On Ramanujan relations between Eisenstein series

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Abstract

The Ramanujan relations between Eisenstein series can be interpreted as an ordinary diferential equation in a parameter space of a family of elliptic curves which is inverse to the Gauss-Manin connection of the corresponding period map constructed by elliptic integrals of first and second kind. In this article we consider a slight modification of elliptic integrals by allowing non-algebraic integrants and we get in a natural way generalizations of Ramanujan relations between Eisenstein series.

1 Introduction

In the inverse of period map of the classical two parameter Weierstrass family of elliptic curves there appaers Eisenstein series of weight 4 and 6, and the Schwarz triangle function with triangular parameters $\frac{1}{p}, \frac{1}{q}, \frac{1}{r}$, where $p, q, r \in \mathbb{N}$, is the inverse of an automorphic function for the triangle group with signature $\langle p, q, r\rangle$. In all these examples the period maps of differential forms of the first kind are considered and considering periods of differential forms of the second kind, one gets differential automorphic functions which are solutions to certain ordinary differential equations (see [11]). Looking in this way, it is not necessary to define (differential) automorphic functions by funcational equations which they satify with respect to a Kleinian group, but as functions which are solutions to certain ordinary differential equations. To explain better this idea, let us state the main results of this paper for an example:

Theorem 1. *Consider the multi-valued function*

$$\mathsf{pm} : \mathbb{C}^3 \backslash \{(t_1, t_2, t_3) \in \mathbb{C}^3 \mid 27t_3^2 - 4t_2^3 = 0\} \to \mathrm{SL}(2, \mathbb{C}) \tag{1}$$

$$t \mapsto \begin{pmatrix} \int_{\delta_1} \frac{dx}{y} & \int_{\delta_1} \frac{xdx}{y} \\ \int_{\delta_2} \frac{dx}{y} & \int_{\delta_2} \frac{xdx}{y} \end{pmatrix},$$

where

$$y = \gamma^{\frac{1}{2}}(27t_3^2 - 4t_2^3)^{\frac{1}{2}(\frac{1}{2}-a)}((x - t_1)^3 - t_2(x - t_1) - t_3)^a \tag{2}$$

and δ_1 and $-\delta_2$ are two straight paths in the x-plane connecting one root of y to the other two roots. Here γ is a complex number depending only on a, b and c, and it is taken so that the image of pm *is in* $\mathrm{SL}(2, \mathbb{C})$*.*

1. *For $a \neq \frac{2}{3}$,* pm *is a local biholomorphism and its local inverse restricted to $\begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix}$, namely $(g_{1,a}(z), g_{2,a}(z), g_{3,a}(z))$, where z is in some small open set U in $\mathbb{C}$, satisfies*

the system of ordinary differential equations:

$$(3) \qquad \begin{cases} \dot{t}_1 = t_1^2 + \frac{3a-1}{9a-6}t_2 \\ \dot{t}_2 = 4t_1t_2 + \frac{3}{3a-2}t_3 \\ \dot{t}_3 = 6t_1t_3 + \frac{2}{9a-6}t_2^2 \end{cases},$$

where dot is the derivation with respect to z.

2. *The integrals $\int_\delta \frac{xdx}{y}$, where δ is a path connecting two roots of y, are constant along the solutions of (3).*

3. *$g_{k,a}$'s with repsect to the group*

$$(4) \qquad \Gamma := \langle M_1, M_2 \rangle \subset \mathrm{SL}(2,\mathbb{C}),$$

$$M_1 := \frac{i}{e^{\pi i a}} \begin{pmatrix} -e^{2\pi i a} & 0 \\ 1 & 1 \end{pmatrix}, M_2 := \frac{i}{e^{\pi i a}} \begin{pmatrix} 1 & e^{2\pi i a} \\ 0 & -e^{2\pi i a} \end{pmatrix},$$

have the following automorphic properties: for every $A = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma$ and $z \in U$ such that $cz + d \neq 0$ there exists an analytic continuation of $g_{k,a},\ k = 1,2,3$ along a path which connects z to Az such that

$$(5) \qquad (cz+d)^{-2k} g_{k,a}(Az) = g_{k,a}(z),\ k = 2,3,$$

$$(6) \qquad (cz+d)^{-2} g_{1,a}(Az) = g_{1,a}(z) + c(cz+d)^{-1}.$$

One can show that

$$(7) \qquad g_{k,\frac{1}{2}} = a_k\Big(1 + (-1)^k \frac{4k}{B_k} \sum_{n\geq 1} \sigma_{2k-1}(n) e^{2\pi i z n}\Big),\quad k = 1,2,3,\ z \in \mathbb{H},$$

is the Eisenstein series of weight $2k$, where $\mathbb{H}$ is the upper half plane, B_k is the k-th Bernoulli number ($B_1 = \frac{1}{6},\ B_2 = \frac{1}{30},\ B_3 = \frac{1}{42},\ \ldots$), $\sigma_i(n) := \sum_{d|n} d^i$ and

$$(8) \qquad (a_1, a_2, a_3) = (\frac{2\pi i}{12}, 3(\frac{2\pi i}{12})^2, 2(\frac{2\pi i}{12})^3)$$

(see [11]). In the case $a = \frac{1}{2}$ the system of ordinary differential equations (3) is known as the Ramanujan relations between $g_{k,a},\ k = 1,2,3$ because he had observed that in this case the series (7) satisfy the differential equation (3) (see for instance [8]). I do not know any explicit expressions like (7) for an arbitrary $a \in \mathbb{C}$. Throughout the text we will consider the family (9) which is a generalization of (2) and it has the advantage that it contains full Gauss hypergeometric functions.

