NONCOMMUTATIVE ARITHMETIC GEOMETRY

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Ce qui a été compris n'existe plus,
L'oiseau s'est confondu avec le vent,
Le ciel avec sa vérité,
L'homme avec sa réalité.
(Paul Eluard)

1. Noncommutative geometry and modular curves

This section describes joint work with Yuri Manin (*cf.* [16]).

When one considers the noncommutative tori $\mathbb{T}^2_\theta$ as degenerations of elliptic curves $E_q = \mathbb{C}^*/q^{\mathbb{Z}}$ for $|q| \to 1$, it becomes natural to introduce a noncommutative version of the tower of modular curves (*cf.* [16]).

In the classical setting, for $G \subset \Gamma = \mathrm{PSL}(2,\mathbb{Z})$ a finite index subgroup and $X_G = G\backslash\mathbb{H}$ the quotient of 2-dimensional real hyperbolic plane, let $\phi : \mathbb{H} \to X_G$ denote the quotient map and $\mathbb{P} = \Gamma/G$ the coset space. The modular curve X_G has a natural algebro-geometric compactification, which consists of adding the cusp points $G\backslash\mathbb{P}^1(\mathbb{Q})$,

$$\overline{X_G} = G\backslash(\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})).$$

Then the natural way to bridge between the algebro–geometric theory of modular curves and noncommutative geometry consists of replacing $\mathbb{P}^1(\mathbb{Q})$ in the classical compactification, which gives rise to the finite set of cusps, with the full boundary $\mathbb{P}^1(\mathbb{R})$ of $\mathbb{H}$. This substitution cannot be done naively, since G does not act discretely on $\mathbb{P}^1(\mathbb{R})$, and the quotient is not even a Hausdorff space. However, the quotient makes perfect sense in noncommutative geometry and is given by the C^*-algebra $C(\mathbb{P}^1(\mathbb{R})) \rtimes G$ or, up to Morita equivalence, by

$$C(\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}) \rtimes \Gamma. \tag{1.1}$$

Various aspects of the classical theory of modular curves can be recast in term of the irrational points in the boundary, invisible to classical algebraic geometry.

An interesting aspect of the theory of noncommutative modular curves defined as in (1.1) is the fact that one obtains an interpretation in noncommutative geometry of the classical modular complex, in terms of K-theory of C^*-algebras (*cf.* [16]). This supports the idea that the noncommutative "boundary" (1.1) of the classical modular curve in fact contains all the interesting arithmetic information of the modular curve itself, which can be regarded as a sort of "holography principle" in the theory of modular curves (*cf.* [15]).

We denote by $\tilde{I}$ and $\tilde{R}$ the elliptic points $\tilde{I} = \Gamma\, i$ and $\tilde{R} = \Gamma\, \rho$ with $\rho = e^{\pi i/3}$, and by I and R the image under the quotient map. In $\mathrm{PSL}(2,\mathbb{Z}) = \mathbb{Z}/2 * \mathbb{Z}/3$, we denote by σ the generator of order two and by τ the generator of order three. To simplify notation we write

$$H_A^B := H_1(\overline{X_G} \setminus A, B; \mathbb{Z}). \tag{1.2}$$

Manin's modular complex (*cf.* [13]) is a complex computing the homology of X_G, induced by a Γ invariant cell decomposition of $\mathbb{H}$,

- 0-cells: the cusps $G\backslash\mathbb{P}^1(\mathbb{Q})$, and the elliptic points I and R.
- 1-cells: the oriented geodesic arcs $G\backslash\, \Gamma\langle i\infty, i\rangle$ and $G\backslash\, \Gamma\langle i, \rho\rangle$.
- 2-cells: $G\backslash\, \Gamma E$, where E is the polygon with vertices $\{i, \rho, 1+i, i\infty\}$ and geodesic arcs as sides.
- Boundary operators $\partial : C_2 \to C_1$ and $\partial : C_1 \to C_0$ given, respectively, by

$$gE \mapsto g\langle i,\rho\rangle + g\langle \rho, 1+i\rangle + g\langle 1+i, i\infty\rangle + g\langle i\infty, i\rangle, \;\; g\in\Gamma$$

$$g\langle i\infty, i\rangle \mapsto g(i) - g(i\infty), \;\; g\langle i,\rho\rangle \mapsto g(\rho) - g(i).$$

