On the resolution of points in generic position

by

M.P. Cavaliere, M.E. Rossi and G. Valla

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

Federal Repuplic of Germany

Dipartimento di Mathematica
Università di Genova
Via L.B. Alberti 4
16132 Genova

Italy

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Introduction.

Let k be an algebraically closed field and let $X = \{P_1, \ldots, P_s\}$ be a set of $s \geq n+1$ distinct points in $\mathbf{P}_k^n$, not contained in any hyperplane. We denote by I the defining ideal of X in the polynomial ring $R = k[X_0, \ldots, X_n]$ and by A the homogeneous coordinate ring of X, $A = R/I = \oplus_{i=0}^{\infty} A_i$.

We say , following Geramita and Orecchia (see [GO]), that the points $P_1, \ldots, P_s$ are in *generic position* if the Hilbert function $H_A(t) := dim_k(A_t)$ satisfies

$$H_A(t) = min\left\{s, \binom{n+t}{n}\right\}.$$

It is well known that almost every set of s points in $\mathbf{P}_k^n$ are in generic position, in the sense that the points in generic position in $\mathbf{P}_k^n$ form a dense open set U of $\mathbf{P}_k^n \times \mathbf{P}_k^n \times \cdots \times \mathbf{P}_k^n$ (s times).

Now for points in generic position the integer t defined by the inequalities

$$\binom{n+t-1}{n} \leq s < \binom{n+t}{n}$$

coincides with the socle degree of A and with the initial degree of A, which is the minimal degree of an hypersurface passing through the points. From this it follows that a minimal graded free resolution for A is given by

$$\begin{aligned} 0 &\to R(-t-n)^{b_n} \oplus R(-t-n+1)^{a_n} \to \ldots \\ &\to R(-t-i)^{b_i} \oplus R(-t-i+1)^{a_i} \to \cdots \to R(-t-1)^{b_1} \oplus R(-t)^{a_1} \to R \to A \to 0 \end{aligned}$$

By the particular Hilbert function of A we get $a_1 = \binom{n+t}{n} - s$ and $b_n = s - \binom{n+t-1}{n}$. It is natural to predict that almost every set of s points in generic position in $\mathbf{P}_k^n$ have the

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same numerical invariants in the resolution. This leads to the following conjecture (see [L] and [BG]).

Minimal resolution conjecture. *There exists a not empty open subset of $(P_k^n)^s$ consisting of sets of points in generic position which have the same numerical resolution.*

The minimal resolution conjecture (MRC for short) has been solved for $n = 2$ (see [GGR] and[GM]), for $n = 3$ (see [BG]) and for any n if $s \geq \binom{n+t}{n} - n$ or $s \leq n+3$ (see [L] and [GL]), while the corresponding Cohen-Macaulay type conjecture has been solved for any n (see [TV]).

The expected integers a_i and b_i have been worked out by A. Lorenzini in her thesis (see [L]) where, even if not explicitely, the following characterization can be found.

Let m be the least integer such that

$$t\left[\binom{n+t}{n} - s\right] \leq sm.$$

Then $m \geq 1$ and A has the expected numerical resolution if and only if $a_{m+1} = b_{m-1} = 0$. More precisely if $j \geq m+1$, A has the wanted numerical invariants a_i and b_i for all $i \geq j$ if and only if $a_j = 0$, while if $j \leq m-1$, A has the wanted numerical invariants a_i and b_i for all $i \leq j$ if and only if $b_j = 0$.

For example, in a recent paper, Green and Lazarsfeld proved that if $s = 2n+1-p$ for some $1 \leq p \leq n$, and the points are in general position then $b_p = 0$ (see [GrL]). This gives the right numerical invariants for the initial part of the resolution.

In this paper we prove that for a general set of points in generic position in $\mathbf{P}_k^n$, we have $a_i = 0$ if $s \geq \binom{n+t}{n} - \binom{i+t-2}{t} - i(n-i+2) + n + 1$ (see Theorem 2). By the above remark this gives the expected numerical invariants for the last part of the resolution. If we apply our result to the case $i = 2$, we get a fresh and easy proof of the MRC for the case $s \geq \binom{n+t}{n} - n$ points in $\mathbf{P}_k^n$. Also, by combining our theorem with the result of Green and Lazarsfeld which takes care of the first part of the resolution, we get the MRC for the case $s = n+4$ points in $\mathbf{P}_k^n$. Finally if we restrict ourselves to the cases $i = 2$ and $t = 2, 3$, then we can improve our result "by one" by showing that $a_2 = 0$ if $s = \binom{n+2}{2} - n - 1$ or $s = \binom{n+3}{3} - n - 1$, thus proving the MRC in these cases too (see Theorem 7 and 8). Some sporadic results are also discussed in the last part of the paper.