The text is organized in the following way: In §2 we consider a more general family of transcendent curves. In §3 and §4 we fix up the paths of integration and calculate the monodromies. In §5 we calculate the derivation of the period map. The calculation is similar to the calculation of Gauss-Manin connections in the algebraic context. In §6 we calculate the determinant of the period map and according to this calculation in §7 we redefine the period map. In §8 we take the inverse of the period map and obtain Ramanujan type relations. §9 is devoted the action of an algebraic group. Finally in §10 we discuss the automorphic properties of the functions which appear in the inverse of the period map.

I would like to thank IMPA in Rio de Janeiro and MPIM in Bonn for their lovely research ambient.

2 Families of transcendent curves

For $a, b, c \in \mathbb{C}$ fixed, we consider the following family of transcendent curves:

$$E_{t,a,b,c} = E_t : y = f(x), \tag{9}$$

$$f(x) := t_0^{\frac{1}{2}}(x-t_1)^a(x-t_2)^b(x-t_3)^c.$$

Here $t = (t_0, t_1, t_2, t_3) \in \mathbb{C}^4$ is a parameter. The discriminant of E_t is defined to be

$$\Delta = \Delta(t) := t_0(t_1-t_2)^2(t_2-t_3)^2(t_3-t_1)^2$$

and we will work with regular parameters, i.e.

$$t \in T := \{t \in \mathbb{C}^4 \mid \Delta(t) \neq 0\}.$$

The parameter t_0 is just for simplifying the calculations related to the Gauss-Manin connection of the family (see §5). If a, b and c are rational numbers then E_t's are algebraic curves. In this case one can use algebro-geometric methods in order to study the periods of E_t, see for instance [14]. In general, E_t is a solution of the following logarithmic differential equation

$$\frac{dy}{y} = \frac{adx}{x-t_1} + \frac{bdx}{x-t_2} + \frac{cdx}{x-t_3}.$$

In order to prove Theorem 1 we also consider the family

$$\tilde{E}_t : y = f(x),\ f(x) = \tilde{t}_0^{\frac{1}{2}}((x-\tilde{t}_1)^3 - \tilde{t}_2(x-\tilde{t}_1) - \tilde{t}_3)^a. \tag{10}$$

In the case $a = b = c$ there is a canonical map from the parameter space of the first curve to the parameter space of the second curve:

$$\tilde{t}_0 = t_0,\ \tilde{t}_1 = \frac{t_1+t_2+t_3}{3},\ \tilde{t}_2 = (\tilde{t}_1-t_1)(\tilde{t}_1-t_2) + (\tilde{t}_1-t_2)(\tilde{t}_1-t_3) + (\tilde{t}_1-t_1)(\tilde{t}_1-t_3),$$

$$\tilde{t}_3 = (\tilde{t}_1-t_1)(\tilde{t}_1-t_2)(\tilde{t}_1-t_3).$$

For simplicity we will also use t instead of $\tilde{t}$; being clear parameters of which family we are talking about.

3 Paths of integration and Pochhammer cycles

We distinguish three, not necessarily closed, paths in E_t. In the x-plane let $\tilde{\delta}_i,\ i = 1, 2, 3$ be the straight path connecting t_{i+1} to t_{i-1}, $i = 1, 2, 3$ (by definition $t_4 := t_1$ and $t_0 = t_3$). There are many paths in E_t which are mapped to $\tilde{\delta}_i$ under the projection on x. We choose one of them and call it δ_i. For the case in which $\mathrm{Re}(a), \mathrm{Re}(b), \mathrm{Re}(c) < 0$ the paths δ_i's and $\tilde{\delta}_i$'s are depicted in Figure 1. We can make our choices so that $\delta_1 + \delta_2 + \delta_3$ is a limit of a closed and homotopic-to-zero path in E_t. For instance, we can take the path $\tilde{\delta}_i$'s in such a way that the triangle formed by them has almost zero area. Now, we have the integral

$$\int_\delta \frac{p(x)dx}{y} = \int_{\tilde{\delta}} \frac{p(x)dx}{f(x)},\ p \in \mathbb{C}[x], \tag{11}$$

where δ is one of the paths explained above. By a linear change in the variable x such integrals can be written in terms of the Gauss hypergeometric function (see [7]).

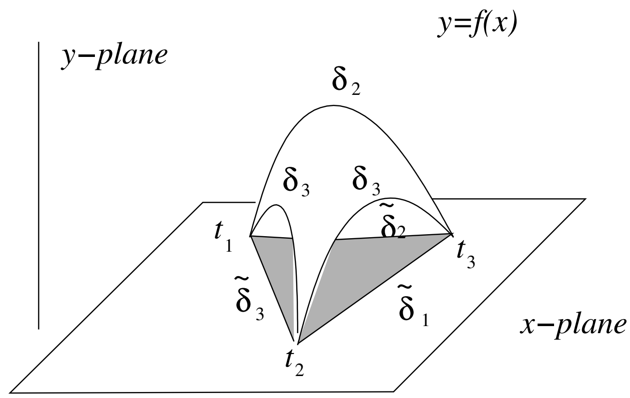

Figure 1: Paths of integration

Another way to study the integrals (11) is by using Pochhammer cycles. For simplicity we explain it for the pairs (t_1, t_2). The Pochhammer cycle associated to the points $t_1, t_2 \in \mathbb{C}$ and the path $\tilde{\delta}_3$ is the commutator

$$\tilde{\alpha}_3 = [\gamma_1, \gamma_2] = \gamma_1^{-1} \cdot \gamma_2^{-1} \cdot \gamma_1 \cdot \gamma_2,$$

where γ_1 is a loop along $\tilde{\delta}_3$ starting and ending at some point in the midle of $\tilde{\delta}_1$ which encircles t_1 once anticlockwise, and γ_2 is a similar loop with respect to t_2. It is easy to see that the cycle $\tilde{\alpha}_3$ lifts up to a closed path α_3 in E_t and if $a, b \notin \mathbb{Z}$ then

$$\int_{\alpha_3} \frac{p(x)dx}{y} = (1 - e^{-2\pi i a})(1 - e^{-2\pi i b}) \int_{\tilde{\alpha}_3} \frac{p(x)dx}{f(x)} dx.$$

(see [7], Proposition 3.3.7). Note that in order to have

$$\int_{\tilde{\delta}_i} d(\frac{p(x)}{f(x)}) = 0, \ \forall p \in \mathbb{C}[x], \ i = 1, 2, 3$$

we have to assume that $\mathrm{Re}(a), \mathrm{Re}(b), \mathrm{Re}(c) < 0$. But this is not necessary if we work with Pochhammer cycles.

