=1}^{5} k_i^2$, by definition, we have

$$
\text{(44)} \qquad \mathrm{K}(n) \leq \sum_{i=1}^{5} k_i.
$$

Furthermore, by Remark 9 with $m := 4$ and $a := n - k_1^2$, we have $\sum_{i=2}^{5} k_i \leq 2\sqrt{n - k_1^2}$. Combining this with the inequality $k_1 > \sqrt{n} - 1$, we obtain

$$
\sum_{i=1}^{5} k_i < k_1 + 2\sqrt{n - (\sqrt{n} - 1)^2} < \sqrt{n} + 2^{\frac{3}{2}} \sqrt[4]{n}.
$$

Combining this with (44), inequality (36) follows.

Inequality (37) follows by taking $\ell := 2n - d$, $k_i := 1$, for $i = 1, \ldots, \ell$, $n_i := 1$, for $i = 1, \ldots, \ell - 1$, and $n_\ell := d - n + 1$.

To show **(38)**, assume that $n \geq 9$ and $n + 6\sqrt{n} - 9 \leq d \leq 2n$. For every number $x \in \mathbb{R}$ we denote by $\lceil x \rceil$ the smallest integer $\geq x$. We define $\ell := 1$, $k_1 := \lceil \sqrt{2n - d} \rceil + 2$, and $n_1 := \lceil \frac{n}{k_1} \rceil$. The claimed inequality is now a consequence of the following claim.

Claim 2. *The conditions (10,11) are satisfied.*

Proof of Claim 2. We prove that **condition (10)** holds. The assumption $d \geq n + 6\sqrt{n} - 9$ implies that $2n - d \leq n - 6\sqrt{n} + 9 = (\sqrt{n} - 3)^2$. Since $n \geq 9$, we have $\sqrt{n} - 3 \geq 0$. It follows that

$$(45) \qquad\qquad (\sqrt{2n - d} + 3)^2 \leq n.$$

On the other hand, we have $k_1 < \sqrt{2n - d} + 3$, and therefore $n_1 \geq \frac{n}{k_1} > \frac{n}{\sqrt{2n-d}+3}$. Combining this with (45), it follows that $n_1 > k_1$. This proves condition (10).

The **first condition in (11)**, $k_1 n_1 \geq n$, follows from the definition of n_1.

To prove the **second condition in (11)**, observe that $2n_1 - k_1 < \frac{2n}{k_1} + 2 - \sqrt{2n - d} - 2$, and therefore

$$k_1(2n_1 - k_1) < 2n - k_1\sqrt{2n - d} \leq 2n - (2n - d) = d.$$

This proves the second condition in (11), and completes the proof of Claim 2, and hence of (38). $\qquad\square$

Inequality (39) follows from a straight-forward argument, and **inequality (40)** follows by choosing $\ell := 1$, $k_1 := k$, and $n_1 := n/k$.

This completes the proof of Proposition 8. $\qquad\square$

3. Proofs of the main results

3.1. Proof of Theorem 1 (Coisotropic intersections). A central ingredient of the proof of Theorem 1 is the following result by Y. Chekanov. Let (M, ω) be a symplectic manifold, Σ a Riemann surface, and $X \subseteq M$ a subset. For every almost complex structure J on M, we define

$$(46) \qquad\qquad A(\Sigma, M, \omega, J, X) :=$$

$$\inf\left(\left\{\int_\Sigma u^*\omega \,\Big|\, u : \Sigma \to M \text{ } J\text{-holomorphic}, \text{ } u(\partial\Sigma) \subseteq X\right\} \cap (0, \infty)\right).$$

Furthermore, we define the *bounded minimal action of (M, ω) relative to X* to be

$$(47) \qquad\qquad A_b(X) = A_b(M, \omega, X) :=$$

$$\sup\left\{\min\{A(S^2, M, \omega, J, \emptyset), A(\mathbb{D}, M, \omega, J, X)\}\right\},$$

where the supremum is taken over all pairs (g, J) that satisfy the conditions of boundedness (see page 4). Here our convention is that $\sup \emptyset = 0$. We define

$$\mathrm{Ham}_c(M, \omega) := \left\{ \varphi_H^1 \,\middle|\, H \in C_c^\infty([0, 1] \times M, \mathbb{R}) \right\},$$
$$\| \cdot \|_\omega^c : \mathrm{Ham}_c(M, \omega) \to [0, \infty),$$
$$\|\varphi\|_\omega^c := \inf \left\{ \|H\| \,\middle|\, H \in C_c^\infty([0, 1] \times M, \mathbb{R}) : \varphi_H^1 = \varphi \right\},$$

where $C_c^\infty([0, 1] \times M, \mathbb{R})$ denotes the space of all $H \in C^\infty([0, 1] \times M, \mathbb{R})$ with compact support.

Theorem 10 ([**Ch**], Main Result). *Let (M, ω) be a symplectic manifold, $L \subseteq M$ a closed Lagrangian submanifold, and $\varphi \in \mathrm{Ham}_c(M, \omega)$. If*

$$(48) \qquad\qquad \|\varphi\|_\omega^c < A_b(M, \omega, L)$$

then $\varphi(L) \cap L \neq \emptyset$.

Remark 11. *The statement of Theorem 10 remains true if φ lies in the bigger group $\mathrm{Ham}(M, \omega)$ and the condition (48) is replaced by the weaker condition*

$$(49) \qquad\qquad \|\varphi\|_\omega < A_b(M, \omega, L)$$

This follows from Theorem 10 and Lemma 35 below. $\square$

Remark. In Chekanov's Main Result it is assumed that (M, ω) is bounded. This is unnecessary, since in the unbounded case we have $A_b(M, \omega, L) = 0$, and hence the statement is void. $\square$

Remark. The definition of (geometric) boundedness in Y. Chekanov's article is slightly stronger, and the number $A_b(M, \omega, L)$ in the hypothesis of the theorem is replaced by a corresponding quantity. However, the proof of the main result in that article goes through with these minor modifications. $\square$

The proof of Theorem 1 also relies on the following construction. Let (M, ω) be a symplectic manifold, and $N \subseteq M$ a coisotropic submanifold. For $x \in N$ we denote by N_x^ω the isotropic leaf of N through x. Furthermore, we denote by N_ω the set of isotropic leaves of N, and by $\pi_N : N \to N_\omega$ the canonical projection.

Assume that N is regular. Then there exists a unique manifold structure on N_ω such that π_N is a smooth submersion. (This follows for example from [**Zi**, Lemma 15, p. 20].) Furthermore, there exists

a unique symplectic structure ω_N on N_ω such that $\pi_N^*\omega_N = \omega|_N$. We define

$$(50) \qquad \widetilde{M} := M \times N_\omega, \quad \widetilde{\omega} := \omega \oplus (-\omega_N),$$

$$(51) \qquad \iota_N : N \to \widetilde{M}, \quad \iota_N(x) := (x, N_x^\omega), \quad \widetilde{N} := \iota_N(N).$$

By a straight-forward argument the set $\widetilde{N}$ is a Lagrangian submanifold of $\widetilde{M}$. The next result is a crucial ingredient in the proof of Theorem 1. It is proved on page 21.

Proposition 12. *If N is closed and regular then*

$$(52) \qquad A_\times(M, \omega, N) \leq A_\times(\widetilde{M}, \widetilde{\omega}, \widetilde{N}).$$

We also need the following. Let (M, ω) be a symplectic manifold and $L \subseteq M$ a Lagrangian submanifold.

Proposition 13. *If M is connected and $L \neq \emptyset$ then we have*

$$A_\times(M, \omega, L) \leq A_b(M, \omega, L).$$

This result will be proved on page 23. We are now ready fo the proof of the key result.

