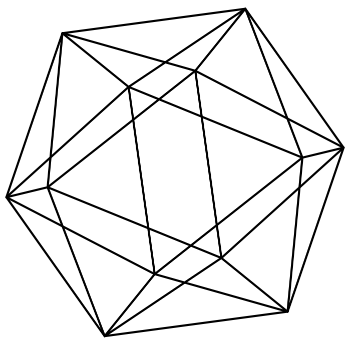
Max-Planck-Institut für Mathematik
Bonn

Mathematische Arbeitstagung 2011

10. Arbeitstagung der zweiten Serie

24. Juni bis 1. Juli 2011



Max-Planck-Institut für Mathematik
Preprint Series 2011 (34)

Mathematische Arbeitstagung 2011

10. Arbeitstagung der zweiten Serie

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MPIM 11-34

Contents

Max-Planck-Institut für Mathematik

Mathematische Arbeitstagung

June 24–30, 2011

Second Announcement

The opening of the Arbeitstagung will take place in the Großer Hörsaal, Mathematisches Institut, Universität Bonn, Wegelerstr. 10, at 3.30 p.m. on Friday, June 24. After the opening and the first program discussion there will be a short break for tea, followed by

THE OPENING LECTURE, Großer Hörsaal, at 5 p.m., by Maxim Kontsevich (IHES).

The other lectures will be decided during the course of the meeting, according to the traditional method of the Arbeitstagung. The lectures continue throughout the week, including Saturday, June 25, and Sunday, June 26. Since Don Zagier will celebrate his 60th birtday on June 29th, the AT will have a number-theoretical emphasis. Apart from the opening lecture, there will be some further invited lectures, by M. Atiyah, H. Cohen, A. Goncharov, B. Gross, T. Ibukiyama and G. van der Geer (amongst others).

From Tuesday June 28 onwards lectures will be given at the Max-Planck-Institute for Mathematics. There are also some non-mathematical events during the week, to which all are invited:

BOAT TRIP ON THE RHINE: on Monday, June 27. The boat trip is to Koblenz and back and will take the whole day. There will be some lectures on the boat.

RECTOR'S PARTY: on Wednesday, June 29, at 8 p.m., Festsaal der Universität

Anyone who wishes to attend the Arbeitstagung and would like us to reserve a hotel room, or who wants to request financial support, should fill in the registration form at our homepage. Your wishes for hotel reservations should arrive here by the beginning of May. Since our resources are rather restricted, applications of requesting financial support will be treated on the basis of need and availability of funds. You will be informed about the financial support as soon as possible. If you have any questions, please contact us by e-mail.

Werner Ballmann *Gerd Faltings* *Peter Teichner* *Don Zagier*

$$h(d_K) > \frac{1}{7000} \prod_{p \mid d_K} \left(1 - \frac{[2\sqrt{p}]}{p+1}\right) \log |d_K|$$

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More information and registration:

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1 Program of the Mathematische Arbeitstagung 2011

Fri, 24 Jun 2011

14:00 - 14:30	MPI Tea Room *Tea*
15:30 - 16:15	Former Mathematical Institute, Großer Hörsaal, Wegelerstr. 10 *Opening and first program discussion*
16:15 - 17:00	Former Mathematical Institute, Wegelerstrasse 10 *Break*
17:00 - 18:00	Former Mathematical Institute, Großer Hörsaal, Wegelerstr. 10 MAXIM KONTSEVICH (IHES) *Opening lecture: Noncommutative identities*

Sat, 25 Jun 2011

10:15 - 11:15	Former Mathematical Institute, Großer Hörsaal, Wegelerstr. 10 TOMOYOSHI IBUKIYAMA (OSAKA UNIVERSITY) *Exact critical values of a symmetric fourth L-function and Zagier's conjecture*
11:15 - 12:00	Former Mathematical Institute, Wegelerstrasse 10 *Break*
12:00 - 13:00	Former Mathematical Institute, Großer Hörsaal, Wegelerstr. 10 MARTIN MÖLLER (UNIVERSITÄT FRANKFURT) *Teichmüller curves*
15:30 - 16:00	MPI Tea Room *Tea*
17:00 - 18:00	Former Mathematical Institute, Großer Hörsaal, Wegelerstr. 10 HIDEKAZU FURUSHO (NAGOYA UNIVERSITY) *Double shuffle for associators*

Sun, 26 Jun 2011

10:15 - 10:30	Former Mathematical Institute, Großer Hörsaal, Wegelerstr. 10 *Program discussion II*
10:30 - 11:30	Former Mathematical Institute, Großer Hörsaal, Wegelerstr. 10 ALEXANDER GONCHAROV (BROWN UNIVERSITY) *Hodge correlators for local systems*
11:30 - 12:00	Former Mathematical Institute, Wegelerstrasse 10 *Break*
12:00 - 13:00	Former Mathematical Institute, Großer Hörsaal, Wegelerstr. 10 VALENTIN BLOMER (UNIVERSITÄT GÖTTINGEN) *Bounding eigenfunctions on arithmetic surfaces*
15:30 - 16:00	MPI Tea Room *Tea*
17:00 - 18:00	Former Mathematical Institute, Großer Hörsaal, Wegelerstr. 10 STAVROS GAROUFALIDIS (GEORGIA INSTITUTE OF TECHNOLOGY) *Quantum knot invariants*

Mon, 27 Jun 2011

09:00 - 14:00	*Boat trip to Koblenz (cast off 09:00 sharp!)*
10:00 - 11:00	BENEDICT GROSS (HARVARD) *Talk on the boat: Don Zagier's work on singular moduli*
12:00 - 13:00	GUNTHER CORNELISSEN (UTRECHT) *Talk on the boat: Classfield theory as a dynamical system*
14:00 - 17:30	*Stopover in Koblenz*
17:30 - 20:00	*Return trip to Bonn*
18:00 - 19:00	GAËTAN BOROT (CEA, FRANCE) *Talk on the boat: Loop equations and spectral curves*

Tue, 28 Jun 2011

10:15 - 10:30	MPI Lecture Hall *Program discussion III*
10:30 - 11:30	MPI Lecture Hall DMITRY LEBEDEV (ITEP MOSCOW AND AARHUS) *Arithmetic geometry and topological strings*
11:30 - 12:00	MPI Tea Room *Tea*
12:00 - 13:00	MPI Lecture Hall ALEXANDER BEILINSON (UNIVERSITY OF CHICAGO) *The p-adic Poincaré lemma and the period map*
15:30 - 16:00	MPI Lecture Hall *The future of the AT (public discussion)*
16:00 - 16:30	MPI Tea Room *Tea*
17:00 - 18:00	MPI Lecture Hall JOACHIM SCHWERMER (UNIVERSITÄT WIEN) *Geometric cycles and discrete groups*

Wed, 29 Jun 2011

10:15 - 11:15	MPI Lecture Hall Gerard van der Geer (University of Amsterdam) *Modular Forms of Genus Three*
11:15 - 12:00	MPI Tea Room *Tea*
12:00 - 13:00	MPI Lecture Hall Tudor Dimofte (Princeton) *Chern-Simons with complex gauge groups*
16:00 - 16:30	MPI Tea Room *Tea*
17:00 - 18:00	MPI Lecture Hall Francis Brown (CNRS, IMJ) *On multiple zeta values*
20:00 - 23:00	*Rector's Party, Festsaal der Universität*

Thu, 30 Jun 2011

10:00 - 11:00	Former Mathematical Institute, Großer Hörsaal, Wegelerstr. 10 Sir Michael Atiyah *Cones and signatures - old formulae revisited*
12:00 - 13:00	MPI Lecture Hall Manjul Bhargava (Princeton) *Selmer groups*
13:00 - 13:30	MPI Tea Room *Unveiling of a piece of art*
15:00 - 16:00	MPI Lecture Hall Anantharam Raghuram (Oklahoma State University) *Special values of L-functions*
16:00 - 16:30	MPI Tea Room *Tea*
17:00 - 18:00	MPI Lecture Hall Majid Hadian (Essen) *Motivic fundamental groups*

Fri, 01 Jul 2011

10:30 - 11:30	MPI Lecture Hall Peter Teichner (MPIM) *Universal elliptic cohomology and modular forms*
12:00 - 13:00	MPI Lecture Hall Martin Raum (MPIM) *Eichler relations*

2 Extended Abstracts of the Talks

The articles below are extended abstracts or short summaries of the talks given at the Mathematische Arbeitstagung 2011. The order is chronologic.

Noncommutative identities

Maxim Kontsevich

June 28, 2011

to Don, on the occasion of his $3 \cdot 4 \cdot 5$ *birthday, with love and admiration*

1 "Characteristic polynomial"

Let us fix an integer $n \geq 1$ and consider the algebra

$$\mathcal{A} := \mathbb{C}\langle X_1^{\pm 1}, \ldots, X_n^{\pm 1} \rangle$$

of noncommutative Laurent polynomials with coefficients in $\mathbb{C}$ in n invertible variables, i.e. the group ring of the free group Free_n in n generators:

$$\mathcal{A} = \mathbb{C}[\mathrm{Free}_n] = \Big\{ \sum_{g \in \mathrm{Free}_n} c_g \cdot g \,|\, c_g \in \mathbb{C},\ c_g = 0 \text{ for almost all } g \in \mathrm{Free}_n \Big\} \ .$$

Define a linear functional "Tr" on $\mathcal{A}$ by taking the constant term,

$$\text{``\`Tr''} : \mathcal{A} \to \mathbb{C} \,, \qquad \sum_g c_g \cdot g \mapsto c_{id} \ .$$

This functional vanishes on commutators, like the trace for matrices. By the analogy with the matrix case, we define the "characteristic polynomial" for any $a \in \mathcal{A}$ as a formal power series in one (central) variable t:

$$P_a = P_a(t) := \text{``det''}(1 - ta) := \exp\left( - \sum_{k \geq 1} \frac{\text{``Tr''}(a^k)}{k} t^k \right) = 1 + \cdots \in \mathbb{C}[[t]] \ .$$

Theorem 1. *For any $a \in \mathcal{A}$ the series P_a is algebraic, i.e.*

$$P_a \in \overline{\mathbb{C}(t)} \cap \mathbb{C}[[t]] \subset \overline{\mathbb{C}((t))} \ .$$

Here are few examples:

- the case $n = 1$ is elementary, follows easily from the residue formula,
- for any $n \geq 1$ and

$$a = X_1 + X_1^{-1} + \cdots + X_n + X_n^{-1}$$

one can show that

$$P_a = \frac{\left(\frac{f+1}{2}\right)^n}{\left(\frac{nf+n-1}{2n-1}\right)^{n-1}}, \quad f = f(t) := \sqrt{1 - 4(2n-1)t^2} = 1 + \cdots \in \mathbb{Z}[[t]] .$$

- if $a = X_1 + \cdots + X_n + (X_1 \dots X_n)^{-1}$ then the series P_a is an algebraic hypergeometric function.

A sketch of the proof:

Let us assume for simplicity that $a \in \mathbb{Z}[\mathrm{Free}_n]$, the general case is just slightly more complicated.

Step 1. For $a = \sum_g c_g \cdot g \in \mathbb{Z}[\mathrm{Free}_n]$ the series P_a also has integer coefficients. Indeed, it is easy to see that

$$P_a = \prod_{k \geq 1} \prod_{(g_1, \dots, g_k)} (1 - c_{g_1} \dots c_{g_k} t^k)$$

where for any $k \geq 1$ we take the product over sequences of elements of Free_n such that $g_1 \dots g_k = id$ and the sequence $(g_1, \dots, g_k)$ is strictly smaller than all its cyclic permutations for the lexicographic order on sequences associated with some total ordering of Free_n considered as a countable set. The similar argument works if we replace Free_n by an arbitrary torsion-free group.

Step 2. Consider series

$$F_a = F_a(t) := \sum_{k \geq 1} \text{“Tr”}(a^k)\, t^k \in \mathbb{Z}[[t]].$$

Then F_a is algebraic. It follows from the theory of noncommutative algebraic series developed by N. Chomsky and M.-P. Schützenberger in 1963 (see [2] and Corollary 6.7.2 in [6]).

Step 3. Recall the Grothendieck conjecture on algebraicity. It says that for any algebraic vector bundle with flat connection over an algebraic variety defined over a number field, all solutions of the corresponding holonomic system of differential equations are algebraic if and only if the p-curvature vanishes for all sufficiently large primes $p \gg 1$. There is a simple sufficient criterion for such a vanishing. Namely, it is enough to assume that there exists a fundamental system of solutions in formal power series at some algebraic point, such that all Taylor coefficients (in some local algebraic coordinate system) of all solutions have in total only finitely many primes in denominators.

The Grothendieck conjecture in its full generality is largely unaccessible by now. The only two general results is a theorem by N. Katz on the validity of the Grothendieck conjecture for Gauss-Manin connections, and a theorem of D. Chudnovsky and G. Chudnovsky [3] in the case of line bundles over algebraic curves. This is exactly our case, by the previous step, and because

$$\frac{d}{dt}P_a = -\frac{F_a}{t}P_a\,.$$

□

2 Noncommutative integrability, the case of two variables

Let now consider the algebra of noncommutative polynomials in two (non-invertible) variables

$$\mathcal{A} = \mathbb{C}\langle X, Y\rangle\ .$$

For any integer $d \geq 1$ we consider the variety $\mathcal{M}_d$ of $\mathrm{GL}_d(\mathbb{C})$-equivalence classes (by conjugation) of d-dimensional representations $\rho : \mathcal{A} \to \mathrm{Mat}_{d\times d}(\mathbb{C})$ of $\mathcal{A}$. More precisely, we are interested only in generic pairs of matrices $(\rho(X), \rho(Y))$ and treat variety $\mathcal{M}_d$ birationally. It has dimension $d^2 + 1$.

For generic ρ we consider the "bi-characteristic polynomial" in two commutative variables

$$P_\rho = P_\rho(x, y) := \det(1 - x\rho(X) - y\rho(Y)) = 1 + \cdots \in \mathbb{C}[x, y]\ .$$

The equation $P_\rho(x, y) = 0$ of degree $\leq d$ defines so-called Vinnikov curve $\mathcal{C}_\rho \subset \mathbb{C}P^2$. The number of parameters for the polynomial P_ρ is $\frac{(d+1)(d+2)}{2} - 1$, and it is strictly smaller than $\dim \mathcal{M}_d$ for $d \geq 3$. The missing parameters correspond to the natural line bundle $\mathcal{L}_\rho$ on $\mathcal{C}_\rho$ (well-defined for generic ρ) given by the kernel of operator $(1 - x\rho(X) - y\rho(Y))$ for $(x, y) \in \mathcal{C}_\rho$. Bundle $\mathcal{L}_\rho$ has the same degree as a square root of the canonical class of $\mathcal{C}_\rho$, and defines a point in a torsor over the Jacobian $\mathrm{Jac}(\mathcal{C}_\rho)$. For any given generic curve $\mathcal{C} \subset \mathbb{C}P^2$ of degree d the line bundle on $\mathcal{C}$ depends on $\mathrm{genus}(\mathcal{C}) = \frac{(d-1)(d-2)}{2}$

parameters. Now the dimensions match:

$$\dim \mathcal{M}_d = d^2 + 1 = \frac{(d+1)(d+2)}{2} - 1 + \frac{(d-1)(d-2)}{2}\,.$$

The conclusion is that $\mathcal{M}_d$ is fibered over the space of planar curves of degree d, with the generic fiber being a torsor over an abelian variety. Hence we have one of simplest examples of an *integrable system*. Any integrable system has a commutative group of discrete symmetries, i.e. birational automorphisms preserving the structure of the fibration, identical on the base, and acting by shifts on the generic fiber. Similarly, one can consider an abelian Lie algebra consisting of vertical rational vector fields which are infinitesimal generators of shifts on fibers.

Now I want to consider *noncommutative* symmetries, i.e. certain universal expressions in free variables (X, Y) which can be specialized and make sense for any $d \geq 1$. An universal discrete symmetry is an automorphism of $\mathcal{A}$ (or maybe of some completion of $\mathcal{A}$) which preserves the conjugacy class of any linear combination $Z(t) := X + tY$, $t \in \mathbb{C}$. Indeed, in this case for any $d \geq 1$ and any representation, the value of the bi-characteristic polynomial P_ρ at any pair of complex numbers $(x, y) \in \mathbb{C}^2$ is preserved, as it can be written as $\det(1 - x\rho(Z(y/x)))$. Hence the automorphism under consideration is inner on both variables X and Y:

$$X \mapsto R \cdot X \cdot R^{-1}, \;\; Y \mapsto R' \cdot Y \cdot (R')^{-1}\,.$$

We are interested in automorphisms of $\mathcal{A}$ only up to inner automorphisms, therefore we may safely assume that $R' = 1$. Thus, the question is reducing to the following one:

$$\boxed{\begin{array}{c}\textit{find } R \textit{ such that for any } t \in \mathbb{C} \textit{ there exists } R_t \textit{ such that}\\ R \cdot X \cdot R^{-1} + tY = R_t \cdot (X + tY) \cdot R_t^{-1}\,.\end{array}}$$

First, let us make calculations on the Lie level. Denote by $\mathfrak{g}$ the Lie algebra of derivations δ of $\mathcal{A}$ of the form

$$\delta(X) = [D, X] \text{ for some } X \in \mathcal{A}, \quad \delta(Y) = 0$$

and such that for any $t \in \mathbb{C}$ there exists $D_t \in \mathcal{A}$ such that

$$\delta(X + tY) = [D_t, X + tY] \iff [D, X] = [D_t, X + tY]\,.$$

It is easy to classify such derivations, and one can check that the following elements form a linear basis of $\mathfrak{g}$:

$$\delta_{n,m}(X) = [c_{n,m}, X], \;\; \delta_{n,m}(Y) = 0, \;\; n \geq 0, \, m \geq 1$$

where for any $n, m \geq 0$ we define

$$c_{n,m} := \sum_{\frac{(n+m)!}{n!m!} \text{ shuffles } w} w \,,$$

i.e. the sum of all words in X, Y containing n copies of X and m copies of Y. Elements $D_t \in \mathcal{A}$ corresponding to the derivation $\delta_{n,m}$ are given by

$$D_t = \sum_{0 \leq k \leq n} c_{n-k,m+k}\, t^k \ .$$

A direct calculation shows that $\mathfrak{g}$ is an abelian Lie algebra.

Let us go now the completions of algebra $\mathcal{A}$, and of Lie algebra $\mathfrak{g}$:

$$\widehat{\mathcal{A}} := \mathbb{C}\langle\langle X, Y\rangle\rangle, \quad \widehat{\mathfrak{g}} := \prod_{n \geq 0, m \geq 1} \mathbb{C} \cdot \delta_{n,m} \ .$$

Then the action of $\widehat{\mathfrak{g}}$ on $\widehat{\mathcal{A}}$ exponentiates a continuous group action

$$\widehat{\mathfrak{g}} \stackrel{\exp}{\simeq} \widehat{G} \subset \mathrm{Aut}(\widehat{\mathcal{A}}) \ .$$

For any $\delta \in \widehat{\mathfrak{g}}$ the corresponding one-parameter group of automorphisms acts by

$$\exp(\tau \cdot \delta) : X \mapsto R(\tau) \cdot X \cdot R(\tau)^{-1}, \quad Y \mapsto Y \qquad \forall\, \tau \in \mathbb{C}$$

for certain invertible element $R(\tau) \in \widehat{\mathcal{A}}^{\times}$. An easy calculation shows that $R(\tau)$ is the unique solution of the differential equation

$$\frac{d}{d\tau} R(\tau) = \delta(R(\tau)) + R(\tau) \cdot D, \qquad R(0) = 1$$

where $D \in \widehat{\mathcal{A}}$ is such that $\delta(X) = [D, X]$. The value $R(\tau)_{|\tau=1}$ gives $\exp(\delta)$.

Now we can start to look for a class of elements $\delta \in \widehat{\mathfrak{g}}$ such that the corresponding automorphism $\exp(\delta)$ is sufficiently nice, e.g. if it makes some sense for $\mathcal{A}$ without passing to the completion.

Let us encode a generic element δ as before by the corresponding generating series in *commutative* variables x, y:

$$\delta = \sum_{n,m} f_{n,m} \delta_{n,m} \in \widehat{\mathfrak{g}} \ \rightsquigarrow \ \widetilde{\delta} := \sum_{n,m} f_{n,m} x^n y^m \in \mathbb{C}[[x, y]] \ .$$

I suggest the following Ansatz:

$$\boxed{\widetilde{\delta} \textit{ is the logarithm of a rational function in } x, y.}$$

Hypothetically, for such δ the corresponding automorphism $\exp(\delta)$ of $\widehat{\mathcal{A}}$ can be extended to certain "algebraic extension" of $\mathcal{A}$. A good indication is

Theorem 2. *For any* $P = P(x,y) = 1 + \cdots \in \mathbb{C}[x,y]$ *expand*

$$\log(P) = \sum_{n,m} f_{n,m} x^n y^m \in \mathbb{C}[[x,y]] \ .$$

Then the series

$$\exp\left(\sum_{n,m} \frac{(n+m)!}{n!m!} f_{n,m} x^n y^m\right)$$

is algebraic.

This result is elementary, and I leave it as an exercise to the reader. (Hint: use the residue formula twice.) It implies that the image of R under the abelianization morphism

$$\mathbb{C}\langle\langle X,Y\rangle\rangle \twoheadrightarrow \mathbb{C}[[x,y]], \ \ X \mapsto x, \, Y \mapsto y$$

is algebraic.

Example. Consider the case

$$\widetilde{\delta} = \log(1 - xy) = -\sum_{k\geq 1} \frac{x^k y^k}{k} \ .$$

Then one can show that

$$R = 1 - YX - C \in \widehat{\mathcal{A}}^{\times}$$

where C is the unique solution of the equation

$$C = X \cdot (1-C)^{-1} \cdot Y \ .$$

It can be written

$$C = XY + XXYY + XXYXYY + XXXYYY + \cdots = () + (()) + (()()) + ((())) + \ldots$$

as the sum of all irreducible bracketings if we replace X by (and Y by).

The equation for C is equivalent to the generic "quadratic equation"

$$T^2 + AT + B = 0$$

by the substitutions

$$A := X^{-1}, \ \ B := -X^{-1}Y, \ \ T := X^{-1}C \ .$$

The invertible elements $R_t \in \widehat{\mathcal{A}}^{\times},\ t \in \mathbb{C}$ are given by

$$R_t := R \cdot (1 - tT^2), \quad R \cdot X \cdot R^{-1} + tY = R_t \cdot (X + tY) \cdot R_t^{-1} \ .$$

I'll finish with another example of an integrable system. Few years ago together with S. Duzhin we discovered numerically that the rational map

$$S_{-1} : (X, Y) \mapsto (XYX^{-1},\, (1 + Y^{-1})\, X^{-1})$$

should be a discrete symmetry of an integrable system, where X, Y are two $d \times d$ matrices for $d \geq 1$. Recently O. Efimovskaya and Th. Wolf found an explanation. Namely, they proved that the conjugacy class of the matrix $Z(t)$ of size $2d \times 2d$, defined as

$$Z(t) := \begin{pmatrix} Y^{-1} + X & tY + Y^{-1}X^{-1} + X^{-1} + 1 \\ Y^{-1} + \frac{1}{t}X & Y + Y^{-1}X^{-1} + X^{-1} + \frac{1}{t} \end{pmatrix}$$

does not change under the discrete symmetry S_{-1} as above, for any $t \in \mathbb{C}$.

3 Noncommutative integrability for many variables

Let $M = (M_{ij})_{1 \leq i,j \leq 3}$ be a matrix whose entries are $9 = 3 \times 3$ free independent noncommutative variables. Let us consider 3 "birational involutions"

$$\begin{aligned} &I_1 : M \mapsto M^{-1} \\ &I_2 : M \mapsto M^t \\ &I_3 : M_{ij} \mapsto (M_{ij})^{-1} \quad \forall i, j \ . \end{aligned}$$

The composition $I_1 \circ I_2 \circ I_3$ commutes with the multiplication on the left and on the right by diagonal 3×3 matrices. We can factorize it by the action of $\mathrm{Diag}_{\mathrm{left}} \times \mathrm{Diag}_{\mathrm{right}}$ and get only 4 independent variables, setting e.g. $M_{ij} = 1$ for $i = 3$ and/or $j = 3$.

Conjecture 1. *The transformation $(I_1 \circ I_2 \circ I_3)^3$ is equal to the identity modulo $\mathrm{Diag}_{\mathrm{left}} \times \mathrm{Diag}_{\mathrm{right}}$-action. In other words, there exists two diagonal 3×3 matrices $D_L(M), D_R(M)$ whose entries are noncommutative rational functions in 9 variables (M_{ij}), such that*

$$(I_1 \circ I_2 \circ I_3)^3(M) = D_L(M) \cdot M \cdot D_R(M) \ .$$

This is a very degenerate case of integrability. The conjecture just means that the finite group $\Sigma_3 \wr (\mathbb{Z}/3\mathbb{Z})$ (the wreath product, with $6^3 \cdot 3 = 648$

elements) acts by noncommutative birational transformations in 4 variables. Similarly, for 4×4 matrices the transformation $I_1\circ I_2\circ I_3$ should give a genuinely nontrivial integrable system. In the simplest case when the entries of this matrix are scalars, the Zariski closure of the generic orbit (modulo the left and the right diagonal actions) is a 8-dimensional abelian variety of the form

$$E^{E_8} = \text{elliptic curve}^{\text{root lattice of } E_8}\,.$$

Finally, I'll present a series of hypothetical discrete symmetries of integrable systems written as recursions. Fix an odd integer $k\geq 3$ and consider sequences $(U_n)_{n\in\mathbb{Z}}$ (of, say, $d\times d$ matrices), satisfying

$$\begin{aligned} U_n &= U_{n-k}^{-1}\,(1+U_{n-1}\,U_{n-k+1}) \quad \text{for } n\in 2\mathbb{Z}\\ U_n &= (1+U_{n-k+1}\,U_{n-1})\,U_{n-k}^{-1} \quad \text{for } n\in 2\mathbb{Z}+1\,. \end{aligned}$$

Then the map $(U_1,\ldots,U_k)\mapsto(U_3,\ldots,U_{k+2})$ is integrable.

4 Noncommutative Laurent phenomenon

In the previous example one observes also the noncommutative Laurent phenomenon:

$$\forall\, n\in\mathbb{Z}\qquad U_n\in\mathbb{Z}\langle U_1^{\pm 1},\ldots,U_k^{\pm 1}\rangle\,.$$

Also with S. Duzhin we discovered that the noncommutative birational map

$$S_l:(X,Y)\mapsto(XYX^{-1},(1+Y^{\,l})\,X^{-1})$$

for $l\geq 1$ satisfies the same Laurent properties, i.e. both components of 2-dimensional vector obtained by an arbitrary number of iterations, belong to the ring $\mathbb{Z}\langle X^{\pm 1},Y^{\pm 1}\rangle$. The case $l=1$ is easy, and the case $l=2$ was studied by A. Usnich (unpublished) and by Ph. Di Francesco and R. Kedem, see [4]. The Laurent property has now three different proofs for the case $l\geq 3$ when the dynamics is *non-integrable*:

- by A. Usnich using triangulated categories, see [7],
- an elementary algebraic proof by A. Berenstein and V. Retakh, see [1],
- a new combinatorial proof of Kyungyong Lee, which also shows that all the coefficients of noncommutative Laurent polynomials obtained by iterations, belong to $\{0,1\}\subset\mathbb{Z}$, see [5].

Finally, recently A. Berenstein and V. Retakh found a large class of noncommutative mutations related with triangulated surfaces, and proved the noncommutative Laurent property for them.

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[4] Ph. Di Francesco, R. Kedem, *Discrete non-commutative integrability: the proof of a conjecture by M. Kontsevich*, arXiv:0909.0615.

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EXACT CRITICAL VALUES OF THE SYMMETRIC FOURTH L FUNCTION AND ZAGIER'S CONJECTURE

ARBEITSTAGUNG, 25 JUNE 2011

TOMOYOSHI IBUKIYAMA

1

We gave a talk on three things.
(1) Exact critical values of the symmetric fourth L function of the Ramanujan Delta function.
(2) A congruence between Hecke eigenvalues of vector valued Siegel modular forms of a lift and a non-lift.
(3) A little survey on why differential opeartors on automorphic forms are interesting. Here (1) and (2) are joint works with H. Katsurada and the details will appear elsewhere (cf. [8]).

2

For any Hecke eigen elliptic modular form $f = \sum_{n=1}^{a_n} q^n \in S_k(\Gamma_1)$ ($\Gamma_1 = SL_2(\mathbb{Z})$) with $a_1 = 1$, the symmetric j-th L function $L(s, f, Sym(j))$ is defined by

$$L(s, f, Sym(j)) = \prod_{p:prime} \prod_{i=0}^{j} (1 - \alpha_p^i \beta_p^{j-i} p^{-s})^{-1}$$

where $1 - a_p p^{-s} + p^{k-1-2s} = (1 - \alpha_p p^{-s})(1 - \beta_p p^{-s})$. In 1977, Don Zagier gave the following conjecture for the Ramanujan Delta function Δ.

Conjecture 2.1 (Zagier [16]). *We have*

$$((2\pi)^{-3s+33}\Gamma(11)^{-1}\Gamma(s)\Gamma(s-11)L(s, \Delta, Sym(4)) = c(s)2^{33}(\Delta, \Delta)^3$$

for $s = 24$, *26, 28, 30, 32 where* $c(s)$ *are given in the following table.*

s	$c(s)$
24	$2^5 \times 3^2$
26	$2^5 \times 3 \times 5$
28	$2^2 \times 23 \times 691/7^2$
30	$2^3 \times 653$
32	$2 \times 3 \times 34981^*/7$

(* He stated 34891 instead of the above prime 34981, but this is an obvious typo since $34891 = 23 \times 37 \times 41$.)

Now, we denote by $S_{13,10}(\Gamma_2)$ the space of vector valued Siegel cusp forms of weight $\det^{13} Sym(10)$ of degree two belonging to the Siegel full modular group $\Gamma_2 = Sp(2,\mathbb{Z})$ (size four). Our main theorems are as follows.

Theorem 2.2. *There exists a vector valued Siegel cusp Hecke eigenform $F \in S_{13,10}(\Gamma_2)$ such that*

$$(2\pi)^{33-3s}\Gamma(11)^{-1}\Gamma(s)\Gamma(s-11)L(s,\Delta,Sym(4)) = c(s)(F,F)$$

for any $s = 24$, 26, 28, 30, 32.

Here $c(s)$ is as in Zagier's conjecture and the above F does not depend on the choice of s and given explicitly by a theta function.

For a Hecke eigenform $f \in S_k(SL_2(\mathbb{Z}))$, denote by $\mathbb{Q}(f)$ the field generated over $\mathbb{Q}$ by all the Hecke eigenvalues of f.

Theorem 2.3. *For any primitive form $f \in S_k(\mathit{\Gamma}_1)$, there exists a constant $c(f)$ depending only on f such that $L(l,f,Sym(4))/\pi^{-3k+3l+3}c(f)$ belongs to $\mathbb{Q}(f)$ for any even integer l such that $2k \le l \le 3k-4$.*

The proof of the above theorem 2.3 is a direct corollary of the results in Ramakrishnan-Shahidi [15] but the proof of Theorem 2.2 is much more difficult. The ingredients of the proof of Theorem 2.2 is as follows.
(1) Kim-Ramakrishnan-Shahidi lifting: Ramakrishnan-Shashidi [15] asserts that there exists a lifting from $f \in S_k(\Gamma_1)$ to a holomorphic vector valued Siegel modular form F of weight $\det^{k+1} Sym(k-2)$ such that

$$L(s,f,Sym(3)) = L(s,F,Sp),$$

where Sp means the spinor L function. (I had also an experiment of this type of lifting with conjecture in [6], as quoted in their paper.) The following fact does not seem to be written in [15] but an easy corollary of their theorem.

$$L(s,f,Sym(4)) = L(s-22,F,St),$$

where St means the standard L function. So the problem is to give F explicitly and calculate critical values of $L(s,F,St)$.
(2) For the calculation of $L(s,F,St)$, we use the pullback formula of Kozima [14], shifting Eisenstein series by certain differential operators.
(3) In the above, we use a general theory on differential opeartors on automorphic forms which behaves well under a certain restriction of the domain.

Practically, we need
(i) Explicit Fourier coefficients of a basis of $S_{13,10}(\Gamma_2)$.
(ii) Explicit Fourier coefficients of Eisenstein series E_l of degree 4 of weight $l = 4$, 6, 8, 10, or 12.
(iii) Explicit holomorphic linear differential operators which map E_l to the sum of tensors of $S_{13,10}(\Gamma_2)$ after the restriction to the diagonal

2×2 blocks from the Siegel upper half space of degree 4.

Among these, the basis in (i) is given by theta functions and Fourier coefficients can be calculated by a computer. As for (ii), it is well known that the Fourier coefficients are written by Siegel series and a fairly precise structural formula for Siegel series is known in [11]. Although this is not completely a closed formula and it is not very easy to calculate Fourier coefficients of E_l, we can do this by computer calculation for each concretely given l.
Finally we need some differential operators. General setting is as follows. We take bounded symmetric domains $\Delta \subset D$ and assume a natural inclusion of the automorphism groups $Aut(\Delta) \subset Aut(D)$. We take finite dimensional vector space V over $\mathbb{C}$ and V-valued automorphy factor J_Δ in $GL(V)$ for Δ and $\mathbb{C}$-valued J_D for D.
We use V-valued differential operators $\mathbb{D}$ such that for any $g \in Aut(\Delta) \subset Aut(D)$, the following diagram is commutative.

$$\begin{tikzcd} Hol(D,\mathbb{C}) \arrow[r, "\mathbb{D}"] \arrow[d, "|_{J_D}[g]"'] & Hol(D,V) \arrow[r, "Res."] & Hol(\Delta,V) \arrow[d, "|_{J_\Delta}[g]"] \\ Hol(D,\mathbb{C}) \arrow[r, "\mathbb{D}"] & Hol(D,V) \arrow[r, "Res."] & Hol(\Delta,V) \end{tikzcd}$$

A general characterization of such differential operators when a pair (Δ, D) consists of products of Siegel upper half spaces is given in [4]. This theory also gives a certain way to calculate operators explicitly, though it is not so easy to execute this in general (cf. also [2]). These operators are interesting at least in the following points.
(1) A source of new special functions (a kind of generalization of Legendre or Gegenbauer) and holonomic systems (cf. the work with Zagier [10] and the work with Kuzumaki and Ochiai [9].)
(2) To obtain new Siegel modular forms by known modular forms which are sometimes difficult by any other method. (cf. [1], [7].)
(3) An application for pullback formulas (e.g. [12], [3])

Here we use (3). Such differential operators are complicated and not easily obtained explicitly, but we can do this by a lengthy computer calculation.

Finally we give a result on congruence. By Tsushima's dimension formula, we have $\dim S_{13,10}(\Gamma_2) = 2$ and the Hecke eigen basis $\{F_{13,10a}, F_{13,10b}\}$ of $S_{13,10}(\Gamma_2)$ is given explicitly by theta functions with harmonic polynomials, one of which (say $F_{13,10a}$) is a Kim-Ramakshnan-Shahidi lift and the other is a non-lift. We can often expect that for any kind of lift, there exists a non-lift congruent to the lift. We denote by $\lambda(n, F_{13,10*})$ the Hecke eigenvalue of $T(n)$ of $F_{13,10*}$. Then we have the following theorem.

Theorem 2.4. *For any natural number n, we have*

$$\lambda(n, F_{13,10a}) \equiv \lambda(n, F_{13,10b}) \bmod 13.$$

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THETA DERIVATIVES AND TEICHMÜLLER CURVES (ARBEITSTAGUNG 2011)

MARTIN MÖLLER

1. Special curves on Hilbert modular surfaces

Consider the Hilbert modular surfaces $X_D = \mathbb{H}^2/\mathrm{SL}(\mathfrak{o}_D \oplus \mathfrak{o}_D^\vee)$ where $\mathfrak{o}$ is the order of discriminant D in $K = \mathbb{Q}(\sqrt{D})$. Clearly the most special algebraic curve in X_D is the diagonal, the image of the composition $z \mapsto (z,z)$ and the projection $\pi : \mathbb{H}^2 \to X_D$. For any matrix $M \in \mathrm{GL}_2^+(K)$ one can consider the twisted diagonal $z \mapsto (Mz, M^\sigma z)$, where σ is the generator of the Galois group. The π-images of these twisted diagonals are still algebraic curves, called special curves, Shimura curves, modular curves or Hirzebruch-Zagier cycles and the literature on them is even longer than the number of names. Note that for these curves both components of the universal covering map are given by Mobius transformations.

An algebraic curve $C \to X_D$ in a Hilbert modular surface is still quite special if one asks just that (at least) *one* of the components of the universal covering map $\mathbb{H} \to \mathbb{H}^2$ should be a Mobius transformation. Equivalently, one may ask that $C \to X_D$ is totally geodesic for the Kobayashi metric and we thus call these curves *Kobayashi geodesics*. Yet equivalently, we can characterize these curves as being everywhere transversal to (at least) one of the two foliations of $\mathbb{H}^2$. See [MV10] for more equivalent conditions.

We will provide examples of these curves soon. We give one number theoretic reason why one might be interested in these curves. Consider the differential equation

$$\begin{aligned} L(y,t) = \big(A(t)\,y'(t)\big)' \,+\, B(t)\,y(t) \;&=\; 0 \\ A(t) \;=\; t\,(t-1)\,(t-\ell)(t-\ell^{-1}) \;&=\; t^4 \,-\, \beta t^3 \,+\, \beta t^2 \,-t\,, \\ B(t) \;=\; \frac{3}{4}\big(3t^2 \,-\, (\beta+\gamma)\,t \,+\, \gamma\big)& \end{aligned} \tag{1}$$

where

$$\ell = \frac{31-7\sqrt{17}}{2}, \quad \beta = \ell + \ell^{-1} + 1 = \frac{1087-217\sqrt{17}}{64}, \quad \gamma = \frac{27-5\sqrt{17}}{4}\,.$$

There is a well-known recursive procedure for finding a solution $y = \sum_{n\geq 0} a_n t^n$ of such a differential equation that involves dividing by $(n+1)^2$ when computing the n-th term. But the solution of this particular differential equation

$$y = 1 + \frac{81-15\sqrt{17}}{16}t + \frac{4845-1155\sqrt{17}}{64}t^2 + \frac{3200225-775495\sqrt{17}}{2048}t^3 + \ldots \tag{2}$$

has coefficients in the ring of integers $\mathfrak{o}_{\sqrt{17}}[1/2]$ ([BM10]). The differential equation is the Picard-Fuchs equation for the curve W_{17} introduced below.

2. Theta derivatives

Let $\Theta_{(m,m')}(v, Z)$ be the usual Siegel theta function on $\mathbb{C}^2 \times \mathbb{H}_2$ with characteristic $(m, m') \in (\frac{1}{2}\mathbb{Z}^2/\mathbb{Z}^2)^2$. A choice of a basis for $\mathfrak{o}_D$ determines a 'Siegel' modular embedding, i.e. map $\psi : \mathbb{H}^2 \to \mathbb{H}_2$ and equivariant with respect to an adapted group homomorphism $\Psi : \mathrm{SL}(\mathfrak{o}_D \oplus \mathfrak{o}_D^\vee) \to \mathrm{Sp}_4(\mathbb{Z})$.
In Siegel upper half space there are no distinguished directions and consequently none of the partial derivatives of Θ with respect to ε_i is distinguished. Altogether the form a vector-valued modular form. But $\mathbb{H}^2$ has two distinguished foliations and thus the restriction of $\Theta(z_1, z_2)$ to the universal covering of X_D has two distinguished partial derivatives. We denote second of these derivatives by $D_2\Theta(z_1, z_2)$. This is a modular form of weight $(1/2, 3/2)$ for some subgroup of $\mathrm{SL}(\mathfrak{o}_D \oplus \mathfrak{o}_D^\vee)$.

Theorem 2.1 ([MZ11]). *The function*

$$D_2\Theta(z_1, z_2) = \prod_{(m,m')\ \text{odd}} D_2\Theta_{(m,m')}(0, \psi(z_1, z_2))$$

is a modular form for the full Hilbert modular group $\mathrm{SL}(\mathfrak{o}_D \oplus \mathfrak{o}_D^\vee)$ *of weight* $(3, 9)$. *Its vanishing locus*

$$W_D = \{D_2\Theta(z_1, z_2) = 0\} \subset X_D$$

is a Kobayashi geodesic.

Sketch of proof. Being transversal to the second of the two foliations means that the derivative in the z_2-direction never vanishes. Using the heat equation this means that the third partial derivative of the theta function never vanishes on W_D (in the interior of X_D). This third derivative is a 'modular form' on W_D. The number of zeros on a compactification of W_D can thus be computed. It suffices thus to list the number of cusps of W_D and show that the vanishing orders of the third derivative at these points add up to the required number. □

3. Connection to Teichmüller curves

Teichmüller curves are algebraic curves in the moduli space of curves $\mathcal{M}_g$ that are totally geodesic for the Kobayashi (equivalently: Teichmüller) metric. In [McM03] McMullen found an interesting series of such curves W_D^{Eig} using eigenforms for real multiplication, see [McM05] for a complete classification. Precisely,

$$\begin{aligned} W_D^{\mathrm{Eig}} =& \{[X] \in \mathcal{M}_2 \,:\, \mathrm{Jac}(X)\,\text{has RM by}\,\mathfrak{o}_D, \\ & \quad \text{a RM-eigenform}\,\omega \in H^0(X, \Omega^1_X)\,\text{has a double zero}\} \end{aligned}$$

Theorem 3.1 ([MZ11]). *These two series of curves coincide, i.e.* $W_D = W_D^{\mathrm{Eig}}$ *when considered in* $\mathcal{A}_2$.

The proof relies on two facts. First, a genus two curve equals the theta divisor in its Jacobian. Second an eigenform has a double zero if and only if the derivative of the theta function in a 'foliation' direction vanishes at a Weierstraß point.
By construction W_D^{Eig} is in $\mathcal{M}_2$, hence disjoint from the locus $P_D \subset X_D$ of reducible abelian surfaces. There are two more proofs of this fact using theta functions only.

Theorem 3.2 ([MZ11]). *The loci* W_D *and* P_D *are disjoint.*

Sketch of proof. The reducible locus is the vanishing locus of the product of all even theta functions. Its restriction to X_D is the vanishing locus of a modular form of weight $(5,5)$. As in the proof of Theorem 2.1 one can thus calculate the number of intersection point of W_D and P_D by intersection theory. Again, a local calculation at the cusps of W_D shows that the intersection points are all located there.
For the second proof, one shows that on the reducible locus the derivatives of theta functions factor as a product of two unary theta series. They are known to vanish only at the cusps. □

Since a Kobayashi geodesic C in X_D is a Teichmüller curve if and only if C is disjoint from P_D, this provides a proof of the property Teichmüller curve using theta functions only.
Disconnecting from the world of Teichmüller curves. Given the univeral covering map $z \mapsto (z, \varphi(z))$ of a Kobayashi geodesic one can obtain more Kobayashi geodesics by twisting, i.e. considering the π-images of $z \mapsto (Mz, M^\sigma \varphi(z))$. For these curves one can ask the same questions as for the Hirzebruch-Zagier curves. Some answers are provided in the forthcoming Ph.D. thesis of C. Weiß. But this is still surely not yet the end the story.
If C is Kobayashi geodesic and $\mathcal{L}_i$ are the classes of the two foliations of X_D, then the quantity $\lambda_2 = (C \cdot \mathcal{L}_1)/(C \cdot \mathcal{L}_2)$ is invariant under twisting. Beside the case $\lambda_2 = 1$ (HZ-cycles) and $\lambda_2 = 1/3$ (from W_D) C. Weiss also showed that the Prym Teichmüller curves of [McM06] give Kobayashi geodesics with $\lambda_2 = 1/7$. A construction of these curves using Θ-functions is in progress.

4. Two compactifications

A list of cusps of W_D was needed in (some of the) proof(s) sketched above. To describe them, there is a very useful compactification $\overline{X_D}^B$ defined by Bainbridge ([Ba07]) as follows. Consider the preimage of X_D in $\mathcal{M}_2$, lift to $\Omega\overline{\mathcal{M}}_2$, the total space of the relative dualizing sheaf over the Deligne-Mumford compactification, and take $\overline{X_D}^B$ to be the normalization of the closure.
On the other hand there is Hirzebruch's compactification $\overline{X_D}^H$, the minimal smooth compactification. This compactification is toroidal, that is given by a fan, a sequence of $\alpha_n \in \mathfrak{o}_D$ totally positive with $\sigma(\alpha_n)/\alpha_n$ decreasing and invariant under multiplication by squares of units in $\mathfrak{o}_D$. The toroidal structure allows to compute easily e.g. if and at which point HZ-cycles meet the boundary.
There is also a way of realizing $\overline{X_D}^B$ as a toroidal compactification. For a fractional $\mathfrak{o}_D$ ideal $\mathfrak{a}$ let $\mathfrak{a}^*[2]$ be the set of non-zero elements in $\frac{1}{2}\mathfrak{a}/\mathfrak{a}$. We let $\widetilde{\mathrm{MM}}(\mathfrak{a},\xi)$ be the set of $\alpha \in K$ such that the quadratic form $F(x) = \mathrm{tr}(\alpha x^2)$ is positive definite and assumes its minimum on $\mathfrak{a} + \xi$ more than once (where x and $-x$ are not distinguished). We define a *multiminimizer for* ξ to be the equivalence classes

$$\mathrm{MM}(\mathfrak{a},\xi) = \widetilde{\mathrm{MM}}(\mathfrak{a},\xi)/\mathbb{Q}^*$$

and we let the set of *multiminimizers* be the union of $\mathrm{MM}(\mathfrak{a},\xi)$ over all $\xi \in a^*[2]$.

Theorem 4.1 ([MZ11])**.** *For any $\mathfrak{a}$, the set of multiminimizers forms a fan. The associated toroidal compactification is Bainbridge's compactification $\overline{X_D}^B$ at the cusp $\mathfrak{a}$.*

This compactification can be calculated by an easy algorithm. In fact, given one multiminimizer, the subsequent ones can be constructed using the 'slow-greater one' continued fraction algorithm. Here 'slow-greater one' continued fraction algorithm means that

$$x_{n+1} = \begin{cases} x_n - 1 & \text{if} \quad x_n > 2 \\ 1/(x_n - 1) & \text{if} \quad 2 > x_n > 1 \end{cases}.$$

Note that Hirzebruch's compactification is driven by the 'fast-minus' continued fraction algorithm

$$x = p_1 - \cfrac{1}{p_2 - \cfrac{1}{\ddots}},$$

where at each step $p_i = \lceil x_i \rceil$.

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Four Groups Related to Associators

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Dedicated to Professor Don Zagier on the occasion of his 60th birthday

1. Associators

We recall the definition of associators [Dr] and explain our main results in [F10a, F11] which are on the defining equations of associators.

Let us fix notations: Let k be a field of characteristic 0 and $\bar{k}$ its algebraic closure. Denote by $U\mathfrak{F}_2 = k\langle\langle X_0, X_1\rangle\rangle$ a non-commutative formal power series ring, a universal enveloping algebra of the completed free Lie algebra $\mathfrak{F}_2$ with two variables X_0 and X_1. Its element $\varphi = \varphi(X_0, X_1)$ is called *group-like* [1] if it satisfies

$$\Delta(\varphi) = \varphi \otimes \varphi \text{ and } \varphi(0,0) = 1 \tag{1}$$

with $\Delta(X_0) = X_0 \otimes 1 + 1 \otimes X_0$ and $\Delta(X_1) = X_1 \otimes 1 + 1 \otimes X_1$. For any k-algebra homomorphism $\iota : U\mathfrak{F}_2 \to S$, the image $\iota(\varphi) \in S$ is denoted by $\varphi(\iota(X_0), \iota(X_1))$.

Definition 1 ([Dr]). A pair (μ, φ) with a *non-zero* element μ in k and a group-like series $\varphi = \varphi(X_0, X_1) \in U\mathfrak{F}_2$ is called an *associator* if it satisfies *one pentagon equation* in $U\mathfrak{a}_4$

$$\varphi(t_{12}, t_{23}+t_{24})\varphi(t_{13}+t_{23}, t_{34}) = \varphi(t_{23}, t_{34})\varphi(t_{12}+t_{13}, t_{24}+t_{34})\varphi(t_{12}, t_{23}) \tag{2}$$

and *two hexagon equations* in $U\mathfrak{a}_3$

$$\exp\{\frac{\mu(t_{13}+t_{23})}{2}\} = \varphi(t_{13}, t_{12}) \exp\{\frac{\mu t_{13}}{2}\}\varphi(t_{13}, t_{23})^{-1} \exp\{\frac{\mu t_{23}}{2}\}\varphi(t_{12}, t_{23}), \tag{3}$$

$$\exp\{\frac{\mu(t_{12}+t_{13})}{2}\} = \varphi(t_{23}, t_{13})^{-1} \exp\{\frac{\mu t_{13}}{2}\}\varphi(t_{12}, t_{13}) \exp\{\frac{\mu t_{12}}{2}\}\varphi(t_{12}, t_{23})^{-1}. \tag{4}$$

Here $U\mathfrak{a}_3$ (resp. $U\mathfrak{a}_4$) means the universal enveloping algebra of the *completed pure braid Lie algebra* $\mathfrak{a}_3$ (resp. $\mathfrak{a}_4$) over k with 3 (resp. 4) strings, generated by t_{ij} ($1 \leqslant i,j \leqslant 3$ (resp. 4)) with defining relations

$$t_{ii} = 0,\ t_{ij} = t_{ji},\ [t_{ij}, t_{ik} + t_{jk}] = 0\ (i,j,k\text{: all distinct})$$
$$\text{and } [t_{ij}, t_{kl}] = 0\ (i,j,k,l\text{: all distinct}).$$

Remark 2. (i). Drinfeld [Dr] proved that such a pair always exists for any filed k of characteristic 0.

(ii). The equations (2)$\sim$(4) reflect the three axioms of braided monoidal categories [JS]. We note that for any k-linear *infinitesimal* tensor category $\mathcal{C}$ each associator gives a structure of braided monoidal category on $\mathcal{C}[[h]]$ (cf.[C, Dr]). Here $\mathcal{C}[[h]]$ means the category whose set of objects is equal to that of $\mathcal{C}$ and whose set of morphism $Mor_{\mathcal{C}[[h]]}(X,Y)$ is $Mor_{\mathcal{C}}(X,Y) \otimes k[[h]]$ (h: a parameter).

[1] It is equivalent to $\varphi \in \exp \mathfrak{F}_2$.

(iii). Associators are essential for construction of quasi-triangular quasi-Hopf quantized universal enveloping algebras [Dr].

(iv). Le and Murakami [LMa] and Bar-Natan [Ba97] gave a reconstruction of universal Vassiliev knot invariant (Kontsevich invariant [K, Ba95]) in a combinatorial way by using associators.

Our first result is the implication of two hexagon equations from one pentagon equation.

Theorem 3 ([F10a])**.** *Let $\varphi = \varphi(X_0, X_1)$ be a group-like element of $U\mathfrak{F}_2$. Suppose that φ satisfies the pentagon equation* (2)*. Then there exists $\mu \in \bar{k}$ (unique up to signature) such that the pair (μ, φ) satisfies two hexagon equations* (3) *and* (4)*.*

Recently several different proofs of the above theorem were obtained (see [AT, BaD, W]).

One of the nice examples of associators is the Drinfeld associator below.

Examples 4. The *Drinfeld associator* $\Phi_{KZ} = \Phi_{KZ}(X_0, X_1) \in \mathbf{C}\langle\langle X_0, X_1\rangle\rangle$ is defined to be the quotient $\Phi_{KZ} = G_1(z)^{-1}G_0(z)$ where G_0 and G_1 are the solutions of the *formal KZ (Knizhnik-Zamolodchikov) equation*, the following differential equation over $\mathbf{C}\backslash\{0,1\}$ with $G(z)$ valued on $\mathbf{C}\langle\langle X_0, X_1\rangle\rangle$

$$\frac{d}{dz}G(z) = (\frac{X_0}{z} + \frac{X_1}{z-1})G(z),$$

such that $G_0(z) \approx z^{X_0}$ when $z \to 0$ and $G_1(z) \approx (1-z)^{X_1}$ when $z \to 1$ (cf.[Dr]). It is shown in [Dr] that the pair $(2\pi\sqrt{-1}, \Phi_{KZ})$ forms an associator for $k = \mathbf{C}$. Namely Φ_{KZ} satisfies (1)$\sim$(4) with $\mu = 2\pi\sqrt{-1}$.

Remark 5. (i). The Drinfeld associator is expressed as follows:

$$\Phi_{KZ}(X_0, X_1) = 1 + \sum (-1)^m \zeta(k_1, \cdots, k_m) X_0^{k_m-1}X_1 \cdots X_0^{k_1-1}X_1 + (\text{regularized terms}).$$

Here $\zeta(k_1, \cdots, k_m)$ is the *multiple zeta value* (MZV in short), the real number defined by the following power series

$$\zeta(k_1, \cdots, k_m) := \sum_{0<n_1<\cdots<n_m} \frac{1}{n_1^{k_1}\cdots n_m^{k_m}} \tag{5}$$

for m, $k_1, \ldots, k_m \in \mathbf{N}(=\mathbf{Z}_{>0})$ with $k_m > 1$ (its convergent condition). All the coefficients of Φ_{KZ} including its regularized terms are explicitly calculated in terms of MZV's in [F03] proposition 3.2.3 by Le-Murakami's method in [LMb].

(ii). Since MZV's are coefficients of Φ_{KZ}, the equations (1)$\sim$(4) for $(\mu, \varphi) = (2\pi\sqrt{-1}, \Phi_{KZ})$ yield algebraic relations among them, which are called *associator relations*. It is expected that the associator relations might produce all algebraic relations among MZV's.

Various relations among MZV's have been found and studied so far. The regularised double shuffle relations which were initially introduced by Zagier and Ecalle in early 90's might be one of the most fascinating ones. To state them let us fix notations again: Let $\pi_Y : k\langle\langle X_0, X_1\rangle\rangle \to k\langle\langle Y_1, Y_2, \ldots\rangle\rangle$ be the k-linear map between non-commutative formal power series rings that sends all the words ending in X_0 to zero and the word $X_0^{n_m-1}X_1 \cdots X_0^{n_1-1}X_1$ $(n_1, \ldots, n_m \in \mathbf{N})$ to $(-1)^m Y_{n_m} \cdots Y_{n_1}$.

Define the coproduct Δ_* on $k\langle\langle Y_1, Y_2, \dots\rangle\rangle$ by

$$\Delta_* Y_n = \sum_{i=0}^{n} Y_i \otimes Y_{n-i}$$

with $Y_0 := 1$. For $\varphi = \sum_{W:\text{word}} c_W(\varphi) W \in U\mathfrak{F}_2 = k\langle\langle X_0, X_1\rangle\rangle$ with $c_W(\varphi) \in k$ (a 'word' means a monic monomial element or 1 in $U\mathfrak{F}_2$), put

$$\varphi_* = \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n} c_{X_0^{n-1}X_1}(\varphi) Y_1^n\right) \cdot \pi_Y(\varphi).$$

The *regularised double shuffle relations* for a group-like series $\varphi \in U\mathfrak{F}_2$ mean

$$\Delta_*(\varphi_*) = \varphi_* \widehat{\otimes} \varphi_*. \tag{6}$$

Remark 6. The *regularised double shuffle relations* for MZV's mean the algebraic relations among them obtained from (1) and (6) for $\varphi = \Phi_{KZ}$ (cf. [IKZ, R]). It is also expected that the relations might produce all algebraic relations among MZV's.

The following is the simplest example of the relations.

Examples 7. For $a, b > 1$,

$$\begin{aligned}\zeta(a)\zeta(b) &= \zeta(a,b) + \zeta(a+b) + \zeta(b,a) \\ &= \sum_{i=0}^{a-1} \binom{b-1+i}{i} \zeta(a-i, b+i) + \sum_{j=0}^{b-1} \binom{a-1+j}{j} \zeta(b-j, a+j).\end{aligned}$$

Our second result here is the implication of the regularised double shuffle relations from the pentagon equation.