The main result

Let k be an algebraically closed field and let $\{u_{ij}\}$, $i = 1, \dots, s-n-1$, $j = 0, \dots, n$, be a set of indeterminates over k. Let K be the field obtained by adjoining these indeterminates to k. Let $Q_0, \dots, Q_n$ be the coordinate points in $\mathbf{P}^n_K$ and let us consider the set $X = \{Q_0, \dots, Q_n, P_1, \dots, P_{s-n-1}\}$ where the P_i are the K-rational points in $\mathbf{P}^n_K$ whose coordinates are given by $P_i := (u_{i0}, \dots, u_{in})$. We denote by R the polynomial ring $K[X_0, \dots, X_n]$ and by I the defining ideal of X in R. The ring $A = R/I$ is the homogeneous coordinate ring of X. It is clear that X is a set of points in generic position (see [TV]), hence a minimal graded free resolution of A as an R-module is

$$
\begin{aligned}
0 \to R(-t-n)^{b_n} \oplus R(-t-n+1)^{a_n} &\to \dots \\
\to R(-t-i)^{b_i} \oplus R(-t-i+1)^{a_i} &\to \dots \to R(-t-1)^{b_1} \oplus R(-t)^{a_1} \to R \to A \to 0
\end{aligned}
$$

where t is the initial degree of A or, which is the same, the integer defined by the inequalities

$$\binom{n+t-1}{n} \le s < \binom{n+t}{n}.$$

Our point of view is to prove numerical properties for the resolution of these points. Since the validity of these properties is equivalent to the fact that certain matrices, whose entries are monomials in the u_{ij}'s have maximal rank, our results prove, after specialisation, that almost every set of s points in $\mathbf{P}^n_k$ which are in generic position has the corresponding property.

Let us consider the graded R-modules $Tor_i^R(A, K)$ which can be computed from the minimal free resolution of A. It is clear that $Tor_i^R(A, K)_j$ is a finite dimensional K-vector space whose dimension is the number of minimal generators in degree j of the $i-th$ free graded module in the resolution. In particular we get

$$dim_K(Tor_i^R(A, K)_{t+i-1}) = a_i$$

and

$$dim_K(Tor_i^R(A, K)_{t+i}) = b_i$$

for every $i = 1, \dots, n$.

We want to compute $Tor_i^R(A,K)$ using a resolution of K which can be obtained by the Koszul complex of $X_0,\dots,X_n$. Let V be a fixed K-vector space of dimension $n+1$ and let $e_0,\dots,e_n$ be a K-vector base of V. The Koszul resolution of K is given by

$$0 \to \overset{n+1}{\Lambda} V \otimes R(-n-1) \xrightarrow{\delta_{n+1}} \overset{n}{\Lambda} V \otimes R(-n) \to \cdots \to \Lambda V \otimes R(-1) \xrightarrow{\delta_1} R \to K \to 0$$

where the δ_i are the usual Koszul maps.

Tensoring by A and taking graded pieces, one finds that $Tor_i^R(A,K)_j$ is given by the homology of the complex of vector spaces

$$\overset{i+1}{\Lambda} V \otimes A(-i-1)_j \to \overset{i}{\Lambda} V \otimes A(-i)_j \to \overset{i-1}{\Lambda} V \otimes A(-i+1)_j.$$

Since $A_p = R_p$ for every $p \le t-1$, we get that $Tor_i^R(A,K)_{i+t-1}$ is the homology of the complex

$$\overset{i+1}{\Lambda} V \otimes R_{t-2} \xrightarrow{\delta_{i+1}} \overset{i}{\Lambda} V \otimes R_{t-1} \xrightarrow{\delta_i \otimes 1} \overset{i-1}{\Lambda} V \otimes A_t$$

PROPOSITION 1. *Let i be any integer, $2 \le i \le n$. With the above assumptions and notations, we have*

$$Tor_i^R(A,K)_{i+t-1} = Ker(\pi) \cap Ker(\delta_{i-1})$$

where π is the canonical map in the following commutative diagram:

$$\begin{tikzcd}
\overset{i+1}{\Lambda} V \otimes R_{t-2} \arrow[r, "\delta_{i+1}"] & \overset{i}{\Lambda} V \otimes R_{t-1} \arrow[r, "\delta_i \otimes 1"] \arrow[d, equal] & \overset{i-1}{\Lambda} V \otimes A_t & \\
 & \overset{i}{\Lambda} V \otimes R_{t-1} \arrow[r, "\delta_i"] & \overset{i-1}{\Lambda} V \otimes R_t \arrow[r, "\delta_{i-1}"] \arrow[u, "\pi"] & \overset{i-2}{\Lambda} V \otimes R_{t+1}
\end{tikzcd}$$

PROOF: We have $Tor_i^R(A,K)_{i+t-1} = Ker(\delta_i \otimes 1)/Im(\delta_{i+1})$. Since $Im(\delta_{i+1}) = Ker(\delta_i)$, δ_i induces an injective map ρ from $Tor_i^R(A,K)_{i+t-1}$ to $\overset{i-1}{\Lambda} V \otimes R_t$. Now it is easy to see that the image of ρ is exactly $Ker(\pi) \cap Ker(\delta_{i-1})$. The conclusion follows.

THEOREM 2. *Let i be any integer, $2 \le i \le n$. If*

$$s \ge \binom{n+t}{t} - \binom{i+t-2}{t} - i(n-i+2) + n + 1$$

then $a_i = dim_K(Tor_i^R(A,K)_{i+t-1}) = 0$.