The modular complex provides a canonical presentation of the homology H^{cusps} via the relative homology sequence

$$0 \to H_{cusp} \to H^{R\cup I}_{cusps} \overset{(\tilde\beta_R,\tilde\beta_I)}{\to} H_0(R)\oplus H_0(I) \to \mathbb{Z} \to 0, \tag{1.3}$$

and the duality $H_A^B \times H_B^A \to \mathbb{Z}$. We have $H_0(R)\oplus H_0(I) = \mathbb{Z}[\mathbb{P}_I]\oplus\mathbb{Z}[\mathbb{P}_R]$, with $\mathbb{P}_I = \langle\sigma\rangle\backslash\mathbb{P}$ and $\mathbb{P}_R = \langle\tau\rangle\backslash\mathbb{P}$, while the homology $H^{R\cup I}_{cusps}$ is generated by the relative cycles $g\gamma_0 := g\langle i,\rho\rangle$, g ranging over representatives of $\mathbb{P}$, so that $H^{R\cup I}_{cusps} = \mathbb{Z}[\mathbb{P}]$. Thus we obtain the exact sequence

$$0 \to H^{cusps} \to \mathbb{Z}^{|\mathbb{P}|} \overset{(\beta_R,\beta_I)}{\to} \mathbb{Z}^{|\mathbb{P}_I|}\oplus\mathbb{Z}^{|\mathbb{P}_R|} \to \mathbb{Z} \to 0. \tag{1.4}$$

The group Γ acts on a tree T with set of edges $T^1 = \Gamma$ and set of vertices T^0 given by the cosets Γ/Γ_0 and Γ/Γ_1, with $\Gamma_0 = \mathbb{Z}/2$ and $\Gamma_1 = \mathbb{Z}/3$. This tree can be realized as a graph in $\mathbb{H}$, with vertices the elliptic points and edges the geodesic segments $\Gamma\langle i,\rho\rangle$. This implies that there is a six term exact sequence (*cf.* [18]), for $X = \mathbb{P}^1(\mathbb{R})\times\mathbb{P}$,

$$\begin{tikzcd}
K_0(C(X)) \arrow[r, "\alpha"] & K_0(C(X)\rtimes\Gamma_0)\oplus K_0(C(X)\rtimes\Gamma_1) \arrow[r, "\tilde\alpha"] & K_0(C(X)\rtimes\Gamma) \arrow[d] \\
K_1(C(X)\rtimes\Gamma) \arrow[u] & K_1(C(X)\rtimes\Gamma_0)\oplus K_1(C(X)\rtimes\Gamma_1) \arrow[l, "\tilde\beta"'] & K_1(C(X)) \arrow[l, "\beta"']
\end{tikzcd}$$

The following result recovers the modular complex from the K-theory of the C^*-algebra (1.1):

Theorem 1.1. *There exists a natural isomorphism of the exact sequence*

$$0 \to \mathrm{Ker}(\beta) \hookrightarrow K_1(C(X)) \xrightarrow{\beta} K_1(C(X) \rtimes \Gamma_0) \oplus K_1(C(X) \rtimes \Gamma_1) \to \mathrm{Im}(\tilde{\beta}) \to 0$$

and the exact sequence (1.4). *These isomorphisms are compatible with the restriction of the group $G' \subset G$, hence they extend compatibly to the full modular tower.*

For $\Gamma = \mathrm{PGL}(2, \mathbb{Z})$, the Γ orbits in $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}$ are the same as equivalence classes

$$(x, s) \sim_T (y, t) \Leftrightarrow \exists n, m : T^n(x, s) = T^m(y, t)$$

where $T : [0, 1] \times \mathbb{P} \to [0, 1] \times \mathbb{P}$ is the shift

$$T(x, s) = \left(\frac{1}{x} - \left[\frac{1}{x} \right], \begin{pmatrix} -[1/x] & 1 \\ 1 & 0 \end{pmatrix} \cdot s \right) \tag{1.5}$$

generalizing the classical shift of the continued fraction expansion. This equivalence relation also gives rise to a noncommutative space, in the form of a corresponding groupoid C^*-algebra. This provides a different approach to noncommutative geometry associated to the boundary of the classical modular curves.