4 The period map and the monodromy group

For a fixed $\mathsf{a} \in T$, the period map is given by:

$$\mathsf{pm} : (T, \mathsf{a}) \to \mathrm{GL}(2, \mathbb{C}), \ t \mapsto \begin{pmatrix} \int_{\delta_1} \frac{dx}{y} & \int_{\delta_1} \frac{xdx}{y} \\ \int_{\delta_2} \frac{dx}{y} & \int_{\delta_2} \frac{xdx}{y} \end{pmatrix}, \tag{12}$$

where (T, a) means a small neighborhood of a in T. The map pm can be extended along any path in T with the starting point $a\mathsf{a}$. We denote by $\mathcal{P}$ the union of images of the extensions of pm and call it the period domain. In order to study the analytic extensions of pm we have to calculate the monodromy group. In what follows we use the following convention: Two paths in E_t are equal if the integration of any differential form $\frac{p(x)dx}{y}$, $p \in \mathbb{C}[x]$ over them is equal. For instance, using this convention we have

$$\delta_1 + \delta_2 + \delta_3 = 0. \tag{13}$$

Let

$$A = e^{2\pi i a},\ B = e^{2\pi i b},\ C = e^{2\pi i c}.$$

We fix t_2 and t_3 and let t_1 turn around t_2 anti clockwise. We obtain three new paths $h_3(\delta_1)$, $h_3(\delta_2)$ and $h_3(\delta_3)$ in E_t such that $h_3(\delta_1)+h_3(\delta_2)+h_3(\delta_3)=0$ (this follows from (13)). Note that in the x-plane (resp. in E_t) the triangle formed by $h_3(\tilde{\delta_i})$'s (resp. $h_3(\delta_i)$'s) does not intersect itself. We have

$$h_3(\delta_2) = \delta_2 + (A - AB)\delta_3,\ h_3(\delta_1) = -\delta_2 - A\delta_3 = \delta_1 + (1 - A)\delta_3,\ h_3(\delta_3) = AB\delta_3$$

(see Figure 2, A). We call h_3 the monodromy around the hyperplane $t_1 = t_2$. These formulas are compatible with the Picard-Lefschetz formula in the case $a = b = c = \frac{1}{2}$. In a similar way

$$h_1(\delta_3) = \delta_3 + B\delta_1 - BC\delta_1, h_1(\delta_2) = -\delta_3 - B\delta_1,\ h_1(\delta_1) = BC\delta_1$$

and

$$h_2(\delta_1) = \delta_1 + C\delta_2 - CA\delta_2, h_2(\delta_3) = -\delta_1 - C\delta_2,\ h_2(\delta_2) = CA\delta_2.$$

Therefore, the monodromies with respect to the basis (δ_1, δ_2) are represented as

$$M_3 = \begin{pmatrix} A & A-1 \\ A(B-1) & A(B-1)+1 \end{pmatrix},\ M_1 = \begin{pmatrix} BC & 0 \\ 1-B & 1 \end{pmatrix}, M_2 = \begin{pmatrix} 1 & C-CA \\ 0 & CA \end{pmatrix}.$$

Note that

$$M_1 M_2 M_3 = \begin{pmatrix} ABC & 0 \\ 0 & ABC \end{pmatrix},$$

and that for $n \in \mathbb{N}$

$$h_3^n(\delta_2) = \delta_2 + (A - AB)\frac{(AB)^n - 1}{AB - 1}\delta_3, h_3^n(\delta_1) = \delta_1 + (1 - A)\frac{(AB)^n - 1}{AB - 1}\delta_3,\ h_3^n(\delta_3) = (AB)^n\delta_3.$$

The monodromy group Γ is defined to be the subgroup of $\mathrm{GL}(2,\mathbb{C})$ generated by $M_i,\ i = 1, 2, 3$. For $a = b = c = \frac{1}{2}$ we have

$$M_3 = \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix},\ M_1 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, M_2 = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

and it is easy to see that $\Gamma = \Gamma(2) := \{A \in \mathrm{SL}(2,\mathbb{Z}) \mid A \equiv_2 \mathrm{Id}\}$.

We discuss the monodromies for the family (10). In this family the monodromies change the place of the roots of f. Therefore, the monodromy h_3 corresponding to the replacement of t_1 and t_2 is given by

$$h_3(\delta_1) = -\delta_2 = \delta_1 + \delta_2,\ h_3(\delta_2) = \delta_2 + A\delta_3,\ h_3(\delta_3) = -A\delta_3.$$

(see Figure 2 B). The other monodromies are

$$h_1(\delta_2) = -\delta_3,\ h_1(\delta_3) = \delta_3 + A\delta_1,\ h(\delta_1) = -A\delta_1.$$

$$h_2(\delta_3) = -\delta_1,\ h_2(\delta_1) = \delta_1 + A\delta_2,\ h(\delta_2) = -A\delta_2.$$

