A NOTE ON THE TWO APPROACHES TO STRINGY FUNCTORS FOR ORBIFOLDS

RALPH M. KAUFMANN

Abstract. In this note, we reconcile two approaches that have been used to construct stringy multiplications. The pushing forward after pulling back that has been used to give a global stringy extension of the functors K_0, K^{top}, A^*, H^*[CR, FG, AGV, JKK2], and the pulling back after having pushed forward, which we have previously used in our (re)-construction program for G–Frobenius algebras, notably in considerations of singularities with symmetries and for symmetric products. A similar approach was also used by [CH] in their considerations of the Chen–Ruan product in a deRham setting for Abelian orbifolds.

We show that the pull–push formalism has a solution by the push–pull equations in two situations. The first is a deRham formalism with Thom push–forward maps and the second is the setting of cyclic twisted sectors, which was at the heart of the (re)-construction program.

We go on to do formal calculations using fractional Euler classes which allows us to formally treat all the stringy multiplications mentioned above in the general setting. The upshot is the formal trivialization of the co–cycles of the reconstruction program using the presentation of the obstruction bundle of [JKK2]. This trivialization can be interpreted in terms of formal twist fields.

Introduction

For global quotients there is a by now standard approach to constructing stringy products via first pulling back and then pushing forward [CR, FG, AGV, JKK2]. We will call this construction the pull–push, which stands for pull after pushing. However, going back to [Ka1, Ka2] we have used a mechanism that first pushes forward and then pulls back to construct and classify G–Frobenius algebra structures. In the same spirit we call this the push–pull — read push after pulling. This approach has been very successful for singularities [Ka2, Ka6] and for special cases of the group, for instance $G = \mathbb{S}_n$, see [Ka4]. The advantage of this approach is that one is left with solving an algebraic co-cycle equation. In many cases this cocycle is unique up to normalized discrete torsion [Ka2, Ka3, Ka4, Ka5, Ka6]. In fact as we argued in [Ka2, Ka4] this mechanism must work if the twisted sectors are cyclic modules over the untwisted sector. Surprisingly a similar technique to ours was used in [CH] where the authors passed to the deRham chains and used formal fractional Thom forms to study the product. Another motivation for this approach is to find the mathematical definition of the notion of twist fields that is prevalent in the physics literature on orbifold conformal field theory.

The goal that is achieved by this paper is the consolidation of these various results and approaches into one big picture:

After reviewing the general setup in §1, we show our first results in §2. Namely, that in the case of cyclic twisted sectors both approaches exist for all the geometric functors considered in [JKK2]. This means that our reconstruction program of [Ka1, Ka2] (see also [Ka3] for a short detailed version) has a solution. It actually then has at least discrete torsion many [Ka5]. The key in this situation is the existence of sections of the pull–back maps which allow us to prove the relevant theorems using only the projection formula.

Next, in §3, we show that in the deRham setting, all the elements of the previous study hold up to homotopy, that is up to exact forms. Hence we can provide a rigorous setting using Thom push–forwards and pull–backs for general global quotients.

In the general setting, see §4, there are some obstacles towards a rigorous calculus of pushing forward and then pulling back. We can make a lot of headway using the excess intersection formula. But then we have to deal with formal fractional Euler classes. There are two types of these classes. The first are formal Euler classes of negative bundles, to be precise the -1 multiples of the normal bundles of the fixed point sets considered to live in K–theory. We can make sense of these as formally defining sections and in the situations above these sections coincide with the ones we constructed. The second type of formal class is that of the fractional Euler class of positive, but fractional classes in rational K–theory. This type of class poses less of a problem and can be treated by adjoining roots to the various rings.

One main result of the formal and rigorous calculations is that in the different situations the classes S_m appearing in the definition of the obstruction bundle produce a co–cycle in the sense of [Ka1, Ka2] that is trivialized by them. Hence one can say that in this setting we have formally identified the twist fields. We close this section with a discussion about possible applications to orbifold Landau–Ginzburg theories that is singularities with symmetries.

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Conventions

Will use at least coefficients in $\mathbb{Q}$ if nothing else is stated. For some applications such as deRham forms we will use $\mathbb{R}$ coefficients. All statements remain valid when passing to $\mathbb{C}$.

1. General setup

We will work in the same setup as in the global part of [JKK2]. That is we simultaneously treat two flavors of geometry, algebraic and differential. For the latter, we consider a stably almost complex manifold X with the action of a finite group G such that the stably almost complex bundle is G equivariant. While for the former X is taken to be a smooth projective variety.

In both situations for $m \in G$ we denote the fixed point set of m by X^m and let

$$I(X) = \amalg_{m\in G} X^m \tag{1.1}$$

be the inertia variety.

We let $\mathcal{F}$ be any of the functors $H^*, K_0, A^*, K^{\text{top}}$, that is cohomology, Grothendieck K_0, Chow ring or topological K–theory with $\mathbb{Q}$ coefficients, and define

$$\mathcal{F}_{stringy}(X,G) := \mathcal{F}(I(X)) = \bigoplus_{m\in G} \mathcal{F}(X^m) \tag{1.2}$$

additively.

We furthermore set

$$\mathrm{Eu}_{\mathcal{F}}(E) = \begin{cases} c_{top}(E) & \text{if } \mathcal{F} = H^* \text{ or } A^* \text{ and } E \text{ is a bundle} \\ \lambda_{-1}(E^*) & \text{if } \mathcal{F} = K \text{ or } K^{top} \end{cases} \tag{1.3}$$

Notice that on bundles Eu is multiplicative. For general K–theory elements we set

$$\mathrm{Eu}_{\mathcal{F},t}(E) = \begin{cases} c_t(E) & \text{if } \mathcal{F} = H^* \text{ or } A^* \\ \lambda_t(E^*) & \text{if } \mathcal{F} = K \text{ or } K^{top} \end{cases} \tag{1.4}$$

Remark 1.1. Notice $\mathrm{Eu}_{\mathcal{F},t}$ is *always* multiplicative and it is a power series that starts with 1 and hence is invertible in $\mathcal{F}(X)[[t]]$.