Theorem 8 ([F11])**.** *Let $\varphi = \varphi(X_0, X_1)$ be a group-like element of $U\mathfrak{F}_2$. Suppose that φ satisfies the pentagon equation* (2)*. Then it also satisfies the regularised double shuffle relations* (6)*.*

This result attains the final goal of the project posed by Deligne-Terasoma [T]. Their idea is to use some convolutions of perverse sheaves, whereas our proof is to use Chen's bar construction calculus.

Remark 9. Our theorem 8 was extended cyclotomically in [F10b].

The following Zagier's relation which is essential for Brown's proof of theorem 14 might be also one of the most fascinating ones. The author does not know if it also follows from our pentagon equation (2).

Theorem 10 ([Z])**.**

$$\zeta(2^{\{a\}}, 3, 2^{\{b\}}) = 2 \sum_{r=1}^{a+b+1} (-1)^r (A^r_{a,b} - B^r_{a,b}) \zeta(2r+1) \zeta(2^{\{a+b+1-r\}})$$

with $A^r_{a,b} = \binom{2r}{2a+2}$ *and* $B^r_{a,b} = (1 - 2^{-2r})\binom{2r}{2b+1}$.

2. Four Groups

We explain recent developments on the four pro-unipotent algebraic groups related to associators; the motivic Galois group, the Grothendieck-Teichmüller group, the double shuffle group and the Kashiwara-Vergne group.

2.1. **Motivic Galois group.** We review on the motivic Galois group, the tannakian dual group of the category of unramified mixed Tate motives.

We work in the triangulated category $DM(\mathbf{Q})_{\mathbf{Q}}$ of *mixed motives* [2] over $\mathbf{Q}$ constructed by Hanamura, Levine and Voevodsky. *Tate motives* $\mathbf{Q}(n)$ $(n \in \mathbf{Z})$ are (Tate) objects of the category. Let $DMT(\mathbf{Q})_{\mathbf{Q}}$ be the triangulated sub-category of $DM(\mathbf{Q})_{\mathbf{Q}}$ generated by Tate motives $\mathbf{Q}(n)$ $(n \in \mathbf{Z})$. By the work of Levine a neutral tannakian $\mathbf{Q}$-category $MT(\mathbf{Q}) = MT(\mathbf{Q})_{\mathbf{Q}}$ of *mixed Tate motives over* $\mathbf{Q}$ is extracted by taking a heart with respect to a t-structure of $DMT(\mathbf{Q})_{\mathbf{Q}}$. Deligne and Goncharov [DeG] introduced the full subcategory $MT(\mathbf{Z}) = MT(\mathbf{Z})_{\mathbf{Q}}$ of *unramified mixed Tate motives* inside. All objects there are mixed Tate motives M (i.e. an object of $MT(\mathbf{Q})$) such that for each subquotient E of M which is an extension of $\mathbf{Q}(n)$ by $\mathbf{Q}(n+1)$ for $n \in \mathbf{Z}$, the extension class of E in

$$Ext^1_{MT(\mathbf{Q})}(\mathbf{Q}(n), \mathbf{Q}(n+1)) = Ext^1_{MT(\mathbf{Q})}(\mathbf{Q}(0), \mathbf{Q}(1)) = \mathbf{Q}^{\times} \otimes \mathbf{Q}$$

is equal to $\mathbf{Z}^{\times} \otimes \mathbf{Q} = \{0\}$.

In the category $MT(\mathbf{Z})$ of unramified mixed Tate motives, the followings hold:

$$\dim_{\mathbf{Q}} Ext^1_{MT(\mathbf{Z})}(\mathbf{Q}(0), \mathbf{Q}(m)) = \begin{cases} 1 \ (m = 3, 5, 7, \dots), \\ 0 \ (m : \text{others}), \end{cases} \tag{7}$$

$$\dim_{\mathbf{Q}} Ext^2_{MT(\mathbf{Z})}(\mathbf{Q}(0), \mathbf{Q}(m)) = 0. \tag{8}$$

The category $MT(\mathbf{Z})$ forms a neutral tannakian $\mathbf{Q}$-category with the fiber functor $\omega_{\mathrm{can}} : MT(\mathbf{Z}) \to Vect_{\mathbf{Q}}$ ($Vect_{\mathbf{Q}}$: the category of $\mathbf{Q}$-vector spaces) sending each motive M to $\oplus_n Hom(\mathbf{Q}(n), Gr^W_{-2n} M)$.

Definition 11. The *motivic Galois group* here is defined to be the pro-$\mathbf{Q}$-algebraic group $\mathrm{Gal}^{\mathcal{M}}(\mathbf{Z}) := \underline{Aut}^{\otimes}(MT(\mathbf{Z}) : \omega_{\mathrm{can}})$.

By tannakian category theory, ω_{can} induces an equivalence of categories

$$MT(\mathbf{Z}) \simeq Rep\,\mathrm{Gal}^{\mathcal{M}}(\mathbf{Z}) \tag{9}$$

where RHS means the category of finite dimensional $\mathbf{Q}$-vector spaces with $\mathrm{Gal}^{\mathcal{M}}(\mathbf{Z})$-action.

Remark 12. The action of $\mathrm{Gal}^{\mathcal{M}}(\mathbf{Z})$ on $\omega_{\mathrm{can}}(\mathbf{Q}(1)) = \mathbf{Q}$ defines a surjection $\mathrm{Gal}^{\mathcal{M}}(\mathbf{Z}) \to \mathbf{G}_m$ and its kernel $\mathrm{Gal}^{\mathcal{M}}(\mathbf{Z})_1$ is the unipotent radical of $\mathrm{Gal}^{\mathcal{M}}(\mathbf{Z})$. There is a canonical splitting $\tau : \mathbf{G}_m \to \mathrm{Gal}^{\mathcal{M}}(\mathbf{Z})$ which gives a negative grading on its associated Lie algebra $Lie\mathrm{Gal}^{\mathcal{M}}(\mathbf{Z})_1$. From (7) and (8) it follows that the Lie algebra is the graded *free* Lie algebra generated by one element in each degree $-3, -5, -7, \dots$. (consult [De] §8 for the full story).

The *motivic fundamental group* $\pi_1^{\mathcal{M}}(\mathbf{P}^1 \backslash \{0, 1, \infty\} : \overrightarrow{01})$ constructed in [DeG] §4 is a (pro-)object of $MT(\mathbf{Z})$. By our tannakian equivalence (9), it gives a (pro-)object of RHS of (9), which induces a (graded) action

$$\Psi : \mathrm{Gal}^{\mathcal{M}}(\mathbf{Z})_1 \to Aut \exp \mathfrak{F}_2. \tag{10}$$

[2] A part of idea of mixed motives is explained [De] §1. According to *Wikipedia*, *"the (partly conjectural) theory of motives is an attempt to find a universal way to linearize algebraic varieties, i.e. motives are supposed to provide a cohomology theory which embodies all these particular cohomologies."*

Remark 13. For each $\sigma \in \mathrm{Gal}^{\mathcal{M}}(\mathbf{Z})_1(k)$, its action on $\exp\mathfrak{F}_2$ is described by $e^{X_0} \mapsto e^{X_0}$ and $e^{X_1} \mapsto \varphi_\sigma^{-1} e^{X_1} \varphi_\sigma$ for some $\varphi_\sigma \in \exp\mathfrak{F}_2$.

The following has been conjectured (Deligne-Ihara conjecture) for a long time and finally proved by Brown by using Zagier's relation (Theorem 10).

Theorem 14 ([Br])**.** *The map Ψ is injective.*

It is a unipotent analogue of the so-called Belyĭ's theorem. The theorem says that all unramified mixed Tate motives are associated with MZV's.

2.2. **Grothendieck-Teichmüller group.** The Grothendieck-Teichmüller group was introduced by Drinfeld [Dr] in his study of deformations of quasi-triangular quasi-Hopf quantized universal enveloping algebras. It was defined to be the set of 'degenerated' associators. The construction of the group was also stimulated by the previous idea of Grothendieck, *un jeu de Teichmüller-Lego*, posed in his article *Esquisse d'un programme* [G].

Definition 15 ([Dr])**.** The *Grothendieck-Teichmüller group GRT_1* is defined to be the pro-algebraic variety whose set of k-valued points consists of group-like series $\varphi \in U\mathfrak{F}_2$ satisfying the defining equations (2)$\sim$(4) of associators with $\mu = 0$.

Remark 16. (i). By our theorem 3, it is reformulated to be the set of group-like series satisfying (2) without quadratic terms.

(ii). It forms a group [Dr] by the multiplication below

$$(11) \qquad \varphi_2 \circ \varphi_1 := \varphi_1(\varphi_2 X_0 \varphi_2^{-1}, X_1) \cdot \varphi_2 = \varphi_2 \cdot \varphi_1(X_0, \varphi_2^{-1} X_1 \varphi_2).$$

(iii). By the map sending $X_0 \mapsto X_0$ and $X_1 \mapsto \varphi^{-1} X_1 \varphi$, the group GRT_1 is regarded as a subgroup of $Aut\exp\mathfrak{F}_2$.

(iii). The cyclotomic analogues of associators and the Grothendieck-Teichmüller group were introduced by Enriquez [E]. Some elimination results on their defining equations in special case were obtained in [EF].

Geometric interpretation (cf. [Dr]) of the equations (2)$\sim$ (4) implies the following

Proposition 17. *$Im\Psi \subset GRT_1$.*

Actually it is expected that they are isomorphic.

Remark 18. (i). The Drinfeld associator Φ_{KZ} is an associator (cf. example 4) but is not a degenerated associator, i.e. $\Phi_{KZ} \notin GRT_1(\mathbf{C})$.

(ii). The p-adic Drinfeld associator Φ_{KZ}^p introduced in [F04] is not an associator but a degenerated associator, i.e. $\Phi_{KZ}^p \in GRT_1(\mathbf{Q}_p)$ (cf. [F07]).

2.3. **Double shuffle group.** The double shuffle group was introduced by Racinet as the set of solutions of the regularised double shuffle relations with 'degeneration' condition (no quadratic terms condition).

Definition 19 ([R])**.** The *double shuffle group DMR_0* is the pro-algebraic variety whose set of k-valued points consists of the group-like series $\varphi \in U\mathfrak{F}_2$ satisfying the regularised double shuffle relations (6) without linear terms and quadratic terms.

Remark 20. (i). We note that DMR stands for *double mélange regularisé* ([R]).

(ii). It was shown in [R] that it forms a group by the operation (11).

(iii). By the same way to remark 16 (iii), the group DMR_0 is regarded as a subgroup of $Aut\exp\mathfrak{F}_2$.

It is also shown that $\mathrm{Im}\Psi$ is contained in DMR_0 (cf.[F07])). Actually it is expected that they are isomorphic. Theorem 8 follows the inclusion between GRT_1 and DMR_0:

Proposition 21. *$GRT_1 \subset DMR_0$.*

It is also expected that they are isomorphic.

Remark 22. (i). The Drinfeld associator Φ_{KZ} satisfies the regularised double shuffle relations (cf. remark 6) but it is not an element of the double shuffle group, i.e. $\Phi_{KZ} \notin DMR_0(\mathbf{C})$, because its quadratic terms are non-zero, actually $\zeta(2)X_1X_0 - \zeta(2)X_0X_1$.

(ii). The p-adic Drinfeld associator Φ^p_{KZ} satisfies the regularised double shuffle relations (cf. [BeF, FJ]) and it is an element of the double shuffle group, i.e. $\Phi^p_{KZ} \in DMR_0(\mathbf{Q}_p)$, which also follows from remark 18.(ii) and proposition 21.

2.4. **Kashiwara-Vergne group.** In [KV] Kashiwara and Vergne proposed a conjecture relating on Campbell-Baker-Hausdorff series which generalises Duflo's theorem (Duflo isomorphism) to some extent. The conjecture was settled generally by Alekseev and Meinrenken [AM]. The Kashiwara-Vergne group was introduced as a 'degeneration' of the set of solution of the conjecture by Alekseev and Torossian in [AT], where they gave another proof of the conjecture by using associators.

The following is one of formulations of the conjecture stated in [AET].

Generalized Kashiwara-Vergne problem: Find a group automorphism $P : \exp\mathfrak{F}_2 \to \exp\mathfrak{F}_2$ such that P belongs to $TAut\exp\mathfrak{F}_2$ (that is,

$$P(e^{X_0}) = p_1 e^{X_0} p_1^{-1} \text{ and } P(e^{X_1}) = p_2 e^{X_1} p_2^{-1}$$

for some p_1, $p_2 \in \exp\mathfrak{F}_2$) and P satisfies

$$P(e^{X_0}e^{X_1}) = e^{(X_0+X_1)}$$

and the coboundary Jacobian condition

$$\delta \circ J(P) = 0.$$

Here J stands for the Jacobian cocycle $J : TAut\exp\mathfrak{F}_2 \to \mathfrak{tr}_2$ and δ means the differential map $\delta : \mathfrak{tr}_n \to \mathfrak{tr}_{n+1}$ for $n = 1, 2, \ldots$ (for their precise definitions see [AT]). We note that P is uniquely determined by the pair (p_1, p_2).

The following is essential for the proof of the conjecture.

Proposition 23 ([AT, AET])**.** *Let (μ, φ) be an associator. Then the pair*

$$(p_1, p_2) = \Big(\varphi(X_0/\mu, X_\infty/\mu),\ e^{X_\infty/2}\varphi(X_1/\mu, X_\infty/\mu)\Big)$$

with $X_\infty = -X_0 - X_1$ gives a solution to the above problem.

The Kashiwara-Vergne group is defined to be the set of solutions of the problem with 'degeneration condition' ('the condition of $\mu = 0$'):

Definition 24 ([AT, AET])**.** The *Kashiwara-Vergne group KRV* is defined to be the set of $P \in Aut\exp\mathfrak{F}_2$ which satisfies $P \in TAut\exp\mathfrak{F}_2$,

$$P(e^{(X_0+X_1)}) = e^{(X_0+X_1)}$$

and the coboundary Jacobian condition $\delta \circ J(P) = 0$.

It forms a subgroup of $Aut \exp \mathfrak{F}_2$. We denote by KRV_0 the subgroup of KRV consisting of P without linear terms in both p_1 and p_2. Proposition 23 yields the inclusion below.

Proposition 25. $GRT_1 \subset KRV_0$.

Actually it is expected that they are isomorphic (cf. [AT]). Recent result of Schneps in [S] also leads

Proposition 26. $DMR_0 \subset KRV_0$.

2.5. **Comparison.** By combining theorem 14 and proposition 17, 21, 25 and 26, we obtain

Proposition 27. $Gal^{\mathcal{M}}(\mathbf{Z})_1 \subseteq GRT_1 \subseteq DMR_0 \subseteq KRV_0$.

Here we come to an interesting question on our four groups.

Question 28. Are they all equal? Namely,

$$\mathrm{Gal}^{\mathcal{M}}(\mathbf{Z})_1 = GRT_1 = DMR_0 = KRV_0 \quad ?$$

Though it might be not so good mathematically to believe such equalities without a strong conceptual support, the author thinks that it might be good at least spiritually to imagine/expect a hidden theory to relate them behind.

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Hidden Hodge symmetries and Hodge correlators

A.B. Goncharov

To Don Zagier for his 60-th birthday

1 Hidden Hodge symmetries

There is a well known parallel between Hodge and étale theories, still incomplete and rather mysterious:

l-**adic Étale Theory**	**Hodge Theory**
Category of l-adic Galois modules	Abelian category $\mathcal{MH}_{\mathbb{R}}$ of real mixed Hodge strucrures
Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$	Hodge Galois group $G_{\mathrm{Hod}} :=$ Galois group of the category $\mathcal{MH}_{\mathbb{R}}$
$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $H^*_{\mathrm{et}}(\overline{X}, \mathbb{Q}_l)$, where X is a variety over $\mathbb{Q}$	$H^*(X(\mathbb{C}), \mathbb{R})$ has a functorial real mixed Hodge structure
étale site	??
$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the étale site, and thus on categories of étale sheaves on X, e.g. on the category of l-adic perverse sheaves	?? ?? ??
$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-equivariant perverse sheaves	Saito's Hodge sheaves

The current absense of the "Hodge site" was emphasized by A.A. Beilinson [B].

The Hodge Galois group. A weight n *pure real Hodge structure* is a real vector space H together with a decreasing filtration $F^\bullet H_{\mathbb{C}}$ on its complexification satisfying

$$H_{\mathbb{C}} = \oplus_{p+q=n} F^p H_{\mathbb{C}} \cap \overline{F^q} H_{\mathbb{C}}.$$

A real Hodge structure is a direct sum of pure ones. The category $\mathcal{PH}_{\mathbb{R}}$ real Hodge structures is equivalent to the category of representations of the real algebraic group $\mathbb{C}^*_{\mathbb{C}/\mathbb{R}}$. The group of complex points of $\mathbb{C}^*_{\mathbb{C}/\mathbb{R}}$ is $\mathbb{C}^* \times \mathbb{C}^*$; the complex conjugation interchanges the factors.

A real *mixed Hodge structure* is given by a real vector space H equipped with the weight filtration $W_\bullet H$ and the Hodge filtration $F^\bullet H_{\mathbb{C}}$ of its complexification, such that the Hodge filtration induces on $\mathrm{gr}^W_n H$ a weight n real Hodge structure. The category

$\mathcal{MH}_{\mathbb{R}}$ of real mixed Hodge structures is an abelian rigid tensor category. There is a fiber functor to the category of real vector spaces

$$\omega_{\rm Hod} : \mathcal{MH}_{\mathbb{R}} \longrightarrow {\rm Vect}_{\mathbb{R}}, \qquad H \longrightarrow \oplus_n {\rm gr}^W_n H.$$

The Hodge Galois group is a real algebraic group given by automorphisms of the fiber functor:

$$G_{\rm H} := {\rm Aut}^{\otimes}\omega_{\rm Hod}.$$

The fiber functor provides a canonical equivalence of categories

$$\omega_{\rm Hod} : \mathcal{MH}_{\mathbb{R}} \stackrel{\sim}{\longrightarrow} G_{\rm Hod} - \text{modules}.$$

The Hodge Galois group is a semidirect product of the unipotent radical $U_{\rm Hod}$ and $\mathbb{C}^*_{\mathbb{C}/\mathbb{R}}$:

$$0 \longrightarrow U_{\rm Hod} \longrightarrow G_{\rm Hod} \longrightarrow {\mathbb{C}^*}_{\mathbb{C}/\mathbb{R}} \longrightarrow 0, \qquad {\mathbb{C}^*}_{\mathbb{C}/\mathbb{R}} \hookrightarrow G_{\rm Hod}. \tag{1}$$

The projection $G_{\rm Hod} \rightarrow {\mathbb{C}^*}_{\mathbb{C}/\mathbb{R}}$ is provided by the inclusion of the category of real Hodge structurs to the category of mixed real Hodge structures. The splitting $s : \mathbb{G}_m \to G_{\rm Hod}$ is provided by the functor $\omega_{\rm Hod}$.

The complexified Lie algebra of $U_{\rm Hod}$ has *canonical* generators $G_{p,q}$, $p, q \geq 1$, satisfying the only relation $\overline{G}_{p,q} = -G_{q,p}$, defined in [G1]. For the subcategory of Hodge-Tate structures they were defined in [L]. Unlike similar but different Deligne's generators [D], they behave nicely in families. So to define an action of the group $G_{\rm Hod}$ one needs to have an action of the subgroup $\mathbb{C}^*_{\mathbb{C}/\mathbb{R}}$ and, in addition to this, an action of a single operator

$$G := \sum_{p,q\geq 1} G_{p,q}.$$

The twistor Galois group. Denote by $\mathbb{C}^*$ the real algebraic group with the group of complex points $\mathbb{C}^*$. The extension induced from (1) by the diagonal embedding $\mathbb{C}^* \subset \mathbb{C}^*_{\mathbb{C}/\mathbb{R}}$ is the *twistor Galois group*. It is a semidirect product of the groups $U_{\rm Hod}$ and $\mathbb{C}^*$.

$$0 \longrightarrow U_{\rm Hod} \longrightarrow G_{\rm T} \stackrel{\longleftarrow}{\longrightarrow} \mathbb{C}^* \longrightarrow 0. \tag{2}$$

It is not difficult to prove

Lemma 1.1 *The category of representations of $G_{\rm T}$ is equivalent to the category of mixed twistor structures defined by Simpson [Si2].*

We suggest the following fills the ??-marks in the dictionary related the Hodge and Galois. Below X is a smooth projective complex algebraic variety.

Conjecture 1.2 *There exists a functorial homotopy action of the twistor Galois group $G_{\rm T}$ by A_∞-equivalences of an A_∞-enhancement of the derived category of perverse sheaves on X such that the category of equivariant objects is equivalent to Saito's category real mixed Hodge sheaves.*[1]

[1]We want to have a natural construction of the action first, and get Saito's category real mixed Hodge sheaves as a consequence, not the other way around.

Denote by $D^b_{\rm sm}(X)$ the category of smooth complexes of sheaves on X, i.e. complexes of sheaves on X whose cohomology are local systems.

Theorem 1.3 *There exists a functorial for pull-backs homotopy action of the twistor Galois group $G_{\rm T}$ by A_∞-equivalences of an A_∞-enhancement of the category $D^b_{\rm sm}(X)$.*

The action of the subgroup $\mathbb{C}^*$ is not algebraic. It arises from Simpson's action of $\mathbb{C}^*$ on semisimple local systems [Si1]. The action of the Lie algebra of the unipotent radical $U_{\rm Hod}$ is determined by a collection of numbers, which we call the *Hodge correlators for semisimple local systems.* Our construction uses the theory of harmonic bundles [Si1]. The Hodge correlators can be interpreted as correlators for a certain Feynman integral. This Feynman integral is probably responsible for the "Hodge site".

For the trivial local system the construction was carried out in [G2]. A more general construction for curves, involving the constant sheaves and delta-functions, was carried out in [G1].

In the case when X is the universal modular curve, the Hodge correlators contain the special values $L(f,n)$ of weight $k \geq 2$ modular forms for $GL_2(\mathbb{Q})$ outside of the critical strip – it turns out that the simplest Hodge correlators in this case coincide with the Rankin-Selberg integrals for the non-critical special values $L(f, k+n)$, $n \geq 0$ – the case $k = 2, n = 0$ is discussed in detail in [G1].

2 Hodge correlators for local systems

2.1 An action of G_T on the " minimal model"of $\mathcal{D}_{\rm sm}(X)$.

Tensor products of irreducible local systems are semisimple local systems. The *category of harmonic bundles* ${\rm Har}_X$ is the graded category whose objects are semi-simple local systems on X and their shifts, and morphisms are given by graded vector spaces

$$\mathrm{Hom}^\bullet_{\mathrm{Har}_X}(V_1, V_2) := H^\bullet(X, V_1^\vee \otimes V_2). \tag{3}$$

Here is our main result.

Theorem 2.1 *There is a homotopy action of the twistor Galois group G_T by A_∞-equivalences of the graded category ${\rm Har}_X$, such that the action of the subgroup $\mathbb{C}^*$ is given by Simpson's action of $\mathbb{C}^*$ on semi-simple local systems.*

This immediately implies Theorem 1.3. Indeed, given a small A_∞-category $\mathcal{A}$, there is a functorial constraction of the triangulated envelope ${\rm Tr}(\mathcal{A})$ of $\mathcal{A}$, the smallest triangulated category containing $\mathcal{A}$. Since $\mathcal{D}^b_{\rm sm}(X)$ is generated as a triangulated category by semi-simple local systems, the category ${\rm Tr}({\rm Har}_X)$ is equivalent to $\mathcal{D}^b_{\rm sm}(X)$ as a triangulated category, and thus is an A_∞-enhancement of the latter. On the other hand, the action of the group G_T from Theorem 2.1 extends by functoriality to the action on ${\rm Tr}({\rm Har}_X)$.

Below we recall what are A_∞-equivalences of DG categories and then define the corresponding data in our case.

2.2 A_∞-equivalences of DG categories

The Hochshild cohomology of a small dg-category $\mathcal{A}$. Let $\mathcal{A}$ be a small dg category. Consider a bicomplex whose n-th column is

$$\prod_{[X_i]} \mathrm{Hom}\Big(\mathcal{A}(X_0, X_1)[1] \otimes \mathcal{A}(X_1, X_2)[1] \otimes \ldots \otimes \mathcal{A}(X_{n-1}, X_n)[1], \mathcal{A}(X_0, X_n)[1]\Big), \qquad (4)$$

where the product is over isomorphism classes $[X_i]$ of objects of the category $\mathcal{A}$. The vertical differential d_1 in the bicomplex is given by the differential on the tensor product of complexes. The horisontal one d_2 is the degree 1 map provided by the composition

$$\mathcal{A}(X_i, X_{i+1}) \otimes \mathcal{A}(X_{i+1}, X_{i+2}) \longrightarrow \mathcal{A}(X_i, X_{i+2}).$$

Let $\mathrm{HC}^*(\mathcal{A})$ be the total complex of this bicomplex. Its cohomology are the Hochshild cohomology $\mathrm{HH}^*(\mathcal{A})$ of $\mathcal{A}$. Let $\mathrm{Fun}_{A_\infty}(\mathcal{A}, \mathcal{A})$ be the space of A_∞-functors from $\mathcal{A}$ to itself. Lemma 2.2 can serve as a definition of A_∞-functors considered modulo homotopy equivalence.

Lemma 2.2 *One has*

$$H^0\mathrm{Fun}_{A_\infty}(\mathcal{A}, \mathcal{A}) = \mathrm{HH}^0(\mathcal{A}). \qquad (5)$$

Indeed, a cocycle in $\mathrm{HC}^0(\mathcal{A})$ is the same thing as an A_∞-functor. Coboundaries corresponds to the homotopic to zero functors.

The cyclic homology of a small rigid dg-category $\mathcal{A}$. Let $(\alpha_0 \otimes ... \otimes \alpha_m)_\mathcal{C}$ be the projection of $\alpha_0 \otimes ... \otimes \alpha_m$ to the coinvariants of the cyclic shift. So, if $\overline{\alpha} := \mathrm{deg}\alpha$,

$$(\alpha_0 \otimes ... \otimes \alpha_m)_\mathcal{C} = (-1)^{\overline{\alpha}_m(\overline{\alpha}_0 + ... + \overline{\alpha}_{m-1})}(\alpha_1 \otimes ... \otimes \alpha_m \otimes \alpha_0)_\mathcal{C}.$$

We assign to $\mathcal{A}$ a bicomplex whose n-th column is

$$\prod_{[X_i]} \Big(\mathcal{A}(X_0, X_1)[1] \otimes \ldots \otimes \mathcal{A}(X_{n-1}, X_n)[1] \otimes \mathcal{A}(X_n, X_0)[1]\Big)_\mathcal{C}.$$

The differentials are induced by the differentials and the composition maps on Hom's. The cyclic homology complex $\mathrm{CC}_*(\mathcal{A})$ of $\mathcal{A}$ is the total complex of this bicomplex. Its homology are the cyclic homology of $\mathcal{A}$.

Assume that there are functorial pairings

$$\mathcal{A}(X, Y)[1] \otimes \mathcal{A}(Y, X)[1] \longrightarrow \mathcal{H}^*.$$

Then there is a morphism of complexes

$$\mathrm{HC}^*(\mathcal{A})^* \longrightarrow \mathrm{CC}_*(\mathcal{A}) \otimes \mathcal{H}. \qquad (6)$$

For the category of harmonic bundles Har_X there is such a pairing with

$$\mathcal{H} := H_{2n}(X)[-2].$$

It provides a map

$$\varphi : \mathrm{Hom}\Big(H_0(\mathrm{CC}_*(\mathrm{Har}_X) \otimes \mathcal{H}, \mathbb{C}\Big) \longrightarrow \mathrm{HH}^0(\mathrm{Har}_X) \stackrel{(5)}{=} H^0\mathrm{Fun}_{A_\infty}(\mathrm{Har}_X, \mathrm{Har}_X). \qquad (7)$$

2.3 The Hodge correlators

Theorem 2.3 *a) There is a linear map, the Hodge correlator map*

$$\mathrm{Cor}_{\mathrm{Har_X}} : H_0(\mathrm{CC}_*(\mathrm{Har}_X) \otimes \mathcal{H}) \longrightarrow \mathbb{C}. \tag{8}$$

Combining it with (7), we get a cohomology class

$$\mathbf{H}_{\mathrm{Har_X}} := \varphi(\mathrm{Cor}_{\mathrm{Har_X}}) \in H^0\mathrm{Fun}_{A_\infty}(\mathrm{Har}_X, \mathrm{Har}_X). \tag{9}$$

b) There is a homotopy action of the twistor Galois group G_{T} *by* A_∞*-autoequivalences of the category* Har_X *such that*

- *Its restriction to the subgroup* $\mathbb{C}^*$ *is the Simpson action [Si1] on the category* Har_X*.*
- *Its restriction to the Lie algebra* $\mathrm{LieU_{Hod}}$ *is given by a Lie algebra map*

$$\mathbb{H}_{\mathrm{Har_X}} : \mathrm{L_{Hod}} \longrightarrow \mathrm{H}^0\mathrm{Fun}_{A_\infty}(\mathrm{Har}_X, \mathrm{Har}_X), \tag{10}$$

uniquely determined by the condition that $\mathbb{H}_{\mathrm{Har_X}}(G) = \mathbf{H}_{\mathrm{Har_X}}$.

c) The action of the group G_{T} *is functorial with respect to the pull backs.*

2.4 Construction.

To define the Hodge correlator map (8), we define a collection of degree zero maps

$$\mathrm{Cor}_{\mathrm{Hod_X}} : \Big(H^\bullet(X, V_0^* \otimes V_1)[1] \otimes \ldots \otimes H^\bullet(X, V_m^* \otimes V_0)[1]\Big)_{\mathcal{C}} \otimes \mathcal{H} \longrightarrow \mathbb{C}. \tag{11}$$

The definition depends on some choices, like harmonic representatatives of cohomology classes. We prove that it is well defined on HC^0, i.e. its resctriction to cycles is independent of the choices, and coboundaries are mapped to zero.

We picture an element in the sourse of the map (11) by a polygon P, see Fig 1, whose vertices are the objects V_i, and the oriented sides V_iV_{i+1} are graded vector space $\mathrm{Ext}^*(V_i, V_{i+1})(1)$.

Figure 1: A decorated plane trivalent tree; V_i are harmonic bundles.

Green currents for harmonic bundles. Let V be a harmonic bundle on X. Then there is a Doulbeaut bicomplex $(\mathcal{A}^\bullet(X,V); D', D'')$ where the differentials D', D'' are provided by the complex structure on X and the harmonic metric on V. It satisfies the D', D''-lemma.

Choose a splitting of the corresponding de Rham complex $\mathcal{A}^\bullet(X,V)$ into an arbitrary subspace $\mathcal{H}ar^\bullet(X,V)$ isomorphically projecting onto the cohomology $H^\bullet(X,V)$ ("harmonic forms") and its orthogonal complement. If $V = \mathbb{C}_X$, we choose $a \in X$ and take the δ-function δ_a at the point $a \in X$ as a representative of the fundamental class.

Let δ_Δ be the Schwarz kernel of the identity map $V \to V$ given by the δ-function of the diagonal, and $P_{\rm Har}$ the Schwarz kernel of the projector onto the space $\mathcal{H}ar^\bullet(X,V)$, realized by an (n,n)-form on $X \times X$. Choose a basis $\{\alpha_i\}$ in $\mathcal{H}ar^\bullet(X,V)$. Denote by $\{\alpha_i^\vee\}$ the dual basis. Then we have

$$P_{\rm Har} = \sum \alpha_i^\vee \otimes \alpha_i, \qquad \int_X \alpha_i \wedge \alpha_j^\vee = \delta_{ij}.$$

Let $p_i : X \times X \to X$ be the projections onto the factors.

Definition 2.4 *A Green current $G(V;x,y)$ is a $p_1^*V^* \otimes p_2^*V$-valued current on $X \times X$,*

$$G(V;x,y) \in \mathcal{D}^{2n-2}(X \times X, p_1^*V^* \otimes p_2^*V), \quad n = \dim_{\mathbb{C}} X,$$

which satisfies the differential equation

$$(2\pi i)^{-1} D'' D' G(V;x,y) = \delta_\Delta - P_{\rm Har}. \tag{12}$$

The two currents on the right hand side of (12) represent the same cohomology class, so the equation has a solution by the $D''D'$-lemma.

Remark. The Green current depends on the choice of the "harmonic forms". So if $V = \mathbb{C}$, it depends on the choise of the base point a. Solutions of equation (12) are well defined modulo ${\rm Im}D'' + {\rm Im}D' + \mathcal{H}ar^\bullet(X,V)$.

Construction of the Hodge correlators. *Trees.* Take a plane trivalent tree T dual to a triangulation of the polygon P, see Fig 1. The complement to T in the polygon P is a union of connected domains parametrized by the vertices of P, and thus decorated by the harmonic bundles V_i. Each edge E of the tree T is shared by two domains. The corresponding harmonic bundles are denoted V_{E-} and V_{E+}. If E is an external edge, we assume that V_{E-} is before V_{E+} for the clockwise orientation.

Given an internal vertex v of the tree T, there are three domains sharing the vertex. We denote the corresponding harmonic bundles by V_i, V_j, V_k, where the cyclic order of the bundles agrees with the clockwise orientation. There is a natural trace map

$${\rm Tr}_v : V_i^* \otimes V_j \otimes V_j^* \otimes V_k \otimes V_k^* \otimes V_i \longrightarrow \mathbb{C}. \tag{13}$$

It is invariant under the cyclic shift.

Decorations. For every edge E of T, choose a graded splitting of the de Rham complex

$$\mathcal{A}^\bullet(X, V_{E-}^* \otimes V_{E+}) = \mathcal{H}ar^\bullet(X, V_{E-}^* \otimes V_{E+}) \bigoplus \mathcal{H}ar^\bullet(X, V_{E-}^* \otimes V_{E+})^\perp.$$

Then a decomposable class in $\Big(\otimes_{i=0}^m H^*(X, V_i^\bullet \otimes V_{i+1})[1]\Big)_{\mathcal{C}}$ has a harmonic representative

$$W = \Big(\alpha_{0,1} \otimes \alpha_{1,2} \otimes \ldots \otimes \alpha_{m,0}\Big)_{\mathcal{C}}.$$

We are going to assign to W a top degree current $\kappa(W)$ on

$$X^{\{\text{internal vertices of } T\}}. \tag{14}$$

Each external edge E of the tree T is decorated by an element

$$\alpha_E \in \mathcal{H}ar^\bullet(X, V_{E-}^* \otimes V_{E+}).$$

Put the current α_E to the copy of X assigned to the internal vertex of the edge E, and pull it back to (14) using the projection p_{α_E} of the latter to the X. Abusing notation, we denote the pull back by α_E. It is a form on (14) with values in the bundle $p_{\alpha_E}^*(V_{E-}^* \otimes V_{E+})$

Green currents. We assign to each internal edge E of the tree T a Green current

$$G(V_{E-}^* \otimes V_{E+}; x_-, x_+). \tag{15}$$

The order of (x_-, x_+) agrees with the one of (V_{E-}^*, V_{E+}) as on Fig 2: the cyclic order of $(V_{E-}^*, x_-, V_{E+}, x_+)$ agrees with the clockwise orientation. The Green current (15) is symmetric:

$$G(V_{E-}^* \otimes V_{E+}; x_-, x_+) = G(V_{E+}^* \otimes V_{E-}; x_+, x_-). \tag{16}$$

So it does not depend on the choice of orientation of the edge E.

Figure 2: Decorations of the Green current assigned to an edge E.

The map ξ. There is a degree zero map

$$\xi : \mathcal{A}^\bullet(X, V_0)[-1] \otimes \ldots \otimes \mathcal{A}^\bullet(X, V_m)[-1] \longrightarrow \mathcal{A}^\bullet(X, V_0 \otimes ... \otimes V_m)[-1]; \tag{17}$$

$$\varphi_0 \otimes \ldots \otimes \varphi_m \longmapsto \mathrm{Sym}_{\{0,\ldots,m\}}\Big(\varphi_0 \wedge D^{\mathbb{C}}\varphi_1 \wedge \ldots \wedge D^{\mathbb{C}}\varphi_m\Big). \tag{18}$$

The graded symmetrization in (18) is defined via isomorphisms $V_{\sigma(0)} \otimes ... \otimes V_{\sigma(m)} \to V_0 \otimes ... \otimes V_m$, where σ is a permutation of $\{0, ..., m\}$. It is essential that $\deg D^{\mathbb{C}}\varphi = \deg\varphi + 1$.

An outline of the construction. We apply the operator ξ to the product of the Green currents assigned to the internal edges of T. Then we multiply on (14) the obtained local system valued current with the one provided by the decoration W, with an appropriate sign. Applying the product of the trace maps (13) over the internal vertices of T, we get a top degree scalar current on (14). Integrating it we get a number assigned to T. Taking the sum over all plane trivalent trees T decorated by W, we get a complex number $\mathrm{Cor}_{\mathrm{Har_X}}(W \otimes \mathcal{H})$. Altogether, we get the map (8). One checks that its degree is zero. The signs in this definition are defined the same way as in [G2].

Theorem 2.5 *The maps (11) give rise to a well defined Hodge correlator map (8).*

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Bounding eigenfunctions on arithmetic surfaces

Valentin Blomer

Let M be a compact Riemannian manifold with Laplace operator Δ, and let ϕ be a normalized eigenfunction,

$$\Delta\phi + \lambda\phi = 0, \quad \|\phi\|_2 = 1.$$

What can we say about $\|\phi\|_\infty$ with respect to λ and M?

This is an interesting question for several reasons: on the one hand, sup-norm bounds give bounds for the multiplicities of eigenvalues:

$$\dim V_\lambda \leq \mathrm{vol}(M) \max_{\substack{\phi \in V_\lambda \\ \|\phi\|_2 = 1}} \|\phi\|_\infty^2.$$

Secondly it is often important and useful to bound an eigenfunction ϕ along certain "periods". Pointwise bounds are in a sense the strongest form of such results, where the period degenerates to a single point. A non-trivial sup-norm bound can therefore be interpreted as a type of equidistribution statement, saying that the mass of ϕ cannot be concentrated too much. We will see that this has connections to the subconvexity problem for L-functions. See [7] for an enlightening discussion.

For $\dim M = 2$ we have the general bound

$$\|\phi\|_\infty \ll (1+\lambda)^{1/4},$$

and this is sharp, as can be seen for $M = S^2$ and ϕ a zonal spherical function. In this talk, we want to consider two examples where the surface is "of arithmetic type". It turns out that the sup-norm problem, originally a purely analytic question, then becomes a problem in diophantine analysis.

Modular curves - the case $SL(2)$

Consider

$$M = X_0(N) = \Gamma_0(N)\backslash\mathbb{H} \to X_0(1), \quad N \text{ squarefree},$$

a cover of the standard modular cover $X_0(1)$. As usual, $\mathbb{H}$ is the upper half plane, a surface with constant negative curvature -1, and $\Gamma_0(N)$ denotes the group 2×2 integer matrix with determinant 1 and left lower corner divisible by N. This example doesn't quite fit into the above framework, as M is non-compact, but it is of finite volume: $\mathrm{vol}(M) = N^{1+o(1)} < \infty$. In cartesian coordinates the Laplacian is given by

$$\Delta = y^2(\partial_x^2 + \partial_y^2).$$

The space $\Gamma_0(N)\backslash\mathbb{H}$ is equipped with arithmetic extra structure, owing to the fact that $\Gamma_0(N)$ is a "congruence subgroup": there is a commutative family of Hecke operators

$$T_n = \frac{1}{\sqrt{n}} \sum_{\substack{ad=n \\ b \,(\mathrm{mod}\ d)}} \begin{pmatrix} a & b \\ & d \end{pmatrix} \in \mathbb{C}[PGL_2]$$

that also commutes with Δ. Assuming that ϕ is a joint eigenfunction of Δ and all Hecke operators, the first breakthrough was obtained by Iwaniec and Sarnak:

Theorem 1. *[6] In the above setting with $N = 1$ one has*

$$\|\phi\|_\infty \ll (1+\lambda)^{5/24+\varepsilon}.$$

The same bound holds for Eisenstein series in compact regions. A direct application is subconvexity for the Dedekind zeta-function of $\mathbb{Q}(i)$: we have a "trivial" period formula $E(1/2+it, i) = \zeta_{\mathbb{Q}(i)}(1/2+it)\times$ some simple factors. Since the eigenvalue of $E(1/2+it, z)$ is $1/4+t^2$, we conclude

$$\zeta_{\mathbb{Q}(i)}(1/2+it) \ll (1+|t|)^{5/12+\varepsilon}.$$

Of course, better bounds are known by other methods (the strongest exponent is slightly better than 1/3), but this shows already a first connection to L-functions.

With a little extra work [1], one can show $\|\phi\|_\infty \ll (1+\lambda)^{5/24}((1+\lambda)N)^\varepsilon$ for any squarefree N. The bound N^ε can be regarded as the trivial bound in the level aspect. We are interested in non-trivial bounds *simultaneously* with respect to the eigenvalue and the volume.

Theorem 2. *[1] Keeping the notation an assumption as above, there is $\delta > 0$ such that*

$$\|\phi\|_\infty \ll \lambda^{1/4}(vol(M)\lambda^{1/2})^{-\delta}$$

One can take $\delta = 1/2500$.

The volume bound holds for automorphic forms ϕ with arbitrary (fixed) K-type, in particular for holomorphic cusp forms. The best possible value for δ would be 1/2 (at least in any fixed compact region of M) which would imply the Lindelöf hypothesis in the level aspect for the standard L-function

$L(1/2, \phi)$.

The proof starts by considering an amplified pre-trace formula. The geometric side of the trace formula leads to a counting problem of the type

$$\#\left\{\gamma = \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \mid \det\gamma = \ell, \mathrm{dist}(z, \gamma z) < \delta\right\},$$

uniformly in $z \in M$, $0 < \delta < 1$, and ℓ an integer of size N^η for some small $\eta > 0$. We see now clearly that a diophantine problem has arisen. An extra difficulty, not existent in the work of Iwaniec-Sarnak occurs: the above count will not produce the desired saving in the volume aspect if $z = x + iy$ is such that x is "well approximable" by rational numbers, that is $x \approx a/q$ with q "small". Reminiscent of the circle method, we therefore split each horocycle into major and minor arcs. The trace formula approach works well for the minor arcs, whereas on the major arcs one has to "pre-condition" the point z by applying a certain Atkin-Lehner involution first. This leads to considerable technical complications that have partly been simplified in [8]. A somewhat different approach can be found in [5].

Unions of ellipsoids - the case $SO(3)$

Next we consider

$$M = \coprod_{j=1}^{h} S^2.$$

which we realize arithmetically as follows: Let B be a positive quaternion algebra over $\mathbb{Q}$ of discriminant D. Then $PB^\times$ is isomorphic to $SO(B^0) \cong SO(3)$ (where B^0 denotes the traceless quaternions) via conjugation. Let $\mathcal{O}$ be an Eichler order in B of level N with $(N, D) = 1$, that is, locally we have $\mathcal{O}_p \cong \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ N\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$. Let $K_f := \mathbb{A}_{\text{fin}}^\times \backslash \mathbb{A}_{\text{fin}}^\times \widehat{\mathcal{O}}^\times$ and fix once and for all a point x_∞ on the ellipsoid $\{x \in B^0 \mid \mathrm{nr}(x) = 1\}$, say the north pole. The stabilizer of x_∞ under the action of $PB^\times$ is isomorphic to $SO(2)$. Therefore we have

$$M := PB^\times \backslash PB^\times(\mathbb{A})/K_f SO(2) = \coprod_{j=1}^{h} S^2$$

where $h \approx DN/12 \approx \mathrm{vol}(M)$ is the class number of B. Note that each connected component is equipped with its own quadratic form, but all these quadratic forms are locally conjugated (in the same genus).

Theorem 3. *[2] Let ϕ be an L^2-normalized Hecke-Laplace eigenform on M of Laplacian eigenvalue $\lambda = k(k+1)$ for some $k \in \mathbb{N}_0$ Then there is a $\delta > 0$ such that*

$$\|\phi\|_\infty \ll (1+\lambda)^{1/4}(vol(M)(1+\lambda)^{1/2})^{-\delta}.$$

One can take $\delta = 1/40$, cf. [3].

In the case of the maximal order in the Hamilton quaternions, that is, $D = 2$, $N = 1$, $h = 1$, a corresponding result was proved in [9]. The proof starts again with an amplified trace formula which leads to

$$\sum_{\substack{\gamma \in \mathcal{O}_j \\ \mathrm{nr}(\gamma) = \ell}} p_k \left(\frac{1}{2} \mathrm{tr}(x, \gamma \cdot x) \right)$$

where p_k is the k-th Legendre polynomial and $\mathcal{O}_j$ is the the order of the j-th ideal class. This amounts to counting integral points on some slightly thickened S^1 inside a slant ellipsoid.

Remark 1: The proof works for a totally positive quaternion algebra B over any totally real number field. This solves automatically the sup-norm problem on $SO(4)$: if the discriminant Δ of the underlying 4-dimensional quadratic space is a square, we need to consider products of functions on $SO(3) \times SO(3)$, otherwise functions on $SO(3)$ over the field extension $\mathbb{Q}(\sqrt{\Delta})$.

Remark 2: Theorem 3 is even non-trivial in the case $\lambda = 0$ (that is, ϕ is locally constant), $D = p$ a prime, $N = 1$. In this case the set of ideal classes of B can be identified with the set of isomorphy classes of supersingular elliptic curves defined over $\overline{\mathbb{F}}_q$, and our theorem shows that a locally constant function on this space cannot accumulate too much of its mass on a single component.

Remark 3: Eichler has shown a correspondence between the eigenspace $V_{k(k+1)}$ and the space of holomorphic cusp forms of weight $2 + k$. Let K be an imaginary quadratic field that embeds into B, and fix a class group character χ viewed as a weight 1 CM form. Gross' [4] formula expresses the special value of the Rankin-Selberg L-function $L(f \times \chi, 1/2)$ where f is a cusp of weight $2 + k$, as the square of a weighted sum of f over Heegner points. Estimating trivially with the sup-norm bound from Theorem 3, one obtains

$$L(f \times \chi, 1/2) \ll k^{6/7}.$$

Again, over $\mathbb{Q}$ there are stronger subconvexity exponents available, but on the one hand our exponent is uniform in the number field, and on the other hand our proof is a sense purely diophantine.

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QUANTUM KNOT INVARIANTS

STAVROS GAROUFALIDIS

To Don Zagier, with admiration

Abstract. This is a survey talk on one of the best known quantum knot invariants, the colored Jones polynomial of a knot, and its relation to the algebraic/geometric topology and hyperbolic geometry of the knot complement. We review several aspects of the colored Jones polynomial, emphasizing modularity, stability and effective computations. The talk was given in the Mathematische Arbeitstagung June 24-July 1, 2011.

Contents

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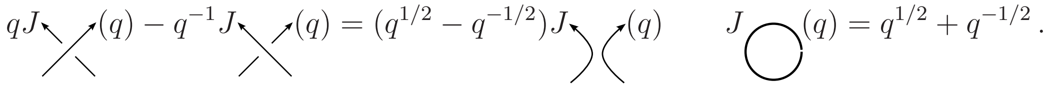
1. The Jones polynomial of a knot

Quantum knot invariants are powerful numerical invariants defined by Quantum Field theory with deep connections to the geometry and topology in dimension three [Wit89]. This is a survey talk on the various limits the colored Jones polynomial [Jon87], one of the best known quantum knot invariants. This is a 25 years old subject that contains theorems and conjectures in disconnected areas of mathematics. We chose to present some old and recent conjectures on the subject, emphasizing two recent aspects of the colored Jones polynomial, Modularity and Stability and their illustration by effective computations. Zagier's influence on this subject is profound, and several results in this talk are joint work with Don. We thank Don Zagier for his generous sharing of his ideas with us.

The Jones polynomial $J_L(q) \in \mathbb{Z}[q^{\pm 1/2}]$ of a link L in 3-space is uniquely determined by the linear relations [Jon87]



The Jones polynomial has a unique extension to a polynomial invariant $J_{L,c}(q)$ of links L together with a coloring c of their components are colored by positive natural numbers that satisfy the following rules

$$
\begin{aligned}
J_{L\cup K, c\cup\{N+1\}}(q) &= J_{L\cup K^{(2)}, c\cup\{N,2\}}(q) - J_{L\cup K, c\cup\{N-1\}}(q), \qquad N \geq 2 \\
J_{L\cup K, c\cup\{1\}}(q) &= J_{L,c}(q) \\
J_{L,\{2,\dots,2\}}(q) &= J_L(q)
\end{aligned}
$$

where $(L \cup K, c \cup \{N\})$ denotes a link with a distinguished component K colored by N and $K^{(2)}$ denotes the 2-parallel of K with zero framing.

Here, a natural number N attached to a component of a link indicates the N-dimensional irreducible representation of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. For a detailed discussion on the polynomial invariants of links that come from quantum groups, see [Jan96, Tur88, Tur94]. The above relations make clear that the colored Jones polynomial of a knot determines and is uniquely determined by the Jones polynomial of a knot and its 0-framed parallels.

2. Three limits of the colored Jones polynomial

In this section we will list three conjectures, the MMR Conjecture (proven), the Slope Conjecture (mostly proven) and the AJ Conjecture (less proven). These conjectures relate the colored Jones polynomial of a knot with the Alexander polynomial, with the set of slopes of incompressible surfaces and with the $\mathrm{PSL}(2,\mathbb{C})$ character variety of the knot complement.

2.1. The colored Jones polynomial and the Alexander polynomial.

We begin by discussing a relation of the colored Jones polynomial with the homology of the universal abelian cover of its complement. The homology $H_1(M,\mathbb{Z}) \simeq \mathbb{Z}$ of the complement $M = S^3 \setminus K$ of a knot K in 3-space is independent of the knot K. This allows us to consider the universal abelian cover $\widetilde{M}$ of M with deck transformation group $\mathbb{Z}$, and with homology $H_1(\widetilde{M},\mathbb{Z})$ a $\mathbb{Z}[t^{\pm 1}]$ module. As is well-known this module is essentially torsion and its order is given by the Alexander polynomial $\Delta_K(t) \in \mathbb{Z}[t^{\pm 1}]$ of K [Rol90]. The Alexander polynomial does not distinguish knots from their mirrors and satisfies $\Delta_K(1) = 1$.

There are infinitely many pairs of knots (for instance $(10_{22}, 10_{35})$ in the Rolfsen table [Rol90, BN]) with equal Jones polynomial but different Alexander polynomial. On the other hand, the colored Jones polynomial determines the Alexander polynomial. This so-called Melvin-Morton-Rozansky Conjecture was proven in [BNG96], and states that

$$\hat{J}_{K,n}(e^{\hbar}) = \sum_{i \geq j \geq 0} a_{K,ij} \hbar^i n^j \in \mathbb{Q}[[n, \hbar]] \tag{1}$$

and

$$\sum_{i=0}^{\infty} a_{K,ij} \hbar^i = \frac{1}{\Delta_K(e^{\hbar})} \in \mathbb{Q}[[\hbar]] \, .$$

Here $\hat{J}_{K,n}(q) = J_{K,n}(q)/J_{\mathrm{Unknot},n}(q) \in \mathbb{Z}[q^{\pm 1}]$ is a normalized form of the colored Jones polynomial. The above conjecture is a statement about formal power series. A stronger analytic version is known [GL11a, Thm.1.3], namely for every knot K there exists an open neighborhood U_K of $0 \in \mathbb{C}$ such that for all $\alpha \in U_K$ we have

$$\lim_n J_{K,n}(e^{\alpha/n}) = \frac{1}{\Delta_K(e^{\alpha})} \, ,$$

where convergence is uniform with respect to compact sets. More is known about the summation of the series (1) along a fixed diagonal $i = j + k$ for fixed k, both on the level of formal power series and on the analytic counterpart. For further details the reader may consult [GL11a] and references therein.

2.2. **The colored Jones polynomial and slopes of incompressible surfaces.** In this section we discuss a conjecture relating the degree of the colored Jones polynomial of a knot K with the set bs_K of boundary slopes of incompressible surfaces in the knot complement $M = S^3 \setminus K$. Although there are infinitely many incompressible surfaces in M, it is known that $\mathrm{bs}_K \subset \mathbb{Q} \cup \{1/0\}$ is a finite set [Hat82]. Incompressible surfaces play an important role in geometric topology in dimension three, often accompanied by the theory of normal surfaces. From our point of view, incompressible surfaces are a tropical limit of the colored Jones polynomial, corresponding to an expansion around $q = 0$ [Gar11c].

The Jones polynomial of a knot is a Laurent polynomial in one variable q with integer coefficients. Ignoring most information, one can consider the degree $\delta_K(n)$ of $\hat{J}_{K,n+1}(q)$ with respect to q. Since $(\hat{J}_{K,n}(q))$ is a q-holonomic sequence [GL05], it follows that δ_K is a quadratic quasi-polynomial [Gar11a]. In other words, we have

$$\delta_K(n) = c_K(n)n^2 + b_K(n)n + a_K(n) \, ,$$

where $a_K, b_K, c_K : \mathbb{N} \longrightarrow \mathbb{Q}$ are periodic functions. In [Gar11b] the author formulated the Slope Conjecture.

Conjecture 2.1. For all knots K we have

$$4c_K(\mathbb{N}) \subset \mathrm{bs}_K \, .$$

The movitating example for the Slope Conjecture was the case of the $(-2,3,7)$ pretzel knot, where we have [Gar11b, Ex.1.4]

$$\delta_{(-2,3,7)}(n) = \left[\frac{37}{8}n^2 + \frac{17}{2}n\right] = \frac{37}{8}n^2 + \frac{17}{2}n - \epsilon(n),$$

where $\epsilon(n)$ is a periodic sequence of period 4 given by $0, 1/8, 1/2, 1/8$ if $n \equiv 0, 1, 2, 3 \bmod 4$ respectively. In addition, we have

$$\mathrm{bs}_{(-2,3,7)} = \{0, 16, 37/2, 20\}\,.$$

In all known examples, $c_K(\mathbb{N})$ consists of a single element, the so-called Jones slope. How the colored Jones polynomial selects one of the finitely many boundary slopes is a challenging and interesting question. The Slope Conjecture is known for all torus knots, all alternating knots and all knots with at most 8 crossings [Gar11b] as well as for all adequate knots [FKP11] and all 2-fusion knots [DG12b].

2.3. **The colored Jones polynomial and the** $\mathrm{PSL}(2,\mathbb{C})$ **character variety.** In this section we discuss a conjecture relating the colored Jones polynomial of a knot K with the moduli space of $\mathrm{SL}(2,\mathbb{C})$-representations of M, restricted to the boundary of M. The latter is a 1-dimensional plane curve (ignoring 0-dimensional components). To formulate the conjecture we need to recall that the colored Jones polynomial $\hat{J}_{K,n}(q)$ is q-holonomic [GL05] i.e., it satisfies a non-trivial linear recursion relation

$$\sum_{j=0}^{d} a_j(q, q^n)\hat{J}_{K,n+j}(q) = 0 \tag{2}$$

for all n where $a_j(u,v) \in \mathbb{Z}[u^{\pm 1}, v^{\pm 1}]$ and $a_d \neq 0$. q-holonomic sequences were introduced by Zeilberger [Zei90], and a fundamental theorem (multisums of q-proper hypergeometric terms are q-holonomic) was proven in [WZ92] and implemented in [PWZ96]. Using two operators M and L which act on a sequence $f(n)$ by

$$(Mf)(n) = q^n f(n), \qquad (Lf)(n) = f(n+1)\,,$$

we can write the recursion (2) in operator form

$$P \cdot \hat{J}_K = 0 \qquad \text{where} \qquad P = \sum_{j=0}^{d} a_j(q, M)L^j\,.$$

It is easy to see that $LM = qML$ and M, L generate the q-Weyl algebra. One can choose a canonical recursion $A_K(M, L, q) \in \mathbb{Z}[q, M]\langle L\rangle/(LM - qML)$ which is a knot invariant [Gar04], the non-commutative A-polynomial of K.

