Continuity of Edge and Corner Pseudo-Differenial Operators

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Abstract. We consider global Fourier and Mellin pseudo-differential operators with operator-valued symbols and extend the Calderón-Vaillancourt theorem to these classes. The composition of each such two operators remains in the class. Moreover, we describe the composition of Mellin pseudo-differential operators, which have symbols that, in addition, extend smoothly up to the origin.

Contents

Introduction

The present paper is concerned with pseudo-differential operators on manifolds with singular geometries in the sense of non-compactness or piecewise smooth Riemannian metrics. An approach to the analysis on manifolds with singularities was developed by SCHULZE [13], [14]. One idea in that calculus is to build up 'higher' singularities by iteration from 'lower' ones. For example, an edge can be written as a product of a Euclidean space and a cone, while a corner is a cone over a base itself having conical singularities. This is reflected in the structure of the corresponding pseudo-differential symbols, which are functions taking values in operator algebras on the 'lower' singular object. In other words, the pseudo-differential calculus is realized by iterating established calculi.

The specific symbolic structure can be interpreted in the framework of pseudo-differential symbols whose values are linear operators between certain Hilbert spaces; these spaces are equipped with strongly continuous groups of isomorphisms that enter in the symbol estimates. This is a non-trivial generalization of the scalar calculus to the operator-valued case. For the (local) calculus see SCHULZE [11], and HIRSCHMANN [4].

An important question in this context is an extension of the Calderón-Vaillancourt theorem in the operator-valued set-up. A first result in this direction was obtained in DORSCHFELDT, GRIEME, SCHULZE [2] under the assumption of the existence of order-reducing operators which reduced the proof to the situation of groups of unitary operators. It turns out that this approach is not convenient for many purposes, e.g., the operator algebra in SEILER [15], where the asymptotic data do not allow reductions to the case of unitary group actions.

The analysis of pseudo-differential operators on non-compact manifolds was studied in the scalar case by PARENTI [7], CORDES [1], SCHROHE [9], and others. The operator-valued analogue appears, in particular, in boundary-value problems on non-compact configurations. Also applications to non-linear problems in connection with travelling waves in infinite cylinders (as recently suggested by VIŠIK) require such tools for the Laplacian with Dirichlet or Neumann conditions. For cylinders with singular cross section, the corresponding local problems were treated by SCHMUTZLER [8], whereas the infinite cylinder is just a case of non-compactness in the present case.

The main purpose of this paper is to give a proof of the Calderón-Vaillancourt theorem in the operator-valued case, admitting arbitrary group actions. To this end we generalize techniques introduced in HWANG [5]. The norm estimates we obtain for pseudo-differential operators show, in particular, the continuity of the operator quantization, i.e., the mapping of the symbols to the associated operators. This becomes important, for example, in parameter-dependent variants of the calculus. Further we obtain natural conditions on the symbols which ensure the compactness of the corresponding pseudo-differential operators. We also perform an analogue for Mellin pseudo-differential operators, for which the Fourier transform on the real axis is replaced by the Mellin transform on the half-line; these operators arise, for instance, in the calculus of corner pseudo-differential operators, cf. SCHULZE [12], and DORSCHFELDT, SCHULZE [3].

In a final Section we apply the results achieved to the global edge algebra of smoothing Mellin and Green operators of SEILER [15] in spaces with asymptotics.

Notation

The real numbers are denoted by $\mathbb{R}$, the complex numbers by $\mathbb{C}$. Furthermore, $\mathbb{R}_+$ are the positive reals, $\overline{\mathbb{R}}_+$ the non-negative reals, and $\Gamma_\beta = \{z \in \mathbb{C};\ \operatorname{Re} z = \beta\}$ for $\beta \in \mathbb{R}$. $\mathbb{N}$ are the positive integers, $\mathbb{N}_0$ the non-negative ones.

In the sequel E and E_j, $j \in \mathbb{N}_0$, are always Hilbert spaces. $\mathcal{L}(E_0, E_1)$ is the space of all linear continuous operators $A : E_0 \to E_1$. The norm of A is denoted by $\|A\|_{E_0,E_1}$.

For a Fréchet space F, the smooth functions on an open set Ω with values in F are denoted by $C^\infty(\Omega, F)$, the compactly supported ones by $C_0^\infty(\Omega, F)$. Moreover, $C^\infty(\overline{\mathbb{R}}_+, F) = C^\infty(\mathbb{R}, F)|_{\overline{\mathbb{R}}_+} = \{u \in C^\infty(\mathbb{R}_+);\ \lim_{t\to 0} \partial^k u(t) \text{ exists } \forall\, k \in \mathbb{N}_0\}$. $\mathcal{A}(\mathbb{C}, F)$ are the entire functions with values in F. The Schwartz space of F-valued rapidly decreasing functions on $\mathbb{R}^q$ is denoted by $\mathcal{S}(\mathbb{R}^q, F)$. If F equals $\mathbb{C}$, it is omitted in the notation.

The Fourier transform of $u \in \mathcal{S}(\mathbb{R}^q, F)$ is

$$\hat{u}(\eta) = \mathcal{F}u(\eta) = \int e^{-iy\eta} u(y)\, dy.$$

The Fourier transform is extended to $\mathcal{S}'(\mathbb{R}^q, F) = \mathcal{L}(\mathcal{S}(\mathbb{R}^q), F)$ in the standard way.

Finally, set $\langle \eta \rangle = (1 + |\eta|^2)^{1/2}$ for $\eta \in \mathbb{R}^q$.

1 Global pseudo-differential operators and Sobolev spaces

Sections 1.1 and 1.2 include basic material concerning global pseudo-differential operators and weighted edge Sobolev spaces. For a detailed exposition of the pseudo-differential calculus we refer to [2]. A more general approach to edge Sobolev spaces can be found in [4].

1.1 Global operator-valued symbols

1.1 Definition. A set $\kappa = \{\kappa_\lambda;\ \lambda > 0\} \subset \mathcal{L}(E, E)$ of isomorphisms is called a (strongly continuous) group action on E if

i) $\kappa_\lambda \kappa_\varrho = \kappa_{\lambda\varrho} \quad \forall\, \lambda, \varrho > 0$ (in particular $\kappa_1 = 1_E$),

ii) For each $e \in E$ the function $\lambda \mapsto \kappa_\lambda e : \mathbb{R}_+ \to E$ is continuous.

Since E is a Hilbert space, also the adjoint group $\kappa^* = \{\kappa_\lambda^*;\ \lambda > 0\}$ is a group action on E. For a group action κ on E one can find non-negative constants c and M such that

$$\|\kappa_\lambda\|_{E,E} \le c \max\{\lambda, \lambda^{-1}\}^M \qquad \forall\, \lambda > 0. \tag{1.1}$$

For abbreviation we set $\kappa(\eta) = \kappa_{\langle\eta\rangle}$. For later reference we observe an easy consequence of Peetre's inequality, namely that there exists a constant c such that

$$\|\kappa^{-1}(\xi)\kappa(\eta)\|_{E,E} \le c\, \langle \xi - \eta \rangle^M \qquad \forall\, \xi, \eta \in \mathbb{R}^q, \tag{1.2}$$

where M is the constant from (1.1).

In following we assume that each E_j is equipped with a fixed group action κ_j.

1.2 Definition. For $\mu, m \in \mathbb{R}$ let $S^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1)$ denote the space of all functions $a \in C^\infty(\mathbb{R}^q \times \mathbb{R}^q, \mathcal{L}(E_0, E_1))$ satisfying

$$\sup_{y,\eta \in \mathbb{R}^q} \left\{ \|\kappa_1^{-1}(\eta)\{\partial_\eta^\alpha \partial_y^\beta a(y,\eta)\}\kappa_0(\eta)\|_{E_0,E_1} \langle\eta\rangle^{|\alpha|-\mu} \langle y\rangle^{|\beta|-m} \right\} < \infty$$

for all multiindices $\alpha, \beta \in \mathbb{N}_0^q$. These semi-norms induce a Fréchet topology on $S^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1)$.

Elementary calculations show that

$$\partial_\eta^\alpha \partial_y^\beta S^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1) \subset S^{\mu-|\alpha|, m-|\beta|}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1),$$

$$S^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_1, E_2) \cdot S^{\mu',m'}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1) \subset S^{\mu+\mu', m+m'}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_2).$$

In case both $E_1 \hookrightarrow E_0$ with $\kappa_{0,\lambda} = \kappa_{1,\lambda}$ on E_1 and $E_2 \hookrightarrow E_3$ with $\kappa_{3,\lambda} = \kappa_{2,\lambda}$ on E_2, the embeddings

$$S^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_2) \hookrightarrow S^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_1, E_2) \hookrightarrow S^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_1, E_3)$$

hold. If M_0, M_1 are the constants corresponding respectively to κ_0, κ_1 via (1.1), then

$$S^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1) \hookrightarrow S^{\mu+M_0+M_1, m}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1)_{(1)},$$

where the subscript (1) means that both E_0 and E_1 are equipped with the trivial group action $\kappa \equiv 1$.

To a given symbol $a \in S^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1)$ we associate a continuous operator

$$\mathrm{op}(a) : \mathcal{S}(\mathbb{R}^q, E_0) \to \mathcal{S}(\mathbb{R}^q, E_1) : u \mapsto [\mathrm{op}(a)u](y) = \int e^{iy\eta} a(y,\eta)\hat{u}(\eta)\, đ\eta.$$

Here, $đ\eta = (2\pi)^{-q}\, d\eta$.

1.3 Theorem. *If $a \in S^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_1, E_2)$ and $b \in S^{\mu',m'}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1)$ then $\mathrm{op}(a)\mathrm{op}(b) = \mathrm{op}(a\#b)$, where the so-called Leibniz product of $a\#b$ is defined by*

$$(a\#b)(y,\eta) = \iint e^{-ix\xi} a(y, \eta+\xi) b(y+x, \eta)\, dx đ\xi$$

(understood as oscillatory integrals). For each $N \in \mathbb{N}$ then the expansion

$$(a\#b) = \sum_{\alpha < N} \frac{1}{\alpha!} (\partial_\eta^\alpha a)(D_y^\alpha b) \; + \; r_N$$

holds, with a remainder r_N in the space $S^{\mu+\mu'-N, m+m'-N}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_2)$, which is given explicitely by the formula

$$r_N(y,\eta) = N \sum_{|\sigma|=N} \int_0^1 \frac{(1-\theta)^{N-1}}{\sigma!} \iint e^{-ix\xi} \partial_\eta^\sigma a(y, \eta+\theta\xi) D_y^\sigma b(y+x, \eta)\, dx đ\xi d\theta.$$

The Leibniz product induces a continuous mapping

$$S^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_1, E_2) \times S^{\mu',m'}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1) \to S^{\mu+\mu', m+m'}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_2).$$

1.2 Weighted abstract edge Sobolev spaces

Let us fix group actions κ and κ_j on the spaces E and E_j, respectively.