Definition 1.2. For a positive element E, i.e. E can be represented by a bundle with rank $r = \mathrm{r}k(E)$, we have that $\mathrm{Eu}_{\mathcal{F}}(E) = \mathrm{Eu}_{\mathcal{F},t}(E)|_{t=-1}$ for $\mathcal{F}$ either K_0 or K^{top} and $\mathrm{Eu}_{\mathcal{F}}(e) = $ Coeff of t^r in $[\mathrm{Eu}_{\mathcal{F},t}(E)]$ if $\mathcal{F}$ is A^* or H^*. To be able to deal with both situations, for E, r as above, we define

$$\mathrm{eval}_{\mathcal{F}|r}(\mathrm{Eu}_{\mathcal{F},t}(E)) = \begin{cases} \mathrm{Eu}_{\mathcal{F},t}(E)|_{t=-1} & \text{if } \mathcal{F} \text{ is } K_0 \text{ or } K^{top} \\ \text{Coeff of } t^r \text{ in } [\mathrm{Eu}_{\mathcal{F},t}(E)] & \text{if } \mathcal{F} \text{ is } A^* \text{ or } H^* \end{cases} \tag{1.5}$$

we then have $\mathrm{eval}_{\mathcal{F}|r}(\mathrm{Eu}_{\mathcal{F},t}(E)) = \mathrm{Eu}_{\mathcal{F}}(E)$

Remark 1.3. Notice that for $\mathcal{F}$ as above and each subgroup $H \subset G$, $\mathcal{F}(X^H)$ is an algebra. We will call the internal product $\mathcal{F}(X^m) \otimes \mathcal{F}(X^H) \to \mathcal{F}(X^H)$ the naïve product. There is however a "stringy–product" which preserves the G–grading. To define it, we recall some definitions from [JKK2].

1.1. **The stringy product via pull–push.** For $m \in G$ we let X^m be the fixed point set of m and for a triple $\mathbf{m} = (m_1, m_2, m_3)$ such that $\prod m_i = \mathbf{1}$ (where $\mathbf{1}$ is the identity of G) we let $X^{\mathbf{m}}$ be the common fixed point set, that is the set fixed under the subgroup generated by them.

In this situation, recall the following definitions. Fix $m \in G$ let $r = ord(m)$ be its order. Furthermore let $W_{m,k}$ be the sub–bundle of $TX|_{X^{\mathbf{m}}}$ on which m acts with character $\exp(2\pi i\frac{k}{r})$, then

$$S_m = \bigoplus_k \frac{k}{r} W_{m,k} \tag{1.6}$$

Notice this formula is invariant under stabilization.

We also wish to point out that using the identification $X^m = X^{m^{-1}}$

$$S_m \oplus (S_{m^{-1}}) = N_{X^m/X} \tag{1.7}$$

where for an embedding $X \to Y$ we will use the notation $N_{X/Y}$ for the normal bundle.

Recall from [JKK2] that in such a situation there is a product on $\mathcal{F}(X, G)$ which is given by

$$v_{m_1} * v_{m_2} := \check{e}_{m_3 *}(e_1^*(v_{m_1}) e_2^*(v_{m_2}) \mathrm{Eu}(\mathcal{R}(\mathbf{m}))) \tag{1.8}$$

where the obstruction bundle $\mathcal{R}(\mathbf{m})$ can be defined by

$$\mathcal{R}(\mathbf{m}) = S_{m_1} \oplus S_{m_2} \oplus S_{m_3} \ominus N_{X^{\mathbf{m}}/X} \tag{1.9}$$

and the $e_i : X^{m_i} \to X$ and $\check{e}_3 : X^{m_3^{-1}} \to X$ are the inclusions. Notice, that as it is written $\mathcal{R}(\mathbf{m})$ only has to be an element of K-theory with rational coefficients, but is actually indeed represented by a bundle [JKK2].

Remark 1.4. The first appearance of a push–pull formula was given in [CR] in terms of a moduli space of maps. The product was for the G invariants, that is for the H^* of the inertia orbifold and is known as Chen–Ruan cohomology. In [FG] the obstruction bundle was given using Galois covers establishing a product for H^* on the inertia variety level, i.e. a G–Frobenius algebra as defined in [Ka1, Ka2], which is commonly referred to as the Fantechi–Göttsche ring. In [JKK1], we put this global structure back into a moduli space setting and proved the trace axiom. The multiplication on the Chow ring A^* for the inertia stack was defined in [AGV]. The representation of the obstruction bundle in terms of the S_m and hence the passing to the differentiable setting as well as the two flavors of K–theory stem from [JKK2].

The following is the key diagram:

$$\begin{array}{ccc} & X & \\ i_1 \nearrow & \uparrow i_2 & \nwarrow \check{i}_3 \\ X^{m_1} & X^{m_2} & X^{m_3^{-1}} \\ e_1 \nwarrow & \uparrow e_2 & \nearrow \check{e}_3 \\ & X^{\mathbf{m}} & \end{array} \tag{1.10}$$

Here we used the notation of [JKK2], where $e_3 : X^{\mathbf{m}} \to X^{m_3}$ and $i_3 : X^{m_3} \to X$ are the inclusion, $\vee : I(X) \to I(X)$ is the involution which sends the component X^m to $X^{m^{-1}}$ using the identity map and $\check{i}_3 = i_3 \circ \vee$, $\check{e}_3 = \vee \circ e_3$. This is short hand notation for the general notation of the inclusion maps $i_m : X^m \to X$, $\check{i}_m := i_m \circ \vee = i_{m^{-1}}$.

1.2. **The $\mathcal{F}(X)$ module structure.** Notice that each $\mathcal{F}(X^m)$ is an $\mathcal{F}(X)$ module in two ways which coincide. First via the naïve product and pull back, i.e. $a \cdot v_m := i_m^*(a)v_m$ and secondly via the stringy multiplication $(a, v_m) \mapsto a * v_m$. Now using (1.7) it is straightforward to check that

$$a \cdot v_m = i_m^*(a)v_m = a * v_m \tag{1.11}$$

2. Pull–push: the cyclic case

The way the product is defined in (1.8) is via first pulling back and then pushing forward using the maps e_k. The aim of this section is to establish rigorous arguments, that one can also first push–forward and then pull back while using the maps i_k. This can be done rigorously using sections and the projection formula. We apply this technique in the current paragraph which treats the cyclic case and in §3 which is devoted to the deRham setting.

2.1. **Sections.** We can realize the (re)–construction program of [Ka1, Ka2, Ka3] in two different situations. First, for any functor $\mathcal{F}$ as above provided there are sections to the pull–back maps i_k^* and secondly in a deRham setting, where these sections exist on the level of forms.

Definition 2.1. We say that $\mathcal{F}$ admits sections for (X, G) if for every map $i_m : X^m \to X$ there are sections $i_{ms} : \mathcal{F}(X^m) \to \mathcal{F}(X)$ of the pull–back maps $i_m^* : \mathcal{F}(X) \to \mathcal{F}(X)$, that is $i_m^* \circ i_{ms} = id : \mathcal{F}(X^m) \to \mathcal{F}^(X^m)$

Examples are for instance given by symmetric products $(X^{\times n}, \mathbb{S}_n)$, see [Ka2, Ka3] or manifolds whose fixed loci are empty or points.

