Group actions on stacks and applications

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Abstract: We provide the correct framework for the treatment of group actions on algebraic stacks (including fixed points and quotients). It is then used to exploit some natural actions on moduli stacks of maps of curves. This leads to the construction of a nice desingularization of the normal stack of curves with level structures considered by Deligne and Mumford ([DM]), and to a presentation of stacks of Galois covers of curves as quotients of a scheme by a finite group.

The motivation at the origin of this article is to investigate some ways in which one can construct moduli for curves and covers above them, using tools from stack theory. This idea arose from reading Bertin-Mézard [BM] (especially its §5) and Abramovich-Corti-Vistoli [ACV]. Our approach is in the spirit of most recent works where one uses the flexibility of the language of algebraic stacks. This language has two (twin) aspects, category-theoretic on one side, and geometric on the other side. Some of our arguments, especially in section 8, are formal arguments involving general constructions concerning group actions on algebraic stacks (this is more on the categoric side). They are, intrinsically, natural enough so they preserve the "modular" aspect. In trying to isolate these arguments, we were led to write results of independant interest. It seemed therefore more adequate to present them in a separated, self-contained part. Thus the article is split into two parts, of comparable size.

Let us motivate the purpose. In general, groups are ubiquitous when one looks at curves and maps between them. For a curve C there is of course its automorphism group $\mathrm{Aut}(C)$, its fundamental group $\pi_1(C)$ with abelian quotients $H^1(C, \mathbb{Z}/n\mathbb{Z})$. For a map $f : C \to D$ there is the group of D-automorphisms of C, the monodromy group, the permutation group of the ramification points,... They all have their own geometric meaning and manifestations. On the level of moduli, these groups often act on the corresponding classifying stacks. This leads, for example, to the presentation $\pi : \mathcal{M}_g(n) \to \mathcal{M}_g$ of the $\mathbb{Q}$-stack of curves, from the moduli space of curves with level n structure. The map π is a quotient by the group $\mathrm{GL}_{2g}(\mathbb{Z}/n\mathbb{Z})$, hence finite étale. The utility of this presentation has been vastly demonstrated in the literature. It is natural to ask : are there such presentations for the various stacks of covers of curves ? Can we handle such group actions on stacks in the same fashion as we do on schemes ? It is expected that the quotient of the stack of curves with ordered marked points $\mathcal{M}_{g,n}$ by the symmetric group should classify curves with unordered marked points ; if G acts on a scheme X then the fixed points of the stack $\mathcal{P}ic(X)$ under G should be related to G-linearized line bundles on X ; the quotient of the modular stack-curve $\mathcal{X}_1(N)$ by $(\mathbb{Z}/N\mathbb{Z})^\times$ should be $\mathcal{X}_0(N)$ (the notations are hopefully well-known to the reader). One should also mention that a few examples appear in the literature : the action of a torus on a stack of stable maps in [Ko], (3.2) and the action of the symmetric group $\mathfrak{S}_d$ on a stack of multisections in [L-MB], (6.6). Our aim is to provide here the material necessary to handle the questions raised above and answer them, as well as to give other applications.

Let us now explain in more detail the structure and the results of the article.

In part A we discuss the notion of a group action on a stack. We are mainly interested in giving general conditions under which the fixed points and the quotient of an algebraic stack are algebraic. In sections 1 and 2 we give definitions and basics on actions. For simplicity let us now consider a flat group

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scheme G and an algebraic stack $\mathcal{M}$, both of finite presentation (abbreviated fp) over some base scheme. Slightly sharper assumptions are in the text. In sections 3 and 4 it is established, respectively, that :

- if G is proper, there is an algebraic stack of fixed points $\mathcal{M}^G$. The map $\mathcal{M}^G \to \mathcal{M}$ is representable, separated and locally fp, with even better properties when G is finite (th. 3.1, prop. 3.5).
- if G is separated, there is a quotient algebraic stack $\mathcal{M}/G$. The map $\mathcal{M} \to \mathcal{M}/G$ is a G-torsor, hence representable, separated and fp (th. 4.1).

Finally in section 5, as a preparation to part B, we recall the concept of rigidification of a stack and look at its behaviour w.r.t. group actions. The result is that rigidification along a group H and quotient by a group G commute under the natural hypotheses (th. 5.1). As far as fixed points are concerned, no such commutation holds in general but we are able to prove a special case, sufficient for what we need later on (prop. 5.3).

In part B we describe applications to moduli of maps of curves. Let G be a given finite monodromy group, with order $|G|$. Our constructions rely on the stacks of curves with level structures. A Teichmüller structure on a curve (in the sense of [DM]) is essentially given by a torsor over the curve, which is itself a curve. This points out a link with the moduli stack of curves with G action, denoted $\mathcal{H}_{g,G}$. We are therefore led to recall, in section 6, what is known about $\mathcal{H}_{g,G}$ in the tame case. In section 7 we derive a definition of a proper $\mathbb{Z}[\frac{1}{|G|}]$-stack $\overline{\mathcal{M}}_g(G)$, that we wish to compare with the normalization of $\overline{\mathcal{M}}_g$ in $\mathcal{M}_g(G)$, introduced by Deligne-Mumford. Denote the latter normalization by $\widetilde{\mathcal{M}}_g(G)$. We show that :

- $\overline{\mathcal{M}}_g(G)$ is a "modular" desingularization of $\widetilde{\mathcal{M}}_g(G)$ over $\mathbb{Z}[\frac{1}{|G|}]$ (th. 7.2.3).

This is, in spirit, close to [ACV], §5.2 but our construction is more straightforward. In section 8, we proceed to find a presentation of $\mathcal{H}_{g,G}$ as a quotient of a scheme by a finite group. We introduce $\mathbb{Z}[\frac{1}{n}]$-stacks denoted $\mathcal{X}^n_{g,G}$ and $\overline{\mathcal{X}}^n_{g,G}$, built up from stacks $\overline{\mathcal{M}}_g(G)$. The parameter n is an arbitrary integer, prime to $|G|$. The meaning of these stacks is more visible on their definition (def. 8.2.1) and we may just mention here that $\mathcal{X}^n_{g,G}$ is a sum of quotients of a scheme by a finite group, as desired. We prove that there is an isomorphism $\mathcal{X}^n_{g,G} \simeq \mathcal{H}_{g,G}$ over $\mathbb{Z}[\frac{1}{n}]$ (th. 8.2.2). In particular, $\mathcal{H}_{g,G}$ is described by a quotient presentation, valid also at characteristics that divide $|G|$. We also show that this isomorphism extends to a morphism $\overline{\mathcal{X}}^n_{g,G} \to \overline{\mathcal{M}}_g\{G\}$ over $\mathbb{Z}[\frac{1}{n}]$, where $\overline{\mathcal{M}}_g\{G\}$ is the stack of stable curves with faithful action of G (prop. 8.3.4).

Notations. Some standard notations were already used above : we note $|E|$ the cardinality of a finite set E. Algebraic stacks are taken in the sense of [L-MB], def. 4.1. If n is a fixed integer and $\mathcal{M}$ is an algebraic stack then we note $\mathcal{M}[\frac{1}{n}] = \mathcal{M} \otimes \mathbb{Z}[\frac{1}{n}]$. For instance we should have written $\mathcal{H}_{g,G}[\frac{1}{n}]$ a few lines above : we will be careful always to do so. The residue field of a point x in a scheme X is denoted by k_x. In part A a scheme S is fixed and we work in the setting of S-stacks, so we often omit S from the notations (for instance in fibred products). The base change of X/S by T/S is written $X_T := X \times_S T$. We often abbreviate "locally of finite presentation" by lfp. We write categories in calligraphic letters (such as $\mathcal{C}$) and 2-categories in fraktur letters (such as $\mathfrak{C}$). A category fibred in groupoids over S is simply called a groupoid over S. In such a groupoid $\mathcal{M}$, the functor of isomorphisms between two objects $x, y \in \mathcal{M}(T)$ is denoted $\mathrm{Isom}_T(x, y)$ or $\mathrm{Isom}_{\mathcal{M}_T}(x, y)$ if mention of $\mathcal{M}$ is needed.

Part A. Group actions on stacks

1 Operations in a 2-category

1.1 We first recall some basics concerning diagrams in a 2-category (in a very sketchy way). Chapter I of [Ha] is a good reference for all we need about this. Loosely speaking, a *diagram* in a 2-category $\mathfrak{C}$ is a set of objects, with a set of 1-morphisms between certain pairs of objects, and a set of 2-morphisms between certain pairs of 1-morphisms with same source and target. We write $\mathcal{D} = \{\mathcal{M}, f, \alpha\}$ to indicate that $\mathcal{M}$ (resp. f, α) ranges through the set of objects (resp. 1-morphisms, 2-morphisms) of the diagram $\mathcal{D}$. The set of i-morphisms ($i = 1, 2$) of the diagram is assumed to be saturated under composition, i.e. including all possible compositions between i-morphisms. We call *i-circuit* a pair of i-morphisms of $\mathcal{D}$ with same source and target ; it *commutes* if its two morphisms coincide. A diagram in $\mathfrak{C}$ is said to be *i-commutative* if any of its i-circuits commutes.

In most diagrams we will consider, to any 1-circuit will be attached a 2-morphism in a natural way, so we will often omit to draw it. For instance in the diagram $\mathcal{D}$ pictured below, with our convention it is understood that there is a 2-morphism $\alpha'' : g'gh \Rightarrow h''f'f$ attached to the exterior rectangular 1-circuit. If $*$ denotes the composition of 2-morphisms between adjoined diagrams ([Ha], I, (1.5)), the 2-circuit pictured on the right is simply written $(\alpha'', \alpha' * \alpha)$. Thus $\mathcal{D}$ is 2-commutative iff $\alpha'' = \alpha' * \alpha$.

$$\begin{tikzcd} \mathcal{M} \arrow[r, "f"] \arrow[d, "h"'] & \mathcal{M}' \arrow[r, "f'"] \arrow[d, "h'"] & \mathcal{M}'' \arrow[d, "h''"] \\ \mathcal{N} \arrow[r, "g"] \arrow[ur, Rightarrow, "\alpha"] & \mathcal{N}' \arrow[r, "g'"] \arrow[ur, Rightarrow, "\alpha'"] & \mathcal{N}'' \end{tikzcd} \qquad \text{together with the 2-circuit} \qquad \begin{tikzcd} & g'h'f \arrow[dr, Rightarrow, "\alpha' f"] & \\ g'gh \arrow[ur, Rightarrow, "g'\alpha"] \arrow[rr, Rightarrow, "\alpha''"'] & & h''f'f \end{tikzcd}$$

1.2 We now look at the 2-category $\mathfrak{C} = \mathfrak{Grpd}/S$ of groupoids over S. In this 2-category all 2-morphisms are isomorphisms so that 2-commutativity of diagrams means "(1-)commutativity up to (given) isomorphisms". Let $\mathcal{M}$ be such a groupoid, and G be a functor in groups over S. We denote by m the multiplication of G, and by e, or sometimes simply 1, its unit section. The weakest possible definition of an action of G on $\mathcal{M}$ is a morphism of groupoids $\mu : G \times \mathcal{M} \to \mathcal{M}$ satisfying 2-commutative diagrams concerning compatibility with respect to the multiplication and the unit section of G :

$$\begin{tikzcd} G \times G \times \mathcal{M} \arrow[r, "m \times \mathrm{id}_{\mathcal{M}}"] \arrow[d, "\mathrm{id}_G \times \mu"'] & G \times \mathcal{M} \arrow[d, "\mu"] \\ G \times \mathcal{M} \arrow[r, "\mu"] \arrow[ur, Rightarrow, "\alpha"] & \mathcal{M} \end{tikzcd} \qquad \begin{tikzcd} G \times \mathcal{M} \arrow[r, "\mu"] \arrow[dr, phantom, "\Searrow \mathfrak{a}"] & \mathcal{M} \\ \mathcal{M} \arrow[u, "e \times \mathrm{id}_{\mathcal{M}}"] \arrow[ur, "\mathrm{id}_{\mathcal{M}}"'] & \end{tikzcd} \tag{1}$$

As usual, given a scheme T, sections $x, y \in \mathcal{M}(T)$, $g \in G(T)$, an arrow $\varphi : x \to y$, we abbreviate $\mu(g, x)$ by $g.x$ and $\mu(g, \varphi)$ by $g.\varphi$. Then α and $\mathfrak{a}$ are merely given by isomorphisms, natural in (g, h, x),

$$\alpha^x_{g,h} : g.(h.x) \xrightarrow{\sim} (gh).x \qquad \text{and} \qquad \mathfrak{a}^x : 1.x \xrightarrow{\sim} x$$

Actually this is not enough to give a reasonable definition because the presence of α and $\mathfrak{a}$ introduces two novelties. Namely, we must make sure, first, that they are compatible with each other, and second, that "higher associativity" (the different ways to pass from an expression such as $g_1.(g_2.(...(g_n.x)))$ to $(g_1...g_n).x$ using α) holds. Similarly, a morphism of G-groupoids $f : \mathcal{M} \to \mathcal{N}$ should be given by a 2-commutative diagram

$$\begin{tikzcd} G \times \mathcal{M} \arrow[r, "\mu"] \arrow[d, "\mathrm{id}_G \times f"'] & \mathcal{M} \arrow[d, "f"] \\ G \times \mathcal{N} \arrow[r, "\nu"] \arrow[ur, Rightarrow, "\sigma"] & \mathcal{N} \end{tikzcd} \tag{2}$$

with some "higher associativity" condition on σ (more concretely we write $\sigma^x_g : g.f(x) \xrightarrow{\sim} f(g.x)$). Sorting all of this out, we arrive at the following definition.

Definition 1.3 Let $\mathcal{M}$ be a groupoid over S and G a functor in groups over S.