Proof of Theorem 1. Let M, ω, N, φ be as in the hypothesis, such that inequality (5) is satisfied. Without loss of generality, we may assume that M is connected. Consider the symplectomorphism

$$\widehat{\varphi} : \widetilde{M} \to \widetilde{M}, \quad (x, x') \mapsto (\varphi(x), x').$$

Claim 1. *We have*

$$(53) \qquad \|\widehat{\varphi}\|_{\widetilde{\omega}} < A_b(\widetilde{M}, \widetilde{\omega}, \widetilde{N}).$$

Proof of Claim 1. By a straight-forward argument we have that

$$(54) \qquad \|\widehat{\varphi}\|_{\widetilde{\omega}} \leq \|\varphi\|_\omega.$$

Since by hypothesis N is regular and closed, we may apply Proposition 12. It follows that inequality (52) holds.

Furthermore, since by assumption M and N are connected, the manifold $\widetilde{M}$ is connected. Therefore, we may apply Proposition 13, to conclude that

$$A_\times(\widetilde{M}, \widetilde{\omega}, \widetilde{N}) \leq A_b(\widetilde{M}, \widetilde{\omega}, \widetilde{N}).$$

Combining this with inequalities (54,5,52), it follows that the inequality (53) holds. This proves Claim 1. $\qquad \square$

Since N is closed, the manifold $\widetilde{N}$ is, as well. It follows that all hypotheses of Theorem 10 are satisfied, with M, ω, φ replaced by $\widetilde{M}, \widetilde{\omega}, \widehat{\varphi}$, and $L := \widetilde{N}$, except for (48). Furthermore, by Claim 1, the inequality (49) is satisfied. Therefore, using Remark 11, it follows that

$$(55) \qquad \widehat{\varphi}(\widetilde{N}) \cap \widetilde{N} \neq \emptyset.$$

We denote by $\mathrm{pr} : \widetilde{M} \to M$ the projection onto the first factor. Then we have

$$\mathrm{pr}\left(\widehat{\varphi}(\widetilde{N}) \cap \widetilde{N}\right) \subseteq \varphi(N) \cap N.$$

Combining this with (55), the statement (6) follows. This proves Theorem 1. $\qquad \square$

Next we will prove Proposition 12. We will use the following construction. Let (M, ω) and $(\widehat{M}, \widehat{\omega})$ be symplectic manifolds, $N \subseteq M$ and $\widehat{N} \subseteq \widehat{M}$ coisotropic submanifolds, and $\varphi : \widehat{M} \to M$ a symplectomorphism satisfying $\varphi(\widehat{N}) = N$. We define

$$(56) \qquad \varphi' : \widehat{N}_{\widehat{\omega}} \to N_\omega, \quad \varphi'(\widehat{N}_{\widehat{x}}^{\widehat{\omega}}) := N_{\varphi(\widehat{x})}^\omega.$$

This map is well-defined. We also define

$$(57) \qquad \widetilde{\varphi} := \varphi \times \varphi' : \widetilde{\widehat{M}} = \widehat{M} \times \widehat{N}_{\widehat{\omega}} \to \widetilde{M} = M \times N_\omega.$$

Remark 14. *Assume that one of the manifolds $\widehat{N}$ or N is regular. Then the other one is, as well, and φ' and hence $\widetilde{\varphi}$ are symplectomorphisms. This follows from a straight-forward argument.* $\square$

The proof of Proposition 12 also uses the following.

Lemma 15. *Let (M, ω) be a symplectic manifold and $N \subseteq M$ a closed, regular coisotropic submanifold. Then we have*

$$A(M, \omega, N) = A(\widetilde{M}, \widetilde{\omega}, \widetilde{N}).$$

Proof of Lemma 15. This is Lemma 10 (Key Lemma) in [**Zi**]. $\qquad \square$

Proof of Proposition 12. Assume that $(M_i, \omega_i, N_i)_{i=1,\dots,k}$ is a bounded splitting of (M, ω, N). We define $(\widetilde{M_i}, \widetilde{N_i})_{i=1,\dots,k}$ as in (50,51) with M replaced by M_i etc. By the definition of a bounded splitting of (M, ω, N), there exists a symplectomorphism $\varphi : \times_i M_i \to M$ such that $\varphi(\times_i N_i) = N$. We define φ' and $\widetilde{\varphi}$ as in (56,57), with

$$(\widehat{M}, \widehat{\omega}, \widehat{N}) := \left(\times_i M_i, \oplus_i \omega_i, \times_i N_i\right).$$

By hypothesis N is regular. Hence by Remark 14, the product $\times_i N_i$ is regular. Applying Lemma 28 below, it follows that N_i is regular, for every i. Since by hypothesis, N is closed, Remark 42 below implies that N_i is closed. We define the symplectic form $\widetilde{\omega}_i$ on $\widetilde{M}_i$ as in (51) with M replaced by M_i etc.

Claim 1. *The tuple $\left(\widetilde{M}_i, \widetilde{\omega}_i, \widetilde{N}_i\right)_i$ is a bounded splitting of $(\widetilde{M}, \widetilde{\omega}, \widetilde{N})$.*

Proof of Claim 1. Let $i \in \{1, \ldots, k\}$. We show that $(\widetilde{M}_i = M_i \times (N_i)_{\omega_i}, \widetilde{\omega}_i)$ **is bounded**: Since N_i is closed, it follows that $(N_i)_{\omega_i}$ is closed. Furthermore, by assumption, (M_i, ω_i) is bounded. It follows that $(\widetilde{M}_i, \widetilde{\omega}_i)$ is bounded.

We now show that **there exists a map** $\widetilde{f} : \times_i \widetilde{M}_i \to \widetilde{M}$ as in the definition of a bounded splitting: By Lemma 27 below the identity map on $\times_i N_i$ descends to a symplectomorphism

$$\psi' : \times_i (N_i)_{\omega_i} \to (\times_i N_i)_{\oplus_i \omega_i}.$$

We denote by

$$\widetilde{\psi} : \times_i \widetilde{M}_i = \times_i (M_i \times (N_i)_{\omega_i}) \to \times_i M_i \times (\times_i N_i)_{\oplus_i \omega_i}$$

the map induced by ψ', and define

$$\widetilde{f} := \widetilde{\varphi} \circ \widetilde{\psi} : \times_i \widetilde{M}_i \to \widetilde{M}.$$

By Remark 14 the map $\widetilde{\varphi}$ is a symplectomorphism. Since ψ' is a symplectomorphism, the same holds for $\widetilde{\psi}$, and hence for $\widetilde{f}$. Furthermore, we have $\widetilde{f}(\times_i \widetilde{N}_i) = \widetilde{N}$. Hence the map $\widetilde{f}$ satisfies the conditions in the definition of a bounded splitting. This proves Claim 1. $\square$

Let $i = 1, \ldots, k$. Since N_i is closed, we may apply Lemma 15, to conclude that $A(\widetilde{M}_i, \widetilde{\omega}_i, \widetilde{N}_i) = A(M_i, \omega_i, N_i)$. Combining this with Claim 1, the inequality (52) follows. This proves Proposition 12. $\square$

For the proof of Proposition 13 we need the following. Recall the definition (46).

Lemma 16. *Let Σ be a Riemann surface, $k \in \mathbb{N}$, and for $i = 1, \ldots, k$ let (M_i, ω_i) be a symplectic manifold, $X_i \subseteq M_i$ a subset, and J_i an ω_i-tame almost complex structure on M_i. Then*

$$(58) \qquad \min_{i=1,\ldots,k} A\big(\Sigma, M_i, \omega_i, J_i, X_i\big) \leq A\big(\Sigma, \times_i M_i, \oplus_i \omega_i, \oplus_i J_i, \times_i X_i\big).$$

In the proof of this lemma we will use the following.