PROOF: By the above proposition we have $Tor_i^R(A,K)_{i+t-1} = Ker(\pi) \cap Ker(\delta_{i-1})$. Now it is clear that $Ker(\pi) = \overset{i-1}{\Lambda} V \otimes I_t$, hence if $\alpha \in Ker(\pi)$ we may write

$$\alpha = \sum_{j=\{j_1,\dots j_{i-1}\}} e_{j_1} \wedge \cdots \wedge e_{j_{i-1}} \otimes F_j$$

with $F_j \in I_t$. We have three important remarks.

1. In F_j there is no term of the form X_p^t for every $p = 0, \dots, n$.

It follows from the fact that F_j must vanish on the coordinate points.

2. In F_j there is no monomial of degree t in the variables $X_{j_1}, \dots, X_{j_{i-1}}$.

Let $\alpha = e_{j_1} \wedge \cdots \wedge e_{j_{i-1}} \otimes M + \dots$ with M monomial of degree t in $X_{j_1}, \dots, X_{j_{i-1}}$. Then $\delta_{i-1}(\alpha) = e_{j_2} \wedge \cdots \wedge e_{j_{i-1}} \otimes X_{j_1} M + \dots$ cannot be zero since to cancel $e_{j_2} \wedge \cdots \wedge e_{j_{i-1}} \otimes X_{j_1} M$ we need in α a non zero term $e_{j_2} \wedge \cdots \wedge e_{j_{i-1}} \wedge e_p \otimes X_{j_1}(M/X_p)$ with p is in the set $\{j_1, \dots j_{i-1}\}$. A contradiction.

3. In F_j there is no monomial of the form $X_p^{t-1} X_q$ with $p \in \{j_1, \dots j_{i-1}\}$.

If for example $\alpha = e_{j_1} \wedge \cdots \wedge e_{j_{i-1}} \otimes X_{j_1}^{t-1} X_q + \dots$ then $\delta_{i-1}(\alpha) = e_{j_2} \wedge \cdots \wedge e_{j_{i-1}} \otimes X_{j_1}^t X_q + \dots$ cannot be zero since to cancel $e_{j_2} \wedge \cdots \wedge e_{j_{i-1}} \otimes X_{j_1}^t X_q$ we need in α a term $e_{j_2} \wedge \cdots \wedge e_{j_{i-1}} \wedge e_q \otimes X_{j_1}^t$. This is impossible by 1.

Using these three facts we see immediately that F_j is the sum of

$$\binom{n+t}{t} - \binom{i-1+t-1}{t} - (i-1)(n-i+2) - (n-i+2)$$

monomials. Thus we have $\binom{n+t}{t} - \binom{i+t-2}{t} - i(n-i+2)$ coefficients for F_j. But F_j must vanish on the set $\{P_1, \dots, P_{s-n-1}\}$ which have generic coordinates; since $s-n-1 \geq \binom{n+t}{t} - \binom{i+t-2}{t} - i(n-i+2)$ this implies $F_j = 0$ and the conclusion follows.

To apply this result we recall that if m is the least integer such that

$$t\left[\binom{n+t}{n} - s\right] \leq sm$$

then for any $j \geq m+1$ the resolution has the expected numerical invariants a_i and b_i for every $i \geq j$ if and only if $a_j = 0$, while for $j \leq m-1$ the resolution has the expected numerical invariants a_i and b_i for every $i \leq j$ if and only if $b_j = 0$.

COROLLARY 3. *Let i be any integer, $2 \leq i \leq n$. If*

$$s \geq \binom{n+t}{t} - \binom{i+t-2}{t} - i(n-i+2) + n + 1$$

then in the resolution of A we have the expected a_k and b_k for every $k \geq i$.

PROOF: We need only to prove that if

$$s \geq \binom{n+t}{t} - \binom{i+t-2}{t} - i(n-i+2) + n + 1$$

then $i \geq m+1$, or , which is the same, that

$$\binom{n+t}{t} - \binom{i+t-2}{t} - i(n-i+2) + n + 1 \geq \frac{t}{t+i-1}\binom{n+t}{n}.$$

This can be easily seen by a direct computation.

We remark that this gives for example a proof of the Cohen-Macaulay type conjecture in the case

$$s \geq \binom{n+t}{t} - \binom{n+t-2}{t} - n + 1.$$

COROLLARY 4. *The MRC holds if $s \geq \binom{n+t}{t} - n$.*

PROOF: If we apply the theorem with $i = 2$ we get $a_2 = dim_K(Tor_2^R(A,K)_{t+1}) = 0$ for $s \geq \binom{n+t}{t} - n$. The conclusion follows.

COROLLARY 5. *The MRC holds for $s = n+4$ points in $\mathbf{P}_k^n$.*

PROOF: Since $s = n+4$ we have $t = 2$, hence if we apply the theorem for $t = 2$ and $i = n-1$ we get $a_{n-1} = dim_K(Tor_{n-1}^R(A,K)_n) = 0$ for $s \geq n+4$. On the other hand we have $n+4 = 2n+1-(n-3)$, hence by the result of Green and Lazersfeld we get $b_{n-3} = 0$. This gives the conclusion.