(i) An *action* of G on $\mathcal{M}$ is a triple $(\mu, \alpha, \mathfrak{a})$ where $\mu : G \times \mathcal{M} \to \mathcal{M}$ is a morphism of groupoids satisfying the above 2-commutative diagrams (1), and such that for all x and g, h, k we have

$$\alpha^x_{g,hk} \circ g.\alpha^x_{h,k} = \alpha^x_{gh,k} \circ \alpha^{k.x}_{g,h} \qquad \text{and} \qquad (\, g.\mathfrak{a}^x = \alpha^x_{g,1} \quad \text{and} \quad \mathfrak{a}^{h.x} = \alpha^x_{1,h} \,)$$

We say that $(\mathcal{M}, \mu, \alpha, \mathfrak{a})$, or simply $\mathcal{M}$, is a *G-groupoid*. If α and $\mathfrak{a}$ are the identity 2-isomorphisms, we say that the action (or the G-groupoid) is *strict*.

(ii) A *morphism of G-groupoids* between $(\mathcal{M},\mu,\alpha,\mathfrak{a})$ and $(\mathcal{N},\nu,\beta,\mathfrak{b})$ is a pair (f,σ) where $f:\mathcal{M}\to\mathcal{N}$ is a morphism of groupoids over S satisfying the above 2-commutative diagram (2), and such that for all x and g,h we have $f(\alpha^x_{g,h})\circ\sigma^{h.x}_g\circ g.\sigma^x_h=\sigma^x_{gh}\circ\beta^{f(x)}_{g,h}$ and $f(\mathfrak{a}^x)\circ\sigma^x_1=\mathfrak{b}^{f(x)}$.
(iii) An *isomorphism of G-groupoids* is a morphism of G-groupoids which is also an equivalence of categories fibred over S.

The 2-category of G-groupoids over S is noted G-$\mathfrak{Grpd}/S$. Expert readers recognized that the data $(\mathcal{M},\mu,\alpha,\mathfrak{a})$ determine exactly what is called a *lax presheaf in groupoids $\mathcal{F}$ over* $\mathcal{C}=B_0G$, where B_0G is the groupoid associated to G, i.e. the groupoid whose fibre over a scheme T/S has only one object, and morphisms the elements of $G(T)$. This lax presheaf (see [Ho], Appendix B) is described as follows :

- To an object of B_0G, i.e. a scheme T over S, is associated the groupoid $\mathcal{F}(T)=\mathcal{M}(T)$.
- To a morphism of B_0G, i.e. a section $g\in G(T)$, is associated the functor $\mu_g:=\mu(g^{-1},.):\mathcal{M}(T)\to\mathcal{M}(T)$.
- For each $g,h\in G(T)$ there is a natural transformation $\mu_g\circ\mu_h\xrightarrow{\sim}\mu_{hg}$ given by $\alpha_{g^{-1},h^{-1}}$.

So now, definitions 1.3(i) and (ii) are just translations of the definitions we find in [Ho], Appendix B. This link with lax presheaves also explains why in fact, given a G-groupoid $\mathcal{M}$, we will always be able to find an equivalent G-groupoid $\mathcal{M}^{\mathrm{str}}$ such that the 2-isomorphisms α and $\mathfrak{a}$ are the identities.

Proposition 1.4 *The inclusion functor of the 2-category of strict G-groupoids as a fully faithful sub-2-category of G-$\mathfrak{Grpd}/S$, is a 2-equivalence. More precisely, there is a "strictification" functor G-$\mathfrak{Grpd}/S\to G$-$\mathfrak{Grpd}/S$ sending any G-groupoid to an isomorphic G-groupoid with strict action.*

Proof : Let $\mathcal{M}$ be a G-groupoid, and define a G-groupoid $\mathcal{M}^{\mathrm{str}}$ in the following way :

- the sections of $\mathcal{M}^{\mathrm{str}}$ over a scheme T are pairs (g,x) with $g\in G(T)$, $x\in\mathcal{M}(T)$,
- the arrows in $\mathcal{M}^{\mathrm{str}}$ bewteen (g,x) and (h,y) are arrows $\varphi:x\to(g^{-1}h).y$ in $\mathcal{M}(T)$,
- composition of two arrows $\varphi:(g,x)\to(h,y)$ and $\psi:(h,y)\to(k,z)$ is given by

$$x\xrightarrow{\varphi}(g^{-1}h).y\xrightarrow{(g^{-1}h).\psi}(g^{-1}h).((h^{-1}k).z)\xrightarrow{\alpha^z_{g^{-1}h,h^{-1}k}}(g^{-1}k).z$$

There is a strict action of G on $\mathcal{M}^{\mathrm{str}}$: an element $\gamma\in G(T)$ sends an object (g,x) to $(\gamma g,x)$, and sends an arrow $\varphi:x\to(g^{-1}h).y$ to the same arrow as a morphism between $(\gamma g,x)$ and $(\gamma h,y)$.

It remains to check that $\mathcal{M}$ and $\mathcal{M}^{\mathrm{str}}$ are isomorphic. We define a morphism of groupoids $u:\mathcal{M}^{\mathrm{str}}\to\mathcal{M}$ by mapping an object (g,x) to $g.x$, and an arrow $(g,x)\to(h,y)$ represented by $\varphi:x\to(g^{-1}h).y$ to $\alpha^y_{g,g^{-1}h}\circ(g.\varphi)$. Clearly, u is a G-morphism. Furthermore it is essentially surjective because any object x in $\mathcal{M}$ is isomorphic via $\mathfrak{a}^x$ to $1.x$. Finally it is straightforward to see that it is fully faithful, so u is an isomorphism. □

Because of this proposition, our point of view in the next section will be to restrict to considering strict actions. From now until section 5 (included), we assume that the scheme S, viewed as the category of S-schemes, is endowed with the étale or fppf topology. As stackification commutes with finite fibred products, it is clear that an action of G on a groupoid $\mathcal{M}$ extends uniquely to an action on the associated stack $\widetilde{\mathcal{M}}$.

2 Definitions

Definition 2.1 Let $\mathcal{M}$ be a stack over S and G a sheaf in groups over S. Let m denote the multiplication of G, and e its unit section. Let T be an S-scheme.

(i) An *action* of G on $\mathcal{M}$ is a *strict* action like in 1.3. It is given by a morphism of stacks $\mu:G\times\mathcal{M}\to\mathcal{M}$ with 1-commutative diagrams (1) with α and $\mathfrak{a}$ equal to identities. The pair $(\mathcal{M},\mu)$ is called a *G-stack*. By abuse of notation, it is sometimes denoted by $\mathcal{M}$.

(ii) A *1-morphism of G-stacks* between $(\mathcal{M}, \mu)$ and $(\mathcal{N}, \nu)$ is a morphism of G-groupoids like in 1.3. It is given by a pair (f, σ) with a 2-commutative diagram (2) such that for all sections $x \in \mathcal{M}(T)$ and $g, h \in G(T)$, we have $\sigma_g^{h.x} \circ g.\sigma_h^x = \sigma_{gh}^x$. We say that the morphism is *strict* if σ is the identity.

(iii) A *2-morphism of G-stacks* between 1-morphisms (f_1, σ_1) and (f_2, σ_2) is a 2-morphism of stacks $\tau : f_1 \Rightarrow f_2$ compatible with the σ_i i.e. such that for all sections $x \in \mathcal{M}(T)$ and $g \in G(T)$ over a scheme T, we have $\sigma_{2,g}^x \circ g.\tau^x = \tau^{g \cdot x} \circ \sigma_{1,g}^x$. In this way we define a 2-category of G-stacks over S, which will be denoted by G-$\mathfrak{St}/S$ or simply G-$\mathfrak{St}$ if the base S is understood. In particular, given two G-stacks $\mathcal{M}, \mathcal{N}$ there is the stack $\mathcal{H}om_{G\text{-}\mathfrak{St}}(\mathcal{M}, \mathcal{N})$ of 1-morphisms and 2-morphisms between them.

(iv) An *isomorphism of G-stacks* is a 1-G-morphism which is also an equivalence of groupoids over S.

It turns out that there is an equivalent way to define morphisms for G-stacks, in the sense of (ii) and (iii) above. This alternative definition requires to introduce in the first place commutative diagrams in the 2-category G-$\mathfrak{St}/S$. The following remark explains this.

Remark 2.2 Let $\mathcal{D} = \{\mathcal{M}, f, \alpha\}$ be a diagram of stacks where $\mathcal{M}$, f, α range through objects, 1-morphisms, and 2-morphisms of $\mathcal{D}$ respectively. Consider

- the diagram $G \times \mathcal{D} := \{G \times \mathcal{M}\,,\ \mathrm{id}_G \times f\,,\ \mathrm{id}_{\mathrm{id}_G} \times \alpha\}$,
- the diagram $G \times G \times \mathcal{D} := \{G \times G \times \mathcal{M}\,,\ \mathrm{id}_{G \times G} \times f\,,\ \mathrm{id}_{\mathrm{id}_{G \times G}} \times \alpha\}$.

Assume that the objects are in fact G-stacks $(\mathcal{M}, \mu)$ and for any objects $(\mathcal{M}, \mu)$, $(\mathcal{N}, \nu)$ and any 1-morphism $f : \mathcal{M} \to \mathcal{N}$ we are given a 2-isomorphism $\sigma : \nu \circ (\mathrm{id}_G \times f) \Rightarrow f \circ \mu$. Then we can form a new diagram $G \times G \times \mathcal{D} \to G \times \mathcal{D} \to \mathcal{D}$. Precisely, at the stage $G \times \mathcal{D} \to \mathcal{D}$ the 1-morphisms are the μ's, the 2-morphisms are the σ's. At the stage $G \times G \times \mathcal{D} \to G \times \mathcal{D}$, the 1-morphisms are the $(\mathrm{id}_G \times \mu)$'s, the 2-morphisms are the $(\mathrm{id}_G \times \sigma)$'s. Then we can *define* a *2-commutative diagram of G-stacks* to be given by data $(\mathcal{D}, \{\mu\}, \{\sigma\})$ such that $G \times G \times \mathcal{D} \to G \times \mathcal{D} \to \mathcal{D}$ is a 2-commutative diagram of stacks. In particular, we can now *define* the notions of *1-morphisms of G-stacks* and *2-morphisms between 1-morphisms of G-stacks*, by the 2-commutativity of the following two elementary diagrams :

$$(\mathcal{M}, \mu) \xrightarrow{(f,\sigma)} (\mathcal{N}, \nu) \qquad \text{and} \qquad \begin{tikzcd} (\mathcal{M}, \mu) \arrow[r, bend left, "{(f_1,\sigma_1)}"] \arrow[r, bend right, "{(f_2,\sigma_2)}"'] \arrow[r, phantom, "\Downarrow\tau"] & (\mathcal{N}, \nu) \end{tikzcd}$$

Checking 2-commutativity means checking 2-commutativity of the following "prisms" in $\mathfrak{St}/S$

$$\begin{tikzcd} G \times G \times \mathcal{M} \arrow[r, "1\times\mu"] \arrow[d, "1\times1\times f"'] & G \times \mathcal{M} \arrow[r, "\mu"] \arrow[d, "1\times f"'] & \mathcal{M} \arrow[d, "f"] \\ G \times G \times \mathcal{N} \arrow[r, "1\times\nu"'] \arrow[ur, phantom, "1\times\sigma"] & G \times \mathcal{N} \arrow[r, "\nu"'] \arrow[ur, phantom, "\sigma"] & \mathcal{N} \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} G \times G \times \mathcal{M} \arrow[r, bend left] \arrow[r, bend right] \arrow[d] & G \times G \times \mathcal{N} \arrow[d] \\ G \times \mathcal{M} \arrow[r, bend left, dotted] \arrow[r, bend right] \arrow[d] & G \times \mathcal{N} \arrow[d] \\ \mathcal{M} \arrow[r, bend left, dotted] \arrow[r, bend right] & \mathcal{N} \end{tikzcd}$$

□

The 2-category G-$\mathfrak{St}/S$ has arbitrary projective and inductive limits. In particular G-$\mathfrak{St}/S$ has fibred products. Any stack $\mathcal{M}$ over S gives a trivial G-stack $(\mathcal{M}, \mathrm{pr}_2)$ and this gives a 2-functor $\imath : \mathfrak{St} \to G$-$\mathfrak{St}$. The invariants and coinvariants are the 2-adjoints of this functor :

Definition 2.3 Let G be a sheaf in groups over S and $\mathcal{M}$ a G-stack over S.

(i) A *stack of fixed points* $\mathcal{M}^G$ is a stack that 2-represents the 2-functor $\mathfrak{St}^\circ \to \mathfrak{Cat}$ defined by

$$F(\mathcal{N}) = \mathcal{H}om_{G\text{-}\mathfrak{St}}(\imath(\mathcal{N}), \mathcal{M})$$

(the latter is the stack of 2.1(iii), and $\mathfrak{Cat}$ is the 2-category of categories).