Remark 17. *Let* (M, ω) *be a symplectic manifold,* J *an* ω*-tame almost complex structure,* Σ *a Riemann surface, and* $u : \Sigma \to M$ *a* J*-holomorphic map. Then* $\int_\Sigma u^* \omega \geq 0$. *This follows from the fact that* $\int_\Sigma u^* \omega$ *is the Dirichlet energy of* u. $\square$

Proof of Lemma 16. Assume that $u := (u_1, \dots, u_k) : \Sigma \to M := \times_i M_i$ is a $J := \oplus_i J_i$-holomorphic map satisfying $u(\partial \Sigma) \subseteq X := \times_i X_i$ and $E := \int_\Sigma u^* \omega > 0$. Let $i = 1, \dots, k$. We denote $E_i := \int_\Sigma u_i^* \omega_i$. Since by assumption J_i is ω_i-tame, and u_i is J_i-holomorphic, by Remark 17 we have $E_i \geq 0$. Combining this with the fact $E = \sum_i E_i$, it follows that

$$(59) \qquad E_i \leq E, \quad \forall i = 1, \dots, k.$$

Since $E > 0$, there exists $i_0 \in \{1, \dots, k\}$ such that $E_{i_0} > 0$. Combining this with inequality (59) and using the fact $u_i(\partial \Sigma) \subseteq X_i$, the inequality (58) follows. This proves Lemma 16. $\square$

In the proof of Proposition 13 we will use the following remark. We define the bounded minimal action as in (47).

Remark 18. *Let* (M, ω) *be a symplectic manifold,* $N \subseteq M$ *a coisotropic submanifold,* M' *a smooth manifold, and* $\varphi : M' \to M$ *a diffeomorphism. Then we have*

$$A_b\big(M', \varphi^* \omega, \varphi^{-1}(N)\big) = A_b(M, \omega, N).$$

Proof of Proposition 13. Let $(M_i, \omega_i, L_i)_{i=1,\dots,k}$ be a bounded splitting of (M, ω, L). The statement of the proposition is a consequence of the following claim:

$$(60) \qquad \min_i A(M_i, \omega_i, L_i) \leq A_b(M, \omega, L).$$

To see that this inequality holds, we choose a map φ as in the definition of a bounded splitting. By Remark 18 we may assume without loss of generality that $(M, \omega) = (\times_i M_i, \oplus_i \omega_i)$ and $\varphi = \mathrm{id}$. For $i = 1, \dots, k$ we choose a pair (g_i, J_i) as in the definition of boundedness of (M_i, ω_i), and we define

$$a_i := \min \big\{ A\big(\mathbb{D}, M_i, \omega_i, J_i, L_i\big), A\big(S^2, M_i, \omega_i, J_i, \emptyset\big)\big\}.$$

Let $i \in \{1, \dots, k\}$. The submanifold $L_i \subseteq M_i$ is Lagrangian. Furthermore, since by hypothesis M is connected, the manifold M_i is connected. Hence it follows from Lemma 29 below that

$$(61) \qquad A(M_i, \omega_i, L_i) \leq a_i.$$

We define $(g, J) := \left(\oplus_i g_i, \oplus_i J_i \right)$. This pair satisfies the conditions of boundedness for (M, ω). Therefore, we have

$$(62) \quad a := \min \left\{ A\left(\mathbb{D}, M, \omega, J, L\right), A\left(S^2, M, \omega, J, \emptyset\right) \right\} \leq A_b(M, \omega, L).$$

It follows from the definition of boundedness that J_i is ω_i-tame, for every i. Therefore, we may apply Lemma 16. It follows that $\min_i a_i \leq a$. Combining this with (62,61), inequality (60) follows. This completes the proof of Proposition 13. $\qquad \square$

3.2. Proofs of Theorem 4 (Regular coisotropic capacity), Corollary 3 (Energy-Gromov-width inequality). We will first prove Theorem 4, since it is used in the proof of Corollary 3. For the proofs of both results we need the following lemma.

Lemma 19. *Let (M, ω) be a bounded and aspherical symplectic manifold of dimension $2n$, $U \subseteq M$ an open subset, and $d \in \{n, \ldots, 2n\}$. Then we have*

$$(63) \qquad\qquad A_\times^{\mathrm{coiso}}(M, \omega, U) \geq A_{\mathrm{coiso}}^d(U, \omega|_U).$$

Proof of Lemma 19. Let $N \subseteq M$ be a regular closed coisotropic submanifold of dimension d such that condition (8) is satisfied. Since by hypothesis (M, ω) is aspherical, Lemma 33 below implies that

$$(64) \qquad\qquad A(M, \omega, N) \geq A(U, \omega|_U, N).$$

Since by hypothesis (M, ω) is bounded, we have $A_\times^{\mathrm{coiso}}(M, \omega, N) \geq A(M, \omega, N)$. Combining this with inequality (64), inequality (63) follows. This proves Lemma 19. $\qquad \square$

The proof of statement (ii) of Theorem 4 involves a certain product of rescaled Stiefel manifolds. These manifolds are given as follows. Let $k, n \in \mathbb{N}$ be such that $k \leq n$, and $a > 0$. We define the *Stiefel manifold of symplectic area a* to be

$$V(k, n, a) := \left\{ \Theta \in \mathbb{C}^{k \times n} \,\big|\, \Theta\Theta^* = \frac{a}{\pi} \mathbf{1}_k \right\}.$$

The proofs of statements (i,iii)) involve the spherical action (or area) spectrum of a symplectic manifold (M, ω). It is given by

$$(65) \quad S(M, \omega) := \inf \left(\left\{ \int_{S^2} u^*\omega \,\big|\, u \in C^\infty(S^2, M) \right\} \cap (0, \infty) \right) \in [0, \infty].$$

Proof of Theorem 4. We start by proving **statement (ii)**. To see that (12) holds, let $\ell \in \mathbb{N}$ and k_i, n_i, $i = 1, \ldots, \ell$ be as in the definition of $\mathrm{k}(n, d)$ (see (9)). We define $a_0 := \pi / \sum_{i=1}^{\ell} k_i$. Let $a \in (0, a_0)$ be a

number. We define $N := \times_{i=1}^{\ell} V(k_i, n_i, a)$. Then N is a closed regular coisotropic submanifold of $B^{2n} \subseteq \mathbb{R}^{2n} = \times_i \mathbb{C}^{k_i \times n_i}$ of dimension d. Since $\mathbb{R}^{2n}$ is simply connected, every loop in an isotropic leaf of N is contractible in $\mathbb{R}^{2n}$.

Hence N satisfies the conditions in the definition of $A_{\text{coiso}}^d(B^{2n}, \omega_0)$. Furthermore, by Remark 31 and Proposition 34 below we have

$$A_{\text{coiso}}^d(B^{2n}, \omega_0) \geq A(B^{2n}, \omega_0, N) = a.$$

Since this holds for arbitrary $a \in (0, a_0)$, it follows that $A_{\text{coiso}}^d(B^{2n}, \omega_0) \geq a_0$. Inequality (12) follows.

To see that (13) holds, note that by Lemma 19, we have

$$A_{\text{coiso}}^d(Z^{2n}, \omega_0) \leq A_{\times}^{\text{coiso}}(\mathbb{R}^{2n}, \omega_0, Z^{2n}).$$

Combining this with Corollary 2 and Remark 38 below, the inequality (13) follows.

We prove **statement (i)**. Let $n \in \mathbb{N}$ and $d \in \{n, \ldots, 2n-1\}$. To prove **monotonicity** of the restriction of A_{coiso}^d, let (M, ω) and (M', ω') be aspherical symplectic manifolds of dimension $2n$, and $\varphi : M' \to M$ a symplectic embedding. In order to prove the inequality $A_{\text{coiso}}^d(M', \omega') \leq A_{\text{coiso}}^d(M, \omega)$, it suffices to show the following. If $N' \subseteq M'$ is a regular closed coisotropic submanifold satisfying (8), then we have

$$(66) \qquad A\big(M, \omega, \varphi(N')\big) \geq A(M', \omega', N').$$

Let N' be such a submanifold. To prove inequality (66), note that by asphericity of (M, ω), we have $S(M, \omega) = \{0\}$. Hence Lemma 33 below implies that

$$S\big(M, \omega, \varphi(N')\big) \subseteq S(M', \omega', N').$$

The inequality (66) follows. This proves (monotonicity).