In fact, the properties of the dynamical system (1.5) can be used to the general purpose of studying the limiting behavior for certain arithmetic invariants defined on modular curves, along a degeneration of elliptic curves to a noncommutative torus, with parameter $\tau \to \theta \in \mathbb{R} \setminus \mathbb{Q}$. An example of this type of result is an extension of the classical theory of modular symbols to homology classes defined by limiting cycles of non-closed geodesics in the modular curves (*cf.* [16], [17]). This is related to the spectral theory of the Ruelle transfer operator of (1.5), which is also closely related to the results of J.Lewis and D.Zagier on Maass wave forms, [12]. The Selberg zeta function of modular curves can be expressed as a Fredholm determinant of this Ruelle transfer operator (*cf.* [1] [16]), a result that can be also seen as a holographic correspondence between different field theories on the bulk space and on its (noncommutative) boundary.

Moreover, it is shown in [16] that cusps forms of weight two for congruence subgroups (or rather their Mellin transforms) can be obtained by integrating along the real axis certain “automorphic series” defined in terms of continued fractions and modular symbols. This type of result may be useful in understanding what function theory for the noncommutative modular curves (1.1) can recover the theory of modular forms on the upper half plane.

2. Noncommutative geometry and arithmetic surfaces

This section describes joint work with Katia Consani, [6] [7], [8].

The fundamental tool from noncommutative geometry is *spectral triples*, a notion introduced by Alain Connes to extend Riemannian geometry to noncommutative spaces (*cf.* [2], [3], [4]). A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ consists of a C*-algebra $\mathcal{A}$ with a representation

$$\rho : \mathcal{A} \to \mathcal{B}(\mathcal{H})$$

in the algebra of bounded operators on a separable Hilbert space $\mathcal{H}$, and an operator D (called the Dirac operator) on $\mathcal{H}$, which satisfies the following properties:

(1) D is self–adjoint.
(2) For all $\lambda \notin \mathbb{R}$, the resolvent $(D-\lambda)^{-1}$ is a compact operator on $\mathcal{H}$.
(3) The commutator $[D, a]$ is a bounded operator on $\mathcal{H}$, for all $a \in \mathcal{A}_0 \subset \mathcal{A}$, a dense involutive subalgebra of $\mathcal{A}$.

The arithmetic setting we consider is the following. For $\mathbb{K}$ a number field and $O_{\mathbb{K}}$ its ring of integers, a smooth proper algebraic curve X over $\mathbb{K}$ determines a smooth minimal model $X_{O_{\mathbb{K}}}$, which defines an arithmetic surface $\mathcal{X}_{O_{\mathbb{K}}}$ over $\mathrm{Sp}(O_{\mathbb{K}})$. The closed fiber $X_\wp$ of $\mathcal{X}_{O_{\mathbb{K}}}$ over a prime $\wp \in O_{\mathbb{K}}$ is given by the reduction mod $\wp$. In the process of adapting classical tools and results of algebraic geometry to this arithmetic context, it becomes clear that some compactification is necessary, at least in order to have a well behaved form of intersection theory. Namely, we need to pass from the "affine" $\mathrm{Sp}(O_{\mathbb{K}})$ to the projective case.

In arithmetic geometry, $\mathrm{Sp}(O_{\mathbb{K}})$ is compactified by adding the *infinite primes* (or *archimedean primes*). These are the embeddings $\alpha : \mathbb{K} \hookrightarrow \mathbb{C}$. For $\mathbb{K}$ a number field with $n = [\mathbb{K} : \mathbb{Q}]$, there are n such embeddings. Among them, there are r embeddings into $\mathbb{R}$, and s pairs of complex conjugate embeddings in $\mathbb{C}$ not contained in $\mathbb{R}$, so that $n = r + 2s$. It is customary to use the notation $\overline{\mathrm{Sp}(O_{\mathbb{K}})}$ for the union of $\mathrm{Sp}(O_{\mathbb{K}})$ and the archimedean primes.