(ii) A *quotient stack* $\mathcal{M}/G$ is a stack that 2-represents the 2-functor $\mathfrak{St} \to \mathfrak{Cat}$ defined by

$$F(\mathcal{N}) = \mathcal{H}om_{G\text{-}\mathfrak{St}}(\mathcal{M}, \imath(\mathcal{N}))$$

Remark 2.4 Assume that $\mathcal{M}$ is a G-stack and $H \lhd G$ is a normal subgroup. When all what follows make sense, we expect G/H to act on $\mathcal{M}/H$ and $\mathcal{M}^H$, and to have transitivity isomorphisms $(\mathcal{M}/H)/(G/H) \simeq \mathcal{M}/G$ and $(\mathcal{M}^H)^{G/H} \simeq \mathcal{M}^G$. It is indeed the case, but one has to be careful and it is the appropriate place to stress the weak-versus-strict problem. Denote by $\pi : \mathcal{M} \to \mathcal{M}/H$ the quotient map and, for each section $g \in G(T)$, by $\mu_g = \mu(g, \cdot)$ the automorphism of $\mathcal{M}$ (in fact $\mathcal{M}_T$). By the 2-universal property we have a 2-commutative diagram

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\mu_g"] \arrow[d, "\pi"'] & \mathcal{M} \arrow[d, "\pi"] \\ \mathcal{M}/H \arrow[r, dashed, "\nu_g"] \arrow[ur, Rightarrow, "\epsilon_g"] & \mathcal{M}/H \end{tikzcd}$$

Here ν_g is unique up to a unique 2-isomorphism, so that for any two sections g_1, g_2 there is a 2-isomorphism $\alpha_{g_1,g_2} : \nu_{g_1} \circ \nu_{g_2} \Rightarrow \nu_{g_1 g_2}$ satisfying the cocyle condition. This makes $(\mathcal{M}/H, \nu, \alpha, \mathfrak{a} = \mathrm{id})$ into a weak G-stack. Of course, for $h \in H$ we can choose $\nu_h = \mathrm{id}$ (with nontrivial ϵ_h) and then by unicity $\alpha_{g_1,g_2} = \mathrm{id}$ as soon as $g_1 \in H$ or $g_2 \in H$. So if we denote by $g \mapsto \overline{g}$ the projection $G \to G/H$, it makes sense to define $\overline{\nu}_{\overline{g}} = \nu_g$ and $\overline{\alpha}_{\overline{g}_1,\overline{g}_2} = \alpha_{g_1,g_2}$. Thus we get a weak G/H-stack $(\mathcal{M}/H, \overline{\nu}, \overline{\alpha}, \mathrm{id})$ which we "strictify" (proposition 1.4) to get a strict G/H-stack and we are done. Note that if we first strictified the G-stack $(\mathcal{M}/H, \nu, \alpha, \mathrm{id})$ we would lose the possibility to choose $\nu_h = \mathrm{id}$ and could not induce a G/H-action any more. It is now a simple matter to derive the transitivity isomorphism. All we said also applies to $\mathcal{M}^H$.

Proposition 2.5 *Let G be a sheaf in groups over S and $\mathcal{M}$ a G-stack over S. Then there exists a stack of fixed points $\mathcal{M}^G$, and its formation commutes with base change on S.*

Proof : From the definition we must have $\mathcal{H}om_{\mathfrak{St}}(\mathcal{N}, \mathcal{M}^G) = \mathcal{H}om_{G\text{-}\mathfrak{St}}(\imath(\mathcal{N}), \mathcal{M})$. From the particular case $\mathcal{N} = S$ we deduce $\mathcal{M}^G = \mathcal{H}om_{\mathfrak{St}}(S, \mathcal{M}^G) = \mathcal{H}om_{G\text{-}\mathfrak{St}}(\imath(S), \mathcal{M})$. This is the stack of G-invariant sections of $\mathcal{M}$, whose objects over a base T are pairs $(x, \{\alpha_g\}_{g \in G(T)})$ where $x \in \mathcal{M}(T)$ and $\alpha_g : x \to g.x$ are isomorphisms such that $g.\alpha_h \circ \alpha_g = \alpha_{gh}$ for all sections $g, h \in G(T)$. $\square$

Proposition 2.6 *Let G be a sheaf in groups over S and $\mathcal{M}$ a G-stack over S. Then there exists a quotient stack $\mathcal{M}/G$, and its formation commutes with base change on S.*

Proof : We define a prestack $\mathcal{P}$ as follows : sections of $\mathcal{P}(T)$ are sections of $\mathcal{M}(T)$, and morphisms in $\mathcal{P}(T)$ between x and y are pairs (g, φ) with $g \in G(T)$ and $\varphi : g.x \to y$ a morphism in $\mathcal{M}(T)$. Let $\mathcal{M}/G$ be the stack associated to $\mathcal{P}$. It is straightforward to check the universal 2-property. $\square$

3 Algebraicity of fixed points

From this section on, we consider only *algebraic* stacks. The category $G\text{-}\mathfrak{AlgSt}/S$ of algebraic G-stacks over S is defined to be the full subcategory of $G\text{-}\mathfrak{St}/S$ of G-stacks whose underlying stack is algebraic. In particular all definitions of 2.1 apply, so we do not have to rewrite them. The definitions of 2.3 carry on in an obvious way, namely the *algebraic stack of fixed points* represents a 2-functor $\mathfrak{AlgSt}^\circ \to \mathfrak{Cat}$, and the *quotient algebraic stack* represents a 2-functor $\mathfrak{AlgSt} \to \mathfrak{Cat}$.

Theorem 3.1 *Let G be a proper, flat group scheme lfp over S. Let $\mathcal{M}$ be an algebraic G-stack, with diagonal lfp over S. Then the fixed point stack $\mathcal{M}^G$ (prop. 2.5) is algebraic (so it is a fixed point stack in $\mathfrak{AlgSt}$). The morphism $\epsilon : \mathcal{M}^G \to \mathcal{M}$ is representable, separated and lfp. The formation of $\mathcal{M}^G$ commutes with base change on S.*

Proof : It is enough to show that the morphism $\mathcal{M}^G \to \mathcal{M}$ is representable with the desired properties. So let $f : T \to \mathcal{M}$ be a 1-morphism, corresponding to an object $x \in \mathcal{M}(T)$. The fibre product $\mathcal{M}^G \times_{\mathcal{M}} T$ is the sheaf whose sections over T'/T are collections of isomorphisms $\{\alpha_g : x \simeq g.x\}_{g\in G(T')}$ such that for all sections $g, h \in G(T')$ we have $g.\alpha_h \circ \alpha_g = \alpha_{gh}$. Denote by x_1 and x_2 the objects of $\mathcal{M}(G \times T)$ corresponding to the 1-morphisms $\mathrm{pr}_2 \circ(\mathrm{id}_G \times f)$ and $\mu \circ (\mathrm{id}_G \times f)$. Reformulating what we said above, there is a closed lfp immersion $\mathcal{M}^G \times_{\mathcal{M}} T \hookrightarrow \mathrm{Hom}_T(G_T, \mathrm{Isom}_{G_T}(x_1, x_2))$. With our assumptions, this Hom sheaf is representable by an open constructible subspace of the Hilbert space, which is algebraic, separated and lfp by a famous theorem of Artin [Ar] cor. 6.2. The result follows. □

Remark 3.2 If $\mathcal{M}$ is representable, then $\mathcal{M}^G$ is representable also, and so the fixed points of $\mathcal{M}$ as a space or as a stack are the same (the Yoneda functor from spaces into stacks commutes with projective limits when they exist, but not with inductive limits).

If we relax the assumption of properness of G, it does not seem plausible that we can say much on representability of $\epsilon : \mathcal{M}^G \to \mathcal{M}$. On the other hand it is possible to give more properties of the morphism $\epsilon : \mathcal{M}^G \to \mathcal{M}$ when the group G is *finite* and $\mathcal{M}$ is separated. Under this assumption, we can not directly deduce that the space $\mathrm{Hom}_T(G_T, \mathrm{Isom}_{G_T}(x_1, x_2))$ in the proof above is finite, or even proper, because G may be ramified. However we will deduce the corresponding property for ϵ by giving a slightly different construction of $\mathcal{M}^G$. We start with a lemma :

Lemma 3.3 *Let Q be a finite flat scheme lfp over S. Let $\mathcal{M}$ be an algebraic stack lfp over S. Then the stack $\mathcal{H}om_S(Q, \mathcal{M})$ of morphisms of stacks from Q to $\mathcal{M}$ is algebraic and lfp over S.*

Proof : Let's note $\mathcal{H} := \mathcal{H}om_S(Q, \mathcal{M})$ and $n = [Q : S]$. Given an S-scheme T, we have $\mathcal{H}(T) = \mathcal{M}(Q\times T)$. From this and the fact that Q is affine, after algebraicity is proved it will follow that $\mathcal{H}$ is lfp over S because given a filtering inductive system of S-algebras A_i, we have isomorphisms

$$\varinjlim \mathcal{H}(A_i) \simeq \varinjlim \mathcal{M}(\mathcal{O}_Q \otimes A_i \simeq \mathcal{M}(\varinjlim \mathcal{O}_Q \otimes A_i \simeq \mathcal{M}(\mathcal{O}_Q \otimes \varinjlim A_i \simeq \mathcal{H}(\varinjlim A_i)$$

Now we show that the diagonal of $\mathcal{H}$ is representable, separated and quasi-compact. It is enough to study the sheaf $\mathrm{Isom}_{\mathcal{H}_T}(x, y)$ for two fixed objects $x, y \in \mathcal{H}(T)$. These correspond to objects $\eta \in \mathcal{M}(Q \times T)$ and $\xi \in \mathcal{M}(Q \times T)$, and

$$\mathrm{Isom}_{\mathcal{H}_T}(x, y) = \mathrm{Hom}_T\left(Q_T, \mathrm{Isom}_{\mathcal{M}_{Q\times T}}(\eta, \xi)\right)$$

Here the sheaf $I := \mathrm{Isom}_{\mathcal{M}_{Q\times T}}(\eta, \xi)$ is representable and of finite presentation over Q_T (it is lfp because $\mathcal{M}$ is, by [EGA], I, 6.2.6. which extends to stacks). It keeps these properties as a T-sheaf. Let us introduce the functor H_n which is the component of the full Hilbert functor of $Q_T \times I$ parametrizing 0-dimensional subspaces of length n. It is representable by a separated algebraic space which is lfp (Artin [Ar] cor. 6.2), and in fact the length n component is quasi-compact because $Q_T \times I$ is. Now, the graph of a morphism $Q_T \to I$ defines a point in H_n (by separation of I), such that the restriction of the first projection $Q_T \times I \to Q_T$ is an isomorphism. The sheaf $\mathrm{Isom}_{\mathcal{H}_T}(x, y)$ is thus isomorphic to the corresponding constructible open subspace of H_n. By constructibility this open immersion is quasi-compact ([EGA], 0_{III}, 9.1.5), and, of course, separated.

Now let $U \to \mathcal{M}$ be an atlas ; we can choose U separated. Then I claim that $V := \mathrm{Hom}_S(Q, U)$ will be an atlas for $\mathcal{H}$. First, by Artin's result again V is representable and lfp. As $\mathcal{H}$ is also lfp this shows that the map $V \to \mathcal{H}$ has the same property. Thus we only have to prove that it is formally smooth and surjective. To prove surjectivity take an algebraically closed field k and a morphism $\mathrm{Spec}(k) \to \mathcal{H}$ i.e. a morphism $f : Q_k \to \mathcal{M}_k$. Then Q_k is an artinian scheme, hence a sum of local artinian k-schemes, so we reduce to the local case. By surjectivity of $U \to \mathcal{M}$, the image of the underlying point of Q_k lifts to U_k, and by smoothness the whole morphism lifts. It remains to prove formal smoothness. Let $A \to A_0$ be a surjection of artinian rings with nilpotent kernel. Assume we have a 2-commutative diagram

$$\begin{tikzcd} V \arrow[r] \arrow[dr, phantom, "\Searrow"] & \mathcal{H} \\ \mathrm{Spec}(A_0) \arrow[u] \arrow[ur] & \end{tikzcd} \qquad \text{meaning that we have} \qquad \begin{tikzcd} U_{A_0} \arrow[r] \arrow[dr, phantom, "\Searrow"] & \mathcal{M}_{A_0} \\ Q_{A_0} \arrow[u] \arrow[ur] & \end{tikzcd}$$

As Q_{A_0} is artinian, by smoothness of $U_A \to \mathcal{M}_A$, the map $Q_{A_0} \to U_{A_0} \to U_A$ immediately lifts to $Q_A \to U_A$, and we are done. $\square$

Remark 3.4 If $Q = S[\varepsilon]/\varepsilon^2$ we recover the tangent stack $T(\mathcal{M}/S)$, and the lemma gives a proof of its algebraicity which is simpler than in [L-MB], chap. 17. If $Q = S[x]/x^n$ we get the stack of n-truncated arcs in $\mathcal{M}$. If Q is sum of $n = [Q : S]$ copies of S then $\mathcal{H}om_S(Q, \mathcal{M}) = \mathcal{M}^n$ so the result is trivial.

Proposition 3.5 *Let G be a finite, flat group scheme lfp over S. Let $\mathcal{M}$ be an algebraic G-stack, lfp over S. Then the morphism $\mathcal{M}^G \to \mathcal{M}$ is furthermore quasicompact, and enjoys any property enjoyed by the diagonal of $\mathcal{M}$, by closed immersions, and stable by composition. In particular it is proper if $\mathcal{M}$ is separated.*

Proof : Throughout, we will omit the description of the morphisms of the different stacks introduced, since they are obvious and quite lengthy to write completely. By the lemma applied to $Q = G$ the stack $\mathcal{H} = \mathcal{H}om(G, \mathcal{M})$ is algebraic. We now define two morphisms $a, b : \mathcal{M} \to \mathcal{H}$. Let $x \in \mathcal{M}(T)$, corresponding to a morphism $f : T \to \mathcal{M}$, and look at the compositions

$$G \times T \xrightarrow{\mathrm{id} \times f} G \times \mathcal{M} \underset{\mathrm{pr}_2}{\overset{\mu}{\rightrightarrows}} \mathcal{M}$$

Then we define $a(x) = (\mu \circ (\mathrm{id}_G \times f))^*(x)$ and $b(x) = (\mathrm{pr}_2 \circ (\mathrm{id}_G \times f))^*(x) = x_{G_T}$. In more naive terms, $a(x) = (g \mapsto g.x)$ and $b(x) = (g \mapsto x)$. Now look at the fibre product defined by the diagram

$$\begin{tikzcd} \mathcal{N} \arrow[r] \arrow[d] & \mathcal{M} \arrow[d, "b"] \\ \mathcal{M} \arrow[r, "a"] & \mathcal{H} \end{tikzcd}$$

An object of $\mathcal{N}$ is a pair $(x, \psi^x : a(x) \simeq b(x))$ where ψ^x consists in isomorphisms $\psi^x_g : g.x \simeq x$. We define a closed substack $\mathcal{Z} \subset \mathcal{H}$ by considering the morphisms $\psi : G \to \mathcal{M}$ such that for all sections $g, h \in G(T)$ we have $g(\psi_h) \circ \psi_g = \psi_{gh}$. The stack $\mathcal{M}^G$ is isomorphic to the preimage of $\mathcal{Z}$ in $\mathcal{N}$. The morphism $\epsilon : \mathcal{M}^G \to \mathcal{M}$ is the first projection. Finally, it is not hard to check that $\mathcal{M}^G$ is lfp, using that it is the case for a and for $\mathcal{H}$ and its diagonal.