Lemma 2.2. *If $\mathcal{F}$ admits sections for (X, G), then $\mathcal{F}(X^m)$ is a cyclic $\mathcal{F}(X)$ module, where the module structure is given by $a \cdot v_m := i^*(a)v_m$. A cyclic generator is 1_m which is the identity element of the algebra $\mathcal{F}(X^m)$ endowed with the naïve product.*

Proof. $v_m = i_m^*(i_{ms}(v_m)) = i_{ms}(v_m) \cdot 1_m$ □

Remark 2.3. We have

$$i_m^*(i_{ms}(a)i_{ms}(b)) = i_m^*(i_{ms}(a))i_m^*(i_{ms}(b)) = ab = i_m^*(i_{ms}(ab)) \tag{2.1}$$

2.2. **A rigorous calculation using sections.**

Proposition 2.4. *If there are sections i_{js} of i_j^* then the following equation holds*

$$\begin{aligned} & \textit{Coeff of } t^r \textit{ in } \{\check{i}_3^*[i_{1s}(v_{m_1})i_{2s}(v_{m_2})\gamma_{m_1,m_2}(t)]\} \\ = \; & \textit{Coeff of } t^r \textit{ in } \left\{\check{i}_3^*[i_{1s}(v_{m_1}\mathrm{Eu}_t(S_{m_1}))i_{2s}(v_{m_2}\mathrm{Eu}_t(S_{m_2}))i_{3s}(\mathrm{Eu}_t(S_{m_3})e_{3*}(\mathrm{Eu}_t(\ominus N_{X^{\mathbf{m}}/X})))]\right\} \\ = \; & v_{m_1} * v_{m_2} \end{aligned} \tag{2.2}$$

where the product $*$ is the product defined in (1.8) and $r = \mathrm{r}k(\mathcal{R}(\mathbf{m}))$ and

$$\begin{aligned} \gamma_{m_1,m_2}(t) &= i_{1s}(\mathrm{Eu}_t(S_{m_1}))i_{2s}(\mathrm{Eu}_t(S_{m_2}))i_{3s}(\mathrm{Eu}_t(S_{m_3})e_{3*}(\mathrm{Eu}_t(\ominus N_{X^{\mathbf{m}}/X}))) \\ &= i_{1s}(\mathrm{Eu}_t(S_{m_1}))i_{2s}(\mathrm{Eu}_t(S_{m_2}))\check{i}_{3s}(\mathrm{Eu}_t(\ominus S_{m_3^{-1}})\check{e}_{3*}(\mathrm{Eu}_t(\ominus N_{X^{\mathbf{m}}/X^{m_3}}))) \end{aligned} \tag{2.3}$$

Proof. Using the projection formula, the defining equation for the sections $i_j^* \circ i_{js} = id$, and the fact that $e_k \circ i_k = j = \check{e}_k \circ \check{i}_k$

$$\begin{aligned} & \check{e}_{3*}[e_1^*(v_{m_1})e_2^*(v_{m_2})\mathrm{Eu}_t(S_{m_1}|_{X^{\mathbf{m}}} \oplus S_{m_2}|_{X^{\mathbf{m}}} \oplus S_{m_3}|_{X^{\mathbf{m}}} \ominus N_{X^{\mathbf{m}}/X})] \\ = \; & \check{e}_{3*}[e_1^*(i_1^*(i_{1s}(v_{m_1}\mathrm{Eu}_t(S_{m_1}))))e_2^*(i_2^*(i_{2s}(v_{m_2}\mathrm{Eu}_t(S_{m_2}))))e_3^*(i_3^*(i_{3s}(\mathrm{Eu}_t(S_{m_3}))))\mathrm{Eu}_t(\ominus N_{X^{\mathbf{m}}/X})] \\ = \; & \check{e}_{3*}[\check{e}_3^*(\check{i}_3^*(i_{1s}(v_{m_1}\mathrm{Eu}_t(S_{m_1}))))\check{e}_3^*(\check{i}_3^*(i_{2s}(v_{m_2}\mathrm{Eu}_t(S_{m_2}))))\check{e}_3^*(\check{i}_3^*(i_{3s}(\mathrm{Eu}_t(S_{m_3}))))\mathrm{Eu}_t(\ominus N_{X^{\mathbf{m}}/X})] \\ = \; & \check{i}_3^*[i_{1s}(v_{m_1}\mathrm{Eu}_t(S_{m_1}))i_{2s}(v_{m_2}\mathrm{Eu}_t(S_{m_2}))i_{3s}(\mathrm{Eu}_t(S_{m_3}))\check{i}_{3s}(\check{e}_{3*}(\mathrm{Eu}_t(\ominus N_{X^{\mathbf{m}}/X})))] \\ = \; & \check{i}_3^*[i_{1s}(v_{m_1}\mathrm{Eu}_t(S_{m_1}))i_{2s}(v_{m_2}\mathrm{Eu}_t(S_{m_2}))i_{3s}((\mathrm{Eu}_t(S_{m_3}))e_{3*}(\mathrm{Eu}_t(\ominus N_{X^{\mathbf{m}}}/X)))] \end{aligned} \tag{2.4}$$

So that taking the coefficient of t^r with $r = \mathrm{r}k(\mathcal{R}(\mathbf{m}))$ we obtain the second claimed equality. For the first equality we can use the fact (2.1)

$$\begin{aligned} & \check{e}_{3*}[e_1^*(v_{m_1})e_2^*(v_{m_2})\mathrm{Eu}_t(S_{m_1}|_{X^{\mathbf{m}}} \oplus S_{m_2}|_{X^{\mathbf{m}}} \oplus S_{m_3}|_{X^{\mathbf{m}}} \ominus N_{X^{\mathbf{m}}/X})] \\ = \; & \check{e}_{3*}[e_1^*(i_1^*(i_{1s}(v_{m_1}\mathrm{Eu}_t(S_{m_1}))))e_2^*(i_2^*(i_{2s}(v_{m_2}\mathrm{Eu}_t(S_{m_2}))))e_3^*(i_3^*(i_{3s}(\mathrm{Eu}_t(S_{m_3}))))\mathrm{Eu}_t(\ominus N_{X^{\mathbf{m}}/X})] \\ = \; & \check{e}_{3*}[e_1^*(i_1^*(i_{1s}(v_{m_1})i_{1s}(\mathrm{Eu}_t(S_{m_1}))))e_2^*(i_2^*(i_{2s}(v_{m_2})i_{2s}(\mathrm{Eu}_t(S_{m_2})))) \\ & \quad e_3^*(i_3^*(i_{3s}(\mathrm{Eu}_t(S_{m_3}))))\mathrm{Eu}_t(\ominus N_{X^{\mathbf{m}}/X})] \end{aligned} \tag{2.5}$$

and proceed as above. Finally, for (2.3), we notice that $N_{X^{\mathbf{m}}/X} = N_{X^{\mathbf{m}}/X^{m_3}} \oplus N_{X^{m_3}/X}|_{X^{\mathbf{m}}}$ and use (1.7). □

Theorem 2.5. *Let $\mathcal{F} \in \{A^*, H^*, K_0, K^*_{\mathrm{top}}\}$ and (X, G) in the appropriate category which admits sections for $\mathcal{F}$ then the equation (2.2) solves the re–construction program of* [Ka2] *with the co–cycles $\gamma_{m_1,m_2} :=$ Coeff of t^r in $(\gamma_{m_1,m_2}(t))$.*

Proof. In this setting the calculation of the Proposition 2.4 applies, which also shows, *a forteriori* that the formulas are independent of the choice of lift and that the $\gamma_{m_1,m_2} :=$ Coeff of t^r in $(\gamma_{m_1,m_2}(t))$ are indeed co–cycles and section independent co–cycles in the sense of [Ka2]. □

Remark 2.6. As we show in §4 below, these co–cycles are formally trivial.