Conformality follows immediately from the definitions.
Non-triviality follows from the inequalities (12,13). Furthermore, inequality (37) in Proposition 8 implies that $\text{k}(n, 2n-1) = 1$. Hence it follows from inequalities (12,13) that A_{coiso}^{2n-1} is normalized. This proves statement (i).

To prove **statement (iii)**, let $N \subseteq M$ be a closed regular coisotropic submanifold of dimension d. We define $\widetilde{N} := N \times M'$. This is a closed and regular coisotropic submanifold of $\widetilde{M} := M \times M'$, of dimension $d + 2n'$. By Lemma 30 below and asphericity of (M', ω') we have

$$S(M', \omega', M') \subseteq S(M', \omega') = \{0\}.$$

Hence Remark 31 below implies that $A(\widetilde{N}) = A(N)$. The inequality (14) follows. This proves statement (iii) and completes the proof of Theorem 4. $\qquad\square$

We are now ready to prove Corollary 3.

Proof of Corollary 3. By Corollary 2 we have

$$(67) \quad e\big(U \times M', M \times M', \omega \oplus \omega'\big) \geq A_{\times}^{\mathrm{coiso}}\big(M \times M', \omega \oplus \omega', U \times M'\big).$$

We denote $2n := \dim M$ and $2n' := \dim M'$. Since by hypothesis (M, ω) and (M', ω') are bounded and aspherical, the same holds for their product. Hence applying Lemma 19, we obtain

$$(68) \quad A_{\times}^{\mathrm{coiso}}\big(M \times M', \omega \oplus \omega', U \times M'\big) \geq A_{\mathrm{coiso}}^{2(n+n')-1}\big(U \times M', \omega|_U \oplus \omega'\big).$$

Using closedness and asphericity of (M', ω'), Theorem 4(iii) implies that

$$(69) \qquad A_{\mathrm{coiso}}^{2(n+n')-1}\big(U \times M', \omega|_U \oplus \omega'\big) \geq A_{\mathrm{coiso}}^{2n-1}(U, \omega|_U).$$

Using asphericity of U, Theorem 4(i) implies that

$$A_{\mathrm{coiso}}^{2n-1}(U, \omega|_U) \geq w(U, \omega|_U).$$

Combining this with inequalities (67,68,69), the inequality (7) follows. This proves Corollary 3. $\qquad\square$

3.3. Proof of Corollary 5 (Skinny non-squeezing).

Proof of Corollary 5. Without loss of generality we may assume that M is connected. We denote

$$\widetilde{M} := \mathbb{R}^{2n} \times M, \quad \widetilde{\omega} := \omega_0 \oplus \omega.$$

Let $\varphi : U \times M \to \widetilde{M}$ be a symplectic embedding. It suffices to prove the following. Assume that $a_0 > 0$ is such that

$$(70) \qquad\qquad \varphi(U \times M) \subseteq Z^{2n}(a_0) \times M.$$

Then we have

$$(71) \qquad\qquad\qquad a_0 > \pi.$$

To see that this inequality holds, we define $N := \times_{i=1}^{k} S^{2n_i-1} \times M$ and $N' := \varphi(N)$.

Claim 1. *Let $a > 0$ be such that $N' \subseteq Z^{2n}(a) \times M$. Then we have $a \geq \pi$.*

Proof of Claim 1. The inclusion $N' \subseteq Z^{2n}(a) \times M$ and Remarks 37 and 38 below imply that

$$e(N', \widetilde{M}, \widetilde{\omega}) \leq a.$$

Therefore, the inequality $a \geq \pi$ is a consequence of the following claim.

Claim 2. *We have*

$$(72) \qquad\qquad e(N', \widetilde{M}, \widetilde{\omega}) \geq \pi.$$

Proof of Claim 2: N is a closed and regular coisotropic submanifold of $U \times M$, and hence N' is a closed and regular coisotropic submanifold of $\widetilde{M}$. We define $n := \sum_{i=1}^{k} n_i$. By Remark 31 below we have

$$
\begin{aligned}
S(U \times M, \widetilde{\omega}, N) \;&\subseteq\; S(\mathbb{R}^{2n} \times M, \widetilde{\omega}, N) \\
(73) \qquad &=\; \left(\sum_i S(\mathbb{R}^{2n_i}, \omega_0, S^{2n_i - 1}) \right) + S(M, \omega, M).
\end{aligned}
$$

Proposition 34 below implies that $S(\mathbb{R}^{2n_i}, \omega_0, S^{2n_i-1}) = \pi\mathbb{Z}$. Furthermore, by Lemma 29(ii) below and the hypothesis (16), we have $S(M, \omega, M) = S(M, \omega) \subseteq \pi\mathbb{Z}$. Combining this with (73), it follows that

$$(74) \qquad\qquad S(U \times M, \widetilde{\omega}, N) \subseteq \pi\mathbb{Z}.$$

By Lemma 26 below the isotropic leaves of N are the products of the isotropic leaves of $\times_i S^{2n_i-1}$ and M (viewed as a coisotropic submanifold of itself). The latter are single points. Furthermore, the hypothesis $n_i \geq 2$, for every i, implies that $\times_i S^{2n_i-1}$ is simply-connected. It follows that every loop in an isotropic leaf of N is contractible in N, and hence in U. Hence we may apply Lemma 33 below, and conclude that

$$(75) \qquad S(\widetilde{M}, \widetilde{\omega}, N') \subseteq S(U \times M, \widetilde{\omega}, N) + S(\widetilde{M}, \widetilde{\omega}).$$

The hypothesis (16) implies that $S(\widetilde{M}, \widetilde{\omega}) \subseteq \pi\mathbb{Z}$. Combining this with (74,75), it follows that $S(\widetilde{M}, \widetilde{\omega}, N') \subseteq \pi\mathbb{Z}$, and therefore,

$$(76) \qquad\qquad A(\widetilde{M}, \widetilde{\omega}, N') \geq \pi.$$

Since by hypothesis M is closed, the symplectic manifold (M, ω) is bounded. Hence the same holds for $(\widetilde{M}, \widetilde{\omega})$. It follows that

$$A_\times(\widetilde{M}, \widetilde{\omega}, N') \geq A(\widetilde{M}, \widetilde{\omega}, N').$$

Combining this with (76) and applying Theorem 1, inequality (72) follows. This proves Claim 2 and hence Claim 1. $\qquad\square$

Using the assumption (70) and compactness of N', there exists $a < a_0$ such that $N' \subseteq Z^{2n}(a) \times M$. By Claim 1, it follows that $a \geq \pi$. Inequality (71) follows. This proves Corollary 5. $\qquad\square$

3.4. Proof of Theorem 6 (Badly squeezable small sets). For the proof of this theorem, we need the following results. Let $d \in [0, \infty)$. For every $n \in \mathbb{N}$ we abbreviate

$$\mathrm{emb}_d := \mathrm{emb}_d^{Z^{2n}, \omega_0}.$$

Let (M, ω) be a symplectic manifold.

Proposition 20. *For every $n \in \mathbb{N}$, we have*

$$(77) \qquad \mathrm{emb}_d(M, \omega) \geq \mathrm{emb}_d(M \times \mathbb{R}^{2n}, \omega \oplus \omega_0).$$

We post-pone the proof of this result to page 29.

Proposition 21. *If $d \geq 2$ then we have*

$$(78) \qquad \mathrm{emb}_d(M, \omega) \geq A_{\mathrm{coiso}}^d(M, \omega).$$

We post-pone the proof of this result to page 30. For the proof of Theorem 6 we also need the following.