1.4 Definition. For $s \in \mathbb{R}$ let $\mathcal{W}^s(\mathbb{R}^q, E_0)$ denote the space of all distributions $u \in \mathcal{S}'(\mathbb{R}^q, E)$ such that $\hat{u}$ is a measurable function and

$$\|u\|_{\mathcal{W}^s(\mathbb{R}^q,E)} = \left(\int \langle\eta\rangle^{2s} \|\kappa^{-1}(\eta)\hat{u}(\eta)\|_E^2 \, d\eta \right)^{1/2} < \infty.$$

For $\delta \in \mathbb{R}$ we have weighted variants of those spaces, namely

$$\mathcal{W}^{s,\delta}(\mathbb{R}^q, E) = \{u \in \mathcal{S}'(\mathbb{R}^q, E);\ \langle\cdot\rangle^{\delta} u \in \mathcal{W}^s(\mathbb{R}^q, E)\},$$

with obvious definition of the corresponding norm. In case of a trivial group action, i.e., $\kappa \equiv 1$, we use notations $H^s(\mathbb{R}^q, E)$ and $H^{s,\delta}(\mathbb{R}^q, E)$.

The spaces $\mathcal{W}^{s,\delta}(\mathbb{R}^q, E)$ are Hilbert spaces, having $C_0^\infty(\mathbb{R}^q, E)$ as a dense subset. If M corresponds to κ via (1.1), obviously

$$\mathcal{W}^{s,\delta}(\mathbb{R}^q, E) \hookrightarrow H^{s-M,\delta}(\mathbb{R}^q, E).$$

If $E_0 \hookrightarrow E_1$ and $\kappa_{1,\lambda} = \kappa_{0,\lambda}$ on E_0 for all $\lambda > 0$, we immediately obtain that

$$\mathcal{W}^{s,\delta}(\mathbb{R}^q, E_1) \hookrightarrow \mathcal{W}^{s,\delta}(\mathbb{R}^q, E_0).$$

If $E_0 \oplus E_1$ is the direct sum equipped with the group action $\{\kappa_{0,\lambda} \oplus \kappa_{1,\lambda}\}$, then

$$\mathcal{W}^{s,\delta}(\mathbb{R}^q, E_0 \oplus E_1) = \mathcal{W}^{s,\delta}(\mathbb{R}^q, E_0) \oplus \mathcal{W}^{s,\delta}(\mathbb{R}^q, E_1).$$

As a consequence of Corollary 1.11 we obtain that

$$\mathcal{W}^{s,\delta}(\mathbb{R}^q, E) \hookrightarrow \mathcal{W}^{s',\delta'}(\mathbb{R}^q, E),$$

whenever $s \geq s'$ and $\delta \geq \delta'$. A motivation for introducing spaces of this kind is the following example.

1.5 Example. For each $\lambda > 0$ define mappings $\kappa_\lambda : \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ by

$$\langle \kappa_\lambda u, \phi \rangle = \langle u, \lambda^{-n/2}\phi(\lambda^{-1}x)\rangle, \qquad \phi(x) \in C_0^\infty(\mathbb{R}^n).$$

(For $u \in L^1_{loc}(\mathbb{R}^n)$ we then have $(\kappa_\lambda u)(x) = \lambda^{n/2}u(\lambda x)$). These mappings induce continuous group actions on each of the spaces $H^s(\mathbb{R}^n)$, the standard Sobolev spaces on $\mathbb{R}^n$. Now it is known (see, e.g. [13], p. 268) that

$$H^s(\mathbb{R}^q \times \mathbb{R}^n) = \mathcal{W}^s(\mathbb{R}^q, H^s(\mathbb{R}^n)).$$

1.3 Continuity of global pseudo-differential operators

1.6 Remark. (Plancherel's formula) Let E be a Hilbert space. Then $\mathcal{F} : L^2(\mathbb{R}^q, E) \to L^2(\mathbb{R}^q, E)$ is an isomorphism, and for each $u \in L^2(\mathbb{R}^q, E)$ we have

$$\|\mathcal{F}u\|_{L^2(\mathbb{R}^q,E)} = (2\pi)^{q/2}\|u\|_{L^2(\mathbb{R}^q,E)}.$$

This is true if and only if E is a Hilbert space.

1.7 Definition. For a multi-index $\alpha \in \mathbb{N}_0^q$ and $y \in \mathbb{R}^q$ we write

$$(i+y)^\alpha = (i+y_1)^{\alpha_1} \cdot \ldots \cdot (i+y_q)^{\alpha_q}, \qquad (i+y)^{-\alpha} = ((i+y)^\alpha)^{-1}.$$

Further, we define the following differential operator

$$(i+D_y)^\alpha = (i+D_{y_1})^{\alpha_1} \cdot \ldots \cdot (i+D_{y_q})^{\alpha_q}.$$

Here, $D_{y_j} = -i\partial_{y_j}$. Then we obtain the relation

$$(i+D_y)^\alpha e^{ixy} = (i+x)^\alpha e^{ixy}. \tag{1.3}$$

1.8 Lemma. *Let E_0, E_1 be Hilbert spaces and κ a group action on E_1. Further let $a \in C^\infty(\mathbb{R}^q \times \mathbb{R}^q, \mathcal{L}(E_0, E_1))$ be such that for some $L \in \mathbb{N}$ the estimate*

$$\pi_L(a) := \sup\{\langle\eta\rangle^{-L}\,\|\partial_\eta^\alpha\partial_y^\beta a(y,\eta)\|_{E_0,E_1};\ y,\eta \in \mathbb{R}^q,\ \alpha \le \alpha_1,\ \beta \le \beta_M\} < \infty$$

holds, where $\alpha_1 = (1,\ldots,1)$, $\beta_M = (M,\ldots,M)$, and M corresponds to κ via (1.1) (here we assume that $M \in \mathbb{N}_0$). Futhermore, let $\phi \in C_0^\infty(\mathbb{R}^{2q})$ with $\phi \equiv 1$ near 0 and set

$$a_\varepsilon(y,\eta) = \phi(\varepsilon y, \varepsilon\eta)a(y,\eta), \qquad 0 < \varepsilon \le 1.$$

Finally, let $u \in \mathcal{S}(\mathbb{R}^q, E_0)$. Then the following statements hold:

- *a)* $\mathrm{op}(a)u \in \mathcal{W}^0(\mathbb{R}^q, E_1)$,
- *b)* $\langle \mathrm{op}(a_\varepsilon)u, v\rangle \to \langle \mathrm{op}(a)u, v\rangle$ *for $\varepsilon \to 0$ and each $v \in \mathcal{S}(\mathbb{R}^q, E_1)$. Here, $\langle\cdot,\cdot\rangle$ is the scalar-product in $\mathcal{W}^0(\mathbb{R}^q, E_1)$.*

PROOF: a) Write $A = \mathrm{op}(a)$. With notation as in Definition 1.7 integration by parts gives

$$Au(y) = (i+y)^{-\alpha_1}\int e^{iy\eta}(i-D_\eta)^{\alpha_1}[a(y,\eta)\hat{u}(\eta)]\,đ\eta. \tag{1}$$

The first factor on the right-hand side is square integrable. Since $\hat{u} \in \mathcal{S}(\mathbb{R}^q, E_0)$ and because of the assumptions on a, the integral, together with its derivatives with respect to y up to order β_M, is bounded. So we obtain that $(i-D_y)^{\beta_M}Au \in L^2(\mathbb{R}^q, E_1)$. Hence we can estimate

$$\begin{aligned}
\|Au\|^2_{\mathcal{W}^0(\mathbb{R}^q,E_1)} &= \int \|\kappa^{-1}(\eta)\widehat{Au}(\eta)\|^2_{E_1}\,d\eta \le c\int \|(i+\eta)^{\beta_M}\widehat{Au}(\eta)\|^2_{E_1}\,d\eta \\
&= c\int \|\mathcal{F}[(i-D_y)^{\beta_M}Au](\eta)\|^2_{E_1}\,d\eta \\
&= c\int \|(i+D_y)^{\beta_M}Au(y)\|^2_{E_1}\,dy < \infty.
\end{aligned}$$

b) Set $b_\varepsilon = a_\varepsilon - a$, $B_\varepsilon = \mathrm{op}(b_\varepsilon)$. We show that $\langle B_\varepsilon u, v\rangle_{\mathcal{W}^0(\mathbb{R}^q,E_1)} \to 0$ for $\varepsilon \to 0$. From the relation

$$\langle B_\varepsilon u, v\rangle_{\mathcal{W}^0(\mathbb{R}^q,E_1)} = \int \langle \widehat{B_\varepsilon u}(\eta), \kappa^{-1}(\eta)^*\kappa^{-1}(\eta)\hat{v}(\eta)\rangle_{E_1}\, d\eta$$

and Plancherel's formula, it is sufficient to show that $B_\varepsilon u \to 0$ in $L^2(\mathbb{R}^q, E_1)$ for $\varepsilon \to 0$. An easy computation, using Leibniz' rule, shows that $\pi_L(b_\varepsilon)$ is uniformly bounded in $0 < \varepsilon \leq 1$. Hence, inserting b_ε instead of a in formula (1), we see that

$$\|B_\varepsilon u(y)\|^2_{E_1} \leq c|(i+y)^{-2\alpha_1}| \in L^1(\mathbb{R}^q),$$

with a constant c independent of ε. In view of Lebesgue's dominated convergence theorem, it remains to show that

$$B_\varepsilon u(y) = \int e^{iy\eta} b_\varepsilon(y,\eta)\hat{u}(\eta)\, đ\eta$$

tends to 0 with ε for all $y \in \mathbb{R}^q$. But this is again an easy consequence of Lebesgue's dominated convergence theorem. ∎

1.9 Theorem. *Let E_0, E_1 be Hilbert spaces with respective group actions κ_0 and κ_1. Further assume that $a \in C^\infty(\mathbb{R}^q \times \mathbb{R}^q, \mathcal{L}(E_0,E_1))$ satisfies*

$$\pi(a) := \sup\{\|\kappa_1^{-1}(\eta)\{\partial_\eta^\alpha \partial_y^\beta a(y,\eta)\}\kappa_0(\eta)\|_{E_0,E_1};\, y,\eta \in \mathbb{R}^q,\ \alpha \leq \alpha_1,\ \beta \leq \beta_M\} < \infty,$$

where $\alpha_1 = (1,\ldots,1)$, $\beta_M = (M+1,\ldots,M+1)$, and $M(\in \mathbb{N}_0)$ corresponds to κ_1 via (1.1). Then a induces a continuous operator

$$A = \mathrm{op}(a) : \mathcal{W}^0(\mathbb{R}^q, E_0) \to \mathcal{W}^0(\mathbb{R}^q, E_1),$$

whose norm can be estimated by

$$\|A\|_{\mathcal{W}^0(\mathbb{R}^q,E_0),\mathcal{W}^0(\mathbb{R}^q,E_1)} \leq c\,\pi(a)$$

with a constant c independent of a.