Correspondingly, one needs to compactify the arithmetic surface $\mathcal{X}_{O_{\mathbb{K}}}$ by adding suitable fibers over the infinite primes. In this way, one can enrich the group of divisors with formal real linear combinations of irreducible "closed vertical fibers at infinity" $\sum_\alpha \lambda_\alpha F_\alpha$. It is important to stress the fact that, in this approach, the fibers F_α are only treated as formal symbols, and no geometric model of such fibers is provided.

Different results in arithmetic geometry point to the fact that these fibers behave like the most degenerate type of closed fibers that occur at finite primes. This *totally degenerate* case is the situation where all the components of the closed fiber are $\mathbb{P}^1$'s and the geometry of the degeneration is entirely described in a purely combinatorial way by the dual graph.

The main idea of Arakelov geometry is that it is sufficient to work with an *infinitesimal neighborhood* of the fibers F_α, and that analysis on the Riemann surface X_α can be used to express this intersection multiplicity in terms of the Green function of the Laplacian,

$$[D_1, D_2]_{inf} = \sum_\alpha \epsilon_\alpha \Big(\sum_{\beta,\gamma} -g_\alpha(P^\alpha_{1,\beta}, P^\alpha_{2,\gamma})\Big), \tag{2.1}$$

where g_α is the Green function on X_α.

A beautiful result of Manin [14] shows that the Arakelov Green function can be computed in terms of geodesics in a hyperbolic handlebody "filling" the Riemann surface. This result admits an interpretation in terms of the holography principle in string theory, which postulates the existence of an explicit correspondence between gravity on a bulk space which is asymptotically anti-de-Sitter and conformal field theory on its boundary at infinity [15].

To begin to understand why noncommutative geometry enters the picture at all, one can analyse more closely the geometry of the dual graph described above. Since the uniformizing group Γ is a free group, there is a very simple coding of the bounded geodesics in $\mathfrak{X}_\Gamma$. These geodesics are parameterized by the set $\Lambda_\Gamma \times_\Gamma \Lambda_\Gamma$, by specifying the endpoints in $\mathbb{H}^3 \cup \mathbb{P}^1(\mathbb{C})$ modulo the action of Γ. This means that the geometric model for the "dual graph" is given by the mapping torus

$$\mathcal{S}_T := \mathcal{S} \times [0,1]/(x,0) \sim (Tx, 1) \tag{2.2}$$

of a discrete dynamical system $(\mathcal{S}, T)$, which is a subshift of finite type in an alphabet of $2g$ letters, describing the coding of geodesics by doubly infinite sequences of generators of Γ and their inverses without cancellations. The quotient by the action of Γ in $\Lambda_\Gamma \times_\Gamma \Lambda_\Gamma$ corresponds here to the action of the invertible shift T on the sequences.

At each place $v \in \mathrm{Sp}(O_\mathbb{K})$ one can consider the local Euler factor

$$L_v(H^1(X), s) = \det\left(1 - Fr_v^* N(v)^{-s} | H^1(\bar{X}_v, \mathbb{Q}_\ell)^{I_v}\right)^{-1}, \tag{2.3}$$

for $X(v)$ the reduction of the curve X at v and $\bar{X}_v = X(v) \otimes_{k(v)} \mathrm{Sp}(\bar{k}(v))$, with $\bar{k}(v)$ an algebraic closure, and Fr_v^* the geometric Frobenius acting on the inertia invariants of ℓ-adic cohomology for $(\ell, q) = 1$.