2.3. **Symmetric Product.** In particular the theorem above applies to symmetric products and gives a new way to show the existence of the unique co–cycles in this situation constructed in [Ka4].

3. The Chain level: a rigorous calculation using deRham Chains

Although it is not true in general that the pull back e_i^* is surjective on cohomology or by the usual Chern isomorphism on K–theory, on the level of deRham chains this is true. Notice that in the proof of Proposition 2.4, we only used the projection formula, the defining equation for the sections and the fact that the pull–back is an algebra homomorphism.

Notation 3.1. In this section, we fix coefficients to be $\mathbb{R}$ and we denote by $\Omega^n(X)$ the n–forms on X. Likewise for a bundle $E \to B$ with compact base we denote $\Omega^n_{cv}(E)$ the n forms on E with compact vertical support and let $H^*_{cv}(E)$ the corresponding cohomology with compact vertical support.

3.1. **DeRham chains and Thom push–forwards.** In this section, we will use deRham chains and the Thom construction [BT]. The advantage is that every form on every X^m is a "pull–back" from a tubular neighborhood.

We recall the salient features adapted to our situation from [BT]. Let $i : X \to Y$ be an embedding, then there is a tubular neighborhood $Tub(N_{X/Y})$ of the zero section of the normal bundle $N_{X/Y}$ which is contained in Y. We let $j : Tub(N_{X/Y}) \to X$ be the inclusion.

Now the Thom isomorphism $\mathcal{T} : H^*(X) \to H^{*+codim(X/Y)}_{cv}(N_{X/Y})$ can be realized on the level of forms via capping with a Thom form Θ: $\mathcal{T}(\omega) = \pi^*(\omega) \wedge \Theta$. The Thom map is inverse to the integration along the fiber π_* and hence $\pi_*(\Theta) = 1$. In fact, the class of this form is the unique class whose vertical restriction is a generator and whose integral along the fiber is 1. For any given tubular neighborhood $Tub(N_{X/Y})$ of the zero section of the normal bundle one can find a form representative Θ such that the $supp(\Theta) \subset Tub(N_{X/Y})$.

3.2. **Push–forward.** In this situation the Thom push-forward $i_* : H^*(X) \to H^*(Y)$ is given by $\mathcal{T}$ followed by the extension by zero j_*. These maps are actually defined on the form level. That is we choose Θ to have support strictly inside the tube, and hence the extension by zero outside the tube is well defined for the forms in the image of the Thom map.

$$i_*(\omega) := j_*(T(\omega)) = j_*(\pi^*(\omega) \wedge \Theta) \tag{3.1}$$

Notice that for two consecutive embeddings $X \xrightarrow{e} Y \xrightarrow{i} Z$, on cohomology we have $e_* \circ i_* = (e \circ i)_* : H^*(X) \to H^*(Z)$. On the level of forms depending on the choice of representatives of the Thom form either the identity holds on the nose, since the Thom classes are multiplicative [BT] or they differ by an exact form $e_* \circ i_*(\omega) = (e \circ i)_* + d\tau$.

3.3. **The projection formula on the level of forms.** The following proposition follows from standard facts [BT]

Proposition 3.2 (Projection formula for forms)**.** *With $i : X \to Y$ and embedding and i_* defined as above, for any form $\omega \in \Omega^*(X)$ and any closed form $\phi \in \Omega^*(Y)$ there is an exact form $d\tau \in \Omega^*(Y)$ such that*

$$i_*(i^*(\omega) \wedge \phi) = \omega \wedge i_*(\phi) + d\tau \tag{3.2}$$

Proof. Denote the zero section by $z : X \to N_{X/Y}$ and projection map of the normal bundle by $\pi : N_{X/Y} \to X$, then $i = j \circ z$.

$$X \quad \overset{\pi|Tub}{\underset{z}{\leftrightarrows}} \quad Tub(N_{X/Y}) \quad \overset{j}{\to} \quad Y \tag{3.3}$$

Since π is a deformation retraction, π^* and z^* are chain homotopic [BT], hence $\pi^* \circ z^*(\omega) = \omega + d\tau$. We can now calculate

$$\begin{aligned} i_*(i^*(\omega) \wedge \phi) &= j_*(\pi^*(i^*(\omega) \wedge \phi) \wedge \Theta) \\ &= j_*(\pi^*(z^*(j^*(\omega)) \wedge \pi^*(\phi) \wedge \Theta)) \\ &= j_*((j^*(\omega) + d\tau) \wedge \pi^*(\phi) \wedge \Theta) \\ &= \omega \wedge j_*(\pi^*(\phi) \wedge \Theta) + j_*(d\tau \wedge \pi^*(\phi) \wedge \Theta) \\ &= \omega \wedge i_*(\phi) + dj_*(\tau \wedge \pi^*(\phi) \wedge \Theta) \end{aligned} \tag{3.4}$$

where the penultimate question holds true, since Θ has support inside $Tub(N_{X/Y})$ and the last equation holds true since d commutes with the extension by zero and pull–back. □

3.4. **Sections.** To construct a section on the level of forms, we first notice that the Thom class can be represented by using a bump function f so that if X^{m_i} is given locally on U by the equations $x_k = \cdots = x_N = 0$

$$\mathcal{T}(1)|_U = f dx_k \wedge \cdots \wedge dx_N \tag{3.5}$$

where f is a bump function along the fiber that can be chosen such that $supp(f)$, the support of f, lies strictly inside the tubular neighborhood and moreover $supp(f)$ lies strictly inside this neighborhood. We consider a "characteristic function" g of an open subset U with $supp(f) \subset U \subset Tub(N)$ inside the tubular neighborhood, see Figure 1. Notice that $fg(x) = f(x)$. We let $\mathbf{g}$ be a 0–form with compact vertical support whose restriction to the fiber is given by g.