The reason for this terminology is the potential relation with the A-polynomial $A_K(M, L)$ of K [CCG+94]. The latter is defined as follows. Let $X_M = \mathrm{Hom}(\pi_1(M), \mathrm{SL}(2,\mathbb{C}))/\mathbb{C}$ denote the moduli space of flat $\mathrm{SL}(2,\mathbb{C})$ connections on M. We have an identification

$$X_{\partial M} \simeq (\mathbb{C}^*)^2/(\mathbb{Z}/2\mathbb{Z}), \qquad \rho \mapsto (M, L)$$

where $\{M, 1/M\}$ (resp., $\{L, 1/L\}$) are the eigenvalues of $\rho(\mu)$ (resp., $\rho(\lambda)$) where (μ, λ) is a meridian-longitude pair on ∂M. X_M and $X_{\partial M}$ are affine varieties and the restriction map $X_M \longrightarrow X_{\partial M}$ is algebraic. The Zariski closure of its image lifted to $(\mathbb{C}^*)^2$, and after removing any 0-dimensional components is a one-dimensional plane curve with defining polynomial $A_K(M, L)$ [CCG+94]. This polynomial plays an important role in the hyperbolic geometry of the knot complement. We are now ready to formulate the AJ Conjecture [Gar04, Gel02]. Let us say that two polynomials $P(M, L) =_M Q(M, L)$ are essentially equal if their irreducible factors with positive L-degree are equal.

Conjecture 2.2. For all knots K, we have $A_K(M^2, L, 1) =_M A_K(M, L)$.

The AJ Conjecture was checked for the 3_1 and the 4_1 knots in [Gar04]. It is known for most 2-bridge knots [Lê06], for torus knots and for the pretzel knots of Section 4; see [LT, Tra].

From the point of view of physics, the AJ Conjecture is a consequence of the fact that quantization and the corresponding quantum field theory exists [Guk05, Dim].

3. The Volume and Modularity Conjectures

3.1. **The Volume Conjecture.** The Kashaev invariant of a knot is a sequence of complex numbers defined by [Kas97, MM01]

$$\langle K\rangle_N = \hat{J}_{K,N}(e(1/N))$$

where $e(\alpha) = e^{2\pi i\alpha}$. The Volume Conjecture concerns the exponential growth rate of the Kashaev invariant and states that

$$\lim_N \frac{1}{N}\log|\langle K\rangle_N| = \frac{\mathrm{vol}(K)}{2\pi}$$

where Vol(K) is the volume of the hyperbolic pieces of the knot complement $S^3 \setminus K$. Among hyperbolic knots, the Volume Conjecture is known only for the 4_1 knot; for a detailed computation see [Mur04]. Refinements of the Volume Conjecture to all orders in N and generalizations were proposed by several authors [DGLZ09, GM08, GL11a, Gar08]. Although proofs are lacking, there appears to be a lot of structure in the asymptotics of the Kashaev invariant. In the next section we will discuss a modularity conjecture of Zagier and some numerical verification.

3.2. **The Modularity Conjecture.** Zagier considered the Galois invariant spreading of the Kashaev invariant on the set of complex roots of unity given by

$$\phi_K : \mathbb{Q}/\mathbb{Z} \longrightarrow \mathbb{C}, \qquad \phi_K\left(\frac{a}{c}\right) = \hat{J}_{K,c}\left(e\left(\frac{a}{c}\right)\right)$$

where $(a, c) = 1$ and $c > 0$. The above formula works even when a and c are not coprime due to a symmetry of the colored Jones polynomial [Hab02]. Moreover ϕ_K determines $\langle K\rangle$ and is uniquely determined by $\langle K\rangle$ via Galois invariance.

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ and $\alpha = a/c$ and $\hbar = 2\pi i/(X + d/c)$ where $X \longrightarrow +\infty$ with bounded denominators. Let $\phi = \phi_K$ denote the extended Kashaev invariant of a hyperbolic knot K and let $F \subset \mathbb{C}$ denote the invariant trace field of $M = S^3 \setminus K$ [MR03]. Let $C(M) \in \mathbb{C}/(4\pi^2\mathbb{Z})$ denote the complex Chern-Simons invariant of M [GZ07, Neu04].

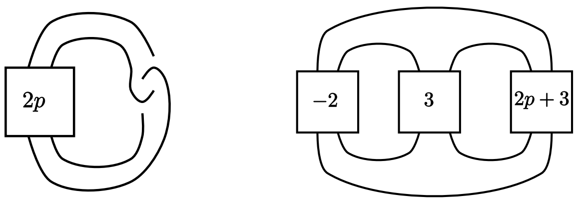
Conjecture 3.1. With the above conventions, there exist $\Delta(\alpha) \in \mathbb{C}$ with $\Delta(\alpha)^{2c} \in F(\epsilon(\alpha))$ and $A_j(\alpha) \in F(e(\alpha))$ such that

$$\frac{\phi(\gamma X)}{\phi(X)} \sim \left(\frac{2\pi}{\hbar}\right)^{3/2} e^{C(M)/\hbar} \Delta(\alpha) \sum_{j=0}^{\infty} A_j(\alpha) \hbar^j \,. \tag{3}$$

When $\gamma = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $X = N - 1$, and with the properly chosen orientation of M, the leading asymptotics of (3) together with the fact that $\Im(C(M)) = \mathrm{vol}(M)$ gives the volume conjecture.

4. Computation of the non-commutative A-polynomial

As we will discuss below, the key to an effective computation the Kashaev invariant is a recursion for the colored Jones polynomial. Proving or guessing such a recursion is at least as hard as computing the A-polynomial of the knot. The A-polynomial is already unknown for several knots with 9 crossings. For an updated table of computations see [Cul10]. The A-polynomial is known for the 1-parameter families of twist knots K_p [HS04] and pretzel knots $KP_p = (-2, 3, 3 + 2p)$ [GM11] depicted on the left and the right part of the following figure



where an integer m inside a box indicates the number of $|m|$ half-twists, right-handed (if $m > 0$) or left-handed (if $m < 0$), according to the following figure

The non-commutative A-polynomial of the twist knots K_p was computed with a certificate by X. Sun and the author in [GS10] for $p = -14, \ldots, 15$. The data is available from

`www.math.gatech.edu/~stavros/publications/twist.knot.data`

The non-commutative A-polynomial of the pretzel knots $KP_p = (-2, 3, 3 + 3p)$ was guessed by C. Koutschan and the author in [GK12a] for $p = -5, \ldots, 5$. The guessing method used an a priori knowledge of the monomials of the recursion, together with computation of the colored Jones polynomial using the fusion formula, and exact but modular arithmetic and rational reconstruction. The data is available from

`www.math.gatech.edu/~stavros/publications/pretzel.data`

For instance, the recursion relation for the colored Jones polynomial $f(n)$ of the $5_2 = (-2, 3, -1)$ pretzel knot is given by

$b(q^n, q) - q^{9+7n}(-1+q^n)(-1+q^{2+n})(1+q^{2+n})(-1+q^{5+2n}) f(n) + q^{5+2n}(-1+q^{1+n})^2(1+q^{1+n})(-1+q^{5+2n})(-1 + q^{1+n} + q^{1+2n} - q^{2+2n} - q^{3+2n} + q^{4+2n} - q^{2+3n} - q^{5+3n} - 2q^{5+4n} + q^{6+5n}) f(1+n) -$

$q(-1+q^{2+n})^2(1+q^{2+n})(-1+q^{1+2n})(-1+2q^{2+n}+q^{2+2n}+q^{5+2n}-q^{4+3n}+q^{5+3n}+q^{6+3n}-q^{7+3n}-q^{7+4n}+q^{9+5n})f(2+n)-(-1+q^{1+n})(1+q^{1+n})(-1+q^{3+n})(-1+q^{1+2n})f(3+n)=0\,,$

where

$$b(q^n,q)=q^{4+2n}(1+q^{1+n})(1+q^{2+n})(-1+q^{1+2n})(-1+q^{3+2n})(-1+q^{5+2n})\,.$$

The recursion relation for the colored Jones polynomial $f(n)$ of the $(-2,3,7)$ pretzel knot is given by

$b(q^n,q)-q^{224+55n}(-1+q^n)(-1+q^{4+n})(-1+q^{5+n})f(n)+q^{218+45n}(-1+q^{1+n})^3(-1+q^{4+n})(-1+q^{5+n})f(1+n)+q^{204+36n}(-1+q^{2+n})^2(1+q^{2+n}+q^{3+n})(-1+q^{5+n})f(2+n)+(-1+q)q^{180+27n}(1+q)(-1+q^{1+n})(-1+q^{3+n})^2(-1+q^{5+n})f(3+n)-q^{149+18n}(-1+q^{1+n})(-1+q^{4+n})^2(1+q+q^{4+n})f(4+n)-q^{104+8n}(-1+q^{1+n})(-1+q^{2+n})(-1+q^{5+n})^3f(5+n)+q^{59}(-1+q^{1+n})(-1+q^{2+n})(-1+q^{6+n})f(6+n)=0\,,$

where

$b(q^n,q)=q^{84+5n}(1-q^{1+n}-q^{2+n}+q^{3+2n}-q^{16+3n}+q^{17+4n}+q^{18+4n}-q^{19+5n}-q^{26+5n}+q^{27+6n}+q^{28+6n}+q^{31+6n}-q^{29+7n}-q^{32+7n}-q^{33+7n}-q^{36+7n}+q^{34+8n}+q^{37+8n}+q^{38+8n}-q^{39+9n}+q^{45+9n}-q^{46+10n}-q^{47+10n}+q^{49+10n}+q^{48+11n}-q^{50+11n}-q^{51+11n}-q^{54+11n}+q^{52+12n}+q^{55+12n}+q^{56+12n}-q^{57+13n}-q^{62+13n}+q^{63+14n}+q^{64+14n}-q^{66+14n}+q^{67+14n}-q^{65+15n}+q^{67+15n}-q^{69+15n}+q^{71+15n}-q^{69+16n}+q^{70+16n}-q^{72+16n}-q^{75+17n}-q^{78+17n}+q^{76+18n}+q^{79+18n}-q^{83+19n}+q^{85+19n}+q^{84+20n}-q^{86+20n}+q^{88+20n}-q^{89+21n}+q^{91+21n}-q^{96+22n}-q^{93+23n}+2q^{98+24n}-q^{99+25n}-q^{108+26n}-q^{107+27n}+q^{109+27n}+q^{108+28n}-q^{110+28n}+q^{112+28n}-q^{113+29n}+q^{115+29n}+q^{112+30n}+q^{115+30n}-q^{117+31n}-q^{120+31n}-q^{117+32n}+q^{118+32n}-q^{120+32n}-q^{119+33n}+q^{121+33n}-q^{123+33n}+q^{125+33n}+q^{123+34n}+q^{124+34n}-q^{126+34n}+q^{127+34n}-q^{123+35n}-q^{128+35n}+q^{124+36n}+q^{127+36n}+q^{128+36n}+q^{126+37n}-q^{128+37n}-q^{129+37n}-q^{132+37n}-q^{130+38n}-q^{131+38n}+q^{133+38n}-q^{129+39n}+q^{135+39n}+q^{130+40n}+q^{133+40n}+q^{134+40n}-q^{131+41n}-q^{134+41n}-q^{135+41n}-q^{138+41n}+q^{135+42n}+q^{136+42n}+q^{139+42n}-q^{133+43n}-q^{140+43n}+q^{137+44n}+q^{138+44n}-q^{142+45n}+q^{135+46n}-q^{139+47n}-q^{140+47n}+q^{144+48n})\,.$

The pretzel knots KP_p are interesting from many points of view. For every integer p, the knots in the pair $(KP_p,-KP_{-p})$ (where $-K$ denotes the mirror of K)

- are geometrically similar, in particular they are scissors congruent, have equal volume, equal invariant trace fields and their Chern-Simons invariant differ by a sixth root of unity,
- their A-polynomials are equal up to a $\mathrm{GL}(2,\mathbb{Z})$ transformation [GM11, Thm.1.4].

Yet, the colored Jones polynomials and the Kashaev invariants of $(KP_p,-KP_{-p})$ are different, and so are the asymptotics of the Kashaev invariant, even $\Delta(0)$ in the modularity conjecture 3.1. An explanation of this puzzle is given in [DG12a].

Zagier posed a question to compare the modularity conjecture for geometrically similar pairs of knots, which was a motivation for many of the computations in Section 5.2.

5. Numerical asymptotics and the Modularity Conjecture

5.1. **Numerical computation of the Kashaev invariant.** To numerically verify Conjecture 3.1 we need to compute the Kashaev invariant efficiently. In this section we discuss this topic.

There are multidimensional R-matrix state sum formulas for the colored Jones polynomial $J_{K,N}(q)$ where the number of summation points are given by a polynomial in N of degree the number of crossings of K minus 1 [GL05]. Unfortunately, this is not practical method even for the 4_1 knot.

An alternative way is to use fusion [KL94, Cos09, GvdV12] which allows one to compute the colored Jones polynomial more efficiently at the cost that the summand is a rational function of q. For instance, the colored Jones polynomial of a 2-fusion knot can be computed in $O(N^3)$ steps using [GK12a, Thm.1.1]. This method works better, but it still has limitations.

A preferred method is to guess a nontrivial recursion relation for the colored Jones polynomial (see Section 4) and instead of using it to compute the colored Jones polynomial, differentiate sufficiently many times and numerically compute the Kashaev invariant. In the efforts to compute the Kashaev invariant of the $(-2, 3, 7)$ pretzel knot, Zagier and the author obtained the following lemma, of theoretical and practical use.

Lemma 5.1. The Kashaev invariant $\langle K\rangle_N$ can be numerically computed in $O(N)$ steps.

A computer implementation of Lemma 5.1 is available.

5.2. **Numerical verification of the Modularity Conjecture.** Given a sequence of complex number (a_n) with an expected asymptotic expansion

$$a_n \sim \lambda^n n^\alpha (\log n)^\beta \sum_{j=0}^{\infty} \frac{c_j}{n^j}$$

how can one numerically compute λ, α, β and several coefficients c_j? This is a well-known numerical problem [BO99]. An acceleration method was proposed in [Zag01, p.954], which is also equivalent to the Richardson transform. For a detailed discussion of the acceleration method see [GIKM, Sec.5.2]. In favorable circumstances the coefficients c_j are algebraic numbers, and a numerical approximation may lead to a guess for their exact value.

A concrete application of the acceleration method was given in the appendix of [GvdV12] where one deals with several λ of the same magnitude as well as $\beta \neq 0$.

Numerical computations of the modularity conjecture for the 4_1 knot were obtained by Zagier around roots of unity of order at most 5, and extended to several other knots in [GZ]. As a sample computation, we present here the numerical data for 4_1 at $\alpha = 0$, computed independently by Zagier and by the first author. The values of A_k in the table below are known for $k = 0, \dots, 150$.

$$\phi_{4_1}(X) = 3^{-1/4} \exp(CX) \left(\sum_{k=0}^{\infty} \frac{A_k}{k! 12^k} h^k \right)$$

$$h = A/X \qquad A = \frac{\pi}{3^{3/2}} \qquad C = \frac{1}{\pi} \mathrm{Li}_2(\exp(2\pi i/3))$$

k	A_k
0	1
1	11
2	697
3	724351/5
4	278392949/5
5	244284791741/7
6	1140363907117019/35
7	212114205337147471/5
8	367362844229968131557/5
9	44921192873529779078383921/275
10	3174342130562495575602143407/7
11	699550295824437662808791404905733/455
12	14222388631469863165732695954913158931/2275
13	5255000379400316520126835457783180207189/175
14	4205484148170089347679282114854031908714273/25
15	16169753990012178960071991589211345955648397560689/14875
16	119390469635156067915857712883546381438702433035719259/14875
17	1116398659629170045249141261665722279335124967712466031771/16625
18	577848332864910742917664402961320978851712483384455237961760783/914375
19	3198465527483558758007094480403141583163892079086635997387742717 83/48125
20	523192890653480894959218049320922357395367170475082317392862964453 8303/67375

21	158555526852538710030232989409745755243229196117995383665148878914255633279/158125
22	2661386877137722419622654464284260776124118194290229321508112749932818157692851/186875
23	1799843320784069980857785293171845353938670480452547724408088829842398128243496119/8125
24	106885707291052039964890652626809747973330411640231418213296228053966317899421094666667 9/284375
25	110385924147117923375631514400725631592106475632597425360858423251905931989136965649581 9559/15925
26	848180221913649277212833106432963449310433483042794323456448440417431293021130955718815 1604709/6125

In addition, we present the numerical data for the 5_2 knot at $\alpha = 1/3$, computed in [GZ].

$$\phi_{5_2}(X/(3X+1))/\phi_{5_2}(X) \sim e^{C/h}(2\pi/h)^{3/2}\Delta(1/3)\left(\sum_{k=0}^{\infty} A_k(1/3)h^k\right)$$

$$h = (2\pi i)/(X+1/3)$$

$$F = \mathbb{Q}(\alpha) \qquad \alpha^3 - \alpha^2 + 1 = 0 \qquad \alpha = 0.877\cdots - 0.744\ldots i$$

$$C = R(1-\alpha^2) + 2R(1-\alpha) - \pi i \log(\alpha) + \pi^2$$

$$R(x) = \mathrm{Li}_2(x) + \frac{1}{2}\log x \log(1-x) - \frac{\pi^2}{6}$$

$$[1-\alpha^2] + 2[1-\alpha] \in \mathcal{B}(F)$$

$$-23 = \pi_1^2\pi_2 \qquad \pi_1 = 3\alpha - 2 \qquad \pi_2 = 3\alpha + 1$$

$$\pi_7 = (\alpha^2-1)\zeta_6 - \alpha + 1 \qquad \pi_{43} = 2\alpha^2 - \alpha - \zeta_6$$

$$\begin{aligned}
\Delta(1/3) &= e(-2/9)\pi_7\frac{3\sqrt{-3}}{\sqrt{\pi_1}} \\
A_0(1/3) &= \pi_7\pi_{43} \\
A_1(1/3) &= \frac{-952+321\alpha-873\alpha^2+(1348+557\alpha+26\alpha^2)\zeta_6}{\alpha^5\pi_1^3}
\end{aligned}$$

One may use the recursion relations [GK12b] for the twisted colored Jones polynomial to expand the above computations around complex roots of unity [DG].

6. Stability

6.1. **Stability of a sequence of polynomials.** The Slope Conjecture deals with the highest (or the lowest) q-exponent in the colored Jones polynomial. In this section we discuss what happens when we shift the colored Jones polynomial and place its lowest q-exponent to 0. Stability concerns the coefficients of the resulting sequence of polynomials in q. A weaker form of stability (0-stability, defined below) for the colored Jones polynomial of an alternating knot was conjectured by Dasbach and Lin, and proven independently by Armond [Arm11].

Stability was observed in some examples of alternating knots by Zagier, and conjectured by the author to hold for all knots, assuming that we restrict the sequence of colored Jones polynomials to suitable arithmetic progressions, dictated by the quasi-polynomial nature of its q-degree [Gar11b, Gar11a]. Zagier asked about modular and asymptotic properties of the limiting q-series.

A proof of stability in full for all alternating links is given in [GL11b]. Besides stability, this approach gives a generalized Nahm sum formula for the corresponding series, which in particular implies convergence in the open unit disk in the q-plane. The generalized Nahm sum formula comes with a computer implementation (using as input a planar diagram of a link), and allows the computation of several terms of the q-series as well as its asymptotics when q approaches radially a root of unity. The Nahm sum formula is reminiscent to the cohomological Hall algebra of motivic Donaldson-Thomas invariants of Kontsevich-Soibelman [KS], and may be related to recent work of Witten [Wit] and Dimofte-Gaiotto-Gukov [DGG].

Let

$$\mathbb{Z}((q)) = \{\sum_{n\in\mathbb{Z}} a_n q^n \,|\, a_n = 0,\, n \ll 0\}$$

denote the ring of power series in q with integer coefficients and bounded below minimum degree.

Definition 6.1. Fix a sequence $(f_n(q))$ of polynomials $f_n(q) \in \mathbb{Z}[q]$. We say that $(f_n(q))$ is 0-*stable* if the following limit exists

$$\lim_n f_n(q) = \Phi_0(q) \in \mathbb{Z}[[q]],$$

i.e. for every natural number $m \in \mathbb{Z}$, there exists a natural number $n(m)$ such that the coefficient of q^m in $f_n(q)$ is constant for all $n > n(m)$.

We say that $(f_n(q))$ is *stable* if there exist elements $\Phi_k(q) \in \mathbb{Z}((q))$ for $k = 0, 1, 2, \dots$ such that for every $k \in \mathbb{N}$ we have

$$\lim_n q^{-nk} \left(f_n(q) - \sum_{j=0}^{k} q^{jn} \Phi_j(q) \right) = 0 \in \mathbb{Z}((q)) .$$

We will denote by

$$F(x, q) = \sum_{k=0}^{\infty} \Phi_k(q) x^k \in \mathbb{Z}((q))[[x]]$$

the corresponding series associated to the stable sequence $(f_n(q))$.

Thus, a 0-stable sequence $f_n(q) \in \mathbb{Z}[q]$ gives rise to a q-series $\lim_n f_n(q) \in \mathbb{Z}[[q]]$. The q-series that come from the colored Jones polynomial are q-hypergeometric series of a special shape, i.e., they are generalized Nahm sums. The latter are introduced in the next section.

6.2. **Generalized Nahm sums.** In [NRT93] Nahm studied q-hypergeometric series $f(q) \in \mathbb{Z}[[q]]$ of the form

$$f(q) = \sum_{n_1, \dots, n_r \geq 0} \frac{q^{\frac{1}{2} n^t \cdot A \cdot n + b \cdot n}}{(q)_{n_1} \dots (q)_{n_r}}$$

where A is a positive definite even integral symmetric matrix and $b \in \mathbb{Z}^r$. Nahm sums appear in character formulas in Conformal Field Theory, and define analytic functions in the complex unit disk $|q| < 1$ with interesting asymptotics at complex roots of unity, and with sometimes modular behavior. Examples of Nahm sums is the famous list of seven mysterious q-series of Ramanujan that are nearly modular (in modern terms, mock modular). For a detailed discussion, see [Zag09]. Nahm sums give rise to elements of the Bloch group, which governs the leading radial asymptotics of $f(q)$ as q approaches a complex root of unity. Nahm's Conjecture concerns the modularity of a Nahm sum $f(q)$, and was studied extensively by Zagier, Vlasenko-Zwegers and others [VZ11, Zag07].

The limit of the colored Jones function of an alternating link leads us to consider generalized Nahm sums of the form

$$\Phi(q) = \sum_{n \in C \cap \mathbb{N}^r} (-1)^{c \cdot n} \frac{q^{\frac{1}{2} n^t \cdot A \cdot n + b \cdot n}}{(q)_{n_1} \dots (q)_{n_r}} \tag{4}$$

where C is a rational polyhedral cone in $\mathbb{R}^r$, $b, c \in \mathbb{Z}^r$ and A is a symmetric (possibly indefinite) symmetric matrix. We will say that the generalized Nahm sum (4) is *regular* if the function

$$n \in C \cap \mathbb{N}^r \mapsto \frac{1}{2} n^t \cdot A \cdot n + b \cdot n$$

is proper and bounded below, where mindeg_q denotes the minimum degree with respect to q. Regularity ensures that the series (4) is a well-defined element of the ring

$$\mathbb{Z}((q)) = \{ \sum_{n \in \mathbb{Z}} a_n q^n \,|\, a_n = 0,\, n \ll 0 \}$$

of power series in q with integer coefficients and bounded below minimum degree. In the remaining of the paper, the term Nahm sum will refer to a generalized Nahm sum.

6.3. **Stability for alternating links.** Let K denote an alternating link. The lowest monomial of $J_{K,n}(q)$ has coefficient ± 1, and dividing $J_{K,n+1}(q)$ by its lowest monomial gives a polynomial $J^+_{K,n}(q) \in 1 + q\mathbb{Z}[q]$. We can now quote the main theorem of [GL11b].

Theorem 6.2. [GL11b] *For every alternating link K, the sequence $(J^+_{K,n}(q))$ is stable and the corresponding limit $F_K(x,q)$ can be effectively computed by a planar projection D of K. Moreover, $F_K(0,q) = \Phi_{K,0}(q)$ is given by an explicit generalized Nahm sum computed by D.*

An illustration of the corresponding q-series $\Phi_{K,0}(q)$ the knots 3_1, 4_1 and 6_3 is given in Section 6.4.

6.4. **Computation of the q-series of alternating links.** Given the generalized Nahm sum for $\Phi_{K,0}(q)$ (a multidimensional sum of as many variables as the number of crossings of K), one may try to guess a formula for $\Phi_{K,0}(q)$. In joint work with Zagier, we computed the first few terms of the corresponding series (an interesting and nontrivial task in itself) and guessed the answer for knots with a small number of crossings. The guesses are presented in the following table

K	c_-	c_+	σ	$\Phi^*_{K,0}(q)$	$\Phi_{K,0}(q)$
$3_1 = -K_1$	3	0	2	h_3	1
$4_1 = K_{-1}$	2	2	0	h_3	h_3
5_1	5	0	4	h_5	1
$5_2 = K_2$	0	5	-2	h^*_4	h_3
$6_1 = K_{-2}$	4	2	0	h_3	h_5
6_2	4	2	2	$h_3 h^*_4$	h_3
6_3	3	3	0	h_3^2	h_3^2
7_1	7	0	6	h_7	1
$7_2 = K_3$	0	7	-2	h^*_6	h_3
7_3	0	7	-4	h^*_4	h_5
7_4	0	7	-2	$(h^*_4)^2$	h_3
7_5	7	0	4	h^*_4	h^*_4
7_6	5	2	2	$h_3 h^*_4$	h_3^2
7_7	3	4	0	h_3^3	h_3^2
$8_1 = K_{-3}$	6	2	0	h_3	h_7
8_2	6	2	4	$h_3 h^*_6$	h_3
8_3	4	4	0	h_5	h_5
8_4	4	4	2	$h^*_4 h_5$	h_3
8_5	2	6	-4	h_3	???
$K_p, p > 0$	0	$2p+1$	-2	h^*_{2p}	h_3
$K_p, p < 0$	$2\lvert p\rvert$	2	0	h_3	$h_{2\lvert p\rvert+1}$
$T(2,p), p > 0$	$2p+1$	0	$2p$	h_{2p+1}	1

where, for a positive natural number b, h_b and h^*_b are the unary theta and false theta series

$$h_b(q) = \sum_{n\in\mathbb{Z}} \varepsilon(n) q^{bn(n+1)/2-n}, \qquad h^*_b(q) = \sum_{n\in\mathbb{Z}} (-1)^n q^{bn(n+1)/2-n}$$

where $\varepsilon(n) = 1, -1$ if $n \geq 0$ or $n < 0$. Observe that

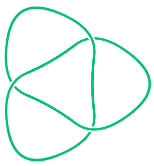
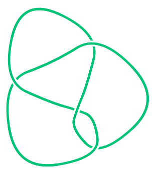
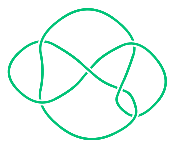
$$h_1(q) = 0, \qquad h_2^*(q) = 1, \qquad h_3(q) = (q)_\infty\,.$$

In the above table, c_+ (resp. c_-) denotes the number of positive (resp., negative) crossings of an alternating knot K, and $\Phi^*_{K,0}(q) = \Phi_{-K,0}(q)$ denotes the q-series of the mirror $-K$ of K, and $T(2,p)$ denotes the $(2,p)$ torus knot.

Concretely, the above table predicts the following identities

$$\begin{aligned}
(q)_\infty^{-2} &= \sum_{a,b,c\geq 0} (-1)^a \frac{q^{\frac{3}{2}a^2+ab+ac+bc+\frac{1}{2}a+b+c}}{(q)_a(q)_b(q)_c(q)_{a+b}(q)_{a+c}} \\
(q)_\infty^{-3} &= \sum_{\substack{a,b,c,d,e\geq 0\\ a+b=d+e}} (-1)^{b+d} \frac{q^{\frac{b^2}{2}+\frac{d^2}{2}+bc+ac+ad+be+\frac{a}{2}+c+\frac{e}{2}}}{(q)_{b+c}(q)_a(q)_b(q)_c(q)_d(q)_e(q)_{c+d}} \\
(q)_\infty^{-4} &= \sum_{\substack{a,b,c,d,e,f\geq 0\\ a+e\geq b,b+f\geq a}} (-1)^{a-b+e} \frac{q^{\frac{a}{2}+\frac{3a^2}{2}+\frac{b}{2}+\frac{b^2}{2}+c+ac+d+ad+cd+\frac{e}{2}+2ae-2be+de+\frac{3e^2}{2}-af+bf+f^2}}{(q)_a(q)_b(q)_c(q)_{a+c}(q)_d(q)_{a+d}(q)_e(q)_{a-b+e}(q)_{a-b+d+e}(q)_f(q)_{-a+b+f}}
\end{aligned}$$

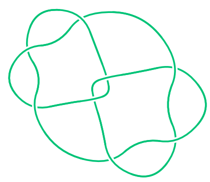
corresponding to the knots

3_1  4_1  6_3 

Some of the identities of the above table have been consequently proven [AD11]. In particular this settles the (mock)-modularity properties of the series $\Phi_{K,0}(q)$ for all but one knot.

The q-series of the remaining knot 8_5 is given by an 8-dimensional Nahn sum

$$\Phi_{8_5,0}(q) = (q)_\infty^8 \sum_{\substack{a,b,c,d,e,f,g,h\geq 0\\ a+f\geq b}} S(a,b,c,d,e,f,g,h)$$

8_5 

where $S = S(a,b,c,d,e,f,g,h)$ is given by

$$S = (-1)^{b+f} \frac{q^{2a+3a^2-\frac{b}{2}-2ab+\frac{3b^2}{2}+c+ac+d+ad+cd+e+ae+de+\frac{3f}{2}+4af-4bf+ef+\frac{5f^2}{2}+g+ag-bg+eg+fg+h+ah-bh+fh+gh}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_h(q)_{a+c}(q)_{a+d}(q)_{a+e}(q)_{a-b+f}(q)_{a-b+e+f}(q)_{a-b+f+g}(q)_{a-b+f+h}}\,.$$

The first few terms are given by

$\Phi_{8_5}(q)/(q)_\infty =$
$1-q+q^2-q^4+q^5+q^6-q^8+2q^{10}+q^{11}+q^{12}-q^{13}-2q^{14}+2q^{16}+3q^{17}+2q^{18}+q^{19}-3q^{21}-2q^{22}+q^{23}+4q^{24}+4q^{25}+5q^{26}+3q^{27}+q^{28}-2q^{29}-3q^{30}-3q^{31}+5q^{33}+8q^{34}+8q^{35}+8q^{36}+6q^{37}+$

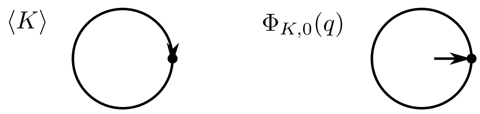
$3q^{38}-2q^{39}-5q^{40}-6q^{41}-q^{42}+2q^{43}+9q^{44}+13q^{45}+17q^{46}+16q^{47}+14q^{48}+9q^{49}+4q^{50}-3q^{51}-8q^{52}-8q^{53}-5q^{54}+3q^{55}+14q^{56}+21q^{57}+27q^{58}+32q^{59}+33q^{60}+28q^{61}+21q^{62}+11q^{63}+q^{64}-9q^{65}-11q^{66}-11q^{67}-2q^{68}+9q^{69}+27q^{70}+40q^{71}+56q^{72}+60q^{73}+65q^{74}+62q^{75}+54q^{76}+39q^{77}+23q^{78}+4q^{79}-9q^{80}-16q^{81}-14q^{82}-3q^{83}+16q^{84}+40q^{85}+67q^{86}+92q^{87}+114q^{88}+129q^{89}+135q^{90}+127q^{91}+115q^{92}+92q^{93}+66q^{94}+35q^{95}+9q^{96}-12q^{97}-14q^{98}-11q^{99}+13q^{100}+O(q)^{101}$.

We were unable to identify $\Phi_{8_5,0}(q)$ with a known q-series. Nor were we able to decide whether it is a mock-modular form [Zag09]. It seems to us that 8_5 is not an exception, and that the mock-modularity of the q-series $\Phi_{8_5,0}(q)$ is an open problem.

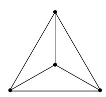
Question 6.3. Can one decide if a generalized Nahm sum is a mock-modular form?

7. Modularity and Stability

Modularity and Stability are two important properties of quantum knot invariants. The Kashaev invariant $\langle K\rangle$ and the q-series $\Phi_{K,0}(q)$ of a knotted 3-dimensional object have some common properties, namely asymptotic expansions at roots of unity approached radially (for $\langle K\rangle$) and on the unit circle (for $\Phi_{K,0}(q)$), depicted in the following figure



The leading asymptotic expansions of $\langle K\rangle$ and $\Phi_{K,0}(q)$ are governed by elements of the Bloch group as is the case of the Kashaev invariant and also the case of the radial limits of Nahm sums [VZ11]. In this section we discuss a conjectural relation, discovered accidentally by Zagier and the author in the spring of 2011, between the asymptotics of $\langle 4_1\rangle$ and $\Phi_{6j,0}(q)$, where $6j$ is the q-6j symbol of the tetrahedron graph whose edges are colored with $2N$ [Cos09, GvdV12]



The evaluation of the above tetrahedron graph $J^{+}_{6j,N}(q)\in 1+q\mathbb{Z}[q]$ is given explicitly by [Cos09, GvdV12]

$$J^{+}_{6j,N}(q)=\frac{1}{1-q}\sum_{n=0}^{N}(-1)^n\frac{q^{\frac{3}{2}n^2+\frac{1}{2}n}}{(q)_n^3}\frac{(q)_{4N+1-n}}{(q)_n^3(q)_{N-n}^4}\,.$$

The sequence $(J^{+}_{6j,N}(q))$ is stable and the corresponding series $F_{6j}(x,q)$ is given by

$$F_{6j}(x,q)=\frac{1}{(1-q)(q)_\infty^3}\sum_{n=0}^{\infty}(-1)^n\frac{q^{\frac{3}{2}n^2+\frac{1}{2}n}}{(q)_n^3}\frac{(xq^{-n})_\infty^4}{(x^4q^{-n+1})_\infty}\in\mathbb{Z}((q))[[x]]\,,$$

where as usual $(x)_\infty=\prod_{k=0}^{\infty}(1-xq^k)$ and $(q)_n=\prod_{k=1}^{n}(1-q^k)$. In particular,

$$\lim_N J^{+}_{6j,N}(q)=\Phi_{6j,0}(q)=\frac{1}{(1-q)(q)_\infty^3}\sum_{n=0}^{\infty}(-1)^n\frac{q^{\frac{3}{2}n^2+\frac{1}{2}n}}{(q)_n^3}\,.$$

Let

$$\phi_{6j,0}(q) = \frac{(q)_\infty^4}{1-q}\Phi_{6j,0}(q) = (q)_\infty \sum_{n=0}^{\infty}(-1)^n \frac{q^{\frac{3}{2}n^2+\frac{1}{2}n}}{(q)_n^3}\,.$$

The first few terms of $\phi_{6j,0}(q)$ are given by

$\phi_{6j,0}(q) = 1-q-2q^2-2q^3-2q^4+q^6+5q^7+7q^8+11q^9+13q^{10}+16q^{11}+14q^{12}+14q^{13}+8q^{14}-12q^{16}-26q^{17}-46q^{18}-66q^{19}-90q^{20}-114q^{21}-135q^{22}-155q^{23}-169q^{24}-174q^{25}-165q^{26}-147q^{27}-105q^{28}-48q^{29}+37q^{30}+142q^{31}+280q^{32}+435q^{33}+627q^{34}+828q^{35}+1060q^{36}+O(q)^{37}\,.$

The next conjecture which combines stability and modularity of two knotted objects has been numerically checked around complex roots of unity of order at most 3.

Conjecture 7.1. As $X \longrightarrow +\infty$ with bounded denominator, we have

$$\phi_{6j,0}(e^{-1/X}) = \phi_{4_1}(X)/X^{1/2} + \overline{\phi_{4_1}(-\bar{X})/(-\bar{X})^{1/2}}\,.$$

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Don Zagier's work on singular moduli

Benedict H. Gross

Singular moduli are the values of the modular function $j(\tau)$ at the points z in the upper half plane that satisfy a quadratic equation with rational coefficients. In other words, they are the j-invariants of elliptic curves with complex multiplication.

These invariants were studied intensively by the leading number theorists of the nineteenth century. They are algebraic integers, which generate certain abelian extensions of the imaginary quadratic fields $\mathbb{Q}(z)$. The theory was believed to have been brought to a very satisfying completion in the early twentieth century. That was before Don got his hands on it.

In early 1983 Don sent me an amazing letter from Japan containing a proof of a factorization formula for the integer which is the norm of the difference of two singular moduli of relatively prime discriminants D and D'. This was a completely new aspect of the theory, which Don had discovered by extensive numerical experimentation. One particularly striking fact (which should have been noticed earlier) is that any prime p dividing this norm must divide an integer of the form $(DD' - x^2)/4$. This letter (in its original handwritten form, as well as a Latex version prepared by Carl Erickson) is reproduced below.

Don's proof involved the study of a Hilbert modular Eisenstein series for the real quadratic field $\mathbb{Q}(\sqrt{DD'})$. At the end of the letter, he challenged me to find an algebraic proof, which I sketched in a letter of reply (also reproduced below) and reproduced in the talk.

In 2002, Don discovered another wonderful formula, relating the integers which are the traces of singular moduli to the Fourier coefficients of a meromorphic modular form of weight 3/2. I will put this result into the context of computing the images of Heegner points in the Jacobians of modular curves. In this case, the Jacobian of the curve of level 1 is trivial, but the generalized Jacobian relative to the divisor $2(\infty)$ is isomorphic to the additive group.

(1)

Monday, Feb. 7 (1983)

Dick,

I've been in Japan for two weeks now and am enjoying it tremendously, both for sightseeing and mathematics. However, telling you about the trip can wait till you get to Germany; I'm writing now for mathematical reasons only. I'd meant not to look at our business until returning to Germany, since I have several other things to finish writing up, but this weekend I returned to it after all, and came up with something.

As you may remember, I had asked you whether our results on $N(j(z)) = N(j(z)-j(\rho))$, $N(j(z)-1728) = N(j(z)-j(i))$ and $N(j(z)-j(z'))$ $(\mathrm{disc}\, z = \mathrm{disc}\, z' = -p)$ might not generalize to results on $N(j(z)-j(z'))$ (or $j(z)-j(z')$) for <u>arbitrary</u> CM points z and z', with unrelated discriminants. You pooh-poohed the idea, explaining why your method applies only to $\mathrm{Aut}(E)$ or to $\mathrm{Hom}(E,E')$ with E, E' having CM by the same order. Nothing daunted (actually, I was: I didn't do the calculations till now), I calculated $j(z)-j(z')$ for $z = \frac{1+i\sqrt{p}}{2}$, $z' = \frac{1+i\sqrt{q}}{2}$ for the primes with class number 1 — a somewhat tricky business, since my HP has only 10 places — and found the values

$p \backslash q$	11	19	43	67	163
7	$7\cdot13\cdot17\cdot19$	$3^7\cdot13\cdot31$	$3^6\cdot5^3\cdot7\cdot19\cdot73$	$3^7\cdot5^3\cdot7\cdot13\cdot61\cdot97$	$3^8\cdot5^3\cdot7\cdot13\cdot17\cdot31\cdot103\cdot229\cdot283$
11		$2^{16}\cdot13$	$2^{15}\cdot7^2\cdot19\cdot29$	$2^{17}\cdot7^2\cdot13\cdot41\cdot43$	$2^{15}\cdot7^3\cdot11\cdot13\cdot17\cdot73\cdot79\cdot107\cdot10$
19			$2^{15}\cdot3^6\cdot37$	$2^{16}\cdot3^7\cdot13\cdot79$	$2^{15}\cdot3^7\cdot13\cdot19\cdot31\cdot37\cdot67\cdot193$
43				$2^{15}\cdot3^6\cdot5^3\cdot7^2$	$2^{19}\cdot3^5\cdot5^3\cdot7^3\cdot433$
67					$2^{15}\cdot3^7\cdot5^3\cdot7^2\cdot13\cdot139\cdot33$

<u>It seemed pretty clear that these numbers were too highly factorised for this</u>

	$p=7$	$p=11$				
$d = 8$	$5^3\cdot7\cdot13$	$2^6\cdot7^2\cdot13$	$2^6\cdot13\cdot29\cdot37$	$2^6\cdot5^3\cdot7^2\cdot37\cdot61$	$2^6\cdot5^4\cdot7^2\cdot13\cdot53\cdot109$	$2^6\cdot5\cdot7$

[Kyoto, Japan]
Monday, Feb. 7 [1983]

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As you may remember, I had asked you whether our results on

$$N(j(z)) = N(j(z) - j(\rho)), N(j(z) - 1728) = N(j(z) - j(i)), \text{ and } N(j(z) - j(z'))$$

($\operatorname{disc} z = \operatorname{disc} z' = -p$) might not generalize to results on $N(j(z) - j(z'))$ (or $j(z) - j(z')$) for arbitrary CM points z and z', with unrelated discriminants. You pooh-poohed the idea, explaining why your method applies only to $\operatorname{Aut}(E)$ or to $\operatorname{Hom}(E, E')$ with E, E' having CM by the same order. Not daunted (actually, I was: I didn't do the calculations till now), I calculated $j(z) - j(z')$ for $z = \frac{1+i\sqrt{p}}{2}, z' = \frac{1+i\sqrt{q}}{2}$ for the primes with class number 1 - a somewhat tricky business, since my HP has only 10 places - and found the values

p	$q = 11$	$q = 19$	$q = 43$
7	$7 \cdot 13 \cdot 17 \cdot 19$	$3^7 \cdot 13 \cdot 31$	$3^6 \cdot 5^3 \cdot 7 \cdot 19 \cdot 73$
11		$2^{16} \cdot 13$	$2^{15} \cdot 7^2 \cdot 19 \cdot 29$
19			$2^{15} \cdot 3^6 \cdot 37$

p	$q = 67$	$q = 163$
7	$3^7 \cdot 5^3 \cdot 7 \cdot 13 \cdot 61 \cdot 97$	$3^8 \cdot 5^3 \cdot 7 \cdot 13 \cdot 17 \cdot 31 \cdot 103 \cdot 229 \cdot 283$
11	$2^{17} \cdot 7^2 \cdot 13 \cdot 41 \cdot 43$	$2^{15} \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 73 \cdot 79 \cdot 107 \cdot 109$
19	$2^{16} \cdot 3^7 \cdot 13 \cdot 79$	$2^{15} \cdot 3^7 \cdot 13 \cdot 19 \cdot 31 \cdot 37 \cdot 67 \cdot 193$
43	$2^{15} \cdot 3^6 \cdot 5^3 \cdot 7^2$	$2^{19} \cdot 3^6 \cdot 5^3 \cdot 7^3 \cdot 37 \cdot 433$
67		$2^{15} \cdot 3^7 \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 139 \cdot 331$

It seemed pretty clear that these numbers were too highly factorized for this to be accidental. However, since we had $\ell \leq p$ for $\ell \mid (j_{-p} - j_{-4})$ and $\ell \leq \frac{3p}{4}$ for $\ell \mid (j_{-p} - j_{-3})$, I had expected $\ell \leq \frac{pq}{4}$ for $\ell \mid (j_{-p} - j_{-q})$, and although this holds in the above table, I was worried by the fact that I never got as *big* as $\frac{pq}{4}$, e.g. for $p = 67, q = 163$ the biggest ℓ is 331, barely bigger than $2q$. From the formulas

$$\ell \mid p - x^2, \quad \ell \mid \frac{3p - x^2}{4} \qquad \text{for} \qquad \ell \mid (j_{-p} - j_{-4}), \quad \ell \mid (j_{-p} - j_{-3}),$$

I expected $\ell \mid \frac{pq - x^2}{4}$; in a typical case this gave

x	$\frac{7\cdot 163-x^2}{4}$	x	$\frac{7\cdot 163-x^2}{4}$	x	$\frac{7\cdot 163-x^2}{4}$	x	$\frac{7\cdot 163-x^2}{4}$
1	$3\cdot 5\cdot 19$	11	$3\cdot 5\cdot 17$	21	$5^2\cdot 7$	31	$3^2\cdot 5$
3	283	13	3^5	23	$3^2\cdot 17$	33	13
5	$3^2\cdot 31$	15	229	25	$3\cdot 43$		
7	$3\cdot 7\cdot 13$	17	$3\cdot 71$	27	103		
9	$5\cdot 53$	19	$3\cdot 5\cdot 13$	29	$3\cdot 5^2$		

All factors $(3,5,7,13,17,31,103,229,283)$ dividing $j_{-7}-j_{-163}$ appear on this list, but so do several others. However, for $\ell \mid j_{-p}-j_{-3}$ we had $\left(\frac{-p}{\ell}\right)=-1$, $\left(\frac{-3}{\ell}\right)=-1$ and similarly for $\ell \mid j_{-p}-j_{-4}$, so here we should have $\left(\frac{-p}{\ell}\right)=\left(\frac{-q}{\ell}\right)=-1$ or $\left(\frac{\ell}{p}\right)=\left(\frac{\ell}{q}\right)=-1$. Moreover, if $\ell \mid \frac{pq-x^2}{4}$ and $\ell\neq p,q$, then $\left(\frac{pq}{\ell}\right)=+1$, so $\left(\frac{-p}{\ell}\right)$ and $\left(\frac{-q}{\ell}\right)$ are always the same. This suggests defining $\chi(d)$ on prime divisors ℓ of $\frac{pq-x^2}{4}$ by

$$\chi(\ell)=\begin{cases}\left(\frac{\ell}{p}\right)=\left(\frac{\ell}{q}\right) & \ell\neq p,q\\ \left(\frac{\ell}{p}\right) & \ell=q\\ \left(\frac{\ell}{q}\right) & \ell=p\end{cases},$$

and extend multiplicatively, setting $R(n)=\sum_{d\mid n}\chi(d)$, and conjecturing

Theorem.

$$\nu_\ell(N(j(\frac{1+i\sqrt{p}}{2})-j(\frac{1+i\sqrt{q}}{2})))=\sum_{\substack{k\in\mathbb{Z}\\ k^2<pq\\ k\text{ odd}}}\ \sum_{\substack{n\geq 1\\ n\text{ odd}}} R(\frac{pq-k^2}{4\ell^n})$$

for $p\equiv q\equiv 3 \pmod 4), p,q>3$, *where* N *is the absolute norm to* $\mathbb{Q}$.

Before trying to prove this, I worked out several examples. In particular, I wanted to understand why so *few* and such *small* primes occur in the above table; in the above theorem you'd expect $\frac{1}{4}$ of all primes $<\frac{pq}{4}$ or about 50 primes going up to 2700 in the case $p=67, q=163$. So I worked out that case:

x	$\frac{67\cdot 163-x^2}{4}$	Contr.	x	$\frac{67\cdot 163-x^2}{4}$	Contr.	x	$\frac{67\cdot 163-x^2}{4}$	Contr.
1	$2\cdot 3\cdot 5\cdot 7\cdot 13$	–	35	$2^3\cdot 3\cdot 101$	–	69	$2^2\cdot 5\cdot 7\cdot 11$	–
3	$2^3\cdot 11\cdot 31$	–	37	$2^2\cdot 3\cdot 199$	3^2	71	$2\cdot 3\cdot 5\cdot 7^2$	–
5	$2^2\cdot 3\cdot 227$	3^2	39	$2\cdot 5^2\cdot 47$	2^2	73	$2\cdot 3\cdot 233$	–
7	$2\cdot 3^2\cdot 151$	2^2	41	$2\cdot 3\cdot 5\cdot 7\cdot 11$	–	75	$2^2\cdot 331$	331
9	$2\cdot 5\cdot 271$	–	43	$2^2\cdot 3^4\cdot 7$	7	77	$2^5\cdot 3\cdot 13$	–
11	$2^2\cdot 3^3\cdot 5^2$	3^2	45	$2^4\cdot 139$	139	79	$2\cdot 3^2\cdot 5\cdot 13$	–
13	$2^7\cdot 3\cdot 7$	–	47	$2\cdot 3^2\cdot 11^2$	2	81	$2\cdot 5\cdot 109$	–
15	$2\cdot 7\cdot 191$	–	49	$2\cdot 3\cdot 5\cdot 71$	–	83	$2^4\cdot 3^2\cdot 7$	7
17	$2\cdot 3\cdot 443$	–	51	$2^5\cdot 5\cdot 13$	–	85	$2^2\cdot 3\cdot 7\cdot 11$	–
19	$2^4\cdot 3\cdot 5\cdot 11$	–	53	$2^2\cdot 3\cdot 13^2$	3	87	$2\cdot 419$	2^2
21	$2^2\cdot 5\cdot 131$	5^2	55	$2\cdot 3\cdot 7\cdot 47$	–	89	$2\cdot 3\cdot 5^3$	–
23	$2\cdot 3\cdot 433$	–	57	$2\cdot 7\cdot 137$	–	91	$2^2\cdot 3\cdot 5\cdot 11$	–
25	$2\cdot 3^2\cdot 11\cdot 13$	–	59	$2^2\cdot 3\cdot 5\cdot 31$	–	93	$2^3\cdot 71$	2^4
27	$2^2\cdot 7^2\cdot 13$	13	61	$2^3\cdot 3^2\cdot 5^2$	2^2	95	$2\cdot 3\cdot 79$	–
29	$2^3\cdot 3^2\cdot 5\cdot 7$	–	63	$2\cdot 11\cdot 79$	–	97	$2\cdot 3^3\cdot 7$	–
31	$2\cdot 3\cdot 5\cdot 83$	–	65	$2\cdot 3^3\cdot 31$	–	99	$2^3\cdot 5\cdot 7$	–
33	$2\cdot 1229$	2^2	67	$2^3\cdot 3\cdot 67$	–	101	$2^2\cdot 3^2\cdot 5$	5
						103	$2\cdot 3\cdot 13$	–

The last column is the contribution of $\frac{pq-x^2}{4}$ to $j(z_p)-j(z_q)$, i.e. it is

$$\ell^{s\cdot R(\frac{pq-x^2}{4\ell^2})} \text{ if } \ell^{2s-1} \mid\mid \frac{pq-x^2}{4}$$

and ℓ is the *only* non-residue dividing $\frac{pq-x^2}{4}$ to an odd power, and 1 (denoted –) if there are several such ℓ. The product of these contributions is

$$2^{15}\cdot 3^7\cdot 5^3\cdot 7^2\cdot 13\cdot 139\cdot 331$$

as required, confirming the conjectured formula; the reason that there are so few contributions is that, since $-p$ and $-q$ have $h=1$, there are exceptionally many ℓ with $\left(\frac{-p}{\ell}\right)=\left(\frac{-q}{\ell}\right)=-1$ (in particular, all $\ell<17$), so almost all $\frac{pq-x^2}{4}$ have more than one such ℓ occurring to an odd power. Indeed, if we fix a prime ℓ then we can do some heuristics on the size of the number ν_ℓ given by the formula in the theorem for $p,q\to\infty$, $\left(\frac{-p}{\ell}\right)=\left(\frac{-q}{\ell}\right)=-1$;

$$\begin{aligned}\nu_\ell &= \sum_{\substack{k^2<pq\\ k \text{ odd}}} \sum_{\substack{n\geq 1\\ \ell^n\mid \frac{pq-k^2}{4}\\ n \text{ odd}}} \sum_{\substack{d\geq 1\\ d\mid \frac{pq-k^2}{4\ell^n}}} \chi(d)\\ &= \sum_{\substack{n\geq 1\\ n \text{ odd}}}\sum_{d\geq 1}\chi(d)\cdot\#\{k\in\mathbb{Z}\mid -\sqrt{pq}<k<\sqrt{pq}, k^2\equiv pq \pmod{4\ell^n d}\}\end{aligned}$$

For d small, $\#\{k\dots\}\approx\frac{\sqrt{pq}}{\ell^n d}N_{pq}(\ell^n d)$, where $N_D(d)=\#\{k \pmod{2d} \mid k^2\equiv D \pmod{4d}\}$, so ν_ℓ looks like

$$\sqrt{pq}\sum_{n\geq 1}\sum_{d\geq 1}\chi(d)\frac{N(\ell^n d)}{d}.$$

But for D a fundamental discriminant (as here) we have $\sum_{d\geq 1}N(d)d^{-s}=\zeta_{\mathbb{Q}(\sqrt{p})}(s)/\zeta(2s)$, and here ($D=pq, p\equiv q\equiv 3 \pmod 4$) we have $N(d)>0 \iff d=N(\mathfrak{a})$ for some primitive ideal $\mathfrak{a}$ of $\mathbb{Q}(\sqrt{\ell})$, $\chi(d)=\chi(\mathfrak{a})$ (genus character corresponding to $D=(-p)\cdot(-q)$,

$$\sum_{d\geq 1}\chi(d)N(d)d^{-s}=\frac{L_{-p}(s)L_{-q}(s)}{\zeta(2s)}.$$

Also, ℓ splits in $\mathbb{Q}(\sqrt{\ell})$, so

$$N(\ell^n d)=N(\ell d)=\begin{cases}2N(d) & \ell\nmid d\\ N(d) & \ell\nmid d\end{cases} \quad\text{for } n\geq 1 \text{ odd};$$

hence

$$\nu_\ell\sim\sqrt{pq}\cdot\sum_{\substack{n\geq 1\\ n \text{ odd}}}\frac{1}{\ell^n}\cdot\frac{L_{-p}(1)L_{-q}(1)}{\zeta(2)}\cdot\frac{2}{1-\ell^{-1}}=\frac{12\ell^2}{(\ell-1)^2(\ell+1)}h(-p)h(-q)$$

where the factor $\frac{2}{1-\ell^{-1}}$ appears becase the Euler factor in

$$\frac{L_{-p}(1)L_{-p}(1)}{\zeta(2)}=\frac{1+\ell^{-1}}{1-\ell^{-1}}=1+\frac{2}{\ell}+\frac{2}{\ell^2}+\cdots$$

gets replaced here by

$$2+\frac{2}{\ell}+\frac{2}{\ell^2}+\cdots=\frac{2}{1-\ell^{-1}}.$$

For $h(-p)=h(-q)=1$ and $\ell=2,3,5,7$ this gives $16, \frac{27}{4}\approx 7, \frac{25}{8}\approx 3, \frac{49}{24}\approx 2$ in accordance with the powers to which these primes occur in the table on page 1 (when they do occur). In any case, we see that the powers of ℓ depend more on $h(-p)$ and $h(-q)$ than on p and q, which explains why they do not grow in the table on page 1.