Therefore, in the basis (δ_1, δ_2), the monodromies has the form

$$M_3 = \begin{pmatrix} 0 & -1 \\ -A & 1-A \end{pmatrix},\ M_1 = \begin{pmatrix} -A & 0 \\ 1 & 1 \end{pmatrix}, M_2 = \begin{pmatrix} 1 & A \\ 0 & -A \end{pmatrix}.$$

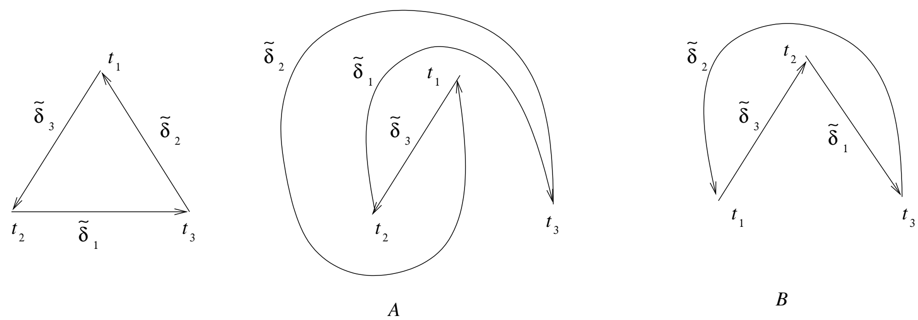

Figure 2: Monodromy

Note that

$$M_3^{-1} M_1 M_3 = M_2$$

For $a = \frac{1}{2}$ we have

$$M_3 = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix},\ M_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, M_2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

and so $\Gamma = \mathrm{SL}(2, \mathbb{Z})$.

Remark 1. In general it is hard to decide for which parameters a, b, c the group Γ is Kleinian , i.e it acts discontinuously in some open subset of $\mathbb{C} \cup \{\infty\}$. There is a necessary condition for such groups called Jorgensen's inequality (see [2]) but it is not sufficient[1]. For $\nu_0 := 1 - a - c = \frac{1}{p},\ \nu_1 := 1 - b - c = \frac{1}{q},\ \nu_\infty := 1 - a - b = \frac{1}{r}$, where p, q, r are positive integers, the group Γ is the triangular group of type $\langle p, q, r\rangle$ and it is Kleinian (see [2, 9, 14]). Despite the fact that $\Gamma\backslash\mathcal{P}$ may not have any reasonable structure, the global period map $\mathsf{pm} : T \to \Gamma\backslash\mathcal{P}$ is well-defined.

5 A kind of Gauss-Manin connection

The Gauss-Manin connection is the art of derivation of differential forms on families of algebraic varieties and then simplifying the result. Despite the fact that the varieties considered in this article are not algebraic, the process of derivation and simplification is similar to the algebraic case (see for instance [10]). In what follows, derivation with respect to x is denoted by $'$.

First of all we have to simplify the integral (11). More precisely we want to reduce it to the integrals with $p = 1, x$. Let $\mathsf{R} = \mathbb{C}(t)$ and

$$g = (x - t_1)(x - t_2)(x - t_3).$$

Proposition 1. *For all $p \in \mathsf{R}[x]$, there is $\tilde{p} \in \mathsf{R}[x]$, $\deg(\tilde{p}) \leq 1$ such that*

$$\int_\delta \frac{p dx}{y} = \int_\delta \frac{\tilde{p} dx}{y},$$

where δ is a path which connects two points of $\{t_1, t_2, t_3\}$ and does not cross it elsewhere.

[1] I would like to thank Katsuhiko Matsuzaki who informed me about the mentioned fact.

Proof. For $n > 1$ modulo exact forms we have

$$0 = d(\frac{x^{n-2}g}{f}) = (-x^{n-2}g\frac{f'}{f} + (x^{n-2}g)')\frac{dx}{f}.$$

Note that $g\frac{f'}{f}$ is a polynomial in x. We set $p_n = a_n x^n + r_n(x),\ a_n \in \mathbb{C},\ \deg(r_n) \leq n-1$ the polynomial in the parenthesis. We have $a_n \neq 0$ and so modulo exact forms we have:

$$x^n \frac{dx}{f} = \frac{-1}{a_n} r_{n-1} \frac{dx}{f}.$$

By various applications of the above equality in $\int_\delta \frac{pdx}{y}$ we finally get the desired equality. $\square$

Let us now differentiate the integrals:

Proposition 2. *Let t be one of the parameters $t_i,\ i = 0, 1, 2, 3$. We have*

$$\frac{\partial}{\partial t}\int_{\tilde{\delta}} \frac{pdx}{f} = \int_{\tilde{\delta}} \nabla_{\frac{\partial}{\partial t}} \frac{pdx}{f},\ p \in \mathbb{C}[x],$$

where

$$\nabla_{\frac{\partial}{\partial t}} \frac{pdx}{f} := \frac{1}{\Delta}((a_1 \frac{-\frac{\partial f}{\partial t} g}{f} p)' + a_2 \frac{-\frac{\partial f}{\partial t} g}{f} p + \Delta \frac{\partial p}{\partial t})\frac{dx}{f}.$$