The contribution of the archimedean primes is expressed in terms of the Hodge structure
$$H_B^* = \oplus_{p,q} H^{p,q}.$$
on the Betti (or deRham) cohomology $H_B^* := H^*_{Betti}(X_\alpha, \mathbb{C})$ of X_α,

(2.4)
$$L_\kappa(H_B^*, s) := \begin{cases} \prod_{p,q} \Gamma_{\mathbb{C}}(s - \min(p,q))^{h^{p,q}} & \kappa = \mathbb{C} \\ \prod_{p<q} \Gamma_{\mathbb{C}}(s-p)^{h^{p,q}} \prod_p \Gamma_{\mathbb{R}}(s-p)^{h^{p+}} \Gamma_{\mathbb{R}}(s-p+1)^{h^{p-}} & \kappa = \mathbb{R} \end{cases}$$

with $h^{p,q} = \dim_{\mathbb{C}} H^{p,q}$ and $h^{p,\pm}$ the dimension of the $(-1)^p$ eigenspace of de-Rham conjugation on $H^{p,p}$, and with
$$\Gamma_{\mathbb{C}}(s) := (2\pi)^{-s}\Gamma(s), \quad \Gamma_{\mathbb{R}}(s) := 2^{-1/2}\pi^{-s/2}\Gamma(s/2).$$

Unlike the Euler factors at the finite primes, the factors at the archimedean primes do not appear at first to have a cohomological interpretation. However, Deninger developed a unified setting where both the archimedean and the non-archimedean Euler factors can be interpreted as infinite determinants. In [10] and [11] he obtained the Euler factor (2.3) as a regularized determinant over an infinite dimensional cohomological theory, while in [9] he gave an analogous formula for (2.4) as regularized determinants,

$$L_\kappa(H_B^q, s)^{-1} = \det_\infty \left(\frac{1}{2\pi}(s - \Phi_q) \mid \mathcal{H}^q \right). \tag{2.5}$$

Consani (*cf.* [5]) gave a cohomological interpretation for the data $(\mathcal{H}^q, \Phi_q)$, where the space $\mathcal{H}^q$ is identified with the kernel of a map N, which represents the logarithm of the local monodromy map at infinity, acting on the hypercohomology $\mathbb{H}^*(K^{\cdot,\cdot})$ of a double complex. The complex $K^{i,j}$ consists of a direct sum of twisted copies of spaces of differential forms on X_α, with suitable cutoffs on the range of the sums, designed in such a way that the monodromy map N, acting as a shift of these twisted copies, will have an interesting kernel and cokernel.

More precisely, one has $K^{i,j} = \oplus_k K^{i,j,k}$, with

(2.6)
$$K^{i,j,k} = \begin{cases} \bigoplus_{\substack{a+b=j+1 \\ |a-b| \le 2k-i}} (\Omega^{a,b} \oplus \Omega^{b,a})_{\mathbb{R}}(\frac{1+j-i}{2}) & \text{if } \begin{matrix} 1+j-i \equiv 0(2) \\ k \ge \max(0,i) \end{matrix} \\ 0 & \text{otherwise,} \end{cases}$$

with
$$(\Omega^{a,b} \oplus \Omega^{b,a})_{\mathbb{R}}(p) := (2\pi\sqrt{-1})^p (\Omega^{a,b} \oplus \Omega^{b,a})_{\mathbb{R}}.$$
the p-th Hodge-Tate twist, $p \in \mathbb{Z}$, of the space of real differential forms on X_α of type $(a,b)+(b,a)$, and differentials $d' : K^{i,j,k} \to K^{i+1,j+1,k+1}$

and $d'' : K^{i,j,k} \to K^{i+1,j+1,k}$,

$$d' = \partial + \overline{\partial} \qquad d'' = \sqrt{-1}\,(\overline{\partial} - \partial).$$

The complex is endowed with monodromy and Lefschetz maps $N : K^{i,j,k} \to K^{i+2,j,k+1}$ and $\ell : K^{i,j,k} \to K^{i,j+2,k}$,

$$N(f) = (2\pi\sqrt{-1})^{-1} f \qquad \ell(f) = (2\pi\sqrt{-1}) f \wedge \omega, \tag{2.7}$$

that commute with d' and d'' and satisfy $[\ell, N] = 0$.