For the next application we need to recall that a set of points in $\mathbf{P}_k^n$ is said to be in general position if no subset of $n+1$ points lies on an hyperplane. The following result has been proved in [TV] and also in [L] with completely different methods.

COROLLARY 6. *Let $\{P_1, \dots, P_s\}$ in $\mathbf{P}_k^n$ be a set of points in generic and general position. If $n+1 < s < \binom{n+2}{2}$, then $a_n = dim_k(Tor_n^R(A,k)_{n+1}) = 0$.*

PROOF: With our assumptions we have $t = 2$. Further we may assume that the first $n+1$ points are the coordinate points. Then it follows from the argument as in the proof of the theorem, applied to the case $i = n$ and $t = 2$, that an element $\alpha \in Ker(\pi) \cap Ker(\delta_{n-1})$ is of the form

$$\alpha = \sum_{0 \le i < j \le n} e_0 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge \hat{e}_j \wedge \cdots \wedge e_n \otimes \lambda_{ij} X_i X_j$$

where $\lambda_{ij} X_i X_j \in I_2$. Since the points are in general position and $s > n+1$, there is at least one point with all coordinates different from zero. This implies $\lambda_{ij} = 0$ for all i and j, hence $\alpha = 0$.

The cases $\mathbf{s} = \binom{n+2}{2} - \mathbf{n} - 1$ and $\mathbf{s} = \binom{n+3}{3} - \mathbf{n} - 1$.

In this section we prove the MRC for $s = \binom{n+2}{2} - n - 1$ and $s = \binom{n+3}{3} - n - 1$ points in $\mathbf{P}_k^n$. We start with $s = \binom{n+2}{2} - n - 1$. Since in $\mathbf{P}_k^2$ and $\mathbf{P}_k^3$ the conjecture holds, we may assume $n \ge 4$. We have $t = 2$ and

$$2\left[\binom{n+2}{2} - s\right] = 2(n+1) \le \binom{n+2}{2} - n - 1 = s$$

hence $m = 1$. The MRC is then a consequence of the following result.

THEOREM 7. *Let $s = \binom{n+2}{2} - n - 1$, then $a_2 = dim_K(Tor_2^R(A,K)_3) = 0$.*

PROOF: As in the above section we have $Tor_2^R(A,K)_3 = Ker(\pi) \cap Ker(\delta_1)$, where δ_1 is the Koszul map $\delta_1 : \overset{1}{\Lambda} V \otimes R_2 \to R_3$ and $Ker(\pi) = \overset{1}{\Lambda} V \otimes I_2$. Let $\alpha = \sum_{i=0}^n e_i \otimes F_i$ be an element of $Tor_2^R(A,K)_3$. Then as in the proof of theorem 2 we may write

$$F_i = \sum_{\substack{0 \le j < k \le n \\ i \ne j,k}} a_{ijk} X_j X_k.$$

We get $(n+1)\binom{n}{2}$ variables $\{a_{ijk}\}_{\substack{0 \le j < k \le n \\ i \ne j,k}}$. Since $\alpha \in \overset{1}{\Lambda} V \otimes I_2$ we get the equations

$$F_i(P_d) : \qquad \sum_{\substack{0 \le j < k \le n \\ i \ne j,k}} a_{ijk} u_{dj} u_{dk} = 0$$

where $i = 0, \dots, n$ and $d = 1, \dots, s - n - 1 = \binom{n}{2} - 1$. On the other hand $\alpha \in Ker(\delta_1)$ hence

$$0 = \delta_1(\alpha) = \sum_{i=0}^{n} X_i F_i = \sum_{i=0}^{n} \sum_{\substack{0 \le j < k \le n \\ i \ne j,k}} a_{ijk} X_i X_j X_k.$$

By imposing the coefficient of $X_p X_q X_r$ to be zero for $0 \le p < q < r \le n$ we get the equations

$$(pqr): \qquad a_{pqr} + a_{qpr} + a_{rpq} = 0.$$

The matrix M associated to this system of homogeneous equations has $(n+1)\left[\binom{n}{2} - 1\right] + \binom{n+1}{3}$ rows and $(n+1)\binom{n}{2}$ columns. This matrix can be described as follows

$$M_{((pqr),a_{ijk})} = \begin{cases} 1 & \text{if } (i,j,k) = (p,q,r),\ (q,p,r) \text{ or } (r,p,q) \\ 0 & \text{otherwise} \end{cases}$$

and

$$M_{(F_t(P_d),a_{ijk})} = \begin{cases} 0 & \text{if } i \ne t \\ u_{dj} u_{dk} & \text{if } i = t \end{cases}.$$

We need to prove that this matrix has maximal rank. We have

$$(n+1)\left[\binom{n}{2} - 1\right] + n + 1 = (n+1)\binom{n}{2}$$

hence it will be enough to prove that we can choose $n+1$ rows among the (pqr) such that the corresponding minor involving these and all the rows $F_i(P_d)$ is non zero. We choose the following rows:

$$(014), (013), (234), (123), (024), (035), \dots, (03n)$$

and call D the corresponding minor.