It remains to prove the properties of the morphism $\mathcal{M}^G \to \mathcal{M}$. First we look at the morphism $b : \mathcal{M} \to \mathcal{H}$. Let $U \to \mathcal{H}$ be a morphism, corresponding to an object $\xi \in \mathcal{M}(G \times U)$. The fibre product $\mathcal{M} \times_{\mathcal{H}} U$ is the stack of triples (T, η, α) composed of a map of schemes $T \to U$, an object $\eta \in \mathcal{M}(T)$ and an isomorphism α between η_{G_T} and ξ_{G_T}. By fppf descent, this is none other than the functor of descent data for ξ with respect to the fppf covering $G_U \to U$. It is represented by a closed sub-algebraic space of $\mathrm{Isom}_{G_U \times_U G_U}(\mathrm{pr}_1^* \xi, \mathrm{pr}_2^* \xi)$. It inherits the properties such as quasi-compactness and separatedness of the diagonal of $\mathcal{M}$. It follows that b has these properties, and similarly for $\mathcal{N}$ and $\mathcal{M}^G$. $\square$

Example 3.6 The morphism $\mathcal{M}^G \to \mathcal{M}$ needs not be a monomorphism of algebraic stacks, although it is a monomorphism of G-algebraic stacks (because of the 2-universal property). Let $\mathcal{M}_{g,2}$ be the stack of smooth 2-pointed curves of genus g. It has an action of the symmetric group $\mathfrak{S}_2$. Let (C, a, b) be a curve over a base S, and suppose that C has two distinct automorphisms σ_1 and σ_2 which exchange the marked points. Then these give two morphisms $S \to (\mathcal{M}_{g,2})^{\mathfrak{S}_2}$, and the compositions $S \to \mathcal{M}_{g,2}$ are equal as morphisms of algebraic stacks. However, they are not equal as morphisms of $\mathfrak{S}_2$-algebraic stacks because the maps σ_1, σ_2 enter in the definition of such a morphism.

Example 3.7 "Fixed points" and "coarse moduli space" do not commute. Assume that $\mathcal{M}$ and $\mathcal{M}^G$ admit coarse moduli spaces M and N. Then M acquires an action of G and there is a map $N \to M^G$. The case where the original action of G on $\mathcal{M}$ is trivial shows that N might be "bigger" than M^G. For a somewhat opposite example, let $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group, of order 8. Its unique involution generates its center Z, and $G = Q/Z \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is not isomorphic to a subgroup of Q. There is a faithful action of G on Q by conjugation, whence an action of G on BQ. Then $(BQ)^G$ is empty, whereas the moduli space of BQ is S and we have $S^G = S$ for the induced action.

Example 3.8 The following example (indicated to me by B. Toen) shows that $\mathcal{M}^G$ may not be algebraic when G is not proper. If H is a commutative group scheme and G a group scheme acting trivially on BH, then an objet of $(BH)^G$ is an H-torsor x together with a morphism $G \to \mathrm{Aut}(x) = H$, so $(BH)^G = BH \times \mathrm{Hom}(G, H)$. This stack is not algebraic in general, though for special groups G, H it may be the case (for instance if both G, H are of multiplicative type — see [SGA3], tome 2).

Finally we mention a simple application of the result above on fixed points. Let G and $\mathcal{M}$ be as in the statement of theorem 3.1, and assume that $\mathcal{M}$ is endowed with the trivial action. Then it is immediate that $\mathcal{M}^G$ is the stack of "G-objects in $\mathcal{M}$" i.e. of objects $x \in \mathcal{M}(T)$ together with a group homomorphism $\rho : G \to \mathrm{Aut}_T(x)$. We denote by $\mathcal{M}\{G\}$ the full faithful subcategory corresponding to the pairs (x, ρ) such that ρ is a faithful action.

Corollary 3.9 *Let $\mathcal{M}$ be an algebraic stack with diagonal lfp over S, and G a flat proper group scheme of finite presentation over S. Then the morphism $\mathcal{M}\{G\} \to \mathcal{M}^G$ defined above is an open quasi-compact immersion, so $\mathcal{M}\{G\}$ is algebraic. In particular, if $\mathcal{M}$ is a smooth (resp. proper) Deligne-Mumford stack and G is a finite étale group such that its order is invertible in $\mathcal{O}_S$, then $\mathcal{M}\{G\}$ is a smooth (resp. proper) Deligne-Mumford stack.*

Proof : To prove that $\mathcal{M}\{G\} \to \mathcal{M}^G$ is an open quasi-compact immersion we take $T \to \mathcal{M}\{G\}$ a morphism from a scheme, corresponding to an object $(x, \rho : G \to \mathrm{Aut}_T(x))$. Under our assumptions, the kernel $K = \ker(\rho)$ is a proper T-group scheme of finite presentation. Then $T^0 := T \times_{\mathcal{M}\{G\}} \mathcal{M}\{G\}$ is the functor $\mathrm{Isom}_{T\text{-gr}}(1_G, K)$, which has an open, quasi-compact immersion into the scheme $\mathrm{Hom}_{T\text{-gr}}(1_G, K) = T$. (Put in other words, T^0 is the locus of points in T such that the unit section $\{1\} \to K$ is an isomorphism; checking openness boils down to proving that if a proper scheme over a discrete valuation ring R has a section, to which it is equal on the special fibre, then it must be $\mathrm{Spec}(R)$.)

The rest of the proposition is classical and we omit the proof. $\square$

Example 3.10 Let $g \geqslant 0$ be an integer and G a finite group. Then $\mathcal{H}_{g,G} := \mathcal{M}_g\{G\}$ is the stack of smooth genus g curves with faithful G action. It is a Deligne-Mumford stack over $\mathrm{Spec}(\mathbb{Z})$, smooth over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{|G|}])$. The stack $\overline{\mathcal{M}}_g\{G\}$ is the stack of stable curves with action. It is proper over $\mathrm{Spec}(\mathbb{Z})$, and smooth over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{|G|}])$.

4 Algebraicity of quotients

Let G be a flat, separated group scheme of finite presentation over S. By a G-torsor over an S-scheme T, we will mean an algebraic space with G-action $p : E \to T$ that locally on T is isomorphic to the trivial G-space $G \times T$. In general such a torsor will not be a scheme, unless if for example G is quasi-affine.

Let $\mathcal{M}$ be a G-algebraic stack over S. In case $\mathcal{M} = X$ is an algebraic space, the quotient of 2.6 is known under the more familiar decription of the stack of G-torsors with an equivariant morphism to X. It is traditionnally denoted $[X/G]$, to avoid confusion with a hypothetical *quotient algebraic space*, but when $\mathcal{M}$ is a general stack no such confusion is possible so it is natural to suppress the brackets.

For general $\mathcal{M}$ we can still define a stack whose objects are G-torsors $p : E \to T$ with an equivariant morphism $(f, \sigma) : E \to \mathcal{M}$. More precisely we define a stack $(\mathcal{M}/G)^*$ whose sections over T are triples $t = (p, f, \sigma)$ as above, and the isomorphisms between t and t' in $(\mathcal{M}/G)^*$ are pairs (u, α) with a G-morphism $u : E \to E'$ and a 2-commutative diagram of G-stacks (see 2.2)

$$\begin{tikzcd} E \arrow[rr, "u"] \arrow[dr, "{(f,\sigma)}"'] & \arrow[d, phantom, "\alpha"] & E' \arrow[dl, "{(f',\sigma')}"] \\ & \mathcal{M} & \end{tikzcd}$$

Theorem 4.1 *Let G be a flat, separated group scheme of finite presentation over S. Let $\mathcal{M}$ be a G-algebraic stack over S. Then the quotient stack $\mathcal{M}/G$ (prop. 2.6) is isomorphic to the stack of G-torsors $(\mathcal{M}/G)^*$, and it is algebraic (so it is a quotient stack in $\mathfrak{AlgSt}$). The canonical morphism $\pi : \mathcal{M} \to \mathcal{M}/G$ is the universal torsor over $\mathcal{M}/G$. The formation of $\mathcal{M}/G$ commutes with base change on S.*

Proof : There are two things to show. First, we explain why $\mathcal{M}/G \simeq (\mathcal{M}/G)^*$. Let $\mathcal{M}/G$ be the quotient as described in 2.6, which is the stack associated to a prestack $\mathcal{P}$. We define a morphism $u : \mathcal{P} \to (\mathcal{M}/G)^*$ by sending an object $x \in \mathcal{M}(T)$ viewed as a map $x : T \to \mathcal{M}$, to the trivial torsor together with the map $G \times T \to \mathcal{M}$ given by $\mu \circ (\mathrm{id} \times x)$, which is clearly equivariant. The image of a morphism $(g, \varphi) : x \to y$ in $\mathcal{P}$ is the multiplication by g (as a map of torsors). This morphism u extends to a morphism of stacks $u' : \mathcal{M}/G \to (\mathcal{M}/G)^*$. It is clearly fully faithful, and also locally essentially surjective by the definition of a torsor. So it is an isomorphism of stacks. From now on we identify $\mathcal{M}/G$ and $(\mathcal{M}/G)^*$.

Second, we prove algebraicity. We keep the above notations of $t = (p, f, \sigma)$ for sections of $\mathcal{M}/G$ and $\varphi = (u, \alpha)$ for morphisms between t and t'. Note that there is a morphism $\omega : \mathcal{M}/G \to BG$ obtained by forgetting the maps to $\mathcal{M}$. To study the diagonal of $\mathcal{M}/G$, we take $t, t' \in (\mathcal{M}/G)(T)$, then ω induces a morphism $\omega_\circ : \mathrm{Isom}_T(t, t') \to \mathrm{Isom}_{BG}(E, E')$ given by $(u, \alpha) \mapsto u$. The target space is algebraic, and the fibre of $\omega_\circ$ over u is the closed (hence algebraic) subspace of $\mathrm{Isom}_{\mathcal{M}_T}(E, u^*E')$ of 2-G-isomorphisms. This shows that $\mathrm{Isom}_T(t, t')$ is representable, separated, quasi-compact. From the fact that the morphism $S \to BG$ is fppf, and from the 2-cartesian diagram

$$\begin{tikzcd}
\mathcal{M} \arrow[r] \arrow[d] & S \arrow[d] \\
\mathcal{M}/G \arrow[r] & BG
\end{tikzcd}$$

we deduce that $\mathcal{M} \to \mathcal{M}/G$ is fppf, and by composition with an atlas of $\mathcal{M}$ we get an fppf presentation of $\mathcal{M}/G$, whence the result by Artin's theorem [L-MB], th. 10.1. $\square$

Remark 4.2 Let $\mathcal{M}$ be a G-algebraic stack as above, and let $x : T \to \mathcal{M}$ be a point. We define the *inertia subgroup* of x as a sheaf on T, by $\mathrm{In}_T(x)(T') = \{(g, \alpha) \,|\, g \in G(T'), \alpha : g(x_{T'}) \simeq x_{T'}\}$ for all T'/T. Note that $\mathrm{In}_T(x) \simeq (G \times T) \times_{\mathcal{M}} T$ so it is algebraic. If we are under the conditions of the theorem, so that a quotient $\pi : \mathcal{M} \to \mathcal{M}/G$ exists, then we have $\mathrm{In}_T(x) \simeq \mathrm{Aut}_T(\pi x)$ canonically. The quotient sheaf $\mathrm{St}_T(x) = \mathrm{In}_T(x)/\mathrm{Aut}_T(x)$, which is not algebraic a priori, measures the default of the action to be free. Locally it is just the set of sections of G such that $gx \simeq x$ (so we might call it the *stabilizer* of x).

Example 4.3 Let $\mathcal{M}$ be a G-algebraic stack over S, so we have morphisms $G \times \mathcal{M} \overset{\mu, \mathrm{pr}_2}{\rightrightarrows} \mathcal{M}$. Given a sheaf F on the smooth-étale site of $\mathcal{M}$, a G-linearization of F is an isomorphism $\alpha : \mu^* F \simeq \mathrm{pr}_2^* F$ which is compatible with associativity : $(m \times \mathrm{id}_{\mathcal{M}})^* \alpha = (\mathrm{id}_G \times \mu)^* \alpha$. We define a (smooth-étale) *G-sheaf* on $\mathcal{M}$ to be a pair (F, α) as above. We can look at the stack of invertible G-sheaves (with obvious isomorphisms of G-sheaves between them), denoted $\mathcal{P}ic^G(\mathcal{M})$, and it is easy to see that we have canonical isomorphisms of stacks $\mathcal{P}ic(\mathcal{M}/G) \simeq \mathcal{P}ic^G(\mathcal{M}) \simeq \mathcal{P}ic(\mathcal{M})^G$. In particular if $\mathcal{P}ic(\mathcal{M})$ is algebraic and G is proper, flat, of finite presentation, we obtain algebraicity of the first two stacks, by theorem 3.1. $\square$

5 Rigidifications

To conclude, we examine how group actions behave with respect to rigidifications. This latter word refers to the situation where there is a fixed flat group lying inside all automorphism groups of objects. Then it can be in some sense removed so as to obtain an algebraic stack $\mathcal{M}/\!\!/H$ called the *rigidification of* $\mathcal{M}$ *along* H, following [ACV].