For any form $\omega \in \Omega^*(X)$, we define

$$i_{ms}(\omega) := j_*(\mathbf{g}\pi_m^*(\omega)) \tag{3.6}$$

Then

$$i_m^*(j_*(\mathbf{g}\pi_m^*(\omega))) = z_m^*(j^*(j_*(\mathbf{g}\pi_m^*(\omega)))) = z_m^*(\mathbf{g})z_m^*(\pi_m^*(\omega)) = \omega + d\tau \tag{3.7}$$

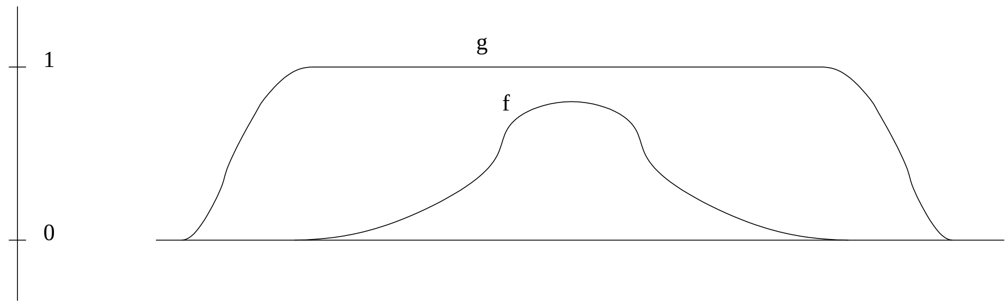

FIGURE 1. A bump function f of the Thom class representative and a characteristic function g

Remark 3.3. Actually $i_*(\omega) := j_*(\mathcal{T}(w))$ is divisible by $j_*(\mathcal{T}(1))$: locally on a coordinate neighborhood U.

$$\begin{aligned} i_{m*}(\omega) &= j_*(\mathcal{T}(w)) = j_*(\pi_m^*(\omega) \wedge \Theta)|_U \\ &= f\pi^*(\omega)|_U \wedge dx_k \wedge \cdots \wedge dx_N \\ &= fg\pi^*(\omega)|_U \wedge dx_k \wedge \cdots \wedge dx_N \\ &= i_{ms}(\omega)|_U \wedge \Theta|_U \end{aligned} \tag{3.8}$$

Theorem 3.4. *With i_{ms} and i_{m*} as defined above the following equation holds on the level of forms.*

$$\begin{aligned} \omega_{m_1} * \omega_{m_2} &:= \check{e}_{m_3*}(e_1^*(\omega_{m_1})e_2^*(\omega_{m_2})\Upsilon(\mathrm{Eu}(\mathcal{R}(\mathbf{m})))) \\ &= \textit{Coeff of } t^r \textit{ in } \{ \check{i}_3^*[i_{s1}(\omega_{m_1})i_{s2}(\omega_{m_2})i_{s1}(\Upsilon(\mathrm{Eu}_t(S_{m_1})))i_{s2}(\Upsilon(\mathrm{Eu}_t(S_{m_1}))) \\ &\quad i_{s3}(\Upsilon(\mathrm{Eu}_t(S_{m_3}))\Upsilon(\mathrm{Eu}_t(\ominus N_{X^{\mathbf{m}}/X}))\Upsilon(\mathrm{Eu}(N_{X^{\mathbf{m}}/X^{m_3}})))]\} + d\tau \end{aligned} \tag{3.9}$$

for some exact form $d\tau$, where $\Upsilon(v)$ is a closed form representative of the class v.

Proof. Completely parallel to the proof of Proposition 2.4, since we have established all equalities up to homotopy, that is up to exact forms. □

Corollary 3.5. *The three point functions coincide with the ones induced by (1.8). That is if Υ denotes the lift of a class to a form and $\Upsilon(v_{m_i}) = \omega_{m_i}$ then*

$$\begin{aligned} \langle \omega_{m_1} * \omega_{m_2}, \omega_{m_3} \rangle &:= \int_X \omega_{m_1} * \omega_{m_2} \wedge \omega_{m_3} \\ &= \int_X \Upsilon(v_{m_1} * v_{m_2}) \wedge \omega_{m_3} \\ &= (v_{m_1} * v_{m_2} \cup v_{m_3}) \cap [X] \qquad (3.10) \\ &= \langle v_{m_1} * v_{m_2}, v_{m_3} \rangle \qquad (3.11) \end{aligned}$$

where $[X]$ is the fundamental class of X and hence the three point functions are independent of the lift.

Proof. Straightforward by Stokes. □

4. Formal and non–formal calculations

In this section, we present some formal and non–formal calculations. This will allow us to make contact with the formal argument of Chen-Hu [CH] who used fractional Thom forms in their arguments in establishing a deRham model for the Chen–Ruan cohomology of Abelian quotients. We have been informed by H.H. Tseng [Ts] that he is working on making the formal part of arguments rigorous in a $\mathbb{C}^*$ equivariant setting, a result which would be great to have.

The ultimate aim would be to rigorously establish the following presentation of the product for the various functors $\mathcal{F}$ without recourse to the deRham theory or sections.

$$\check{i}^{3*}(i_{1*}(v_{m_1}\sigma_1)i_{2*}(v_{m_2}\sigma_2)\check{i}_{3*}(\tilde{\sigma}_3)) = \check{e}_{3*}(e_1^*(v_{m_1})e_2^*(v_{m_2})\mathrm{Eu}(\mathcal{R}(\mathbf{m}))) \tag{4.1}$$

This is: can one answer the following questions?

Question 4.1. Can one find elements σ_i, $\tilde{\sigma}_i$ such that the equation (4.1) holds?

Question 4.2. Is there a setting in which the $\sigma_1, \sigma_2, \tilde{\sigma}_3$ form a co–cycle or better even a trivial co–cycle?

In a formal sense this can be done as we show below, but we are still lacking a rigorous setting. Of course the preceding paragraphs do give rigorous results using the existence of sections for the deRham and the cyclic setting.

The motivation for this comes among other things from physics, where the σ_i or better the σ_g are the twist fields of orbifold conformal field theory.

We first notice that for $r = \mathrm{r}k(\mathcal{R}(\mathbf{m}))$ the r.h.s. of (4.1) is

$$\mathrm{eval}_{\mathcal{F}|r}[e_{3*}(e_1^*(v_1)e_2^*(v_2)\mathrm{Eu}_t(\mathcal{R}(\mathbf{m})))] \tag{4.2}$$

4.1. **A rigorous excess intersection calculation.** We calculate in $\mathcal{F}(X,G)[[t]]$. Using the excess intersection formula [FL, Qu] on

$$\begin{array}{ccc} X^{\mathbf{m}} & \xrightarrow{\check{e}_3} & X^{m_3^{-1}} \\ (e_1,e_2,\check{e}_3)\circ(\Delta,id)\circ\Delta \downarrow & & \downarrow (\check{i}_3,\check{i}_3,\check{i}_3)\circ(\Delta,id)\circ\Delta \\ X^{m_1}\times X^{m_2}\times X^{m_3^{-1}} & \xrightarrow{(i_1,i_2,\check{i}_3)} & X\times X\times X \end{array} \tag{4.3}$$

which has excess bundle $N_{X^{m_1}/X}\oplus N_{X^{m_2}/X}\oplus N_{X^{m_3^{-1}}/X}|X^{\mathbf{m}}\ominus N_{X^{\mathbf{m}}/X^{m_3^{-1}}}$ we can transform the l.h.s. of equation (4.1) as follows:

$$\begin{aligned} l.h.s.(4.1) &= \check{i}_3^*[i_{1*}(v_{m_1}\sigma_1)i_{2*}(v_{m_2}\sigma_2)\check{i}_{3*}(\tilde{\sigma}_3)] \\ &= e_{3*}[e_1^*(v_{m_1}\sigma_1\mathrm{Eu}(N_{X^{m_1}/X}))e_2^*(v_{m_2}\sigma_2\mathrm{Eu}(N_{X^{m_2}/X})) \\ &\qquad e_3^*(\tilde{\sigma}_3\mathrm{Eu}(N_{X^{m_3^{-1}}/X}))\mathrm{Eu}(\ominus N_{X^{\mathbf{m}}/X^{m_3^{-1}}})] \\ &= \mathrm{eval}_{\mathcal{F}|k}\Big\{\check{e}_{3*}[e_1^*(v_{m_1}\sigma_1\mathrm{Eu}_t(N_{X^{m_1}/X}))e_2^*(v_{m_2}\sigma_2\mathrm{Eu}_t(N_{X^{m_2}/X})) \\ &\qquad \check{e}_3^*(\tilde{\sigma}_3\mathrm{Eu}_t(N_{X^{m_3^{-1}}/X}))\mathrm{Eu}_t(\ominus N_{X^{X^{\mathbf{m}}/m_3^{-1}}})]\Big\} \end{aligned} \tag{4.4}$$

$k = \mathrm{r}k(N_{X^{m_1}/X} \oplus N_{X^{m_2}/X} \oplus N_{X^{m_3}/X}|X^{\mathbf{m}} \ominus N_{X^{\mathbf{m}}/X^{m_3^{-1}}})$.

While the r.h.s. can be transformed to

$$\begin{aligned}
r.h.s.(4.1) &= \mathrm{eval}_{\mathcal{F}|r}\{\check{e}_{3*}[e_1^*(v_{m_1}\mathrm{Eu}_t(S_{m_1}))e_2^*(v_{m_2}\mathrm{Eu}_t(S_{m_2})) \\
&\qquad e_3^*(v_{m_3}\mathrm{Eu}_t(S_{m_3}))\mathrm{Eu}_t(\ominus N_{X^{\mathbf{m}}/X})]\} \\
&= \mathrm{eval}_{\mathcal{F}|r}\{\check{e}_{3*}[e_1^*(v_{m_1}\mathrm{Eu}_t(S_{m_1})\mathrm{Eu}_t(N_{X^{m_1}/X})\mathrm{Eu}_t(\ominus N_{X^{m_1}/X})) \\
&\qquad e_2^*(v_{m_2}\mathrm{Eu}_t(S_{m_2})\mathrm{Eu}_t(N_{X^{m_2}/X})\mathrm{Eu}_t(\ominus N_{X^{m_2}/X})) \\
&\qquad e_3^*(\mathrm{Eu}_t(S_{m_3}))e_3^*(\mathrm{Eu}_t(\ominus N_{X^{m_3}/X}))\mathrm{Eu}_t(\ominus N_{X^{\mathbf{m}}/X^{m_3^{-1}}})]\Big\} \\
&= \mathrm{eval}_{\mathcal{F}|r}\Big\{\check{e}_{3*}[e_1^*(v_{m_1}\mathrm{Eu}_t(\ominus S_{m_1^{-1}})\mathrm{Eu}_t(N_{X^{m_1}/X})) \\
&\qquad e_2^*(v_{m_2}\mathrm{Eu}_t(\ominus S_{m_2})\mathrm{Eu}_t(N_{X^{m_2}/X}))\check{e}_3^*(\mathrm{Eu}_t(\ominus S_{m_3^{-1}}))\mathrm{Eu}_t(\ominus N_{X^{\mathbf{m}}/X^{m_3^{-1}}})]\Big\}
\end{aligned} \tag{4.5}$$

$r = \mathrm{r}k(\mathcal{R}(m))$.

4.2. **A formal solution using fractional Euler classes.** Comparing the two sides that is equations (4.4) and (4.5) one is tempted to set:

$$\begin{aligned}
\sigma_{1,t} &= \mathrm{Eu}_t(\ominus S_{m_1^{-1}}) = \mathrm{Eu}_t(S_{m_1})\mathrm{Eu}_t(\ominus N_{X^{m_1}/X}) \\
\sigma_{2,t} &= \mathrm{Eu}_t(\ominus S_{m_2^{-1}}) = \mathrm{Eu}_t(S_{m_2})\mathrm{Eu}_t(\ominus N_{X^{m_2}/X}) \\
\tilde{\sigma}_{3,t} &= \mathrm{Eu}_t(\ominus S_{m_3^{-1}} \ominus N_{X^{m_3^{-1}}/X}) = \vee^*(\mathrm{Eu}_t(S_{m_3})\mathrm{Eu}_t(\ominus N_{X^{m_3}/X})^2)
\end{aligned} \tag{4.6}$$

and then use a kind of evaluation map that is set $\sigma_i = \mathrm{eval}_{\mathcal{F}|\mathrm{v}r(\sigma_i)}(\sigma_{i,t})$ and $\tilde{\sigma}_3 := \mathrm{eval}_{\mathcal{F}|\mathrm{v}r(\tilde{\sigma}_3)}(\sigma_{3,t})$ where $\mathrm{v}r$ denotes the virtual rank. This is, however, not possible, since it is not clear that the respective power series converges for -1 nor is it clear what the coefficient at a rational power or a negative virtual rank means.

4.3. **Adjoining formal symbols.** Let $\mathfrak{S}$ be a collection of elements of rational K–theory $K_{\mathbb{Q}}(Y)$.

We will think of the formulas first in $\mathcal{F}(Y)[\mathfrak{S}]$ and write elements $\mathfrak{S}$ by using the formal symbols $\mathfrak{Eu}(x)$ (one should think "$\mathfrak{Eu}(x) = \mathrm{eval}_{\mathcal{F}|\mathrm{v}r(x)}(x)$")

We can see that we can "solve" the equation (4.1) if we *formally* set

$$\begin{aligned}
\sigma_i &= \mathfrak{Eu}(S_{m_i})\mathfrak{Eu}(\ominus N_{X^{m_i}/X}) \\
\tilde{\sigma}_3 &= \vee^*(\mathfrak{Eu}(S_{m_3})\mathfrak{Eu}(\ominus N_{X^{m_3}/X})^2)
\end{aligned} \tag{4.7}$$

(4.8)

as we explain in the following.

One would like to add certain relations of the form

(1) Enlarging $\mathfrak{S}$ to the semi–group it generates

$$\mathfrak{Eu}(x)\mathfrak{Eu}(y) - \mathfrak{Eu}(x \oplus y) \tag{4.9}$$

(2) If $x+y=E$ with E a bundle

$$\mathfrak{Eu}(x)\mathfrak{Eu}(y)-\mathrm{Eu}(E) \tag{4.10}$$

We denote by $\mathcal{F}_{\mathfrak{S}}(Y)$ the ring obtained by modding out by the relations above.