Proof: First, assume that a is compactly supported in (y,η). Further, let $u \in C_0^\infty(\mathbb{R}^q, E_0)$ and $v \in \mathcal{S}(\mathbb{R}^q, E_1)$. Then

$$\begin{aligned}\langle Au, v\rangle_{\mathcal{W}^0(\mathbb{R}^q,E_1)} &= \int \left\langle \kappa_1^{-1}(\eta)\, \widehat{Au}(\eta),\, \kappa_1^{-1}(\eta)\hat{v}(\eta)\right\rangle_{E_1} d\eta \\ &= \iiiint e^{i(\xi-\eta)y - i\xi x} \left\langle \kappa_1^{-1}(\eta)a(y,\xi)u(x), \kappa_1^{-1}(\eta)\hat{v}(\eta)\right\rangle_{E_1} dy dx đ\xi d\eta.\end{aligned}$$

Now relation (1.1) yields that the latter expression equals, after a twofold integration by parts,

$$\iiiint e^{i(\xi-\eta)y - i\xi x}(i+y-x)^{-\alpha_1}(i-D_\xi)^{\alpha_1}$$
$$\left\{(i+\xi-\eta)^{-\beta_M}\left\langle \kappa_1^{-1}(\eta)\ (i-D_y)^{\beta_M} a(y,\xi)u(x),\, \kappa_1^{-1}(\eta)\hat{v}(\eta)\right\rangle_{E_1}\right\} dy dx đ\xi d\eta,$$

and from this we get that $\langle Au, v\rangle_{\mathcal{W}^0(\mathbb{R}^q,E_1)}$ equals

$$\sum_{\gamma\leq\alpha_1} \iint e^{iy\xi}\left\langle [\kappa_1^{-1}(\xi)D_\xi^\gamma (i-D_y)^{\beta_M} a(y,\xi)\kappa_0(\xi)] f(y,\xi), g_\gamma(y,\xi)\right\rangle_{E_1} dy đ\xi,$$

with functions

$$f(y,\xi) = \kappa_0^{-1}(\xi) \int e^{-i\xi x}(i+y-x)^{-\alpha_1} u(x)\,dx,$$
$$g_\gamma(y,\xi) = (-1)^{\alpha_1-\gamma} \int e^{iy\eta}(\kappa_1^{-1}(\eta)\kappa_1(\xi))^*(i-D_\xi)^{\alpha_1-\gamma}(i+\xi-\eta)^{\beta_M}\kappa^{-1}(\eta)\hat{v}(\eta)\,d\eta.$$

Hence we obtain the estimate

$$|\langle Au, v\rangle_{\mathcal{W}^0(\mathbf{R}^q,E_1)}| \le c\,\pi(a) \sum_{\gamma\le\alpha_1} \|f\|_{L^2(\mathbf{R}^{2q},E_0)}\|g_\gamma\|_{L^2(\mathbf{R}^{2q},E_1)}.$$

Using Plancherel's formula, we get

$$\|g_\gamma\|_{L^2}^2 = (2\pi)^q \iint \|(\kappa_1^{-1}(\eta)\kappa_1(\xi))^*(i-D_\xi)^{\alpha_1-\gamma}(i+\xi-\eta)^{-\beta_M}\kappa^{-1}(\eta)\hat{v}(\eta)\|_{E_1}^2\,d\eta d\xi.$$

In view of (1.2) we can estimate

$$\begin{aligned}\|(\kappa_1^{-1}(\eta)\kappa_1(\xi))^*(i-D_\xi)^{\alpha_1-\gamma}(i+\xi-\eta)^{-\beta_M}\|_{E_1,E_1} &\le c\,\langle\xi-\eta\rangle^M\,|(i+\xi-\eta)^{-\beta_M}| \\ &\le c\,\langle\xi_1-\eta_1\rangle^M\cdot\ldots\cdot\langle\xi_q-\eta_q\rangle^M\,|(i+\xi-\eta)^{-\beta_M}| = c|(i+\xi-\eta)^{-\alpha_1}|,\end{aligned}$$

and this implies

$$\|g_\gamma\|_{L^2}^2 \le c\Big\{\int |(i+\xi)^{-2\alpha_1}|\,d\xi\Big\}\,\|v\|_{\mathcal{W}^0(\mathbf{R}^q,E_1)}^2.$$

For abbreviation now set $h(z) = (i-z)^{-\alpha_1}$, and $h_y(z) = h(z-y)$. Then note that

$$\hat{h}_y(\xi) = e^{-i\xi y}\hat{h}(\xi), \qquad \langle\xi\rangle^{2L}\,\hat{h}(\xi) = \mathcal{F}[(1-\Delta)^L h](\xi). \tag{1}$$

Because of $\mathcal{F}(\phi\psi) = (2\pi)^{-q}\hat{\phi}*\hat{\psi}$ and the identities from (1), we obtain

$$\begin{aligned} f(y,\xi) &= \kappa_0^{-1}(\xi)(2\pi)^{-q}\int \hat{h}_y(\xi-x)\hat{u}(x)\,dx \\ &= (2\pi)^{-q}e^{-i\xi y}\int e^{ixy}\frac{\kappa_0^{-1}(\xi)\kappa_0(x)}{\langle\xi-x\rangle^{2M}}\mathcal{F}[(1-\Delta)^M h](\xi-x)\kappa_0^{-1}(x)\hat{u}(x)\,dx.\end{aligned}$$

Now Plancherel's formula and (1.2) yield

$$\begin{aligned}\|f\|_{L^2}^2 &= \iint \Big\|(2\pi)^{-q}\int e^{ixy}\frac{\kappa_0^{-1}(\xi)\kappa_0(x)}{\langle\xi-x\rangle^{2M}}\mathcal{F}[(1-\Delta)^M h](\xi-x)\kappa_0^{-1}(x)\hat{u}(x)\,dx\Big\|_{E_0}^2 dyd\xi \\ &= (2\pi)^{-q}\iint \Big\|\frac{\kappa_0^{-1}(\xi)\kappa_0(x)}{\langle\xi-x\rangle^{2M}}\mathcal{F}[(1-\Delta)^M h](\xi-x)\kappa_0^{-1}(x)\hat{u}(x)\Big\|_{E_0}^2 dxd\xi \\ &\le c\iint \|\mathcal{F}[(1-\Delta)^M h](\xi-x)\kappa_0^{-1}(x)\hat{u}(x)\|_{E_0}^2\,d\xi dx \\ &= c\int |\mathcal{F}[(1-\Delta)^M h](\xi)|^2 d\xi\;\int \|\kappa_0^{-1}(x)\hat{u}(x)\|_{E_0}^2 dx \\ &= c\Big\{\int |(1-\Delta)^M h(\xi)|^2 d\xi\Big\}\|u\|_{\mathcal{W}^0(\mathbf{R}^q,E_0)}^2\end{aligned}$$

Altogether, we now have verified that

$$|\langle Au, v\rangle_{\mathcal{W}^0(\mathbb{R}^q,E_1)}| \le c\,\pi(a)\|u\|_{\mathcal{W}^0(\mathbb{R}^q,E_0)}\|v\|_{\mathcal{W}^0(\mathbb{R}^q,E_1)}, \tag{2}$$

with a certain constant c independent of a, u, and v.

Finally, consider the general case. Choose $\phi \in C_0^\infty(\mathbb{R}^{2q})$ with $\phi \equiv 1$ near 0. For $0 < \varepsilon \le 1$ define $a_\varepsilon(y,\eta) = \phi(\varepsilon y, \varepsilon\eta)a(y,\eta)$. Using Leibniz' rule it is easy to verify that $\pi(a_\varepsilon) \le c\pi(a)$ with a constant c independent of ε. Then, according to Lemma 1.8.b) and estimate (2),

$$|\langle Au, v\rangle_{\mathcal{W}^0(\mathbb{R}^q,E_1)}| \xleftarrow{\varepsilon\to 0} |\langle \mathrm{op}(a_\varepsilon)u, v\rangle_{\mathcal{W}^0(\mathbb{R}^q,E_1)}| \le c\,\pi(a)\|u\|_{\mathcal{W}^0(\mathbb{R}^q,E_0)}\|v\|_{\mathcal{W}^0(\mathbb{R}^q,E_1)}.$$

In view of the density of $\mathcal{S}(\mathbb{R}^q, E_1)$ in $\mathcal{W}^0(\mathbb{R}^q, E_1)$ we get

$$\|Au\|_{\mathcal{W}^0(\mathbb{R}^q,E_1)} \le c\,\pi(a)\|u\|_{\mathcal{W}^0(\mathbb{R}^q,E_0)}.$$

This implies the assertion. ∎

1.10 Remark. Let E_0, E_1 be Hilbert spaces with arbitrary group actions.

a) Each symbol $a \in S^\mu(\mathbb{R}^q_\eta; E_0, E_1)$, i.e., a is a symbol independent of the variable y, induces for each $s \in \mathbb{R}$ continuous operators

$$\mathrm{op}(a) : \mathcal{W}^s(\mathbb{R}^q, E_0) \to \mathcal{W}^{s-\mu}(\mathbb{R}^q, E_1).$$

b) For $\mu, \delta \in \mathbb{R}$ set

$$\Lambda^{\mu,\delta} = \mathrm{op}(\langle\eta\rangle^\mu \# \langle y\rangle^\delta), \qquad P^{\mu,\delta} = \mathrm{op}(\langle y\rangle^\delta \langle\eta\rangle^\mu).$$

These operators obviously induce for each $s \in \mathbb{R}$ isometric isomorphisms

$$\Lambda^{\mu,\delta} : \mathcal{W}^{s,\delta}(\mathbb{R}^q, E_j) \to \mathcal{W}^{s-\mu}(\mathbb{R}^q, E_j),\ P^{\mu,\delta} : \mathcal{W}^s(\mathbb{R}^q, E_j) \to \mathcal{W}^{s-\mu,-\delta}(\mathbb{R}^q, E_j)$$

with $\Lambda^{\mu,\delta}P^{-\mu,-\delta} = P^{-\mu,-\delta}\Lambda^{\mu,\delta} = 1$.

1.11 Corollary. *Let E_0, E_1 be Hilbert spaces with arbitrary group actions. Further let $a \in S^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1)$. Then a induces for each $s, \delta \in \mathbb{R}$ continuous operators*

$$\mathrm{op}(a) : \mathcal{W}^{s,\delta}(\mathbb{R}^q, E_0) \to \mathcal{W}^{s-\mu,\delta-m}(\mathbb{R}^q, E_1),$$

and the mapping

$$a \mapsto \mathrm{op}(a) : S^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1) \to \mathcal{L}(\mathcal{W}^{s,\delta}(\mathbb{R}^q, E_0), \mathcal{W}^{s-\mu,\delta-m}(\mathbb{R}^q, E_1))$$

is continuous.