Remark 22. *Let $n \in \mathbb{N}$, $d \in [0, \infty)$, $r > 0$, and $U \subseteq \mathbb{R}^{2n}$ an open subset. Then*

$$\mathrm{emb}_d(rU, \omega_0) = r^2 \, \mathrm{emb}_d(U, \omega_0).$$

This follows from a straight-forward argument. $\square$

For $k, n \in \mathbb{N}$ satisfying $k \leq n$ we denote by

$$V(k, n) := \left\{ \Theta \in \mathbb{C}^{k \times n} \,\middle|\, \Theta\Theta^* = \mathbf{1}_k \right\}$$

the Stiefel manifold of unitary k-frames in $\mathbb{C}^n$.

Proof of Theorem 6. Let $n \in \{2, 3 \ldots\}$ and $d \in [n, \infty)$. By Remark 22 it suffices to prove that

$$(79) \qquad \mathrm{emb}_d(B^{2n}(a), \omega_0) \geq \pi, \quad \forall a > \pi \, \mathrm{k}_{\geq}(n, d).$$

To show that this condition holds, let $a > \pi \, \mathrm{k}_{\geq}(n, d)$. We choose $\ell \in \mathbb{N}$ and $k_1, \ldots, k_\ell \in \mathbb{N}$ such that $\sum_{i=1}^{\ell} k_i = \mathrm{k}_{\geq}(n, d)$. We also choose $n_1, \ldots, n_\ell$ satisfying (10,19). We define $n' := \sum_i k_i n_i - n$. By the first inequality in (19) we have $n' \geq 0$. Propositions 20 and 21 imply that

$$\begin{aligned}
\mathrm{emb}_d(B^{2n}(a), \omega_0) \;&\geq\; \mathrm{emb}_d(B^{2n}(a) \times \mathbb{R}^{2n'}, \omega_0) \\
(80) \qquad\qquad\qquad &\geq\; A_{\mathrm{coiso}}^d(B^{2n}(a) \times \mathbb{R}^{2n'}, \omega_0).
\end{aligned}$$

By the inequalities (10) the Stiefel manifolds $V(k_i, n_i)$ are well-defined. We define $N := V(k_1, n_1) \times \ldots \times V(k_\ell, n_\ell)$.

Claim 1. *We have*

$$A_{\text{coiso}}^d(B^{2n}(a) \times \mathbb{R}^{2n'}, \omega_0) \geq A(\mathbb{R}^{2(n+n')}, \omega_0, N). \tag{81}$$

Proof of Claim 1. Note that N is a regular closed coisotropic submanifold of $B^{2n}(a) \times \mathbb{R}^{2n'}$ of dimension $\sum_i k_i(2n_i - k_i)$. Hence by the second inequality in (19) we have $\dim N \leq d$. Furthermore, since $B^{2n}(a) \times \mathbb{R}^{2n'}$ is contractible, condition (8) is satisfied. Therefore, by definition, we have

$$A_{\text{coiso}}^d(B^{2n}(a) \times \mathbb{R}^{2n'}, \omega_0) \geq A(B^{2n}(a) \times \mathbb{R}^{2n'}, \omega_0, N).$$

The right hand side is bounded below by $A(\mathbb{R}^{2(n+n')}, \omega_0, N)$. Inequality (81) follows. This proves Claim 1. □

By Remark 31 and Proposition 34 below we have

$$A(\mathbb{R}^{2(n+n')}, \omega_0, N) = \pi.$$

Combining this with (80) and Claim 1, inequality (79) follows. This proves Theorem 6. □

Proof of Proposition 20. Let $\widetilde{X} \subseteq M \times \mathbb{R}^{2n}$ be a compact subset of Hausdorff dimension at most d. We denote $2m := \dim M$, $\widetilde{M} := M \times \mathbb{R}^{2n}$, $\widetilde{\omega} := \omega \oplus \omega_0$, by $\text{pr} : \widetilde{M} \to M$ the projection onto the first component, and $X := \text{pr}(\widetilde{X})$. Then X is a compact subset of M. Furthermore, by standard results (cf. [**Fed**]), the Hausdorff dimension of X does not exceed that of $\widetilde{X}$, and thus is at most d.

Claim 1. *We have*

$$\text{emb}^{Z^{2m}, \omega_0}(M, \omega, X) \geq \text{emb}^{Z^{2(m+n)}, \omega_0}(\widetilde{M}, \widetilde{\omega}, \widetilde{X}). \tag{82}$$

Proof of Claim 1. Let $a > 0$. Assume that there exists a pair (U, φ), where $U \subseteq M$ is an open neighborhood of X and $\varphi : U \to Z^{2m}(a)$ is a symplectic embedding. We define $\widetilde{U} := U \times \mathbb{R}^{2n}$ and $\widetilde{\varphi} := \varphi \times \text{id}_{\mathbb{R}^{2n}}$. Then $\widetilde{X} \subseteq \widetilde{U}$ and $\widetilde{\varphi}$ is a symplectic embedding of $\widetilde{U}$ into $Z^{2(m+n)}(a)$. The inequality (82) follows. This proves Claim 1. □

Taking the supremum over all compact sets $\widetilde{X} \subseteq M \times \mathbb{R}^{2n}$ of Hausdorff dimension at most d, Claim 1 implies inequality (77). This completes the proof of Proposition 20. □

The idea of proof of Proposition 21 is the following. Let $N \subseteq M$ be a d-dimensional closed regular coisotropic submanifold satisfying (8). The idea is to glue finitely many disks in such a way that every loop in an isotropic fiber of N is contractible in the resulting subset of M.

This is possible because of (8) and regularity and closedness of N. The statement of Proposition 21 will then be a consequence of Theorem 1, Lemma 33, and Remark 38 below.

Proof of Proposition 21. Let $N \subseteq M$ be a non-empty closed regular coisotropic submanifold of dimension at most d, satisfying (8). Inequality (78) is a consequence of the following claim.

Claim 1. *We have*

$$\text{(83)} \qquad \qquad \text{emb}_d(M, \omega) \geq A(N).$$

Proof of Claim 1: Without loss of generality, we may assume that N is connected. We choose an isotropic leaf $F \subseteq N$ and a point $x_0 \in F$. Regularity of N implies that F is a smooth submanifold of N. It is closed, since N is closed. It follows that the fundamental group of F with base point x_0 is finitely generated. Therefore, there exists a finite set S of smooth loops $x : S^1 \subseteq \mathbb{C} \to F$ satisfying $x(1) = x_0$, whose continuous homotopy classes with fixed base point generate $\pi_1(F, x_0)$.

The assumption (8) implies that for every $x \in S$ there exists a smooth map $u_x : \mathbb{D} \to M$ satisfying $u_x|_{S^1} = x$. We choose such a collection of maps $(u_x)_{x \in S}$ and define

$$X := N \cup \bigcup_{x \in S} u_x(\mathbb{D}) \subseteq M.$$

This set is compact. Furthermore, a standard result (cf. [**Fed**, p. 176]) implies that $u_x(\mathbb{D})$ has Hausdorff dimension at most 2. Since by hypothesis $d \geq 2$, it follows that X has Hausdorff dimension at most d. We denote $2n := \dim M$. Assume that $a > 0$ is such that there exists a pair (U, φ), where $U \subseteq M$ is an open neighborhood of X and $\varphi : U \hookrightarrow Z^{2n}(a)$ is a symplectic embedding. Using the fact $A(N) = A(M, \omega, N) \leq A(U, \omega|_U, N)$, inequality (83) is a consequence of the following claim.

Claim 2. *We have*

$$\text{(84)} \qquad \qquad a \geq A(U, \omega|_U, N).$$

Proof of Claim 2: We choose a pair (U, φ) as above.

Claim 3. *Every continuous loop in an isotropic leaf of N is contractible in X.*

In the proof of this claim we use the following notation. Let X be a set and $x : S^1 \to X$ a map. We define $x^{-1} : S^1 \to X$ by $x^{-1}(z) := x(\overline{z})$.