In the formula given on page 2, I wrote $N(j_{-p}-j_{-q})$ although all j-values so far have been in $\mathbb{Q}$. Although this was the obvious conjecture, I thought I should test one case of $h>1$. The first one is $p=7, q=23$, where we get
(here every x contributes, not like the $67,163$ - case!), i.e. we should have

$$N(j(\frac{1+i\sqrt{7}}{2})-j(\frac{1+i\sqrt{23}}{2}))=5^9\cdot 7^3\cdot 17\cdot 19.$$

x	$\frac{7\cdot 23-x^2}{4}$	Contribution	x	$\frac{7\cdot 23-x^2}{4}$	Contribtuion
1	$2^3\cdot 5$	5^4	7	$2^2\cdot 7$	7^3
3	$2\cdot 19$	19^2	9	$2^2\cdot 5$	5^3
5	$2\cdot 17$	17^2	11	$2\cdot 5$	5^2

From Berwick we have (with $\theta^3-\theta-1$)

$$\begin{aligned} j(\frac{1+i\sqrt{7}}{2})-j(\frac{1+i\sqrt{23}}{2}) &= -\,3^3\cdot 5^3+5^3(5\theta^2+11\theta+7)^3 \\ &= 5^3\cdot[(5\theta^2+11\theta+7)-3] \\ &\quad \cdot[(5\theta^2+11\theta+7)^2+3(5\theta^2+11\theta+7)+3^2] \\ &= 5^3\cdot 7\cdot(5\theta^2+11\theta+4)(33\theta^2+46\theta+27) \end{aligned}$$

Now

$$x=5\theta^2+11\theta+4 \implies \begin{aligned} &x^2=186\theta^2+223\theta+126, \\ &x^3=4757\theta^2+6369\theta+3665=22x^2+133x+361 \end{aligned}$$

and

$$y=33\theta^2+46\theta+27 \implies \begin{aligned} &y^2=4987\theta^2+6609\theta+3765, \\ &y^3=727479\theta^2+963703\theta+549154=147y^2-170y+289 \end{aligned}$$

so $N(x)=19^2, N(y)=17^2$, and the formula works. In fact, we have $x=\frac{19}{\pi_{19}}\theta^7, y=\frac{17}{\pi_{17}}\theta^{14}$ where $\pi_{17}=3\theta+2, \pi_{19}=3\theta+1$, so $j(\frac{1+i\sqrt{7}}{2})-j(\frac{1+i\sqrt{23}}{2})=5^3\cdot 7\cdot\pi_{17}^*\cdot\pi_{19}^*\cdot\theta^{21}$ where $\pi_\ell^*=\frac{\ell}{\pi_\ell}$ with norm ℓ^2. This corresponds to the prime factorization you'd expect from the analogue of your results on $N(j), N(j-1728), N(j-j')$, viz.

Conjecture. *Let $K=\mathbb{Q}(\sqrt{-p}), j=j(\frac{1+i\sqrt{p}}{2}), h=h(-p)$, $A_0, A_{\pm 1},\dots,A_{\pm\frac{h-1}{2}}$ the ideal classes of $K, (\ell)=\ell_0\ell_1\cdots\ell_{\frac{h-1}{2}}$ the correspondingly numbered decomposition of (ℓ) in $\mathbb{Q}(j)$ ("correspondingly" means as in your paper, i.e. via the Artin symbol twisted by $\mathfrak{a}\mapsto\mathfrak{a}^2$) with $N\ell_0=\ell, N\ell_j=\ell^2$ (here $\left(\frac{\ell}{p}\right)=0$ or -1). Then*

$$\prod_{\mathrm{disc}(z)=q}\left(j(\frac{1+i\sqrt{p}}{2})-j(z)\right)=\prod_{j=0}^{\frac{h-1}{2}}\ell_j^{\sum_{k^2<pq}\sum_{n\geq 1, n\ \text{odd}} R_j(\frac{pq-x^2}{4\ell^n})}.$$

($R_j(n)=\#\{\mathfrak{a}\in A_j\mid N\mathfrak{a}=n\}$).

Presumably a clever fellow like you will be able to prove the theorem on page 2 by supersingular methods, and then your proof will automatically give this; you should also be able to work out the full splitting of $j(z_{-p})-j(z_{-q})$ in the composition of $\mathbb{Q}(j_{-p})$ and $\mathbb{Q}(j_{-q})$. However, I have an analytic proof of the theorem (hence theorem & not conjecture) and, as in the cases we studied already, it gives only the norm. On the other hand, it works for $q=-3$ or -4 (or any fund. disc. $-q$ prime to p), so that I now have an analytic proof of our results for A and B separately rather than just A^2B, making me a fully justified author of our future j-paper.

Before describing my proof, let me describe a different method of getting at the above result by using the results we already have. This result strongly supports the formula given on page 2, but does not quite prove it (unless you can think of an improvement); on the other hand, it gets at $(j(z_{-p}) - j(z_{-q}))$ rather than just the norm.

Let $h_D(X)$ and $\mathcal{H}_D(X)$ be the near-polynomials

$$h_D(X) = \begin{cases} X^{1/3} & D = -3 \\ (X - 1728)^{1/2} & D = -4 \\ \prod_{\substack{\mathrm{disc}\, z = D \\ \mathrm{mod}\ \mathrm{SL}_2(\mathbb{Z})}} (X - j(z)) & D < -4, \end{cases}$$

$$\mathcal{H}_N(X) = \prod_{f^2 | N} h_{-N/f^2}(X) \qquad (N > 0, N \equiv 0, 3 \pmod 4)$$

so that $\deg h_D = h(D)/\frac{1}{2}w(D), \deg \mathcal{H}_N = H(N)$ (Hurwitz-Kronecker notation). If $\Phi_m(X,Y)$ is the usual modular polynomial, then, as is well known

$$\Phi_m(X,X) = \prod_{x^2 < 4m} \mathcal{H}_{4m-x^2}(X) \qquad (m \neq \square);$$

this is an actual polynomial because the multiplicity of, say, $h_{-3}(X)$ is

$$\#\{x, y \mid 4m - x^2 = 3y^2\} \equiv 0 \pmod 3.$$

For $m = \square$ we still have this formula for $\Phi_m(j,j)$ if we replace the term $\Phi_1(X,Y) = X - Y$ dividing $\Phi_m(X,Y) \pmod \square$ by $j'/2\pi i \eta(z)^4 = j^{2/3}(j - 1728)^{1/2} = \prod_{x^2<4} \mathcal{H}_{4-x^2}(j)$. Then our old formula was (roughly; there are some twists for $\ell \mid m$ or $\ell = p$)

$$\nu_\ell\left(N_{\mathbb{Q}(j)/\mathbb{Q}} \Phi_m(j,j)\right) = \sum_{\substack{n \geq 1 \\ n \text{ odd}}} \sum_{\substack{x,y \in \mathbb{Z} \\ Q(x,y) < mp}} R\left(\frac{mp - Q(x,y)}{\ell^n}\right)$$

(where $j = j(\frac{1+i\sqrt{p}}{2}), Q(x,y) =$ principal form $= (x^2 + py^2)/4$) while the new formula we want is

$$\nu_\ell\left(N_{\mathbb{Q}(j)/\mathbb{Q}} \mathcal{H}_N(j)\right) = \sum_{\substack{n \geq 1 \\ n \text{ odd}}} \sum_{x^2 < Np} R\left(\frac{Np - x^2}{\ell^n}\right) \qquad (N > 0, N \equiv 0, 3 \pmod 4).$$

But $\Phi_m(j,j) = \prod_{y^2<4m} \mathcal{H}_{4m-y^2}$, as stated, so the first formula can be written

$$\sum_{y^2<4m} \nu_\ell(N(\mathcal{H}_{4m-y^2}(j))) = \sum_{y^2<4m} \sum_{n \geq 1} \sum_{x^2 < (4m-y^2)p} R\left(\frac{(4m-y^2)p - x^2}{4\ell^n}\right).$$

In other words, if $(*)_N$ is the desired identity $(N > 0, N \equiv 0, 3 \pmod 4)$, then our result on Φ_m proves $\sum_{y^2<4m} (*)_{4m-y^2}$. Unfortunately, this is not quite enough; for each new m we get $(*)_{4m}$ and $(*)_{4m-1}$ together, so we can prove the result we need by induction. If we could prove, say, $\sum_{y^2<4m} y^2 \cdot (*)_{4m-y^2}$, then at each new stage we'd get $(*)_{4m-1}$ and $(*)_{4m}$ separately, which would suffice; however, I see no way to get this. (Notice, however, that the identity

$$\sum_{y^2 - 4m} H(4m - y^2) = \sum_{d|m} \max(d, \frac{m}{d}) + \begin{cases} \frac{1}{6} & m = \square \\ 0 & m \neq \square \end{cases}$$

obtained by taking the degrees of $\Phi_m(X,X) = \prod \mathcal{H}_{4m-y^2}$ is the first of an infinite series of identities giving

$$\sum_{y^2<4m} p_\nu(m,y^2)H(4m-y^2)$$

in terms of $\mathrm{tr}(T(m), S_{2\nu+2}(\mathrm{SL}_2\mathbb{Z}))$ for certain homogenous polynomials p_ν of degree ν; can these be sharpened to express $\prod \mathcal{H}_{4m-y^2}(X)^{p_\nu(m,y^2)}$ as a polynomial in X which for $X = j(E)$ relates somehow to $\mathrm{End}(E)$?)

Enough digressions; let me show you my proof of the theorem. Strangely enough, almost all of the ingredients — finding an Eisenstein series which vanishes at $s = 0$ and computing $\frac{\partial}{\partial s}|_{s=0}$ of its coefficients, using Sturm's holomorphic projection in weight 2, and expressing $\log(j(z) - j(z'))$ as $\lim_{s\to 0}\left(\sum_{\gamma\in\Gamma}\cdots - \text{pole}\right)$ — are the same as in the analytic proof of the result on $N(j(z) - j(z'))$ for $\mathrm{disc}(z) = \mathrm{disc}(z') = -p$, but the starting point is completely different: instead of using Rankin's method, one uses Siegel's way (actually due to Eichler, as I think I once told you) of computing L-series of real number fields by restricting Hilbert Eisenstein series to the diagonal. More precisely, let us rewrite our conjectural result

$$\log|N(j(z_1) - j(z_2))| = \sum_{\substack{\ell \text{ prime} \\ \left(\frac{D_1}{\ell}\right)=\left(\frac{D_2}{\ell}\right)=-1}} \Bigg(\sum_{\substack{|k|<\sqrt{D} \\ k\equiv D \pmod 2}} \sum_{\ell^n|\frac{D-k^2}{4}} \sum_{d|\frac{D-k^2}{4\ell^n}} \chi(d)\Bigg)\log\ell$$

(where we have replaced $-p$ and $-q$ by arbitrary coprime fundamental discriminants $\mathrm{disc}\, z_1 = D_1, \mathrm{disc}\, z_2 = D_2 < 0$ and set $D = D_1D_2$) as

$$\begin{aligned}\sum_{\substack{\mathrm{disc}\, z_i=D_i \pmod{\mathrm{SL}_2(\mathbb{Z})} \\ i=1,2}} \log|j(z_1) - j(z_2)| &= \sum_{\substack{|k|<\sqrt{D} \\ k\equiv D \pmod 2}} \Bigg(\sum_{d|\frac{D-k^2}{4}} \chi(d)\log d\Bigg) \\ &= \sum_{\substack{\nu\in\mathcal{D}^{-1} \\ \upsilon\gg 0 \\ \mathrm{tr}(\nu)=1}} \Bigg(\sum_{\mathfrak{a}|(\nu)\mathcal{D}} \chi(\mathfrak{a})\log N(\mathfrak{a})\Bigg);\end{aligned}$$

here $\mathcal{D}^{-1}$ = inverse different of $\mathbb{Q}(\sqrt{D}), \nu = \frac{k+\sqrt{D}}{2\sqrt{D}}$, and $\chi(\mathfrak{a})$ in the inner sum is the genus character associated to the decomposition $D = D_1D_2$. Note that $\sum\chi(\mathfrak{a})\log N(\mathfrak{a}) = \frac{d}{ds}\left(\sum\chi(\mathfrak{a})N(\mathfrak{a})^2\right)|_{s=0}$ and that $\sum\chi(\mathfrak{a})N(\mathfrak{a})^s$ vanishes at $s = 0$ ($(\nu)\mathcal{D}$ is a principal ideal with a generator $\nu\sqrt{D}$ of negative norm, and the character χ is of norm signature type since $D_1, D_2 < 0$). In other words, we are looking at the number

$$\frac{d}{ds}\sum_{\substack{\nu\gg 0 \\ \mathrm{tr}(\nu)=1}} \sigma_{s,\chi}\left((\nu)\mathcal{D}\right),$$

where

$$\sigma_{s,\chi}(\mathfrak{a}) = \sum_{\substack{\mathfrak{b}|\mathfrak{a} \\ \mathfrak{b} \text{ integral}}} \chi(\mathfrak{b})N(\mathfrak{b})^s.$$

for an (integral) ideal $\mathfrak{a}$.

Now in Siegel's paper, the number

$$\sum_{\substack{\nu\gg 0\\ \mathrm{tr}\nu=m}} \sigma_{k-1,\chi}\left((\nu)\mathcal{D}\right)$$

occurs as the m^{th} Fourier coefficient of the restriction to $\mathrm{SL}_2(\mathbb{Z})$ of the Eisenstein series of weight k on $\mathrm{SL}_2(\mathcal{O}_D)$ corresponding to the character χ. Siegel looked at the case $\chi =$ wide ideal class character (i.e. $\chi\left((\lambda)\right)\ \forall\ \lambda\in\mathcal{O}$), k even, but his method works equally well for χ of norm signature type (i.e. $\chi\left((\lambda)\right)=\mathrm{sign}(N(\lambda))\ \forall\ \lambda\in\mathcal{O}$) and k odd. However, for $k=1$ the corresponding Eisenstein series, which can be defined despite non-convergence by Hecke's method, vanishes identically. It is interesting that Hecke studied these series but failed to notice their vanishing — in fact, he claimed to show they weren't 0 — so that his whole paper was invalidated (as pointed out by Schoenberg in his footnotes to H.'s *Werke*). Van der Geer and I in our paper on $\mathbb{Q}(\sqrt{13})$ pointed out that Hecke's *method* was correct and that one could get examples of non-vanishing Eisenstein series of weight 1 on the Hilbert modular group by going to congruence subgroups. However, what I (unfortunately) didn't think of doing then was to take Hecke's series that vanish at $s=0$ and look at their derivatives there.

Enough talk; let's calculate. Let $K=\mathbb{Q}(\sqrt{D})$ $(D>0)$ be a real quadratic field, χ a narrow ideal class character of K of norm signature type (later, χ will be a genus character). Set

$$\begin{aligned}E(z,z';s) &= E_{k,\chi,1}(z,z',s)\\ &= \sum_{[\mathfrak{a}]}\overline{\chi(\mathfrak{a})}N(\mathfrak{a})^{1+2s}\sum_{(m,n)\in(\mathfrak{a}\times\mathfrak{a}-(0,0))/\mathcal{O}^\times}\frac{y^s y'^s}{(mz+n)(m'z'+n')(mz+n)^{2s}(m'z'+n')^{2s}}\end{aligned}$$

$(z=x+iy, z'=x'+iy'\in\mathcal{H}, s\in\mathbb{C}, \Re(s)\gg 0)$, where $[\mathfrak{a}]$ runs over all *wide* ideal classes (the summand is unchanged by $\mathfrak{a}\mapsto(\lambda)\mathfrak{a}$); this is a non-holomorphic Eisenstein series for $\mathrm{SL}_2\mathcal{O}$ and transforms like a holomorphic Hilbert modular form of weight 1. (Such forms needn't be 0, since $\mathcal{O}_K$ cannot contain a unit of norm -1.) The usual Fourier coefficient calculation gives

$$\begin{aligned}E(z,z';s) =& L_K(1+2s,\chi)y^s y'^s + D^{-1/2}L_K(2s,\chi)\Phi_s(0)^2 y^{-s}y'^{-s}\\ &+ D^{-1/2}y^{-s}y'^{-s}\sum_{\substack{\nu\in\mathcal{D}^{-1}\\ \nu\neq 0}}\sigma_{-2s,\chi}((\nu)\mathcal{D})\Phi_s(2\pi\nu y)\Phi_s(2\pi\nu' y)e^{2\pi i(\nu x+\nu' x')}\end{aligned}$$

where

$$\Phi_s(t)=\int_{-\infty}^{\infty}\frac{e^{-ixt}}{(x+i)(x^2+1)^s}dx \qquad (t\in\mathbb{R}).$$

Now $\Phi_s(t)$ has an analytic continuation to all s (so $E(z,z';s)$ also does) with

$$\Phi_0(t)=\begin{cases}-2\pi i e^{-t} & t>0\\ -\pi i & t=0\\ 0 & t<0\end{cases}$$

Hence if $\chi=\bar\chi$ (i.e. χ is a genus character), then

$$E(z,z';0)=L_K(1,\chi)-\pi^2 D^{-1/2}L_K(0,\chi)-4\pi^2 D^{-1/2}\sum_{\nu\gg 0}\sigma_{0,\chi}((\nu)\mathcal{D})e^{2\pi i(\nu z+\nu' z')}\equiv 0$$

by the functional equation of $L_K(s,\chi)$ and the fact that $\chi((\nu)\mathcal{D}) = -1$ for $\nu \gg 0$. (This was the vanishing that Hecke failed to notice.) In this case we look at $\frac{d}{ds}|_{s=0}$. For $\nu \gg 0$ the factor $\Phi_s(2\pi\nu y), \Phi_s(2\pi\nu' y)$ are $\neq 0$ at $s = 0$, so we replace $\sigma_{-2s,\chi}$ by its derivative and Φ_s by Φ_0. For $N(\nu) < 0$, $\sigma_{0,\chi}((\nu)\mathcal{D})$ is non-0 but one of the Φ_s vanishes. For $\nu \ll 0$, all 3 factors $\sigma_{-2,\chi}(\nu\mathcal{D}), \Phi_s(2\pi\nu y), \Phi_s(2\pi\nu' y')$ vanish, so they don't contribute. Hence

$$\frac{\partial}{\partial s}E(z,z';s)|_{s=0} = 2L_K(1,\chi)\log(yy') + 4C_\chi + 8\pi^2 D^{-1/2}\sum_{\substack{\nu\in\mathcal{D}^{-1}\\ \nu\gg 0}} \sigma'_\chi((\nu)\mathcal{D})e^{2\pi i(\nu z+\nu' z')}$$
$$-2\pi i D^{-1/2}\sum_{\substack{\nu\in\mathcal{D}^{-1}\\ \nu>0>\nu'}} \sigma_{0,\chi}((\nu)\mathcal{D})\Phi(2\pi\,|\nu'|\,y)e^{2\pi i(\nu z+\nu' z')} - (\text{same with } \nu\leftrightarrow\nu'),$$

where

$$\begin{aligned} C_\chi &= L'_K(1,\chi) + L_K(1,\chi)(\text{constant expression involving } \Gamma'(\tfrac{1}{2}),\ \text{etc.}),\\ \sigma'_\chi(\mathfrak{a}) &= \frac{\partial}{\partial s}\sigma_{s,\chi}(\mathfrak{a})|_{s=0} = \sum_{\mathfrak{b}|\mathfrak{a}}\chi(\mathfrak{b})\log N(\mathfrak{b}) \qquad (\mathfrak{a}\text{ integral}),\\ \Phi(t) &= e^t\cdot\frac{\partial}{\partial s}\Phi_s(-t)|_{s=0} = -2\pi i\int_1^\infty e^{-2tx}\frac{dx}{x} \qquad (t>0), \end{aligned}$$

and the term $(\nu\leftrightarrow\nu')$ is like its predecessor with ν,ν' and y,y' interchanged. Setting $z = z'$, we deduce that the function

$$E(z) = L_K(1,\chi)\log y + C_\chi + \frac{2\pi^2}{\sqrt{D}}\sum_{m=1}^\infty\Big(\sum_{\substack{\nu\in\mathcal{D}^{-1}\\ \nu\gg 0\\ \mathrm{tr}(\nu)=m}}\sigma'_\chi((\nu)\mathcal{D})\Big)e^{2\pi imz}$$
$$-\frac{\pi i}{\sqrt{D}}\sum_{m=1}^\infty\Big(\sum_{\substack{\nu\in\mathcal{D}^{-1}\\ \nu>0>\nu'\\ \mathrm{tr}(\nu)=m}}\sigma_{0,\chi}((\nu)\mathcal{D})\Phi(2\pi\,|\nu'|\,u)\Big)e^{2\pi imz}$$

transforms under $\mathrm{SL}_2(\mathbb{Z})$ like a modular form of weight 2. (Here we have divided by 4; the calculation is a little cleaner if we replace $E(z,z';s)$ by

$$E^*(z,z';s) = \pi^{-2s}D^s\Gamma(s+1)^2E(z,z';s) = -E^*(z,z';-s)$$

and work with $\Lambda_K(s,\chi)$ instead of $L_K(s,\chi)$.) Applying the holomorphic projection lemma of our paper, we deduce

$$\sum_{\substack{\nu\in\mathcal{D}^{-1}\\ \nu\gg 0\\ \mathrm{tr}(\nu)=m}}\sigma'_\chi((\nu)\mathcal{D}) = \lim_{s\to 0}\Big[2im\sum_{\substack{\nu\in\mathcal{D}^{-1}\\ \nu>0>\nu'\\ \mathrm{tr}(\nu)=m}}\sigma_{0,\chi}((\nu)\mathcal{D})\int_0^\infty\Phi(2\pi\,|\nu'|\,y)e^{-2\pi my}y^s dy + \frac{12i}{m}\frac{\sigma_1(m)}{m}\frac{L_K(1,\chi)}{s}\Big]$$
$$+\frac{12i}{\pi}\frac{\sigma_1(m)}{m}C_\chi + (\text{elementary expression})\cdot L_K(1,\chi)$$

We want to show that for $m = 1$ this reduces to

$$\sum_{\substack{\operatorname{disc} z_1 = D_1 \\ \operatorname{disc} z_2 = D_2 \\ (\bmod \mathrm{SL}_2(\mathbb{Z}))}} \log |j(z_1) - j(z_2)| \qquad (\chi \leftrightarrow D = D_1 \cdot D_2);$$

the result for higher m will correspond to non-maximal orders. The calculation is exactly analogous to the one in our paper: one shows that the integral

$$\int_0^\infty \Phi(2\pi |\nu'| y) e^{-2\pi m y} y^s dy \quad \left(= \frac{-2\pi i \Gamma(s+1)}{(2\pi m)^{s+1}} \int_1^\infty \frac{dx}{x(1 + 2|\nu'| x)^{s+1}} \right)$$

in the above expression can be replaced by (elementary factor)$Q_s(1 + \frac{2|\nu'|}{m})$ without changing the value of the limit; one then observes that

$$\sum_{\substack{\nu > 0 > \nu' \\ \operatorname{tr}(\nu) = m}} \sigma_{0,\chi}((\nu)\mathcal{D}) Q_s(1 + \frac{2|\nu'|}{m}) = \sum_{\substack{n > m\sqrt{D} \\ n \equiv mD \pmod 2}} \sigma_{0,\chi}(\frac{n + m\sqrt{D}}{2}) Q_s(\frac{n}{m\sqrt{D}}).$$

Hence (for $m = 1$)

$$\sum_{\substack{\nu \gg 0 \\ \operatorname{tr}(\nu) = 1}} \sigma'_\chi((\nu)\mathcal{D}) = (\text{elem.}) \cdot \lim_{s \to 0} \Big(\sum_{\substack{n > \sqrt{D} \\ n \equiv D \pmod 2}} R(\frac{n^2 - D}{4}) Q_s(\frac{n}{\sqrt{D}}) - \frac{\text{const.}}{s} \Big) + \text{ const.}$$

where the first constant is (elem.) $\cdot\, L_K(1, \chi)$ = (elem.) $\cdot\, h(D_1)h(D_2)$ and the second is (elem.) $\cdot\, L'_K(1, \chi)$ + (elem.) $\cdot\, L_K(1, \chi)$. On the other hand,

$$\sum_{\substack{z_1 \in \mathcal{H}/\Gamma \\ \operatorname{disc} z_1 = D_1}} \sum_{\substack{z_2 \in \mathcal{H}/\Gamma \\ \operatorname{disc} z_2 = D_2}} \log |j(z) - j(z')| =$$

$$\lim_{s \to 0} \Big(\sum_{\substack{(z_1, z_2) \in \mathcal{H}^2/\Gamma \\ \operatorname{disc} z_j = D_j (j=1,2)}} Q_s\Big(1 + \frac{(z_1 - z_2)^2}{2y_1 y_2}\Big) - \frac{\text{const.}}{s} \Big) + (\text{const.})$$

and one easily checks $1 + \frac{(z_1 - z_2)^2}{2y_1 y_2} = \frac{n}{\sqrt{D}}$ for some $n > \sqrt{D}$ with $\frac{n^2 - D}{4} = N\mathfrak{a}$ and that this is a 1-1 correspondence. Modulo details, that completes the proof.

This letter is getting very long and I should sign off, especially as it's 4:45 A.M. and I have a Japanese lesson today and am supposed to go skiing tomorrow. There was one other thing I wanted to mention, though. I always liked the higher Green's functions $R_k(z, z')$, whereas you prefer to stick to $j(z) - j(z')$ (or its analogues for $\Gamma_D(N)$) since you can only make sense of the finite heights in that case. However, I urge you to think seriously about R_k for $k > 1$. Our result shows that, for instance, the function $R_k(z) = \lim_{z \to z'} (R_k(z, z') - \text{singularity})$ satisfies

$$\sum_{\operatorname{disc} z = -p \pmod \Gamma} R_k(z) = \sum_{0 < n < p} \left(\sum_{d | n} \left(\frac{d}{p} \right) \log d \right) R_{Q_0}(p - n) \cdot (\frac{2n}{p} - 1)$$

if $S_{2k} = 1$ ($k = 2, 3, 4, 5, 7$). I had checked this numerically for $k = 2$ and $h(-p) = 1$, using

$$R_2(z) = \frac{\pi}{3} y + \frac{119\zeta(3)}{4\pi^2} y^{-2} - \left(4 - \frac{240}{\pi y} - \frac{120}{\pi^2 y^2} \right) e^{-2\pi y} \cos 2\pi x + \dots$$

and got agreement (not perfect, since I don't know the coefficient of $e^{-4\pi y}$). I now looked at $p=23$ and $p=31$ and found (to accuracy $e^{-2\pi\sqrt{p}}$, i.e. very nearly exactly on my HP-37)

$$R_2\left(\frac{1+i\sqrt{23}}{2}\right)=\frac{1}{23}[21\log 11+15\log(3\theta+1)+5\log 7-14\log(\theta+2)+22\log(3\theta+2)$$
$$+15\log 5-40\log(2-\theta)-23\log(2\theta-1)-250\log\theta]+\frac{1}{2}\log\theta+\frac{1}{2}\log(3-\theta)$$
$$(\theta^3-\theta-1=0)$$

$$R_2\left(\frac{1+i\sqrt{31}}{2}\right)=\frac{1}{31}[30\log(-\theta^2+2\theta+2)+31\log 3+23\log(\theta+1)-31\log(3\theta-4)$$
$$+6\log(3-\theta)+13\log 11-181\log\theta]+\frac{1}{2}\log(3\theta+1)$$
$$(\theta^3-\theta^2-1=0)$$

which except for the coefficients 250 and 181 of $\log\theta$ (i.e. the choice of generator of a principal ideal) is what you would get by supposing that $pR_2(z)$ is the log of a number in $\mathbb{Q}(j)$ of the appropriate norm, found by splitting up the norm in the same way as you did for $\log N(j(z)-j(z'))$. So $R_2(z)$ (and presumably also R_3, R_4, R_5, R_7) can be used just as well as $j(z)$ to generate class fields and hence is worthy of your algebraically oriented attention; moreover, the wealth of such functions suggests that there may be canonical generators for a great many ideals in $\mathbb{Q}(j)$ or $K(j)$, so that one gets relations in the class group à la Stickelberger.

Yours, Don

[Providence, RI]
Feb 18, 1983

Dear Don,

Mea culpa – this letter is intended as my repentance. Let $p \equiv 3 \pmod 4$ be prime with $p > 3$, $K = \mathbb{Q}(\sqrt{-p}), j = j(\frac{1+\sqrt{-p}}{2}), H = K(j)$ as usual. Let N be a positive integer with $N \equiv 0, 3 \pmod 4$, so $-N$ is a discriminant of a positive definite binary quadratic form. Assume further that N is prime to p, and define $\mathcal{H}_N(x) = \prod_{f^2|N} h_{-N/f^2}(x)$ as in page 6 of your letter. For example

$$\begin{aligned} \mathcal{H}_4(x) &= (x-1728)^{1/2} \\ \mathcal{H}_{12}(x) &= x^{1/3}(x - 2^4 3^3 5^3) \\ &\vdots \quad \text{etc.} \end{aligned}$$

The value $\mathcal{H}_N(j)$ is an algebraic integer in H, and the following theorem gives its prime factorization.

Proposition 1. *Let λ be a finite prime of H dividing the rational prime ℓ.*

(1) *If* $\left(\frac{\ell}{p}\right) = +1$ *then* $\mathrm{ord}_\lambda(\mathcal{H}_N(j)) = 0$

(2) *If* $\left(\frac{\ell}{p}\right) = -1$ *and* $\lambda = \lambda_\tau$, *then*

$$\mathrm{ord}_\lambda(\mathcal{H}_N(j)) = \sum_{z \geq 0} \sum_{k \geq 1} \delta(z) r_{\tau^2}\left(\frac{Np - z^2}{4\ell^k}\right)$$

where $r_{\tau^2}(m)$ is the number of integral ideals of norm m of K in the class of τ^2 ($\tau \in \mathrm{Gal}(H/K)$, and where $\delta(z) = 2$ if $z > 0$ and $z \equiv 0 \pmod p$, and $\delta(z) = 1$ otherwise.

Before the proof, some more examples:

$p = 11$	$\mathcal{H}_{12}(j) = 2^9 \cdot 11 \cdot 17 \cdot 29$
$p = 7$	$\mathcal{H}_{12}(j) = 3^4 \cdot 5^4 \cdot 17$
$p = 11$	$\mathcal{H}_{28}(j) = 7^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 41 \cdot 61 \cdot 73$

and an obvious *corollary*: if ℓ divides $\mathbb{N}_{H/\mathbb{Q}}\mathcal{H}_N(j)$ then $\ell \leq Np/4$.

Now a sketch of the proof. If $\left(\frac{\ell}{p}\right) = +1$, the elliptic curve E with invariant j has good *ordinary* reduction $\pmod \ell$. Let E' be any curve with multiplication by an order containing $\mathcal{O}_{-N} = \mathbb{Z} + \frac{N+\sqrt{-N}}{2}\mathbb{Z}$; then $j' \neq j$ in characteristic zero, and by Deuring's theorem on the reduction of singular moduli at ordinary primes $\therefore j' \not\equiv j \pmod \lambda$.

If $\left(\frac{\ell}{p}\right) \neq +1$ the curve E has supersingular reduction $\pmod \ell$. Let W denote the integers in the maximal unramified extension of the completion H_2 and to a prime of W. We'll assume $\ell > 3$ and $\ell \neq p$ for simplicity, but everything works in those cases too. By the results in singular moduli, $\mathrm{End}_{W/\pi^k}(\tilde{E}) = R(\mathfrak{a})_k$, where $\mathfrak{a}$ is an ideal with class τ in G. If $\tilde{E}$ is isomorphic to any $\tilde{E}'$ as above, $R(\mathfrak{a})_k$ must contain an element $[\alpha, \beta]$ which satisfies the same characteristic polynomial as $\frac{N+\sqrt{-N}}{2}$. That is:

$$\operatorname{Tr}\alpha \Rightarrow \alpha = \frac{x+N\sqrt{-p}}{2\sqrt{-p}} \quad \text{with } x \in \mathbb{Z}$$

$$\mathbb{N}[\alpha,\beta] = \alpha\bar{\alpha} + \ell^{2k-1}\beta\bar{\beta} = \frac{N^2+N}{4}.$$

But $\alpha\bar{\alpha} = \frac{x^2+pN^2}{4p}$ and $\beta = \gamma/\sqrt{-p}$ with $\gamma \in \bar{\mathfrak{a}}/\mathfrak{a}$. Thus we get a solution to the equation:

$$(*) \qquad x^2 + 4\ell^{2k-1}\mathbb{N}\mathfrak{b} = Np$$

with $\mathfrak{b} = (\gamma)\mathfrak{a}/\bar{\mathfrak{a}}$ an integral ideal in the class of τ^2. Conversely, if we *have* a solution $(x, \mathfrak{b})$ to $(*)$, we can reverse the process to recover $\pm\mathfrak{b}$. The insistence that $x \geq 0$ fixes the sign of β whenever $x \not\equiv 0 \pmod{p}$, as we must have the congruence $\alpha \equiv \mu\beta \pmod{\mathcal{O}_{\sqrt{-p}}}$. If $x \equiv 0 \pmod{p}$ we get two possibilities (but $x = 0$ really only contributes *one*). Furthermore, if we *have* any $[\alpha, \beta]$ in $R(\mathfrak{a})_k$ satisfying the equation of $\left(\frac{N+\sqrt{-N}}{2}\right)$, by Deuring's theory we can lift the curve *together* with this endomorphism to characteristic zero. This gives a curve E' over W with $\operatorname{End}_W(E') \supseteq \mathcal{O}_{-N}$. Putting all this together in the right order gives the proof. To check the $\delta = 2$ business, try $p = 11$ and $N = 43$.

Sorry I didn't see this before - it's really identical with the formulae for $j^{1/3}$ and $(j - 1728)^{1/2}$, where I was looking for elements like i or $\rho = \frac{1+\sqrt{-3}}{2}$ in $R(\mathfrak{a})_k$. I think it should definitely go in the paper on singular moduli.

Your idea about relations in the class group had occurred to me before, but then I saw only a *finite* number of relations for each p. Now each choice of N gives a *new* principal ideal, so it's probably worth looking into carefully. But I'm worried that the primes of residue characteristic $\left(\frac{\ell}{p}\right) = +1$ never enter in ....

Best, Dick

Class field theory as a dynamical system

by Gunther Cornelissen (Utrecht)
at the Arbeitstagung 2011

To Don Zagier, on his 60th birthday

Counting points. Let X denote a smooth projective curve over a finite field $k = \mathbf{F}_q$. Is X determined (up to isomorphism) from counting its points over finite extensions of k, i.e., by the numbers $N_n := |X(\mathbf{F}_{q^n})|$, i.e., by knowing its *zeta function*

$$\zeta_X(s) := \exp\left(\sum_{n\geq 1} N_n \frac{q^{-sn}}{n}\right) ?$$

The answer is **no** in general. Tate (1966) and Turner (1978) proved that for two curves X, Y over k, the equality $\zeta_X = \zeta_Y$ is equivalent to their respective Jacobians $\mathrm{Jac}(X) \sim \mathrm{Jac}(Y)$ being k-isogenous. The following example of E. Howe from 1996 illustrates this phenomenon: let $X_\pm \colon y^2 = x^5 \pm x^3 + x^2 - x - 1$ over $\mathbf{F}_3$. Then

$$\zeta_{X_\pm} = \frac{1 - T + T^2 - 3T^3 + 9T^4}{(1-T)(1-3T)} \text{ with } T = q^{-s},$$

and here are the first few point counts (for this occasion done independently in Sage):

n	1	2	3	4	5	6	7 …
N_n	3	11	21	107	288	719	2271 …

Can we remedy this?

Number fields. Now consider the same problem for a number field K, with its Dedekind zeta function

$$\zeta_K(s) := \sum_{0\neq\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s},$$

where the sum runs over all non-zero ideals $\mathfrak{a}$ of the ring of integers of K. Knowing ζ_K is the same as knowing $f(\mathfrak{p}|p)$ for all prime ideals $\mathfrak{p}$. A Theorem of Mihály Bauer (1903) says that if K, L are two number fields *that are Galois over* $\mathbf{Q}$, then $K \cong L$ is equivalent to $\zeta_K = \zeta_L$. However, a result of Gaßmann from 1926 says that in general, there do exist non-isomorphic number fields K, L with $\zeta_K = \zeta_L$. Actually, he proves that $\zeta_K = \zeta_L$ is equivalent to the following statement: fix a common extension N of K and L that is Galois over $\mathbf{Q}$ with Galois group G, and let H_K and H_L denote the Galois groups of N/K and N/L, respectively. Then $\zeta_K = \zeta_L$ if and only if each G-conjugacy class intersects H_K and H_L in the same number of elements. A result from Perlis from 1977 says that the first example with $\zeta_K = \zeta_L$ but $K \ncong L$ occurs in degree 7 over $\mathbf{Q}$, and an example is given by $K = \mathbf{Q}(\alpha), L = \mathbf{Q}(\beta)$ with

$$\alpha^7 - 7\alpha + 3 = 0 \text{ and } \beta^7 + 14\beta^4 - 42\beta^2 - 21\beta + 9 = 0.$$

Can we remedy this?

Historical aside: internal/external = failure/success. Here are some further attempts at finding objects that determine isomorphism of number fields K and L: an *isomorphism of adele rings* $\mathbf{A}_K \cong \mathbf{A}_L$ is stronger than equality of zeta functions (strictly stronger for number fields, equivalent for function fields), but still does not imply field isomorphism (Komatsu, 1976); an example is $K = \mathbf{Q}(\sqrt[8]{18})$and $L = \mathbf{Q}(\sqrt[8]{288})$. An isomorphism of abelian Galois groups $G_K^{\mathrm{ab}} \cong G_L^{\mathrm{ab}}$ is not enough, either: Kubota determined the isomorphism type of G_K^{ab} (its *Ulm invariants*) in terms of K, and Onabe (1976) gave explicit examples, such as $G^{\mathrm{ab}}_{\mathbf{Q}(\sqrt{-2})} \cong G^{\mathrm{ab}}_{\mathbf{Q}(\sqrt{-3})}$. At the other side of the spectrum, an isomorphism of absolute Galois groups $G_K \cong G_L$ does imply that $K \cong L$! This is due to Neukirch (1969) when K, L are Galois over $\mathbf{Q}$ and Uchida (1976) in general. This last theorem is the first manifestation of what Grothendieck called **an**abelian theorems. We conclude that the objects listed above, that are *internal* to a number field K (i.e., can be described in terms of ideals of K), such as ζ_K, $\mathbf{A}_K$ or G_K^{ab} (which is internal by class field theory), lead to *failure*, whereas a mysterious object G_K, that is *external* to K (described in terms of extensions of K, or via the Langlands program in terms of automorphic forms), leads to *success* … Can we do better, and have internal success?

Method: class field theory as (noncommutative) dynamical system. Let J_K denote the group of fractional ideals of K, J_K^+ the semigroup of integral ideals of K, $\vartheta_K\colon \mathbf{A}_K^* \to G_K^{\mathrm{ab}}$ the Artin reciprocity map and $\hat{\mathscr{O}}_K$ the integral finite adeles of K. Choose a section s of the natural map $\mathbf{A}_{K,f}^* \to J_K\colon (x_\mathfrak{p})_\mathfrak{p} \mapsto \prod \mathfrak{p}^{v_\mathfrak{p}(x_\mathfrak{p})}$.

These objects were used by Ha and Paugam in 2005 to construct a dynamical system associated to K (for $K = \mathbf{Q}$, this is the famous Bost-Connes system), as follows: we make a *topological space*

$$X_K = G_K^{\mathrm{ab}} \times_{\hat{\mathscr{O}}_K^*} \hat{\mathscr{O}}_K,$$

consisting of classes $[(\gamma, \rho)]$ for $\gamma \in G_K^{\mathrm{ab}}$ and $\rho \in \hat{\mathscr{O}}_K$, defined by the equivalence

$$(\gamma, \rho) \sim (\vartheta_K(u^{-1}) \cdot \gamma, u\rho) \text{ for all } u \in \hat{\mathscr{O}}_K^*.$$

Then we consider the *action* of $\mathfrak{n} \in J_K^+$ on X_K given by

$$\mathfrak{n} * [(\gamma, \rho)] := [(\vartheta_K(s(\mathfrak{n}))^{-1}\gamma, s(\mathfrak{n})\rho)].$$

In this way, we get a dynamical system (X_K, J_K^+).

Main Theorem. (C-Matilde Marcolli, arxiv:1009.0736) *For two number fields K and L, an isomorphism $K \cong L$ is equivalent to a norm-preserving isomorphism of dynamical systems $(X_K, J_K^+) \cong (X_L, J_L^+)$.*

By *isomorphism of dynamical systems*, we mean a homeomorphism $\Phi\colon X_K \xrightarrow{\sim} X_L$ and a group homomorphism $\varphi\colon J_K^+ \xrightarrow{\sim} J_L^+$ such that $\Phi(\mathfrak{n} * x) = \varphi(\mathfrak{n}) * \Phi(x)$ for all $x \in X_K$ and $\mathfrak{n} \in J_K^+$; and *norm-preserving* means that $N_L(\varphi(\mathfrak{n})) = N_K(\mathfrak{n})$ for all $\mathfrak{n} \in J_K^+$.

In a sense, this theorem shows that a *suitable combination of failure* (ζ_K, which will be the partition function of the system, G_K^{ab} and $\mathbf{A}_K$, which occur in the system) *may lead to success.* It gives an "internal" description of the isomorphism type of a number field. It also holds in a function field, with a slightly different, easier proof.

The proof is really to "hit the dynamical system with a hammer until enough isomorphic objects jump out".

Reformulation using Quantum Statistical Mechanics. There is a way to reformulate the main theorem by encoding the dynamics in Banach algebra language. We set $A_K := C(X_K) \rtimes J_K^+$ to be the semigroup crossed product C^*-algebra corresponding to the dynamical system. Physically, it corresponds to the *algebra of observables*. If we let $\mu_\mathfrak{n}$ and $\mu_\mathfrak{n}^*$ denote the partial isometries of the algebra corresponding to $\mathfrak{n} \in J_K^+$, then we also need the non-involutive subalgebra $A_K^\dagger$ of A_K generated by $C(X)$ and $\langle \mu_\mathfrak{n} \rangle_{\mathfrak{n} \in J_K^+}$ (but not the $\mu_\mathfrak{n}^*$). We also consider a one-parameter subgroup of automorphisms of A_K, denoted $\sigma_K\colon \mathbf{R} \hookrightarrow \mathrm{Aut}(A_K)$, defined by $\sigma_K(t)(f) = f$ and $\sigma_K(t)(\mu_\mathfrak{n}) = N_K(\mathfrak{n})^{it}\mu_\mathfrak{n}$. The algebra with this so-called *time evolution* is an abstract *quantum statistical mechanical system.* A slightly stronger statement than the main theorem is the following: $K \cong L$ is equivalent to an isomorphism of $(A_K, \sigma_K) \xrightarrow{\sim} (A_L, \sigma_L)$ that maps $A_K^\dagger$ to $A_L^\dagger$.

From the main theorem, we can deduce our answer to the problems outlined before:

Theorem. *If K and L are global fields (number fields, or function fields of curves over finite fields), then $K \cong L$ (which, in the case of function fields of curves is equivalent to isomorphism of the curves up to twists over the ground field) is equivalent to the existence of an isomorphism $\psi\colon G_K^{\mathrm{ab}} \xrightarrow{\sim} G_L^{\mathrm{ab}}$, such that **all** abelian L-series match: $L_K(\chi) = L_L((\psi^{-1})^*\chi)$ for all $\chi \in \mathrm{Hom}(G_K^{\mathrm{ab}}, S^1)$.*

We discovered this theorem because L-series occur as evaluations of low temperature equilibrium states of the system at particular test functions related to the character. Our proof of this theorem is to deduce from L-series equality an isomorphism of dynamical systems, which basically boils down to a bit of character theory, and then using the main theorem. In the meanwhile, Bart de Smit has discovered a purely number theoretical proof of the theorem for L-series for number fields, and has actually proven something much stronger: for every number field K, there is a character of order 3, such that $L_K(\chi) \neq L_{K'}(\chi')$ for *every* number field $K' \not\cong K$ and character for $G_{K'}^{\mathrm{ab}}$. This proof does not seem to transfer readily to function fields.

Final remark: the theorem is not really an analytic statement. It suffices to have equality of L-series at sufficiently large integers. Hence the theorem also holds with p-adic L-functions. One may read it as an equivalence of rank-one motives over K and L.

An analog in Riemannian geometry. The isospectrality problem has a long history, that can be traced back at least to the Wolfskehl lecture of the dutch physicist Lorentz in Göttingen in 1910, where he asked whether the spectrum of the Laplacian on a domain (with suitable boundary conditions) determines the volume. He refers to the Leiden PhD thesis of Johanna Reudler, that very cleverly computes several convincing examples (published in 1912). Hermann Weyl proved the general case in 1911, and much later Mark Kac popularized the question whether the entire shape of the region (so up to euclidean transformations) is determined by the spectrum, as "Can you hear the shape of a drum?"(this formulation is due to Bers, the problem was originally posed by Bochner). The first counterexample was the construction of two non-isometric Riemannian manifolds with the same spectrum by Milnor, based on Witt's theory of quadratic forms. Then even came non-homeomorphic isospectral manifolds in the work of Ikeda (lens spaces) and Vignéras (3-manifolds).

Let (X, g) denote a closed Riemannian manifold with Laplace operator Δ_X. The question whether or not the spectrum (with multiplicities) determines the isometry type of X is the same as that whether or not the *spectral zeta function*

$$\zeta_X(s) = \sum_{\lambda \neq 0} \frac{1}{\lambda^s} = \mathrm{tr}(\Delta_X^{-s})$$

(sum over the non-zero eigenvalues of the Laplace operator, with multiplicities) does so. Can we do better? This time, our "remedy" is the following: for $a \in C(X)$, set $\zeta_{X,a}(s) = \mathrm{tr}(a\Delta_X^{-s})$, and for $a \in W(X)$ (Lipschitz functions) set $\tilde{\zeta}_{X,a} = \mathrm{tr}(a[\Delta_X, a]\Delta_X^{-s})$. Then:

Theorem. (C-Jan Willem de Jong; arXiv:1007.0907) *Let X and Y denote two closed RIemannian manifolds, and $\varphi\colon X \to Y$ a C^1-bijective map. Then φ being an isometry is equivalent to the following two properties holding simultaneously*

(a) $\zeta_{Y,a_0} = \zeta_{X,\varphi^*(a_0)}$ *for all* $a_0 \in C(Y)$*, and*

(b) $\tilde{\zeta}_{Y,a_1} = \tilde{\zeta}_{X,\varphi^*(a_1)}$ *for all* $a_1 \in W(Y)$*.*

The proof is a rather formal computation with residues. Various analytically more challenging amplifications are possible, for example, condition (a) alone suffices when the spectrum is simple (which is the generic case by a result of Uhlenbeck). In the above theorem, one can also restrict to a countable dense subset of functions, and to sufficiently large integral values of the zeta functions, so the characterisation is really by countably many values.

Lengths of maps. One may now define the *length of a map* $\varphi\colon X \to Y$ as the "distance between the (meromorphic) zeta functions that occur in the theorem". The usual distance of meromorphic functions doesn't quite work, but the following does: The length $\ell(\varphi)$ of φ of Riemannian manifolds of dimension n is

$$\ell(\varphi) := \sup_{\substack{a_0 \in C(Y, \mathbf{R}_{\geq 0}) - \{0\} \\ a_1 \in W^1(Y) - \mathbf{R}}} \sup_{n \leq s \leq n+1} \max\{|\log\left|\frac{\zeta_{X,a_0^*}(s)}{\zeta_{Y,a_0}(s)}\right||, |\log\left|\frac{\tilde{\zeta}_{X,a_1^*}(s)}{\tilde{\zeta}_{Y,a_1}(s)}\right||\}.$$

This then satisfies $\ell(\varphi) = 0$ if and only if φ is an isometry, and $\ell(\psi \circ \varphi) \leq \ell(\psi) + \ell(\varphi)$. One can also show that

$$d(X, Y) := \max\{\inf_{C^1(X \stackrel{\varphi}{\to} Y)} \ell(\varphi), +\infty\}$$

defines an extended metric between isometry classes of Riemannian manifolds.

As an example especially for Don Zagier, we bound the distance d between two tori, corresponding to i and $\rho = (1 + \sqrt{-3})/2$ in the upper half plane. This will satisfy

$$e^d \leq \frac{E(i, 2)}{E(\rho, 2)} = \frac{\zeta_{m^2+n^2}(2)}{\zeta_{m^2-mn+n^2}(2)} = \frac{3\sqrt{3}}{4} \cdot \frac{D(i)}{D(\rho)} = 1.17235730884473\ldots,$$

where E is an Eisenstein series, ζ_Q (with Q a binary quadratic form) is the Epstein zeta function, and D is the Bloch-Wigner dilogarithm function.

Pluralizing zeta. ZETA counts things (points, ideals, geodesics, spectra, ...) — it is beautiful, but sometimes lonely, it can fail as an *individual.* But it will be happy and succeed as *part of a family* of ZETAS. This statement applies with the author substituted for ZETA (maybe not the part about beauty ...) and the MPIM substituted for 'family'. Thank you, Don. $\square$

Loop equations and spectral curves

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Schwinger-Dyson equations are well-known relations in any field theory, which follow from the diffeomorphism invariance of an integration or by integration by parts. They are very useful in probability, in quantum field theory, string theory, ... because they give a large amount (even if not a complete set) of relations between expectations values, i.e. the observables of the theory. Those arising from large matrix integrals, with a unitary invariant weight, can be recast into a universal form, which we call *loop equations*. I do not intend to be precise here, but we will use the name "loop equations" as soon as we deal with a family of observables $W_n(x_1, \ldots, x_n)$, which are analytic multivalued 1-forms in each complex variable x_i ($1 \leq i \leq n$), and satisfy:

$$(\star_n) \qquad \sum_a W_n({}^a x, x_I) = \delta_{n,1}\, \nu^{-1}\, P_1(x_1) + \delta_{n,2}\, P_2(x_1, x_2)$$

$$(\diamondsuit_n) \qquad \sum_{a<b} W_{n+1}({}^a x, {}^b x, x_I) + \sum_{J \subseteq I} W_{|J|+1}({}^a x, x_J) W_{n-|J|}({}^b x, x_{I\setminus J}) = \mathrm{d}x\, \nu^{-2}\, Q_n(x)$$

where P_i and Q_i are analytic 1-forms whose singularities are known and fixed *a priori*, and in particular which are univalued around the points where W_n become multivalued. The meaning of this property should become clear to the reader after some examples. ν is an extra parameter. The purpose of this text is to illustrate the ubiquity of loop equations, and review briefly the existing technology to solve them.

1. The topological recursion

Loop equations have many solutions, which are not always matrix integrals. After precursor works by Ambjørn, Chekhov, Makeenko and others [2, 1], Eynard and Orantin have described a complete family of solutions of loop equations [3], for which (i) the W_n's have a expansion of the form $W_n = \sum_{g \geq 0} \nu^{-(2-2g-n)}\, W_n^{(g)}$ (it can be either a formal generating series with parameter ν, or an asymptotic series when $\nu \to 0$) ; (ii) where $\omega({}^a x)$ is interpreted as the value taken by ω in different sheets of a covering $X : \Sigma \to \mathbb{P}_1$; (iii) $W_n^{(g)}$ is an element of $\Omega^1(\Sigma)^{\otimes n}$ (i.e. a 1-form in each of its n variables), whose singularities are located at the branchpoints of X only.

Definition

It can be presented as an axiomatic construction, which associates to a *spectral curve* $\mathcal{S}$, a family of forms $\omega_n^{(g)}[\mathcal{S}] \in \Omega^1(\Sigma)^{\otimes n}$, and of numbers $\mathcal{F}^{(g)}[\mathcal{S}] \in \mathbb{C}$. We need some preliminary definitions. For us, a spectral curve is the data of a plane curve and a *Bergman kernel*. A plane curve is defined by an equation $E(X,Y)=0$ in $\mathbb{C}^2$, or equivalently by the data of a Riemann surface Σ and two analytic functions X and Y defined on Σ. A Bergman kernel is an element $B\in\Omega^1(\Sigma)\otimes\Omega^1(\Sigma)$ (i.e a bi-differential form), whose singularity locus is the diagonal $\Delta=\{(z,z)\quad z\in\Sigma\}$, and such that, in any coordinate patch ξ, $B(\xi(z_1),\xi(z_2))=\frac{\mathrm{d}\xi(z_1)\mathrm{d}\xi(z_2)}{(\xi(z_1)-\xi(z_2))^2}+O(1)$ when $z_2\to z_1$. When $\Sigma\simeq\mathbb{C}$ or $\widehat{\mathbb{C}}$, we can take for sure $B(\xi_1,\xi_2)=\frac{\mathrm{d}\xi_1\mathrm{d}\xi_2}{(\xi_1-\xi_2)^2}$ where ξ is a global coordinate. On the torus $\Sigma\simeq\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$, another example of Bergman kernel is provided by the Weierstraß elliptic function, namely $B(u_1,u_2)=\mathrm{d}u_1\mathrm{d}u_2\,(\wp(u_1-u_2|\tau)+\mathrm{cte})$. The *ramification points* $a_i\in\Sigma$ are the zeroes of $\mathrm{d}X$. For simplicity, we shall assume that these are only simple zeroes. So, in a small neighborhood $U_i\subseteq\Sigma$ of a_i, a given value $x\in X(U_i)$ has two preimages in U_i. We define a local involution by associating to $z\in U_i$ the point $\overline{z}\in X^{-1}(X(U_i))$ such that $X(z)=X(\overline{z})$ but $z\neq\overline{z}$ unless $z=a_i$. We then define the *adapted Cauchy kernel*, which is an element of $\Omega^1(\Sigma)\otimes\Omega^{-1}\Big(\coprod_i U_i\Big)$:

$$R(z_0,z)=-\frac{1}{2}\,\frac{\int_{\overline{z}}^{z}B(z_0,\cdot)}{\big(Y(z)-Y(\overline{z})\big)\mathrm{d}X(z)}$$

We are now in position to give the definition of $\omega_n^{(g)}$ (since the spectral curve $\mathcal{S}$ is fixed, we omit to precise it in bracket), by induction on $-\chi_{n,g}=2g+2-n$. We set

$$\omega_1^{(0)}(z)=-Y\mathrm{d}X(z)$$

by convention, in order to have later uniform formulas for all n and g. We set for initialization:

$$\omega_2^{(0)}(z_1,z_2)=B(z_1,z_2)$$

and then:

$$\omega_n^{(g)}(z_1,\underbrace{z_2\ldots,z_n}_{z_I})=\sum_i\operatorname*{Res}_{z\to a_i}R(z_1,z)\Big[\omega_{n+1}^{(g-1)}(z,\overline{z},z_I)+\sum_{\substack{J\subseteq I\\0\leq h\leq g}}{}^{'}\,\omega_{|J|+1}^{(h)}(z,z_J)\,\omega_{n-|J|}^{(g-h)}(\overline{z},z_{I\setminus J})\Big]$$

It is a sum over residues at ramifications points. $\sum'$ means that the terms involving $\omega_1^{(0)}$ should be discarded. Though one has to choose one variable (here z_1) to write the formula, one can prove by induction that $\omega_n^{(g)}(z_1,\ldots,z_n)$ is symmetric in all the z_i's. This formula has a diagrammatic representation: $\omega_n^{(g)}$ is a weighted sum over trivalent graphs $\mathcal{G}$ of genus g with n external legs. Recursively, following an external leg of $\mathcal{G}$ up to a trivalent vertex and cutting it, we are left either with a graph $\mathcal{G}'$ which is either connected but with one handle less (first term), or disconnected so that the handles and the remaining external legs are shared out between the two connected components (second term). This construction is a recursion on the Euler characteristics of the graphs $\mathcal{G}$, hence the name *topological recursion*. Eventually, we define for $g \geq 2$ the complex numbers:

$$\omega_0^{(g)} \equiv \mathcal{F}^{(g)} = \frac{1}{2g-2} \sum_i \operatorname*{Res}_{z \to a_i} \omega_1^{(g)}(z) \int^z \omega_1^{(0)}(z)$$

We shall not address here the definition of $\mathcal{F}^{(0)}$ (the so-called *prepotential* of the spectral curve [4]) nor of $\mathcal{F}^{(1)}$ (which is related [5] to the spectral determinant of the Laplacian on Σ endowed with the metrics $|Y\mathrm{d}X|^2$), but we point out that these notions exist and complete harmoniously the construction. I have not told yet what does the $\omega_0^{(g)} = \mathcal{F}^{(g)}$ represent from the point of view of the loop equations. Formula 0-1 for $n = 0$ provides the explanation. Notice that the stable topologies $\chi_{g,n} < 0$ are in some sense more uniform to compute than the unstable topologies $\chi_{g,n} \geq 0$ (occuring for $(g,n) = (0,0),(1,0),(0,1),(0,2)$).