Proof. We can find two polynomials $a_1, a_2 \in \mathsf{R}[x]$ such that

$$g\frac{f'}{f} a_1 + g a_2 = \Delta.$$

We have

$$\begin{aligned}
\frac{\partial}{\partial t}\int_{\tilde{\delta}} \frac{pdx}{f} &= \int_{\tilde{\delta}} (\frac{-\frac{\partial f}{\partial t} p}{f} + \frac{\partial p}{\partial t})\frac{dx}{f} \\
&= \int_{\tilde{\delta}} (\frac{\frac{-\frac{\partial f}{\partial t} gp}{f}}{g} + \frac{\partial p}{\partial t})\frac{dx}{f} \\
&= \int_{\tilde{\delta}} (\frac{\tilde{p}}{g} + \frac{\partial p}{\partial t})\frac{dx}{f}, \qquad \tilde{p} = \frac{-\frac{\partial f}{\partial t} g}{f} p \\
&= \frac{1}{\Delta}\int_{\tilde{\delta}} (\frac{(g\frac{f'}{f} a_1 + g a_2)\tilde{p}}{g} + \Delta \frac{\partial p}{\partial t})\frac{dx}{f} \\
&= \frac{1}{\Delta}\int_{\tilde{\delta}} (\frac{df}{f^2} a_1 \tilde{p} + a_2 \tilde{p}\frac{dx}{f} + \Delta \frac{\partial p}{\partial t}\frac{dx}{f}) \\
&= \frac{1}{\Delta}\int_{\tilde{\delta}} (\frac{1}{f} d(a_1\tilde{p}) + a_2 \tilde{p}\frac{dx}{f} + \Delta \frac{\partial p}{\partial t}\frac{dx}{f}) \\
&= \frac{1}{\Delta}\int_{\tilde{\delta}} ((a_1\tilde{p})' + a_2\tilde{p} + \Delta \frac{\partial p}{\partial t})\frac{dx}{f}.
\end{aligned}$$

$\square$

For the implementation of the algorithms of this section in SINGULAR [3] see the author's web page. For the family (9) we have used these algorithms and we have obtained:
(14)

$$\nabla_{\frac{\partial}{\partial t_1}}\omega = \frac{1}{(t_1-t_2)(t_1-t_3)}\begin{pmatrix} -at_1+(a+c-1)t_2+(a+b-1)t_3 & -a-b-c+2 \\ at_2t_3+(b-1)t_1t_3+(c-1)t_1t_2 & (-a-b-c+2)t_1 \end{pmatrix}\omega$$

where

$$\omega = \begin{pmatrix} \frac{dx}{y} \\ \frac{xdx}{y} \end{pmatrix}.$$

The derivations with respect to t_2 (resp t_3) is obtained by permutation of t_1 with t_2 and a with b (resp. t_1 with t_3 and a with c). It is also easy to check by hand that

$$\nabla_{\frac{\partial}{\partial t_0}}\omega = \frac{1}{t_0}\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}\omega$$

We will simply denote by $\nabla_{\frac{\partial}{\partial t_i}}$, $i=0,1,2,3$ the matrix A in $\nabla_{\frac{\partial}{\partial t_i}}\omega = A\omega$.

For the family (10) we use $g=(x-t_1)^3-t_2(x-t_1)-t_3$ and $\Delta = t_0(4t_2^3-27t_3^2)$ and we have

$$\nabla_{\frac{\partial}{\partial t_0}} = \frac{1}{t_0}\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix},\ \nabla_{\frac{\partial}{\partial t_1}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$\nabla_{\frac{\partial}{\partial t_2}} = \frac{1}{(4t_2^3-27t_3^2)}\begin{pmatrix} -27at_1t_3-6at_2^2+18t_1t_3+2t_2^2 & 27at_3-18t_3 \\ -27at_1^2t_3+9at_2t_3+18t_1^2t_3-2t_1t_2^2-3t_2t_3) & 27at_1t_3-6at_2^2-18t_1t_3+4t_2^2 \end{pmatrix},$$

$$\nabla_{\frac{\partial}{\partial t_3}} = \frac{1}{(4t_2^3-27t_3^2)}\begin{pmatrix} 18at_1t_2+27at_3-12t_1t_2-9t_3 & -18at_2+12t_2 \\ 18at_1^2t_2-6at_2^2-12t_1^2t_2+9t_1t_3+2t_2^2 & -18at_1t_2+27at_3+12t_1t_2-18t_3 \end{pmatrix}.$$

6 Determinant of the period matrix

¿From Proposition 2 it follows that the period map satisfies the differential equation

$$d(\mathsf{pm}) = \mathsf{pm}A^{\mathsf{tr}},\ \text{where } A=\sum_{i=0}^{3}(\nabla_{\frac{\partial}{\partial t_i}})dt_i.$$

This and (14) imply that $\det := \det(\mathsf{pm})$ satisfies

$$\frac{\partial \det}{\partial t_1} = \frac{1}{(t_1-t_2)(t_1-t_3)}((a+c-1)t_2+(a+b-1)t_3+(-2a-b-c+2)t_1)\det.$$

Solving this differential equation one concludes that det is of the form $C(t_1-t_3)^{1-a-c}(t_1-t_2)^{1-a-b}$, where C does not depend on t_1. Repeating the same argument for t_0,t_2,t_3 we conclude that

$$\det(\mathsf{pm}) = \gamma\cdot t_0^{-1}(t_1-t_3)^{1-a-c}(t_1-t_2)^{1-a-b}(t_2-t_3)^{1-b-c}, \tag{15}$$

where γ is a constant depending only on a,b and c. For the family (10) in a similar way we get

$$\det(\mathsf{pm}) = \gamma\cdot t_0^{-1}(27t_3^2-4t_2^3)^{\frac{1}{2}-a}.$$

7 Redefining the period map and the monodromy group

We have calculated the determinant of the period map in (15). It depends on t_1, t_2, t_3 except for the case $a = b = c = \frac{1}{2}$. In order that the determinant of the period map to be equal to t_0^{-1} and the monodromy group to be a subgroup of $\mathrm{SL}(2, \mathbb{C})$, we have to multiply (12) by

$$p := \gamma^{-\frac{1}{2}}(t_1 - t_3)^{-\frac{1}{2}(1-a-c)}(t_1 - t_2)^{-\frac{1}{2}(1-a-b)}(t_2 - t_3)^{-\frac{1}{2}(1-b-c)}.$$