Proof of Claim 3. Let x be such a loop. **Assume first that** $x(S^1) \subseteq F$. It follows from our choice of the set S that there exist $\ell \in \mathbb{N}_0$, $x_1, \ldots, x_\ell \in S$, and $\varepsilon_1, \ldots, \varepsilon_\ell \in \{1, -1\}$, such that x is continuously homotopic inside F to $x_1^{\varepsilon_1} \cdots x_\ell^{\varepsilon_\ell}$. Since X contains the images $u_{x_i}(\mathbb{D})$, for $i = 1, \ldots, \ell$, it follows that x is contractible in X.

Consider now the general situation. Since N is path-connected, the same holds for N_ω. Hence there exists a path $\overline{y} \in C([0,1], N_\omega)$ such that $\overline{y}(0)$ is the leaf through x_0, and $\overline{y}(1)$ is the leaf containing $x(S^1)$.

By regularity of N there exists a unique manifold structure on the set of isotropic leaves N_ω of N, such that the canonical projection $\pi_N : N \to N_\omega$ is a smooth submersion. (See Lemma 15 in [**Zi**].) Since N is closed, C. Ehresmann's Theorem implies that π_N is a smooth (locally trivial) fiber bundle. (See the proposition on p. 31 in [**Eh**].) It follows that π_N has the continuous homotopy lifting property. Hence there exists $u \in C([0,1] \times S^1, N)$ such that $\mathrm{pr}_N \circ u(t, z) = \overline{y}(t)$, for every $t \in [0,1]$ and $z \in S^1$, and $u(1, \cdot) = x$. By what we already proved, the loop $u(0, \cdot)$ is contractible in X. It follows that the same holds for $u(1, \cdot) = x$. This proves Claim 3. $\qquad\square$

Using Claim 3 and asphericity of $(\mathbb{R}^{2n}, \omega_0)$, Lemma 33 implies that

(85) $$A(U, \omega|_U, N) \leq A(\mathbb{R}^{2n}, \omega_0, \varphi(N)).$$

Furthermore, the coisotropic submanifold $\varphi(N) \subseteq \mathbb{R}^{2n}$ is non-empty, closed, and regular. Hence Theorem 1 implies that

(86) $$A(\mathbb{R}^{2n}, \omega_0, \varphi(N)) \leq e\big(\varphi(N), \mathbb{R}^{2n}, \omega_0\big).$$

By Remark 38, we have

$$e\big(\varphi(N), \mathbb{R}^{2n}, \omega_0\big) \leq e\big(Z^{2n}(a), \omega_0, \mathbb{R}^{2n}, \omega_0\big) \leq a.$$

Combining this with (85,86), inequality (84) follows. This proves Claim 2 and hence Claim 1, and concludes the proof of Proposition 21. $\quad\square$

3.5. Proof of Corollary 7 (Stably exotic form). We need the following results, in which (M, ω) and (M', ω') are symplectic manifolds.

Corollary 23. *Assume that (M, ω) is (geometrically) bounded and every compact subset of M is Hamiltonianly displaceable. Then (M, ω) is* not *(coisotropically) infinite.*

Proof. This is a direct consequence of Theorem 1 and Lemma 35. $\quad\square$

Proposition 24. *The following statements hold.*

(i) *(Aspherical manifold) If (M, ω) is closed and aspherical then it is strongly infinite.*

(ii) *(Product) The product of two infinite/ strongly infinite symplectic manifolds is infinite/ strongly infinite.*

(iii) *(Embedding) Assume that (M,ω) is aspherical and (M',ω') is strongly infinite and embeds into (M,ω). Then (M,ω) is strongly infinite.*

Proof of Proposition 24. **Statement (i)** follows from Lemma 29(ii) below, using that $N := M$ is a regular coisotropic submanifold of itself and the fact that the isotropic leaves of M are single points. **Statement (ii)** follows from Remark 31 and Lemma 26 below. **Statement (iii)** follows from Lemma 33 below. $\qquad\square$

Proof of Corollary 7. Assume that ω is a (coisotropically) infinite form on $\mathbb{R}^{2n}$. Since $\mathbb{R}^{2n}$ is simply connected, it follows that ω is strongly infinite. Let $X, \sigma, \sigma', \varphi$ be as in the definition of stable exoticness. We show that condition (20) holds. Consider first the **case** in which σ' is *not* aspherical. Then (20) holds, since $\sigma \oplus \omega_0$ is aspherical.

Consider now the **case** in which σ' is aspherical. Then it follows from Proposition 24(i,ii) that $(M,\widetilde{\Omega}) := \big(X \times \mathbb{R}^{2n}, \sigma' \oplus \omega\big)$ is strongly infinite. We define $\Omega := \sigma \oplus \omega_0$. The symplectic manifold (M,Ω) is bounded, since it is the product of two bounded symplectic manifolds. Furthermore, every compact subset of M is displaceable in an Ω-Hamiltonian way, since $(\mathbb{R}^{2n},\omega_0)$ has this property. Therefore, by Corollary 23, (M,Ω) is not infinite.

Since, by assumption, σ is aspherical, Ω is aspherical. Combining these facts and using that $(M,\widetilde{\Omega})$ is strongly infinite, it follows from Proposition 24(iii) that $(M,\widetilde{\Omega})$ does not embed into (M,Ω). Therefore, Condition (20) holds. This completes the proof of Corollary 7. $\qquad\square$

Appendix A. Auxiliary results

A.1. (Pre-)symplectic geometry. The next result is used in the proofs of Lemmas 26 and 28 below.

Let V be a finite dimensional vector space and ω a skew-symmetric 2-form on V. We define

$$V^\omega := \big\{v \in V \,\big|\, \omega(v,w) = 0,\ \forall w \in V\big\}, \quad \operatorname{corank}\omega := \dim V^\omega.$$

By a *presymplectic structure* on a manifold M we mean a closed two-form ω on M such that $\operatorname{corank}\omega_x$ does not depend on $x \in M$. Note that if (M,ω) is a symplectic manifold and $N \subseteq M$ is a coisotropic submanifold then $\omega|_N$ is a presymplectic structure on N of corank equal to the codimension of N in M.

For a presymplectic manifold (M, ω) we denote by $R^{M,\omega} \subseteq M \times M$ its isotropy relation. By definition, this is the set of all pairs $(x(0), x(1))$, where $x \in C^\infty([0,1], M)$ is a path satisfying $\dot{x}(t) \in T_{x(t)} M^\omega$, for every $t \in [0,1]$. For $x \in M$ we denote by $M_x^\omega \subseteq M$ the isotropic leaf through x, i.e., the $R^{M,\omega}$-equivalence class of x.

We call (M, ω) *regular* if $R^{M,\omega}$ is a closed subset and a submanifold of $M \times M$. Equivalently, there exists a smooth structure on the set of isotropic leaves M_ω for which the canonical projection $\pi : M \to M_\omega$ is a smooth submersion. In this case we define ω_M to be the unique two-form on M_ω such that $\pi^* \omega_M = \omega$. This is a symplectic form.

For $i = 0, 1$ let (M_i, ω_i) be a presymplectic manifold. We define the swap map

(87) $S : M_1 \times M_1 \times M_2 \times M_2 \to M_1 \times M_2 \times M_1 \times M_2$

by $S(x_1, y_1, x_2, y_2) := (x_1, x_2, y_1, y_2)$.

Lemma 25. *We have that $R^{M_1 \times M_2, \omega_1 \oplus \omega_2} = S\big(R^{M_1, \omega_1} \times R^{M_2, \omega_2}\big)$.*

Proof of Lemma 25. This follows from a straight-forward argument. $\square$

The next result is an immediate consequence of Lemma 25. It is used in the proofs of Corollary 5, Proposition 24, and Lemma 27 and Remark 31 below.

Lemma 26. *For every pair $(x_1, x_2) \in M_1 \times M_2$, we have*

$$(M_1 \times M_2)_{(x_1, x_2)}^{\omega_1 \oplus \omega_2} = (M_1)_{x_1}^{\omega_1} \times (M_2)_{x_2}^{\omega_2}.$$

The next two results are used in the proof of Proposition 12.