More precisely, let $\mathcal{M}$ be an algebraic stack and let H be a group scheme which is flat, separated and of finite presentation over the base. We assume that for any object $x \in \mathcal{M}(T)$, there is an injective morphism $i_x : H_T \hookrightarrow \mathrm{Aut}_T(x)$ whose formation is compatible with base change. In this situation, for any $x, y \in \mathcal{M}(T)$, the group H_T acts on the left and on the right on the sheaf $\mathrm{Hom}_T(x, y)$ by the rule $h_1.u.h_2 := i_y(h_1) \circ u \circ i_x(h_2)$. We assume (with self-explanatory notations) that for any T'/T,

$$\text{for any } h \in H(T') \text{ and } u \in \mathrm{Hom}(x_{T'}, y_{T'}), \text{ we have } u^{-1}hu \in H(T') \tag{3}$$

and we say that H *is normal in the sheaves* $\mathrm{Hom}_{\mathcal{M}}(x, y)$.

Theorem 5.1 *With the assumptions above, there exists a stack $\mathcal{M}/\!\!/H$ and a smooth surjective morphism of finite presentation $f : \mathcal{M} \to \mathcal{M}/\!\!/H$ whose formation commutes with base change, such that*

(i) *via f, elements of $H \subset \mathrm{Aut}_T(x)$ map to the identity, and f is universal for this property.*

(ii) *f is a gerbe ; if $\mathcal{M}$ is Deligne-Mumford then $\mathcal{M}/\!\!/H$ also is, and then f is étale.*

(iii) *if $\mathcal{M}$ is separated or proper, then $\mathcal{M}/\!\!/H$ also is.*

(iv) *$\mathcal{M}/\!\!/H$ has a coarse moduli space iff $\mathcal{M}$ has one (then they are the same).*

Now let G be also a flat, separated group scheme of finite presentation. Assume moreover that $\mathcal{M}$ is endowed with an action of G such that the subgroups $H_T \hookrightarrow \mathrm{Aut}_T(x)$ are stable. Then

(v) *the injections $j_x : H_T \hookrightarrow \mathrm{Aut}_{\mathcal{M}_T}(x) \hookrightarrow \mathrm{Aut}_{(\mathcal{M}/G)_T}(x)$ make H normal in the sheaves $\mathrm{Hom}_{\mathcal{M}/G}(x, y)$,*

(vi) *the action of G on $\mathcal{M}$ induces an action on $\mathcal{M}/\!\!/H$,*

and we have a canonical isomorphism $\imath : (\mathcal{M}/G)/\!\!/H \xrightarrow{\sim} (\mathcal{M}/\!\!/H)/G$.

Proof : Points (i) to (iv) are proved in [ACV], 5.1.5 (with complements in [Ro], I, §3 for the normality assumption on H). For (v) one needs just express the group law of the sheaves $\mathrm{Aut}_{(\mathcal{M}/G)_T}(x)$, using the description via the inertia sheaf $\mathrm{In}_T(x)$ of remark 4.2. Point (vi) is straightforward. It remains to check the final statement : from the universal property (i) we get morphisms in both ways that are obviously inverse to each other. □

Remarks 5.2 (i) If k is an algebraically closed field, then the k-points of $\mathcal{M}/\!\!/H$ are the same as those of $\mathcal{M}$, and for x one such point we have $\mathrm{Aut}_{\mathcal{M}/\!\!/H}(f(x)) = \mathrm{Aut}_{\mathcal{M}}(x)/H$.

(ii) As an example it is clear that all objects of the stack $\mathcal{M}\{G\}$ of 3.9 have the center $Z = Z(G)$ in their automorphism group, so we can rigidify and get $\mathcal{M}\{G\}/\!\!/Z$.

Rigidification and fixed points do not commute in general : since the map $\epsilon : \mathcal{M}^G \to \mathcal{M}$ is representable it simply doesn't make sense in general to try to form something such as $\mathcal{M}^G/\!\!/H$. However in the example of a stack of the form $\mathcal{M}\{G\}$ as above, we may get such a result :

Proposition 5.3 *Let $\mathcal{M}$ be an algebraic stack with diagonal lfp over S. Let G be a finite (ordinary) group and H a G-module which is a finite abelian group. Then G acts on $\mathcal{M}\{H\}$ by twisting the actions, namely by the rule $g.(x, \rho) = (x, \rho \circ g^{-1})$. Let $\mathcal{N}$ be any algebraic substack of $\mathcal{M}\{H\}$, stable under G. Then there is a morphism $\imath : \mathcal{N}^G/\!\!/H^G \to (\mathcal{N}/\!\!/H)^G$ and*

(i) *if $H^1(G, H) = 0$ it is a monomorphism,*

(ii) *if $H^2(G, H) = 0$ it is an epimorphism.*

Proof : Given T/S, a section of $\mathcal{N}^G(T)$ is a triple $\mathsf{x} = (x, \rho, \{\alpha_g\}_{g\in G})$. Here ρ is a faithful action of H on x and the $\alpha_g : x \simeq x$ satisfy $\alpha_g \circ h = g^{-1}(h) \circ \alpha_g$; the cocycle condition reduces to $\alpha_{g_1 g_2} = \alpha_{g_1} \circ \alpha_{g_2}$. A morphism $\mathsf{x} \to \mathsf{x}'$ is an H-equivariant map $u : x \to x'$ that commutes with the α_g and α'_g, in particular, elements of H^G give automorphisms of the objects of $\mathcal{N}^G$ and this gives the existence of $\imath$.

To check (i) we may work locally over the base T, so that two sections x, x' of $\mathcal{N}^G/\!\!/H^G$ enjoy a description as above and we have to check that the map from

$$\mathrm{Hom}_{\mathcal{N}^G/\!\!/H^G}(\mathsf{x}, \mathsf{x}') = \left\{ \begin{array}{l} u : x \to x' \text{ such that} \\ u \circ \alpha_g = \alpha'_g \circ u \ \ (\forall g) \end{array} \right\} \Big/ H^G$$

to

$$\mathrm{Hom}_{(\mathcal{N}/\!\!/H)^G}(\imath(\mathsf{x}), \imath(\mathsf{x}')) = \left\{ \begin{array}{l} \overline{u} = u \bmod H, \text{ for } u : x \to x', \\ \text{such that } \overline{u} \circ \overline{\alpha}_g = \overline{\alpha}'_g \circ \overline{u} \ \ (\forall g) \end{array} \right\}$$

is bijective. It is injective since if u, v have the same image, i.e. $v = hu$ for some $h \in H$, then $v\alpha_g = hu\alpha_g = h\alpha'_g u = \alpha'_g g^{-1}(h)u$ and on the other hand it is also equal to $\alpha'_g v = \alpha'_g hu$. It follows that $g^{-1}(h) = h$

for all $g \in G$, so $h \in H^G$. It is surjective because given $\overline{u}$ the class of u modulo H, there exists $h_g \in H$ such that $u\alpha_g = h_g\alpha'_g u$. A straightforward computation gives that h_g is a 1-cocyle. By the assumption $H^1(G,H) = 0$ there is $k \in H$ such that $h_g = g(k)k^{-1}$ and then $u^* = ku$ lifts $\overline{u}$, i.e. $u^* \circ \alpha_g = \alpha'_g \circ u^*$.

To check (ii), locally on T, an object of $(\mathcal{N}/\!\!/H)^G$ is a triple $(x, \rho, \overline{\alpha}_g)$ where $\overline{\alpha}_g$ is the class of an isomorphism $\alpha_g : (x, \rho \circ g^{-1}) \to (x, \rho)$ and satisfies $\overline{\alpha}_{g_1g_2} = \overline{\alpha}_{g_1} \circ \overline{\alpha}_{g_2}$. This means there exist elements h_{g_1,g_2} such that we have $\alpha_{g_1g_2} = h_{g_1,g_2}\alpha_{g_1}\alpha_{g_2}$. Computing $\alpha_{g_1g_2g_3}$ in two different ways, one checks that $h = \{h_{g_1,g_2}\}$ is a 2-cocycle. As $H^2(G,H) = 0$ by assumption we get that h is a coboundary i.e. it has the form $h_{g_1,g_2} = k_{g_1}g_1(k_{g_2})k^{-1}_{g_1g_2}$ for some collection $\{k_g\}_{g\in G} \in H$. It follows that we can consider $\alpha^*_g = k_g\alpha_g$ which satisfies $\alpha^*_{g_1g_2} = \alpha^*_{g_1}\alpha^*_{g_2}$. Then $\mathsf{x} = (x, \rho, \alpha^*_g)$ defines a preimage in $\mathcal{N}^G/\!\!/H^G$. □

Part B. Applications : moduli for covers of curves

Here we give applications of the machinery of part A to simple constructions of moduli stacks. We build on a first construction which is a smooth compactification for the stack of curves with level structures. Then we use fixed points and quotients of various finite group actions, in order to obtain the stack $\mathcal{H}_{g,G} = \mathcal{M}_g\{G\}$ (see 3.10) as a quotient stack.

6 Preliminaries on Hurwitz theory of tame covers

Here we review shortly the construction of Hurwitz stacks of Galois covers. The theory aims at providing a natural compactification of the stack of tame covers of smooth curves $\mathcal{H}_{g,G} \otimes \mathbb{Z}[\frac{1}{|G|}]$, as defined in 3.10. We just wish to present the material necessary to state theorem 6.5, to be used in section 7.

Here are, in more detail, the relevant definitions concerning curves and actions. We fix an integer $g \geqslant 2$, a finite group G and a scheme S. All along this section, we will always assume that $|G| \in \mathcal{O}_S^\times$. By a *curve over* S we mean a proper flat morphism of finite presentation $f : C \to S$ with 1-dimensional reduced and connected fibres. By a G-*curve over* S we mean a curve together with a fibrewise faithful action $\rho : G \to \mathrm{Aut}_S(C)$. The mention of f and ρ is often understood and we simply write C. Finally we say that a curve (or a G-curve) is *smooth*, resp. *nodal*, when the geometric fibres of $C \to S$ are smooth, resp. have only ordinary double point singularities.

Definition 6.1 Let C be a nodal G-curve over S. For $x \in C$ we denote by G_x its stabilizer. Assume that S is the spectrum of an algebraically closed field. We say that the action is *stable* if for every node $x \in C$, and every $g \in G_x$, we have $\det(g) = 1$ if g respects the branches, and -1 otherwise (the determinant is computed with respect to the natural representation of G_x in the 2-dimensional cotangent space T^*_x). For arbitrary S, we say that the action is *stable* if it is the case on each geometric fibre of $C \to S$.

When the base is an algebraically closed field, it is shown that C has a universal equivariant deformation with smooth generic fibre if and only if the action of G is stable ([Be], [E]). Thus, the obstructions to the possibility of deforming the pair (C, ρ) smoothly are all localized at the nodes. When the action is stable, we can distinguish between two kinds of fixed points :

- a fixed point x is *of cyclic type* if either it is smooth, or it is a node and no element of G_x permutes the branches at x. Then G_x is isomorphic to the cyclic group $\mathbb{Z}/e\mathbb{Z}$.
- a fixed point is *of dihedral type* if it is a node and one element of G_x permutes the branches at x. Then G_x is isomorphic to the group $\mathbb{D}_e = \langle a, b \,|\, a^2 = (ab)^2 = b^e = 1\rangle$ of order $2e$. (This is the dihedral group for $e \geqslant 2$, while $\mathbb{D}_1 \simeq \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{D}_2 \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.)

Let S be arbitrary again. Let $\pi : C \to C/G$ be the quotient map. From the assumption $|G| \in \mathcal{O}_S^\times$ it follows that formation of π commutes with base change ([KaMa] A.7.1.3). As usual, we denote by ω the dualizing sheaves.

Definition 6.2 The *ramification divisor* is defined by $R = \mathrm{Div}(\mathcal{O}_R)$ where $\mathcal{O}_R := \mathrm{coker}(\pi^*\omega_{C/G} \to \omega_C)$ (see [GIT] chap. 5, §3 for the operation Div).

The ramification divisor is flat on the base, and its formation commutes with base change. Locally it has an equation given by the determinant of the map $\pi^*\omega_{C/G} \to \omega_C$. A local computation shows that its support $|R|$ is the set of fixed points of the action, excluding nodal points of cyclic type. For instance, look at the family $xy = tz$ over the base $\mathbb{C}[[t]]$. Let $\mathbb{Z}/n\mathbb{Z}$ act by $\sigma(x:y:z) = (e^{2i\pi/n}x : e^{-2i\pi/n}y : z)$. The fixed point scheme is not flat, it has degree 2 on the generic fibre and 3 on the special fibre. If we remove the nodal fixed point from the special fibre, the resulting scheme is flat, it is the support of R.

Definitions 6.3 (i) A *nodal r-marked curve* over a base S is a tuple $(C, x_1, \ldots, x_r)$ where C is a nodal curve and the $x_i : S \to C$ are disjoint sections that land in the smooth locus of C. We often note $\underline{x} := (x_1, \ldots, x_r)$ and we call the relative divisor $\sum x_i$ the *unordered mark*. The morphisms between $(C, \underline{x})$ and $(C', \underline{x}')$ are morphisms of S-curves $u : C \to C'$ such that $ux_i = x_i'v$ for all i. We say that $(C, \underline{x})$ is *stable* if its automorphism group is finite.