Properties

This intrinsic construction has several interesting properties, the first one being that it satisfies $\star_n$ and $\diamondsuit_n$, order by order in powers of ν. The index a labels the sheets of the covering $X : \Sigma \to \widehat{\mathbb{C}}$. The other most important properties regarding algebraic geometry are:

- $\diamond$ *Symplectic invariance* [6]. $\mathcal{F}^{(g)}[\mathcal{S}]$ are invariant under the transformations $(X,Y) \to (X',Y')$ such that $\mathrm{d}X' \wedge \mathrm{d}Y' = \pm\mathrm{d}X \wedge \mathrm{d}Y$. Although $\omega_n^{(g)}[\mathcal{S}]$ are themselves not invariant, their cohomology class is invariant.

- $\diamond$ *Special geometry*. Imagine that we have a smooth family of curves $\mathcal{S}_\varepsilon = (\Sigma, X + \varepsilon(\delta X), Y + \varepsilon(\delta Y), B)$. We introduce $\delta\Omega = (\delta Y)\mathrm{d}X - (\delta X)\mathrm{d}Y$, which can always be represented in the form $\Omega(z_0) = \int_{z \in \Omega^*} \Lambda_\Omega(z)\, B(z_0, z)$, where $\Omega^* \subseteq \Sigma$ is a cycle and Λ_Ω is a germ of holomorphic function on Ω^*. Then:

$$\forall n,g \in \mathbb{N}^2 \qquad \partial_{\varepsilon=0}\, \omega_n^{(g)}[\mathcal{S}_\varepsilon](z_1,\ldots,z_n) = \int_{z \in \Omega^*} \Lambda_\Omega(z)\, \omega_{n+1}^{(g)}[\mathcal{S}_0](z,z_1,\ldots,z_n) \quad \text{(0-1)}$$

- $\diamond$ *Modular properties*. Assume that Σ is a compact Riemnn surface of genus $\mathfrak{g} > 0$. $H_1(\Sigma,\mathbb{Z}) \simeq \mathbb{Z}^{2\mathfrak{g}}$, and it admits a *symplectic basis* $(\mathcal{A}_\mathfrak{h}, \mathcal{B}_\mathfrak{h})_{1 \leq \mathfrak{h} \leq \mathfrak{g}}$. Such a choice determines a basis of holomorphic forms $\mathrm{d}u_\mathfrak{h}$ which are dual to the $\mathcal{A}$-cycles, an Abel map $\mathbf{u}$, a matrix of periods $\tau_{\mathfrak{h}\mathfrak{h}'} = (2i\pi)^{-1} \oint_{\mathcal{B}_\mathfrak{h}} \mathrm{d}u_{\mathfrak{h}'}$ and thus a Siegel theta function. Then, the admissible Bergman kernels on Σ are of the form:

$$B_\kappa(z_1,z_2) = \mathrm{d}_{z_1}\mathrm{d}_{z_2} \ln \Theta(\mathbf{u}(z_1) - \mathbf{u}(z_2) + \mathbf{c}|\tau) + 2i\pi \sum_{\mathfrak{h},\mathfrak{h}'=1}^{\mathfrak{g}} \kappa_{\mathfrak{h}\mathfrak{h}'}\, \mathrm{d}u_\mathfrak{h}(z_1)\, \mathrm{d}u_{\mathfrak{h}'}(z_2)$$

where $\mathbf{c}$ is a nonsingular odd characteristics. In turn, the choice of a Bergman kernel allows to construct the $\omega_n^{(g)}[\mathcal{S}_\kappa]$ and the $\mathcal{F}^{(g)}[\mathcal{S}_\kappa]$. We have emphasized the dependence in κ, and actually $\omega_n^{(g)}$ is a polynomial of degree $3g-3+2n$ in κ. The modular group $\mathrm{Sp}(2\mathfrak{g},\mathbb{Z})$ acts on all these objects by change of symplectic basis. Remarkably, for $\kappa = \kappa^0 = \frac{i}{2\,\mathrm{Im}\,\tau}$, B_{κ^0} is invariant under the full modular group. This implies that all $\omega_n^{(g)}[\mathcal{S}_{\kappa^0}]$ with $\chi_{g,n} < 0$ are also invariant. But they are obviously not holomorphic in the moduli of the curve: they satisfy *holomorphic anomaly equations* [7]. Conversely, if κ is chosen such that $\omega_n^{(g)}[\mathcal{S}_\kappa]$ is holomorphic in the moduli of the curve, the $\omega_n^{(g)}$ are no more modular invariant. This kind of phenomena has been highlighted earlier by Kaneko and Zagier in the context of modular forms [8], and Aganagic, Bouchard and Klemm in the context of topological strings [9]

This axiomatic is robust, in the sense that it gives solutions to many different-looking loop equations. The curve $\mathcal{S}$ need not be algebraic, nor compact. The notion of Bergman kernel can also be twisted [10]. The topological recursion has many applications, to compute "weighted sum over surfaces" [11], understood widely: number of discretized surfaces, intersection numbers in the moduli space of curves [12], Gromov-Witten invariants (cf. below) and instanton counting in string theory, ... For each of these problems, there exists a spectral curve $\mathcal{S}$ for which the $\omega_n^{(g)}$'s produces (provably, or conjecturally for Gromov-Witten invariants) the desired numbers or generating series of them. Nevertheless, we do not understand at present its true nature in algebraic geometry. In particular, though the $\mathcal{F}^{(g)}[\mathcal{S}]$ are symplectic invariants, their meaning is unclear.

2. Some instances of loop equations

The lesson to draw at the end of each of the examples below is that (a) if you have quantities W_n which satisfy loop equations and have (provably or by construction) an expansion of the form $\sum_{g\geq 0} \nu^{-(2-2g-n)}\, W_n^{(g)}$, where $W_n^{(g)}(x_1,\ldots,x_n)$ for $\chi_{g,n} < 0$ have singularities only at branchpoints ; (b) and if you can determine explicitly the corresponding spectral curve $\mathcal{S}$; then, $W_n^{(g)}$ is given by the topological recursion applied to $\mathcal{S}$.

N dimensional integrals and matrix integrals

Consider a measure of the form:

$$\mathrm{d}\mu_{N,\gamma^N}(\lambda) = \prod_{i=1}^{N} \mathrm{d}\lambda_i\, e^{-NV(\lambda_i)} \prod_{1\leq i<j\leq N} K(\lambda_i,\lambda_j) \tag{0-2}$$

where we assume for our example $K(\lambda_i,\lambda_j) = |\lambda_i-\lambda_j|^{\beta_{\mathrm{id}}} \prod_f (\lambda_i - f(\lambda_j))^{\beta_f}$, and also that $f(\gamma)\cap\gamma=\emptyset$ for any f appearing there. We call $Z_{N,\gamma}=\int \mathrm{d}\mu_{N,\gamma^N}(\lambda)$ the partition function, and we define the *correlators*:

$$\begin{aligned} W_1(x) & = & \Big\langle \sum_{i=1}^{N} \frac{\mathrm{d}x}{x-\lambda_i}\Big\rangle \\ W_2(x_1,x_2) & = & \Big\langle \sum_{i_1=1}^{N}\sum_{i_2=1}^{N} \frac{\mathrm{d}x_1}{x_1-\lambda_{i_1}}\frac{\mathrm{d}x_2}{x_2-\lambda_{i_2}}\Big\rangle - \Big\langle \sum_{i=1}^{N} \frac{\mathrm{d}x_1}{x_1-\lambda_{i_1}}\Big\rangle\Big\langle \sum_{i_2=1}^{N} \frac{\mathrm{d}x_2}{x_2-\lambda_{i_2}}\Big\rangle \quad \text{etc.} \end{aligned}$$

If we express the invariance of the integral (up to boundary terms) under all possible change of variables generated by $\lambda_i\mapsto\lambda_i+\varepsilon\,\lambda_i^p$ to first order in ε, we arrive to a relation [10]:

$$\begin{aligned} & (\beta_{\mathrm{id}}-2)\mathrm{d}_x[W_1(x)]+W_2(x,x)+\beta\big(W_1(x)\big)^2 \\ & +\sum_f \big(\beta_f W_2(x,f(x))+W_1(x)W_1(f(x))\big)-N\,\mathrm{d}V(x)\,W_1(x) \quad = \quad \mathrm{d}x\,Q_1(x) \end{aligned}$$

While the W_n's are holomorphic differential forms in the domain $(\mathbb{C}\setminus\gamma)^n$ but probably will have a discontinuity on γ, the main information hidden in this equation is that $Q_1(x)$ is a holomorphic 1-form in the neighborhood on γ. When $\beta_{\mathrm{id}}=2$, note the similarity with $\diamondsuit_1$, where $\nu^{-1}=N$ is the number of random variables. By the same method, one can derive relations of the form $\diamondsuit_n$ for the n-point correlators W_n. Computing then the discotinuity of the left hand side, we arrive also to $\star_n$. Sticking to $\beta_{\mathrm{id}}=2$, let us mention various realizations of (0-2):

- ◇ For $K(\lambda_i,\lambda_j)=|\lambda_i-\lambda_j|^2$, $\mathrm{d}\mu_{N,\mathbb{R}^N}(\lambda)$ is the measure on eigenvalues on a $N\times N$ hermitian matrix M equipped with a measure $\mathrm{d}\mu(M)=\mathrm{d}M\,e^{-N\,\mathrm{Tr}\,V(M)}$.
- ◇ The partition function of $\mathrm{U}(N)$ Chern-Simons theory for the Seifert manifolds $X(p_1/q_1,\ldots,p_L/q_L)$ [13] can be written as before with
$$K(\lambda_i,\lambda_j)=\prod_{l=1}^{L}\frac{\lambda_i^{p_l}-\lambda_j^{p_l}}{\lambda_i-\lambda_j}\,\frac{\lambda_i^{q_l}-\lambda_j^{q_l}}{\lambda_i-\lambda_j}$$
- ◇ If V and K are polynomials, $\beta_f\in -a\mathbb{N}$ for some number a independent of f, and (0-2) is considered as a formal measure, the standard mapping of Brézin, Itzkyson, Parisi and Zuber [14] shows that Z is a generating series for discretized surfaces $\mathcal{M}$, carrying self-avoiding loops going through the faces of $\mathcal{M}$. The weight of $\mathcal{M}$ in this sum is proportional to N^{χ} (χ being the Euler characteristics of $\mathcal{M}$) and $a^{\#\mathrm{loops}}$.

In this context, the spectral curve is the plane curve of equation $y=\lim_{N\to\infty}N^{-1}\,W_1(x)$, endowed with a Bergman kernel related to $\lim_{N\to\infty}W_2(x_1,x_2)$.

It has also been shown that loop equations of the form $(\star_n, \diamondsuit_n)$ arise in the chain of hermitian matrices with external field, i.e. for a measure on $\mathcal{H}_N(\mathbb{R})^k$:

$$\mathrm{d}\mu(M_1, \ldots, M_k) = \mathrm{d}M_1 \cdots \mathrm{d}M_k \, \exp\, N\Big(\sum_{l=1}^{k} \mathrm{Tr} V_l(M_k) - \sum_{l} c_{l,l+1}\, \mathrm{Tr} M_l M_{l+1} + \mathrm{Tr} M_1 \mathbf{R}\Big)$$

where $\mathbf{R}$ is a fixed matrix (cf. [6] for the chain of 2 matrices). Again, the n-point correlators W_n are defined from expectations values of $\prod_{j=1}^{n} \mathrm{Tr}\, \frac{1}{x_j - M_l}$ (for any fixed index $l \in \{1, \ldots, k\}$). Unlike the case of (0-2), several non trivial steps are involved on top of the derivation of Schwinger-Dyson equations before arriving to $(\star_n, \diamondsuit_n)$, and in particular they rely on the existence of a $1/N$ expansion.

Integrable systems

Given any $d \times d$ linear system with rational coefficients

$$\nu\, \partial_x \Psi(x) = \mathbf{L}(x)\Psi(x) \tag{0-3}$$

one may define the transition matrix $\mathbf{K}$ and the correlators W_n:

$$\begin{aligned}
\mathbf{K}(x_1, x_2) &= \frac{\Psi^{-1}(x_1)\Psi(x_2)}{x_1 - x_2} \sqrt{\mathrm{d}x_1\, \mathrm{d}x_2} \\
W_1(x; a) &= \lim_{x' \to x} \Big(\mathbf{K}_{aa}(x, x') - \frac{1}{x - x'}\Big) \\
W_2(x_1, x_2; a_1, a_2) &= -\mathbf{K}_{a_1,a_2}(x_1, x_2)\mathbf{K}_{a_2,a_1}(x_2, x_1) - \frac{\delta_{a_1,a_2}}{(x_1 - x_2)^2} \\
W_n(x_1, \ldots, x_n; a_1, \ldots, a_n) &= (-1)^{n+1} \sum_{\substack{\sigma \text{ cycle of} \\ \text{length } n}} \mathbf{K}_{a_i, a_{\sigma(i)}}(x_i, x_{\sigma(i)})
\end{aligned}$$

Bergère and Eynard observed that [15]

$$\begin{aligned}
\sum_a W_1(x; a) &= -(\mathrm{d}x)\, \nu^{-1}\, \mathrm{Tr}\, \mathbf{L}(x) \\
\sum_{a<b} W_2(x, x; a, b) + W_1(x; a)W_1(x; b) &= \frac{(\mathrm{d}x)^2\, \nu^{-2}}{2} \Big[\big(\mathrm{Tr}\, \mathbf{L}(x)\big)^2 - \mathrm{Tr}\, \mathbf{L}^2(x)\Big]
\end{aligned} \tag{0-4}$$

Whatever complicated the analytical properties of W_1 and W_2 may be, the right hand side is a rational function whose singularities are fixed from the beginning. These equations look like $(\star_1, \diamondsuit_1)$, where a is an index attached to independent vector solutions ψ_a of $\partial_x \psi_a(x) = \mathbf{L}(x)\psi_a(x)$ (there are d of them).

In general, a solution to (0-3) has essential singularities at the poles p of $\mathbf{L}$:

$$\Psi(x) \underset{x \to p}{\sim} (x - p)^{\mathbf{S}_p} \exp\Big(\nu^{-1} \sum_{a,b=1}^{d} \sum_{j \geq 1} \frac{t_{p,j,[a,b]}}{(x - p)^j}\, \mathbf{E}_{a,b}\Big)$$

We can study *isomonodromic deformations* $\Psi(x,\vec{t}_{p_0})$ of that solution, namely we choose a pole p_0 and an index $a \in \{1,\dots,d\}$ and consider the flows $(\partial_{t_{p_0,j,[a,a]}})_{j\geq 1}$ for a given pole p_0, without changing the monodromy matrices $\mathbf{S}_p$ and the other $t_{p,k,[a',b']}$. The result is that there exists a family $[\mathbf{M}_{(p_0,j,a)}(x,\vec{t}_{p_0})]_{j\geq 1}$ and $L(x,\vec{t}_{p_0})$ of $d\times d$ matrices, whose coefficients are rational functions of x having well controlled singularities, and such that:

$$\begin{cases} \nu\,\partial_x \Psi(x,\vec{t}_{p_0}) = L(x,\vec{t}_{p_0})\,\Psi(x,\vec{t}_{p_0}) \\ \nu\,\partial_{t_I}\Psi(x,\vec{t}_{p_0}) = \mathbf{M}_I(x,\vec{t}_{p_0})\,\Psi(x,\vec{t}_{p_0}) \end{cases}$$

The system above expresses the existence of a family of commuting flows, and is called a *(classical) integrable system*. Such systems are important because they have many links with geometry, and we have many techniques to say something about them [16]. Since the compatibility equations of the systems yields interesting nonlinear differential equations on the coefficients of $\mathbf{L}$ and $\mathbf{M}$, they play also an important role in the study of such nonlinear equations. The relevant fact immediately useful to us is that the solution $\Psi(x,\vec{t}_{p_0})$ can be in principle reconstructed from a function $\tau(\vec{t}_{p_0})$, the so called *tau function* of Jimbo, Miwa and Ueno [17]. If we build the formal operator $D^{(p_0)}_{(x,a)} = \sum_{j\geq 1}(x-p_0)^{j-1}\,\partial_{t_{p_0,j,[a,a]}}$, we claim [10] that the Taylor series around around p_0 of the n-point correlators W_n encode the n-th derivatives of the tau function:

$$W_n(x_1,\dots,x_n;a_1,\dots,a_n) = \mathrm{d}x_1,\dots\,\mathrm{d}x_n\,\nu^n\,D^{(p_0)}_{(x_1,a_1)}\cdots D^{(p_0)}_{(x_n,a_n)}\ln\tau(\vec{t}_{p_0}) - \frac{\delta_{n,2}\,\delta_{a_1,a_2}}{(x_1-x_2)^2}$$

Applying the operators $(n-1)$ operators $D^{(p_0)}_{(x_i,a_i)}$ to (0-4) yields relations similar to $\diamondsuit_n$. In this context, the spectral curve is the plane curve given by the semiclassical dispersion relation $\Sigma\,:\,\det(y-\lim_{\nu\to 0}\mathbf{L}(x,\vec{t}))=0$ of the differential system, endowed with a Bergman kernel related to $\lim_{\nu\to 0}W_2(x_1,x_2;a_1,a_2)$. $a_i\in\{1,\dots,d\}$ is also in relation with the different sheets of the covering $x\,:\,\Sigma\to\widehat{\mathbb{C}}$.

Combinatorics of surfaces

In a large variety of problems in enumerative geometry, we want to compute a generating series $W_n^{(g)}$ for connected orientable surfaces of genus g, and with n marked points (or n boundaries). For this, we rather build a generating series $W_n = \sum_{g\geq 0}\nu^{-(2-2g-n)}\,W_n^{(g)}$ where ν is a formal parameter. One might be able to establish relations of combinatorial nature between the $W_n^{(g)}$'s, which in some cases are equivalent to loop equations expanded order by order in powers of ν. The two kind of terms in $\diamondsuit_n$ would come from the geometric alternative of "losing one handle, adding a boundary", or "disconnect the surface". *Simple Hurwitz numbers* $H_{g,\mu}$ have been studied successfully with this method. It is not useless to consider this example, since it is the simplest case of the next, important application of loop equations. By definition, if μ is a partition, $H_{g,\mu}$ is the number of (topological classes of) coverings

$\pi : \Sigma_g \to \widehat{\mathbb{C}}$ of the Riemann sphere by a compact Riemann surface of genus g, such that all but one branchpoints are simple, and the special branchpoint has $n = \ell(\mu)$ preimages in Σ_g with respective multiplicities $\mu_1, \ldots, \mu_n$. The conjecture of Bouchard and Mariño [18], which we proved in [19], is that:

$$\sum_{\mu \,/\, \ell(\mu)=n} t^{|\mu|}\, \mu_1 \cdots \mu_n\, M_\mu(v_1, \ldots, v_n)\, H_{g,\mu} = \frac{\omega_n(g)[\mathcal{S}_{\mathrm{L},t}]}{\mathrm{d}X(v_1) \cdots \mathrm{d}X(v_n)} \tag{0-5}$$

where $\mathcal{S}_{\mathrm{L},t}$ is the Lambert curve:

$$X(v) = \ln v = -Y(v) + \ln[Y(v)/t], \qquad B(v_1, v_2) = \frac{\mathrm{d}Y(v_1)\,\mathrm{d}Y(v_2)}{(Y(v_1) - Y(v_2))^2}$$

Actually, loop equations are in that case equivalent [20] to the *cut-and-join* recursion relations obtained by Goulden, Jackson, Vakil [21] by looking at all possible ways of merging a simple branchpoint to the special branchpoint.

The BKMP conjecture

When $\mathfrak{X}$ is a symplectic manifold, the Gromov-Witten invariants $N_{g,\beta}(\mathfrak{X})$ can be defined as integration of the virtual fundamental class over the moduli space of stable maps $\phi ; \Sigma_g \to \mathfrak{X}$ with fixed degree $\beta \in H_2(\mathfrak{X}, \mathbb{Z})$. They are rational numbers, which count intuitively how many ways are there to embed a Riemann surface of genus g in $\mathfrak{X}$, up to automorphisms. These invariants are the most interesting when $\mathfrak{X}$ has complex dimension 3, and are the better understood when $\mathfrak{X}$ is a toric Calabi-Yau, thanks to the method of the *topological vertex* [22]. Yet, an important problem is to compute the generating series:

$$F_g(\mathfrak{X}; \vec{t}) = \sum_{\beta \in H_2(\mathfrak{X})} e^{-\beta \cdot t}\, N_{g,\beta}(\mathfrak{X})$$

which is non-perturbative in the Kähler radii t. There exists also open Gromov-Witten invariants, which count in a certain sense embeddings of a Riemann surface of genus g with n boundaries in $\mathfrak{X}$. They can be packed into a generating series $W_n^{(g)}(\mathfrak{X}; \vec{t}; z_1, \ldots, z_n)$, the variables z_i's being coupled to the configuration of the boundaries in $\mathfrak{X}$. Supported by many numerical evidence, Bouchard, Klemm, Mariño and Pasquetti [23] have formulated the conjecture that $W_n(\mathfrak{X}; \vec{t}; z_1, \ldots, z_n) = \sum_{g \geq 0} \nu^{-(2-2g-n)}\, W_n^{(g)}(\mathfrak{X}; \vec{t}; z_1, \ldots, z_n)$ satisfies loop equations, hence $W_n^{(g)}$'s are computed by the topological recursion. The spectral curve should be the singular locus of the mirror Calabi-Yau, which has an equation of the form $\mathrm{Polynomial}(e^X, e^Y) = 0$ depending also on $\vec{t}$. The appropriate choice of functions X and Y and of Bergman kernel is prescribed by the *mirror map*.

When $\mathfrak{X} = \mathbb{C}^3$ for a special configuration of boundaries[1], Gromov-Witten invariants can be expressed in terms of simple Hurwitz numbers [24], and this led Bouchard and

[1]The infinite framing limit

Mariño to conjecture (0-5). Today, the only cases where the BKMP conjecture has been proved are $\mathbb{C}^3$ for general configuration of boundaries [25], and $\mathfrak{X}_p = \mathcal{O}(p) \oplus \mathcal{O}(-p-2) \to \mathbb{P}_1$ [26]. Although progress has been made toward the full conjecture, we still do not understand at a fundamental level why the topological recursion would compute Gromov-Witten invariants.

Apart from asymptotic power series expansions

There exists a way to exhibit beautiful[2] solutions of the loop equations starting from the topological recursion. These solutions do not have an expansion in powers of ν, but they rather feature pseudo-periodic behavior when $\nu \to 0$ [27, 28]. It is conjectured that they give the correct answer to the large N asymptotic expansions of multi-cut matrix models, to the asymptotics of biorthogonal polynomials, to the non perturbative completion problem in string theory, ... but unfortunately, no general theorems are available yet.

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Archimedean Langlands correspondence and Quantum Field Theory

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1 Introduction

In the first part of this note we review an interpretation of the Archimedean Langlands correspondence via mirror symmetry in two-dimensional topological field theories [1], [2], [3]. In the second part we review a simplified version of the Archimedean Langlands correspondence that allows a similar interpretation but in terms of finite-dimensional symplectic geometry [4] (topological theory in zero dimension).

To present a motivation of further presentation we start with a discussion of the Riemann ζ-function. The Riemann ζ-function is defined as the analytic continuation of the series

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_{p\in\mathcal{P}}\frac{1}{1-p^{-s}},\qquad \mathrm{Re}(s)>1.$$

Here the product goes over the set $\mathcal{P}$ of prime numbers. The analytic continuation satisfies the functional equation

$$\zeta^*(s)=\zeta^*(1-s),$$

where

$$\zeta^*(s)=\zeta^*(s)=\zeta(s)\pi^{-s/2}\Gamma(\frac{s}{2})=\prod_{p\in\mathcal{P}\cup\infty}\zeta_p(s),$$

$$\zeta_\infty(s)=\pi^{-s/2}\Gamma(\frac{s}{2}),\qquad \zeta_p(s)=\frac{1}{1-p^{-s}}.$$

The form of the functional equation implies that $\zeta^*(s)$ is a more fundamental object then $\zeta(s)$. According to A.Weil the meaning of the additional factor $\zeta_\infty(s)$ can be understood as follows.

Recall that an exponential valuation (norm) $|\ |:K\to\mathbb{R}_+$ is defined by the properties:

- $|x\,y|=|x|\,|y|$,
- $|x|=0\leftrightarrow x=0$,
- $|x+y|\leq|x|+|y|,\quad Archimedean,$

  $|x+y|\leq max(|x|,|y|)\quad non-Archimedean.$

Essentially different norms on $\mathbb{Z}$ are given by:

- non-Archimedean norm: for each prime p

$$|a|_p=p^{-n}\quad \text{iff}\quad a=p^na_0,\qquad (p,a_0)=1.$$

- Archimedean norm:

$$|a|_\infty = |a|.$$

The corresponding completions lead to inclusions $\mathbb{Q} \subset \mathbb{Q}_p$, $\mathbb{Q} \subset \mathbb{R}$.

One can reformulate the product formula for $\zeta^*(s)$ as

$$\zeta^*(s) = \prod_{p \in \overline{\mathrm{Spec}(\mathbb{Z})}} \zeta_p(s), \tag{1.1}$$

where local zeta-functions are associated with local completions of $\mathbb{Q}$

$$\zeta_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(\frac{s}{2}), \qquad \zeta_{\mathbb{Q}_p}(s) = \frac{1}{1-p^{-s}}, \qquad p \neq \infty.$$

The nontrivial form of the additional factor $\zeta_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(\frac{s}{2})$ *manifests some hidden structure of the field* $\mathbb{R}$ *of real numbers.*

2 Archimedean Langlands correspondence via 2d topological field theory

To better understand the form of the local Archimedean contributions in various product formulas, analogous to (1.1), we consider Whittaker functions associated with local fields $\mathbb{Q}_p$, $\mathbb{R}$. The Whittaker functions over $\mathbb{Q}_p$ allow two completely different realizations manifesting local non-Archimedean Langlands duality.

The Archimedean version of the Langlands duality for the Whittaker functions over $\mathbb{R}$ is more involved. Its proper formulation provides a hint on what is the right point of view on the field of real numbers leading to involved expressions like $\zeta_{\mathbb{R}}(s)$.

The Whittaker function is defined as a matrix element of an infinite-dimensional representation $\pi_\lambda : G \to End(\mathcal{V})$ of a reductive group G

$$\Psi_\lambda(g) =< \psi_L|\, \pi_\lambda(g)\, |\psi_R\rangle, \qquad g \in G,$$

such that

$$\Psi_\lambda(n_- g n_+) = \chi_L(n_-)\chi_R(n_+)\, \Psi_\lambda(g), \qquad n_\pm \in N_\pm,$$

where $N_\pm$ - opposite maximal unipotent subgroups and χ_L/χ_R - non-trivial characters of N_- and N_+.

It has the following basic properties.

- The Whittaker function $\Psi_\lambda(g)$ reduces to a function $\Psi_\lambda(a)$ on a factor $A = N_-\backslash G/N_+$ (in split case A is a diagonal subgroup).

- The Whittaker functions have natural integral representations arising from explicit realizations of the pairing in representation $\pi_\lambda : G \to End(\mathcal{V})$.

- Irreducibility of the representation $\pi_\lambda : G \to End(\mathcal{V})$ leads to a system of difference/differential equations on Ψ_λ

$$\mathcal{H}_r \Psi_\lambda(a) = c_r(\lambda)\Psi_\lambda(a), \qquad a \in A.$$

Non-Archimedean Langlands duality pattern for the Whittaker functions can be succinctly described via the Shintani-Casselman-Shalika formula. The Whittaker function for $G(\mathbb{Q}_p)$ can be expressed as a character of a finite-dimensional representation of the Langlands dual group $G^{\vee}$. For $G(\mathbb{Q}_p) = GL(\ell+1,\mathbb{Q}_p)$ let $\mathcal{V}_{\gamma_1,\ldots,\gamma_{\ell+1}}$ be an irreducible representation induced from a generic character $\chi^B_{(p^{\gamma_1},\ldots,p^{\gamma_{\ell+1}})}(g) = \prod_{j=1}^{\ell+1}|g_{jj}|^{\gamma_j}$ of the Borel subgroup $B \subset GL(\ell+1,\mathbb{Q}_p)$. Let $V_{n_1,n_2,\ldots,n_{\ell+1}}$ be a finite-dimensional irreducible representation of $GL(\ell+1,\mathbb{C})$ corresponding to a partition $(n_1 \geq n_2 \geq \ldots \geq n_{\ell+1})$, then

$$\Psi_{(\gamma_1,\ldots,\gamma_{\ell+1})}(\mathrm{diag}(p^{n_1},\ldots,p^{n_{\ell+1}})) = \mathrm{Tr}_{\,V_{n_1,\ldots,n_{\ell+1}}}\,\mathrm{diag}(p^{\gamma_1},\ldots,p^{\gamma_{\ell+1}}). \tag{2.1}$$

Now let us look for an analog of this relation for the Archimedean Whittaker functions. The $\mathfrak{gl}_{\ell+1}$-Whittaker function over $\mathbb{R}$ is a common eigenfunction of a family of commuting differential operators which can be identified with quantum Hamiltonians of $\mathfrak{gl}_{\ell+1}$-Toda chain. The simplest non-trivial quantum Hamiltonian acting on Whittaker function is given by

$$\mathcal{H} = -\frac{\hbar^2}{2}\sum_{i=1}^{\ell+1}\frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{\ell} e^{x_{i+1}-x_i}.$$

Explicit solution of this system of equations is given by a matrix element written in the integral form using a realization of principle series representation as a space of (twisted) functions on a G-homogeneous space.

Due to Givental we know the following integral representation of $\mathfrak{gl}_{\ell+1}$-Whittaker function:

$$\Psi_{\lambda_1,\ldots,\lambda_{\ell+1}}(x_1,\ldots,x_{\ell+1};\hbar) = \tag{2.2}$$

$$= \int\limits_{\mathcal{C}} \prod_{k=1}^{\ell}\prod_{i=1}^{k} dT_{k,i}\,\exp\left(\frac{\imath}{\hbar}\sum_{k=1}^{\ell+1}\lambda_k\left(\sum_{i=1}^{k}T_{k,i}-\sum_{i=1}^{k-1}T_{k-1,i}\right)\right)$$
$$\times\exp\left\{\sum_{k=1}^{\ell}-\frac{1}{\hbar}\left(\sum_{i=1}^{k}e^{T_{ki}-T_{k+1,i}}+\sum_{i=1}^{k}e^{T_{k+1,i+1}-T_{k,i}}\right)\right\},$$

where $(x_1,\ldots,x_{\ell+1}) = (T_{\ell+1,1},\ldots,T_{\ell+1,\ell+1})$. This integral arises naturally as a matrix element in principle series representation of $GL_{\ell+1}(\mathbb{R})$.

To formulate the Archimedean Langlands correspondence for the Whittaker functions (i.e. Archimedean analog of Shintani - Casselman - Shalika formula) one shall define another representation for Archimedean Whittaker function analogous to trace representation in the right hand side of (2.1).

In [1]-[3] it was shown that along with finite-dimensional integral representations the Whittaker functions have also *infinite-dimensional* integral representations. Recall that given a finite-dimensional symplectic manifold (M,ω) and the Hamiltonian action of a compact Lie group G one can define a G-equivariant symplectic volume of M as:

$$Z_{(M,\omega)}(\lambda) = \int_M e^{\omega_G}.$$

Here ω_G is G-equivariant extension of the symplectic form ω depending on an element $\lambda \in \mathfrak{g}^*$ of the dual Lie algebra. This construction can be also extended to a class of infinite-dimensional manifolds M.

The $\mathfrak{gl}_{\ell+1}$-Whittaker function allows the following infinite-dimensional integral representation

$$\Psi_\lambda(x) = \int_{\mathcal{M}(D,GL_{\ell+1}/B)} e^{\omega_G(x,\lambda)}, \tag{2.3}$$

where $\mathcal{M}(D,GL_{\ell+1}(\mathbb{C})/B)$ is an infinite-dimensional spaces of holomorphic maps of a two-dimensional disk $D=\{z\in\mathbb{C}|\,|z|\leq 1\}$ into symplectic $U_{\ell+1}$-spaces $GL_{\ell+1}(\mathbb{C})/B$. There is a natural Hamiltonian action of $G=S^1\times U_{\ell+1}$ on $\mathcal{M}(D,GL_{\ell+1}(\mathbb{C})/B)$, where S^1 acts by rotations of D and the action of $U_{\ell+1}$ is induced from the action on $GL_{\ell+1}(\mathbb{C})/B$. Thus the equivalence of *finite-dimensional* and *infinite-dimensional* integral representations of the Whittaker functions is an Archimedean counterpart of local Langlands correspondence over $\mathbb{Q}_p$ identifying $\mathbb{Q}_p$-Whittaker functions given by matrix element with characters of irreducible finite-dimensional representations of dual Lie groups. The equivariant symplectic volume of the space $\mathcal{M}(D,GL_{\ell+1}(\mathbb{C})/B)$ of holomorphic maps can be interpreted as a correlation function in a two-dimensional equivariant topological sigma model on a disk D with the target space $GL_{\ell+1}(\mathbb{C})/B$. In these terms the local Archimedean Langlands correspondence is an instance of *mirror symmetry.*

We illustrate this interpretation for the most simple case of local Archimedean L-factors associated with principle series representations of $GL_{\ell+1}(\mathbb{R})$. We construct two different integral representations of local Archimedean L-factors as correlation functions in Topological Field Theories (TFT) of types A and B. In terms of type A TFT local Archimedean L-factors are given by equivariant symplectic volumes of spaces of holomorphic maps of D into $V=\mathbb{C}^{\ell+1}$. In Type B TFT local Archimedean L-factors are given by periods of holomorphic forms over middle-dimensional cycles. Thus we demonstrate that the mirror symmetry relating underlying topological field theories of type A and B at the same time relates infinite-dimensional and finite-dimensional integral representations of Archimedean L-factors. Let us stress that this is an Archimedean analog of "Period=Trace" relation (2.1) for non-Archimedean case (in Archimedean case the trace is replaced by its classical limit - equivariant volume of the underlying symplectic manifold).

Let $D=\{z|\,|z|\leq 1\}$ be a disk with a flat metric

$$h=\frac{1}{2}(dzd\bar{z}+d\bar{z}\,dz)=(dr)^2+r^2(d\sigma)^2, \qquad z=re^{\imath\sigma}.$$

Lie group S^1 acts by rotations on D. We supply $\mathbb{C}^{\ell+1}$ with the following Kähler form and the Kähler metric

$$\omega=\frac{\imath}{2}\sum_{j=1}^{\ell+1}\,d\varphi^j\wedge d\bar{\varphi}^j, \qquad g=\frac{1}{2}\sum_{j=1}^{\ell+1}\,(d\varphi^j\otimes d\bar{\varphi}^j+d\bar{\varphi}^j\otimes d\varphi^j).$$

Lie group $U_{\ell+1}$ acts on $\mathbb{C}^{\ell+1}$ via standard representation.

Type A topological sigma model.

Let K and $\bar{K}$ be canonical and anti-canonical bundles on world-sheet D and $T_{\mathbb{C}}X = T^{1,0} \oplus T^{0,1}$ be a decomposition of the complexified tangent bundle of the target space $X = \mathbb{C}^{\ell+1}$.

The quantum field content of the model is as follows.

Commuting fields: φ, $\bar{\varphi}$- describe maps $\Phi : D \to X$, F, $\bar{F}$ - sections of $K \otimes \Phi^*(T^{0,1})$, $\bar{K} \otimes \Phi^*(T^{1,0})$.

Anticommuting fields: χ, $\bar{\chi}$ - sections of $\Phi^*(\Pi T^{1,0})$, $\Phi^*(\Pi T^{0,1})$, ψ, $\bar{\psi}$ - sections of $K \otimes \Phi^*(\Pi T^{0,1})$, $\bar{K} \otimes \Phi^*(\Pi T^{1,0})$.

Metrics g and h induce the Hermitian parings $\langle\, , \,\rangle$

$$\langle \chi, \chi \rangle = \sum_{j=1}^{\ell+1} g_{i\bar{j}}\, \bar{\chi}^{\bar{j}}\, \chi^i, \qquad \langle F, F \rangle = \sum_{j=1}^{\ell+1} h^{z\bar{z}} g_{i\bar{j}} \bar{F}_z^{\bar{j}}\, F_{\bar{z}}^i.$$

The $S^1 \times U_{\ell+1}$-equivariant BRST transformation is defined as follows:

$$\delta_G \varphi = \chi, \qquad \delta_G \chi = -(\imath \Lambda \varphi + \hbar\, \mathcal{L}_{v_0} \varphi),$$

$$\delta_G \psi = F, \qquad \delta_G F = -(\imath \Lambda \psi + \hbar\, \mathcal{L}_{v_0} \psi),$$

where Λ is an element of $\mathrm{Lie}(U_{\ell+1})$, $v_0 = \frac{\partial}{\partial \sigma}$ is a generator of $\mathrm{Lie}(S^1)$ and $\mathcal{L}_{v_0} = d\, i_{v_0} + i_{v_0}\, d$ is the Lie derivative. Equivariant BRST operator satisfies

$$\delta_G^2 = -(\text{inf. symmetry transformation}).$$

Consider a linear sigma model with the action functional given by

$$S_D = \int_D d^2 z\, \delta_G(\imath \langle \psi, \overline{\partial} \varphi \rangle + \imath \langle \bar{\psi}, \partial \bar{\varphi} \rangle) =$$

$$\imath \int_D d^2 z\, \Big(\langle F, \overline{\partial} \varphi \rangle + \langle \bar{F}, \partial \bar{\varphi} \rangle + \langle \bar{\psi}, \partial \bar{\chi} \rangle + \langle \psi, \overline{\partial} \chi \rangle \Big).$$

The δ_G-invariant observable is defined by :

$$\mathcal{O} = \frac{\imath}{\pi} \int_0^{2\pi} d\sigma\, (-\langle \chi(e^{\imath\sigma}), \chi(e^{\imath\sigma}) \rangle + \langle \varphi(e^{\imath\sigma}), (\imath \Lambda + \hbar \mathcal{L}_{v_0}) \varphi(e^{\imath\sigma}) \rangle).$$

Theorem A [1] *In $S^1 \times U_{\ell+1}$-equivariant Type A topological linear sigma model on D with the target space $V = \mathbb{C}^{\ell+1}$ one has the following expression for a correlation function of* $\exp(\mathcal{O})$*:*

$$\left\langle e^{\mathcal{O}} \right\rangle = \hbar^{-\frac{\ell+1}{2}}\, \det{}_V \, \left(\frac{\pi}{\hbar} \right)^{-\Lambda/\hbar} \Gamma(\Lambda/\hbar).$$

By taking $\hbar = 1$ and changing the variables $\Lambda \to (s \cdot \mathrm{id} - \Lambda)/2$ the correlation function turns into local Archimedean L-factor. The left hand side is an integral over a symplectic space of holomorphic maps $D \to \mathbb{C}^{\ell+1}$ and is given by inverse ζ-function regularized infinite-dimensional determinant.

Type B topological Landau-Ginzburg theory

Type B linear topological sigma model is associated with a pair $(\mathbb{C}^{\ell+1}, W)$, $W \in H^0(\mathbb{C}^{\ell+1}, \mathcal{O})$. Let us specify the standard field content of the model.

Commuting fields: ϕ, $\bar{\phi}$ - describe maps $\Phi : D \to \mathbb{C}^{\ell+1}$, $\bar{G}$, G - sections of $\Phi^*(T^{0,1})$, $K \otimes \bar{K} \otimes \Phi^*(T^{1,0})$.

Anticommuting fields: η, θ - sections of $\Phi^*(\Pi T^{0,1})$, ρ - sections of $(K \oplus \bar{K}) \otimes \Phi^*(\Pi T^{1,0})$.

Real structure. Topological linear sigma model allows a non-standard real structure

$$(\phi^i)^\dagger = \phi^i, \qquad (\bar{\phi}^i)^\dagger = -\bar{\phi}^i, \qquad (\theta_i)^\dagger = -\theta_i,$$
$$(\bar{\eta}^i)^\dagger = -\bar{\eta}^i, \qquad (\rho^i)^\dagger = \rho^i, \qquad (G^i)^\dagger = G^i, \qquad (\bar{G}^i)^\dagger = -\bar{G}^i.$$

This real structure is imposed by the condition on Type B topological sigma model to be a mirror dual to the Type A topological sigma model discussed previously.

The S^1-equivariant BRST transformation δ_{S^1} is defined as follows:

$$\delta_{S^1}\phi^i_- = \eta^i, \qquad \delta_{S^1}\eta^i = \hbar\iota_{v_0} d\phi^i_-,$$
$$\delta_{S^1}\theta^i = G^i_-, \qquad \delta_{S^1}G^i_- = \hbar\iota_{v_0} d\theta^i,$$
$$\delta_{S^1}\rho^i = -d\phi^i_+ - \hbar\iota_{v_0}G^i_+, \qquad \delta_{S^1}\phi^i_+ = \hbar\iota_{v_0}\rho^i, \qquad \delta_{S^1}G^i_+ = d\,\rho^i.$$

Define the δ_{S^1}-invariant observable inserted at the center of the disk D:

$$\mathcal{O} = \prod_{i=1}^{\ell+1} \delta(\phi^i_-(0))\,\eta^i(0).$$

The action functional for the nonstandard real structure is given by

$$S = -\imath \sum_{j=1}^{\ell+1} \int_D \Big((d\phi^j_+ + \hbar\iota_{v_0}G^j_+) \wedge * d\phi^j_- + \rho^j \wedge * d\eta^j - \theta_j d\rho^j$$
$$+ G^j_+ G^j_- \Big) + \sum_{i,j=1}^{\ell+1} \int_D d^2z\sqrt{h} \left(-\frac{\partial^2 W_-(\phi_-)}{\partial\phi^i_-\partial\phi^j_-}\eta^i\theta^j - \imath\frac{\partial W_-(\phi_-)}{\partial\phi^i_-}\,G^i_- \right)$$
$$+ \sum_{i,j=1}^{\ell+1} \int_D \left(-\frac{1}{2}\frac{\partial^2 W_+(\phi_+)}{\partial\phi^i_+\partial\phi^j_+}\rho^i \wedge \rho^j + \frac{\partial W_+(\phi_+)}{\partial\phi^i_+}\,G^i_+ \right)$$
$$-\frac{1}{\hbar}\int_{S^1=\partial D} d\sigma W_+(\phi_+),$$

where W_+ and W_- are arbitrary independent regular functions on $\mathbb{R}^{\ell+1}$.

Theorem B [2] *The correlation function of* $\exp\mathcal{O}$ *in the type B topological* S^1*-equivariant Landau-Ginzburg sigma model on D with*

$$W_+(\phi_+) = \sum_{j=1}^{\ell+1}(\lambda_j\phi^j_+ - e^{\phi^j_+}), \qquad W_-(\phi_-) = 0,$$

is given by

$$\langle \mathcal{O} \rangle = \int_{\mathbb{R}^{\ell+1}} \prod_{j=1}^{\ell+1} dt^j \quad e^{\frac{1}{\hbar}\sum_{j=1}^{\ell+1}(\lambda_j t^j - e^{t^j})} = \prod_{j=1}^{\ell+1} \hbar^{\frac{\lambda_j}{\hbar}}\, \Gamma\left(\frac{\lambda_j}{\hbar}\right).$$

This coincides with the correlation function calculated in Type A TFT. The underlying reason is the mirror duality between considered Type A and Type B TFT. *In this terms the local Archimedean Langlands correspondence is an instance of mirror symmetry.*

Similar considerations applied to type A topological field theory with the target space being flag space and its type B Landau-Ginzburg dual lead to Archimedean version of Shintani-Casselman-Shalika formula (that is to the identity (2.3)=(2.2)) and thus to the Archimedean analog of the local Langlands duality on the level of the Whittaker functions. We can draw the following conclusions from this result.

• Archimedean geometry arises as a symplectic geometry of infinite-dimensional spaces of holomorphic maps of two-dimensional disks into a target space.

• Mirror duality between holomorphic periods and infinite-dimensional symplectic volumes/traces is a guiding principle to construct Archimedean analogs of all standard notions of algebraic geometry.

• S^1-equivariant topological sigma model is a way to describe topological sigma model coupled with topological gravity. Thus *topological string theory* is a proper setting for geometry over real numbers.

3 Elementary Archimedean Langlands correspondence

Formulation in terms of two-dimensional topological field theories leads to a natural question - what do we get by replacing the symplectic volume of $\mathcal{M}(D, GL_{\ell+1}(\mathbb{C})/B)$ by the symplectic volume of $GL_{\ell+1}/B$. Effectively this is achieved by taking the limit $\hbar \to \infty$ where $\hbar$ is the equivariant parameter corresponding to S^1-rotations of the disk D. We call the resulting expression elementary Whittaker function. The same limit taken in the finite-dimensional integral representation (2.2) of Whittaker function leads to interpretation of elementary $\mathfrak{gl}_{\ell+1}$-Whittaker function as a matrix element of a representation of the monoid $GL_{\ell+1}(\mathcal{R})$ where $\mathcal{R}$ is a tropical semifield. The elementary version of the Archimedean Langlands correspondence relates finite-dimensional symplectic geometry of flag spaces G/B and representation theory of tropical monoids associated with dual reductive groups $G^{\vee}$. Below we consider the elementary version of Archimedean Langlands correspondence following [4].

We define the elementary $\mathfrak{gl}_{\ell+1}$-Whittaker function as $U_{\ell+1}$-equivariant volume of $GL_{\ell+1}/B$:

$$\Psi^{(0)}_{\lambda_1,\cdots,\lambda_{\ell+1}}(x_1,\cdots,x_{\ell+1}) = \int_{GL_{\ell+1}/B} e^{\omega_{U_{\ell+1}}(x,\lambda)}. \tag{3.1}$$

The elementary Whittaker function is a limit of the classical Whittaker function

$$\Psi^{(0)}_{\lambda}(x) = \lim_{\hbar\to\infty} (\hbar)^{-\ell(\ell+1)/2}\Psi_{\lambda}(\hbar x, \hbar). \tag{3.2}$$

The integral (3.1) can be explicitly calculated in many ways e.g via explicit calculation or via the Duistermaat-Heckman localization formula. One of the calculations is based on using the Gelfand-Zetlin type parametrization of an open part of $GL_{\ell+1}/B$ using the Darboux coordinates $\{T_{ij}, \theta_{ij}\}$, such that the symplectic form ω is given by

$$\omega = \sum_{i \geq j} \delta T_{ij} \wedge \delta \theta_{ij}.$$

Here θ_{ij} are periodic coordinates $\theta_{ij} \sim \theta_{ij} + 1$ and $T_{ij} \in \mathbb{R}$, $1 \leq j \leq i < \ell + 1$ satisfy the Gelfand-Zetlin conditions

$$T_{i+1,j} \geq T_{i,j} \geq T_{i+1,j+1}, \qquad 1 \leq j \leq i \leq \ell + 1.$$

This defines the convex Gelfand-Zetlin polytop $\mathcal{D}_{\ell+1}$ in $\mathbb{R}^{\ell(\ell-1)/2}$ which can be interpreted as an image of $GL_{\ell+1}/B$ under moment map with respect to $U(1)^{\ell(\ell-1)/2}$.

The elementary Whittaker function given by $U_{\ell+1}$-equivariant volume of $GL_{\ell+1}/B$ has the following integral representation:

$$\Psi_{\lambda}^{(0)}(x) = \int\limits_{\mathcal{D}_{\ell+1}} \exp\{\imath \sum_{k=1}^{\ell+1} \lambda_k \left(\sum_{i=1}^{k} T_{k,i} - \sum_{i=1}^{k-1} T_{k-1,i} \right)\} \prod_{k=1}^{\ell} \prod_{i=1}^{k} dT_{k,i},$$

$$T_{\ell+1,1} \geq \ldots \geq T_{\ell+1,\ell+1},$$

where $x_i = T_{\ell+1,i}, i = 1, \ldots, \ell+1$. The elementary analog of Toda chain is given by a quantum billiards i.e. the elementary $\mathfrak{gl}_{\ell+1}$-Whittaker function is a common eigenfunction of the elementary Toda chain Hamiltonians

$$P_i(\partial_x)\, \Psi_{\lambda}(x) = P_i(\lambda)\, \Psi_{\lambda}(x), \qquad P_i(y) \in \mathbb{C}[y_1, \ldots, y_{\ell+1}]^{\mathfrak{S}_{\ell+1}},$$

where $x \in \overline{\mathcal{D}}_{\ell+1} = \{x = (x_1, \ldots, x_{\ell+1}) \in \mathbb{R}^{\ell+1} | x_i \geq x_{i+1}\}$ is a compactification of the fundamental domain of the action of $\mathfrak{S}_{\ell+1}$ in $\mathbb{R}^{\ell+1}$ and the Dirichlet boundary conditions are imposed

$$\Psi_{\lambda}(x)|_{x_j = x_{j+1}} = 0.$$

The existence of the nontrivial limit (3.2) on the level of the integral representation (2.2) is related with the deep *positivity property* of the integral representation (2.2). The integral representation of $\mathfrak{gl}_{\ell+1}$-Whittaker function over $\mathbb{R}$ is obtained via explicit realization of the pairing on infinite - dimensional principal series representation [5] . The integral is over an open part of the flag space $GL_{\ell+1}/B$ corresponding to positive elements of the maximal unipotent subgroup N_+ of $GL_{\ell+1}$. Positive elements of N_+ are the elements realized in the standard matrix representation by positive matrices i.e. the matrices with all non-trivial minors being positive. The product of two positive elements of $GL_{\ell+1}(\mathbb{R})$ is again a positive element and thus the space of positive elements $GL_{\ell+1}^{>0}(\mathbb{R})$ is a monoid (i.e. one can multiply elements but not divide in general). Note that according to Lusztig this property allows a generalization to the case of an arbitrary reductive group.

The $\mathfrak{gl}_{\ell+1}$-Whittaker function can be lifted to a function on the monoid $GL_{\ell+1}^{>0}$ such that the following functional equation holds:

$$\Psi_{\lambda}(fgn) = \psi^{+}(n)\psi^{-}(f)\Psi_{\lambda}(g),$$

where

$$g \in GL_{\ell+1}^{>}, \quad n \in N_{+}^{>}, \quad f \in N_{-}^{>},$$

and $\psi^{\pm}$ are characters of $N_{\pm}^{>}$. This is a key property that allows to take the limit $\hbar \to \infty$ retaining the property of the Whittaker function to be a matrix element. The resulting Whittaker function is a matrix element of a monoid over *tropical semifield*.

Tropical semifield $\mathcal{R}$ is a set $\mathbb{R}$ with the following operations

$$\alpha \dot{\times} \beta = \alpha + \beta, \qquad \alpha \dot{+} \beta = \min(\alpha, \beta).$$

The tropical semifield $\mathcal{R}$ can be understood as a degeneration of the standard semifield structure on the positive subset $\mathbb{R}_+ \subset \mathbb{R}$ of real numbers written in the following form

$$a \times_\hbar b = a \times b, \qquad a +_\hbar b = (a^\hbar + b^\hbar)^{1/\hbar}.$$

This semifield is isomorphic to $\mathbb{R}_+$ with the standard addition and multiplication via the map $a \to a^{\frac{1}{\hbar}}$.

Consider $a = e^{-\alpha}$, $b = e^{-\beta}$, $\alpha, \beta \in \mathbb{R}$ and take the limit $\hbar \to +\infty$. In the result we obtain tropical structure

$$\alpha \dot{\times} \beta := -\lim_{\hbar \to +\infty} \log \left(e^{-\alpha} \times_\hbar e^{-\beta} \right) = \alpha + \beta,$$

$$\alpha \dot{+} \beta = -\lim_{\hbar \to +\infty} \log \left(e^{-\alpha} +_\hbar e^{-\beta} \right) = \min(\alpha, \beta).$$

The set of matrices $\mathrm{Mat}_{\ell+1}(\mathcal{R})$ has a monoid structure arising in the limit $\hbar \to +\infty$ from the monoid structure on the subset of positive elements $GL_{\ell+1}^{>}(\mathbb{R}_+^{(\hbar)})$.

Theorem [4]: *(i) The elementary $\mathfrak{gl}_{\ell+1}$-Whittaker function allows the following matrix element representation:*

$$\Psi_\lambda^{(0)}(x) = \langle \psi_L, \, \pi_\lambda^{(0)}(g(x)) \, \psi_R \rangle, \qquad g(x) = \mathrm{diag}(x_1, \cdots, x_{\ell+1}),$$

where ψ_L and ψ_R are left and right Whittaker vectors.

(ii) The function $\Psi_\lambda^{(0)}(x)$ can be naturally lifted to a function on the monoid $GL_{\ell+1}(\mathcal{R})$ satisfying the functional relations

$$\Psi_\lambda^{(0)}(fgn) = \psi_0^+(n)\psi_0^-(f)\Psi_\lambda^{(0)}(g), \qquad g \in GL_{\ell+1}(\mathcal{R}), \quad n \in N_+(\mathcal{R}), \quad f \in N_-(\mathcal{R}_+).$$

The elementary analog of the Archimedean Langlands correspondence can be formulated as follows:

$U_{\ell+1}$-equivariant volume of $GL_{\ell+1}/B$

$=$

Matrix element of the monoid $GL_{\ell+1}(\mathcal{R})$ where $\mathcal{R}$ is the tropical semifield.

4 $\mathbb{Q}_1$ as a common degeneration of $\mathbb{Q}_p$ and $\mathbb{R}$

The appearance of the tropical semifield in formulation of the elementary version of the Archimedean Langlands correspondence propose the following question.

Question: Does the tropical limit have a meaning from arithmetic point of view?

To try to answer this question let us recall that $GL_{\ell+1}(\mathbb{Q}_p)$-Whittaker functions are given by characters of finite-dimensional irreducible representations of $\mathfrak{gl}_{\ell+1}$

$$\Psi_{p^{-\lambda_1},\cdots,p^{-\lambda_{\ell+1}}}(n_1,\cdots,n_{\ell+1}) = \mathrm{Tr}\,_{V_{n_1,\cdots,n_{\ell+1}}} p^{-\lambda_1 E_{11}} \cdots p^{-\lambda_{\ell+1}E_{\ell+1,\ell+1}},$$

where the partition $n_1 \geq \ldots \geq n_{\ell+1}$ corresponds to a finite-dimensional irreducible representation.

One can write the character explicitly using the Gelfand-Zetlin bases in finite-dimensional irreducible representations of $\mathfrak{gl}_{\ell+1}$. Denote $\mathcal{P}^{\ell+1}$ a set of Gelfand-Zetlin patterns, that is a set of collections $\underline{q} = \{q_{i,j}\}$, $i = 1,\ldots,\ell+1$, $j = 1,\ldots,j$ of integers satisfying the conditions

$$q_{i+1,j} \geq q_{i,j} \geq q_{i+1,j+1}.$$

An irreducible finite-dimensional representation can be realized in a vector space with the basis $v_{\underline{p}}$ enumerated by the Gelfand-Zetlin patterns $\underline{q}$ with fixed $n_i = q_{\ell+1,i}$, $i = 1,\ldots,\ell+1$. This leads to the following expression for the $GL_{\ell+1}(\mathbb{Q}_p)$-Whittaker function of $GL_{\ell+1}$ corresponding to a partition $(n_1 \geq \ldots \geq n_{\ell+1})$

$$\Psi_{p^{-\lambda_1},\ldots,p^{-\lambda_{\ell+1}}}(n_1,\ldots,n_{\ell+1}) \;=\; \sum_{q_{k,i}\in\mathcal{P}^{\ell+1}} \prod_{k=1}^{\ell+1} p^{-\lambda_k\,(\sum_{i=1}^{k} q_{k,i} - \sum_{i=1}^{k-1} q_{k-1,i})}.$$

Let $n_i(p,x_j)$ be integer parts of $x_j/\log p$, $x_j \in \mathbb{R}$. In the formal limit $p \to 1$ the non-Archimedean Whittaker function reduces to the elementary Whittaker function

$$\Psi^{(0)}_{\lambda_1,\ldots,\lambda_{\ell+1}}(x_1,\ldots,x_{\ell+1}) \tag{4.1}$$

$$= \lim_{p\to 1}\,(\log p)^{\ell(\ell+1)/2}\;\Psi_{p^{-\lambda_1},\cdots,p^{-\lambda_{\ell+1}}}(n_1(p,x_1),\cdots,n_{\ell+1}(p,x_{\ell+1})).$$

This is a manifestation of the well-known fact that characters of irreducible representations turn in appropriate limit into equivariant volumes of coadjoint orbits. The sum over the set $\mathcal{P}_{\ell+1}$ of Gelfand-Zetlin patterns turns into the integral over $\mathcal{D}_{\ell+1}$ thus reproducing integral representation of the elementary Whittaker function.