PROOF: Let $s, \delta \in \mathbb{R}$ be fixed. By Theorem 1.3 there is a unique element $\tilde{a} \in S^{0,0}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1)$ such that

$$\mathrm{op}(a) = P^{\mu-s,m-\delta}\mathrm{op}(\tilde{a})\Lambda^{s,\delta},$$

and the mapping $a \mapsto \tilde{a} : S^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1) \to S^{0,0}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1)$ is continuous. Now from Theorem 1.9 and Remark 1.10.b) we obtain

$$\|\mathrm{op}(a)\|_{\mathcal{W}^{s,\delta}(\mathbb{R}^q,E_0),\mathcal{W}^{s-\mu,\delta-m}(\mathbb{R}^q,E_1)} \le c\,\pi(\tilde{a}).$$

This clearly implies the assertion in view of the continuity of the map $a \mapsto \tilde{a}$. ∎

1.12 Proposition. *Let E_0, E_1 be Hilbert spaces with arbitrary group actions. Further let $a \in S^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1)$ with $\mu, m < 0$ such that $a(y,\eta) : E_0 \to E_1$ is a compact operator for all $y, \eta \in \mathbb{R}^q$. Then*

$$\operatorname{op}(a) : \mathcal{W}^{s,\delta}(\mathbb{R}^q, E_0) \to \mathcal{W}^{s,\delta}(\mathbb{R}^q, E_1),$$

is compact for each $s, \delta \in \mathbb{R}$.

PROOF: Using the order reductions $\Lambda^{\mu,\delta}$, $P^{\mu,\delta}$ from Remark 1.10.b), we can assume that $s = \delta = 0$. Now assume that a is compactly supported in (y,η). Then from [14], Theorem 1.3.54, we know that $\operatorname{op}(a) : \mathcal{W}^0(\mathbb{R}^q, E_0) \to \mathcal{W}^0(\mathbb{R}^q, E_1)$ is compact. Finally, for general a, set

$$a_\varepsilon(y,\eta) = \phi(\varepsilon y, \varepsilon\eta) a(y,\eta), \quad \varepsilon > 0,$$

where $\phi \in C_0^\infty(\mathbb{R}^{2q})$ with $\phi \equiv 1$ near 0. Since a has negative order, $a_\varepsilon \to a$ in $S^{0,0}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1)$ for $\varepsilon \to 0$. By Corollary 1.11 then follows that

$$\operatorname{op}(a_\varepsilon) \to \operatorname{op}(a) \quad \text{in } \mathcal{L}(\mathcal{W}^0(\mathbb{R}^q, E_0), \mathcal{W}^0(\mathbb{R}^q, E_1)) \text{ for } \varepsilon \to 0.$$

Hence $\operatorname{op}(a)$ is compact as a limit of compact operators. ∎

1.13 Corollary. *Let $E_0 \hookrightarrow E_1$ be Hilbert spaces with arbitrary group actions such that the embedding is compact, and the group action on E_1 induces (by restriction) the group action on E_0. Then for $s > s'$, $\delta > \delta'$ we have compactness of the embedding*

$$\mathcal{W}^{s,\delta}(\mathbb{R}^q, E_0) \hookrightarrow \mathcal{W}^{s',\delta'}(\mathbb{R}^q, E_1).$$

1.14 Remark. Let$S^\mu(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1)$ denote the space of all $a \in C^\infty(\mathbb{R}^q \times \mathbb{R}^q, \mathcal{L}(E_0, E_1))$ satisfying

$$\sup_{y,\eta \in \mathbb{R}^q} \left\{ \|\kappa_1^{-1}(\eta)\{\partial_\eta^\alpha \partial_y^\beta a(y,\eta)\}\kappa_0(\eta)\|_{E_0,E_1} \langle\eta\rangle^{|\alpha|-\mu} \right\} < \infty$$

for all $\alpha, \beta \in \mathbb{N}_0^q$. The associated pseudo-differential operators act from $\mathcal{W}^s(\mathbb{R}^q, E_0)$ to $\mathcal{W}^{s-\mu}(\mathbb{R}^q, E_1)$. Then the obvious analogues of Theorem 1.3 and Corollary 1.11 are valid.

2 Calculi for Mellin pseudo-differential operators

Again, let E and E_j be Hilbert spaces with respective group actions κ and κ_j.

2.1 The Mellin transform

For $u(t) \in t^{\gamma+1/2}L^1(\mathbb{R}_+, E)$ we define the weighted Mellin transform

$$(\mathcal{M}_\gamma u)(z) = \int_0^\infty t^z u(t) \, \frac{dt}{t}, \qquad z \in \Gamma_{1/2-\gamma}.$$

A straightforward calculation shows that

$$(\mathcal{M}_\gamma u)(1/2 - \gamma + i\tau) = (\mathcal{F} S_\gamma u)(\tau) \tag{2.4}$$

if we define

$$(S_\gamma u)(r) = e^{(\gamma-1/2)r} u(e^{-r}), \quad r \in \mathbb{R}.$$

From this relation we clearly obtain that

$$\mathcal{M}_\gamma : \mathcal{T}_\gamma(\mathbb{R}_+, E) \to \mathcal{S}(\Gamma_{1/2-\gamma}, E)$$

is bijective, if we set $\mathcal{T}_\gamma(\mathbb{R}_+, E) = S_\gamma^{-1}(\mathcal{S}(\mathbb{R}, E))$. The space $\mathcal{T}_{1/2}(\mathbb{R}_+, E)$ equals

$$\Big\{u \in C^\infty(\mathbb{R}_+, E);\, p_N(u) = \sup_{k,l \le N,\, t>0} \|(\log t)^k (t\partial_t)^l u(t)\|_E < \infty \quad \forall N \in \mathbb{N}_0\Big\}.$$

The system of norms $p_N(\cdot)$ gives a Fréchet topology on $\mathcal{T}_{1/2}(\mathbb{R}_+, E)$, which induces the topology on $\mathcal{T}_\gamma(\mathbb{R}_+, E) = t^{\gamma-1/2}\mathcal{T}_{1/2}(\mathbb{R}_+, E)$. For abbreviation we set $\mathcal{T}_\gamma(\mathbb{R}_+) = \mathcal{T}_\gamma(\mathbb{R}_+, \mathbb{C})$.
Let $\mathcal{T}_\gamma'(\mathbb{R}_+, E) = \mathcal{L}(\mathcal{T}_{-\gamma}(\mathbb{R}_+), E)$ be the space of all continuous linear operators $\mathcal{T}_{-\gamma}(\mathbb{R}_+) \to E$. We regard $\mathcal{T}_\gamma(\mathbb{R}_+, E)$ as a subset of $\mathcal{T}_\gamma'(\mathbb{R}_+, E)$ via

$$\langle f, u\rangle = \int_0^\infty f(t)u(t)\, dt, \qquad u \in \mathcal{T}_{-\gamma}(\mathbb{R}_+).$$

For $F \in \mathcal{T}_\gamma'(\mathbb{R}_+, E)$ define

$$\langle \mathcal{M}_\gamma F, v\rangle = \langle F, t^{-1}(\mathcal{M}_\gamma^{-1} v)(t^{-1})\rangle, \qquad \langle S_\gamma F, \tilde{v}\rangle = \langle F, S_{-\gamma}^{-1}\tilde{v}\rangle$$

for $v \in \mathcal{S}(\Gamma_{1/2-\gamma})$ and $\tilde{v} \in \mathcal{S}(\mathbb{R})$. This yields mappings

$$\mathcal{M}_\gamma : \mathcal{T}_\gamma'(\mathbb{R}_+, E) \to \mathcal{S}'(\Gamma_{1/2-\gamma}, E), \qquad S_\gamma : \mathcal{T}_\gamma'(\mathbb{R}_+, E) \to \mathcal{S}'(\mathbb{R}, E)$$

that extend $\mathcal{M}_\gamma$ and S_γ from $\mathcal{T}_\gamma(\mathbb{R}_+, E)$ to $\mathcal{T}_\gamma'(\mathbb{R}_+, E)$. Formula (2.4) is valid in the distributional sense.
The multiplication of a distribution $F \in \mathcal{T}_\gamma'(\mathbb{R}_+, E)$ with a function g of tempered growth, i.e., $g \in C^\infty(\mathbb{R}_+)$, and all derivatives $|(t\partial_t)^k g|$ are majorized by a power of $|\log t|$, is defined by

$$\langle gF, u\rangle = \langle F, gu\rangle, \qquad u \in \mathcal{T}_{-\gamma}(\mathbb{R}_+).$$

2.2 Mellin symbols and Mellin edge Sobolev spaces

2.1 Definition. For $s, \delta, \gamma \in \mathbb{R}$ let $\mathcal{V}_\gamma^s(\mathbb{R}_+, E)$ denote the space of all $u \in \mathcal{T}_\gamma'(\mathbb{R}_+, E)$ such that $\mathcal{M}_\gamma u$ is a measurable function and

$$\|u\|_{\mathcal{V}_\gamma^s(\mathbb{R}_+,E)} = \left(\int \langle\tau\rangle^{2s} \|\kappa^{-1}(\tau)(\mathcal{M}_\gamma u)(1/2-\gamma+i\tau)\|_E^2\, d\tau\right)^{1/2} < \infty.$$

Further we set

$$\mathcal{V}_\gamma^{s,\delta}(\mathbb{R}_+, E) = \{u \in \mathcal{T}_\gamma'(\mathbb{R}_+, E);\, \langle\log t\rangle^\delta u \in \mathcal{V}_\gamma^s(\mathbb{R}_+, E)\},$$

with obvious definition of the corresponding norm.

From (2.4) we see that S_γ induces an isomorphism

$$S_\gamma : \mathcal{V}_\gamma^{s,\delta}(\mathbb{R}_+, E) \to \mathcal{W}^{s,\delta}(\mathbb{R}, E).$$

Hence the functional analytical properties of $\mathcal{W}^{s,\delta}(\mathbb{R}, E)$ carry over to $\mathcal{V}_\gamma^{s,\delta}(\mathbb{R}_+, E)$. In particular, $\mathcal{V}_\gamma^{s,\delta}(\mathbb{R}_+, E)$ is a Hilbert space having $C_0^\infty(\mathbb{R}_+, E)$ as a dense subset.
To a function $h \in C^\infty(\mathbb{R}_+ \times \Gamma_{1/2-\gamma}, \mathcal{L}(E_0, E_1))$ associate (formally) a Mellin pseudo-differential operator by

$$\mathrm{op}_M^\gamma(h)u(t) = \int t^{-(1/2-\gamma+i\tau)} h(t, 1/2-\gamma+i\tau)(\mathcal{M}_\gamma u)(1/2-\gamma+i\tau)\, d\!\!{}^-\tau$$

for $u \in \mathcal{T}_\gamma(\mathbb{R}_+, E_0)$. Then it is not difficult to verify that

$$\mathrm{op}_M^\gamma(h) = S_\gamma^{-1}\,\mathrm{op}(h_\gamma)\, S_\gamma \quad \text{with} \quad h_\gamma(y,\eta) = h(e^{-y}, 1/2-\gamma+i\eta), \tag{2.5}$$

where $\mathrm{op}(h_\gamma)$ is the usual Fourier pseudo-differential operator. Thus the question of investigating the appropriate symbol classes in the Mellin set-up is reduced to looking at the image of the Fourier symbol classes introduced in Definition 1.2, and Remark 1.14, under the transformation $y = -\log t$.