But one has to be careful with negative bundles, i.e. $\mathfrak{Eu}(\ominus E)$, since these will be morally the inverses to nilpotent elements and hence if we were to localize, we would obtain the zero ring.

Looking at our equations, we would like to have $Y=I(X)$ with the maps i^*_m $\mathfrak{S}=\{S_m, \ominus N_{X^m/X} | m \in G\}$ but then using the relations above, we would get into trouble with $\mathfrak{Eu}(S_m)\mathfrak{Eu}(S_{m^{-1}})\mathfrak{Eu}(\ominus N_{X^m/X})$.

What we will formally do is to view $\mathfrak{Eu}(\ominus N_{X^m/X})$ as division, when it is possible, as we demonstrate below.

On the other hand, there is no problem adjoining only the S_m.

Therefore we will consider adjoining two sets of variables $\mathfrak{S}_1 := \{S_m\}$ and $\mathfrak{S}_2 := \{\ominus N_{X^m/X}\}$. Then we will consider the formulas to live in $\mathcal{F}_{\mathfrak{S}_1}(Y)[\mathfrak{S}_2]$, where we think of $\mathfrak{S}_2$ as formal division operators when defined and also use the convention:

(1) Pull–back: if $i^*(x)$ and x in $K_{\mathbb{Q}}(Y)$ for some morphism i: $i^*(\mathfrak{Eu}(x)) = \mathfrak{Eu}(i^*(x))$

4.4. **Formal manipulations: divisions give formal sections.** For a given inclusion $i: Y \to X$, we sometimes can construct sections i_s of i^*. Notice that i_* is not quite a section due to the self intersection formula:

$$i^*(i_*(a)) = a\mathrm{Eu}(N_{X/Y}) \tag{4.11}$$

this is why we formally set

$$\text{“}i_s(a) := i_*(a\mathfrak{Eu}(\ominus N_{X/Y}))\text{”} \tag{4.12}$$

Indeed, then

$$\text{“}i^*(i_s(a)) := i^*(i_*(a\mathfrak{Eu}(\ominus N_{X/Y}))) = a\mathrm{Eu}(N_{X/Y})\mathfrak{Eu}(\ominus N_{X/Y}) = a\text{”} \tag{4.13}$$

Notice that if i_s is indeed a section:

$$i_*(ab) = i_*(i^*(i_s(a))b) = i_s(a)i_*(b) \tag{4.14}$$

and hence

$$i_*(a) = i_s(a)i_*(1) \tag{4.15}$$

So that we see that if there are sections:

$$i_s(a) = i_*(a)/i_*(1) \tag{4.16}$$

and hence indeed the division operation is well justified and the formal calculation is valid. This was the case in §2 and §3, see in particular equation (3.8).

In the notation above, the l.h.s. of (4.1) after substitution of (4.7) becomes

$$i_3^*[i_{s1}(v_{m_1})i_{s2}(v_{m_2})i_{s1}(\mathfrak{Eu}(S_{m_1}))i_{s2}(\mathfrak{Eu}(S_{m_1}))i_{s3}(\mathfrak{Eu}(S_{m_3})\mathfrak{Eu}(\ominus N_{X^{m_3}/X}))] \tag{4.17}$$

while the r.h.s. of (4.1)can be written as

$$\check{e}_{3*}(e_1^*(v_{m_1})e_2^*(v_{m_2})e_1^*(\mathfrak{Eu}(S_{m_1}))e_2^*(\mathfrak{Eu}(S_{m_2}))e_3^*(\mathfrak{Eu}(S_{m_3}))\mathfrak{Eu}(\ominus N_{X^{\mathbf{m}}/X})) \quad (4.18)$$

4.4.1. **Cocycles.** Notice since $S_m \oplus \vee^*(S_{m^{-1}}) = N_{X^m/X}$ we formally have that

$$\text{“}\check{i}_{3s}(\mathfrak{Eu}(S_{m_3^{-1}})i_{3s}(\mathfrak{Eu}(S_{m_3})\mathfrak{Eu}(\ominus N_{X^{m_3}/X})))$$
$$= i_{s3}(\mathrm{Eu}(N_{X^{m_3}/X})\mathfrak{Eu}(\ominus N_{X^{m_3}/X})) = 1\text{”} \quad (4.19)$$

This if we set $s(m) := i_{ms}(\mathfrak{Eu}(S_m))$ and let $\gamma := ds$, that is $\gamma(m_1, m_2) = s(m_1)s(m_2)/s(m_1m_2)$, then the l.h.s. of (4.1) becomes

$$i_3^*[i_{s1}(v_{m_1})i_{s2}(v_{m_2})\gamma(m_1,m_2)] \quad (4.20)$$

with a trivial co-cycle

$$\text{“}\gamma(m_1,m_2) = i_{1s}(\mathfrak{Eu}(S_{m_1}))i_{2s}(\mathfrak{Eu}(S_{m_2}))\check{i}_{3s}(\mathfrak{Eu}(S_{m_3^{-1}}))^{-1}\text{”} \quad (4.21)$$

This formal equation is very important, since it makes contact with the algebraic problem posed and studied in [Ka2] called the re–construction problem in [Ka1, Ka2, Ka3]. This program has previously been very useful for symmetric products [Ka4] and singularities with symmetries [Ka6].

4.5. **Positive fractional Euler-classes.** Unlike the negative fractional Euler classes $\mathfrak{Eu}(\ominus N)$, we can make the Euler classes of positive rational combinations of bundles rigorous.

First we notice that by the splitting principle [H, FL], we can make a ring extension in which all the constituent bundles split. Then we are left with classes of the form $\mathfrak{Eu}(\frac{k}{r}\mathscr{L})$ that is fractional line bundles. Let $1+u = \mathrm{Eu}(\mathscr{L})$ then we can easily adjoin r–th roots to the extension of R' of $R := \mathcal{F}(X,G)$ in which all isotypical components of the $N_{X^m/X}$ split by passing to $R'[w]/(w^r - u)$. After adjoining all $|m|$–th roots of the various $\mathscr{L}_{m,k,i}$, where the $\mathscr{L}_{m,k,i}$ are the bundles that split $W_{m,k}$, we can simply set

$$\mathfrak{Eu}(S_m) := \prod_{k\neq 0,i} w_{m,k,i}^{k/|m|} \quad (4.22)$$

In this large ring R is a subring and hence we can read off formulae on this subring analogously to the procedure used in the splitting principle.

4.6. **Admissible functors.** Here we collect the formal properties of the functors $\mathcal{F}$ we used in our formal calculations.

Definition 4.3. Let $\mathcal{F}$ be a functor together with an Euler-class Eu_t which has the following properties

(1) $\mathcal{F}$ The Euler class Eu_t is defined for elements of rational K-theory and is multiplicative and takes values in $\mathcal{F}(X)[[t]]$.
(2) $\mathcal{F}$ is contravariant, i.e. it has pullbacks and the Euler-class is natural with respect to these.
(3) $\mathcal{F}$ has push-forwards i_* for closed embeddings $i : X \hookrightarrow Y$.