The hypercohomology $\mathbb{H}^*(\mathrm{Cone}(N), d)$ is a pre-Hilbert space and an $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ module, where the inner product and the group action are inherited from the structure of a *polarized bigraded Lefschetz module*, in the sense of Deligne and Saito [19], of the complex $K^{\cdot,\cdot,\cdot}$. This structure is specified by the data of a representation

$$\sigma : \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \to \mathrm{Aut}(K^{\cdot,\cdot,\cdot}) \tag{2.8}$$

$$\sigma\left\{ \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}, \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix} \right\}(x) = u^i v^j x \qquad \text{for } x \in K^{i,j}$$

$$d\sigma\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, 0 \right\} = N \qquad d\sigma\left\{ 0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\} = \ell,$$

and a polarization $\psi : K^{-i,-j,k} \otimes K^{i,j,k+i} \to \mathbb{R}(1)$,

$$\psi(x,y) := \left(\frac{1}{2\pi\sqrt{-1}} \right) \epsilon(1-j)(-1)^k \int_{X_\alpha} x \wedge Cy,$$

with $\epsilon(m) := (-1)^{\frac{m(m+1)}{2}}$, for $m \in \mathbb{Z}$, and $C(x) := (\sqrt{-1})^{a-b} x$ the Weil operator, for x a differential form of type (a,b). The polarization defines an inner product

$$\langle \cdot,\cdot \rangle : K \otimes K \to \mathbb{R}(1), \qquad \langle x, y \rangle := \psi\left(x, \sigma(w,w) y\right), \tag{2.9}$$

with

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The action of the right $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{H}^*(\mathrm{Cone}(N))$ extends to an action by bounded operators on the Hilbert space completion in the inner product (2.9). The weight operator $\Phi = \oplus_q \Phi_q$ gives an unbounded self-adjoint operator with $\mathrm{Sp}(\Phi) = \mathbb{Z}$, with compact resolvents $(\Phi - \lambda)^{-1}$, $\lambda \notin \mathbb{R}$, and with the the property that the commutators $[a, \Phi]$, with $a = \sigma(1, g)$ and $g \in \mathrm{SL}(2,\mathbb{R})$, are bounded operators. Thus, the weight operator has the correct properties of a Dirac operator. If the archimedean prime is a real embedding, so that the Riemann surface X_α has a real structure, then the spectral data $(\mathbb{H}^*(\mathrm{Cone}(N)), \Phi)$ are also enriched with a real structure given by the involution $J = \bar{F}_\infty$ given by the real Frobenius.

There is an extra piece of structure on the spectral data

$$(\mathcal{H}, D, J) = (\mathbb{H}^*(\mathrm{Cone}(N)), \Phi, \bar{F}_\infty), \tag{2.10}$$

which comes from a duality isomorphism δ on the hypercohomology $\mathbb{H}^*(\mathrm{Cone}(N))$, which interchanges the graded pieces corresponding to the hypercohomologies of $\mathrm{Ker}(N)$ and $\mathrm{Coker}(N)$. This duality map defines an involution γ on $\mathcal{H}$ satisfying $\gamma^2 = 1$, $\gamma^* = \gamma$, $[\gamma, a] = 0$ for $a = \sigma(1, g)$ and $g \in \mathrm{SL}(2, \mathbb{R})$, and

$$(\Phi\gamma + \gamma\Phi)|_{\mathbb{H}^q(\mathrm{Cone}(N))} = q, \tag{2.11}$$

this last relation being a modified version, depending on the degree q of the cohomology, of the relation $D\gamma + \gamma D = 0$ of even spectral triples.

The extra geometric datum of a uniformization of X_α provides a choice of an algebra of operators $\mathcal{A}$ that completes the data (2.10) to a spectral triple, by restricting the $\mathrm{SL}(2, \mathbb{R})$ representation corresponding to the Lefschetz operator (the one that gives rise to bounded operators) to the list of a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$. In the case of Schottky uniformization, a Fuchsian Schottky group is obtained via a quasi-circle and Bers simultaneous uniformization (the quasi circle is just $\mathbb{P}^1(\mathbb{R})$ in tha case with a real structure).