2.2 Definition. The space $S^{\mu,m}(\mathbb{R}_+ \times \Gamma_{1/2-\gamma}; E_0, E_1)$ consists of all functions $h \in C^\infty(\mathbb{R}_+ \times \Gamma_{1/2-\gamma}, \mathcal{L}(E_0, E_1))$ such that

$$\sup_{t>0, \tau\in\mathbf{R}} \left\{ \|\kappa_1^{-1}(\tau)\{\partial_\tau^l (t\partial_t)^k h(t, 1/2-\gamma+i\tau)\}\kappa_0(\tau)\|_{E_0,E_1} \langle\tau\rangle^{l-\mu} \langle\log t\rangle^{k-m} \right\} < \infty$$

for all $k, l \in \mathbb{N}_0$. These semi-norms induce a Fréchet topology on $S^{\mu,m}(\mathbb{R}_+ \times \Gamma_{1/2-\gamma}; E_0, E_1)$. By $S^\mu(\Gamma_{1/2-\gamma}; E_0, E_1)$ denote the closed subspace of $S^{\mu,0}(\mathbb{R}_+ \times \Gamma_{1/2-\gamma}; E_0, E_1)$ of all symbols h that are independent of t.

An essential point is that

$$h \in S^{\mu,m}(\mathbb{R}_+ \times \Gamma_{1/2-\gamma}; E_0, E_1) \iff h_\gamma \in S^{\mu,m}(\mathbb{R} \times \mathbb{R}; E_0, E_1),$$

where h_γ is defined as in (2.5), and the mapping $h \mapsto h_\gamma$ is an isomorphism. This basically relies on the fact that the push-forward of ∂_t under $t \mapsto e^{-t} : \mathbb{R} \to \mathbb{R}_+$ is $(-t\partial_t)$. Now we obtain the following results:

2.3 Theorem. *Let $g \in S^{\mu,m}(\mathbb{R}_+ \times \Gamma_{1/2-\gamma}; E_1, E_2)$ and $h \in S^{\mu',m'}(\mathbb{R}_+ \times \Gamma_{1/2-\gamma}; E_0, E_1)$. Then $\mathrm{op}_M^\gamma(g)\mathrm{op}_M^\gamma(h) = \mathrm{op}_M^\gamma(g\#h)$, where the Leibniz product of g and h is defined by*

$$(g\#h)(t,z) = \iint_0^\infty s^{i\xi} g(t, z+i\xi) h(st, z)\, \frac{ds}{s} d\!\!{}^-\xi, \qquad z \in \Gamma_{1/2-\gamma}.$$

For each $N \in \mathbb{N}$ we obtain the expansion

$$(g\#h) = \sum_{k<N} \frac{1}{k!}(\partial_z^k g)(-t\partial_t)^k h \;+\; r_N$$

with a remainder r_N in the space $S^{\mu+\mu'-N,m+m'-N}(\mathbb{R}_+ \times \Gamma_{1/2-\gamma}; E_0, E_2)$, which is given explicitely by the formula

$$r_N(t,z) = \int_0^1 \frac{(1-\theta)^{N-1}}{(N-1)!} \iint_0^\infty s^{i\xi} \partial_z^N g(t, z+i\theta\xi)(-t\partial_t)^N h(st,z) \frac{ds}{s} d\xi d\theta,$$

$z \in \Gamma_{1/2-\gamma}$. The Leibniz product induces a continuous mapping

$$S^{\mu,m}(\mathbb{R}_+ \times \Gamma_{1/2-\gamma}; E_1, E_2) \times S^{\mu',m'}(\mathbb{R}_+ \times \Gamma_{1/2-\gamma}; E_0, E_1) \to S^{\mu+\mu',m+m'}(\mathbb{R}_+ \times \Gamma_{1/2-\gamma}; E_0, E_2).$$

For later purpose we want to clarify the meaning of the oszillatory integrals in the previous theorem. Let $\{\chi_\varepsilon;\ 0 < \varepsilon < 1\}$ be a family of functions with

i) $\chi_\varepsilon \in \mathcal{S}(\mathbb{R}^2)$ for each ε,

ii) $\sup_{x\in\mathbb{R}^2, 0<\varepsilon<1} |\partial_x^\alpha \chi_\varepsilon(x)| < \infty$ for all $\alpha \in \mathbb{N}_0^2$,

iii) $\partial_x^\alpha \chi_\varepsilon(x) \longrightarrow \begin{cases} 1, & \alpha = 0 \\ 0, & |\alpha| > 0 \end{cases}$ for $\varepsilon \to 0$ pointwise for each $x \in \mathbb{R}^2$.

If we set $\chi_\varepsilon^+(s,\xi) = \chi_\varepsilon(-\log s, \xi)$, then $g\#h = \lim_{\varepsilon\to 0} I_\varepsilon$ with

$$I_\varepsilon(t,z) = \iint_0^\infty s^{i\xi} \chi_\varepsilon^+(s,\xi) g(t, z+i\xi) h(st,z) \frac{ds}{s} d\xi. \tag{2.6}$$

We also have $\partial_t^k \partial_z^l (g\#h) = \lim_{\varepsilon\to 0} \partial_t^k \partial_z^l I_\varepsilon$.

2.4 Theorem. *Let $h \in S^{\mu,m}(\mathbb{R}_+ \times \Gamma_{1/2-\gamma}; E_0, E_1)$. Then h induces for each $s, \delta \in \mathbb{R}$ continuous operators*

$$\mathrm{op}_M^\gamma(h) : \mathcal{V}_\gamma^{s,\delta}(\mathbb{R}_+, E_0) \to \mathcal{V}_\gamma^{s-\mu,\delta-m}(\mathbb{R}_+, E_1),$$

and we obtain the continuous mapping

$$h \mapsto \mathrm{op}_M^\gamma(h) : S^{\mu,m}(\mathbb{R}_+ \times \Gamma_{1/2-\gamma}; E_0, E_1) \to \mathcal{L}(\mathcal{V}_\gamma^{s,\delta}(\mathbb{R}_+, E_0), \mathcal{V}_\gamma^{s-\mu,\delta-m}(\mathbb{R}_+, E_1)).$$

2.5 Proposition. *Let $h \in S^{\mu,m}(\mathbb{R}_+ \times \Gamma_{1/2-\gamma}; E_0, E_1)$ with $\mu, m < 0$ such that $h(t,z) : E_0 \to E_1$ is a compact operator for all $(t,z) \in \mathbb{R}_+ \times \Gamma_{1/2-\gamma}$. Then*

$$\mathrm{op}_M^\gamma(h) : \mathcal{V}_\gamma^{s,\delta}(\mathbb{R}_+, E_0) \to \mathcal{V}_\gamma^{s,\delta}(\mathbb{R}_+, E_1),$$

is compact for each $s, \delta \in \mathbb{R}$.

2.6 Corollary. *Let E_0 be compactly embedded in E_1, and assume that the group action on E_1 induces (by restriction) the group action on E_0. Then also the embedding*

$$\mathcal{V}_\gamma^{s,\delta}(\mathbb{R}_+, E_0) \hookrightarrow \mathcal{V}_\gamma^{s',\delta'}(\mathbb{R}_+, E_1)$$

is compact whenever $s > s'$ and $\delta > \delta'$.

2.7 Remark. The space $S^\mu(\mathbb{R}_+ \times \Gamma_{1/2-\gamma}; E_0, E_1)$, consisting of all functions $h \in C^\infty(\mathbb{R}_+ \times \Gamma_{1/2-\gamma}, \mathcal{L}(E_0, E_1))$ such that

$$\sup_{t>0, \tau\in\mathbb{R}} \left\{ \|\kappa_1^{-1}(\tau)\{\partial_\tau^l (t\partial_t)^k h(t, 1/2-\gamma+i\tau)\}\kappa_0(\tau)\|_{E_0,E_1} \langle\tau\rangle^{l-\mu} \right\} < \infty$$

for all $k, l \in \mathbb{N}_0$, corresponds under the mapping $h \mapsto h_\gamma$ to $S^\mu(\mathbb{R} \times \mathbb{R}; E_0, E_1)$, cf. Remark 1.14. The analogues of the Theorems 2.3, 2.4, are valid for this symbol class.

2.3 Mellin symbols with smoothness up to the origin

A crucial role in calculi for manifolds with singularities play Mellin symbols, which are smooth not only in $\mathbb{R}_+$ but on $\overline{\mathbb{R}}_+$. These symbols are, in particular, Mellin symbols in the above sense. We consider the behaviour of these symbols under the Leibniz product. In general, the smoothness up to $t = 0$ is preseved only modulo certain smoothing remainders, which are described in the present section. No remainders occur if we require the symbols to extend holomorphically in the covariable.

Let us define

$$S^\mu(\overline{\mathbb{R}}_+ \times \Gamma_{1/2-\gamma}; E_0, E_1) = C^\infty(\overline{\mathbb{R}}_+, S^\mu(\Gamma_{1/2-\gamma}; E_0, E_1)).$$

If $h \in S^\mu(\overline{\mathbb{R}}_+ \times \Gamma_{1/2-\gamma}; E_0, E_1)$ is independent of t for large t, then $h \in S^{\mu,0}(\mathbb{R}_+ \times \Gamma_{1/2-\gamma}; E_0, E_1)$, and $(t\partial_t)^k h \in S^{\mu,-\infty}(\mathbb{R}_+ \times \Gamma_{1/2-\gamma}; E_0, E_1)$ for all $k \in \mathbb{N}$. The last statement holds, since $(t\partial_t)^k$ generates a zero in $t = 0$, which dominates logarithmic growth.

2.8 Proposition. *Let $g \in S^\mu(\overline{\mathbb{R}}_+ \times \Gamma_{1/2-\gamma}; E_1, E_2)$ and $h \in S^{\mu'}(\overline{\mathbb{R}}_+ \times \Gamma_{1/2-\gamma}; E_0, E_1)$, both independent of t for large t. Then there exists a symbol $f \in S^{-\infty,-\infty}(\mathbb{R}_+ \times \Gamma_{1/2-\gamma}; E_0, E_2)$ such that*

$$g\#h - f \in S^{\mu+\mu'}(\overline{\mathbb{R}}_+ \times \Gamma_{1/2-\gamma}; E_0, E_2).$$

PROOF: For abbreviation write $S^{\mu,m}(\mathbb{R}_+ \times \Gamma) = S^{\mu,m}(\mathbb{R}_+ \times \Gamma_{1/2-\gamma}; E_0, E_1)$, and analogously for the classes on $\overline{\mathbb{R}}_+$. For $k \in \mathbb{N}$ define

$$c_k = \frac{1}{k!}(\partial_z^k g)(-t\partial_t)^k h \quad \in S^{\mu+\mu'-k}(\overline{\mathbb{R}}_+ \times \Gamma) \cap S^{\mu+\mu'-k,-\infty}(\mathbb{R}_+ \times \Gamma).$$