Lemma 27. *The identity map on $M_1 \times M_2$ descends to a bijection*

$$(M_1)_{\omega_1} \times (M_2)_{\omega_2} \to (M_1 \times M_2)_{\omega_1 \oplus \omega_2}.$$

If (M_i, ω_i) is regular for $i = 1, 2$, then this map is a symplectomorphism with respect to $(\omega_1)_{M_1} \oplus (\omega_2)_{M_2}$ and $(\omega_1 \oplus \omega_2)_{M_1 \times M_2}$.

Proof of Lemma 27. This follows from a straight-forward argument, using Lemma 26 and the definitions of the smooth and symplectic structures on the quotients. $\square$

Lemma 28. *If the presymplectic manifold $\big(M_1 \times M_2, \omega_1 \oplus \omega_2\big)$ is regular then (M_i, ω_i) is also regular, for $i = 1, 2$.*

Proof of Lemma 28. It follows from Lemma 25 and Remark 43 below that R^{N_i,ω_i} is a closed subset of $N_i \times N_i$, for $i = 1, 2$. Furthermore, Lemmas 25 and 44 imply that R^{N_i,ω_i} is a submanifold of $N_i \times N_i$, for $i = 1, 2$. It follows that (M_i, ω_i) is regular, for $i = 1, 2$. This proves Lemma 28. □

The next lemma is used in the proofs of Propositions 13 and 24. Let (M, ω) be a symplectic manifold. Recall the definitions (31,65).

Lemma 29. *Assume that M is connected. Then the following statements hold:*

(i) For every non-empty coisotropic submanifold $N \subseteq M$ we have

$$(88) \qquad S(M, \omega, N) \supseteq S(M, \omega).$$

(ii) We have

$$S(M, \omega, M) = S(M, \omega).$$

For the proof of this lemma, we need the following result, which was also used in the proof of Theorem 4.

Lemma 30. *We have*

$$(89) \qquad S(M, \omega, M) \subseteq S(M, \omega).$$

Proof of Lemma 30. Let $u \in C^\infty(\mathbb{D}, M)$ be such that $u(S^1)$ is contained in some leaf of M. This leaf consists of a single point $x_0 \in M$. We identify $S^2 \cong \mathbb{R}^2 \cup \{\infty\}$ and choose a map $f \in C^\infty(\mathbb{D}, S^2)$ that restricts to an orientation preserving diffeomorphism from B^2 to $\mathbb{R}^2$. We also choose a map $\rho \in C^\infty([0, 1], [0, 1])$ such that $\rho \equiv 1$ in a neighborhood of 0 and $\rho(r) = 1/r$ in a neighborhood of 1. We define

$$u' : \mathbb{D} \to M, \quad u'(z) := u\big(\rho(|z|)z\big).$$

This map is constantly equal to x_0 in a neighborhood of S^1. Hence there exists a unique smooth map $v : S^2 \to M$ satisfying $v \circ f = u'|_{B^2}$. We have

$$\int_{S^2} v^*\omega = \int_{\mathbb{D}} u'^*\omega = \int_{\mathbb{D}} u^*\omega.$$

Here in the second equality we used the fact that u' is homotopic to u with fixed restriction to S^1. The inclusion (89) follows. This proves Lemma 30. □

Proof of Lemma 29. We prove **statement (i)**: Let $u \in C^\infty(S^2, M)$ be a map. We identify S^2 with $\mathbb{R}^2 \cup \{\infty\}$. By Lemma 39 below there exists

a map $v \in C^\infty(S^2, M)$ that is smoothly homotopic to u and satisfies $v(\infty) \in N$.

We choose a smooth map $f : \mathbb{D} \to S^2$ that maps the interior $B_1 \subseteq \mathbb{D}$ diffeomorphically and in an orientation preserving way onto $\mathbb{R}^2$. Then the map $v \circ f : \mathbb{D} \to M$ satisfies $v \circ f(S^1) \subseteq N$. Furthermore,

$$\int_{S^2} u^* \omega = \int_{S^2} v^* \omega = \int_{\mathbb{D}} (v \circ f)^* \omega.$$

The inclusion (88) follows. This proves (i).

Statement (ii) follows from statement (i) and Lemma 30. This proves Lemma 29. $\square$

The next remark is used in the proofs of most main results of this paper.

Remark 31. *Let (M, ω) and (M', ω') be symplectic manifolds, and $N \subseteq M$ and $N' \subseteq M'$ coisotropic submanifolds. Then*

$$S\big(M \times M', \omega \oplus \omega', N \times N'\big) = S(M, \omega, N) + S(M', \omega', N').$$

This follows from a straight-forward argument, using Lemma 26. $\square$

Remark 32. *Let (M, ω) and (M', ω') be symplectic manifolds of the same dimension, $N' \subseteq M'$ a coisotropic submanifold, and $\varphi : M' \to M$ a symplectic embedding. The action spectrum $S(M', \omega', N')$ is contained in $S(M, \omega, \varphi(N'))$. This follows from a straight-forward argument.* $\square$

The next lemma gives a condition under which the opposite inclusion holds up to a correction term. It is used in the proofs of Theorem 4, Corollary 5, and Propositions 24 and 21.

Lemma 33. *If every continuous loop in a leaf of N' is contractible in M' then we have*

$$S(M, \omega, \varphi(N')) \subseteq S(M', \omega', N') + S(M, \omega),$$

where the action spectrum $S(M, \omega)$ is defined as in (65).

Proof of Lemma 33. Let $u \in C^\infty(\mathbb{D}, M)$ be a map such that $u(S^1)$ is contained in some isotropic leaf of $N := \varphi(N')$. It suffices to prove that

(90)
$$\int_{\mathbb{D}} u^* \omega \in S(M', \omega', N') + S(M, \omega).$$

To see that this condition is satisfied, note that by our hypothesis the loop $x' := \varphi^{-1} \circ (u|_{S^1}) : S^1 \to M'$ is contractible in M'. It follows that

there exists a map $u' \in C^\infty(\mathbb{D}, M')$ such that $u'|_{S^1} = x'$. We denote by $\overline{\mathbb{D}}$ the disk with the reversed orientation and by $\mathbb{D}\#\overline{\mathbb{D}}$ the smooth oriented manifold obtained by concatenating the two disks along their boundary. We define $f : \mathbb{D}\#\overline{\mathbb{D}} \to M$ to be the concatenation of u and $\varphi \circ u'$. It follows that

$$\text{(91)} \qquad \int_{\mathbb{D}\cup\overline{\mathbb{D}}} f^*\omega = \int_{\mathbb{D}} u^*\omega - \int_{\mathbb{D}} (\varphi \circ u')^*\omega = \int_{\mathbb{D}} u^*\omega - \int_{\mathbb{D}} u'^*\omega'.$$

Since $\mathbb{D}\cup\overline{\mathbb{D}}$ is diffeomorphic to S^2, we have $\int_{\mathbb{D}\cup\overline{\mathbb{D}}} f^*\omega \in S(M,\omega)$. Combining this with (91), the inclusion (90) follows. $\qquad\square$

The next result is used in the example on page 11 and the proofs of Corollaries 3 and 5, inequality (12) in Theorem 4, and Theorem 6. For $k, n \in \mathbb{N}$ satisfying $k \leq n$ we denote by

$$V(k,n) := \left\{ \Theta \in \mathbb{C}^{k\times n} \,\middle|\, \Theta\Theta^* = \mathbf{1}_k \right\}$$

the Stiefel manifold of unitary k-frames in $\mathbb{C}^n$.

Proposition 34. *We have*

$$A(\mathbb{C}^{k\times n}, \omega_0, V(k,n)) = \pi.$$

Proof. For a proof we refer to [**Zi**, Proposition 1.3]. $\qquad\square$

The next lemma is used in Remark 11.