(ii) A *stable G-curve r-marked by ramification* is a triple $(C, \rho, \underline{x})$ where $(C, \underline{x})$ is a stable marked curve and ρ is a fibrewise faithful stable action on C, such that the unordered mark and the ramification divisor are equal. A morphism between such objects is a morphism of marked curves that is G-equivariant. Given an integer $g \geqslant 2$, we denote by $\overline{\mathcal{I}}_{g,G,r}$ (resp. $\mathcal{I}_{g,G,r}$) the stack of stable (resp. smooth) G-curves, r-marked by ramification, whose fibres all have arithmetic genus g.

Of course, when C is a smooth G-curve over an algebraically closed field, the conditions on $\sum x_i = R$ in definition 6.3(i) are empty so it brings nothing new to look at the G-curve marked by ramification, or to look at the G-curve alone (except that one must choose an ordering of the ramification points). But when C is singular, this point of view is very convenient :

- it forbids fixed points of dihedral type (but there may be nodes of cyclic type).
- the nodes of C must then lie above nodes of C/G (consequence of the point above).
- the ramification divisor $|R|$ is then étale.

Definition 6.4 Let $(C, \rho, \underline{x})$ be a stable G-curve marked by ramification over a base S. Then the isomorphism class of the representation of G in the cotangent space of the unordered marked sections $\bigoplus_{1 \leqslant i \leqslant r} T_{x_i}^*$ is called the *ramification datum*. We denote it by $\xi = \xi(C, \rho, \underline{x})$.

Note that $\oplus T_{x_i}^*$ is a free $\mathcal{O}_S$-module of rank r. One can show that ξ is locally constant on the base, i.e. there is an integral representation V (a projective module of finite type over $\mathbb{Z}[\frac{1}{|G|}]$) such that $\xi \simeq V \otimes \mathcal{O}_S$ as representations. So we have splittings into open and closed substacks,

$$\mathcal{H}_{g,G}[\tfrac{1}{|G|}] = \amalg\, \mathcal{H}_{g,G,\xi} \quad , \quad \mathcal{I}_{g,G,r} = \amalg\, \mathcal{I}_{g,G,r,\xi} \quad , \quad \overline{\mathcal{I}}_{g,G,r} = \amalg\, \overline{\mathcal{I}}_{g,G,r,\xi}$$

In fact ξ is clearly induced, in the sense of representations, by the list of pairs (G_x, χ_x) where x ranges through a set of representatives $\{x_{i_1}, \ldots, x_{i_b}\}$ of the G-orbits in $\underline{x}$, and χ_x is the character of the representation of G_x on T_x^*. One can equivalently define ξ to be this list, like in [BR].

For a G-curve C, arbitrary automorphisms of C may permute the ramification points. More precisely, we can consider the group $\mathfrak{S}_r^\xi$ of permutations of the r ramification points that preserve the ramification datum ξ. Then there is a natural map $\mathcal{I}_{g,G,r,\xi}/\mathfrak{S}_r^\xi \xrightarrow{\sim} \mathcal{H}_{g,G,\xi}$ which is an isomorphism (see 4.1 for the procedure of quotient). Therefore, we define $\overline{\mathcal{H}}_{g,G,\xi} = \overline{\mathcal{I}}_{g,G,r,\xi}/\mathfrak{S}_r^\xi$. Observe that ξ determines two invariants, namely the genus g' of the quotient curve C/G (by the Riemann-Hurwitz formula), and the number b of G-orbits in $\underline{x}$. The following statement summarizes and completes our discussion :

Theorem 6.5 ([BR]) *The stack $\overline{\mathcal{H}}_{g,G,\xi}$ is a smooth proper stack over the ring $\mathbb{Z}[\frac{1}{|G|}]$. It is equidimensional of dimension $3g' - 3 + b$. The open substack $\mathcal{H}_{g,G,\xi}$ is fibrewise dense.* □

Note that we can view $\overline{\mathcal{H}}_{g,G,\xi}$ as a closed substack of $\overline{\mathcal{M}}_g\{G\} \otimes \mathbb{Z}[\frac{1}{|G|}]$, defined in 3.10. Therefore, we recover most assertions of the theorem as formal consequences of corollary 3.9 (and the dimension may be computed by looking at the map $\overline{\mathcal{H}}_{g,G,\xi} \to \overline{\mathcal{M}}_{g',b}$ taking $(C, \rho, \underline{x})$ to C/G + branch locus).

7 Level structures on curves

In this section we define a stack of stable curves with level structure. It will be used and studied further in the sequel (section 8). We first recall briefly the setting from [DM], including the normal stack $\widetilde{\mathcal{M}}_g(G)$. Then we define the proper stack $\overline{\mathcal{M}}_g(G)$ and show that is desingularizes $\widetilde{\mathcal{M}}_g(G)$ (theorem 7.2.3 is little more precise). At last we prove that stable curves admit level structures locally for the fppf topology (corollary 7.2.4).

Fix an integer $g \geqslant 2$ and denote by Π the fundamental group of a compact Riemann surface of genus g. Choose a finite group G such that there exists a surjection $\Pi \to G$ and put $n := |G|$ its order.

7.1 Level structures on smooth curves

Deligne and Mumford defined a stack $\mathcal{M}_g(G)$ over $\mathbb{Z}[\frac{1}{n}]$, parametrizing smooth curves C/S of genus g together with a *Teichmüller structure* of level G. We briefly recall that such a structure is given by a global section of the sheaf of exterior surjective group homomorphisms from $\pi_1(C, *)$ to G, denoted $\mathcal{H}om^{\text{ext}}(\pi_1(C), G)$. We refer to [DM], (5.5) for the details. Note that the base point $*$ can be chosen arbitrarily because we mod out by inner automorphisms of π_1. When $G = (\mathbb{Z}/n\mathbb{Z})^{2g}$ we recover the stack of curves with full level n structure, usually denoted $\mathcal{M}_g(n)$. Indeed, a surjective morphism $\pi_1(C, *) \to (\mathbb{Z}/n\mathbb{Z})^{2g}$ determines a torsor over C (up to isomorphism), and in turn this gives a basis for $H^1(C, \mathbb{Z}/n\mathbb{Z})$, which is a full level n structure in the classical sense.

As a matter of notation, we will use the letter L and its variants to denote level structures, like $\ell : \pi_1(C, *) \to G$ or $(L \to C, \lambda : G \to \mathrm{Aut}(L))$ for the G-torsor it determines.

Deligne and Mumford also considered the normalization of $\overline{\mathcal{M}}_g$ with respect to the forgetful morphism $\mathcal{M}_g(G) \to \mathcal{M}_g$ (see [DM]). We denote it by $\widetilde{\mathcal{M}}_g(G)$. We recall :

Theorem 7.1.1 ([DM]) *The $\mathbb{Z}[\frac{1}{n}]$-stacks $\mathcal{M}_g(G)$ and $\widetilde{\mathcal{M}}_g(G)$ are algebraic, separated and of finite type. The stack $\widetilde{\mathcal{M}}_g(G)$ is proper. It contains $\mathcal{M}_g(G)$ as a smooth, fibrewise dense substack.* □

In general $\widetilde{\mathcal{M}}_g(G)$ is only normal, but some special cases where it is smooth have been studied in [PdJ] (see also proof of 7.2.4 below). As a complement to the theorem, Deligne extended Serre's lemma to stable curves and proved that if there exists a surjection $G \to (\mathbb{Z}/m\mathbb{Z})^{2g}$ with $m \geqslant 3$, then both stacks are representable ([D], lemma 3.5.1).

7.2 Level structures on stable curves

We now derive from Hurwitz theory a stack which should be viewed as a resolution of singularities of $\widetilde{\mathcal{M}}_g(G)$. The data g, G, n are as above.

The genus of a Galois étale cover of a curve of genus g, with Galois group equal to G, is equal to $n(g-1)+1$. We denote by γ this number. We refer to section 6 for the presentation of the Hurwitz stack with no ramification $\xi = \emptyset$, and to section 5 for the operation of rigidification denoted by a $/\!\!/$ sign. Observe that the center $Z(G) \subset G$ is present in all automorphism groups of curves with G action, so we may rigidify the Hurwitz stack along $Z(G)$ (see 3.9 and 5.2(ii)).

Definition 7.2.1 We define the stack of *stable curves with (Teichmüller) structure of level* G by

$$\overline{\mathcal{M}}_g(G) := \overline{\mathcal{H}}_{\gamma,G,\emptyset}/\!\!/Z(G)$$

By theorem 5.1(ii)(iii), it is a smooth and proper stack over $\mathbb{Z}[\frac{1}{n}]$. We recall from 5.2(i) that for any algebraically closed field k, the objects of $\overline{\mathcal{M}}_g(G)$ over k are the same as those of $\overline{\mathcal{H}}_{\gamma,G,\emptyset}$, but automorphism groups are divided by $Z(G)$. On the open substack of smooth curves we have :

Proposition 7.2.2 *There is an isomorphism of $\mathbb{Z}[\frac{1}{n}]$-stacks $\mathcal{H}_{\gamma,G,\emptyset}/\!\!/Z(G) \xrightarrow{\sim} \mathcal{M}_g(G)$.*

Proof : An object of $\mathcal{H}_{\gamma,G,\emptyset}$ is a smooth curve L/S with fixed-point free action of the group G. We define a morphism $\mathcal{H}_{\gamma,G,\emptyset} \to \mathcal{M}_g(G)$ by sending L to the base $C = L/G$ of the cover, together with the exterior morphism $\ell : \pi_1(C,*) \to G$ which this cover determines by the theory of the fundamental group (we need to have sections of C, which we have locally on the base). It is clear that this gives a morphism as announced by the universal property of rigidification (see 5.1).

Given a curve C/S of genus g, with a Teichmüller structure ℓ, what we said above shows that étale locally on S, this provides a Galois étale cover L of group G. So our morphism is essentially surjective.

It remains to check that it is also a monomorphism. To this aim it is enough to look at a base equal to a field $S = \mathrm{Spec}(k)$, to take L, L' two k-curves with action and to show that the map $\mathrm{Hom}_G(L,L')/Z(G) \to \mathrm{Hom}_k((C,\ell),(C',\ell'))$ is bijective. Injectivity comes from Galois theory of smooth curves, which asserts that the group of automorphisms of L that induce the identity on C is just G. Surjectivity is straightforward from the definitions. $\square$

We now come to the main point :

Theorem 7.2.3 *There is a proper, birational morphism $\overline{\mathcal{M}}_g(G) \to \widetilde{\mathcal{M}}_g(G)$ which is an isomorphism on $\mathcal{U} = \mathcal{M}_g(G)$. The two stacks have the same coarse moduli spaces; in particular, when $\widetilde{\mathcal{M}}_g(G)$ is representable, it is the moduli space of $\overline{\mathcal{M}}_g(G)$.*

Proof : We consider the stack $\mathcal{X}$ defined by the following fibred product:

$$\begin{tikzcd} \mathcal{X} \arrow[r, "w"] \arrow[d, "v"] & \widetilde{\mathcal{M}}_g(G) \arrow[d, "u"] \\ \overline{\mathcal{M}}_g(G) \arrow[r] & \overline{\mathcal{M}}_g \end{tikzcd}$$

The stacks $\overline{\mathcal{M}}_g(G)$ and $\widetilde{\mathcal{M}}_g(G)$ share the common open substack $\mathcal{U} = \mathcal{M}_g(G)$, which therefore embeds diagonally in $\mathcal{X}$. Let $\mathcal{Y}$ be the reduced closed substack structure on the closure of the image of $\mathcal{U}$ in $\mathcal{X}$. The restriction $v|_{\mathcal{Y}} : \mathcal{Y} \to \overline{\mathcal{M}}_g(G)$ is (representable) finite because it is the composition of a closed immersion with the pullback of u. Also it is birational. It follows that it is an isomorphism : indeed we can work étale locally on $\overline{\mathcal{M}}_g(G)$, this brings us back to the case of a morphism of schemes, and then Zariski's main theorem [EGA4], (8.12.10) yields the result. Therefore $\pi = w \circ (v|_{\mathcal{Y}})^{-1}$ is a morphism as announced.

We now show that the two stacks have the same moduli space. Let $\overline{M}$, $\widetilde{M}$ be the respective moduli spaces, which exist by [KeMo]. The induced map $\overline{M} \to \widetilde{M}$ is again finite, birational landing in a normal space. By Zariski's main theorem this is an isomorphism. $\square$

The stack $\widetilde{\mathcal{M}}_g(G)$ is not smooth (in general) and not modular (i.e. one can hardly say what its objects stand for). Therefore, $\overline{\mathcal{M}}_g(G)$ corrects these defficiencies. This allows deeper study as will be done in 8.3. We should emphasize that the map $\overline{\mathcal{M}}_g(G) \to \overline{\mathcal{M}}_g[\frac{1}{n}]$ is not representable, whereas it is the case for the map $\widetilde{\mathcal{M}}_g(G) \to \overline{\mathcal{M}}_g[\frac{1}{n}]$. Most of the time this is a minor inconvenient ; for instance, in the following result of essential interest, the problem is easily overcome.