Then, by Theorem 2.3, $g\#h = gh + \sum_{k=1}^N c_k + r_N$ with $r_N \in S^{\mu+\mu'-(N+1),-\infty}(\mathbb{R}_+ \times \Gamma)$. Let $\phi \in C^\infty(\mathbb{R})$ such that $\phi \equiv 0$ near $\tau = 0$, and $\phi \equiv 1$ for large τ. Then one can choose a sequence $(d_k) \subset \mathbb{R}$, tending to infinity with k, such that

$$f_1(t, 1/2 - \gamma + i\tau) = \sum_{k=1}^{\infty} \phi(d_k^{-1}\tau)c_k(t, 1/2 - \gamma + i\tau)$$

converges in $S^{\mu+\mu'-1}(\overline{\mathbb{R}}_+ \times \Gamma) \cap S^{\mu+\mu'-1,-\infty}(\mathbb{R}_+ \times \Gamma)$ and

$$f_1 - \sum_{k=1}^{N} c_k \in S^{\mu+\mu'-(N+1),-\infty}(\mathbb{R}_+ \times \Gamma).$$

In particular, $(gh + f_1) \in S^{\mu+\mu'}(\overline{\mathbb{R}}_+ \times \Gamma)$, and for each $N \in \mathbb{N}$

$$f := g\#h - (gh + f_1) = f_1 - \sum_{k=1}^{N} c_k \; - r_N \in S^{\mu+\mu'-(N+1),-\infty}(\mathbb{R}_+ \times \Gamma).$$

∎

2.9 Remark. Let $h \in S^{-\infty,-\infty}(\mathbb{R}_+ \times \Gamma_{1/2-\gamma}; E_0, E_1)$. Then $\mathrm{op}_M^\gamma(h)$ can be written as an integral operator (with respect to the measure $\frac{dt}{t}$) with a kernel $k \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+, \mathcal{L}(E_0, E_1))$ satisfying

$$e^{(1/2-\gamma)(t-t')}k(e^t, e^{t'}) \in \mathcal{S}(\mathbb{R}_t \times \mathbb{R}_{t'}, \mathcal{L}(E_0, E_1)).$$

Vice versa, each kernel having this property corresponds to a symbol h of infinitely negative order. These results hold in view of (2.5) and the known facts for Fourier symbols from $S^{-\infty,-\infty}(\mathbb{R} \times \mathbb{R}; E_0, E_1)$.

2.10 Definition. A function $h \in C^\infty(\mathbb{R}_+, \mathcal{A}(\mathbb{C}; \mathcal{L}(E_0, E_1)))$ is, by definition, an element of $S^{\mu,m}(\mathbb{R}_+ \times \mathbb{C}; E_0, E_1)$ if

$$\sup_{c<\beta<c'} p\Big((t, i\tau) \mapsto h(t, \beta + i\tau)\Big) < \infty$$

for each semi-norm $p(\cdot)$ of $S^{\mu,m}(\mathbb{R}_+ \times \Gamma_0; E_0, E_1)$ and all reals $c < c'$. Further let $S^\mu(\overline{\mathbb{R}}_+ \times \mathbb{C}; E_0, E_1) = C^\infty(\overline{\mathbb{R}}_+, S^\mu(\mathbb{C}; E_0, E_1))$.

Then the analogue of Theorem 2.3 is valid for the class $S^{\mu,m}(\mathbb{R}_+ \times \mathbb{C}; E_0, E_1)$ (where the Leibniz product corresponds to a fixed weight γ).
To a symbol $h \in S^\mu(\overline{\mathbb{R}}_+ \times \mathbb{C}; E_0, E_1)$ we associate its conormal symbol of order $\mu - k$, given by

$$\sigma^{\mu-k}(h)(z) = \frac{1}{k!}(\partial_t^k h)(0, z).$$

2.11 Theorem. *Let $g \in S^\mu(\overline{\mathbb{R}}_+ \times \mathbb{C}; E_1, E_2)$ and $h \in S^{\mu'}(\overline{\mathbb{R}}_+ \times \mathbb{C}; E_0, E_1)$, both independent of t for large t. Then*

$$g\#h \in S^{\mu+\mu'}(\overline{\mathbb{R}}_+ \times \mathbb{C}; E_0, E_2),$$

and the conormal symbols of the Leibniz product equal

$$\sigma^{\mu+\mu'-k}(g\#h) = \sum_{l+m=k} \Big(T^{-l}\sigma^{\mu-m}(g)\Big)\, \sigma^{\mu'-l}(h).$$

Here, $(T^\varrho f)(z) = f(z + \varrho)$ for a function f defined on $\mathbb{C}$.

Proof: Clearly, the z-derivatives of $g\#h$ can be pulled under the oscillatory integral. Hence, by induction, it suffices to show that

$$\partial_t(g\#h) = (T^{-1}g)\#(\partial_t h) + (\partial_t g)\#h \tag{1}$$

and that this derivative extends continuously to $\overline{\mathbb{R}}_+$. For convenience we assume that g is independent of t. The general case is proved in completely the same manner, but is awkward to write down in view of the Leibniz rule. Now let χ_1 be holomorphic in $\{-2 < \operatorname{Re} z < 2\}$) such that $\chi_1(0) = 1$ and $\beta \mapsto \chi_1(\beta + i\cdot) :]-2, 2[\to \mathcal{S}(\mathbb{R})$ is bounded (for example, χ_1 can be chosen as the Mellin transform of a function from $C_0^\infty(\mathbb{R}_+)$). Further let $\chi_2 \in \mathcal{S}(\mathbb{R})$, and

$$\chi_\varepsilon^+(s, \xi) = \chi_1(\varepsilon i\xi)\chi_2(-\varepsilon \log s).$$

Now associate I_ε to $g\#h$ as in (2.6). Then

$$\partial_t I_\varepsilon(t, z) = \iint_0^\infty s^{i\xi+1}\chi_\varepsilon^+(s, \xi) g(z + i\xi)(\partial_t h)(st, z)\, \frac{ds}{s} d\!\!{}^-\xi.$$

Write the integration in ξ as an integral over the curve Γ_0. For fixed s, t and z, the integrand extends holomorphically to $\{-2 < \operatorname{Re} w < 2\}$ and decreases as a Schwartz function uniformly on parallel lines of the imaginary axis. Hence, by Cauchy's integral formula, we can replace Γ_0 by Γ_{-1}. This yields

$$\partial_t I_\varepsilon(t,z) = \iint_0^\infty s^{i\xi}\tilde{\chi}_\varepsilon^+(s,\xi)(T^{-1}g(z+i\xi))(\partial_t h)(st,z)\,\frac{ds}{s}\,đ\xi$$

with $\tilde{\chi}_\varepsilon^+(s,\xi) = \chi_1(\varepsilon i\xi - \varepsilon)\chi_2(-\varepsilon\log s)$. Taking the limit $\varepsilon \to 0$ implies (1). It remains to verify that $\lim_{\varepsilon\to 0}(g\#h)(t,\cdot) = g(0,\cdot)h(0,\cdot)$ in $S^{\mu+\mu'}(\mathbb{C}; E_0, E_2)$. But this is true because of $g\#h = gh + c$ with a certain $c \in S^{\mu+\mu',-1}(\mathbb{R}_+ \times \mathbb{C}; E_0, E_2)$, in view of Theorem 2.3. Obviously, $\lim_{\varepsilon\to 0} c(t,\cdot) = 0$ in $S^{\mu+\mu'}(\mathbb{C}; E_0, E_2)$. ∎

Let us finally mention, that the part of the previous theorem concerning the Leibniz product was obtained in [7] for scalar-valued Mellin symbols, i.e. $E_0 = E_1 = E_2 = \mathbb{C}$ equipped with the trivial group action $\kappa \equiv 1$.

3 Application to operators on an infinite wedge

We illustrate an application of the results from Section 1 to an algebra of pseudo-differential operators on an infinite wedge, introduced in [16]. Here the wedge is the product of the edge $\mathbb{R}^q$ and a (stretched) cone $X^\wedge = \mathbb{R}_+ \times X$, where X is a smooth closed manifold. The Sobolev spaces of the wedge are of the form $\mathcal{W}^{s,\delta}(\mathbb{R}^q, E)$ for certain spaces E of distributions on $X^\wedge$. The operators have symbols taking values in a certain class of operators (namely the cone algebra with asymptotics in the sense of SCHULZE) that act between the Sobolev spaces on the cone.

Let us fix some notation. By $L^\mu(X)$ we denote the space of all pseudo-differential operators of order μ on X, and by $L^\mu(X;\mathbb{R})$ the parameter-dependent ones with parameter $\tau \in \mathbb{R}$, which is treated as an additional covariable. The standard Sobolev spaces on X are $H^s(X)$. On $C_0^\infty(X^\wedge) = C_0^\infty(\mathbb{R}_+, C^\infty(X))$ we define the Mellin transform as in Section 2.1, now acting on functions taking values in $C^\infty(X)$.

In the following, $\omega(t) \in C_0^\infty(\overline{\mathbb{R}}_+)$ is a cut-off function with respect to 0, i.e., $\omega(t) = 1$ for small t. For a Fréchet space E, which is a left module over an algebra A, and $a \in A$ we set $[a]E = \overline{\{ae;\ e \in E\}}$, where the closure is taken in the topology of E.

3.1 Cone Sobolev spaces and spaces with asymptotics

For $s,\gamma \in \mathbb{R}$ let $\mathcal{H}^{s,\gamma}(X^\wedge)$ be the completion of $C_0^\infty(X^\wedge) = C_0^\infty(\mathbb{R}_+, C^\infty(X))$ with respect to the norm

$$\|u\|^2 = \int_{\Gamma_{\frac{n+1}{2}-\gamma}} \|R^s(z)\mathcal{M}_{\gamma-n/2}u(z)\|^2_{L^2(X)}\,đz.$$

Here, $R^s(z) \in L^s(X;\Gamma_{\frac{n+1}{2}-\gamma})$ is a parameter-dependent pseudo-differential operator that induces isomorphisms $H^{s'}(X) \to H^{s'-s}(X)$ for each $s' \in \mathbb{R}$ and $z \in \Gamma_{\frac{n+1}{2}-\gamma}$. Moreover, $đz = (2\pi i)^{-1}dz$.

To an $f \in L^\mu(X;\Gamma_{1/2-\gamma})$ we associate a Mellin pseudo-differential operator

$$\left[\mathrm{op}_M^\gamma(f)u\right](t) = \int_{\Gamma_{1/2-\gamma}} t^{-z}f(z)(\mathcal{M}_\gamma u)(z)\,đz, \qquad u \in C_0^\infty(X^\wedge).$$

This extends for each $s \in \mathbb{R}$ by continuity to operators

$$\operatorname{op}_M^\gamma(f) : \mathcal{H}^{s,\gamma+n/2}(X^\wedge) \to \mathcal{H}^{s-\mu,\gamma+n/2}(X^\wedge).$$

Let $\{U_1, \ldots, U_N\}$ be a covering of X and $\chi_j : U_j \to V_j \subset S^n$ be diffeomorphisms. Here S^n is the unit sphere in $\mathbb{R}^{1+n}$. To the mappings χ_j associate

$$\tilde{\chi}_j : \mathbb{R}_+ \times U_j \to \mathbb{R}^{1+n} : (t, x) \mapsto t\chi_j(x).$$

Let $\{\phi_1, \ldots, \phi_N\}$ be a partition of unity on X with $\phi_j \in C_0^\infty(U_j)$. Then $[1-\omega]H^{s,\delta}_{cone}(X^\wedge)$ denotes the closure of $C_0^\infty(X^\wedge)$ with respect to the norm

$$\|u\|^2 = \sum_{j=1}^{N} \|((1-\omega)\phi_j u) \circ \tilde{\chi}_j^{-1}\|^2_{H^{s,\delta}(\mathbb{R}^{1+n})}.$$

The *cone Sobolev spaces* are defined as

$$\mathcal{K}^{s,\gamma}(X^\wedge) = [\omega]\mathcal{H}^{s,\gamma}(X^\wedge) + [1-\omega]H^{s,0}_{cone}(X^\wedge).$$

Note that $\mathcal{K}^{s,\gamma}(X^\wedge) \hookrightarrow \mathcal{K}^{s',\gamma'}(X^\wedge)$ if $s \geq s'$ and $\gamma \geq \gamma'$.