Thus the elementary Whittaker function can be understood not only as a limit of the Whittaker function over $\mathbb{R}$ (see (3.2)) but also as a limit of the Whittaker function over $\mathbb{Q}_p$. Optimistically one can consider (4.1) as an indication on the possible interpretation of the elementary Whittaker functions/ Whittaker functions over tropical semifield as *the Whittaker functions over a yet to be defined field* $\mathbb{Q}_1$. Let us stress that the similar relation holds between elementary version of L-factors

$$L(s) = \prod_{j=1}^{\ell+1} \frac{1}{s-\lambda_j}$$

and non-Archimedean local L-factors. Thus elementary L-factors shall be considered as local L-factors corresponding to $\mathbb{Q}_1$. Note that these local L-factors were introduced by Kurokawa as L-factors over a field with one element $\mathbb{F}_1$. In our approach these local L-factors are interpreted in terms of tropical semifield.

To clarify the relation between tropical semifield $\mathcal{R}$, $\mathbb{Q}_1$ and $\mathbb{Q}_p$ recall that the valuation on a local non-Archimedean field K is a map $\nu :\; K \to \mathbb{R}$ such that

$$\nu(x) = 0 \leftrightarrow x = 0,$$

$$\nu(x \cdot y) = \nu(x) + \nu(y),$$

$$\nu(x + y) \geq \min(\nu(x), \nu(y)).$$

The non-Archimedean valuation is a morphism $\nu : K \to \mathcal{R}$ of K considered as a semifield (i.e taking into account only addition, multiplication and division operations) to the tropical semifield $\mathcal{R}$

$$\nu_p(p^n a) = n, \qquad (p, a) = 1.$$

Thus the image of $\nu_p : \mathbb{Q}_p \to \mathcal{R}$ is a semifield $(\mathbb{Z}, \dot{\times}, \dot{+}) \in \mathcal{R}$ and ν_p has a big kernel $\mathbb{Z}_p^*$ consisting of invertible p-adic integers.

The limit $\mathbb{Q}_1$ of the field $\mathbb{Q}_p$ when $p \to 1$ is a strange creature but at least it shall allow a *surjective* valuation map ν_1 onto the tropical semifield $\mathcal{R}$. What is the kernel of the evaluation map ν_1 is not quite clear yet.

Fortunately many constructions over $\mathbb{Q}_p$ can be reformulated in terms of the semifield $(\mathbb{Z}, \dot{\times}, \dot{+})$ considered as an image domain of the valuation map ν_p. These constructions then allow a limit $p \to 1$ formulated in terms of the tropical semifield $\mathcal{R}$.

An example of such construction is the integral representation of $\mathbb{Q}_p$-Whittaker functions and its $p \to 1$ limit. This explains the meaning of the tropical Whittaker function as the Whittaker function over $\mathbb{Q}_1$.

To conclude this Section we stress the following points.

• The construction of the elementary Langlands correspondence reveals a fundamental role of the geometry over tropical semifields as a mirror dual description of finite-dimensional symplectic geometry.

• In elementary setting many difficult problems related with Langlands correspondence (e.g. functoriality) have a chance to be solved in full generality.

• Positivity is an arithmetic property.

• The natural appearance of the tropical field as an image of $\mathbb{Q}_1$ under the norm map rises several fundamental questions about the mysterious field $\mathbb{Q}_1$. What is a proper description of $\mathbb{Q}_1$ (what is a kernel of the valuation map ν_1)? What is the residual field $\mathbb{F}_1$ of $\mathbb{Q}_1$?

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The Poincaré lemma and the period map for p-adic varieties

Alexander A. Beilinson

For an algebraic variety X over a field of characteristic 0 we have its algebraic de Rham cohomology $H^{\cdot}_{\mathrm{dR}}(X) := H^{\cdot}(X_{\mathrm{Zar}}, \Omega^{\cdot}_X)$. If the base field is $\mathbb{C}$, then one has the Betti cohomology $H^{\cdot}_{\mathrm{B}}(X) := H^{\cdot}(X_{\mathrm{cl}}, \mathbb{Q})$ and a canonical period isomorphism ("integration of algebraic differential forms over topological cycles") $\rho : H^{\cdot}_{\mathrm{dR}}(X) \xrightarrow{\sim} H^{\cdot}_{\mathrm{B}}(X) \otimes \mathbb{C}$ compatible with the $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-conjugation. To define ρ, consider the analytic de Rham cohomology $H^{\cdot}_{\mathrm{dR}}(X_{\mathrm{an}})$. One has evident maps $H^{\cdot}_{\mathrm{dR}}(X) \xrightarrow{\alpha} H^{\cdot}_{\mathrm{dR}}(X_{\mathrm{an}}) \xleftarrow{\beta} H^{\cdot}_{\mathrm{B}}(X) \otimes \mathbb{C}$. Then β is an isomorphism due to the Poincaré lemma, and $\rho := \beta^{-1}\alpha$ (the fact that ρ is an isomorphism was established by Grothendieck).

Suppose our base field is an algebraic closure of $\mathbb{Q}_p$. The role of $H^{\cdot}_{\mathrm{B}}(X)$ is played now by the p-adic étale cohomology $H^{\cdot}_{\text{ét}}(X, \mathbb{Q}_p)$. Following Tate and Grothendieck, one can ask for an analog of the period isomorphism in this setting. The precise conjecture was made by Fontaine in the beginning of 80s: he introduced a remarkable p-adic periods field B_{dR}, which is a twisted version of the Laurent power series field $\mathbb{C}_p((\pi))$, where $\mathbb{C}_p$ is Tate's field (the p-adic completion of $\bar{\mathbb{Q}}_p$) and π stands for the Tate twist $\mathbb{C}_p(1)$, and conjectured that there should be a natural p-adic period isomorphism $\rho_p : H^{\cdot}_{\mathrm{dR}}(X) \otimes_{\bar{Q}_p} \mathrm{B}_{\mathrm{dR}} \xrightarrow{\sim} H^{\cdot}_{\text{ét}}(X, \mathbb{Q}_p) \otimes \mathrm{B}_{\mathrm{dR}}$ compatible with the Galois symmetries. Moreover, as was envisioned later by Fontaine and Jannsen, the matrix coefficients of ρ should lie in the subring $\bar{K}\mathrm{B}_{\mathrm{st}}$ of B_{dR}, and ρ be compatible with the extra symmetries of log crystalline story.

The p-adic period map was defined in three very different ways in works of, respectively, Faltings, Niziol, and Tsuji. In the talk, based on recent papers "p-adic periods and derived de Rham cohomology" and "On the crystalline period map", I sketched another construction of ρ_p which is fairly direct and has the same flavor as the classical picture. Its key ingredient is the p-adic Poincaré lemma, which tells that natural complexes of presheaves on the category $\bar{\mathbb{Q}}_p$-varieties, defined by the derived de Rham or log crystalline cohomology of log schemes over $\bar{\mathbb{Z}}_p$, reduce to constants when viewed locally in the h-topology. The proofs are simple applications of de Jong's alterations technique.

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GEOMETRIC CYCLES AND DISCRETE GROUPS

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Talk at the Arbeitstagung 2011, Bonn

Let G be a connected semi-simple real Lie group with finite center, and let $K \subset$ be a maximal compact subgroup. The homogenous space $K\backslash G = X$ attached to the Riemannian symmetric pair (G, K) is a Riemannian symmetric space, diffeomorphic to some $\mathbb{R}^n$. Suppose that $\Gamma \subset G$ is a torsion free discrete subgroup (such that G/Γ has finite volume with respect to some non-zero G-invariant measure). Then Γ acts freely on X, and the quotient space X/Γ is a Riemannian locally symmetric space. Our object of concern is the cohomology space $H^*(X/\Gamma, \mathbb{C})$, viewed as singular cohomology or, by use of the deRham theorem, as the cohomology attached to the complex of Γ-invariant differential forms on X. We have to distinguish two cases: Γ is uniform (or cocompact) if G/Γ is compact, nonuniform otherwise. For example, the principal congruence subgroups $\Gamma(m) \subset SL_n(\mathbb{Z})$ of level m are torsion free subgroups of finite index for $m > 4$, and $\Gamma(m)$ are nonuniform discrete subgroups of $SL_n(\mathbb{R})$. It is a more difficult task to construct uniform discrete subgroups in a given G. However, by a number theoretical approach, a connected semi-simple Lie group always has discrete subgroups Γ so that G/Γ is compact. Arithmetically defined subgroups in semi-simple algebraic groups defined over some algebraic number field give rise to such uniform examples.

The cohomology groups attached to arithmetic groups Γ in reductive algebraic groups defined over an algebraic number field k can be interpreted in terms of the automorphic spectrum of the underlying arithmetic group. This context in place, it is the main objective of this talk to discuss a geometric approach to construct non-trivial cohomology classes (in particular, special cycles following Millson-Raghunathan resp. Rohlfs-Schwermer) for arithmetically defined groups and draw some consequences for the existence of certain automorphic representations in these cases. In particular, we discuss the case of uniform discrete subgroups of the real Lie group $SU^*(2n)$, the special linear group over the Hamilton quaternions.

1. Geometric construction of cohomology classes

We briefly review the general construction of geometric cycles in arithmetic quotients X/Γ as outlined in [8, Sections 6 and 9]. In the next section, in the specific case of interest for us, we use one of the techniques developed in [7] to show that certain geometric cycles exist and represent non-zero homology classes for the underlying manifold X/Γ. This relies on the approach via "excess intersections".

1.1. **Generalities.** Let G be a connected reductive algebraic group defined over an algebraic number field k. We choose an embedding $\rho : G \to GL_N$ and write $G_{\mathcal{O}_k} = G(k) \cap GL_N(\mathcal{O}_k)$ for the group of integral points with respect to ρ.

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For every archimedean place $v \in V_\infty$ corresponding to the embedding $\sigma_v : k \to \overline{k}$ there are given a local field $k_v = \mathbb{R}$ or $\mathbb{C}$ and a real Lie group $G_v = G^{\sigma_v}(k_v)$. The group $G_\infty = \prod_{v \in V_\infty} G_v$, viewed as the topological product of the groups G_v, $v \in V_\infty$, is isomorphic to the group of real points $G'(\mathbb{R})$ of the algebraic $\mathbb{Q}$-group $G' = \mathrm{Res}_{k/\mathbb{Q}} G$ obtained from G by restriction of scalars. In G_∞, we identify $G(k)$ resp. $G_{\mathcal{O}_k}$ with the set of elements $(g^{\sigma_v})_{v \in V_\infty}$ with $g \in G(k)$ resp. $g \in G_{\mathcal{O}_k}$. If Γ is an arithmetic subgroup of G then Γ is a discrete subgroup in G_∞.

Each of the groups G_v has finitely many connected components. The factor G_v has maximal compact subgroups, and any two of these are conjugate by an inner automorphism. Thus, if K_v is one of them, the homogeneous space $K_v \backslash G_v = X_v$ may be viewed as the space of maximal compact subgroups of G_v. Since X_v is diffeomorphic to $\mathbb{R}^{\mathrm{d}(G_v)}$, where $\mathrm{d}(G_v) = \dim G_v - \dim K_v$, the space X_v is contractible. Notice that, if G is semi-simple, the space X_v is the symmetric space associated to G_v. We let $X = \prod_{v \in V_\infty} X_v$ (or we write X_G emphasizing the underlying k-group G) resp. $\mathrm{d}(G) = \sum_{v \in V_\infty} \mathrm{d}(G_v)$.

A torsion-free arithmetic subgroup Γ of G acts properly discontinously and freely on X and the quotient X/Γ is a smooth manifold of dimension $\mathrm{d}(G)$. The space X/Γ has finite volume if and only if G has no non-trivial rational character, and it is compact if and only if, in addition, every rational unipotent element belongs to the radical of G.

1.2. **The construction of geometric cycles.** Let G denote a connected semi-simple algebraic group defined over an algebraic number field k, $\Gamma \subset G(k)$ an arithmetic subgroup. Let H be a reductive k - subgroup of G, let K_H be a maximal compact subgroup of the real Lie group H_∞, and let $X_H = K_H \backslash H_\infty$ be the space associated to H_∞. If $x_0 \in X$ is fixed under the natural action of $K_H \subset G_\infty$ on X, then the assignment $h \mapsto x_0\, h$ defines a closed embedding $X_H = K_H \backslash H_\infty \longrightarrow X$, that is, the orbit map identifies X_H with a totally geodesic submanifold of X. Thus, we also have a natural map $j_{H|\Gamma} : X_H/\Gamma_H \longrightarrow X/\Gamma$, where $\Gamma_H = \Gamma \cap H(k)$. It is known [8, Sect. 6] that the map $j_{H|\Gamma}$ is proper.

Now we are interested in situations in which for a given subgroup H and a torsion free arithmetic subgroup Γ of G, the corresponding map $j_{H|\Gamma}$ is an injective immersion. Thus, by being proper, $j_{H|\Gamma}$ is an embedding, and the image $j_H(X_H/\Gamma_H)$ of X_H/Γ_H is a submanifold in X/Γ. This submanifold is totally geodesic, to be called a *geometric cycle* in X/Γ. The following Theorem, stated in [8, Sect. 6, Thm. D] with an outline of its proof, is a combination of a result by Raghunathan [1, Sect. 2] and a result in [7].

Theorem. *Let G be a connected semi-simple algebraic k-group, let $H \subset G$ be a connected reductive k-subgroup, and let Γ be an arithmetic subgroup of $G(k)$. Then there exists a subgroup Γ' of finite index in Γ such that if Γ is replaced by Γ' the map*

$$j_{H|\Gamma'} : X_H/\Gamma'_H \longrightarrow X/\Gamma'$$

is a proper, injective, closed embedding, and so that each connected component of the image is an orientable, totally geodesic submanifold of X/Γ'.

For example, such geometric cycles naturally arise as fixed point components of an automorphism μ of finite order on X/Γ which is induced by a rational automorphism of G. It is known (see e.g. [8, 6.4]) that the connected components of the fixed point set $Fix(\langle\mu\rangle, X/\Gamma)$ are totally geodesic closed submanifolds in X/Γ of the form $F(\gamma) = X(\gamma)/\Gamma(\gamma)$ where γ ranges over a set of representatives for the classes in the non-abelian cohomology set $H^1(\langle\mu\rangle, \Gamma)$. Such a connected compenent is of the form $X(\gamma)/\Gamma(\gamma)$ where $X(\gamma)$ is the set of fixed points of the action of μ on X twisted by the cocycle γ. The component originates with the group $G(\gamma)$ of elements fixed by the γ-twisted μ-action on G. Occasionally one also writes $X_{G(\gamma)}$ for $X(\gamma)$. As first noted in [5] resp. [6] in specific cases, the map $j_{G(\gamma)|\Gamma}$ is injective in such a case.

In general, we are interested in cases where a geometric cycle Y is orientable and its fundamental class is not homologous to zero in X/Γ, in singular homology or homology with closed supports, as necessary. As stated, there exists a subgroup of finite index in Γ such that the corresponding cycles are orientable.

One way to go about the second question is to construct an orientable submanifold Y' of complementary dimension such that the intersection product (if defined) of its fundamental class with that of Y is non-zero. In the case of classical groups, this idea was exploited in the work of Millson-Raghunathan [4]. In doing so, if X/Γ is non-compact, we have to assume that at least one of the cycles Y, Y' is compact, while the other need not be. A simple method to prove that such a geometric cycle represents a nontrivial homology class is by showing that the cycle intersects a second geometric cycle, of complementary dimension, in a single point with multiplicity ± 1. However, geometric cycles of complementary dimension usually intersect in a quite complicated set, possibly of dimension greater than zero. The theory of "excess intersections" as developed in [7, Sects. 3 and 4], is helpful in such a situation. In particular, it provides a formula for the intersection number of a pair of two such geometric cycles Y and Y' which intersect perfectly. We will use this technique in the specific case we are interested in.

By definition, Y and Y' intersect perfectly if the connected components of the intersection are immersed submanifolds in X/Γ and for each of the components F of $Y \cap Y'$ the tangent bundle TF of F coincides with the intersection of the restriction of the tangent bundles of Y and Y' to F, that is, $TF = TY_{|F} \cap TY'_{|F}$. If the intersection is compact the intersection number of two such cycles can be expressed as the sum of the Euler numbers of the excess bundles corresponding to the connected components of the intersection [7, Prop. 3.3]. A detailed analysis of the intersection number might then enable us to show that the underlying geometric cycles are indeed non-bounding cycles. In order to find a non-zero intersection product, if at all possible, it is often necessary to replace the arithmetic group Γ by a suitable subgroup of finite index.

Following the recent work [9], we discuss the geometric construction of non-bounding cycles in the case of arithmetic subgroups in algebraic groups of type ${}^2A_{2n-1}$.

2. Arithmetic subgroups in algebraic groups of type ${}^2A_{2n-1}$

Starting off with a totally real number field F, a quaternion algebra Q'/F, a specific quadratic extension field L of F, the associated quaternion algebra $Q = Q' \otimes L$ which admits an involution τ of the second kind we attach to a suitable Hermitian form on Q the corresponding simple connected algebraic group $\mathbf{G}' = \mathbf{SU}(H, Q, \tau)$ defined over F. For an appropriate choice of our data the group $\mathbf{G}'(\mathbb{R})$ is the real Lie group $SU^*(2n)$. Arithmetic subgroups of $\mathbf{G}'$ give rise to discrete subgroups in

$SU^*(2n)$. Then one can define various (families of) rational F-automorphisms of finite order on $\mathbf{G}'$ and determine their corresponding groups of fixed points.

2.1. **The algebraic group.** Let F a totally real number field of degree $[F:\mathbb{Q}] =: r \geq 1$. Denote by $V_\infty = \{s : F \hookrightarrow \mathbb{R}\}$ the set of real places of F. Instead of $s(F)$ we write F_s. Let $d \in F$, $d > 0$ such that $s(d) < 0$, for all $s \in V_\infty - \{\mathrm{id}\}$. It is a consequence of the weak approximation theorem for number fields that such a number exists. Let $L = F(\sqrt{d})$ and let σ denote the unique non-trivial Galoisautomorphism of the extension L/F. Furthermore, let Q' be a quaternion algebra over F, endowed with the canonical conjugation τ_c. We suppose that Q' does not split over $\mathbb{R}$. The quaternion algebra $Q := Q' \otimes L$ admits the involution $\tau := \tau_c \otimes \sigma$ of the second kind, that is, the restriction of τ to the center $Z(Q) = L$ of Q acts as the Galoisautomorphism σ. All involutions of the second kind on a quaternion algebra are obtained in this way (cf. [2, 2.22]). We denote the involution $\tau_c \otimes 1$ (resp. $1 \otimes \sigma$) on Q simply by τ_c (resp. σ). Note that σ induces an F-algebra automorphism, also denoted by σ, of the matrix algebra $M_n(Q)$ of order two. By our choice of L we see that Q does not split over $\mathbb{R}$ and that all other conjugates $s(Q)$, $s \in V_\infty - \{\mathrm{id}\}$, split over $\mathbb{R}$.

Now, we choose $h_1, \ldots, h_n \in F$, such that $s(h_i) > 0$, for all $i = 1, \ldots, n$, $s \in V_\infty$ and define $H = \mathrm{diag}(h_1, \ldots, h_n)$, $H_k = \mathrm{diag}(h_1, \ldots, h_k)$ and $H^k = \mathrm{diag}(h_{k+1}, \ldots, h_n)$, $k = 1, \ldots, n-1$. Then we have the unitary group

$$\mathbf{U}(H, Q, \tau) \quad = \quad \{g \in \mathbf{GL}(n, Q) | \tau({}^t g) H g = H\}$$

and the special unitary group $\mathbf{SU}(H, Q, \tau) = \mathbf{U}(Q, H) \cap \mathbf{SL}(n, Q)$. The algebraic group $\mathbf{G}' = \mathbf{SU}(H, Q, \tau)$ is a simple, simply connected, connected F-group of type ${}^2A_{2n-1}$. Let $\mathbf{G} := Res_{F/\mathbb{Q}}\mathbf{G}'$ be the algebraic $\mathbb{Q}$-group obtained from $\mathbf{G}'$ by restriction of scalars. By our choice of Q' and L, we have $G' := \mathbf{G}'(\mathbb{R}) \cong SU^*(2n)$ and

$$G := \mathbf{G}(\mathbb{R}) \quad \cong \quad SU^*(2n) \times SU(2n) \times \cdots \times SU(2n).$$

Hence $\mathbf{G}'$ is F-anisotropic resp. $\mathbf{G}$ is $\mathbb{Q}$-anisotropic. Note, that in the case $F = \mathbb{Q}$, $G = SU^*(2n)$ and $\mathbf{G}' = \mathbf{G}$ is $\mathbb{Q}$-anisotropic too, since H is positive definite.

2.2. **Rational automorphisms of order two.** The F-rational involution $\boldsymbol{\theta} : \mathbf{G}' \to \mathbf{G}'$, $g \mapsto H^{-1}\tau_c({}^t g)^{-1}H$ induces the ordinary Cartan involution $\theta : g \mapsto {}^t\bar{g}^{-1}$ on $SL(n, \mathbb{H})$, because the real group $\mathbf{G}'(\boldsymbol{\theta})(\mathbb{R})$ given by the fixed points of $\boldsymbol{\theta}$ in $\mathbf{G}'$ is $Sp(n)$. Given an index $k = 1, \ldots, n-1$, one has the F-rational involution $\nu_k : \mathbf{G}' \to \mathbf{G}'$, $g \mapsto \mathbb{I}_{k,n-k} g \mathbb{I}_{k,n-k}$, where $\mathbb{I}_{k,n-k}$ denotes the diagonalmatrix with 1 at the first k entries and -1 at the last $n-k$ entries. The involutions ν_k and $\boldsymbol{\theta}$ commute with one another.

Let $\mathrm{Skew}(Q, \tau_c) = \{x \in Q | \ \tau_c(x) = -x\}$ be the set of skew-symmetric elements in Q. For any invertible element $u \in \mathrm{Skew}(Q, \tau_c)$, we get an involution $\mathrm{conj}_u \circ \tau_c$ on Q (cf. [2, 2.21]). The algebra Q has an L-basis given by elements $1, i, j, k$ such that $i^2 =: a_0, j^2 =: b_0 \in F$ and $ij = k = -ji$, for short, $Q = Q(a_0, b_0|L)$. Obviously, $i, j \in \mathrm{Skew}(Q, \tau_c)$. Define $J_1 := j\mathbb{I}_n$, $J_2 := i\mathbb{I}_n$,. Then $\mu_s : \mathbf{G}' \to \mathbf{G}'$, $g \mapsto J_s g J_s^{-1}$, $s = 1, 2$ is an F-rational automorphism. Moreover, μ_s is an involution and commutes with $\boldsymbol{\theta}$. Note that $\tau_{r_s} : Q \to Q$, $\tau_{r_1}(x) = j\tau_c(x)j^{-1}$, $\tau_{r_2}(x) = i\tau_c(x)i^{-1}$ is an antiinvolution of Q (called the *reversion*).

The following table lists all the subgroups of $\mathbf{G}'$ obtained as groups of fixed points as described above. It also indicates the notation for the corresponding geometric cycles discussed in the next section. Recall that the range of the index k is $k = 1, \ldots, n-1$.

$\mathbf{B}$	$\mathbf{B} \cong$	$\mathbf{B}(\mathbb{R}) \cong$	C
$\mathbf{G}'$	$\mathbf{SU}(H, Q, \tau)$	$SL(n, \mathbb{H})$	X/Γ
$\mathbf{G}'(\nu_k)$	$\mathbf{S}(\mathbf{U}(H_k, Q, \tau) \times \mathbf{U}(H^k, Q, \tau))$	$S(GL(k, \mathbb{H}) \times GL(n-k, \mathbb{H}))$	$C(\nu_k)$
$\mathbf{G}'(\nu_k \boldsymbol{\theta})$	$\mathbf{SU}(H\mathbb{I}_{k,n-k}, Q', \tau_c)$	$Sp(k, n-k)$	$C(\nu_k \boldsymbol{\theta})$
$\mathbf{G}'(\mu_1)$	$\mathbf{SU}(H, L[j], \tau)$	$SL(n, \mathbb{C})$	$C(\mu_1)$
$\mathbf{G}'(\mu_2)$	$\mathbf{SU}(H, L[i], \tau)$	$SL(n, \mathbb{C})$	$C(\mu_2)$
$\mathbf{G}'(\mu_1 \boldsymbol{\theta})$	$\mathbf{SU}(H, Q_1, \tau_{r_1})$	$SO(n, \mathbb{H})$	$C(\mu_1 \boldsymbol{\theta})$
$\mathbf{G}'(\mu_2 \boldsymbol{\theta})$	$\mathbf{SU}(H, Q_2, \tau_{r_2})$	$SO(n, \mathbb{H})$	$C(\mu_2 \boldsymbol{\theta})$

3. Non-bounding cycles

3.1. **Special cycles.** We consider an algebraic F-group $\mathbf{G}' = \mathbf{SU}(H, Q, \tau)$ and the corresponding algebraic $\mathbb{Q}$-group $\mathbf{G} := Res_{F/\mathbb{Q}} \mathbf{G}'$ obtained from $\mathbf{G}'$ by restriction of scalars. By our choice of Q' and L, we have $G' := \mathbf{G}'(\mathbb{R}) \cong SU^*(2n)$ and

$$G := \mathbf{G}(\mathbb{R}) \quad \cong \quad SU^*(2n) \times SU(2n) \times \cdots \times SU(2n).$$

A torsion free arithmetic subgroup of $\mathbf{G}'$ gives rise to a discrete subgroup of the real Lie group $SU^*(2n)$. By the very construction of $\mathbf{G}'$, using the compactness criterion of Borel and Harish-Chandra, these discrete subgroups are cocompact. Consequently, the arithmetic quotient X/Γ is compact.

On one hand, we have the family $\{\nu_k\}_{k=1,\ldots,n-1}$ of F - rational automorphisms of $\mathbf{G}'$ of order two. On the other, we have the two F - rational automorphisms $\mu_s : \mathbf{G}' \to \mathbf{G}'$. In both cases the automorphisms commute with the Cartan involution $\boldsymbol{\theta}$. Therefore, following the general construction of geometric cycles, a torsion free arithmetic subgroup of $\mathbf{G}'$ gives rise to the family $\{C(\nu_k)\}_{k=1,\ldots,n-1}$ of cycles. They come with the family $\{C(\nu_k \boldsymbol{\theta})\}_{k=1,\ldots,n-1}$ where the cycles $C(\nu_k)$ and $C(\nu_k \boldsymbol{\theta})$, $k = 1, \ldots, n-1$, have complementary dimension in X/Γ. Similarly, there are cycles $C(\mu_s)$, $s = 1, 2$, with the cycles $C(\mu_1 \boldsymbol{\theta})$, $s = 1, 2$ of complementary dimension.

3.2. **Theorem.** *There exists a uniform discrete arithmetically defined subgroup Γ of the real Lie group $SU^*(2n)$ so that the cohomology $H^j(X/\Gamma, \mathbb{R})$ contains a non-trivial cohomology class for any integer*

$$j = 4k(n-k), \quad \text{and} \quad j = \dim X/\Gamma - 4k(n-k), \qquad [k = 1, \ldots, n-1]$$

respectively

$$j = (n+1)(n-1) \quad \text{and} \quad j = n(n-1).$$

By duality, these classes are detected by the fundamental classes of a totally geodesic submanifold, a so called geometric cycle, of the form $C(\nu_k)$ resp. $C(\nu_k \boldsymbol{\theta})$ in the first case, and $C(\mu_s)$ resp. $C(\mu_s \boldsymbol{\theta})$ in the second case. These classes cannot be obtained as the restriction of a continuous class from the underlying Lie group $SU^*(2n)$.

3.3. **Comments.** Since the cohomology of an arithmetically defined group Γ is strongly related to the theory of automorphic forms with respect to Γ this geometric construction of non-vanishing classes leads to results concerning the existence of specific automorphic forms. The deRham cohomology groups $H^*(X/\Gamma, \mathbb{C})$ can be interpreted as the relative Lie algebra cohomology groups $H^*(\mathfrak{g}, K, C^\infty(G/\Gamma)_K \otimes \mathbb{C})$ where $\mathfrak{g}$ denotes the complexified Lie algebra of G. Since Γ is a uniform discrete group, by a result of Matsushima, the study of this cohomology amounts to the study of the finite algebraic sum

$$\bigoplus\nolimits_{\pi \in \hat{G}} m(\pi, \Gamma)\ H^*(\mathfrak{g}, K, H_{\pi,K} \otimes \mathbb{C})$$

where the sum ranges over all irreducible unitary representations (π, H_π) of G which occur with non-zero multiplicity $m(\pi, \Gamma)$ in the spectral decomposition of the space of square integrable functions $L^2(G/\Gamma) = \hat{\bigoplus}_{\pi \in \hat{G}} m(\pi, \Gamma)\ H_\pi$ and have non-vanishing relative Lie algebra cohomology. One makes explicit the general classification, due to Vogan-Zuckerman [10], of unitary representations with non-vanishing cohomology in the case of the real Lie group $SU^*(2n)$ (see [9, Sect. 3]).

In view of this representation theoretical interpretation of the cohomology groups, the existence of non-vanishing geometric cycles implies the existence of certain automorphic forms. However, on one hand, a direct comparison of the various families of non-vanishing classes for X/Γ with the family $\{A_{\mathfrak{q}}\}_{\mathfrak{q}}$ of irreducible unitary representations of $SU^*(2n)$ (up to infinitesimal equivalence) with non-zero cohomology shows that the cardinality of the latter exceeds by far the range of geometrically constructed cycles. Therefore the geometric construction misses possible cohomological degrees. On the other hand, in some cases one can "identify" an automorphic form which corresponds to a non-bounding geometric cycle but, in all generality, this is an enticing open problem. It might be that the theory of period integrals is of some help in a structural characterization.

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MODULAR FORMS FOR GENUS THREE

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This is a report of joint work with Jonas Bergström and Carel Faber.

1. Happy Birthday to Don

2. Introduction

For an algebraic geometer modular forms live on moduli spaces; the moduli spaces in question are the moduli space A_g of principally polarized abelian varieties of dimension g and the moduli space M_g of curves of genus g. Over $\mathbb{C}$ we have $A_g(\mathbb{C}) = \mathrm{Sp}(2g, \mathbb{Z})\backslash\mathcal{H}_g$, with $\mathcal{H}_g$ the Siegel upper half space. Let ρ be an irreducible complex representation $\rho : \mathrm{GL}(g, \mathbb{C}) \to \mathrm{Aut}(W)$. A Siegel modular form of weight ρ is a holomorphic map $f : \mathcal{H}_g \to W$ such that $f((a\tau + b)(c\tau + d)^{-1}) = \rho(c\tau + d)f(\tau)$ for all $(a, b; c, d) \in \mathrm{Sp}(2g\mathbb{Z})$. The space of cusp forms is denoted by S_ρ. We want to calculate the trace of the Hecke operators on S_ρ, and by this we mean calculating it as explicitly as Don would do that. Our tool is the cohomology of local systems.

Recall that these moduli spaces are defined over $\mathbb{Z}$ and the idea is that one can study the cohomology over $\mathbb{Q}$ by looking at the fibre $M_g \otimes \mathbb{F}_p$ with $\mathbb{F}_p$ a finite field and using comparison theorems; we get information about the ℓ-adic étale cohomology ($\ell \neq p$) of $M_g \otimes \overline{\mathbb{F}}_p$ by counting points over finite fields.

3. Genus one

Let us start with $g = 1$. The space S_k of cusp forms of weight k on $\mathrm{SL}(2, \mathbb{Z})$ has a cohomological interpretation: consider the universal elliptic curve $\pi : \mathcal{X}_1 \to A_1$ and the local system $V = R^1\pi_*\mathbb{Q}$ of rank 2. For $a \in \mathbb{Z}_{\geq 1}$ we have the local system $V_a = \mathrm{Sym}^a(V)$ of rank $a + 1$. We look at the 'motivic' Euler characteristic

$$e_c(A_1, V_a) = \sum_{i=0}^{2}(-1)^i[H_c^i(A_1, V_a)] \, ,$$

where the subindex c refers to compactly supported cohomology and the square brackets indicate that we consider the cohomology in an appropriate Grothendieck group of mixed Hodge modules or Galois representations (for the ℓ-adic counterpart $V_a^{(\ell)}$). Remark that the cohomology vanishes for a odd.

Then we have $e_c(A_1, V_a) = -S[a+2] - 1$ for even $a \geq 2$ with $S[k]$ the motive associated to the space of cusp forms S_k as constructed by Scholl. The

Eichler-Shimura congruence relation then implies that the trace of Frobenius on $H^1_c(A_1 \otimes \overline{\mathbb{F}}_p, V_a^{(\ell)})$ equals $1 + \mathrm{tr}(T(p), S_{a+2})$, that is, 1 plus the trace of the Hecke operator $T(p)$ on S_{a+2}.

We then can calculate the trace of $T(p)$ on all spaces S_{a+2} if we

1) make a list of all elliptic curves defined over $\mathbb{F}_p$ up to $\cong_{\mathbb{F}_p}$;

2) determine for all E in the list $\#\mathrm{Aut}_{\mathbb{F}_p}(E)$ and $\#E(\mathbb{F}_p) = p+1-\alpha-\bar{\alpha}$.

Then the formula is

$$\mathrm{Tr}(T(p), S_{a+2}) + 1 = -\sum_E \frac{\alpha^a + \alpha^{a-1}\bar{\alpha} + \cdots + \bar{\alpha}^a}{\#\mathrm{Aut}_{\mathbb{F}_p}(E)}$$

4. Genus Two

We extended this approach to genus 2 by looking at the universal abelian surface $\pi : \mathcal{X}_2 \to A_2$, the local system $V = R^1\pi_*\mathbb{Q}$ and the symplectic local systems V_λ with $\lambda = (a, b)$ associated to a representation of $\mathrm{Sp}(4, \mathbb{Q})$ of highest weight $a - b, b$. We write $e_c(A_2, V_\lambda) = \sum_i (-1)^i [H^i_c(A_2, V_\lambda)]$ for the Euler characteristic. i

Note that the cohomology vanishes if $a+b$ is odd. A result of Faltings tells us that $H^i(A_2, V_\lambda)$ and H^i_c have mixed Hodge structures and $H^i_! = \mathrm{Im}(H^i_c \to H^i)$ has a pure Hodge structure. Moreover, if λ is regular, i.e., $a > b > 0$, then if $H^i_!(A_2, V_\lambda) \neq (0)$ we have $i = 3$. The first step in the Hodge filtration $F^{a+b+3} \subset F^{a+2} \subset F^{b+1} \subset F^0 = H^3_!(A_2, V_\lambda)$ can be interpreted as a space of vector-valued Siegel modular cusp forms:

$$F^{a+b+3} \cong S_{a-b,b+3},$$

with the factor of automorphy being $\mathrm{Sym}^{a-b}(C\tau + D)\det(C\tau + D)^{b+3}$ for a matrix $\tau = (A, B; C, D) \in \mathrm{Sp}(2g, \mathbb{Z})$.

If we want to use the traces of Frobenius obtained by counting over finite fields to calculate the traces of the Hecke operators as we did for $g = 1$ we face for $g = 2$ two problems. First we must calculate the Eisenstein cohomology, that is, the kernel $\sum(-1)^i \ker(H^i_c \to H^i)$; this we did in [6, 4]. Second, there are contributions that do not see the first and the last part of the Hodge filtration (endoscopy). We gave a conjectural formula for this in [4]. In [8], Weissauer shows that the conjecture (in the case of a regular weight) can be deduced from earlier work of his. Assuming this the formula for the trace of the Hecke operator $T(p)$ on $S_{a-b,b+3}$ is

$$-\text{trace of } F_p \text{ on } e_c(A_2 \otimes \mathbb{F}_p, V^\ell_{a,b}) + \text{trace of } F_p \text{ on } e_{2,\mathrm{extra}}(a, b)$$

with F_p Frobenius at p and $e_{2,\mathrm{extra}}(a, b)$ given by

$$s_{a-b+2} - s_{a+b+4}(S[a-b+2]+1)L^{b+1} + \begin{cases} S[b+2]+1 & a \equiv 0 (\mathrm{mod}\, 2) \\ -S[a+3] & a \equiv 1 (\mathrm{mod}\, 2), \end{cases}$$

and L the Lefschetz motive and $s_k = \dim S_k$. With this formula and our counting (using that A_2 is the moduli space of curves of genus 2 of compact type) we can calculate the trace of $T(p)$ on the spaces $S_{j,k}$ for all j and k for

$p \leq 37$. The results agree with everything we know about $g = 2$ modular forms.

For example, let $a = b = 32$. Then the space $S_{0,35}$ is 1-dimensional and generated by Igusa's χ_{35} and we we find for the eigenvalue for $p = 37$

$$\lambda_{37}(\chi_{35}) = -47788585641545948035267859493926208327050656971703460.$$

Inspired by our results Harder formulated a conjecture about congruences between $g = 1$ and $g = 2$ modular forms and we obtained a lot of numerical evidence for this, see [7, 6]. All of these things have been generalized to $g = 2$ and level 2 in [2].

5. Genus Three

What about $g = 3$? There we have a degree 2 map of stacks $M_3 \to A_3$. We now have local systems $V_{a,b,c}$ parametrized by triples (a, b, c) with $a \geq b \geq c \geq 0$. We are interested in vector-valued Siegel modular cusp forms of weight $(a - b, b - c, c + 4)$, i.e. holomorphic functions $f : \mathcal{H}_3 \to W$ on the Siegel upper half space $\mathcal{H}_3$ to a finite-dimensional complex vector space W satisfying

$$f((a\tau + b)(c\tau + d)^{-1}) = \rho(c\tau + d)f(\tau)$$

where ρ is the irreducible representation of $\mathrm{GL}(3, \mathbb{C})$ on W of highest weight $a - b, b - c, c + 4$.

We now have the following *conjectural formula* for the trace of the Hecke operator $T(p)$ on the space of cusp forms $S_{a-b,b-c,c+4}$:

$$\text{trace of Frobenius on } e_c(A_3 \otimes \mathbb{F}_p), V_{a,b,c}) - e_{3,\mathrm{extra}}(a, b, c),$$

with $e_{3,\mathrm{extra}}(a, b, c)$ given by

$$\begin{aligned} &-e_c(A_2, V_{a+1,b+1}) - e_{2,\mathrm{extra}}(a + 1, b + 1) \otimes S[c + 2] \\ &\quad +e_c(A_2, V_{a+1,c}) + e_{2,\mathrm{extra}}(a + 1, c) \otimes S[b + 3] \\ &\qquad -e_c(A_2, V_{b,c}) - e_{2,\mathrm{extra}}(b, c) \otimes S[a + 4] \end{aligned}$$

The evidence we have is overwhelming and includes the following. It fits all the calculations we did over finite fields. The numerical Euler characteristic

$$\sum (-1)^i \dim H^i_c(A_3, V_{a,b,c})$$

is known by [3, **?**] and this fits the results. As a corollary we get a formula for the dimension of the space of cusp forms $S_{a-b,b-c,c+4}$. We find that for $a + b + c \leq 60$ the space $S_{a-b,b-c,c+4}$ contributes to the cohomology a rank that is always divisible by 8. For $a = b = c$ it fits with the dimension formula for $\dim S_{0,0,c+4}$ for scalar-valued modular forms due to Tsuyumine. Moreover, we observed various sorts of Harder-type congruences between $g = 1$ and $g = 3$ modular forms.

We also have a precise conjectural formula for all the lifts from $g = 1$ to $g = 3$.

Gross and Savin predicted that there should be Siegel modular forms of genus 3 with motivic Galois group of type G_2. We found examples of these, for example in the space $S_{3,3,7}$.

To illustrate our results, assuming the conjecture we find for the eigenvalues of $T(p)$ with $p = 2, 3, 5$ and 7 on $S_{3,3,7}$ the values $2^3 \cdot 3^3 \cdot 5$, $2^6 \cdot 3^4 \cdot 5 \cdot 7$, $2^3 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 9749$ and $2^8 \cdot 5^3 \cdot 7^2 \cdot 8887$.

Or taking $(a, b, c) = (11, 5, 2)$ we find that the space of Siegel modular cusp forms $S_{6,3,6}$ is 1-dimensional, say generated by F with eigenvalue for $T(17)$

$$\lambda_{17}(F) = -107529004510200.$$

One can also look at the cohomology of M_3 instead of A_3. The degree 2 covering $M_3 \to A_3$ is ramified along the hyperelliptic locus. Unlike A_3 the moduli space M_3 can have cohomology for $a + b + c$ odd. This is related to Teichmüller modular forms that do not come from Siegel modular forms. An example is the modular form $\chi_9 = \sqrt{\chi_{18}}$ on M_3 that vanishes on the hyperelliptic locus and was studied by Ichikawa; we see it occurring in the cohomology of the local system $V_{5,5,5}$ on M_3. We also could detect vector-valued Teichmüller modular forms that do not come from Siegel modular forms.

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Arbeitstagung, June 2011

T. Dimofte: ***Chern-Simons Theory with Complex Gauge Group***

1 Introduction

I will be discussing a circle of ideas that started out with a paper of Sergei Gukov in 2003 [1], and have since appeared in papers of Sergei Gukov, Don Zagier, myself, and others [2, 3, 4, 5, 6].

I will start with an overview of classical and quantum Chern-Simons theory with a complex gauge group. I will give a few mathematical definitions and a few "physical" definitions. While these physical definitions may initially involve objects like path integrals, I must stress that it should be possible to define all "quantum," "physical" objects quite rigorously — just as in the well known case of Chern-Simons theory with compact gauge group.

Next, I will give some examples of the standard objects one computes in complex Chern-Simons theory: partition functions, A-polynomials (and their generalizations), and quantum $\hat{A}$-polynomials. I will then try to give some motivation for why these objects might be of interest in a wider mathematical setting.

Finally, I will discuss some basic details of the actual quantization procedure used in complex Chern-Simons theory. I hope to give a flavor of how computations actually work, and to emphasize that the process of computation really boils down to rigorous, well defined algebra and combinatorics.

2 Classical and quantum Chern-Simons theory

Classically, Chern-Simons theory is a theory of flat connections on a 3-manifold. It takes as inputs

- a 3-manifold M, possibly with boundary; and
- a gauge group $G_\mathbb{C}$, which will be complex for us, *e.g.* $G_\mathbb{C} = SL(2,\mathbb{C})$;

and considers flat $G_\mathbb{C}$ connections on M. Note that locally, a connections can be described as a Lie algebra ($\mathfrak{g}_\mathbb{C}$)–valued one-form $\mathcal{A}$. The condition for flatness then becomes

$$\mathcal{F} := d\mathcal{A} + \mathcal{A}\wedge\mathcal{A} = 0\,. \tag{1}$$

Given such a connection, Chern-Simons theory computes its "volume" on $\mathcal{M}$:

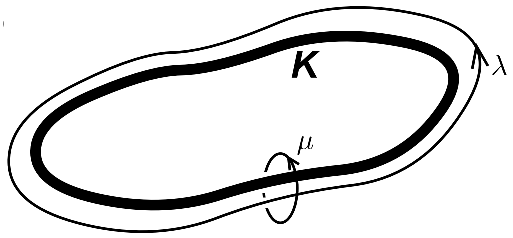
$$\text{"Vol}(\mathcal{A})\text{"} \;=\; S_{\text{CS}}(\mathcal{A}) \;= -\frac{1}{2}\int_M \text{Tr}\big(\mathcal{A}\wedge d\mathcal{A} + \tfrac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\big)\,. \tag{2}$$

Now and for the rest of the talk, I will specialize to $\boxed{G_\mathbb{C} = SL(2,\mathbb{C})}$, which is by no means necessary, but is useful for building intuition. When $G_\mathbb{C} = SL(2,\mathbb{C})$, flat connections can be rewritten as hyperbolic metrics, and then we are just talking about classical hyperbolic geometry. In particular,

$$S_{\text{CS}}(\mathcal{A}) \;=\; i(\text{Vol}(M) + i\text{CS}(M))\,, \tag{3}$$

where $\text{Vol}(M)$ and $\text{CS}(M)$ are the classical hyperbolic volume and Chern-Simons invariants of a 3-manifold.

The story becomes a bit more interesting when M has boundaries. Then the volumes above depend on boundary conditions. For example, if we consider a knot complement $M = S^3\backslash K$, there is a torus boundary $\partial M = T^2$. As good boundary conditions for the differential equation (1), we must then specify the eigenvalues of the holonomy of $\mathcal{A}$ around *one* cycle on this torus. For a knot complement, a canonical choice of cycle is the so-called "meridian" μ of the knot. This is a small loop that links the knot once, as in Figure 1, and generates $H_1(M,\mathbb{Z}) \simeq \mathbb{Z}$.



$$\mu \sim \begin{pmatrix} m & * \\ 0 & m^{-1} \end{pmatrix}$$

$$\lambda \sim \begin{pmatrix} \ell & * \\ 0 & \ell^{-1} \end{pmatrix}$$

Figure 1: Meridian (μ) and longitude (λ) holonomies for a knot complement K.

Suppose that we fix the holonomy to be conjugate to $\left(\begin{smallmatrix} m & * \\ 0 & m^{-1} \end{smallmatrix}\right)$, as in the figure. Let's set

$$m = e^u\,. \tag{4}$$

Then, geometrically, $\text{Im}(u)$ can be identified with the cusp angle at K of a hyperbolic metric on M; and $\text{Re}(u)$ can be identified with the torsion in the metric as one circles around K. The classical hyperbolic invariants in (3) (equivalently, the volume of a flat connection (2)) become generalized to

$$S_{\text{CS}}(\mathcal{A}) \;\to\; S_{\text{CS}}(\mathcal{A};u) \;=\; i(\text{Vol}(M;u) + i\text{CS}(M;u))\,. \tag{5}$$

Now, there is another cycle on $T^2 = \partial M$, dual to the meridian. It is typically called the longitude λ of the knot, and can be described as a loop parallel to K and having zero linking number with K (the longitude is null-homologous in M). It has its own holonomy eigenvalues $(\ell, \ell^{-1}) = (e^v, e^{-v})$. However, these are not independent of (m, m^{-1}). *Indeed, in order to define good boundary conditions for a flat connection on M, we can specify either m or ℓ, but not both.*

There is some interesting symplectic geometry hidden in this statement. If we define

$$\mathcal{P}_{\partial M} = \mathcal{M}_{\text{flat}}(SL(2,\mathbb{C}),T^2) = \{(\ell,m)\}/\mathbb{Z}^2 \simeq (\mathbb{C}^* \times \mathbb{C}^*)/\mathbb{Z}_2 \tag{6}$$

to be the space of 2d flat $G_{\mathbb{C}}$ connections on the boundary, then the subset of 2d connections that can be extended as 3d flat connections throughout the bulk of M (*a.k.a.* the subset of good boundary conditions) forms a mid-dimensional submanifold

$$\mathcal{L}_M = \{\text{connections that extend to } M\} \;\subset\; \mathcal{P}_{\partial M}\,. \tag{7}$$

Indeed, since $G_{\mathbb{C}} = SL(2,\mathbb{C})$ is an algebraic group, it turns out that $\mathcal{L}_M = \{A(\ell, m) = 0\}$ is cut out by an algebraic equation; the equation is the well known A-polynomial of M [7].

Moreover, notice that $\mathcal{L}_M$ is in fact Lagrangian in $\mathcal{P}_{\partial M}$ with respect to the symplectic structure

$$\omega = 2\,\frac{d\ell}{\ell} \wedge \frac{dm}{m} = 2\,dv \wedge du\,. \tag{8}$$

This symplectic structure is induced naturally on $\mathcal{P}_{\partial M}$ by the Chern-Simons action (2). This is equivalent to saying that the classical Chern-Simons action itself (or rather, its dependence on $u = \log m$) can be written as

$$S_{\rm CS}(\mathcal{A}; u) = -2 \int_{\mathcal{L}_M}^{u} \theta\,, \tag{9}$$

where θ is a one-form that satisfies $d\theta = \omega$ (for example $\theta = 2vdu$), and we integrate along $\mathcal{L}_M$. Note that on $\mathcal{L}_M$ itself, θ is closed. In terms of hyperbolic geometry, formula (9) is precisely the variation in $S_{\rm CS}(\mathcal{A}; u) = i(\mathrm{Vol}(M; u) + i\mathrm{CS}(M; u))$ uncovered by Neumann-Zagier and Yoshida [8, 9].

Having described many of classical quantities associated to complex Chern-Simons theory, we can consider what it means to quantize them. Physically, quantum Chern-Simons theory arises by putting the Chern-Simons action (2) in a path integral, or *partition function*:

$$\mathcal{Z}(M; u; \hbar) = \int D\mathcal{A}\; e^{\frac{1}{\hbar} S_{\rm CS}(\mathcal{A}; u)}\,. \tag{10}$$

The functional integral here is over *all* connections on M (modulo gauge equivalence), not just the flat connections. When M has a boundary, we must still impose boundary conditions "u," of exactly the same type discussed in the classical scenario above.[1]

In the semi-classical limit $\hbar \to 0$, the leading contribution to the partition function (10) comes from critical points of the classical action — and these are precisely the classical flat connections. Then we expect the partition function to have an (asymptotic) expansion of the form

$$\mathcal{Z}(M; u; \hbar) \overset{\hbar \to 0}{\sim} \exp\left[\tfrac{1}{\hbar} S_{CS}(\mathcal{A}^{\rm flat}; u) + \hbar^{n \geq 0}\ \text{corrections}\right] + \exp^{\rm y}\ \text{small corrections}\,, \tag{11}$$

where $\mathcal{A}^{\rm flat}$ is the flat connection on M with the largest volume.[2] When M is a hyperbolic manifold, this is the same as the connection associated to the hyperbolic metric. Therefore, we can think of quantum Chern-Simons theory with gauge group $G_{\mathbb{C}} = SL(2,\mathbb{C})$ as a theory that lets metrics on M fluctuate away from being purely hyperbolic, but suppresses their fluctuations by a factor of $(-i)\hbar$. We see explicitly that in (11), the contribution from non-hyperbolic metrics (non-flat connections) comes with factors of $\hbar$.

The partition function (10) is supposed to be well defined, and I will try to explain later (Section 5) how it can be made so, in terms of a simple combinatorial algorithm. For now, let

[1] In particular, the notion of longitude and meridian "holonomy eigenvalues" on a torus (say) can be made sense of even when connections $\mathcal{A}$ are not flat.

[2] More precisely, the flat connection with largest real volume $\mathrm{Im}\, S_{\rm CS}(\mathcal{A}; u)$ is the dominant contribution when $-i\hbar$ is real and positive.

me note that one way to characterize $\mathcal{Z}(M;u;\hbar)$ is as being a solution to a certain difference equation of the form

$$\hat{A}(\hat{\ell},\hat{m};q)\,\mathcal{Z}(M;u;\hbar) \;=\; 0\,. \tag{12}$$

Here $A(\hat{\ell},\hat{m};q)$ is a quantization of the classical A-polynomial (defining the Lagrangian $\mathcal{L}_M$) in the following sense. The canonically conjugate coordinates ℓ and m on $\mathcal{P}_{\partial M}$ are promoted to quantum operators that act on $\mathcal{Z}(M;u;\hbar)$ (now viewed as a wavefunction) as

$$\hat{m}\,\mathcal{Z}(u)=e^{\hat{u}}\,\mathcal{Z}(u)=e^{u}\,\mathcal{Z}(u)\,,\qquad \hat{\ell}\,\mathcal{Z}(u)=e^{\hat{v}}\,\mathcal{Z}(u)=e^{\frac{\hbar}{2}\partial_u}\,\mathcal{Z}(u)=\mathcal{Z}(u+\hbar/2)\,. \tag{13}$$

These operators "q–commute,"

$$\hat{\ell}\hat{m}=q^{\frac{1}{2}}\hat{m}\hat{\ell}\,,\qquad \text{with}\qquad q:=\exp(\hbar)\,. \tag{14}$$

In promoting the classical polynomial $A(\ell,m)$ to an operator $\hat{A}(\hat{\ell},\hat{m};q)$, severe q–dependent ordering ambiguities can arise. These have all been resolved, and there is a unique way to fix them.

3 Examples

As a first example of a partition function in quantum Chern-Simons theory, we can consider the complement of the trefoil knot $M=S^3\backslash\mathbf{3_1}$. The classical A-polynomial is

$$A_{\mathbf{3_1}}=m^6\ell+1\,, \tag{15}$$

and the quantum partition function turns out to be

$$\mathcal{Z}(\mathbf{3_1};u;\hbar)=e^{-\frac{1}{\hbar}6u^2+\frac{2\pi i+\hbar}{\hbar}u}\,. \tag{16}$$

The trefoil is not a hyperbolic manifold in the classical sense — in particular, its hyperbolic volume at $u=0$ is zero — it still admits flat $SL(2,\mathbb{C})$ connections. In fact, there is a unique irreducible flat $SL(2,\mathbb{C})$ connection, and (16) calculates fluctuations around it. In this case, the $\hbar$-expansion in (16) is fairly trivial, due mainly to the fact that the trefoil is simply a torus knot.

It is easy to check that (16) is annihilated by the operator

$$\hat{A}_{\mathbf{3_1}}=q\hat{m}^6\hat{\ell}+1\,, \tag{17}$$

which is clearly a quantization of (15). Moreover, the partition function has a symmetry

$$\mathcal{Z}(\mathbf{3_1};u;\hbar)\;=\;Z\big(\mathbf{3_1};\tfrac{2\pi i}{\hbar}u;-\tfrac{4\pi^2}{\hbar}\big)\,, \tag{18}$$

which is the first hint of modular-like behavior in Chern-Simons theory.

A more complicated, and much more typical example is the complement of the figure-eight knot, $M=S^3\backslash\mathbf{4_1}$. The classical A-polynomial in this case is[3]

$$A_{\mathbf{4_1}}=\ell-(m^4-m^2-2-m^{-2}+m^{-4})+\ell^{-1}\,, \tag{19}$$

[3]Since this is defining equation of a variety in $\mathbb{C}^*\times\mathbb{C}^*$, we are free to multiply A by factors of $\ell^{\pm1}$ and $m^{\pm1}$. Here it's written as a Laurent polynomial.

and it becomes quantized as

$$\hat{A}_{\mathbf{4_1}} = (q^{-1}\hat{m}^2 - \hat{m}^{-2})\hat{\ell} - (\hat{m}^2 - \hat{m}^{-2})(\hat{m}^4 - \hat{m}^2 - q - q^{-1} - \hat{m}^{-2} + \hat{m}^{-4}) + (q\hat{m}^2 - \hat{m}^{-2})\hat{\ell}^{-1}\,. \tag{20}$$

This quantization is far from obvious. The quantum $\hat{A}$ operator annihilates a partition function

$$\mathcal{Z}(\mathbf{4_1}; u; \hbar) = \frac{1}{\sqrt{\pi\hbar}} e^{-\frac{4u^2+2\pi i u}{\hbar}} \int dp\, \frac{\Phi_\hbar(p-2u)}{\Phi_\hbar(-p)} e^{\frac{2pu}{\hbar}}\,, \tag{21}$$

where $\Phi_\hbar(p)$ is the "noncompact" quantum dilogarithm function [10, 11], given (*e.g.*) as

$$\Phi_\hbar(p) := \prod_{r=1}^{\infty} \frac{1 + e^{(r-\frac{1}{2})\hbar + p}}{1 + e^{(r-\frac{1}{2})\frac{4\pi^2}{\hbar} + \frac{2\pi i}{\hbar}p}}\,, \qquad (\mathrm{Re}\,\hbar < 0)\,, \tag{22}$$

with a similar formula when $\mathrm{Re}\,\hbar > 0$.