Corollary 7.2.4 *Let C/S be a stable curve of genus $g \geqslant 2$. Assume that $n = |G| \geqslant 3$ (with $n \in \mathcal{O}_S^\times$ as always). Then C has a level G structure after a finite faithfully flat extension $S' \to S$.*

Proof : We first provide a finite flat cover of $\overline{\mathcal{M}}_\gamma[\frac{1}{n}]$ by a scheme ($\gamma = n(g-1)+1$). For a well-chosen group Γ, the stack $\widetilde{\mathcal{M}}_\gamma(\Gamma)[\frac{1}{n}]$ will answer the question. Let $\Gamma = \pi/\pi^{(k)}\pi^m$ be the quotient of π by the characteristic subgroup generated by $(k+1)$-th commutators and m-th powers, see [PdJ]. We have to make sure that the order of Γ is invertible on $\mathrm{Spec}(\mathbb{Z}[\frac{1}{n}])$. So if n is a power of 2 we choose $(k,m) = (3,4)$ and if not, we pick an odd prime factor p of n and choose $(k,m) = (3,p)$. By [PdJ], th. 3.11, the normalization $\widetilde{\mathcal{M}}_\gamma(\Gamma)[\frac{1}{n}]$ is a smooth proper scheme so the covering map to $\overline{\mathcal{M}}_\gamma[\frac{1}{n}]$ is finite and flat.

By pullback along $\overline{\mathcal{H}}_{\gamma,G,\emptyset} \to \overline{\mathcal{M}}_\gamma[\frac{1}{n}]$ this provides a scheme U with a finite flat cover to both stacks $\overline{\mathcal{H}}_{\gamma,G,\emptyset}$ and $\overline{\mathcal{M}}_g(G)$. Now the curve C gives a morphism $S \to \overline{\mathcal{M}}_g[\frac{1}{n}]$. Consider the following diagram of fibred products :

$$\begin{tikzcd} S' \arrow[r] \arrow[d] & \mathcal{V} \arrow[r] \arrow[d] & S \arrow[d] \\ U \arrow[r] & \overline{\mathcal{M}}_g(G) \arrow[r] & \overline{\mathcal{M}}_g[\frac{1}{n}] \end{tikzcd}$$

The stack $\mathcal{V}$ is not representable, whence the need for U. Clearly $S' \to S$ is finite and flat. □

7.2.5 Let us finally describe functoriality with respect to the group G. Let $N \lhd G$ be a normal subgroup, and $G' = G/N$. Let $n' = |G'|$ and $\gamma' = n'(g-1)+1$. For classical level structures, there is a map $\mathcal{M}_g(G) \to \mathcal{M}_g(G')[\frac{1}{n}]$ given by the composition $\pi_1(C) \to G \to G'$. The analogue for our compactifications goes as follows. Start from

$$\overline{\mathcal{H}}_{\gamma,G,\emptyset} \longrightarrow \overline{\mathcal{H}}_{\gamma',G',\emptyset}$$
$$(L,\lambda) \mapsto (L/N, \lambda/N)$$

where $\lambda/N : G/N \to \mathrm{Aut}(L/N)$ is the induced action. As the center $Z(G)$ maps to the center $Z(G')$ we deduce a morphism after rigidifications, $\overline{\mathcal{M}}_g(G) \to \overline{\mathcal{M}}_g(G')[\frac{1}{n}]$. This map is proper, quasi-finite and flat. It is ramified on the boundary and étale on the locus of smooth curves.

Notation 7.2.6 To conform with tradition (see 7.1), when $G = (\mathbb{Z}/n\mathbb{Z})^{2g}$ we note $\overline{\mathcal{M}}_g(n)$ for $\overline{\mathcal{M}}_g(G)$.

8 A presentation of $\mathcal{H}_{g,G}$ using level structures

In this section we wish to show how the tools developped in the previous sections relate to the stack $\mathcal{H}_{g,G} = \mathcal{M}_g\{G\}$ (see 3.10). Note that here, we include the prime divisors of $|G|$ among the characteristics (precisions on this appear below). The first step is to get to the definition of certain stacks over $\mathbb{Z}[\frac{1}{n}]$, denoted $\mathcal{X}^n_{g,G}$ and $\overline{\mathcal{X}}^n_{g,G}$. Then we show that there is an isomorphism $\mathcal{X}^n_{g,G} \simeq \mathcal{H}_{g,G}[\frac{1}{n}]$ which gives the desired presentation (theorem 8.2.2). In the last subsection we extend this morphism to $\overline{\mathcal{X}}^n_{g,G}$ (proposition 8.3.4). This involves an interesting deeper study of automorphisms of level structures on stable curves.

From now until the end, we fix a finite group G and an integer $n \geqslant 3$ prime to $|G|$. In particular we must warn the reader that n is *not* any more the order of G. The reason is that G will come up as a group acting on curves of genus g, and *not* as a level structure any more. The integer n will come up as the abelian level when we will use the smooth stack $\overline{\mathcal{M}}_g(n)$ defined in section 7.

8.1 G-module structures on the n-torsion

Assume that G acts faithfully on a curve C of genus $g \geqslant 2$, defined over an algebraically closed field of characteristic prime to n. Then it acts (faithfully, by Serre's lemma) on the n-torsion of the jacobian of the curve $C[n] \simeq (\mathbb{Z}/n\mathbb{Z})^{2g}$. We denote by H_n the abstract group $(\mathbb{Z}/n\mathbb{Z})^{2g}$. A G-module structure on H_n is just a group homomorphism $i : G \to \mathrm{Aut}(H_n) = \mathrm{GL}_{2g}(\mathbb{Z}/n\mathbb{Z})$, and we say that H_n is a faithful module if i is injective.

Notation 8.1.1 We introduce the following finite set :

$$\mathcal{I}_n = \left\{ \begin{array}{c} \text{faithful } G\text{-module structures on } H_n \text{ such that for all divisors } m|n \\ \text{with } m \geqslant 3, \text{ the induced } G\text{-module } H_m = H_n \otimes \mathbb{Z}/m\mathbb{Z} \text{ is faithful} \end{array} \right\}$$

It is clearly enough to check the condition for prime divisors m. By its very definition, $\mathcal{I}$ is a functor from $\mathbb{Z}_{\geqslant 3}$ anti-ordered by divisibility, to the category of finite sets. Namely for $m|n$ with $m \geqslant 3$, we have a morphism $\mathcal{I}_n \to \mathcal{I}_m$ given by $i \mapsto i \otimes \mathbb{Z}/m\mathbb{Z}$.

Remark that if n is a power of a prime $\ell \geqslant 3$, then the kernel of the reduction map $\mathrm{GL}_{2g}(\mathbb{Z}/n\mathbb{Z}) \to \mathrm{GL}_{2g}(\mathbb{Z}/\ell\mathbb{Z})$ is an ℓ-group. So in that case, the condition in the definition of $\mathcal{I}_n$ is automatically verified because G can't meet this kernel ($|G|$ and n are coprime).

8.1.2 The stack of stable curves of genus g with level n structure has been defined in 7.2.1 by rigidification of the stack $\overline{\mathcal{H}}_{\gamma,H,\emptyset}$ with $H=(\mathbb{Z}/n\mathbb{Z})^{2g}$ and $\gamma=n^{2g}(g-1)+1$. We fix $i\in\mathcal{I}_n$ as introduced above. Then there is an action μ_i of G on $\overline{\mathcal{H}}_{\gamma,H,\emptyset}$ (see definition 2.1 for the relevant concepts). It goes as follows : for any $g\in G$,

- $g.(L,\lambda):=(L,g.\lambda)$ for an H-curve (L,λ), where $g.\lambda=\lambda\circ g^{-1}$,
- $g.u:=u$ for a map $u:(L,\lambda)\to(L',\lambda')$.

We note simply $\overline{\mathcal{H}}^i_{\gamma,H,\emptyset}$ for the G-stack $(\overline{\mathcal{H}}_{\gamma,H,\emptyset},\mu_i)$. Now let us describe the fixed point stack (see proposition 2.5 for a general description and theorem 3.1 as well as proposition 3.5 for its properties). Let S be a base scheme over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{n}])$.

(i) An object of $(\overline{\mathcal{H}}^i_{\gamma,H,\emptyset})^G$ over S is a triple $t=(L,\lambda,\{\alpha_g\}_{g\in G})$. The pair (L,λ) is a curve with action of H, and $\alpha_g:(L,g.\lambda)\to(L,\lambda)$ is an isomorphism such that $\alpha_{g_1g_2}=\alpha_{g_1}\circ g_1.\alpha_{g_2}=\alpha_{g_1}\circ\alpha_{g_2}$ because of the second bullet above. In particular, α_g is H-equivariant which means that for all $h\in H$,

$$\alpha_g\circ g^{-1}(h)=h\circ\alpha_g$$

This is the same thing to say that α_g belongs to the normalizer of H in $\mathrm{Aut}_S(L)$.

(ii) A morphism between two objects $t=(L,\lambda,\alpha_g)$ and $t'=(L',\lambda',\alpha'_g)$ is an H-equivariant morphism $u:L\to L'$ such that $u\circ\alpha_g=\alpha'_g\circ u$ for all $g\in G$.

8.2 Stacks $\mathcal{X}^n_{g,G}$

Clearly the action μ_i induces an action on $\overline{\mathcal{M}}_g(n)$, defined in 7.2.6, because of the universal property of rigidification. We write $\overline{\mathcal{M}}^i_g(n)$ for the resulting G-stack. The fixed point stack $\overline{\mathcal{M}}^i_g(n)^G$ enjoys a similar description as $(\overline{\mathcal{H}}^i_{\gamma,H,\emptyset})^G$ above (8.1.2). The following definition is justified by the usual fact that if we have a group G and a normal subgroup H, then whenever G acts on a "structure" X, G/H acts on the fixed points X^H. For stacks, this is made precise in remark 2.4.

Definition 8.2.1 Fix $i\in\mathcal{I}_n$. Let $Z^i_n(G)$ denote the centralizer of G in $\mathrm{GL}_{2g}(\mathbb{Z}/n\mathbb{Z})$. We define the following $\mathbb{Z}[\frac{1}{n}]$-stacks :

(1) $\mathcal{X}^n_{g,G}:=\coprod_{i\in\mathcal{I}_n}\frac{\mathcal{M}^i_g(n)^G}{Z^i_n(G)}$.

(2) $\overline{\mathcal{X}}^n_{g,G}:=\coprod_{i\in\mathcal{I}_n}\frac{\overline{\mathcal{M}}^i_g(n)^G}{Z^i_n(G)}$.

The formation of $\overline{\mathcal{X}}^n_{g,G}$ is functorial with respect to n. Indeed, let $m|n$ with $m,n\geqslant 3$. We pick $i\in\mathcal{I}_n$ and denote $j=i\otimes\mathbb{Z}/m\mathbb{Z}\in\mathcal{I}_m$. By 7.2.5 there is a morphism $\overline{\mathcal{M}}_g(n)\to\overline{\mathcal{M}}_g(m)[\frac{1}{n}]$ and this is equivariant for the actions induced by i and j. So we derive morphisms $\overline{\mathcal{M}}^i_g(n)^G/Z^i_n(G)\to\overline{\mathcal{M}}^j_g(m)^G/Z^j_m(G)$ and $\overline{\mathcal{X}}^n_{g,G}\to\overline{\mathcal{X}}^m_{g,G}[\frac{1}{n}]$. This map is proper and quasi-finite, but not representable in general.

Our interest for these stacks comes from the following result. Its virtue is that as $\mathcal{M}^i_g(n)^G$ is a scheme, then away from the characteristics that divide n, we express explicitly the stack of curves with action as a quotient of a scheme by a finite group (or more precisely, as a sum of such quotients).

Theorem 8.2.2 *Let $n\geqslant 3$ be prime to $|G|$. We have an isomorphism of $\mathbb{Z}[\frac{1}{n}]$-stacks $\mathcal{X}^n_{g,G}\xrightarrow{\sim}\mathcal{H}_{g,G}[\frac{1}{n}]$.*

Proof : It is more convenient here to think of $\mathcal{M}_g(n)$ in its classical definition. That is to say, its objects are, locally on the base, pairs (C,ℓ) with a curve and an isomorphism $\ell:H^1(C,\mathbb{Z}/n\mathbb{Z})\to(\mathbb{Z}/n\mathbb{Z})^{2g}$. Morphisms are maps $u:C\to C'$ such that $u^*\ell'=\ell$. Given $i\in\mathcal{I}_n$, through which G acts on $(\mathbb{Z}/n\mathbb{Z})^{2g}$, the action on $\mathcal{M}_g(n)$ is given by $g(C,\ell)=(C,g\circ\ell)$. We now define maps from each summand of $\mathcal{X}^n_{g,G}$ to $\mathcal{H}_{g,G}$. Starting from $\mathcal{M}^i_g(n)^G$, the map is just the forgetful morphism that forgets the level structure (note that the G-linearization provides a faithful action of G on the curve). The fact that it passes to the quotient by $Z^i_n(G)$ is obvious. It remains to check that the result is an isomorphism.

The fact that the map is an epimorphism is clear because, given a curve C with action of G, we can choose any level structure ℓ on C and look at how it pulls back via $g : C \to C$. This corresponds to an action $i \in \mathcal{I}_n$. Thus we get a preimage in the i-th component.

To check that the map is a monomorphism, we may again work locally over the base S. We pick two objects in the same i-component : an object is a triple $t = (C, \ell, \alpha_g)$ over a base E which is a $Z_n^i(G)$-torsor over S (4.1). Localizing again to trivialize the torsors, we have

$$\mathrm{Hom}(t, t') = \left\{ (z, u) \;\middle|\; \begin{array}{c} z \in Z_n^i(G) \text{ and } u : (C, z \circ \ell) \to (C', \ell') \\ \text{is such that } u \circ \alpha_g = \alpha'_g \circ u \ \ (\forall g) \end{array} \right\}$$

(see prop. 2.6 and rem. 4.2). The map to $\mathrm{Hom}((C, \alpha_g), (C', \alpha'_g))$ forgets z. So it is bijective because, obviously, u determines uniquely the element $z \in Z_n^i(G)$ by $z = (u^*\ell') \circ \ell^{-1}$. □

8.3 Extension to $\overline{\mathcal{X}}_{g,G}^n$

It is "almost" clear from the proof of theorem 8.2.2 that the definition of the morphism $\mathcal{X}_{g,G}^n \to \mathcal{H}_{g,G}[\frac{1}{n}]$ extends to $\overline{\mathcal{X}}_{g,G}^n$. It should then land in the stack of stable curves with faithful action of G. Our last aim is to provide the extension of this morphism. It requires two preparatory results, given below. The hard part is to check that we arrive in the stack of stable curves with *faithful* action.