We recall that if A is a square matrix of size say v, an A-product is an element $(-1)^{sgn(\sigma)} a_{1\,\sigma(1)} \cdots a_{v\,\sigma(v)}$ where σ is a permutation of $\{1, 2, \dots, v\}$. Thus $det(A)$ is the sum of the A-products. Let Δ be the product of the following monomials:

$$
\begin{aligned}
m_1 &= X_0^n X_1^n X_2^2 \\
m_2 &= X_0^2 X_2^n X_3^n \\
m_3 &= X_0^n X_3^2 X_4^n \\
m_4 &= X_1^2 X_2^n X_4^n \\
m_5 &= X_1^n X_3^n X_4^2 \\
b_{k1} &= X_0^n X_2^2 X_k^n && k = 5, \dots, n \\
b_{k2} &= X_3^n X_4^2 X_k^n && k = 5, \dots, n \\
b_{k3} &= X_1^n X_2^2 X_k^n && k = 5, \dots, n \\
d_k &= X_1 X_2^2 X_4^{n-1} X_k^n && k = 5, \dots, n \\
c_{kj} &= X_2^2 X_j^n X_k^n && k = 6, \dots, n \qquad j = 5, \dots, k-1
\end{aligned}
$$

The cardinality of this finite set of monomials is

$$5 + 3(n-4) + (n-4) + 1 + \cdots + (n-5) = \binom{n}{2} - 1.$$

Hence if we order this set as we like, we can rewrite the monomials as $M_1, \dots, M_{\binom{n}{2}-1}$ and consider the monomial

$$\Delta = \prod_{i=1}^{\binom{n}{2}-1} M_i(P_i).$$

It is then clear that we get the conclusion by proving that Δ is a D-product which can be obtained in a unique way with the following strategy.

Step 1 A monomial $u_{.l}^n u_{.m}^n u_{.r}^2$ can be obtained in a unique way by choosing the following elements:

$$(F_l(P_.), a_{l(mr)}) \qquad (F_m(P_.), a_{m(lr)}) \qquad (F_i(P_.), a_{i(lm)}) \qquad i \neq l, m$$

where we use the following convention

$$a_{p(qr)} = \begin{cases} a_{pqr} & \text{for } q < r \\ a_{prq} & \text{for } r < q \end{cases}$$

It is clear that we must use rows $F_i(P_.)$ for $i = 0, \ldots, n$. Since on each row $F_i(P_.)$ we do not have $u_{.i}$, on the row $F_l(P_.)$ we must choose the element $u_{.m}u_{.r}$ which is on the column $a_{l(mr)}$ and on $F_m(P_.)$ the element $u_{.l}u_{.r}$ which is on the column $a_{m(lr)}$. At this point from each row $F_i(P_.)$ with $i \neq l, m$ we must pick up the element $u_{.l}u_{.m}$ which is on the columns $a_{i(lm)}$. We remark that we used in this way all the rows corresponding to the point $P_.$.

Step 2 The monomials d_k can be obtained in a unique way by choosing the following elements :

$$(F_3(P_.), a_{32k}) \qquad (F_4(P_.), a_{42k}) \qquad (F_k(P_.), a_{k14}) \qquad (F_i(P_.), a_{i4k}) \qquad i \neq 3, 4, k.$$

It is clear that we must use the rows $F_i(P_.)$ for $i = 0, \ldots, n$. Further the monomial $u_{.k}$ must be choosen on the rows $F_i(P_.)$ for all $i \neq k$. Hence on the row $F_3(P_.)$ we must choose or $u_{.1}u_{.k}$, or $u_{.2}u_{.k}$ or $u_{.4}u_{.k}$ which are respectively on the columns a_{31k}, a_{32k} and a_{34k}. But the column a_{31k} must be choosen for the monomial $b_{k\,3}$ and the column a_{34k} for the monomial $b_{k\,2}$. Hence we must choose the entry $(F_3(P_.), a_{32k})$. Similarly on the row $F_4(P_.)$ we must choose or $u_{.1}u_{.k}$ or $u_{.2}u_{.k}$ which are respectively on the columns a_{41k} and a_{42k}. But the column a_{41k} must be choosen for the monomial $b_{k\,3}$, hence we must choose the entry $(F_4(P_.), a_{42k})$. At this point $u_{.2}$ is nomore available, hence we must pick up from the row $F_k(P_.)$ the monomial $u_{.1}u_{.4}$ which corresponds to the entry $(F_k(P_.), a_{k14})$ and from each row $F_i(P_.)$ with $i \neq 3, 4, k$ the monomial $u_{.4}u_{.k}$ which corresponds to the entry $(F_i(P_.), a_{i4k})$. We remark that also in this case we used all the rows corresponding to the point $P_.$.