Theorem 2.1. *For a smooth compact Riemann surface X of genus $g \geq 2$ The following data determine a 1-summable spectral triple:*

$$(\mathcal{A}, \mathcal{H}, D) = (\mathcal{A}_\Gamma, \mathbb{H}^*(\mathrm{Cone}(N)), \Phi)$$

where $\mathcal{A}_\Gamma$ is the algebra associated to a choice of uniformization by the group Γ. If X has a real structure, then the spectral triple has a real structure $J = \bar{F}_\infty$.

The local factor (2.4) also admit a natural interpretation in terms of the spectral geometry. Consider the zeta function

$$\zeta_{a, P_-\Phi}(s, z) := \sum_{\lambda \in \mathrm{Sp}(P_-\Phi)} \mathrm{Tr}(a\Pi(\lambda, P_-\Phi))(s - \lambda)^{-z}, \tag{2.12}$$

with $a = \sigma(1, -\mathrm{id}) \in \mathcal{A}_\Gamma$ and the operator $P_-\Phi = \Phi_{|\mathbb{H}^*(K^\cdot, d)^{N=0}}$ given by the compression of Φ by the projection on the -1 eigenspace of the involution γ. Notice that, regardless of the choice of the algebra $\mathcal{A}_\Gamma$ of the uniformization, the element $\sigma(1, -\mathrm{id})$ is always there, by the fact of lifting from $\mathrm{PSL}(2, \mathbb{R})$ to $\mathrm{SL}(2, \mathbb{R})$. However, according to the choice of $\mathcal{A}_\Gamma$, (2.12) is included in a larger family of zeta functions $\zeta_{a,D}(s, z)$, that also may yield interesting arithmetic information. We have the result:

Theorem 2.2. *The regularized determinant of the zeta function (2.12),*

$$\det_{\infty,a,P_-\Phi}(s) := \exp\left(-\frac{d}{dz}\zeta_{a,P_-\Phi/(2\pi)}(s/(2\pi),z)|_{z=0}\right),$$

gives the alternating product of the local factors (2.4)

$$\det_{\infty,a,P_-\Phi}(s)^{-1} = \frac{L_{\mathbb{C}}(H^1(X,\mathbb{C}),s)}{L_{\mathbb{C}}(H^0(X,\mathbb{C}),s)\cdot L_{\mathbb{C}}(H^2(X,\mathbb{C}),s)}.$$

In the case of curves, the interesting information comes especially from the first cohomology of the dual graph. In this case, we show that there is a subspace of $\mathbb{H}^*(\mathrm{Cone}(N))$ that embeds, via a map defined by the Schottky uniformization, into a larger vector space related to the first homology and cohomology of the dual graph of the closed fiber F_α, interpreted as in (2.2) as the solenoid $\mathcal{S}_T$.

There is a filtration on the cohomology $H^1(\mathcal{S}_T)$, with $\mathrm{Rank}V_n = 2g(2g-1)^{n-1}(2g-2)+1$. Associated to this filtration, one can consider the graded space $\mathcal{H}^1 := \oplus_{p\leq 0} gr^\Gamma_{2p}\mathcal{H}^1$, with

$$gr^\Gamma_{2p}H^1 := (V_{-p}/V_{-p-1})\otimes_{\mathbb{R}}\mathbb{R}(p).$$

There is a preferred choice of a graded subspace $\mathcal{V}\subset\mathcal{H}^1$, spanned by the classes of characteristic functions $\chi_{n,k}$ of the set of infinite words without cancellations in the generators of Γ and their inverses, which start with the word given by repeating n-times the same g_k. Since an infinite periodic sequence in one of the generators represents one of the fundamental closed geodesics in the handlebody, the graded pieces $gr^\Gamma_{2p}\mathcal{V}$ can be regarded as the "best approximation" within $gr^\Gamma_{2p}\mathcal{H}^1$ to cohomology classes supported on the core handles of the handlebody. The map

$$g_i \mapsto \omega_{g_i} = \sum_{h\in C(|g_i)} d\,\log\langle hz^+(g_i), hz^-(g_i), z, z_0\rangle,$$

which gives a basic of holomorphic differentials from the generators of Γ (*cf.* [14]), induces an isomorphism

$$U : gr^\Gamma_{2p}\mathcal{V} \to gr^w_{2p}\mathbb{H}^1(\mathrm{Cone}(N)). \tag{2.13}$$

Let $\mathcal{H}_1 = \oplus_{p\geq 1} gr^\Gamma_{2p}\mathcal{H}_1$ denote the graded space associated to the first homology group $H_1(\mathcal{S}_T)$ filtered by lengths of the closed geodesics.