(4) $\mathcal{F}$ has an excess intersection formula for closed embeddings. That is we have an evaluation morphism $\mathrm{Eu} := \mathrm{eval}_{\mathcal{F}|r}(\mathrm{Eu}_{\mathcal{F},t}) : \mathcal{F}(X)[[t]] \to \mathcal{F}(X)$ such that for the Cartesian squares

$$\begin{array}{ccc} Z & \stackrel{e_2}{\to} & Y_2 \\ \downarrow e_1 & & \downarrow i_2 \\ Y_1 & \stackrel{i_1}{\to} & X \end{array} \tag{4.23}$$

we have the following formula

$$i_2^*(i_{1*}(a)) = e_{2*}(e_1^*(a)\epsilon_j) \tag{4.24}$$

where $\epsilon := \mathrm{Eu}(E)$ with E the excess bundle $E := N_{Y_1/X}|_Z \ominus N_{Z/Y_2}$ and r is its rank.

We call such a functor *admissible*.

All the functors $\mathcal{F}$ studied above are admissible and the calculations of this section —formal and non–formal— carry over to admissible functors. Actually deRham forms are admissible up to homotopy, see below, so that *mutatis mutandis* we can use the same formal arguments on the level of forms.

4.7. **Forms as an admissible functor, fractional Thom classes.** In this case, we have an Euler class and all the properties of an admissible functor are valid on the chain level - up to homotopy, that is up to closed forms.

(1) The Thom push–forward on the chain level induces the push–forward in cohomology induced by the Poincaré pairing, since the Thom class and the Poincaré dual can be represented by the same form [BT].
(2) The projection formula holds, since the pull–back of the Thom class is the Euler class of the normal bundle [BT].
(3) The excess intersection formula holds up to homotopy. Since it holds in cobordism theory and cohomology [Qu] we know that for closed ω the two forms $i_2^* i_{1*}(\omega)$ and $e_{2*}e_1^*(\omega\Upsilon(\mathrm{Eu}(E)))$ differ by a closed form.
(4) In particular, we can use the Thom pushforward and then the divisibility of the push–forward by the Thom class gives us sections.

Hence we can make the same formal calculations as above. Notice that since we indeed have sections as explained in §3, we can avoid mention of $\mathfrak{Eu}(\ominus N_{X^m/X})$ and only have to deal with $\mathfrak{Eu}(S_m)$. In particular equation (3.8) shows that we can indeed divide $i_{m*}(a)$ by the Thom form $i_{m*}(1) = \Theta_m$, which is how we defined i_{ms}.

Now using the formalism of §4.5 and passing to a local trivializing neighborhood U, where the line bundles $\mathscr{L}_{m,k}$ have first Chern class represented by the forms $dx_l, \dots dx_N$, then we get a form representative of $\mathfrak{Eu}(S_m)$

$$\Upsilon(\mathfrak{Eu}(S_m))|_U = f^{k/|m|} \prod_{k \neq 0,i} (dx)^{k/|m|} \tag{4.25}$$

which is the expression for the fractional Thom form that was used in [CH] in their study of Abelian quotients.

What we have now is the generalization to an arbitrary group as well as a trivialization of the co–cycles in terms of roots, thus completing that (re)–construction program of [Ka1, Ka2] in the deRham setting of global quotients. The surprising answer is that there is always a stringy multiplication arising from a co–cycle that is trivializable in a ring extension obtained by adjoining roots. In particular the formulas (4.17) and (4.18) can be made sense of and the co–cycle that appears yields the stringy orbifold product.

4.8. **Remarks about singularities with symmetries aka. orbifold Landau–Ginzburg theories.** In conclusion, we wish to make some remarks about singularities with symmetries as regarded in [Ka2, Ka6]. It is tempting to produce a solution to the stringy multiplication problem in this setting using the formula (1.8). We recall that the relevant data is a pair (f, G) of a singularity $f : \mathbb{C}^n \to \mathbb{C}$ with an isolated critical point at zero and a finite group G with embedding into $Gl(n, \mathbb{C})$ such that $g^*(f) = f$.

To establish this we would need a theory of Chern classes. In the quasi–homogenous case, we have a candidate for the top Chern class associated to the singularity f which is

$$\mathrm{Eu}(f) := hess(f) = det(Hess(f)) \tag{4.26}$$

Using the basic principles of Chern–Weil theory we can even define a total Chern class in this situation:

$$\mathrm{Eu}_t(f) := \sum_i tr(\Lambda^i Hess(f))t^i \tag{4.27}$$

We furthermore need the definition of Chern classes or at least the top Chern class of the obstruction bundle. This is more or less straightforward in the Abelian case. For this we first notice that the role of the tangent space is played by $\mathbb{C}^n$ together with its G action. Each subgroup $\langle g\rangle, g \in G$ then defines a representation and we can define $S_g \in Rep(G) \otimes \mathbb{Q}$ by the formula by noticing that the Eigenbundles in this case are just subrepresentations.

For any subrepresentation V of G we define

$$\mathrm{Eu}_t(V) = \sum_i tr(\Lambda^i Hess(f|_V))t^i \tag{4.28}$$

In the Abelian case it is easy to figure out that indeed $\mathcal{R}(\mathbf{m})$ is the subrepresentation V which is given as follows (cf. [CH, JKK2]). Simultaneously diagonalize the action of G. Let $g = diag(\exp(2\pi i \lambda_j(g))$, with $\lambda_j(g) \in [0, 1)$ then V is spanned by the simultaneous Eigenvectors e_j whose log–Eigenvalues satisfy

$$\lambda_j(g) + \lambda_j(h) = \lambda_j(gh) + 1$$

Hence (1.8) defines a multiplication on the orbifold Milnor ring $\bigoplus_{g\in G} M(f|_{Fix(g)})$ (cf.[Ka2, Ka6]) where M denotes the Minor ring. Pull–back is the restriction of functions and push–forward is the adjoint map. There are even sections $i_g s$ which are given is given by considering a function of fewer variables to be a function of more variables (cf.[Ka2, Ka6, Ka4]). It turns out that this multiplication does not respect the bi–grading in general.

If we furthermore assume that $Hess(f)$ is diagonal, we can even give the fractional Euler classes. They are

$$\mathfrak{Eu}(S_g) = \prod_j (1 + \partial^2/\partial z_j^2(f)t)^{\lambda_j(g)} \tag{4.29}$$

It is at present not clear what geometry this describes. In the case $f_n = z_0^n + \dots z_{n-1}^n$ this gives a multiplication which is part A–model and part B–model. The untwisted sector behaving like the B–side and the twisted sectors behaving like the A–side.

We plan to return to this problem in a subsequent paper.

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E-mail address: kaufmann@math.uconn.edu

University of Connecticut, Department of Mathematics, Storrs, CT 06269 and Purdue University Department of Mathematics, West Lafayette, IN 47907