Step 3 We used all the columns but

$$a_{014}, a_{103}, a_{234}, a_{312}, a_{402}, a_{503}, \ldots, a_{n03}.$$

This can be easily seen by looking at the following table where on each row one can find the monomial and the columns used to get it.

m_1 :	a_{012}	a_{102}	a_{i01}			
m_2 :	a_{203}	a_{302}	a_{i23}			
m_3 :	a_{034}	a_{403}	a_{i04}			
m_4 :	a_{214}	a_{412}	a_{i24}			
m_5 :	a_{134}	a_{314}	a_{i13}			
b_{k1} :	$a_{0\,2\,k}$	$a_{k\,0\,2}$	$a_{i\,0\,k}$		$k=5,\dots,n$	
b_{k2} :	$a_{3\,4\,k}$	$a_{k\,3\,4}$	$a_{i\,3\,k}$		$k=5,\dots,n$	
b_{k3} :	$a_{1\,2\,k}$	$a_{k\,1\,2}$	$a_{i\,1\,k}$		$k=5,\dots,n$	
d_k :	a_{32k}	a_{42k}	a_{k14}	a_{i4k}	$k=5,\dots,n$	
c_{kj} :	$a_{j\,2\,k}$	$a_{k\,2\,j}$	$a_{i\,j\,k}$		$k=6,\dots,n$	$j=5\dots,k-1$,

Step 4 Conclusion

We get the conclusion by choosing the entries

$$((014),a_{014}),((013),a_{103}),((234),a_{234}),((123),a_{312}),$$
$$((024),a_{402}),((035),a_{503}),\dots,((03n),a_{n03})$$

which are all equal to 1.

Now we come to the case of $s=\binom{n+3}{3}-n-1$ points in $\mathbf{P}_k^n$. Since in $\mathbf{P}_k^2$ and $\mathbf{P}_k^3$ the conjecture holds, we may assume $n\geq 4$. We have $t=3$ and

$$3\left[\binom{n+3}{3}-s\right]=3(n+1)\leq\binom{n+3}{3}-n-1=s$$

hence $m=1$. The MRC is then a consequence of the following result.

THEOREM 8. *Let $s=\binom{n+3}{3}-n-1$, then $a_2=dim_K(Tor_2^R(A,K)_4)=0$.*

PROOF: By Proposition 1 we have $Tor_2^R(A,K)_4=Ker(\pi)\cap Ker(\delta_1)$, where δ_1 is the Koszul map $\delta_1:\overset{1}{\Lambda}V\otimes R_3\rightarrow R_4$ and $Ker(\pi)=\overset{1}{\Lambda}V\otimes I_3$. Let $\alpha=\sum_{i=0}^n e_i\otimes F_i$ be an element of $Tor_2^R(A,K)_4$. From the remarks made in the proof of Theorem 2 we may write

$$F_i = \sum_{(j,k,h)\in S_i} a_{ijkh} X_j X_k X_h$$

where S_i is the set of triplets defined by

$$S_i := \{0 \le j < k < h \le n\} \cup \{0 \le j = k < h \le n, j \ne i\} \cup \{0 \le j < k = h \le n, k \ne i\}.$$

Since $\alpha \in \overset{1}{\Lambda} V \otimes I_3$ we get the equations

$$F_i(P_d): \qquad \sum_{(j,k,h)\in S_i} a_{ijkh} u_{dj} u_{dk} u_{dh} = 0$$

where $i = 0, \ldots, n$ and $d = 1, \ldots, s-n-1 = \binom{n+3}{3} - 2n - 2$. On the other hand $\alpha \in Ker(\delta_1)$ hence

$$0 = \delta_1(\alpha) = \sum_{i=0}^{n} X_i F_i = \sum_{i=0}^{n} \sum_{(j,k,h)\in S_i} a_{ijkh} X_i X_j X_k X_h.$$

By imposing the coefficient of $X_p X_q X_r X_t$ to be zero for $0 \le p < q < r < t \le n$ we get the equations

$$(pqrt): \qquad a_{pqrt} + a_{qprt} + a_{rpqt} + a_{tpqr} = 0.$$

By imposing the coefficient of $X_p^2 X_q X_r$ to be zero for $0 \le p < q < r \le n$ we get the equations

$$(ppqr): \qquad a_{ppqr} + a_{qppr} + a_{rppq} = 0.$$

By imposing the coefficient of $X_p X_q^2 X_r$ to be zero for $0 \le p < q < r \le n$ we get the equations

$$(pqqr) := \qquad a_{pqqr} + a_{qpqr} + a_{rpqq} = 0.$$

By imposing the coefficient of $X_p X_q X_r^2$ to be zero for $0 \le p < q < r \le n$ we get the equations

$$(pqrr) := \qquad a_{pqrr} + a_{qprr} + a_{rpqr} = 0.$$

Finally by imposing the coefficient of $X_p^2 X_q^2$ to be zero for $0 \le p < q \le n$ we get the equations

$$(ppqq) := a_{ppqq} + a_{qppq}.$$

The matrix M associated to this system of homogeneous equations has

$$(n+1)\left[\binom{n+3}{3}-2n-2\right]+\binom{n+1}{4}+3\binom{n+1}{3}+\binom{n+1}{2}$$

rows and

$$(n+1)\left[\binom{n+3}{3}-2n-1\right]$$

columns. This matrix can be described as follows

$$M_{((pqrt),a_{ijkh})}=\begin{cases}1 & \text{if } (i,j,k,h)=(p,q,r,t),\ (q,p,r,t)\ ,\ (r,p,q,t) \text{ or } (t,p,q,r)\\ 0 & \text{otherwise}\end{cases}$$