In the classical limit $\hbar \to 0$, we have $\Phi_\hbar(p) \sim e^{\frac{1}{\hbar}\mathrm{Li}_2(-e^p)}$, so

$$\mathcal{Z}(\mathbf{4_1}; u; \hbar) \;\sim\; \frac{1}{\sqrt{\pi\hbar}} e^{-\frac{4u^2+2\pi i u}{\hbar}} \int dp\, \exp\frac{1}{\hbar}\big[\mathrm{Li}_2(-e^{p-2u}) - \mathrm{Li}_2(-e^{-p}) + 2pu\big]\,. \tag{23}$$

The integrand has two critical points, each corresponding to a flat connection on the $\mathbf{4_1}$ knot complement. In particular, the flat connection that provides the dominant asymptotic in (23) is the one giving the hyperbolic metric. We find, as expected, that $\mathcal{Z}(\mathbf{4_1}; u; \hbar) \sim \exp\frac{i}{\hbar}\big[\mathrm{Vol}(\mathbf{4_1}; u) + i\mathrm{CS}(\mathbf{4_1}; u)\big]$. It is useful to observe that hyperbolic volumes are typically expressed as a sum of classical dilogarithms; whereas quantum Chern-Simons partition functions are typically integrals of products of quantum dilogarithms.

The inversion symmetry (18) also holds for the figure-eight partition function (up to a factor of $e^{\frac{4\pi i}{\hbar}u}$). More precisely, I should say this holds formally for the integral (21), and will hold more concretely once specific integration contour(s) are chosen.

4 Why is this interesting?

Chern-Simons theory with complex gauge group has deep and interesting connections to other subjects in geometry, topology, and number theory. I'll list four of them.

4.1 Knot polynomials

First, there is a relation between partition functions $\mathcal{Z}(M; u; \hbar)$ and more common, *compact* knot and 3-manifold invariants, such as Jones polynomials.

For example, the compact version of the $SL(2, \mathbb{C})$ invariants discussed here are the colored Jones polynomials. As Witten explained 20 years ago, the colored Jones polynomials of a knot K arise by considering Chern-Simons theory with compact gauge group $SU(2)$. Specifically, if we put $SU(2)$ Chern-Simons theory on a manifold $M = S^3 \backslash K$, set $\hbar = \frac{2\pi i}{k}$, or q to be a root of unity

$$q = e^{\frac{2\pi i}{k}}\,, \tag{24}$$

and fix boundary conditions[4]

$$m = e^u \equiv e^{\frac{i\pi N}{k}} = q^{N/2} \tag{25}$$

for $N \in \mathbb{N}$, then the compact Chern-Simons partition function becomes the colored Jones polynomial $J_N(K,q)$.

In 2003, Gukov argued that $SL(2,\mathbb{C})$ Chern-Simons theory should be thought of as an analytic continuation of $SU(2)$ theory. In particular, the complex partition functions $\mathcal{Z}(M\backslash K;u;\hbar)$ should have the same semi-classical asymptotics at fixed u and $\hbar \to 0$ as the colored Jones. This generalized and gave a physical motivation for the Volume Conjecture [12, 13], which states that the asymptotics of colored Jones polynomials are governed by the hyperbolic volume of a knot complement.

Gukov also argued that there should exist a quantized version of the A-polynomial, our "$\hat{A}$" operator, that annihilates the complex Chern-Simons partition function — and that this *same* operator provides a recursion relation for the colored Jones polynomial. By translating the action of $\hat{\ell}$ and $\hat{m}$ operators (13) to compact notation, we see that they should act on $J_N(K;q)$ as

$$\hat{m}\, J_N(K;q) = q^{N/2} J_N(K;q)\,, \qquad \hat{\ell}\, J_N(K;q) = J_{N+1}(K;q)\,. \tag{26}$$

Thus, an equation

$$\hat{A}(\hat{\ell},\hat{m};q)\, J_N(K;q) = 0 \tag{27}$$

becomes a recursion relation.

Luckily, Garoufalidis and Le [14, 15] had already discovered that colored Jones polynomials obey such recursions, and Garoufalidis had conjectured that the recursion operators reduce to the A-polynomial in the classical limit $q \to 1$. We now have a way of unambiguously quantizing $\hat{A}$-polynomials directly in complex Chern-Simons theory, and all tests so far show that they are exactly the same[5] operators appearing in $J_N(K;q)$ recursions. A similar exact relation between compact and complex partition functions (rather than just the operators that annihilate them) has *yet* to be fully understood; the analytic continuation that takes $J_N(K;q)$ to $\mathcal{Z}(M;u;\hbar)$ is highly subtle.

4.2 Lifting quantum Teichmüller theory

We have already discussed the fact that Chern-Simons theory with complex gauge group $SL(2,\mathbb{C})$ provides a quantization of 3d hyperbolic geometry. Thus, one might think of it as a 3d lift of 2d quantum Teichmüller theory, which quantizes 2d hyperbolic geometry on surfaces.

For example, in 2d quantum Teichmüller theory [16, 17], one quantizes the space $\mathcal{P}_\Sigma$ of hyperbolic metrics on a surface Σ, obtaining an infinite-dimensional Hilbert space $\mathcal{H}_\Sigma$. The quantization is done with respect to the Weil-Petersson symplectic form ω_{WP} on $\mathcal{P}_\Sigma$. Then one asks how elements φ of the mapping class group of Σ act on $\mathcal{H}_\Sigma$.

The question of mapping class group is answered in Chern-Simons theory by considering 3-manifolds that are mapping cylinders, twisted by φ: $M_\varphi = \Sigma \times_\varphi I$. Topologically, this

[4]More commonly, the integer label N of the colored Jones polynomials is associated to the dimension of a representation of $SU(2)$ that "colors" the knot. This description is equivalent to fixing discrete boundary conditions at the knot, as in (25).

[5]With one well understood caveat; interested readers are referred to Section 2 of [4] for further discussion.

manifold is just a cylinder, but we choose boundary conditions at the two ends of the interval I in a manner consistent with the twisting. (For example, if $\Sigma = T^2$ and $\phi = S$, we would fix the meridian at one end of $T^2 \times I$ and the longitude at the other end.) Since the total boundary of M_φ is $\overline{\Sigma} \sqcup \Sigma$, the Chern-Simons partition function $\mathcal{Z}(M_\varphi)$ — depending on boundary conditions at both ends — is an element of[6] $\mathcal{H}_\Sigma^* \otimes \mathcal{H}_\Sigma$, and it naturally provides an isomorphism

$$\mathcal{Z}(M_\varphi) \; : \quad \mathcal{H}_\Sigma \xrightarrow{\sim} \mathcal{H}_\Sigma \,. \tag{28}$$

This isomorphism is the sought after mapping class group action. Moreover, the quantized Lagrangian $\hat{\mathcal{L}}_{M_\varphi}$ that annihilates $\mathcal{Z}(M_\varphi)$ provides a corresponding isomorphism in the algebra of operators on $\mathcal{H}_\Sigma$.

More generally, Chern-Simons theory provides a map $\mathcal{H}_\Sigma \to \mathcal{H}_{\Sigma'}$ associated to any 3d cobordism M between surfaces Σ and Σ'. In fact, it must do so in order to be a good TQFT (topological quantum field theory). The relation between Chern-Simons and Teichmüller theories was first discussed in [18]; explicit details of partition functions and cobordisms in the present $SL(2,\mathbb{C})$ setting appear in [5, 19, 20].

4.3 K-theory and quantization

Quantization of the A-polynomial $\hat{A}(\hat{\ell}, \hat{m}; q)$, and quantization of more general Lagrangians, may have applications and implications in many other disciplines.

Algebraically, quantization is closely related to K-theory of function fields. Specifically, one can think of ℓ and m as two rational functions on the classical A-polynomial curve $\mathcal{L}_M = \{A(\ell, m) = 0\}$. There is a strong physical constraint on the form the A-polynomial can take in order for it to be quantizable — the constraint is essentially equivalent to the integral (9) being well defined. It then turns out (*cf.* [6]) that this physical constraint is equivalent to the class $\{\ell, m\}$ being torsion in the K-theory group $K_2(\mathcal{L}_M)$.

Another method for quantizing curves, which was presented earlier in this Arbeitstagung by G. Borot, is the "topological recursion" of Eynard and Orantin [21]. The topological recursion can take an abstract curve $A(\ell, m) = 0$ (not necessarily the A-polynomial of any knot) and compute an operator $\hat{A}(\ell, m; q)$ order by order in $\hbar = \log(q)$. It is widely believed that this quantization should be the same as the one given by Chern-Simons theory, and this idea has recently been investigated in [22, 23, 6].

4.4 Modularity?

Finally, we hope that complex Chern-Simons theory will provide a window into the modular behavior of quantum topological invariants. There have been many hints that various versions or limits of Chern-Simons theory should be modular — starting with the work of Lawrence and Zagier 11 years ago [24], and leading up to Zagier's recent discoveries of "quantum modular forms" [25]. Further modularity for colored Jones polynomials was discussed at this Arbeitstagung by S. Garoufalidis.

[6] Implicit in this statement is the fact that the Hilbert space of Chern-Simons theory is the *same* as quantum Teichmüller space. This turns out to be the case! In particular, Chern-Simons phase spaces $\mathcal{P}_{\partial M}$ are complexifications of classical Teichmüller spaces $\mathcal{P}_\Sigma$, and the form (8), generalized to arbitrary surfaces Σ, is just an analytic continuation of the Weil-Petersson form.

In complex Chern-Simons theory, it has been shown quite generally that the partition functions always have a symmetry [4, 5]

$$\mathcal{Z}(M;u;\hbar) = \mathcal{Z}(M;\tfrac{2\pi i}{\hbar}u;-\tfrac{4\pi^2}{\hbar})\,. \tag{29}$$

Upon associating $\hbar \sim \tau$ (up to a scaling), this should be reminiscent of a Jacobi-type modular S transformation $(z,\tau) \to (z/\tau, -1/\tau)$. Usually, S is generator of the modular group whose action is nontrivial (on, say, Jacobi forms); the action of $T\,:\,\tau \to \tau+1$ is very easy to see. In constrast, in complex Chern-Simons theory, the situation is maximally reversed: S is immediate, but T is not a symmetry of the partition function at all. Whether $\mathcal{Z}(M)$ can be modified or redefined in some way so as to make it fully modular is still an open and interesting question.

5 Rudiments of quantization

In the remainder of these notes, let me give a brief taste of how quantization and computations actually work. The basic idea is to take a potentially complicated 3-manifold M and cut it into ideal tetrahedra,

$$M \quad\rightsquigarrow\quad \bigcup_{i=1}^{N} \Delta_i\,. \tag{30}$$

Then each tetrahedron Δ_i is quantized in a canonical way, obtaining *both* a partition function $\mathcal{Z}(\Delta_i)$ and an operator $\hat{A}_{\Delta_i}$ that annihilates it; and the tetrahedra are glued back together appropriately to produce the partition function and operator(s) associated to M. This method of cutting and gluing uses the basic properties of Chern-Simons theory as a TQFT.

Such a construction applies to a wide class of 3-manifolds. In [4], I recently showed how to use it for any knot or link complements in S^3 aside from the unknot.[7] In upcoming work [20] with D. Gaiotto, S. Gukov, and R. van der Veen, the construction will be extended to 3-manifolds with arbitrary Riemann surface boundaries.

Let me consider then a single ideal tetrahedron Δ. Topologically, it is best to think of it as a tetrahedron whose vertices have been truncated (Figure 2). In terms of hyperbolic geometry, the vertices lie off at infinity, at the boundary of hyperbolic three-space, and are not part of the tetrahedron itself. Several of these ideal tetrahedra can be glued together to form any knot complement M (aside from the unknot), where little truncated vertex boundaries come together to form the torus boundary of M. For example, the figure-eight knot complement can be decomposed into two ideal tetrahedra.[8]

The hyperbolic (or $SL(2,\mathbb{C})$ structure) on an ideal tetrahedron can be described in terms of three complex parameters z, z', z''. These are complexified dihedral angles (in fact, $\exp[\text{torsion} + i(\text{angle})]$) on pairs of opposite edges, as in Figure 2, and satisfy

$$zz'z'' = -1 \tag{31}$$

[7] A slightly different cutting and gluing construction in much the same spirit, based on [26], was first used in [2] to obtain Chern-Simons partition functions.

[8] The idea of 3d ideal hyperbolic triangulations was pioneered by W. Thurston.

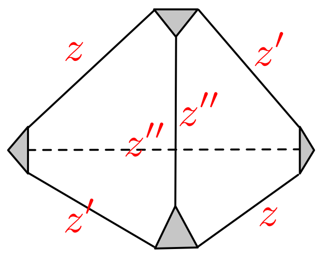


Figure 2: An ideal hyperbolic tetrahedron, with vertices truncated

and

$$z + z'^{-1} - 1 = 0 \qquad (\Leftrightarrow \; z' + z''^{-1} - 1 = 0 \;\Leftrightarrow\; z'' + z^{-1} - 1 = 0\,)\,. \tag{32}$$

Therefore, only a single one of the z, z', z'' is truly independent. I claim that the space of flat $SL(2,\mathbb{C})$ connections on the *boundary* of a tetrahedron should be constrained by the first equation:

$$\mathcal{P}_{\partial\Delta} \;\simeq\; \{(z,z',z'')\in(\mathbb{C}^*)^3\,|\,zz'z''=-1\}\,, \tag{33}$$

whereas the space of flat connections that can extend from the boundary to the bulk[9] is cut out by the second equation:

$$\mathcal{L}_\Delta = \{z + z'^{-1} - 1 = 0\} \quad \subset\; \mathcal{P}_{\partial\Delta}\,. \tag{34}$$

This is a Lagrangian submanifold with respect to the symplectic structure

$$\omega_{\partial\Delta} = \frac{dz}{z}\wedge\frac{dz'}{z'}\,. \tag{35}$$

In order to quantize, z, z', z'' should be promoted to operators $\hat z, \hat z', \hat z''$ that q-commute,

$$\hat z\hat z' = q\hat z'\hat z\,,\qquad \hat z'\hat z'' = q\hat z''\hat z'\,,\qquad \hat z''\hat z = q\hat z\hat z''\,,\qquad (q=e^\hbar) \tag{36}$$

and have satisfy a central constraint $\hat z\hat z'\hat z'' = -q$. Moreover, the classical Lagrangian $\mathcal{L}_\Delta$ should be promoted to an operator $\hat{\mathcal{L}}_\Delta = \hat z + \hat z'^{-1} - 1$ that annihilates the partition function of a tetrahedron:

$$(\hat z + \hat z'^{-1} - 1)\,\mathcal{Z}(\Delta; Z'; \hbar) = 0\,, \tag{37}$$

where $Z' := \log(z')$ is the equivalent of the meridian boundary parameter "u" for an ideal tetrahedron, and we define an action $\hat z'\mathcal{Z}(Z') = e^{Z'}\mathcal{Z}(Z')$, while $\hat z\,\mathcal{Z}(Z') = \mathcal{Z}(Z'+\hbar)$. It is fairly easy to see that a solution to (37) (that also happens to satisfy the symmetry (29)) is

$$\mathcal{Z}(\Delta; Z'; \hbar) = \Phi_\hbar(-Z' + i\pi + \tfrac{\hbar}{2})\,. \tag{38}$$

Thus, the partition function of an ideal tetrahedron is precisely a quantum dilogarithm function.

[9] To be completely rigorous: $\mathcal{P}_{\partial\Delta}$ is the moduli space of $SL(2,\mathbb{C})$ structures on $\partial\Delta$ viewed as a three-punctures sphere, with a requirement that the holonomy at each puncture be unipotent. The Lagrangian $\mathcal{L}_\Delta$ then describes the subset of connections whose holonomy at the punctures is *trivial*.

The partition function has several wonderful properties, including invariance under cyclic permutations $z \to z' \to z'' \to z$, which is clearly a symmetry of the phase space $\mathcal{P}_{\partial\Delta}$ and the Lagrangian $\mathcal{L}_\Delta$. These are discussed in [4].

The most important and interesting part of the quantization construction is the gluing of tetrahedra together to form a nontrivial 3-manifold, and unfortunately there is no time to do justice to it presently. Details of this gluing are also discussed in [4]. Very roughly, gluing involves a generalized symplectic reduction, both classically and quantum mechanically. For example, the phase space $\mathcal{P}_{\partial M}$ for a glued manifold is the symplectic reduction of a product of phase spaces $\mathcal{P}_{\partial\Delta_1}\times\cdots\times\mathcal{P}_{\partial\Delta_n}$ for component tetrahedra. Classically, this basically follows from the work of Neumann and Zagier [8]. A product of classical Lagrangians $\mathcal{L}_{\Delta_i}$ can be pulled through the symplectic reduction to yield a Lagrangian $\mathcal{L}_M \subset \mathcal{P}_{\partial M}$ for M, as can quantum Lagrangians and quantum partition functions. The final result is an operator $\hat{A}_M$ and a partition function $\mathcal{Z}(M)$ that it annihilates; and because individual tetrahedra have $\mathcal{Z}(\Delta_i)$ equal to quantum dilogarithms (38), the total partition function $\mathcal{Z}(M)$ generically becomes an integral of a product of quantum dilogarithms. All ordering ambiguities and factors of 'q' in the operator $\hat{A}_M$ are *completely* and *uniquely* fixed by requiring that the final answer is independent of how a 3-manifold M is actually triangulated — *i.e.* that the final answer is a topological invariant of M, precisely as complex Chern-Simons theory should provide.

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ON MULTIPLE ZETA VALUES

It is a great pleasure, and no small honour, to give this talk on the occasion of Don Zagier's 60th birthday. I shall report on the recent proof of some conjectures on multiple zeta values, in which Don played a crucial role.

1. INTRODUCTION

Let $n_1, \ldots, n_{r-1} \geq 1$, $n_r \geq 2$ be integers. The *multiple zeta value* is defined by

$$\zeta(n_1, \ldots, n_r) = \sum_{0<k_1<\ldots<k_r} \frac{1}{k_1^{n_1} \ldots k_r^{n_r}} \in \mathbb{R} \,. \tag{1.1}$$

The *weight* of a tuple $(n_1, \ldots, n_r)$ is the quantity $n_1 + \ldots + n_r$, its *depth* is the integer r. These numbers were first defined by Euler for $r = 2$, and were popularized by Don Zagier in the 90's, who discovered that they satisfy vast numbers of relations. For example, there are *a priori* $2^{13} = 8192$ such numbers in weight 15, but in reality they form a vector space over $\mathbb{Q}$ of dimension at most 28.

Let $\mathcal{Z}$ denote the $\mathbb{Q}$-vector space spanned by the numbers (1.1). It is relatively easy to show that $\mathcal{Z}$ is closed under multiplication. The purpose of this talk is to outline a proof of the following two theorems:

Theorem 1. *The periods of mixed Tate motives over $\mathbb{Z}$ lie in $\mathcal{Z}[2i\pi^{-1}]$.*

An obvious question is whether there is a vector space (or algebra) basis for $\mathcal{Z}$ over $\mathbb{Q}$, and one can try to write down a conjectural basis in each weight and low depth. There were good reasons for thinking (see §3.2) that such a basis might have consisted of the set of $\zeta(n_1, \ldots, n_r)\pi^{2m}$, where all n_i are odd. This approach is quickly scuppered by the existence of exceptional relations such as

$$28\,\zeta(3,9) + 150\,\zeta(5,7) + 168\,\zeta(7,5) = \frac{5197}{691}\zeta(12) \,. \tag{1.2}$$

It is the first in an infinite series of identities amongst double zetas which were discovered by Gangl, Kaneko and Zagier, and are related to the period polynomials for cusp forms of weight k for $\mathrm{PSL}(2,\mathbb{Z})$. This is the first inkling of the shadow cast by the world of elliptic motives on the multiple zeta values. In order to circumvent this problem, one can instead try to find a conjectural basis in *high* depth, and indeed this had previously done by M. Hoffman in 1997, who conjectured the following theorem:

Theorem 2. *Every multiple zeta value of weight N is a $\mathbb{Q}$-linear combination of*

$$\{\zeta(a_1, \ldots, a_r) : \text{ where } a_i = 2 \text{ or } 3 \,, \ \text{ and } a_1 + \ldots + a_r = N\} \,.$$

Theorems 1 and 2 are proved simultaneously using motivic multiple zeta values, which are closely related to the motivic fundamental group of $\mathbb{P}^1\backslash\{0,1,\infty\}$. Deligne has recently proved analogous results for $\mathbb{P}^1\backslash\{0,\mu_N,\infty\}$, where μ_N is the group of N^{th} roots of unity and $N = 2,3,4,6,8$. The situation is rather different, since for these exceptional values of N, exotic relations such as (1.2) do not arise.

2. MZV'S AS PERIODS

In order to see why multiple zeta values are periods, consider the following example:

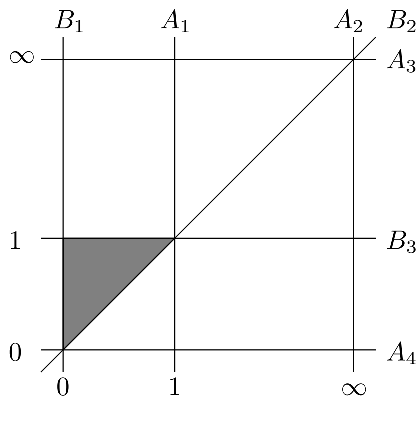
$$\zeta(2) = \int_{0\leq t_1 \leq t_2 \leq 1} \frac{dt_1}{1-t_2}\frac{dt_2}{t_2} \qquad \text{(Leibniz)} \tag{2.1}$$

It is a period of the moduli space of genus 0 curves with 5 marked points:

$$\mathfrak{M}_{0,5} = \big(\mathbb{P}^1\backslash\{0,1,\infty\} \times \mathbb{P}^1\backslash\{0,1,\infty\}\big)\backslash\Delta$$

Date: 29 June 2011.

where Δ denotes the diagonal. Its real points are pictured here.



Let

$$\omega = \frac{dt_1}{1-t_1}\frac{dt_2}{t_2} \in \Omega^2(\mathfrak{M}_{0,5}) ,$$

which has singularities contained in $\mathcal{A} = \bigcup_{i=1}^4 A_i$, and let

$$X = \{0 \leq t_1 \leq t_2 \leq 1\} \subset \mathbb{P}^1 \times \mathbb{P}^1$$

whose boundary is contained in $\mathcal{B} = \bigcup_{i=1}^3 B_i$. They define (co-)homology classes:

$$\begin{array}{rcl} [\omega] & \in & H^2_{DR}(\mathbb{P}^1 \times \mathbb{P}^1 \backslash \mathcal{A}) \\ [X] & \in & H_{2,B}(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{B}) \cong H^2(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{B})^\vee \end{array}$$

As a first approximation, one would like to consider the motive $H^2(\mathbb{P}^1 \times \mathbb{P}^1 \backslash \mathcal{A}, \mathcal{B}\backslash \mathcal{B} \cap \mathcal{A})$ which is of mixed Tate type. However, for technical reasons (the boundary of X meets the boundary of $\mathcal{A}$ at the points $(0,0)$ and $(1,1)$) this is not the correct object. Instead, one must consider

$$M = H^2(\overline{\mathfrak{M}}_{0,5} \backslash \mathcal{A}', \mathcal{B}' \backslash \mathcal{B}' \cap \mathcal{A}')$$

where $\overline{\mathfrak{M}}_{0,5}$ is the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ in (x,x), where $x = 0, 1, \infty$, and $\mathcal{A}', \mathcal{B}'$ are slightly larger sets of boundary divisors which include the exceptional divisors. One verifies this time that $[\omega] \in M_{DR}$ and $[X] \in M_B^\vee$. Thus the integral (2.1), and hence the number $\zeta(2)$, is a period of M.

Idea 1: *Replace the number $\zeta(2) \in \mathbb{R}$ with the triple $\zeta^{\mathfrak{m}}(2) \overset{\text{def}}{=} (M, [\omega], [X])$, or 'motivic $\zeta(2)$'. The period $\zeta(2)$ can be retrieved from this data simply by integrating $[\omega]$ over $[X]$, by (2.1)*

2.1. **Generalization.** In general, let $\varepsilon_1, \ldots, \varepsilon_n \in \{0,1\}$, where $\varepsilon_1 = 1$ and $\varepsilon_n = 0$. Let

$$I(0; \varepsilon_1, \ldots, \varepsilon_n; 1) = \int_{0 \leq t_1 \leq \ldots \leq t_n \leq 1} \frac{dt_1}{t_1 - \varepsilon_1} \ldots \frac{dt_n}{t_n - \varepsilon_n} . \tag{2.2}$$

It was observed by Kontsevich that (recall $n_r \geq 2$):

$$\zeta(n_1, \ldots, n_r) = (-1)^r I(0; 10^{n_1-1} 10^{n_2-1} \ldots 10^{n_r-1}; 1)$$

where 0^k denotes a sequence of k consecutive zeros. If $N = n_1 + \ldots + n_r$, then Goncharov and Manin showed as a consequence that (2.2) is a period of $H^{N+3}(\overline{\mathfrak{M}}_{0,N+3} \backslash A, B \backslash (A \cap B))$, where A, B are unions of distinct boundary divisors of $\overline{\mathfrak{M}}_{0,N+3}$, the Deligne-Mumford compactification of $\mathfrak{M}_{0,N+3}$, and furthermore that this defines an element in the category $MT(\mathbb{Z})$ of mixed Tate motives over $\mathbb{Z}$.

2.2. **Regularization.** One can define $I(\varepsilon_0; \varepsilon_1, \ldots, \varepsilon_n; \varepsilon_{n+1})$ for any $\varepsilon_i \in \{0,1\}$, where the integral (2.2) formally diverges. It is easily expressible as a $\mathbb{Z}$-linear combination of multiple zeta values (1.1).

3. MIXED TATE MOTIVES

3.1. **Structure of $MT(\mathbb{Z})$.** Let $MT(\mathbb{Z})$ denote the category of mixed Tate motives over $\mathbb{Z}$. It is an abelian tensor category whose simple objects are the Tate motives $\mathbb{Q}(n)$, indexed by $n \in \mathbb{Z}$, and which have weight $-2n$. The structure of $MT(\mathbb{Z})$ is determined by the data:

$$\mathrm{Ext}^1_{MT(\mathbb{Z})}(\mathbb{Q}(0), \mathbb{Q}(n)) \cong \begin{cases} \mathbb{Q} & \text{if } n \geq 3 \text{ is odd ,} \\ 0 & \text{otherwise ,} \end{cases} \tag{3.1}$$

and the fact that the Ext^2's vanish. The dimensions (3.1) come from Borel's computation of the ranks of the rational algebraic K-theory of $\mathbb{Q}$. A better way to think about it is to observe that $MT(\mathbb{Z})$ is a Tannakian category with a canonical fiber functor. Thus $MT(\mathbb{Z})$ is equivalent to the category of representations of an affine group scheme $\mathcal{G}_{MT}$ over $\mathbb{Q}$, which is a semi-direct product

$$\mathcal{G}_{MT} \cong \mathcal{G}_{\mathcal{U}} \rtimes \mathbb{G}_m \ , \tag{3.2}$$

where $\mathcal{G}_{\mathcal{U}}$ is the prounipotent algebraic group over $\mathbb{Q}$ whose Lie algebra is the free Lie algebra with one generator σ_{2n+1} in degree $-(2n+1)$. The generators correspond to (3.1), and the freeness follows from the vanishing of the Ext^2's. A variant of Idea 1 is the following rough statement:

Idea 2: *A period, e.g. a multiple zeta value, defines a function on $\mathcal{G}_{\mathcal{U}}$.*

3.2. **Functions on the motivic Galois group.** Let $\mathcal{A}^{MT}$ denote the graded ring of affine functions on $\mathcal{G}_{\mathcal{U}}$ over $\mathbb{Q}$. It is a commutative graded Hopf algebra. It follows from the remarks above that $\mathcal{A}^{MT}$ is non-canonically isomorphic to the cofree Hopf algebra on cogenerators f_{2r+1} in degree $2r+1 \geq 3$:

$$\mathcal{A}^{MT} \cong \mathbb{Q}\langle f_3, f_5, f_7, \ldots \rangle \ .$$

This has a basis consisting of non-commutative words in the f_{odd}'s. The multiplication on $\mathcal{A}^{MT}$ is given by the shuffle product, and the coproduct $\Delta : \mathcal{A}^{MT} \to \mathcal{A}^{MT} \otimes_{\mathbb{Q}} \mathcal{A}^{MT}$ is given by deconcatenation:

$$\Delta(f_{i_1} \ldots f_{i_r}) = \sum_{k=0}^{r} f_{i_1} \ldots f_{i_k} \otimes f_{i_{k+1}} \ldots f_{i_r}$$

Define the following trivial comodule over $\mathcal{A}^{MT}$:

$$\mathcal{H}^{MT_+} = \mathcal{A}^{MT} \otimes_{\mathbb{Q}} \mathbb{Q}[f_2] \ , \tag{3.3}$$

where f_2 is of degree 2, and commutes with the f_{odd}. We also write the coaction:

$$\Delta : \mathcal{H}^{MT_+} \longrightarrow \mathcal{A}^{MT} \otimes_{\mathbb{Q}} \mathcal{H}^{MT_+} \ .$$

It is determined by its restriction to $\mathcal{A}^{MT}$ and the formula $\Delta(f_2) = 1 \otimes f_2$. If we set $d_k = \dim_{\mathbb{Q}} \mathcal{H}^{MT_+}_k$, then one readily verifies that $d_0 = 1, d_1 = 0, d_2 = 1$, and

$$d_k = d_{k-2} + d_{k-3} \text{ for } k \geq 3 \ . \tag{3.4}$$

Here, a subscript (e.g. $\mathcal{Z}_k, \mathcal{H}^{MT_+}_k$) denotes the subspace spanned by elements of weight k.

Example 3. $\mathcal{H}^{MT_+}_8$ is of dimension 4, spanned by $f_5 f_3$, $f_3 f_5$, $f_3^2 f_2$, and f_2^4. Compare the space $\mathcal{Z}_8$ of MZV's of weight 8, which is spanned by $\zeta(3,5)$, $\zeta(3)\zeta(5)$, $\zeta(3)^2\zeta(2)$, and $\zeta(8)$.

4. MOTIVIC MULTIPLE ZETA VALUES

The idea is to lift the multiple zeta values $\zeta(n_1, \ldots, n_r)$ to motivic versions $\zeta^{\mathfrak{m}}(n_1, \ldots, n_r)$, in such a way that the standard relations hold. Using the theory of the motivic fundamental group of $\mathbb{P}^1 \backslash \{0, 1, \infty\}$, one can show that there exists a graded algebra $\mathcal{H}$, generated by elements

$$I^{\mathfrak{m}}(\varepsilon_0; \varepsilon_1, \ldots, \varepsilon_n; \varepsilon_{n+1}) \in \mathcal{H} \quad \text{for all} \quad \varepsilon_0, \ldots, \varepsilon_{n+1} \in \{0, 1\} \ ,$$

which we call motivic iterated integrals, such that all the usual properties of iterated integrals hold (shuffle product, reflection formulae, etc). There is a well-defined map called the *period*,

$$\begin{aligned} per : \mathcal{H} &\rightarrow \mathbb{R} \\ I^{\mathfrak{m}}(\varepsilon_0; \varepsilon_1, \ldots, \varepsilon_n; \varepsilon_{n+1}) &\rightarrow I^{\mathfrak{m}}(\varepsilon_0; \varepsilon_1, \ldots, \varepsilon_n; \varepsilon_{n+1}) \,. \end{aligned}$$

We define the *motivic multiple zeta value* to be $\zeta^{\mathfrak{m}}(n_1, \ldots, n_r) = (-1)^r I^{\mathfrak{m}}(0; 10^{n_1-1} \ldots 10^{n_r-1}; 1)$. Its period is $\zeta(n_1, \ldots, n_r)$. The coalgebra $\mathcal{H}$ admits a coaction of $\mathcal{A}^{MT}$ which we describe in §5 .

Proposition 4. *There is a non-canonical embedding of algebra-comodules over $\mathcal{A}^{MT}$*

$$\mathcal{H} \hookrightarrow \mathcal{H}^{MT_+} \tag{4.1}$$

which maps $\zeta^{\mathfrak{m}}(2)$ to f_2, and $\zeta^{\mathfrak{m}}(2n+1)$ to f_{2n+1} for all $n \geq 1$.

The motivic formalism is very powerful. For instance, the proposition immediately implies that

$$\dim_{\mathbb{Q}} \mathcal{Z}_k \leq \dim_{\mathbb{Q}} \mathcal{H}_k \leq \dim_{\mathbb{Q}} \mathcal{H}_k^{MT_+} = d_k$$

where the numbers d_k are defined by (3.4). This upper bound was first proved independently by Goncharov and Terasoma, proving one half of Zagier's conjecture, which states that $\dim_{\mathbb{Q}} \mathcal{Z}_k = d_k$.

5. The coaction

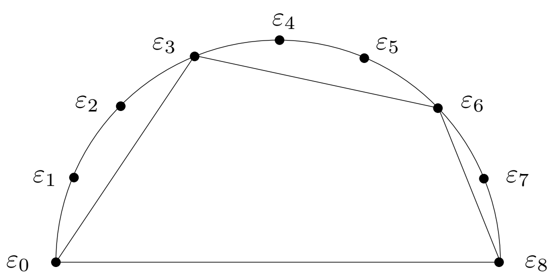
What we gain in passing to motivic multiple zeta values is the coaction of $\mathcal{A}^{MT}$. Let $\mathcal{A} = \mathcal{H}/\zeta^{\mathfrak{m}}(2)$, and denote the quotient map by $\pi : \mathcal{H} \rightarrow \mathcal{A}$. The following formula is a refinement of a formula due to Goncharov, which is in turn dual to a formula computed by Y. Ihara many years earlier.

Proposition 5. *The coaction $\Delta : \mathcal{H} \rightarrow \mathcal{A} \otimes_{\mathbb{Q}} \mathcal{H}$ can be computed explicitly as follows. For any $a_0, \ldots, a_{n+1} \in \{0,1\}$, the image of a generator $\Delta\, I^{\mathfrak{m}}(a_0; a_1, \ldots, a_n; a_{n+1})$ is given by*

$$\sum_{i_0 < i_1 < \ldots < i_k < i_{k+1}} \pi\Big(\prod_{p=0}^{k} I^{\mathfrak{m}}(a_{i_p}; a_{i_p+1}, . . , a_{i_{p+1}-1}; a_{i_{p+1}}) \Big) \otimes I^{\mathfrak{m}}(a_0; a_{i_1}, . . , a_{i_k}; a_{n+1}) \tag{5.1}$$

where the sum is over indices satisfying $i_0 = 0$ and $i_{k+1} = n+1$, and all $0 \leq k \leq n$. The left-hand side of the coproduct is viewed modulo $\zeta^{\mathfrak{m}}(2)$. Note that $I^{\mathfrak{m}}(a; b)$ is defined to be 1 for all $a, b \in \{0,1\}$.

The following diagram illustrates a typical term in the formula:



$$\pi\big(I^{\mathfrak{m}}(\varepsilon_0; \varepsilon_1, \varepsilon_2; \varepsilon_3) I^{\mathfrak{m}}(\varepsilon_3; \varepsilon_4, \varepsilon_5; \varepsilon_6) I^{\mathfrak{m}}(\varepsilon_6; \varepsilon_7; \varepsilon_8)\big) \otimes I^{\mathfrak{m}}(\varepsilon_0; \varepsilon_3, \varepsilon_6; \varepsilon_8)$$

6. The Hoffman Basis

Main Theorem 6.1. *The following elements are linearly independent:*

$$\{\zeta^{\mathfrak{m}}(a_1, \ldots, a_r),\ \textit{where } a_i = 2 \textit{ or } 3\} \subset \mathcal{H} \,. \tag{6.1}$$

Let $\mathcal{H}^{2,3}$ denote the $\mathbb{Q}$-linear span of the elements (6.1). We have

$$\mathcal{H}^{2,3} \subseteq \mathcal{H} \subseteq \mathcal{H}^{MT_+} \tag{6.2}$$

The main theorem implies that

$$\dim_{\mathbb{Q}} \mathcal{H}^{2,3}_N = \#\{(a_1,\ldots,a_r) : a_i = 2 \text{ or } 3 \text{ and } a_1+\ldots+a_r = N\}$$

The number on the right-hand side is clearly equal to the integer $d_N = \dim_{\mathbb{Q}} \mathcal{H}^{MT_+}_N$, by (3.4). It follows that $\dim_{\mathbb{Q}} \mathcal{H}^{2,3}_N = \dim_{\mathbb{Q}} \mathcal{H}^{MT_+}_N$ and we have equalities in (6.2). There are two consequences:

Corollary 6. $\mathcal{H}^{2,3} = \mathcal{H}$. *In other words, every motivic multiple zeta value* $\zeta^{\mathfrak{m}}(n_1,\ldots,n_r)$ *is a* $\mathbb{Q}$*-linear combination of elements of the form* (6.1) *with indices* 2 *or* 3*.*

By taking the period map, this implies that every multiple zeta value is a $\mathbb{Q}$-linear combination of Hoffman elements, and hence implies theorem 2.

Corollary 7. $\mathcal{H} = \mathcal{H}^{MT_+}$.

Equivalently, the category of mixed Tate motives over $\mathbb{Z}$ is spanned by the motivic fundamental group of $\mathbb{P}^1\backslash\{0,1,\infty\}$, as conjectured by Deligne and Ihara. On taking the period map, it implies theorem 1.

7. Some ideas of the proof of the main theorem

(1) Introduce the *level filtration* on $\mathcal{H}^{2,3}$, defined by

$$F_\ell \mathcal{H}^{2,3} = \mathbb{Q}\langle \zeta^{\mathfrak{m}}(a_1,\ldots,a_n) : a_i = 2 \text{ or } 3, \text{ and at most } \ell \text{ indices } a_i = 3\rangle \ .$$

The proof of the independence of (6.1) is by induction on the level.

(2) Surprisingly, the subspace $\mathcal{H}^{2,3}$, and the level filtration, are motivic. In other words:

$$\Delta : F_\ell \mathcal{H}^{2,3} \to \mathcal{A} \otimes_{\mathbb{Q}} F_\ell \mathcal{H}^{2,3} \ .$$

(3) The formula (5.1) for the coaction is unwieldy and complicated. It is considerably simplified if one passes to the infinitesimal coaction. For this, let $\mathcal{L} = \mathcal{A}_{>0}/\mathcal{A}_{>0}\mathcal{A}_{>0}$, and set

$$D : \mathcal{H} \xrightarrow{\Delta} \mathcal{A} \otimes_{\mathbb{Q}} \mathcal{H} \longrightarrow \mathcal{L} \otimes_{\mathbb{Q}} \mathcal{H} \tag{7.1}$$

(4) Analyze what happens in levels 0 and 1. In level 0, we have $F_0\mathcal{H}^{2,3} = \{\zeta^{\mathfrak{m}}(2,\ldots,2)\}$. One shows that $\zeta^{\mathfrak{m}}(\underbrace{2,\ldots,2}_{n}) = \frac{6^n}{(n+1)!}\zeta^{\mathfrak{m}}(2)^n$ and so maps to 0 in $\mathcal{A}$. In level 1, the elements are

$$F_1\mathcal{H}^{2,3} = \{\zeta^{\mathfrak{m}}(\underbrace{2,\ldots,2}_{a},3,\underbrace{2,\ldots,2}_{b})\}$$

One shows that there exist constants $c_{a,b} \in \mathbb{Q}$ and $\alpha_i \in \mathbb{Q}$ such that

$$\zeta^{\mathfrak{m}}(\underbrace{2,\ldots,2}_{a},3,\underbrace{2,\ldots,2}_{b}) = c_{a,b}\zeta^{\mathfrak{m}}(2a+2b+3) + \sum_{i=1}^{a+b} \alpha_i \zeta^{\mathfrak{m}}(2i+1)\zeta^{\mathfrak{m}}(\underbrace{2,\ldots,2}_{2(a+b+1-i)})$$

The *coefficients* $c_{a,b}$ have to be computed analytically: see §8 for this part of the story.

(5) Look at the infinitesimal coaction on the associated graded of $\mathcal{H}^{2,3}$ for the level filtration. In each weight N, and level ℓ, (7.1) defines an operator *which lowers the level*:

$$D_{N,\ell} : \mathrm{gr}^F_\ell \, \mathcal{H}^{2,3}_N \longrightarrow \bigoplus_{i\geq 1} \mathrm{gr}^F_{\ell-1} \, \mathcal{H}^{2,3}_{N-2i-1}$$

The bulk of the work consists in showing that $D_{N,\ell}$ is injective. This follows from 2-adic properties of the coefficients $c_{a,b}$, which follow from Zagier's theorem. Theorem 6.1 follows from the injectivity of the $D_{N,\ell}$: take a non-trivial relation between the elements (6.1) which is of minimal level. Applying $D_{N,\ell}$ gives a non-trivial relation of smaller level, which gives a contradiction.

8. ZAGIER'S THEOREM

To see why the coaction alone is insufficient to determine the full structure of the motivic multiple zeta values, consider the following example in weight 5. The vector space $\mathcal{H}_5^{MT_+}$ is spanned by two elements: f_5, and $f_3 f_2$, and likewise $\mathcal{H}_5$ is also of dimension two, spanned by $\zeta^{\mathfrak{m}}(5)$ and $\zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(2)$. The Hoffman elements of weight 5 are $\zeta^{\mathfrak{m}}(2,3)$ and $\zeta^{\mathfrak{m}}(3,2)$, so we know that

$$\begin{aligned}
\zeta^{\mathfrak{m}}(3,2) &= c_{32}\,\zeta^{\mathfrak{m}}(5) + d_{32}\,\zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(2)\\
\zeta^{\mathfrak{m}}(2,3) &= c_{23}\,\zeta^{\mathfrak{m}}(5) + d_{23}\,\zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(2)
\end{aligned}$$

for some coefficients $c_{23}, c_{32}, d_{23}, d_{32} \in \mathbb{Q}$. The coaction tells us that $d_{23} = 3$, $d_{32} = -2$ but gives us no information about the coefficients c_{23}, c_{32}. They can be computed by taking a regulator map.

Thus to determine c_{23}, for example, take the period map, which gives:

$$c_{23} = \frac{\zeta(2,3) - 3\,\zeta(3)\zeta(2)}{\zeta(5)} = \frac{-11}{2}\ .$$

By a similar computation, one can show that $c_{32} = 9/2$. The injectivity of $D_{5,1}$ in this case is equivalent to the invertibility of the following matrix:

$$\begin{pmatrix} c_{32} & d_{32} \\ c_{23} & d_{23} \end{pmatrix} = \begin{pmatrix} \frac{9}{2} & -2 \\ \frac{-11}{2} & 3 \end{pmatrix}$$

8.1. **Zagier's theorem.** The missing ingredient is provided by the following theorem.

Theorem 8. *(Don Zagier 2010). Let $a, b \geq 0$. Then*

$$\zeta(\underbrace{2,\ldots,2}_{a},3,\underbrace{2,\ldots,2}_{b}) = 2\sum_{r=1}^{a+b+1}(-1)^r\left(\binom{2r}{2a+2} - \left(1-2^{-2r}\right)\binom{2r}{2b+1}\right)\zeta(2r+1)\,\zeta(\underbrace{2,\ldots,2}_{a+b+1-r})\ .$$

His proof is quite remarkable. First he defines the two generating series

$$\begin{aligned}
F(x,y) &= \sum_{a,\,b\geq 0}(-1)^{a+b+1}Z(a,b)\,x^{2a+2}\,y^{2b+1}\\
F^*(x,y) &= \sum_{a,\,b\geq 0}(-1)^{a+b+1}Z^*(a,b)\,x^{2a+2}y^{2b+1}\ ,
\end{aligned}$$

where $Z(a,b)$ denotes the left-hand side of the equation in theorem 8, and $Z^*(a,b)$ is the right-hand side. He then shows: first, that the generating function $F(x,y)$ can be expressed as the product of a sine function and the derivative of an ${}_3F_2$-hypergeometric function, and second, that the generating function $F^*(x,y)$ is a linear combination of fourteen functions which are products of the sine function and a digamma function. These two expressions are seemingly totally unrelated. Nevertheless, he shows that

- $F(x,x) = F^*(x,x)$ for all $x \in \mathbb{C}$
- $F(n,y) = F^*(n,y)$ for all $n \in \mathbb{N}$ and $y \in \mathbb{C}$
- $F(x,n) = F^*(x,n)$ for all $n \in \mathbb{N}$ and $x \in \mathbb{C}$

The last two identities are completely non-trivial. Finally, by bounding the growth of the functions $F(x,y)$ and $F^*(x,y)$, one can show using a variant of the Phragmén-Lindelöf theorem that these properties are enough to conclude that $F(x,y) = F^*(x,y)$ for all $x, y \in \mathbb{C}$, which completes the proof.

CONES + SIGNATURES

OLD FORMULAE REVISITED

1968 ZAGIER MIT $\to$ OXFORD

INDEX THEORY

EQUIVARIANT SIGNATURE THEOREM

G FINITE GROUP ACTING ON 4R-dim COMPACT ORIENTED MANIFOLD X'

$$H^{2R}(X', \mathbb{R}) = H_+ \oplus H_- \quad \text{(G-MODULES)}$$

$$\operatorname{SIGN}(g, X) = \operatorname{Trace}(g|H_+) - \operatorname{Trace}(g|H_-)$$

FOR $g = 1$ HIRZEBRUCH FORMULA IN TERMS OF PONTRJAGIN CLASSES

FOR $g \neq 1$ FORMULA INVOLVES FIXED-POINT SET OF g

$X = X'/G$ RATIONAL HOMOLOGY MANIFOLD

$H^*(X'/G) = H^*(X')^G$ (INVARIANTS)

FORMULA FOR SIGN(X'/G) INVOLVING FIXED-POINTS

VIEW AS HIRZEBRUCH FORMULA FOR X

+ CORRECTIONS DUE TO SINGULARITIES (FROM FIXED-POINTS)

"SIGNATURE DEFECTS"

dim $X' = 4$

ISOLATED FIXED-POINT (FOR ALL $g \neq 1$)

SIGNATURE DEFECT IS (ESSENTIALLY)

DEDEKIND SUM $\sum_{k=1}^{N-1} \cot \frac{\pi k}{N} \cdot \cot \frac{\pi q k}{N}$ $(q, N) = 1$, $g^N = 1$

⇒ INTERESTING CONNECTIONS

BETWEEN GEOMETRY & NUMBER THEORY

HIRZEBRUCH "THE SIGNATURE THEOREM:

REMINISCENCES & RECREATION"

PROSPECTS IN MATHEMATICS

ANN. MATH. STUDY 70(1971) 3-31

1970 ZAGIER → BONN

⇒ COLLABORATION WITH

HIRZEBRUCH

MORE NUMBER THEORY

FOCUS ON dim 4 NO ISOLATED POINTS

ONLY FIXED SURFACE $Y' \subset X'$

G CYCLIC ORDER N

$X' \to X$ N-FOLD BRANCHED COVER

BRANCH LOCUS $Y' = Y \subset X$

$X = X'/G$ IS TOPOLOGICALLY A

MANIFOLD

Co-dim 2 SITUATION PARALLELS

dim 2 RIEMANN SURFACES

SIGNATURE FORMULA

$$* \quad \boxed{\mathrm{SIGN}(X) = \frac{1}{N}\,\mathrm{SIGN}(X') + \left(\frac{N^2-1}{3N^2}\right) Y^2}$$

(SELF. INTERSECTION OF Y IN X)

COMPARE MORE ELEMENTARY FORMULA FOR EULER NUMBER χ

$$\boxed{\chi(X) = \frac{1}{N}\,\chi(X') + \left(\frac{N-1}{N}\right)\chi(Y)}$$

(COUNT SIMPLICES)

GENERAL G-SIGNATURE THEOREM GIVES FORMULA FOR "DEFECT" OF Y AS

$$\left\{ \sum_{k=1}^{N-1} \frac{1}{\sin^2\left(\frac{\pi k}{N}\right)} \right\} \frac{Y^2}{N^2}$$

"EASY TO VERIFY" $\{ \quad \} = \dfrac{N^2-1}{3}$ (†)

SLICK PROOF (F.H)

APPLY G-SIGNATURE THEOREM TO G CYCLIC ORDER N ACTING ON X'_N

$$Z_0^N + Z_1^N + Z_2^N + Z_3^N = 0 \quad (\text{IN } CP_3)$$

BY $Z_0 \to \zeta^{-1} Z_0 \quad Z_j \to Z_j \quad (j \neq 0)$

$$\zeta = \exp\left(\frac{2\pi i}{N}\right)$$

$X = X'_N / G = CP_2 \qquad Y' = Y \{ Z_0 = 0, Z_1^2 + Z_2^2 + Z_3^2 = 0 \}$

$\text{SIGN}(X) = 1 \qquad Y^2 = N^2$

$\text{SIGN}(X'_N) = \dfrac{N(4-N^2)}{3}$ (USE CHERN CLASSES)

VERIFIES * $\quad 1 = \dfrac{4-N^2}{3} + \dfrac{(N^2-1)}{3N^2} \cdot N^2$

PROVES (†)

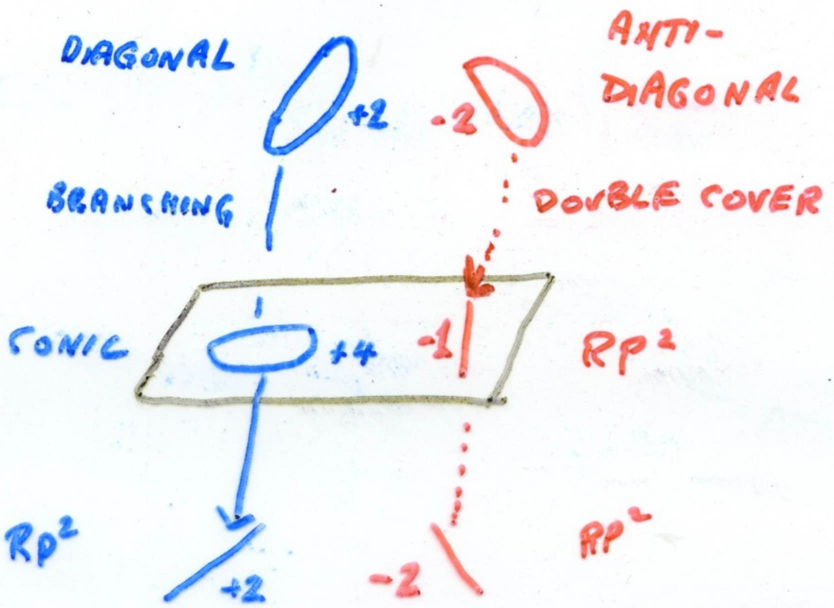
NOTE $Y \subset X$ CAN BE NON-ORIENTABLE

BUT Y^2 STILL DEFINED (LOCAL COEFFTS)

& FORMULA * STILL HOLDS

EXAMPLE

$$S^2 \times S^2 \xrightarrow{\text{SYMMETRIZE}} CP_2 \xrightarrow{\text{DIVIDE BY COMPLEX CONJUGATION}} S^4$$



$$\boxed{\begin{gathered} Y = RP^2 \subset X = CP_2 \\ Y^2 = -1 \end{gathered}}$$

3x3 REAL SYMMETRIC MATRICES A Trace $A = 0$

Trace $A^2 = 1$

EIGENVALUES

$\lambda_1 \leq \lambda_2 \leq \lambda_3$

$\sum \lambda_i = 0 \quad \sum \lambda_i^2 = 1$

PARAMETRIZE $SO(3)$-ORBITS

GENERIC ORBIT dim 3

2 SPECIAL ORBITS dim 2 RP^2

$\lambda_1 = \lambda_2$ OR $\lambda_2 = \lambda_3$

CHECK FORMULAE FOR SIGN , χ

1) $S^2 \times S^2 \longrightarrow CP_2$

$$\mathrm{SIGN}(CP_2) = \frac{1}{2}\,\mathrm{SIGN}(S^2 \times S^2) + \frac{4-1}{3.4} \cdot 4$$

$$1 = 0 + 1$$

$$\chi(CP_2) = \frac{1}{2}\,\chi(S^2 \times S^2) + \frac{1}{2} \cdot 2$$

$$3 = 2 + 1$$

2) $CP_2 \longrightarrow S^4$

$$\mathrm{SIGN}(S^4) = \frac{1}{2}\,\mathrm{SIGN}(CP_2) + \frac{4-1}{3.4}(-2)$$

$$0 = \frac{1}{2} - \frac{1}{2}$$

$$\chi(S^4) = \frac{1}{2}\,\chi(CP_2) + \frac{1}{2} \cdot 1$$

$$2 = \frac{3}{2} + \frac{1}{2}$$

DIFFERENTIAL GEOMETRY

$\dim X = 4$ $\quad$ $\text{SIGN}(X) = \frac{1}{3}\int_X p_1$

p_1 PONTRJAGIN FORM

$\chi(X) = \int_X e$ $\quad$ e EULER FORM

FOR X HAVING CONICAL SINGULARITY

ANGLE $2\pi/N$ ALONG Y

$$\boxed{\text{SIGN}(X) = \frac{1}{3}\int_X p_1 + \frac{1}{3}\left(1 - \frac{1}{N^2}\right) Y^2}$$

$$\boxed{\chi(X) = \int_X e + \left(1 - \frac{1}{N}\right)\chi(Y)}$$

MAKES SENSE & HOLDS EVEN WHEN

X IS NOT GLOBALLY A QUOTIENT X/G.

NEEDS INDEX THEORY FOR ORBIFOLDS

RIEMANN CURVATURE

Riem. = Ricci + Weyl — (conformal invariant)

$\quad\quad\quad\quad\ \ \parallel$

$\quad\quad$ Scalar + Ricci$'$

EINSTEIN CONDITION $\quad$ Ricci$' = 0$

(Ricci = Scalar)

dim 4 $\quad W = W^+ \oplus W^-$

SELF-DUAL $\quad$ ANTI-SELF-DUAL

FOR EINSTEIN 4-MANIFOLD

$$e = \frac{1}{8\pi^2} \|\text{Riem}\|^2$$

$$\frac{p_1}{3} = \frac{1}{4\pi^2}\left\{ |W^+|^2 - |W^-|^2 \right\}$$

SELF-DUAL IF $W^- = 0$

$$\Rightarrow \frac{p_1}{3} = \frac{1}{4\pi^2} |W|^2$$

FOR X dim 4, EINSTEIN SELF-DUAL CONICAL ALONG Y ANGLE $2\pi/N$

$$\boxed{\operatorname{SIGN}(X) = \frac{1}{4\pi^2}\|W\|^2 + \frac{1}{3}\left(1-\frac{1}{N^2}\right)Y^2}$$

$$\boxed{\chi(X) = \frac{1}{8\pi^2}\|\mathrm{Riem}\|^2 + \left(1-\frac{1}{N}\right)\chi(Y)}$$

HITCHIN MANIFOLDS

SEQUENCE $H(N)$ dim 4 COMPACT

SELF-DUAL EINSTEIN

DEFINED ON S^4 CONICAL ALONG RP^2 ANGLE $2\pi/N$

$N=1$ $\quad H(1) = S^4$ STANDARD METRIC

$N=2$ $\quad H(1)$ DOUBLE COVERED BY CP^2 BRANCHED ALONG RP^2 FUBINI-STUDY METRIC

FOR $N \geq 3$ ONLY <u>ORBIFOLDS</u>

<u>NOTE</u> THEOREM COMPACT SELF-DUAL EINSTEIN OF POSITIVE SCALAR CURVATURE $= S^4$ OR CP^2

USE GEOMETRIC FORMULAE FOR SIGN$(H(N))$ AND $\chi(H(N))$ TO COMPUTE
$\|Riem\|^2$ AND $\|W\|^2$

TOPOLOGICALLY $H(N) = S^4$ ALL N

$\chi = 2$ SIGN $= 0$

AND $Y = RP^2$ $Y^2 = -2$ $\chi(Y) = 1$

$$\|W(N)\|^2 = \frac{8\pi^2}{3}\left(1 - \frac{1}{N^2}\right)$$

$$\|Riem(N)\|^2 = 8\pi^2\left(1 + \frac{1}{N}\right)$$

<u>LIMIT $N \to \infty$</u> (SOME SUBTLE POINTS)

$H(N) \longrightarrow H(\infty) =$ ATIYAH-HITCHIN METRIC ON $S^4 - RP^2$

SO(3)-INVARIANT

$$\|W(\infty)\|^2 = \frac{8\pi^2}{3} \qquad \|Riem(\infty)\|^2 = 8\pi^2$$

MONOPOLE MODULI SPACES

A_H = MODULI SPACE OF (CENTRED)

SU(2) MONOPOLES OF CHARGE 2 ON $\mathbb{R}^3$

[WITH NATURAL HYPERKAHLER METRIC

? $H(N)$ = MODULI SPACE OF MONOPOLES ON

HYPERBOLIC 3-SPACE OF

CURVATURE $\left(-\frac{4}{N^2}\right)$

NATURAL METRIC DIVERGES

HITCHIN METRIC ?