Before we plunge into the details we must note that, because the order of G is prime to n, we are in one of the remarkable cases, studied in section 5, where the rigidification procedure at the origin of the definition of $\overline{\mathcal{M}}_g(n)$ goes through the successive operations on $\overline{\mathcal{H}}_{\gamma,H,\emptyset}^i$. More precisely,

Lemma 8.3.1 *The morphism $\overline{\mathcal{H}}_{\gamma,H,\emptyset}^i \to \overline{\mathcal{M}}_g^i(n)$ induces an isomorphism $\frac{(\overline{\mathcal{H}}_{\gamma,H,\emptyset}^i)^G}{Z_n^i(G)} /\!\!/ H^G \xrightarrow{\sim} \frac{\overline{\mathcal{M}}_g^i(n)^G}{Z_n^i(G)}$. (This is also true for the substacks parametrizing smooth curves.)*

Proof : As $|G|$ is prime to n, the abelian group H is a weakly injective G-module, so the higher cohomology groups $H^i(G, H)$, $i \geqslant 1$, vanish (see [CE], chap. XII, 2.2 or 2.7). In particular we have $H^1(G, H) = H^2(G, H) = 0$. Then we apply proposition 5.3 to the stack $\mathcal{N} = \overline{\mathcal{H}}_{\gamma,H,\emptyset}$ which is a substack of the stack of stable curves with faithful action. It follows that $(\overline{\mathcal{H}}_{\gamma,H,\emptyset}^i)^G /\!\!/ H^G \simeq (\overline{\mathcal{H}}_{\gamma,H,\emptyset}^i /\!\!/ H)^G$. Then commutation of $/\!\!/ H^G$ with quotients is a general fact proven in theorem 5.1. □

One must be careful that the subgroup H^G depends on $i \in \mathcal{I}_n$ (even though the notation does not make it explicit). Thanks to this lemma, we can work directly with the curves L (standing for level structures) to define $\overline{\mathcal{X}}_{g,G}^n \to \overline{\mathcal{M}}_g\{G\}[\frac{1}{n}]$.

Now we give some preparatory results on the automorphisms of such level n structure curves. Let C be a stable curve over k algebraically closed, and let $n \geqslant 3$ be prime to $\mathrm{char}(k)$. Let $\pi : L \to C$ represent a level n structure on C. Inside the group $\mathrm{Aut}(L)$, we can single out the group H, its centralizer Z and its normalizer N. Also there is the subgroup $\mathrm{Aut}^*(L) = \{f \text{ such that } \pi f = \pi\}$ of automorphisms that induce the identity on C. Let $N^* = N \cap \mathrm{Aut}^*(L) = \ker(N \to \mathrm{Aut}(C))$. In these terms the first result is that actually $Z \subset N^*$, so that we have the picture $H \subset Z \subset N^* \subset N \subset \mathrm{Aut}(L)$:

Proposition 8.3.2 *Let C be a stable curve of genus g over an algebraically closed field k. Let $n \geqslant 3$ be an integer prime to the characteristic of k. Let $L \to C$ be a level n structure on C (definition 7.2.1). Then an H-equivariant automorphism of L induces the identity on C.*

Proof : Let $f : L \to L$ be an automorphism commuting to the action of H. It induces an automorphism $\overline{f} : C \to C$. As f preserves the decomposition $\pi_*\mathcal{O}_L = \oplus \mathcal{F}_\chi$ into isotypical factors, provided we show that $\pi_*\mathcal{O}_L \supset C[n]$, it will follow that $\overline{f}$ acts trivially on $C[n]$, and by Serre's lemma we will have that $\overline{f}$ is the identity on C. So we must show that $C[n] \subset \pi_*\mathcal{O}_L$. Let $\mathcal{F}$ be an invertible sheaf of C such that $\mathcal{F}^{\otimes n} \simeq \mathcal{O}_C$, and let $d|n$ be its order. Let $C^* = \mathrm{Spec}(\oplus_{i=0}^{d-1} \mathcal{F}^{\otimes i})$ which is an étale cover of C.

Let $\mathfrak{L}$ be an equivariant deformation of L to a complete discrete valuation ring R of residue field k and fraction field K. It has an action of H, and $\mathfrak{C} = \mathfrak{L}/H$ is a deformation of C. By the properties of étale morphisms there is a deformation $\mathfrak{C}^*$ of C^* and an étale map $\mathfrak{C}^* \to \mathfrak{C}$ ([EGA], IV$_4$, 18.3.4). On the generic fibre we have a factorization $\mathfrak{L}_K \to \mathfrak{C}^*_K \to \mathfrak{C}_K$ (because the curves are smooth), which extends to all of R because $\mathfrak{C}^* \to \mathfrak{C}$ is finite and $\mathfrak{L}$ has reduced special fibre. Now, the sheaf $\mathcal{F}$ pulls back to $\mathcal{O}_{C^*}$ on C^*, so it pulls back to $\mathcal{O}_L$ on L.

Therefore, the isomorphism $\pi^*\mathcal{F} \simeq \mathcal{O}_L$ defines a linearization of $\mathcal{O}_L$ as an H-sheaf. Such a linearization is just multiplication by the values of a character χ of H. Let V_χ be an irreducible representation corresponding to χ. Then it follows that $\pi^*\mathcal{F} \simeq \mathcal{O}_L \otimes V_\chi$ as H-sheaves. Therefore $\mathcal{F} \simeq \pi_*^H(\pi^*\mathcal{F}) \simeq \pi_*^H(\mathcal{O}_L \otimes V_\chi)$ which is just the χ^{-1} factor of the isotypical decomposition. This is what we wanted. $\square$

The reader should keep in mind that in the case where L is a smooth curve, all inclusions in the chain $H \subset \cdots \subset \mathrm{Aut}(L)$ are equalities. For the second result we will just quote [BR] where it is established. This result gives a measure of the gap $H \subset N^*$. It needs more preparation and a detailed analysis of the role of the nodes of cyclic type in stable curves. The output is that, essentially, among the automorphisms of L that induce the identity on C, the extra ones (compared to the smooth case) are purely of combinatorial nature, in a very nice way :

Proposition 8.3.3 *Let C be a stable curve of genus g over an algebraically closed field k. Let $n \geqslant 3$ be an integer prime to the characteristic of k. Let $L \to C$ be a level n structure on C. Then $N^*/H \simeq (\mathbb{Z}/n\mathbb{Z})^m$ where m is the number of nodes of C that do not disconnect it.* $\square$

Here is now the result of these considerations put together.

Proposition 8.3.4 *There is a morphism $\theta : \overline{\mathcal{X}}^n_{g,G} \to \overline{\mathcal{M}}_g\{G\}[\frac{1}{n}]$.*

Proof : Using lemma 8.3.1 we first define a map from $(\overline{\mathcal{H}}^i_{\gamma,H,\emptyset})^G$. Take a curve $(L, \lambda, \{\alpha_g\}_{g\in G})$. For each g the morphism $\alpha_g : L \to L$, being H-invariant, induces a morphism on C which we denote by ρ_g. This gives an action of G on the curve C. The main point is to see that this action is faithful on each fibre over the base. This comes from the fact that the group N^* of automorphisms that become trivial on C, has exponent a power of n (proposition 8.3.3). As G has order prime to n, we are done. $\square$

This morphism is not representable, unless we look at very special cases. We now proceed to find two such cases. The representability of θ is equivalent to the injectivity of the morphism $\mathrm{Aut}(x) \to \mathrm{Aut}(\theta(x))$, for all geometric points $x : \mathrm{Spec}(k) \to \overline{\mathcal{X}}^n_{g,G}$. We may choose a representative of x in $(\overline{\mathcal{H}}^i_{\gamma,H,\emptyset})^G$, i.e. a triple $(L, \lambda, \{\alpha_g\}_{g\in G})$. The α_g's give an action of G on H such that $H \rtimes G \subset \mathrm{Aut}(L)$. In these terms, with the notations above, we can use lemma 8.3.1 and get

$$\mathrm{Aut}(x) = \frac{N \cap \mathrm{Cent}(G)}{H \cap \mathrm{Cent}(G)} \qquad \text{and} \qquad K(x) := \ker\left(\mathrm{Aut}(x) \to \mathrm{Aut}(\theta(x))\right) = \frac{N^* \cap \mathrm{Cent}(G)}{H \cap \mathrm{Cent}(G)}$$

where $\mathrm{Cent}(G)$ is the centralizer of G in $\mathrm{Aut}(L)$. We obtain :

- θ is of course representable above $\mathcal{M}_g\{G\}[\frac{1}{n}] = \mathcal{H}_{g,G}[\frac{1}{n}]$ and in fact it remains so above the open substack of $\overline{\mathcal{M}}_g\{G\}$ consisting of curves such that any double point of the curve disconnects the curve (i.e. smooth or "tree-like" curves). Indeed, $K(x)$ is a subgroup of N^*/H, so it is trivial by proposition 8.3.3.
- $\theta \otimes \mathbb{Q} : \overline{\mathcal{X}}^n_{g,G} \otimes \mathbb{Q} \to \overline{\mathcal{M}}_g\{G\} \otimes \mathbb{Q}$ is representable if $|G|$ is equal to $84(g-1)$, the Hurwitz bound for curves of genus g. Indeed, in this case we obtain $84n^{2g}(g-1) = |H \rtimes G| \leqslant |\mathrm{Aut}(L)| \leqslant 84(\gamma-1) = 84n^{2g}(g-1)$ by the Hurwitz bound for curves of genus γ. So we have equality $\mathrm{Aut}(L) = H \rtimes G$, whence $N^* = H$ and so $K(x)$ is trivial.

8.3.5 Final remark. If $n = \ell$ is equal to a prime, then there is a tower of stacks over $\mathbb{Z}[\frac{1}{\ell}]$

$$\cdots \to \overline{\mathcal{X}}^{\ell^3}_{g,G} \to \overline{\mathcal{X}}^{\ell^2}_{g,G} \to \overline{\mathcal{X}}^{\ell}_{g,G}$$

It would be interesting to find out whether this tower has properties of stabilization, such as isomorphism (resp. homeomorphism) sufficiently high in the tower — or if the corresponding moduli spaces enjoy such stabilization properties.

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References

[ACV] D. Abramovich, A. Corti, A. Vistoli, *Twisted bundles and admissible covers*, electronic preprint arXiv: math.AG/0106211.

[Ar] M. Artin, *Algebrization of formal moduli I*, Global Analysis, papers in honor of K. Kodaira, Spencer and Iyanaga Eds, Univ. Tokyo Press, 21-71 (1969).

[BM] J. Bertin, A. Mézard, *Déformations formelles des revêtements sauvagement ramifiés de courbes algébriques*, Invent. Math. 141, 195-238 (2000).

[BR] J. Bertin, M. Romagny, *Compactification des schémas de Hurwitz et applications*, in preparation.

[CE] H. Cartan, S. Eilenberg, *Homological Algebra*, Princeton University Press (1956).

[D] P. Deligne, *Le lemme de Gabber*, in Szpiro (ed.) Séminaire sur les pinceaux arithmétiques: La conjecture de Mordell. Astérisque 127. Société Mathématique de France (1985).

[DM] P. Deligne, D. Mumford, *The irreducibility of the space of curves of given genus*, Publ. Math. IHÉS 36, 75-109 (1969).

[EGA] A. Grothendieck, *Eléments de géométrie algébrique*, Publ. Math. IHÉS 4 (I), 8 (II), 11, 17 (III), 20 , 24, 28, 32 (IV) (1961-1967).

[E] T. Ekedahl, *Boundary behaviour of Hurwitz schemes*, in Resolution of singularities, Hauser et al. (ed.), tribute to Oscar Zariski, Obergurgl, Austria, 1997. Birkhäuser, PM 181, 285-298 (2000).

[Ha] M. Hakim, *Topos annelés et schémas relatifs*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 64. Springer (1972).

[Ho] S. Hollander, *A Homotopy Theory for Stacks*, electronic preprint arXiv: math.AT/0110247.

[KaMa] N. Katz, B. Mazur, *Arithmetic moduli of elliptic curves*, Ann. of Math. Stud. 108, Princeton University Press (1985).

[Ko] M. Kontsevitch, *Enumeration of rational curves via torus actions*, Dijkgraaf et al. (ed.), The moduli space of curves, Texel Island, Netherlands, April 1994. Birkhäuser. PM 129, 335-368 (1995).

[L-MB] G. Laumon, L. Moret-Bailly, *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. 39. Springer (2000).

[PdJ] M. Pikaart, J. de Jong, *Moduli of curves with non-abelian level structure*, Dijkgraaf et al. (ed.), The moduli space of curves. Proc. Conf. Texel Island, Netherlands, April 1994. Birkhäuser. Prog. Math. 129, 335-368 (1995).

[Ro] M. Romagny, *Sur quelques aspects des champs de revêtements de courbes algébriques*, Thesis, Institut Fourier, Univ. Grenoble 1 (2002), available at http://www-fourier.ujf-grenoble.fr/∼romagny.

[SGA3] M. Demazure, A. Grothendieck, *Schémas en groupes* (SGA3), LNM 152. Springer (1970).

[We] S. Wewers, *Construction of Hurwitz spaces*, thesis, Universität GH Essen, preprint 21 (1998).