For $\gamma \in \mathbb{R}$ and an intervall $\Theta =]\vartheta, 0]$, $\vartheta < 0$, we call Q a *discrete asymptotic type* with respect to (γ, Θ), and write $Q \in \operatorname{As}(\gamma, \Theta)$, if

$$Q = \left\{(q_j, m_j) \in \mathbb{C} \times \mathbb{N}_0;\ \tfrac{n+1}{2} - \gamma + \vartheta < \operatorname{Re} q_j < \tfrac{n+1}{2} - \gamma,\ j = 0, \ldots, N\right\}$$

for some $N \in \mathbb{N}_0$. With such a type Q associate spaces

$$\mathcal{E}_Q^s(X^\wedge) = \left\{(t,x) \mapsto \omega(t) \sum_{j=0}^{N} \sum_{k=0}^{m_j} \xi_{jk}(x) t^{-q_j} \log^k t;\ \xi_{jk} \in H^s(X)\right\}, \qquad \mathcal{E}_Q(X^\wedge) = \cap_{s\in\mathbb{R}} \mathcal{E}_Q^s(X^\wedge),$$

which are canonically isomorphic to a finite product of $H^s(X)$ and $C^\infty(X)$, respectively. Writing $\mathcal{K}_\Theta^{s,\gamma}(X^\wedge) = \cap_{\varepsilon>0} \mathcal{K}^{s,\gamma-\vartheta-\varepsilon}(X^\wedge)$ we then set

$$\mathcal{K}_Q^{s,\gamma}(X^\wedge) = \mathcal{K}_\Theta^{s,\gamma}(X^\wedge) + \mathcal{E}_Q(X^\wedge), \qquad \mathcal{K}_Q^{\infty,\gamma}(X^\wedge) = \cap_{s\in\mathbb{R}} \mathcal{K}_Q^{s,\gamma}(X^\wedge),$$

which are Fréchet spaces that can be written as projective limits of Hilbert spaces

$$\begin{aligned}
\mathcal{K}_Q^{s,\gamma}(X^\wedge) &= \operatorname*{proj-lim}_{k\in\mathbb{N}} \left\{[\omega]\{\mathcal{K}^{s,\gamma-\vartheta-c_k}(X^\wedge) + \mathcal{E}_Q^k(X^\wedge)\} + [1-\omega]H^{s,0}_{cone}(X^\wedge)\right\}, \\
\mathcal{K}_Q^{\infty,\gamma}(X^\wedge) &= \operatorname*{proj-lim}_{k\in\mathbb{N}} \left\{[\omega]\{\mathcal{K}^{k,\gamma-\vartheta-c_k}(X^\wedge) + \mathcal{E}_Q^k(X^\wedge)\} + [1-\omega]H^{k,0}_{cone}(X^\wedge)\right\},
\end{aligned}$$

where $c_k = c_Q/k$ and c_Q is chosen in a way that $\operatorname{Re} q_j > \frac{n+1}{2} - \gamma + \vartheta + c_Q$ for all j. Finally, we define the space of rapidly decreasing functions on $X^\wedge$ as $\mathcal{S}(X^\wedge) = \mathcal{S}(\mathbb{R}, C^\infty(X))|_{\mathbb{R}_+}$, and set

$$\mathcal{S}_Q^\gamma(X^\wedge) = [\omega]\mathcal{K}_Q^{\infty,\gamma}(X^\wedge) + [1-\omega]\mathcal{S}(X^\wedge),$$

which is a projective limit of the Hilbert spaces

$$[\omega]\{\mathcal{K}^{k,\gamma-\vartheta-c_k}(X^\wedge) + \mathcal{E}_Q^k(X^\wedge)\} + [1-\omega]H^{k,k}_{cone}(X^\wedge),$$

with c_k as above. The corresponding group actions on all these spaces now are induced by the mapping $C_0^\infty(X^\wedge) \to C_0^\infty(X^\wedge)$ defined by

$$(\kappa_\lambda u)(t,x) = \lambda^{\frac{n+1}{2}} u(\lambda t, x). \tag{3.7}$$

On $\mathcal{K}^{0,0}(X^\wedge)$ this is, in particular, a group of unitary operators.

3.2 Weighted edge Sobolev spaces

First we generalize the material from Section 1 to Fréchet spaces, which are projective limits of Hilbert spaces. More precisely, let F_j $(j = 0, 1)$ be Fréchet spaces that can be written as

$$F_j = \operatorname*{proj-lim}_{k \in \mathbb{N}} E_j^k$$

with Hilbert spaces $E_j^1 \hookleftarrow E_j^2 \hookleftarrow \dots$, such that the group action on E_j^1 induces (by restriction) the corresponding group action on each E_j^k. We then set

$$\mathcal{W}^{s,\delta}(\mathbb{R}^q, F_j) = \operatorname*{proj-lim}_{k \in \mathbb{N}} \mathcal{W}^{s,\delta}(\mathbb{R}^q, E_j^k). \tag{3.8}$$

Furthermore, we define

$$S^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q; F_0, F_1) = \cap_{l \in \mathbb{N}} \Big\{ \cup_{k \in \mathbb{N}} S^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_0^k, E_1^l) \Big\}, \tag{3.9}$$

i.e., $a \in S^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q; F_0, F_1)$ if and only if to each $l \in \mathbb{N}$ there is a $k \in \mathbb{N}$ such that $a \in S^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_0^k, E_1^l)$. By the definition of projective limits and Corollary 1.11 it is then obvious that each $a \in S^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q; F_0, F_1)$ induces continuous operators

$$\operatorname{op}(a) : \mathcal{W}^{s,\delta}(\mathbb{R}^q, F_0) \to \mathcal{W}^{s-\mu,\delta-m}(\mathbb{R}^q, F_1)$$

for arbitrary $s, \delta \in \mathbb{R}$. This abstract setting can be applied to the cone Sobolev spaces.

3.1 Definition. Let $s, \delta, \gamma \in \mathbb{R}$ and $Q \in \operatorname{As}(\gamma, \Theta)$, $\Theta =]\vartheta, 0]$. The weighted edge Sobolev spaces are defined as

$$\mathcal{W}_\gamma^{s,\delta}(\mathbb{R}^q \times X^\wedge) := \mathcal{W}^{s,\delta}(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge)), \qquad \mathcal{W}_Q^{s,\delta}(\mathbb{R}^q \times X^\wedge) := \mathcal{W}^{s,\delta}(\mathbb{R}^q, \mathcal{K}_Q^{s,\gamma}(X^\wedge)).$$

3.2 Remark. As a consequence of Example 1.5 we obtain

$$H^s_{comp}(\mathbb{R}^q \times X^\wedge) \subset \mathcal{W}_Q^{s,\delta}(\mathbb{R}^q \times X^\wedge) \subset \mathcal{W}_\gamma^{s,\delta}(\mathbb{R}^q \times X^\wedge) \subset H^s_{loc}(\mathbb{R}^q \times X^\wedge)$$

for each $s, \delta, \gamma \in \mathbb{R}$ and $Q \in \operatorname{As}(\gamma, \Theta)$ (for more details see, e.g. [14]).

3.3 Example: Mellin and Green pseudo-differential operators

Throughout this section let data $\underline{g} = (\gamma, \gamma - \nu, \Theta)$ with weight-intervall $\Theta =]-k, 0]$, $k \in \mathbb{N}$, be fixed. Further let $N_+, N_- \in \mathbb{N}$.

3.3 Definition. For $\mu, m \in \mathbb{R}$ the space $R_G^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}; N_-, N_+)$ consists of all symbols $\mathbf{g}(y, \eta) \in \cap_{s \in \mathbb{R}} S^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge) \oplus \mathbb{C}^{N_-}, \mathcal{K}^{\infty,\gamma-\nu}(X^\wedge) \oplus \mathbb{C}^{N_+})$ satisfying

$$\mathbf{g} \in \cap_{s \in \mathbb{R}} S^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge) \oplus \mathbb{C}^{N_-}, \mathcal{S}_{Q_1}^{\gamma-\nu}(X^\wedge) \oplus \mathbb{C}^{N_+}),$$

$$\mathbf{g}^* \in \cap_{s \in \mathbb{R}} S^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{K}^{s,\nu-\gamma}(X^\wedge) \oplus \mathbb{C}^{N_+}, \mathcal{S}_{Q_2}^{-\gamma}(X^\wedge) \oplus \mathbb{C}^{N_-})$$

for certain asymptotic types $Q_1 \in \operatorname{As}(\gamma - \nu, \Theta)$, $Q_2 \in \operatorname{As}(-\gamma, \Theta)$ depending on $\mathbf{g}$. Here the involved group actions are $\{\kappa_\lambda \oplus 1\}$, where $\{\kappa_\lambda\}$ is the standard group action from (3.7). Further * means the pointwise formal adjoint in the sense of

$$(gu, v)_{\mathcal{K}^{0,0}(X^\wedge) \oplus \mathbb{C}^{N_+}} = (u, g^* v)_{\mathcal{K}^{0,0}(X^\wedge) \oplus \mathbb{C}^{N_-}}$$

for all $u \in C_0^\infty(X^\wedge) \oplus \mathbb{C}^{N_-}$ and $v \in C_0^\infty(X^\wedge) \oplus \mathbb{C}^{N_+}$.