$$M_{((ppqr),a_{ijkh})}=\begin{cases}1 & \text{if}(i,j,k,h)=(p,p,q,r),\ (q,p,p,r) \text{ or } (r,p,p,q)\\ 0 & \text{otherwise}\end{cases}$$

$$M_{((pqqr),a_{ijkh})}=\begin{cases}1 & \text{if } (i,j,k,h)=(p,q,q,r),\ (q,p,q,r) \text{ or } (r,p,q,q)\\ 0 & \text{otherwise}\end{cases}$$

$$M_{((pqrr),a_{ijkh})}=\begin{cases}1 & \text{if } (i,j,k,h)=(p,q,r,r),\ (q,p,r,r) \text{ or } (r,p,q,r)\\ 0 & \text{otherwise}\end{cases}$$

$$M_{((ppqq),a_{ijkh})}=\begin{cases}1 & \text{if } (i,j,k,h)=(p,p,q,q) \text{ or } (q,p,p,q)\\ 0 & \text{otherwise}\end{cases}$$

$$M_{(F_t(P_d),a_{ijkh})}=\begin{cases}0 & \text{if } i\neq t\\ u_{dj}u_{dk}u_{dh} & \text{if } i=t\end{cases}$$

We need to prove that this matrix has maximal rank. We have

$$(n+1)\left[\binom{n+3}{3}-2n-2\right]+n+1=(n+1)\left[\binom{n+3}{3}-2n-1\right]$$

hence it will be enough to prove that we can choose $n+1$ rows among the $(pqrt)$ such that the corresponding minor involving these and all the rows $F_i(P_d)$ is non zero. We choose the following rows:

$$(0122),(1123),(0223),(1223),\ldots,(122n)$$

and call D the corresponding minor.

Let us consider the following monomials of degree $3n+3$.

$$
\begin{aligned}
b_{ij} &= X_i^{2n+1}X_j^{n+2} && 0 \leq i < j \leq n\\
m_{01} &= X_0^{n+1}X_1^{2n}X_2^2 &&\\
m_{03} &= X_0^{n+1}X_2^2X_3^{2n} &&\\
m_{13} &= X_0^2X_1^{n+1}X_3^{2n} &&\\
m_{23} &= X_1^2X_2^{n+1}X_3^{2n} &&\\
m_{i\,i+1} &= X_i^{n+1}X_{i+1}^{2n}X_0^2 && i \geq 4,\ i\ even\\
m_{ij} &= X_i^{n+1}X_j^{2n}X_{i+1}^2 && i\ even,\ j \geq max(4, i+2)\\
p_{02} &= X_0^{n+1}X_2^{2n-1}X_1^2X_3 &&\\
p_{1j} &= X_1^{n+1}X_j^{2n}X_0X_3 && j \geq 4\\
p_{ij} &= X_i^{n+1}X_j^{2n}X_0X_{i-1} && 3 \leq i, i\ odd, j \geq max(4, i+1)\\
c &= X_1^nX_2^nX_3^nX_0^3 &&\\
q_{13j} &= X_1^{n+1}X_3^{n+1}X_j^nX_0 && j \geq 4\\
q_{03j} &= X_0^{n+1}X_3^{n+1}X_j^nX_2 && j \geq 4\\
q_{0ij} &= X_0^{n+1}X_i^{n+1}X_j^nX_1 && i\ odd,\ 5 \leq i < j \leq n\\
r_{0i\,i+1} &= X_0^{n+1}X_i^nX_{i+1}^nX_1^2 && i\ even,\ 4 \leq i < n\\
s_{i\,i+1\,j} &= X_i^{n-1}X_{i+1}^{n-2}X_j^nX_0^3X_1^3 && i\ even,\ 2 \leq i < i+1 < j \leq n\\
d_{ijk} &= X_i^{n+1}X_j^{n+1}X_k^{n+1} && (i,j,k) \in F
\end{aligned}
$$

where F is the set of triplets different from $(1,2,3), (0,2,3), (0,1,j)$, $j \geq 2$ and the ones used as indexes of the last 5 sets of monomials.

It is clear that the cardinality of this set of monomials is

$$\binom{n+1}{2} + \binom{n+1}{2} - 6 + 5 + 1 + \binom{n+1}{3} - (n+1) = \binom{n+3}{3} - 2n - 2$$

hence if we order this set as we like, we can rewrite the monomials as

$$M_1, \ldots, M_{\binom{n+3}{3}-2n-2}$$

and consider the monomial

$$\Delta = \prod_i M_i(P_i).$$

It is then clear that we get the conclusion by proving that Δ is a D-product which can be obtained in a unique way. In the following table one can find for each monomial the entries we use to get it. The table has to be red according to the following convention. If one finds for the monomial say M the column say a_{ijkh}, this means that we use the entries $(F_i(P_.), a_{ijkh})$ where $P_.$ is the point choosen for M.