In other words, we have to redefine

$$f(x) := \gamma^{\frac{1}{2}} t_0^{\frac{1}{2}}(t_1 - t_3)^{\frac{1}{2}(1-a-c)}(t_1 - t_2)^{\frac{1}{2}(1-a-b)}(t_2 - t_3)^{\frac{1}{2}(1-b-c)}(x - t_1)^a(x - t_2)^b(x - t_3)^c$$

for the family (9). We have to recalculate the Gauss-Manin connection. By Leibniz rule we have

$$\nabla(p\omega) = (dp) \cdot \omega + p \cdot A \otimes \omega = (\frac{dp}{p} I_{2\times 2} + A) \otimes (p\omega)$$

and

$$\frac{dp}{p} = \frac{1}{2}(a + b - 1)\frac{dt_1 - dt_2}{t_1 - t_2} + \cdots = (\frac{1}{2}(a + b - 1)\frac{1}{t_1 - t_2} + \frac{1}{2}(a + c - 1)\frac{1}{t_1 - t_3})dt_1 + \cdots$$

After redefining the period map the monodromy matrices are changed as follows:

$$M_3 = \frac{-1}{\sqrt{AB}}\begin{pmatrix} A & A-1 \\ A(B-1) & A(B-1)+1 \end{pmatrix},\ M_1 = \frac{-1}{\sqrt{BC}}\begin{pmatrix} BC & 0 \\ 1-B & 1 \end{pmatrix}, M_2 = \frac{-1}{\sqrt{CA}}\begin{pmatrix} 1 & C-CA \\ 0 & CA \end{pmatrix}$$

Note that $\sqrt{A} = e^{\pi i a}$, $\cdots$ are well-defined and $\Gamma := \langle M_1, M_2, M_3\rangle = \langle M_1, M_2\rangle \subset \mathrm{SL}(2, \mathbb{C})$.

For the family (10) we redefine

$$f(x) = \gamma^{\frac{1}{2}} t_0^{\frac{1}{2}}(27t_3^2 - 4t_2^3)^{\frac{1}{2}(\frac{1}{2}-a)}((x - t_1)^3 - t_2(x - t_1) - t_3)^a,\ t = (t_0, t_1, t_2, t_3) \in \mathbb{C}^4$$

which is the one in (1) with $t_0 = 1$. For $p = (27t_3^2 - 4t_2^3)^{-\frac{1}{2}(\frac{1}{2}-a)}$ we have

$$\frac{dp}{p} = \frac{1}{2}(a - \frac{1}{2})\frac{54t_3dt_3 - 12t_2^2dt_2}{27t_3^2 - 4t_2^3}$$

The new monodromy group is (4). For both families we conclude that $\det(\mathsf{pm}) = t_0^{-1}$.

Remark 2. A subgroup A of $\mathrm{SL}(2, \mathbb{R})$ is called arithmetic if A is commensurable with $\mathrm{SL}(2, \mathbb{Z})$, i.e. $A \cap \mathrm{SL}(2, \mathbb{Z})$ has a finite index in both $\mathrm{SL}(2, \mathbb{Z})$ and A. It is a natural question to classify all the cases such that the monodromy group is conjugated with an arithmetic group. For the case $a = b = 1 - c = \frac{1}{6}$ the monodromy group is conjugated with $\mathrm{SL}(2, \mathbb{Z})$ (see [12]).

8 The inverse of the period map

First we note that the period map is a local biholomorphism. We consider pm as a map sending the point (t_0, t_1, t_2, t_3) to (x_1, x_2, x_3, x_4). Its derivative at t is a 4×4 matrix whose i-th column constitutes of the first and second row of $x(\nabla_{\frac{\partial}{\partial t_i}})^{\mathsf{tr}}$. For $s := a + b + c - 2 \neq 0$, this is an invertible matrix. More precisely, we have

$$(dF)_x = (d\mathsf{pm})_t^{-1} = \frac{1}{\det(x)}\cdot$$

$$\left(\begin{matrix} -t_0x_4 & t_0x_3 \\ (at_1t_2x_3+at_1t_3x_3-at_1x_4-at_2t_3x_3+bt_1^2x_3-bt_1x_4+ct_1^2x_3-ct_1x_4-t_1^2x_3-t_1t_2x_3-t_1t_3x_3+2t_1x_4+t_2t_3x_3)/s & (-t_1x_3+x_4) \\ (at_2^2x_3-at_2x_4+bt_1t_2x_3-bt_1t_3x_3+bt_2t_3x_3-bt_2x_4+ct_2^2x_3-ct_2x_4-t_1t_2x_3+t_1t_3x_3-t_2^2x_3-t_2t_3x_3+2t_2x_4)/s & (-t_2x_3+x_4) \\ (at_3^2x_3-at_3x_4+bt_3^2x_3-bt_3x_4-ct_1t_2x_3+ct_1t_3x_3+ct_2t_3x_3-ct_3x_4+t_1t_2x_3-t_1t_3x_3-t_2t_3x_3-t_3^2x_3+2t_3x_4)/s & (-t_3x_3+x_4) \end{matrix}\right.$$

$$\left.\begin{matrix} t_0x_2 & -t_0x_1 \\ (-at_1t_2x_1-at_1t_3x_1+at_1x_2+at_2t_3x_1-bt_1^2x_1+bt_1x_2-ct_1^2x_1+ct_1x_2+t_1^2x_1+t_1t_2x_1+t_1t_3x_1-2t_1x_2-t_2t_3x_1)/s & (t_1x_1-x_2) \\ (-at_2^2x_1+at_2x_2-bt_1t_2x_1+bt_1t_3x_1-bt_2t_3x_1+bt_2x_2-ct_2^2x_1+ct_2x_2+t_1t_2x_1-t_1t_3x_1+t_2^2x_1+t_2t_3x_1-2t_2x_2)/s & (t_2x_1-x_2) \\ (-at_3^2x_1+at_3x_2-bt_3^2x_1+bt_3x_2+ct_1t_2x_1-ct_1t_3x_1-ct_2t_3x_1+ct_3x_2-t_1t_2x_1+t_1t_3x_1+t_2t_3x_1+t_3^2x_1-2t_3x_2)/s & (t_3x_1-x_2) \end{matrix}\right)$$