Lemma 35. *Let (M,ω) be a symplectic manifold, $K \subseteq M$ a compact subset, $\varphi \in \mathrm{Ham}(M,\omega)$, and $\varepsilon > 0$. Then there exists $\psi \in \mathrm{Ham}_c(M,\omega)$ such that*

$$\text{(92)} \qquad\qquad \psi|_K = \varphi|_K, \quad \|\psi\|_\omega^c \leq \|\varphi\|_\omega + \varepsilon.$$

(Here our convention is that $\infty + \varepsilon := \infty$.)

For the proof of this lemma, we need the following.

Remark 36. Let (M,ω) be a symplectic manifold, $H_0, H \in C^\infty(M,\mathbb{R})$, $U \subseteq M$ an open subset, and $a > 0$. Assume that $\varphi_{H_0}^t$ (the Hamiltonian time t-flow of H_0) is well-defined on U for $t \in [0,a]$, and that there exists a function $c \in C^\infty([0,1],\mathbb{R})$ such that

$$H \circ \varphi_{H_0}^t = H_0 \circ \varphi_{H_0}^t + c(t)$$

on U. Then φ_H^t is well-defined on U and $\varphi_H^t = \varphi_{H_0}^t$ on U, for $t \in [0,a]$. This follows from a straight-forward argument. $\square$

Proof of Lemma 35. Without loss of generality we may assume that M is connected. We choose a function $H_0 \in C^\infty([0,1] \times M, \mathbb{R})$ such that $\varphi_{H_0}^1 = \varphi$ and $\|H_0\| \leq \|\varphi\|_\omega + \varepsilon$. Since the set $K_0 := \bigcup_{t \in [0,1]} \varphi_{H_0}^t(K) \subseteq M$ is compact, it has an open neighborhood $U_0 \subseteq M$ with compact closure. We choose an open neighborhood $U_1 \subseteq M$ of $\overline{U_0}$ with compact closure.

We choose a function $f \in C^\infty(M, [0,1])$ such that $f|_{M \setminus U_1} \equiv 0$ and $f|_{\overline{U_0}} \equiv 1$. We fix a point $x_0 \in M$ and define

$$H : [0,1] \times M \to \mathbb{R}, \quad H(t,x) := f(x)\big(H_0(t,x) - H_0(t,x_0)\big).$$

Then the support of H is contained in $\overline{U_1}$ and hence compact. Furthermore, for $t \in [0,1]$ and $x \in U_0$ we have $H(t, \varphi_{H_0}^t(x)) = H_0(t, \varphi_{H_0}^t(x)) - H_0(t,x_0)$. Therefore, by Remark 36 we have $\varphi_H^1(x) = \varphi_{H_0}^1(x)$, for every $x \in U_0$, and therefore the first condition in (92) holds.

Finally, observe that

$$\max_{x \in M} H(t,x) \leq \sup_{x \in M} H_0(t,x) - H_0(t,x_0).$$

Combining this with a similar inequality for $\min_{x \in M} H(t,x)$, it follows that $\|H\| \leq \|H_0\|$. Since $\|H_0\| \leq \|\varphi\|_\omega + \varepsilon$, the second condition in (92) follows. This proves Lemma 35. $\square$

The next two remarks were used in the example on page 11 and the proofs of Theorem 4(ii), Proposition 21, and Corollary 7.

Remark 37. *Let (M, ω) and (M', ω') be symplectic manifolds and $X \subseteq M$ a subset. Then we have*

$$e\big(X \times M', M \times M'\big) \leq e(X, M).$$

This follows from a straight-forward argument. $\square$

Remark 38. *For every $n \in \mathbb{N}$ and $a > 0$ we have*

$$e(Z^{2n}(a), \mathbb{R}^{2n}) \leq a.$$

This follows from a straight-forward argument. $\square$

A.2. Topology and manifolds. The next result was used in the proof of Lemma 29.

Lemma 39. *Let M and M' be manifolds, $x_0 \in M$, $x_0' \in M'$, and $u \in C^\infty(M', M)$. If M is connected then u is smoothly homotopic to a map $v \in C^\infty(M', M)$ satisfying $x_0 = v(x_0')$.*

For the proof of this lemma we need the following two results. Let $n \in \mathbb{N}$, M be a manifold, and $u \in C^\infty(\mathbb{R}^n, M)$.

Lemma 40. *There exists a map $h \in C^\infty([0,1] \times \mathbb{R}^n, M)$ satisfying*

$$h(1,x) = u(0), \quad \forall x \in B_1^n,$$

$$(93) \quad h(t,x) = u(x), \quad \forall (t,x) \in (\{0\} \times \mathbb{R}^n) \cup ([0,1] \times (\mathbb{R}^n \setminus B_2^n)).$$

Proof of Lemma 40. We choose a function $f \in C^\infty([0,1] \times [0,\infty), [0,\infty))$ satisfying

$$f(1,a) = 0, \quad \forall a \in [0,1],$$

$$f(t,a) = 1, \quad \forall (t,a) \in (\{0\} \times [0,\infty)) \cup ([0,1] \times [4,\infty)).$$

We define $h(t,x) := u(f(t,|x|^2)x)$, for every $t \in [0,1]$ and $x \in \mathbb{R}^n$. This map has the required properties. This proves Lemma 40. $\qquad\square$

Lemma 41. *Let $\gamma \in C^\infty([0,1], M)$ be a path. Assume that $u(x) = \gamma(0)$, for every $x \in B_1^n$. Then there exists a map $h \in C^\infty([0,1] \times \mathbb{R}^n, M)$ satisfying $h(1,0) = \gamma(1)$ and (93).*

Proof of Lemma 41. We choose a function $f \in C^\infty([-1,1], [0,1])$ satisfying $f(t) = 0$ for $t \leq 0$ and $f(1) = 1$. We define $h : [0,1] \times \mathbb{R}^n \to M$ by

$$h(t,x) := \begin{cases} \gamma \circ f(t - |x|^2), & \text{if } |x| \leq 1, \\ u(x), & \text{otherwise.} \end{cases}$$

This map has the required properties. This proves Lemma 41. $\qquad\square$

Proof of Lemma 39. Since M is connected, there exists a path $\gamma \in C^\infty([0,1], M)$ satisfying $\gamma(0) = u(x_0')$ and $\gamma(1) = x_0$. The statement of the lemma follows from an argument using a chart around x_0' and Lemmas 40 and 41. (We concatenate the homotopies provided by these Lemmas and smoothen the resulting homotopy.) This proves Lemma 39. $\qquad\square$

The next two remarks and Lemma 44 below were used in the proof of Proposition 12. These statements follow from elementary arguments.

Remark 42. *If X_1 and X_2 are non-empty topological spaces such that $X_1 \times X_2$ is compact then X_1 and X_2 are compact.* $\square$

Remark 43. *For $i = 1, 2$ let X_i be a topological space and $\emptyset \neq A_i \subseteq X_i$ a subset. If $A_1 \times A_2 \subseteq X_1 \times X_2$ is closed then A_1 and A_2 are closed.* $\square$

Lemma 44. *For $i = 1, 2$ let M_i be a manifold and $X_i \subseteq M_i$ a non-empty subset. Assume that $X_1 \times X_2 \subseteq M_1 \times M_2$ is a submanifold. Then X_i is a submanifold of M_i, for $i = 1, 2$.*

The next lemma was used to ensure that the Hofer norm $\| \cdot \|_\omega$ (see (28)) is well-defined.

Lemma 45. *Let X be a topological space and $f : [0,1] \times X \to \mathbb{R}$ a continuous function. Assume that there exists a sequence of compact subsets $K_\nu \subseteq X$, $\nu \in \mathbb{N}$ such that $\bigcup_\nu K_\nu = X$. Then the map*

$$[0,1] \ni t \mapsto \sup_{x \in X} f(t,x)$$

is Borel measurable.

Proof. This follows from an elementary argument. $\square$

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