Theorem 2.3. *The map* (2.13) *induces a compatible embedding of spectral data*

$$(\mathbb{H}^1(\mathrm{Ker}(N))\oplus\mathbb{H}^3(\mathrm{Coker}(N)),\Phi,\bar{F}_\infty,\gamma)\subset(\mathcal{H}^1\oplus\mathcal{H}_1,D,\bar{F}_\infty,\tilde{\gamma}),$$

with identifications

$$\mathbb{H}^1(\mathrm{Ker}(N)) = \oplus_{p\le 0} gr^w_{2p}\mathbb{H}^1(\mathrm{Cone}(N))$$
$$\mathbb{H}^3(\mathrm{Coker}(N)) = \oplus_{p\le 0} gr^w_{2(-p+2)}\mathbb{H}^2(\mathrm{Cone}(N)),$$

and in the right hand side D is the weight operator, $\bar{F}_\infty$ is the involution given by change of orientation, and $\tilde{\gamma}$ is the duality isomorphism induced by the pairing of homology and cohomology of $\mathcal{S}_T$.

The embedding of spectral data maps the terms corresponding to kernel and cokernel of the monodromy map N respectively to terms corresponding to cokernel and kernel of the map $1-T$, where T is the discrete dynamical system defining the solenoid $\mathcal{S}_T$. This suggests the presence of a duality between the monodromy N and the map $1-T$.

One advantage of the spectral data $(\mathcal{H}^1 \oplus \mathcal{H}_1, D, \bar{F}_\infty)$ is that they extend naturally to the case of a non-archimedean prime of $k(v)$-split reduction, for Mumford curves, where the solenoid $\mathcal{S}_T$ is replaced by a space of doubly infinite walks in the quotient Δ_K/Γ of the Bruhat–Tits tree by the action of a p-adic Schottky group (*cf.* [8]).

There is a noncommutative space associated to the larger spectral data $(\mathcal{H}^1 \oplus \mathcal{H}_1, D, \bar{F}_\infty)$, which encodes the geometry of the "reduction mod infinity", namely the analog of the reductions mod p^m realized by walks on the Bruhat–Tits tree that emanate from the dual graph Δ_Γ/Γ towards the ends Ω_Γ/Γ. In terms of the formula of [14] for the Green function, these correspond to the terms involving the geodesics that wander off at infinity, away from the convex core of the handlebody. The noncommutative space is described by Morita equivalent algebras

$$(\mathrm{C}(\Lambda_\Gamma) \otimes \mathrm{C}_0(\mathbb{H}^3)) \rtimes \Gamma \simeq \mathrm{C}_0(\mathfrak{X}_\Gamma, \mathcal{E}), \tag{2.14}$$

where $\mathcal{A} = \mathrm{C}_0(\mathfrak{X}_\Gamma, \mathcal{E})$ denotes sections of the bundle

$$\mathcal{E} = (\mathrm{C}(\Lambda_\Gamma) \times \mathbb{H}^3)/\Gamma \longrightarrow \mathfrak{X}_\Gamma,$$

with Γ acting diagonally on $\mathrm{C}(\Lambda_\Gamma) \times \mathbb{H}^3$.

Theorem 2.4. *The data $(\mathcal{A}, \mathcal{H}, D)$, with $\mathcal{A}$ as above, where $\mathcal{H}$ is the space $\mathcal{H}^1 \oplus \mathcal{H}_1$ (up to tensoring with the bimodule of the Morita equivalence) and D is the weight operator, define a spectral triple.*

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