and

$$F=(F_0,F_1,F_2,F_3):(\mathcal{P},x_0)\to(T,\mathsf{a})$$

is the local inverse of pm, where $x_0=\mathsf{pm}(\mathsf{a})$. ¿From $\det(\mathsf{pm})=t_0^{-1}$ it follows that $F_0(x)=\det(x)^{-1}$. Let us take a in such a way that x_0 is of the form $\begin{pmatrix} z_0 & -1 \\ 1 & 0 \end{pmatrix}$. Let $g_i(z)$ be the restriction of F_i to $\begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix}$, where z is in a neighborhood of z_0 in $\mathbb{C}$. Considering the equations related to the entries $(i,1), i=2,3,4$, we conclude that $(g_1(z),g_2(z),g_3(z))$ satisfies the ordinary differential equation: [2]

$$\left\{\begin{array}{l} \dot{t}_1=\frac{a-1}{a+b+c-2}(t_1t_2+t_1t_3-t_2t_3)+\frac{b+c-1}{a+b+c-2}t_1^2 \\ \dot{t}_2=\frac{b-1}{a+b+c-2}(t_2t_1+t_2t_3-t_1t_3)+\frac{a+c-1}{a+b+c-2}t_2^2 \\ \dot{t}_3=\frac{c-1}{a+b+c-2}(t_3t_1+t_3t_2-t_1t_2)+\frac{a+b-1}{a+b+c-2}t_3^2 \end{array}\right. . \tag{16}$$

In a similar way for the family (10), we get (3) and so the first part of Theorem 1 is proved. Let Ra be the vector field in $\mathbb{C}^4$ corresponding to (16) together with $\dot{t}_0=0$. It is a mere calculation to see that

$$\nabla_{\mathrm{Ra}}=\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}.$$

This means that $d(\mathsf{pm})(\mathrm{Ra})=\begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix}$ and so $\int_\delta \frac{xdx}{y}$ is constant along the leaves of $\mathcal{F}(\mathrm{Ra})$. The similar argument work for the family (10) and so the second part of Theorem 1 is proved.

9 Action of an algebraic group

The algebraic group

$$G_0=\left\{\begin{pmatrix} k_1 & k_3 \\ 0 & k_2 \end{pmatrix} \mid k_3\in\mathbb{C}, k_1,k_2\in\mathbb{C}^*\right\} \tag{17}$$

acts on the period domain $\mathcal{P}\subset \mathrm{GL}(2,\mathbb{C})$ from the right by the usual multiplication of matrices. It acts also in $\mathbb{C}^4$ as follows:

$$t\bullet g:=(t_0(k_1k_2)^{-1},t_1k_1^{-1}k_2+k_3k_1^{-1},t_2k_1^{-1}k_2+k_3k_1^{-1},t_3k_1^{-1}k_2+k_3k_1^{-1})$$

$$t=(t_0,t_1,t_2,t_3)\in\mathbb{C}^4, g=\begin{pmatrix} k_1 & k_3 \\ 0 & k_2 \end{pmatrix}\in G_0. \tag{18}$$

The relation between these two actions of G_0 is given by:

[2]When the paper was finished, I found that such a differential equation was already discovered by G. Halphen [6, 5, 4] in his study of hyper-geometric functions. However, the geometric interpretation and the automorphic properties of its solutions are new in this paper .

Proposition 3. *We have*

$$\mathsf{pm}(t\bullet g)=\mathsf{pm}(t)\cdot g,\ \ t\in T,\ \ g\in (G_0,I) \tag{19}$$

where (X,x) means a small neighborhood of x in X and I is the identity 2×2 matrix.

If $t_s, s\in[0,1]$ is a path in T and $g_s,\ s\in[0,1]$ is a path in G_0 which connects I to $g\in G_0$, by analytic continuation of the equality (19), it makes sense to say that (19) is true for an arbitrary $g\in G_0$.

Proof. Let

$$\alpha:\mathbb{C}^2\to\mathbb{C}^2,\ (x,y)\mapsto (k_2^{-1}k_1x-k_3k_2^{-1}, k_2^{-1}k_1^2y).$$

Then

$$k_2k_1^{-2}\alpha^{-1}(y-f(x))=y-(8\pi it_0)^{\frac{1}{2}}k_2k_1^{-2}(t_2-t_3)^{\frac{1}{2}(1-b-c)}(k_2^{-1}k_1x-k_3k_2^{-1}-t_1)^a\cdots=y-$$

$$(8\pi it_0)^{\frac{1}{2}}k_2^{1-a-b-c}k_1^{a+b+c-2}(k_2^{-1}k_1)^{\frac{1}{2}(3-2(a+b+c))}(k_2k_1^{-1}t_2-k_2k_1^{-1}t_3)^{\frac{1}{2}(1-b-c)}(x-(k_2k_1^{-1}t_1+k_3k_1^{-1}))^a$$

$$\cdots=y-(8\pi it_0)^{\frac{1}{2}}(k_2k_1)^{-\frac{1}{2}}(k_2k_1^{-1}t_2+k_3k_1^{-1}-(k_2k_1^{-1}t_3+k_3k_1^{-1}))^{\frac{1}{2}(1-b-c)}(x-(k_2k_1^{-1}t_1+k_3k_1^{-1}))^a\cdots$$