A set P is called *discrete asymptotic type for Mellin symbols* if

$$P = \{(p_j, m_j) \in \mathbb{C} \times \mathbb{N}_0;\ j \in \mathbb{Z}\}, \quad \operatorname{Re} p_j \to \pm\infty \text{ for } j \to \mp\infty.$$

The projection of P to the complex plane is denoted by $\pi_{\mathbb{C}} P = \{p_j;\ j \in \mathbb{Z}\}$. Then the space $M_P^{\mu,m}(X)$ consists, roughly speaking, of all functions $h(y) \in C^\infty(\mathbb{R}^q, \mathcal{A}(\mathbb{C}\backslash\pi_{\mathbb{C}} P, L^\mu(X)))$, where the poles in $p_j \in \pi_{\mathbb{C}} P$ are at most of order $m_j + 1$, and the Laurent coefficients $\sigma_{jk}(y)$ of the principal part of $h(y)$ in p_j being elements of $L^{-\infty}(X)$, and

$$\sup_{y \in \mathbb{R}^q} \langle y \rangle^{|\alpha|-m} q\left(\partial_y^\alpha \sigma_{jk}(y)\right) < \infty$$

for all $\alpha \in \mathbb{N}_0^q$ and semi-norms $q(\cdot)$ of $L^{-\infty}(X)$. Further one requires, that $h(y, z + i\beta)$ is "outside $\pi_{\mathbb{C}} P$" an element of $L^\mu(X; \Gamma_\beta)$ such that

$$\sup\left\{\langle y \rangle^{|\alpha|-m} q\left(\partial_y^\alpha [h(y, \beta + i\varrho)]\right);\ c_1 \leq \beta \leq c_2,\ y \in \mathbb{R}^q\right\} < \infty$$

for all $c_1 < c_2 \in \mathbb{R}$, each semi-norm $q(\cdot)$ of $L^\mu(X; \mathbb{R}_\varrho)$, and all $\alpha \in \mathbb{N}_0^q$. In particular, $h(y,z) \in L^\mu(X; \Gamma_\beta)$ for each $\beta \in \mathbb{R}$ such that $\pi_{\mathbb{C}} P \cap \Gamma_\beta = \emptyset$.

For a precise definition of these spaces we refer to [16], Section 4.

3.4 Definition. Let $\mu \in \mathbb{R}$ with $\nu - \mu \in \mathbb{N}_0$. A function

$$m(y,\eta) = \omega(t[\eta]) \sum_{j=0}^{k+\mu-\nu-1} t^{-\mu+j} \sum_{|\alpha| \leq j} \operatorname{op}_M^{\gamma_{j\alpha}}(h_{j\alpha})(y) \eta^\alpha \omega(t[\eta]) \tag{3.10}$$

with

$$h_{j\alpha} \in M_{P_{j\alpha}}^{-\infty,m}(X^\wedge), \quad \pi_{\mathbb{C}} P_{j\alpha} \cap \Gamma_{1/2-\gamma_{j\alpha}} = \emptyset, \quad \gamma - n/2 - (\nu - \mu) - j \leq \gamma_{j\alpha} \leq \gamma - n/2$$

is called smoothing Mellin symbol (of order (μ, m) with respect to $\underline{g}$). Note that $m \equiv 0$ if $\nu - \mu \geq k$. Hereby, $\omega(t[\eta])$ and $t^{-\mu+j}$ have to be understood as (parameter-dependent) operators of multiplication between the cone Sobolev spaces, and $\eta \mapsto [\eta]$ is a smooth strictly positive function that equals $|\eta|$ for large $|\eta|$.

Now the space $R_{M+G}^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}; N_-, N_+)$ denotes all functions of the form

$$(\mathbf{m} + \mathbf{g})(y,\eta) = \begin{pmatrix} m(y,\eta) & 0 \\ 0 & 0 \end{pmatrix} + \mathbf{g}(y,\eta) : \begin{array}{c} \mathcal{K}^{s,\gamma}(X^\wedge) \\ \oplus \\ \mathbb{C}^{N_-} \end{array} \longrightarrow \begin{array}{c} \mathcal{K}^{\infty,\gamma-\nu}(X^\wedge) \\ \oplus \\ \mathbb{C}^{N_+} \end{array}$$

with $\mathbf{g} \in R_G^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}; N_-, N_+)$ and $m(y,\eta)$ as in the above Definition 3.4.

3.5 Theorem. *(cf. [16], Proposition 2.19) Let* $\mathbf{m}(y,\eta)$ *as above. Then*

$$\mathbf{m} \in \cap_{s \in \mathbb{R}} S^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge) \oplus \mathbb{C}^{N_-}, \mathcal{K}^{\infty,\gamma-\nu}(X^\wedge) \oplus \mathbb{C}^{N_+}).$$

Furthermore, to each $Q \in As(\gamma, \Theta)$ *exists a* $Q' \in As(\gamma - \nu, \Theta)$ *such that*

$$\mathbf{m} \in \cap_{s \in \mathbb{R}} S^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{K}_Q^{s,\gamma}(X^\wedge) \oplus \mathbb{C}^{N_-}, \mathcal{K}_{Q'}^{\infty,\gamma-\nu}(X^\wedge) \oplus \mathbb{C}^{N_+}).$$

Together with Corollary 1.11 we now get the following theorem.

3.6 Theorem. *Let* $\mathbf{m}+\mathbf{g} \in R^{\mu,m}_{M+G}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}; N_-, N_+)$ *and let asymptotic types* Q_1, Q_2 *be associated with* $\mathbf{g}$. *Then for each* $s, \delta \in \mathbb{R}$ *we obtain continuous operators*

$$\mathrm{op}(\mathbf{g}) : \begin{array}{c} \mathcal{W}^{s,\delta}_{\gamma}(\mathbb{R}^q \times X^\wedge) \\ \oplus \\ H^{s,\delta}(\mathbb{R}^q, \mathbb{C}^{N_-}) \end{array} \longrightarrow \begin{array}{c} \mathcal{W}^{s-\mu,\delta-m}_{Q_2}(\mathbb{R}^q \times X^\wedge) \\ \oplus \\ H^{s-\mu,\delta-m}(\mathbb{R}^q, \mathbb{C}^{N_+}) \end{array},$$

$$\mathrm{op}(\mathbf{m}+\mathbf{g}) : \begin{array}{c} \mathcal{W}^{s,\delta}_{\gamma}(\mathbb{R}^q \times X^\wedge) \\ \oplus \\ H^{s,\delta}(\mathbb{R}^q, \mathbb{C}^{N_-}) \end{array} \longrightarrow \begin{array}{c} \mathcal{W}^{s-\mu,\delta-m}_{\gamma-\nu}(\mathbb{R}^q \times X^\wedge) \\ \oplus \\ H^{s-\mu,\delta-m}(\mathbb{R}^q, \mathbb{C}^{N_+}) \end{array}.$$

Further, to each $Q \in As(\gamma, \Theta)$ *there is a* $Q' \in As(\gamma-\nu, \Theta)$, *depending on* $\mathbf{m}$ *and* $\mathbf{g}$, *such that*

$$\mathrm{op}(\mathbf{m}+\mathbf{g}) : \begin{array}{c} \mathcal{W}^{s,\delta}_{Q}(\mathbb{R}^q \times X^\wedge) \\ \oplus \\ H^{s,\delta}(\mathbb{R}^q, \mathbb{C}^{N_-}) \end{array} \longrightarrow \begin{array}{c} \mathcal{W}^{s-\mu,\delta-m}_{Q'}(\mathbb{R}^q \times X^\wedge) \\ \oplus \\ H^{s-\mu,\delta-m}(\mathbb{R}^q, \mathbb{C}^{N_+}) \end{array}$$

continuously.

Let 1 denote the identity operator $E \to E$ and $\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, viewed as an operator $E \oplus F \to E \oplus G$ for various spaces E, F, G.
Further let from now on $\underline{g} = (0, 0, \Theta)$. In the following we consider the algebra of operators

$$\mathrm{op}(\mathbf{1}+\mathbf{m}+\mathbf{g}), \quad \text{with} \quad \mathbf{m}+\mathbf{g} \in R^{0,0}_{M+G}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}; N_-, N_+).$$

3.7 Definition. A symbol $\mathbf{1}+\mathbf{m}+\mathbf{g}$ and the corresponding operator $\mathrm{op}(\mathbf{1}+\mathbf{m}+\mathbf{g})$ are called *elliptic* if

i) there exists an asymptotic type P with $\pi_{\mathbb{C}} P \cap \Gamma_{\frac{n+1}{2}} = \emptyset$ such that $(1+h_{00})^{-1} \in M^{0,0}_P(X)$, where h_{00} corresponds to $\mathbf{m}$ via (3.10),

ii) for large $|(y, \eta)|$

$$(\mathbf{1}+\mathbf{m}+\mathbf{g})(y,\eta) : \begin{array}{c} \mathcal{K}^{0,0}(X^\wedge) \\ \oplus \\ \mathbb{C}^{N_-} \end{array} \longrightarrow \begin{array}{c} \mathcal{K}^{0,0}(X^\wedge) \\ \oplus \\ \mathbb{C}^{N_+} \end{array}$$

is invertible and the inverse is uniformly bounded in (y, η).

From Theorem 3.6 and [16], Theorem 3.10 we can conclude:

3.8 Theorem. *Let* $\mathbf{A} = \mathrm{op}(\mathbf{1}+\mathbf{m}+\mathbf{g})$ *with* $\mathbf{m}+\mathbf{g} \in R^{0,0}_{M+G}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}; N_-, N_+)$ *be elliptic. Then there exists a* $\mathbf{B} = \mathrm{op}(\mathbf{1}+\mathbf{m}'+\mathbf{g}')$ *with* $\mathbf{m}'+\mathbf{g}' \in R^{0,0}_{M+G}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}; N_+, N_-)$ *such that*

$$\mathbf{AB} - 1 = \mathrm{op}(\mathbf{g}_+), \qquad \mathbf{BA} - 1 = \mathrm{op}(\mathbf{g}_-)$$

for certain $\mathbf{g}_\pm \in R_G^{-\infty,-\infty}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}; N_\pm, N_\pm)$. *Further,*

$$\mathbf{A} : \begin{array}{c} \mathcal{W}_0^{s,\delta}(\mathbb{R}^q \times X^\wedge) \\ \oplus \\ H^{s,\delta}(\mathbb{R}^q, \mathbb{C}^{N_-}) \end{array} \longrightarrow \begin{array}{c} \mathcal{W}_0^{s,\delta}(\mathbb{R}^q \times X^\wedge) \\ \oplus \\ H^{s,\delta}(\mathbb{R}^q, \mathbb{C}^{N_+}) \end{array}$$

is Fredholm for each $s, \delta \in \mathbb{R}$. *Let* $u \in \mathcal{W}_0^{-\infty,-\infty}(\mathbb{R}^q \times X^\wedge) \oplus H^{-\infty,-\infty}(\mathbb{R}^q \oplus \mathbb{C}^{N_-})$ *and* $\mathbf{A}u = f$. *Then*

a) $f \in \mathcal{W}_0^{s,\delta}(\mathbb{R}^q \times X^\wedge) \oplus H^{s,\delta}(\mathbb{R}^q \oplus \mathbb{C}^{N_+})$ *implies*

$$u \in \mathcal{W}_0^{s,\delta}(\mathbb{R}^q \times X^\wedge) \oplus H^{s,\delta}(\mathbb{R}^q \oplus \mathbb{C}^{N_-}),$$

b) $f \in \mathcal{W}_{Q_1}^{s,\delta}(\mathbb{R}^q \times X^\wedge) \oplus H^{s,\delta}(\mathbb{R}^q \oplus \mathbb{C}^{N_+})$ *implies the existence of a* $Q_2 \in As(0, \Theta)$ *such that*

$$u \in \mathcal{W}_{Q_2}^{s,\delta}(\mathbb{R}^q \times X^\wedge) \oplus H^{s,\delta}(\mathbb{R}^q \oplus \mathbb{C}^{N_-}).$$

The Fredholmness of $\mathbf{A}$ is a consequence of Proposition 1.12.

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