(NOT UNDERSTOOD)

MONOPOLES IN $\mathbb{R}^3$ PARAMETRIZED

BY SPECTRAL CURVE Γ ON

QUADRIC CONE

(CHARG 2 $\Rightarrow$ ELLIPTIC CURVE)

HYPERBOLIC MONOPOLES $\rightarrow$ SPECTRAL

CURVE Γ_N ON NON-SINGULAR QUADRIC

(CHARG 2 $\Rightarrow$ ELLIPTIC CURVE)

CONSTRAINT ON Γ_N

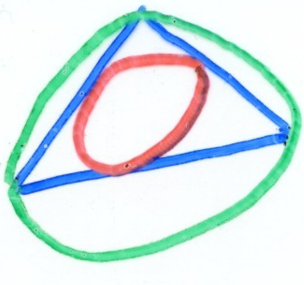
D_1 , D_2 DIVISORS (OF DEG 2) CUT OUT BY P_1 FACTORS OF QUADRIC

CONSTRAINT

$$(2N+2)(D_1 - D_2) \sim 0$$

$\Longrightarrow$ PONCELET

POLYGONS INSCRIBED IN ONE CONIC & CIRCUMSCRIBED TO ANOTHER



N. J. HITCHIN A NEW FAMILY OF EINSTEIN METRICS

MANIFOLDS + GEOMETRY PISA 1993

(190-222) Sympos. Math. XXXVI

CUP 1996

M. F. ATIYAH MAGNETIC MONOPOLES IN HYPERBOLIC SPACE

PROC. BOMBAY COLL. 1984 ON VECTOR BUNDLES ON ALGEBRAIC VARIETIES

OUP (1987) 1-34

Average sizes of Selmer groups and ranks of elliptic curves

Manjul Bhargava
Princeton University

Arbeitstagung for Don
MPI, Bonn

30 June 2011

Average rank?

Q: What is the rank of elliptic curves on average?

In order to ask this question more precisely, we need a natural way to measure the size of elliptic curves, so that we can order them by size.

We use the simplest such measure, called the naive height, which is basically a measure of the size of the coefficients of the defining equation of the elliptic curve.

A canonical representation of rational elliptic curves

To define the naive height, we use the following

Fact: Any elliptic curve E over $\mathbb{Q}$ is isomorphic to a cubic curve in the plane of the form

$$E_{A,B} : y^2 = x^3 + Ax + B.$$

In fact, any $E/\mathbb{Q}$ is isomorphic to a *unique* $E_{A,B}$ such that

$$\text{for all primes } p,\ \ p^4 \mid A \ \Rightarrow\ p^6 \nmid B.$$

The reason is: if $p^4 \mid A$ and $p^6 \mid B$, then $E_{A,B} \cong E_{A/p^4,B/p^6}$ via $x \mapsto p^2x'$ and $y \mapsto p^3y'$.

The height of an elliptic curve

Thus we have a canonical representation of any $E/\mathbb{Q}$ as

$$E_{A,B} : y^2 = x^3 + Ax + B.$$

We may thus define the height of E by the size of the coefficients of the defining equation.

If $E = E_{A,B}$, then $H(E_{A,B}) := \max\{4|A|^3, 27B^2\}$. This is called the (naive) height of E.

The naive height is essentially the exponential of what is called the "Faltings height".

Another related measure of the size of $E_{A,B}$ is called the discriminant $\Delta(E_{A,B}) := -4A^3 - 27B^2$.

Finally, there is a measure of size called the conductor $N(E)$ of E.

These various measures are conjectured to be about the same order of magnitude for all but a negligible proportion of elliptic curves!

Average rank

Q: If all elliptic curves over $\mathbb{Q}$ are ordered by their heights (or discriminants, etc.), what is the average size of the rank?

Conjecture (Goldfeld, Katz-Sarnak): 1/2. (*More precisely, one expects* 50% *of curves to have rank* 0, *and* 50% *to have rank* 1.)

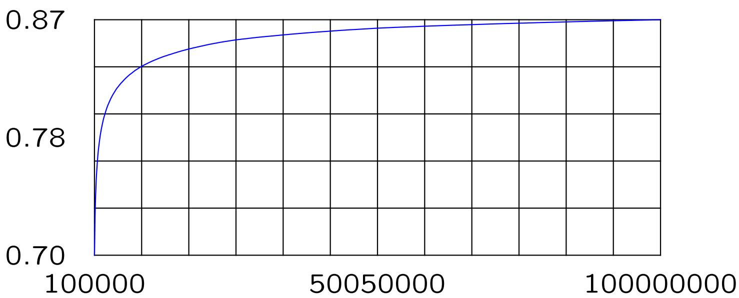
However, previously this average has not even been known to be finite (let alone 1/2)! (at least not unconditionally!)

Computations do not currently give much support to the conjecture either.

It was observed by Brumer and McGuinness in their 1990 computations that rank 2 curves seem to occur surprisingly often, and with *increasing* frequency! These computations were extended recently by Bektemirov, Stein, and Watkins:

All Curves Ordered By Conductor

The average rank of all curves of conductor $\leq 10^8$ is $0.8664\ldots$.
A graph of the average rank as a function:



We created this graph by computing the average rank of curves of conductor up to $n \cdot 10^5$ for $1 \leq n \leq 1000$.

35

A special family

In a well-known work, Zagier and Kramarz (1987) did rank computations in the family of elliptic curves $E_k : x^3 + y^3 = m$.

They found that this apparent overabundance of rank 2 and higher rank curves is even more pronounced in this family!

GRH + BSD

The first theoretical result, towards the boundedness of average rank of all elliptic curves, are due to Brumer.

In 1992, Brumer showed that the Generalized Riemann Hypothesis (GRH) and the Birch and Swinnerton-Dyer Conjecture (BSD) together imply that the average rank is bounded. (in fact, bounded by 2.3.)

In 2004, Heath-brown (still assuming GRH + BSD) improved this to average rank ≤ 2.0.

In 2009, Young further improved this (again assuming GRH + BSD) to $\leq \frac{25}{14} \approx 1.79$.

The main theorem

Theorem. *When elliptic curves $E/\mathbb{Q}$ are ordered by height, the average rank is bounded; in fact, it is bounded by* 1.5*.*

We prove something stronger, namely:

Theorem. *The same is true for the 2-Selmer rank, i.e., the average 2-Selmer rank is bounded by* 1.5*.*

Recall that the 2-Selmer group $S^{(2)}(E)$ of an elliptic curve $E/\mathbb{Q}$ fits into an exact sequence

$$0 \to E(\mathbb{Q})/2E(\mathbb{Q}) \to S^{(2)}(E) \to \text{Ш}_E[2] \to 0.$$

So $r_2(S^{(2)}(E)) = r_2(E(\mathbb{Q})[2]) + r_2(\text{Ш}_E[2]) + r(E) \leq 1.5$ on average.

We actually prove something even stronger, namely:

Theorem. *When elliptic curves $E/\mathbb{Q}$ are ordered by height, the average size of the 2-Selmer group $S^{(2)}(E)$ is exactly* 3*.*

The main theorem

Theorem. *When elliptic curves $E/\mathbb{Q}$ are ordered by height, the average rank is bounded; in fact, it is bounded by* 1.5*.*

We prove something stronger, namely:

Theorem. *The same is true for the 2-Selmer rank, i.e., the average 2-Selmer rank is bounded by* 1.5*.*

Recall that the 2-Selmer group $S^{(2)}(E)$ of an elliptic curve $E/\mathbb{Q}$ fits into an exact sequence

$$0 \to E(\mathbb{Q})/2E(\mathbb{Q}) \to S^{(2)}(E) \to \text{Ш}_E[2] \to 0.$$

So $r_2(S^{(2)}(E)) = r_2(E(\mathbb{Q})[2]) + r_2(\text{Ш}_E[2]) + r(E) \leq 1.5$ on average.

We actually prove something even stronger, namely:

Theorem. *When all elliptic curves $E/\mathbb{Q}$ in any family defined by finitely many congruence conditions are ordered by height, the average size of the 2-Selmer group $S^{(2)}(E)$ is exactly* 3*.*

Proof of theorem

To get a hold of 2-Selmer groups of elliptic curves, we use a correspondence between 2-Selmer elements and integral binary quartic forms, which was first introduced and used in the original computations of Birch and Swinnerton-Dyer.

To state the result, recall that the action of $\mathrm{GL}_2(\mathbb{Z})$ on binary quartic forms, by linear substitution of variable, has two independent polynomial invariants, traditionally denoted I and J, respectively. The invariant I has degree 2 and the invariant J has degree 3 in the coefficients of the binary quartic form.

Theorem. (Birch & Swinnerton-Dyer) *There is an injective map from $S^{(2)}(E_{A,B})$ to the set of $\mathrm{GL}_2(\mathbb{Z})$-equivalence classes of integral binary quartic forms having invariants $I = -2^4 \cdot 3 \cdot A$ and $J = -2^4 \cdot 3 \cdot B$.*

BSD's theorem yields an efficient method for rank computations of elliptic curves. This method has been further refined by Cremona, and implemented in his well-known `mwrank` program.

Counting binary forms

Disquisitiones Arithmeticae (1801)

Binary quadratic form:

$$Q(x, y) = ax^2 + bxy + cy^2 \;\; (a, b, c \in \mathbb{Z})$$

$\mathrm{SL}_2(\mathbb{Z})$ acts on the set of binary quadratic forms (by linear substitution).

$\mathrm{Disc}(Q) = b^2 - 4ac$. (unique SL_2-polynomial invariant)

It is known that there are only finitely many $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of binary quadratic forms with given value of discriminant D.

How many classes h_D are there with discriminant D, or with D at most X?

Theorem. (Gauss 1801/Mertens 1874/Siegel 1944)

$$\sum_{-X<D<0} h_D \sim \frac{\pi}{18} \cdot X^{3/2}; \qquad \sum_{0<D<X} h_D \log \epsilon_D \sim \frac{\pi^2}{18} \cdot X^{3/2}.$$

Counting binary forms: cubic forms

The next natural case is that of binary cubic forms $f(x,y) = ax^3 + bx^2y + cxy^2 + dy^3, \ a,b,c,d \in \mathbb{Z}$.

$\mathrm{GL}_2(\mathbb{Z})$ acts naturally on such forms.

There is again just one polynomial invariant for this action, namely the discriminant $\mathrm{Disc}(f)$ of f, given by

$$\mathrm{Disc}(f) = b^2c^2 + 18abcd - 4ac^3 - 4b^3d - 27a^2d^2.$$

As before there exist only finitely many $\mathrm{GL}_2(\mathbb{Z})$-equivalence classes of binary cubic forms with given value of discriminant D.

How many classes $h(D)$ of irreducible binary cubic forms are there with discriminant D, or with D at most X?

Theorem. (Davenport 1951)

$$\sum_{-X<D<0} h(D) \sim \frac{\pi^2}{24}\cdot X; \qquad \sum_{0<D<X} h(D) \sim \frac{\pi^2}{72}\cdot X.$$

Counting binary forms: quartic forms

The next natural case is that of binary quartic forms $f(x,y) = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4, \ a,b,c,d,e \in \mathbb{Z}$.

$\mathrm{GL}_2(\mathbb{Z})$ again acts on these forms by linear substitution.

There are now two polynomial invariants for this action, traditionally denoted I and J, where:

$$I(f) = 12ae - 3bd + c^2,$$
$$J(f) = 72ace + 9bcd - 27ad^2 - 27eb^2 - 2c^3.$$

Again, if you fix both I and J, then there exist only finitely many $\mathrm{GL}_2(\mathbb{Z})$-equivalence classes of integral binary quartic forms having this value of (I,J).

On average, how many classes $h_{I,J}$ of irreducible binary quartic forms are there having given invariants I and J? Equivalently, how many equivalence classes of binary quartic forms are there having bounded I and J?

Counting binary quartic forms

We define the height $H(f)$ of a binary quartic form f by:

$$H(f) := H(I,J) := \max\{|I^3|, J^2/4\}$$

How many equivalence classes of quartics f have $H(f) < X$?

Works of Julia, Cremona, Stoll, Yukie, Yang each imply that this number is $O(X^{5/6+\epsilon})$. Almost any reduction theory method implies this immediately.

Theorem.

(a) $$\sum_{\substack{H(I,J)<X \\ \mathrm{Disc}(I,J)>0}} h(I,J) \sim \frac{12}{135}\zeta(2)\cdot X^{5/6};$$

(b) $$\sum_{\substack{H(I,J)<X \\ \mathrm{Disc}(I,J)<0}} h(I,J) \sim \frac{32}{135}\zeta(2)\cdot X^{5/6}.$$

How many classes do we get per (I,J)?

Eligible (I,J)

We say that a pair $(I,J) \in \mathbb{Z}\times\mathbb{Z}$ is eligible if it occurs as the invariants of some integer binary quartic form. In fact, the set of eligible (I,J) is defined purely by congruences.

These congruence conditions are:

(a) $I \equiv 0 \pmod 3$ and $J \equiv 0 \pmod{27}$,

(b) $I \equiv 1 \pmod 9$ and $J \equiv \pm 2 \pmod{27}$,

(c) $I \equiv 4 \pmod 9$ and $J \equiv \pm 16 \pmod{27}$,

(d) $I \equiv 7 \pmod 9$ and $J \equiv \pm 7 \pmod{27}$.

The number of eligible (I,J) having height less than X is thus a constant times $X^{5/6}$. (In fact, $\frac{8}{27}\cdot X^{5/6}$.)

The average number of binary quartic forms per (I, J)

We may thus average the number of $\mathrm{GL}_2(\mathbb{Z})$-orbits of binary quartics over eligible pairs (I, J).

Theorem.

(a) The average number of positive discriminant binary quartic forms per eligible (I, J) is $3\zeta(2)/2$.

(b) The average number of negative discriminant binary quartic forms per eligible (I, J) is $\zeta(2)$.

The analogous theorems can be proven for equivalence classes of binary quartic forms satisfying any desired finite set of congruence conditions.

Back to elliptic curves!

To prove the main theorem, about the average size of the 2-Selmer group being 3:

- Given $A, B \in \mathbb{Z}$, choose an integral binary quartic form f for each element of $S^{(2)}(E_{A,B})$, such that
  - $y^2 = f(x)$ gives the desired 2-covering over $\mathbb{Q}$;
  - the invariants $(I(f), J(f))$ agree with the invariants (A, B) of the elliptic curve (at least away from 2 and 3);

  The construction of such a set of binary quartic forms follows from the work of Birch and Swinnerton-Dyer.
- Count these integral binary quartic forms. These are defined by infinitely many congruence conditions, so a sieve has to be performed. A uniformity estimate must be proven to perform this sieve, and that is by far the most technical part of this work. It involves counting integral points in much bigger spaces than binary quartic forms!

Average Size of 2-Selmer

In particular, we must count points of bounded invariants in a certain nonreductive coregular space of dimension 12.

Once this count is performed, the uniformity estimate proven, and then the sieve carried out, we finally obtain:

Theorem. *When all elliptic curves $E/\mathbb{Q}$ (in any family defined by finitely many congruence conditions) are ordered by height, the average size of the 2-Selmer group $S^{(2)}(E)$ is 3.*

Corollary. *When all elliptic curves $E/\mathbb{Q}$ (in any family defined by finitely many congruence conditions) are ordered by height, the average rank is at most 1.5.*

What about 3-Selmer?

We may also determine the average size of the 3-Selmer group of elliptic curves!

The set of 3-Selmer elements of elliptic curves is parametrized by 3-coverings, which may in turn be parametrized by appropriate $\mathrm{GL}_3(\mathbb{Q})$-orbits of integer ternary cubic forms. (This follows from a result of Cassels.)

The analogous "minimization" results of BSD over the integers have been proven by Cremona, Fisher, and Stoll in this case.

Proceeding in an analogous way (though now the dimension of the basic space is much bigger!), we show:

Theorem. *When all elliptic curves $E/\mathbb{Q}$ (in any family defined by finitely many congruence conditions) are ordered by height, the mean size of $S^{(3)}(E)$ is 4.*

Corollary. *When all elliptic curves $E/\mathbb{Q}$ (in any family defined by finitely many congruence conditions) are ordered by height, the average rank is less than 1.17.*

Some consequences

Consider the family $\mathcal{F}$ of elliptic curves E that satisfy the following mild conditions.

- The curve E and its twist by -1 both have additive reduction at 2.
- The j-invariant of the curve E is a 2-adic unit.
- The curve E has good ordinary reduction at 3.
- The odd part of the discriminant of E is squarefree and congruent to 1 mod 4.

It is easy to show that curves satisfying these conditions consist of a positive proportion of all elliptic curves.

Furthermore, our results about 3-Selmer also apply to this family.

Suppose $E \in \mathcal{F}$. Then E twisted by -1 is also in $\mathcal{F}$, and furthermore, the analytic root numbers of E and its twist by -1 are different. Therefore, exactly half the root numbers of curves in $\mathcal{F}$ are $+1$.

Parity of p-Selmer rank

A recent result of Tim and Vladimir Dokchitser states that the parity of the p-Selmer rank of E is even iff the root number of E is $+1$!

Combining this with the fact that the 3-Selmer average is at most 4 in any family (e.g., $\mathcal{F}$), we are able to prove:

Theorem. *When all elliptic curves $E/\mathbb{Q}$ are ordered by height, a positive proportion of them have rank* 0.

Indeed, as the average number of 3-Selmer elements of curves in $\mathcal{F}$ is at most 4, it is not possible for all the curves with even 3-Selmer rank to have rank greater than 0. At least half of them must have rank 0!

A similar argument gives:

Theorem. *Assume $\text{Ш}(E)$ is finite for all E. When all elliptic curves $E/\mathbb{Q}$ are ordered by height, a positive proportion of them have rank* 1.

Nonvanishing of elliptic curve L-functions

What about analytic rank?

A recent result of Skinner–Urban states that if the L-function of an elliptic curve E vanishes at $s = 1$ and E has good ordinary reduction at 3, then the 3-Selmer group of E is nontrivial.

Combining this with the fact that the 3-Selmer average is at most 4 in any family (e.g., $\mathcal{F}$), we are able to prove:

Theorem. *When all elliptic curves $E/\mathbb{Q}$ are ordered by height, a positive proportion of them have* analytic rank 0*; that is, a positive proportion of elliptic curves have nonvanishing L-function at $s = 1$.*

Corollary. *A positive proportion of elliptic curves satisfy BSD.*

What about 4-Selmer and 5-Selmer?

Elements in 4-Selmer and 5-Selmer groups of elliptic curves can be mapped to integer points, up to equivalence, having the corresponding invariants in the spaces

$$\mathbb{Z}^2 \otimes \mathrm{Sym}^2(\mathbb{Z}^4) \ \text{ and } \ \mathbb{Z}^5 \otimes \wedge^2\mathbb{Z}^5,$$

respectively. (This again can be deduced from work of Cassels, Cremona–Fisher–Stoll, and Fisher.)

Counting points in these spaces should thus similarly lead to the analogous results for 4-Selmer and 5-Selmer. However, cusps are extremely complicated. (These spaces are 20- and 50-dimensional, respectively, with about 1000 cuspidal regions to deal with!)

What about 4-Selmer and 5-Selmer?

Dealing with these issues, we are finally able to prove:

Theorem. *When all elliptic curves $E/\mathbb{Q}$ (in any family defined by finitely many congruence conditions) are ordered by height, the mean size of $S^{(4)}(E)$ is 7.*

Theorem. *When all elliptic curves $E/\mathbb{Q}$ (in any family defined by finitely many congruence conditions) are ordered by height, the mean size of $S^{(5)}(E)$ is 6.*

Using the last theorem, together with a more careful analysis of changing of root numbers under twisting, we can now prove:

Corollary. *When all elliptic curves $E/\mathbb{Q}$ are ordered by height, the average rank is less than 1.*

Some extensions

Similar counting techniques applied to various other (coregular) spaces lead to densities of other data associated to elliptic curves and related algebraic and geometric objects.

There are at least 50 such spaces that parametrize genus one curves with extra data.

There are also several further spaces of forms that parametrize various data corresponding to higher genus curves and higher dimensional varieties.

This allows one to compute average Selmer group sizes, and thus bound average ranks, for various families of elliptic curves with marked points, and also for Jacobians of various families of higher genus curves (joint work with Wei Ho and Dick Gross respectively).

Last week: What about special families like Don's?

Note that the Zagier–Kramarz family

$$x^3 + y^3 = m \sim y^2 = x^3 - 432m^2$$

is contained in the larger family of curves

$$E_k : y^2 = x^3 + k$$

having j-invariant zero.

There is a rational 3-isogeny $\phi : E_k \to E_{-27k}$. Is there an expected size of the associated Selmer group?

Last week: What about special families like Don's?

In joint work with Elkies, we recently proved:

Theorem. *As the elliptic curves E_k vary ($k \to \infty$), the average size c of the ϕ-Selmer group exists.*

We are in the midst of determining the value of c explicitly. (We have $c \approx 2$.)

This theorem, a root number analysis, and the results of Dokchitser–Dokchitser then imply

Corollary. *The average rank of the curves E_k is less than one.*

Corollary. *A positive proportion of the curves E_k have rank zero.*

Last week: What about special families like Don's?

However, Zagier–Kramarz considered only the curves

$$x^3 + y^3 = m \sim y^2 = x^3 - 432m^2.$$

In general, we can consider $E_{D,m} : y^2 = x^3 + Dm^2$ for any fixed D, with m varying.

There is a geometric method (joint work w/ Shnidman) that allows one to treat the ϕ-Selmer group for these cubic twist families. We prove:

Theorem. *The average size of the ϕ-Selmer group $S^{(\phi)}(E_{D,m})$ is*

$$\begin{cases} < \infty & \text{if } D \neq -432 \\ \infty & \text{if } D = -432. \end{cases}$$

!!!

Wishes to Don

Happy Birthday!!!!!

SPECIAL VALUES OF RANKIN–SELBERG L-FUNCTIONS

A. RAGHURAM

Abstract. This write-up is the abstract for my talk on 30th June at the 2011-Arbeitstagung.

1. A cohomological approach to critical values

A classical theorem due to Manin and Shimura (independently) says that to a primitive holomorphic cusp form φ of weight k on the upper half plane, one can attach two numbers $u^{\pm}(\varphi)$ such that for any integer m with $1 \leq m \leq k-1$ and any Dirichlet character χ one has

$$L(m, \varphi, \chi) \ \sim \ (2\pi i)^m u^{\pm}(\varphi)\mathcal{G}(\chi).$$

Here $\mathcal{G}(\chi)$ is the Gauß sum of χ, $\pm = \epsilon_m \epsilon_\chi := (-1)^m \chi(-1)$, and by $\sim$ one means up to quantities in a suitable rationality field; which here is the field generated by the Fourier coefficients of φ and the values of χ.

The same theorem can be rephrased in a 'neo-classical' language as follows: Let Π be a cohomological cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$. One can attach two numbers $p^{\pm}(\Pi)$, which may be called Betti-Whittaker periods of Π, which capture, up to powers of $(2\pi i)$, the possibly transcendental parts of the critical values of the standard L-function $L(s, \Pi)$.

(1) If $s = 1/2$ is critical, then $L(1/2, \Pi) \ \sim \ (2\pi i)^{d_\infty} p^{+}(\Pi)$.

(2) For any critical point $s = 1/2 + m$, and any finite order Hecke character χ we have

$$L(1/2+m, \Pi) \ \sim \ (2\pi i)^{d_\infty + m} p^{\pm}(\Pi)\mathcal{G}(\chi).$$

This reformulation suggests that there are three aspects to proving algebraicity results on critical values:

I. *Identify periods*; such as $p^{\pm}(\Pi)$. Often these periods arise via a comparison of two entirely different rational structures on the same representation space.

II. *Prove period relations*; such as $p^{\epsilon}(\Pi \otimes \chi) \sim p^{\epsilon\epsilon_\chi}(\Pi)\mathcal{G}(\chi)$ for any algebraic Hecke character.

III. *Prove a theorem for one critical value;* such as for $L(1/2, \Pi)$. This step usually involves giving a cohomological interpretation to some analytic theory of L-functions.

2. Ratios of critical values for Rankin-Selberg L-functions (Joint work with Günter Harder)

Observe the following consequence of Manin/Shimura's result. Given φ, there exists $\Omega(\varphi) \in \mathbb{C}^\times$ such that for $1 \leq m \leq k-2$ we have

$$\frac{\Lambda(m, \varphi, \chi)}{\Lambda(m+1, \varphi, \chi)} \ \sim \ \Omega(\varphi)^{\epsilon_m \epsilon_\chi}$$

with $\epsilon_m = (-1)^m$ and $\epsilon_\chi = \chi(-1)$. Such a result can be generalized to ratios of critical values for Rankin-Selberg L-functions for $\mathrm{GL}_n \times \mathrm{GL}_{n'}$ with n-even and n'-odd.

Date: July 4, 2011.

2.1. **The relative periods.** Let $\sigma_f \in \mathrm{Coh}(\mathrm{GL}_n, \lambda)$, by which we mean that σ_f is a $\mathrm{GL}_n(\mathbb{A}_f)$-summand of the inner cohomology $H_!^\bullet(S^{\mathrm{GL}_n}, \mathcal{E}_\lambda)$ of a locally symmetric space S^{GL_n} of GL_n with coefficients in a sheaf $\mathcal{E}_\lambda$ coming from an algebraic absolutely irreducible finite-dimensional representation of GL_n over $\mathbb{Q}$. Assume n is even. Suppose σ_f is the finite part of a cuspidal automorphic representation σ of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$, then one knows that σ_f appears twice in the inner cohomology for degree $\bullet = b_n := n^2/4$. Denote these two copies by $\sigma_f^\pm$. There are two intertwining operators between $\sigma_f^\pm$; one concerning inner cohomology and this is defined over the rationality field of σ_f, and the other using a transcendental description over $\mathbb{C}$ of cuspidal cohomology. Comparing these we get a relative period $\Omega(\sigma_f)$.

2.2. **Period relations.** By analyzing the behaviour of cohomology groups under Tate-twists of the coefficient system, one can prove the relation: $\Omega(\sigma_f \otimes |\ |^m) \ \sim\ \Omega(\sigma_f)^{\epsilon_m}$.

2.3. **The main theorem.** Let $\sigma_f \in \mathrm{Coh}(\mathrm{GL}_n, \lambda)$ and $\sigma'_f \in \mathrm{Coh}(\mathrm{GL}_{n'}, \lambda')$. Assume that n is even and n' is odd. Let $m = 1/2 + m_0 \in 1/2 + \mathbb{Z}$ be a half-integer such that both m and $m+1$ are critical for $L(s, \sigma_f \times \sigma_f'^\vee)$. Then the complex number

$$\frac{1}{\Omega(\sigma_f)^{\epsilon_m \epsilon_{\sigma'}}} \frac{\Lambda(m, \sigma_f \times \sigma_f'^\vee)}{\Lambda(m+1, \sigma_f \times \sigma_f'^\vee)}$$

is algebraic, and is equivariant under the action of the automorphism group of $\mathbb{C}$. Here $\epsilon_{\sigma'}$ is a sign determined by σ', $\epsilon_m = (-1)^{m_0}$ and $\Lambda(s, \sigma_f \times \sigma_f'^\vee)$ is the completed Rankin–Selberg L-function. In particular,

$$\frac{\Lambda(m, \sigma_f \times \sigma_f'^\vee)}{\Lambda(m+1, \sigma_f \times \sigma_f'^\vee)} \ \sim\ \Omega(\sigma_f)^{\epsilon_m \epsilon_{\sigma'}},$$

where, by $\sim$, we mean up to an element of any number field containing the rationality fields $\mathbb{Q}(\sigma_f)$ and $\mathbb{Q}(\sigma'_f)$.

2.4. **Eisenstein cohomology.** Our main tool to proving such a result on ratios of critical values is the theory of Eisenstein cohomology. Put $N = n + n'$. This theory gives a description of the image of the total cohomology $H^\bullet(S^{\mathrm{GL}_N}, \mathcal{E}_\mu)$ in the cohomology $H^\bullet(\partial S^{\mathrm{GL}_N}, \mathcal{E}_\mu)$ of the Borel-Serre boundary $\partial S^{\mathrm{GL}_N}$. This boundary is stratified as $\partial S^{\mathrm{GL}_N} = \cup_P \partial_P S^{\mathrm{GL}_N}$ for P-running through a suitable class of parabolic subgroups of GL_N. Let $P = P_{(n,n')}$ be a maximal parabolic subgroup with Levi quotient $\mathrm{GL}_n \times \mathrm{GL}_{n'}$ and let $Q = P_{(n',n)}$ be its associate parabolic. The result on ratios of critical values falls out of a cohomological interpretation to Langlands's constant term theorem by considering the image of $H^\bullet(S^{\mathrm{GL}_N}, \mathcal{E}_\mu)$ in $H^\bullet(\partial_P S^{\mathrm{GL}_N}, \mathcal{E}_\mu) \oplus H^\bullet(\partial_Q S^{\mathrm{GL}_N}, \mathcal{E}_\mu)$, where some very interesting Weyl-group combinatorics forces us to look at

(1) cohomology degree $\bullet = (N^2-1)/4 = n^2/4 + (n'^2-1)/4 + nn'/2$; and
(2) highest weight μ which is built out of the weights λ and λ'.

* * * * * * *

Motivic Fundamental Groups and Integral Points

Majid Hadian

Introduction

This short note is an extended abstract of a lecture given by the author at the Mathematische Arbeitstagung 2011, Bonn, Germany. Our goal is to give a brief exposition on the theory of motivic fundamental groups and their application to Diophantine geometry. Being so, we try to avoid technicalities and sometimes we are even inaccurate to some extend. Interested readers can find solid and more detailed treatments in [5], [6], and [4].

1 Motivic Fundamental Groups ...

Let k be a number field and fix an embedding $k \subset \mathbb{C}$ of k into the field of complex numbers. Let $\bar{k}$ be the algebraic closure of k in $\mathbb{C}$ and denote the absolute Galois group $\mathrm{Gal}(\bar{k}/k)$ by G_k. There are different natural cohomology functors defined on the category $\mathfrak{Var}_k$ of algebraic varieties over k. Let X be an object in $\mathfrak{Var}_k$, and for simplicity assume that X is the complement of a strict normal crossing divisor in a smooth projective variety $\overline{X}$ (resolution of singularities is available in characteristic zero). One can consider:

- the singular cohomology $H^*_s(X(\mathbb{C}), \mathbb{Z})$ of the associated complex variety $X(\mathbb{C})$, which is formed of $\mathbb{Z}$-modules;
- the étale cohomology $H^*_{\text{ét}}(X_{\bar{k}}, \mathbb{Q}_p)$ of $X_{\bar{k}} := X \otimes_k \bar{k}$, which is formed of $\mathbb{Q}_p$-vector spaces equipped with canonical Galois action by G_k;
- the algebraic de Rham cohomology $H^*_{dR}(X, \mathcal{O}_X)$ of X, which is formed of k-vector spaces equipped with Hodge filtration and Frobenius action (the Frobenius action is induced from comparing with the crystalline cohomology).

The very interesting fact is that these cohomology functors are not independent, and can be compared after suitable extension of scalars. Now, the general yoga of motives, roughly speaking, begins by the following question proposed by Grothendieck. Is there a universal cohomology functor H^*_M (called motivic cohomology) which gives rise to all the above cohomology functors as different realizations? There is a huge literature in the direction of this question and we are not going to go into it. Instead, we are going to consider the analogue question concerning the unipotent fundamental group.

Note that there are various constructions of the unipotent fundamental group of X parallel to the above cohomology theories (one should of course fix a base point which is denoted by $*$ in the sequel). Namely, one can consider:

- the Malčev completion of the topological fundamental group of the complex variety $X(\mathbb{C})$, which is a pro-unipotent algebraic group scheme over $\mathbb{Q}$ and will be denoted by $\pi_1^{top}(X,*)$;
- the étale unipotent fundamental group of $X_{\bar{k}}$, whose category of representations is the category of unipotent smooth p-adic étale sheaves over $X_{\bar{k}}$ and will be denoted by $\pi_1^{\text{ét}}(X,*)$. $\pi_1^{\text{ét}}(X,*)$ is a pro-unipotent algebraic group scheme over $\mathbb{Q}_p$ and admits a canonical Galois action by G_k.
- the de Rham unipotent fundamental group, whose category of representations is the category of vector bundles over $\overline{X}$ equipped with unipotent integrable connection with logarithmic poles at the divisor $\overline{X} \setminus X$. The de Rham unipotent fundamental group, which is a pro-unipotent algebraic group scheme over k, will be denoted by $\pi_1^{dR}(X,*)$ and can be furnished with Hodge filtration and Frobenius action.

All these fundamental groups, being pro-unipotent, admit exhaustive descending central series. The associated algebraic quotients will be denoted by an extra superscript, which shows the unipotent level of the quotient. Now, one can raise the similar question and ask whether or not there exists a universal unipotent fundamental group which encompasses all these versions as different realizations. Although, one expects to have an affirmative answer in general, it is only verified in the case where X is a unirational variety. The idea of constructing the motivic unipotent fundamental group, denoted by $\pi_1^M(X,*)$ from now on, for a unirational variety is as follows:

First of all, since any unirational variety admits a dominant morphism from an open subscheme of a projective space $\mathbb{P}^N$ and unipotent group schemes over fields of characteristic zero are torsion free, one reduces to the case of open subschemes of $\mathbb{P}^N$. Then, by taking a generic hyperplane section by a line in $\mathbb{P}^N$ and using Lefschetz hyperplane section theorem, one reduces to the case of a punctured projective line. Finally, for a punctured projective line, one can explicitly construct the motivic unipotent fundamental group as a pro-unipotent group scheme over the Tannakian category of mixed Tate motives (see [2]).

2 ... and Integral Points

Now we want to show how motivic unipotent fundamental groups can be applied in Diophantine geometry. For motivation, let us recall an idea, due to Chabauty, which leads to a partial solution for Mordell's conjecture (see [1]). Roughly speaking, the idea is as follows. Let C be a smooth projective hyperbolic curve over the number field k, and let $j : C \to J$ be the Abel-Jacobi map from C to its Jacobian. Now, if the arithmetic rank of J is strictly less that its dimension, one can find a nonzero invariant differential 1-form ω on J which vanishes on all but finitely many k-rational points. The pullback $j^*(\omega)$ of such a form is a nonzero 1-form on C which vanishes on all but finitely many k-rational points. But this shows the finiteness of $C(k)$.

This reduces Mordell's conjecture for a smooth projective hyperbolic curve C, to finding a tower of étale coverings $C_n \twoheadrightarrow C$ whose genus grows faster than the arithmetic rank of Jacobian. Although this project is never accomplished and Mordell's conjecture has been solved by Faltings by a very deep study of Galois representations associated to Abelian varieties, Kim has picked Chabauty's idea up more recently, and developed a non-abelian version of it, which seems to be very interesting and powerful (see [7]). Let us briefly sketch Kim's idea in the case of a punctured projective line (for the positive genus case, see [3]).

Let S be a finite set of finite places of k, $\mathcal{O}_S$ be the ring of S-integers in k, and let X be a punctured projective line over $\mathcal{O}_S$ (to avoid empty statements, assume that the number of punctures is at least three). Let T be the saturation of S, consisting of all finite places of k whose residue characteristic appears as the residue characteristic of at least one place in S. Finally, fix a finite place v of k in the complement of T and denote the residue characteristic of v by p. Let k_v be the completion of k at v and $\mathcal{O}_v$ be the normalization of $\mathbb{Z}_p$ in k_v.

We want to show that the Diophantine set $X(\mathcal{O}_S)$ is finite. If it is empty, there is nothing to show, otherwise fix a base point $x \in X(\mathcal{O}_S)$. Now by studying étale and de Rham path torsors for the curves X_k and X_{k_v}, and by comparing the étale and the de Rham versions using higher dimensional p-adic Hodge theory, one obtains the following commutative diagram:

$$\begin{tikzcd}
 & (\pi_1^{dR,n}/F^0)(k_v) \arrow[dr, bend left, "="] & \\
X(\mathcal{O}_T)^\circ \arrow[r, "i"] \arrow[d, "p_{\text{ét}}^{\mathrm{gl},(n)}"] & X_{k_v}(\mathcal{O}_v) \cap \mathcal{D}_1^\circ(x_v) \arrow[u, twoheadrightarrow, "p_{dR}^{(n)}"'] \arrow[r, dotted] \arrow[d, "p_{\text{ét}}^{\mathrm{loc},(n)}"] & W_{k_v/\mathbb{Q}_p}(\pi_1^{dR,n}/F^0)(\mathbb{Q}_p) \arrow[dl, bend left, "c"] \\
H^1(G_T, \pi_1^{\text{ét},n})(\mathbb{Q}_p) \arrow[r, "\mathrm{res}"] & H^1(G_v, \pi_1^{\text{ét},n})(\mathbb{Q}_p) &
\end{tikzcd}$$

Let us explain the notations in the above diagram. The maps denoted by the letter p are different versions of the period map, constructed by studying the variation of the path torsor when one varies the end point of the path. The map i is the inclusion map and the map c is the comparison map which can be constructed by p-adic Hodge theory techniques. G_v denotes the absolute Galois group of the local field k_v, and G_T is the absolute Galois group of the maximal extension k_T of k unramified outside T. The map "res" is the usual restriction map between Galois cohomologies. $\mathcal{D}_1^\circ(x_v)$ denotes the open p-adic unit disk in X_{k_v}, centered at the point x_v induced by the base point x. $X(\mathcal{O}_T)^\circ$ is the intersection of $X(\mathcal{O}_T)$ and $\mathcal{D}_1^\circ(x_v)$. $F^\bullet$ denotes the Hodge filtration on the de Rham fundamental group, and finally, $W_{k_v/\mathbb{Q}_p}$ is the Weil restriction functor.

Very crucial are the following properties of the maps appeared in the above diagram. The comparison map c is induced by an algebraic map which is injective and whose image is Zariski close, and the de Rham period map $p_{dR}^{(n)}$ is induced by a p-adic analytic map whose image is Zariski dense. Using these properties and the commutativity of the above diagram, one can reduce the finiteness of $X(\mathcal{O}_T)$ to a strict inequality between dimensions of the global and the local Galois cohomologies appeared in the last row of the above diagram. This can be thought of as the non-abelian version of Chabauty's hypothesis.

In order to proceed further, in the above line of ideas, one needs to estimate the dimension of the global Galois cohomology in the above diagram. But this is a very hard problem. Alternatively, we suggest to use the motivicity of unipotent fundamental group and path torsors to replace this global Galois cohomologies with motivic cohomologies of X, which can be related to rational K-groups of the base number field k (see [5] or [6]). Following these ideas, and using Borel's calculation of rational K-groups of number fields, we obtain a motivic proof of the following special case of Siegel's finiteness theorem for S-integral points:

Theorem 1. *Let X be the punctured projective line with $d \geq 3$ punctures, and let k be a totally real number field of degree at most $d-1$. Then $X(\mathcal{O}_S)$ is finite.*

Remark 2. *Using the motivic version of Lefschetz hyperplane section theorem, the above result can be generalized to higher dimensional unirational varieties, which gives nondensity of S-integral points for such varieties in the p-adic topology (see [5] or [6]).*

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UNIVERSAL ELLIPTIC COHOMOLOGY AND MODULAR FORMS

PETER TEICHNER, AFTER HOPKINS, MILLER AND LURIE

This unexpected talk at the Arbeitstagung 2011 was about the homotopy groups of the spectrum tmf of *topological modular forms*. Its existence was proven by Hopkins and Miller [H], with a conceptual interpretation in terms of derived algebraic geometry given later by Lurie [L]. The references [T, G, H, L] contain excellent surveys on the topic, that's why these notes are very brief.

Let $\mathcal{M}_{\mathrm{ell}}$ be the Deligne-Mumford moduli stack of (pointed) elliptic curves with nodal singularities (over arbitrary commutative rings). It comes equipped with the determinant line bundle ω and the global sections

$$\mathrm{MF}_k := H^0(\mathcal{M}_{\mathrm{ell}}; \omega^{\otimes k})$$

are the *integral modular forms* of weight k. Deligne computed this ring to be presented as follows:

$$\mathrm{MF}_* = \mathbb{Z}[c_4, c_6, \Delta]/c_4^3 - c_6^2 = 12^3 \cdot \Delta$$

where c_k have weight k and Δ is the *discriminant* whose non-vanishing characterizes smooth elliptic curves. The following result was attributed to Hopkins, Miller and Goerss in [L].

Theorem 1. *There exists a sheaf $\mathcal{O}$ of commutative ring spectra on the Deligne-Mumford stack $\mathcal{M}_{\mathrm{ell}}$, characterized up to homotopy by the following properties:*

(1) *$\pi_0(\mathcal{O})$ are the functions on $\mathcal{M}_{\mathrm{ell}}$, $\pi_{2k+1}(\mathcal{O}) = 0$ and*
(2) *$\pi_{2k}(\mathcal{O}) \cong \omega^{\otimes k}$ as sheaves of $\pi_0(\mathcal{O})$-modules on $\mathcal{M}_{\mathrm{ell}}$.*

Definition 2. *The commutative ring spectrum* tmf *of* topological modular forms *is defined as the global sections of $(\mathcal{M}_{\mathrm{ell}}, \mathcal{O})$. It turns out to be* connective*, i.e. $\pi_k(\mathrm{tmf}) = 0$ for $k < 0$.*

By construction, there is a *descent spectral sequence*, converging to the homotopy groups of tmf:

$$H^s(\mathcal{M}_{\mathrm{ell}}; \pi_t(\mathcal{O})) \Longrightarrow \pi_{t-s}(\mathrm{tmf})$$

The indexing is compatible with that of the Adams spectral sequence, even though the E_2-term is purely algebraic. The edge homomorphism is given by maps

$$e_{2k} : \pi_{2k}(\mathrm{tmf}) \to \mathrm{MF}_k$$

Since $\mathcal{M}_{\mathrm{ell}}$ has no higher cohomology away from 6, these maps become isomorphisms after tensoring with $\mathbb{Z}[1/6]$. From the above presentation, it follows that MF_* is torsionfree and as a consequence, $\pi_*(\mathrm{tmf})$ only has p^n-torsion for the primes $p = 2$ and 3.

Amazingly, this torsion is completely known and most of it comes from the unit map $u : \mathbb{S}^0 \to \mathrm{tmf}$ of ring spectra. For summaries of these computations, see [B] or Henriques' article in [T]. For example,

- The unit u induces isomorphisms $\pi_k(\mathbb{S}^0) \to \operatorname{tors} \pi_k(\mathrm{tmf})$ for $k = 1, 2, \ldots, 24$.
- c_6 and $c_4 c_6$ generate the cokernel $(\mathbb{Z}/2)^2$ of the edge homomorphism up to dimension 23 (in particular, c_4 and $2c_6$ are in its image).
- 24 is the smallest multiple of Δ in the image of e_{24} and 24 is also the smallest power of Δ in the image of the edge homomorphism.
- There is a unique class $P \in \pi_{24^2}(\mathrm{tmf})$ with $e_{24^2}(D) = \Delta^{24}$. Multiplication by P induces a "periodicity" isomorphism

$$\operatorname{tors} \pi_k(\mathrm{tmf}) \cong \operatorname{tors} \pi_{k+24^2}(\mathrm{tmf}) \quad \text{for } k \geq 0$$

Inverting P leads to the spectrum TMF whose homotopy groups are 24^2-periodic. This is the spectrum that Stolz and the author [ST] believe to give the classifying space of super symmetric Euclidean field theories of dimension $2|1$. One reason is that the partition function of such a theory is a weak integral modular form, just like the image of the edge homomorphism on $\pi_*(\mathrm{TMF})$.

The above statements follow from a complete knowledge of the differentials in the descent spectral sequence. These cannot be derived algebraically but come from a comparison with the Adams-Novikov spectral sequence as follows.

One can formulate Quillen's result on the relation between (1-dim.) formal groups and unitary bordism by saying that the Hopf algebroid $\mathrm{MU}_* \mathrm{MU}$ represents the stack $\mathcal{M}_{\mathrm{fg}}$ of formal groups (over arbitrary commutative rings). In particular, this leads to an isomorphism

$$H^s(\mathcal{M}_{\mathrm{fg}}; \omega^{\otimes t}) \cong \mathrm{Ext}^{s,2t}_{\mathrm{MU}_* \mathrm{MU}}(\mathrm{MU}_*, \mathrm{MU}_*)$$

The right hand side is the E^2-term of the Adams-Novikov spectral sequence converging to $\pi_{2t-s}(\mathbb{S}^0)$, whereas the left hand side can be compared to the E^2-term of the above descent spectral sequence: The formal groups associated to elliptic curves lead to a stack morphism $\mathcal{M}_{\mathrm{ell}} \to \mathcal{M}_{\mathrm{fg}}$ that's covered by a morphism of line bundles ω.

By playing out the two E_2-terms and the homotopy theoretic knowlegde of the differentials in the Adams-Novikov spectral sequence, one arrives at quite a miraculous computation of all differentials in the descent spectral sequence and hence at a complete understanding of $\pi_*(\mathrm{tmf})$.

If G is a compact Lie group of dimension n, its right invariant framing gives an element in framed bordism $\Omega_n^{fr} \cong \pi_n(\mathbb{S}^0)$. One of the nice features of tmf is the fact that many of these elements map nontrivially under the unit map to $\pi_n(\mathrm{tmf})$ [H]. In fact, the map $u_* : \Omega_*^{fr} \to \pi_*(\mathrm{tmf})$ factors through the *string orientation map*

$$\sigma : \Omega_*^{String} \to \pi_*(\mathrm{tmf})$$

which gives the *Witten genus* Φ_W after composition with the edge homomorphism to $\mathrm{MF}_*, e \circ \sigma = \Phi_W$. Recall that a manifold X is *string* if it is spin and has vanishing characteristic class $\frac{p_1}{2}(TX) \in H^4(X)$.

The Witten genus [W] of a closed string manifold X is the partition function of a Euclidean field theory of super dimension $2|1$, the so called super symmetric Sigma model of X. This field theory is mathematically only defined on the classical level (where it actually is conformal and extends to all genera), the quantization requires a (yet non-existent) measure on the space of maps from a complex elliptic curve to X. In our approach to Euclidean field theories [ST], we hope to circumvent this measure by cutting the torus into finer and finer triangles and and using the locality (or gluing) properties of the field theory to get a well defined partition function. This would then lead to a string orientation, strengthening our conjectured relation between Euclidean field theories and TMF.

To get a mathematically defined expression, Witten used the circle action on the free loop space LX to compute this partition function via the S^1-equivariant index theorem (non-existent in this infinite dimensional setting) which predicts a localization to the fixed point set X. A well defined genus with values in $\mathbb{Q}[[q]]$ resulted for all oriented manifolds. For a spin manifold X, the coefficients are indices of twisted Dirac operators and hence the Witten genus of X lies in $\mathbb{Z}[[q]]$. Don Zagier[1] showed in [Z] that this power series is the q-expansion of a modular form if $p_1(X)$ is torsion, completing the existence proof for the map

$$\Phi_W : \Omega_*^{String} \to \mathrm{MF}_*$$

[1]Happy Birthday, Don

It is believed that the string orientation σ (which refines Φ_W to include interesting torsion information) is induced by a map of commutative ring spectra on the Thom spectrum for string manifolds:

$$\mathrm{MString} \to \mathrm{tmf}$$

Important partial results have been obtained in [AHS] via the *theorem of the cube*.

It is proposed in [L] that such a ring map can be constructed canonically from the interpretation of the structure sheaf $\mathcal{O}$ in terms of derived (oriented) elliptic curves and the resulting 2-equivariance properties of tmf. Such a characterization of maps of commutative ring spectra into tmf would be very exciting, also in view of our conjectured relation to super symmetric Euclidean field theories.

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MOCK EICHLER–SHIMURA RELATIONS

MARTIN RAUM

We establish a connection between two seemingly disparate topics and techniques: mock modular forms (holomorphic parts of holomorphic Maaß forms) and noncritical values of L-functions of cusp forms.

Given a cuspidal weight k modular form $f \in S_k$ for $\mathrm{SL}_2(\mathbb{Z})$, Choie and Diamantis deduced a function $r_{f,2}$ playing the role of a generating function for the noncritical values of the L-function associated to f. More precisely, the Taylor expansion at $z \searrow 0$ equals

$$\sum_{m=0}^{\infty} i^{k+m} \frac{(m+k-1)!m!}{(k-1)(2\pi)^{m+k}} L_f(k+m).$$

This function can be incorporated into Eichler cohomology after adding a simple, nonholomorphic correction term $\tilde{r}_{f,2}$. This correction term is an almost holomorphic polynomial in the sense of Kaneko-Zagier. The powers of y that occur, moreover, have only negative exponents and can be thus considered purely nonholomorphic. Denoting the completion $\hat{r}_{f,2} := r_{f,2} + \tilde{r}_{f,2}$, a first theorem that we prove says

$$\hat{r}_{f,2}\big|_k (1+S) = \hat{r}_{f,2}\big|_k (1+U+U^2) = 0.$$

The slash action $f|_k\, \gamma(z) = (cz+d)^{-k} f((az+b)/(cz+d))$ is the usual one, and the matrices $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $U = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ are generators of $\mathrm{SL}_2(\mathbb{Z})$. That is, the completed generating function $\hat{r}_{f,2}$ satisfies the Eichler–Shirmura relations.

This observation leads us to the next definition. We write V_{k-2} for the space of polynomials in z of degree less than or equal to $k-2$. The space

$$W_{k,2} := \big\{ \mathcal{P} : \mathbb{H} \to \mathbb{C} \,:\, \xi_k(\mathcal{P}) \in V_{k-2};\, \mathcal{P}|_k\,(1+S) = \mathcal{P}\big|_k\,(1+U+U^2) = 0 \big\}.$$

Definition. *A holomorphic function $p_2 : \mathbb{H} \to \mathbb{C}$ is called a* mock period function *if there exists a $\widetilde{p}_2 \in \oplus_{j=1}^{k-1} y^{-j} V_{k-2}$ such that*

$$p_2 + \widetilde{p}_2 \in W_{k,2}.$$

A desirable statement would be that any of these function can be completed to Eichler cocycle. This is actually not quite possible, since, denoting the space of holomorphic functions on $\mathbb{H}$ by $O(\mathbb{H})$, the Eichler cohomology $H^1(\mathrm{SL}_2(\mathbb{Z}), O(\mathbb{H}))$ vanishes by a theorem of Knopp. The best possible that we can hope for, though, turns out to be true:

Theorem. *Every $P \in W_{k,2}$ can be written as*

$$P = \widehat{r}_{f,2} + \widehat{r}^{*}_{g,2} + aF|_k\,(S-1)$$

for unique $f, g \in S_k$ and an $F \in O(\mathbb{H})$. Here, $\widehat{r}^{*}_{g,2}$ is a completed mock period function associated $r_g(-X)$.

---

work joint with Kathrin Bringmann and Nikolaos Diamantis.

A reformulation of this theorem reveals the striking analogy with the now classical Eichler-Shimura theorem. Set

$$U_{k,2} := \Big(\mathcal{O}(\mathbb{H}) + \big\{f \in \oplus_{j=1}^{k-1} y^{-j} V_{k-2} \,:\, \xi_k(f) \in V_{k-2}\big\}\Big) \cap \{f : \mathbb{H} \to \mathbb{C};\, f|_k\, T = f\}.$$

Theorem. *The map $\phi : S_k \oplus S_k \to W_{k,2}$ defined by*

$$\phi(f, g) := \widehat{r}_{f,2} + \widehat{r}^{*}_{g,2}$$

induces an isomorphism

$$\bar{\phi} :\; S_k \oplus S_k \cong_{\mathbb{R}} W_{k,2}/V_{k,2},$$

where $V_{k,2} := U_{k,2}\big|_k\,(S-1)$.

The study of mock period function is intimately connected to the study of sesquiharmonic function, first used in work by Duke and Imamoḡlu. We write Δ_k for the weight k Laplacian and $\xi_k := 2iy^k\overline{\partial_{\overline{z}}\,\cdot}$ for the usual elliptic ξ-operator.

Definition. *A real-analytic function $\mathcal{F} : \mathbb{H} \to \mathbb{C}$ is called a* sesquiharmonic Maaß form of weight k*, if the following conditions are satisfied:*

i) We have for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ that $\mathcal{F}|_k\,\gamma = \mathcal{F}$.
ii) We have that $\xi_k \Delta_k(\mathcal{F}) = 0$.
iii) The function $\mathcal{F}$ has at most linear exponential growth at infinity.

In the world of mock modular forms, period polynomials arise as deformation deficits under $(S-1)$ of the holomorphic parts of harmonic (weak) Maaß forms. The behavior of sesquiharmonic modular forms parallels this. Their Fourier expansion can be split into a holomorphic, a harmonic and a nonharmonic part:

$$\mathcal{F}(z) = \sum_{n \gg -\infty} a(n) q^n + \sum_{n \gg -\infty} b(n)\Gamma(1-k, 4\pi n y) q^{-n} + \sum_{n>0} c(n)\boldsymbol{\Gamma}_{k-1}(4\pi n y) q^n$$

for any $\mathcal{F}$ satisfying $\partial_z^{k-1}\xi_k\mathcal{F} \in S_k$. Here,

$$\boldsymbol{\Gamma}_s(y) := \int_y^\infty \Gamma(s,t) t^{-s} e^t \,\frac{dt}{t}.$$

Denoting the middle part of that Fourier expansion by $\mathcal{F}^{+-}$, we find that the deformation deficit $\mathcal{F}^{+-}\big|_k(S-1)$ is a completed mock period function, establishing the connection announced at the beginning.

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Strooker	Jan	(Utrecht)
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Strömberg	Fredrik	(TU Darmstadt)
Swoboda	Jan	(MPI)
Tarizadeh	Abolfazl	(Essen)
Teicher	Mina	(Bar Ilan)
Teichner	Peter	(MPI)
Teymuri Garakani	Mahdi	(Univ. Federal, Rio de Janeiro)
Thiele	Christoph	(UCLA/ Universität Bonn)
Ugarcovici	Ilie	(DePaul University, Chicago)
van der Geer	Gerard	(Universiteit van Amsterdam)
Vallette	Bruno	(MPI)
Varchenko	Alexander	(MPI)
Vdovina	Alina	(Newcastle University, UK)
Vershinin	Vladimir	(University of Montpellier 2)
Viazovska	Maryna	(MPI)
Virdol	Christian	(MPI)
Vishe	Pankaj	(MPI)
Vlasenko	Masha	(MPI)
Waldherr	Matthias	(Universität Köln)
Weiß	Christian	(Universität Frankfurt)
Weiße	Alexander	(MPI)
Weschmann	Uwe	(Universität Heidelberg)
Yagunov	Serge	(MPI)
Yakimova	Oksana	(MPI)
Yang	Jae-Hyun	(Inha University, Korea)
Yassemi	Siamak	(University of Tehran)
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Zeidler	Eberhard	(MPIMN, Leipzig)
Zudilin	Wadim	(NSW, Australia)
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