This implies that α induces an isomorphism

$$\alpha: E_{t\bullet g}\to E_t.$$

Now

$$\alpha^{-1}\omega=\begin{pmatrix}k_1^{-1} & 0\\ -k_3k_2^{-1}k_1^{-1} & k_2^{-1}\end{pmatrix}\omega=\begin{pmatrix}k_1 & 0\\ k_3 & k_2\end{pmatrix}^{-1}\omega,$$

where $\omega=(\frac{dx}{y},\frac{xdx}{y})^{\mathsf{tr}}$, and so

$$\mathsf{pm}(t)=\mathsf{pm}(t\bullet g).g^{-1}$$

which proves (19). □

In a similar way for the family (10) we have the action

$$t\bullet g:=(t_0k_1^{-1}k_2^{-1}, t_1k_1^{-1}k_2+k_3k_1^{-1}, t_2k_1^{-3}k_2, t_3k_1^{-4}k_2^2)$$

$$t=(t_0,t_1,t_2,t_3)\in\mathbb{C}^4, g=\begin{pmatrix}k_1 & k_3\\ 0 & k_2\end{pmatrix}\in G_0 \tag{20}$$

with the property (19).

Remark 3. The rational map

$$\alpha:\mathbb{C}^2\to\mathbb{C}^2,\ (x,y)\mapsto (x,\frac{y}{(x-t_1)^{[a]}(x-t_2)^{[b]}(x-t_3)^{[c]}})$$

sends $E_{t,a,b,c}$ biholomorphically to $E_{t,\langle a\rangle,\langle b\rangle,\langle c\rangle}$. We use Proposition 1 and write

$$\alpha^*\omega^{\mathsf{tr}}=\omega^{\mathsf{tr}}C,\ C\in\mathrm{Mat}(2,\mathbb{Q}[t]).$$

The period map associated to $E_{t,\langle a\rangle,\langle b\rangle,\langle c\rangle}$ is the multiplication of the period map associated to $E_{t,a,b,c}$ with C. For this reason it is sometimes practical to assume that $0\leq \mathrm{Re}(a),\mathrm{Re}(b),\mathrm{Re}(c)<1$.

10 Automorphic properties of g_i's

We continue the notation introduced in §8. We denote by

$$F = (F_0, F_1, F_2, F_3) : (\mathcal{P}, x_0) \to (T, \mathsf{a})$$

the local inverse of the period map. Taking F of (19) we conclude that

$$F(xg) = F(x) \bullet g, \ g \in (G_0, I). \tag{21}$$

We get

$$F_0(xg) = F_0(x) k_1^{-1} k_2^{-1},$$

$$F_i(xg) = F_1(x) k_1^{-1} k_2 + k_3 k_1^{-1}, \ i = 1, 2, 3 \tag{22}$$

The first equality also follows from $F_0(x) = \det(x)^{-1}$.

For any $A = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma$ there is a path $\gamma \in \pi_1(T, \mathsf{a})$ such that if $\tilde{\mathsf{pm}} : (T, \mathsf{a}) \to \mathcal{P}$ is the analytic continuation of pm along γ then

$$\tilde{\mathsf{pm}}(t) = A\mathsf{pm}(t), \ \forall t \in (T, \mathsf{a}).$$

This implies that the analytic continuation of F along the path $\delta = \mathsf{pm}(\gamma)$, which connects $\mathsf{pm}(\mathsf{a})$ to $A\mathsf{pm}(\mathsf{a})$ satisfies

$$F(x) = F(Ax), \ x \in (\mathcal{P}, x_0). \tag{23}$$

Using the Schwarz function

$$D(t) = \frac{\int_{\delta_1} \frac{dx}{f}}{\int_{\delta_2} \frac{dx}{f}}$$

we define the path $\sigma = D(\gamma)$. If $cz_0 + d \neq 0$ then Az_0 is well-defined and the path σ connects z_0 to Az_0 in $\mathbb{C}$. We claim that there is an analytic continuations of g_i's along σ such that

$$(cz+d)^{-2} g_i(Az) = g_i(z) + c(cz+d)^{-1}, \ i = 1, 2, 3, \ A = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma, z \in (\mathbb{C}, z_0).$$

We have

$$\begin{aligned}
(1, g_1, g_2, g_3) &= F \begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix} \\
&\overset{(23)}{=} F\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix} \right) \\
&= F\left(\begin{pmatrix} Az & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} cz+d & -c \\ 0 & (cz+d)^{-1}\det(A) \end{pmatrix} \right) \\
&\overset{(21)}{=} F\left(\begin{pmatrix} Az & -1 \\ 1 & 0 \end{pmatrix} \right) \bullet \begin{pmatrix} cz+d & -c \\ 0 & (cz+d)^{-1} \end{pmatrix} \\
&= (1, (cz+d)^{-2} g_1(Az) - c(cz+d)^{-1}, \cdots)
\end{aligned}$$

The fourth equality makes sense in the following way: Let

$$x_s := \begin{pmatrix} D(\gamma_s) & -1 \\ 1 & 0 \end{pmatrix} \in \mathcal{P},\ \tau_s := x_s^{-1}\mathsf{pm}(\gamma_s) \in G_0,\ s \in [0,1].$$

τ is a path in G_0 which connects I to $\begin{pmatrix} cz+d & -c \\ 0 & (cz+d)^{-1} \end{pmatrix}$. For s near enough to 0 we have $F(x_s\tau_s) = F(x_s) \bullet \tau_s$ and so by analytic continuation we have it for $s = 1$.

In a similar way we prove the third part of Theorem 1. Note that for the family (10), F_2 and F_3 satisfy:

$$F_2(xg) = F_2(x)k_1^{-3}k_2,\ F_3(xg) = F_3(x)k_1^{-4}k_2^2,\ \forall x \in \mathcal{L},\ g \in G_0.$$

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