b_{ij} :	a_{iijj}	a_{liij}			
m_{01} :	a_{0012}	a_{1012}	a_{l011}		
m_{03} :	a_{0023}	a_{3023}	a_{l033}		
m_{13} :	a_{1013}	a_{3013}	a_{l133}		
m_{23} :	a_{2123}	a_{3123}	a_{l233}		
$m_{i\,i+1}$:	$a_{i0i\,i+1}$	$a_{i+1\,0i\,i+1}$	$a_{li\,i+1\,i+1}$		
m_{ij} :	$a_{ii\,i+1\,j}$	$a_{ji\,i+1\,j}$	a_{lijj}		
p_{02} :	a_{0013}	a_{2012}	a_{l022}		
p_{1j} :	a_{101j}	a_{j13j}	a_{l1jj}		
p_{ij} :	$a_{i\,i-1\,ij}$	a_{j0ij}	a_{lijj}		
q_{13j} :	a_{j013}	a_{l13j}			
q_{03j} :	a_{j023}	a_{l03j}			
q_{0ij} :	a_{j01i}	a_{l0ij}			
c :	a_{0123}	a_{1023}	a_{2013}	a_{3012}	a_{l123}
d_{ijk} :	a_{lijk}				
$r_{0i\,i+1}$:	$a_{i01\,i+1}$	$a_{i+1\,01i}$	$a_{l0i\,i+1}$		
$s_{i\,i+1\,j}$:	a_{i01j}	$a_{i+1\,01j}$	a_{j01i}	$a_{li\,i+1\,j}$	

Looking at this table it is easy to see that we used all the columns but

$$a_{0122}, a_{1123}, a_{2023}, a_{3122}, a_{4122}, \ldots, a_{n122}.$$

Hence if we choose now the entries

$$(0122, a_{1122}), (1123, a_{1123}), (0223, a_{2023}), \dots, (122n, a_{n122})$$

which are all equal to 1 we get Δ. The proof that this is the unique way to get Δ is similar, even if more complicate, to the analogous given in theorem 7. We omit the tedious details.

Using theorem 7 we can get some sporadic solution of the MRC.

PROPOSITION 9. *The MRC holds in* $\mathbf{P}^4_k$ *if* $t = 2$.

PROOF: Since $t = 2$ we have $5 \le s < 15$. Now if $s \le 8$ or $s \ge 10$ the conclusion follows by Corollary 5, Corollary 4 and Theorem 7. It remains to consider the case of 9 points in $\mathbf{P}^4_k$. We have in this case $m = 2$, hence we need to prove $a_3 = b_1 = 0$. By Theorem 2 we have $a_3 = 0$. Hence the resolution is

$$0 \to R(-6)^{b_4} \to R(-5)^{b_3} \to R(-4)^{b_2} \oplus R(-3)^{a_2} \to R(-3)^{b_1} \oplus R(-2)^{a_1} \to R \to 0$$

By the alternating sum of the homogeneous pieces we get $a_1 = 6$ and $a_2 = 4 + b_1$. The conclusion follows if we can prove $a_2 \le 4$. As in the proof of Theorem 7 we have a matrix M with $(4+1)(9-5) + \binom{4+1}{3} = 30$ rows and $(4+1)\binom{4}{2} = 30$ columns and we need to prove that its rank is ≥ 26. Using the same notations as above we choose all the rows $F_i(P_d)$ for $i = 0, \dots, 4$ and $d = 1, \dots, 4$ and further the rows $(013), (014), (024), (123), (134), (234)$. As for the columns we delete the following four: $a_{103}, a_{314}, a_{413}, a_{312}$. In this way we get a square matrix D whose determinant is not zero since the monomial $\Delta = m_1 m_2 m_3 m_4$ is a D-product which can be obtained in a unique way. To prove this, we remark that m_1, m_2, m_3 and m_4 can be obtained in a unique way using the lines $F_i(P_d)$ with $i = 0, \dots, 4$ and $d = 1, \dots, 4$ and the columns

$$a_{012}, a_{102}, a_{i01}, a_{203}, a_{302}, a_{i23}, a_{034}, a_{403}, a_{i04}, a_{214}, a_{412}, a_{i24}.$$

Now we delete the four columns $a_{103}, a_{314}, a_{413}, a_{312}$ and we get the following six columns left: $a_{013}, a_{014}, a_{134}, a_{213}, a_{234}, a_{402}$. As in the proof of the theorem we get the conclusion by choosing the entries

$$((013), a_{013}), ((014), a_{014}), ((134), a_{134}), ((123), a_{213}), ((024), a_{402}), ((234), a_{234}).$$

If we pass to $\mathbf{P}_k^5$ we get the MRC for $s \leq 9$ and $s \geq 15$. For $s = 11,12$ and 13 we get $m = 2$ and $a_3 = 0$ by Theorem 2. As in the proof of Proposition 9 we can also prove $b_1 = 0$ and we get the MRC also in these cases. For $s = 10$ we have $m = 3$, $a_4 = 0$ by Theorem 2, $b_1 = 0$ by the result of Green and Lazarsfeld but we cannot prove that $b_2 = 0$. Finally for $s = 14$ we have $m = 1$ and we cannot prove that $a_2 = 0$.

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Dipartimento di Matematica, Università di Genova, Via L.B.Alberti 4, 16132 Genova, Italy