An Operator Algebra on Manifolds with Cusp-Type Singularities

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Abstract

Equations on manifolds with cusp-type singularities are investigated. The corresponding operator algebra is constructed and finiteness theorems (Fredholm property) are established. The resurgent character of solutions is proved for equations with infinitely flat right-hand side.

Keywords: elliptic theory, manifolds with singularities, casp, noncommutative analysis, left ordered representation, local casp algebra, ellipticity, regularizer, finiteness theorem, Fredholm property, Borel-Laplace transform, resurgent analysis, asymptotic expansions, endless continuability

Contents

Introduction

In this paper, we consider elliptic differential equations on manifolds with singularities of cusp-type.

In our previous paper [1], we investigated asymptotic solutions to the corresponding homogeneous equations and found, in particular, that these solutions are functions of exponential growth of order k near each cusp point of the same order. Hence, it is natural to construct the elliptic theory on manifolds with cusp-type singularities in weighted Sobolev spaces with weight exponential of order k in a neighborhood of a cusp point of the same order.

In fact, it is convenient to construct the elliptic theory in the framework of an algebra, so that elliptic operators form the subgroup of invertible elements (we carry out our considerations modulo compact operators). If such an algebra is constructed, the desired finiteness theorem (that is, the Fredholm property) is a direct consequence of the existence of near-inverses (regularizers) for elliptic elements of the algebra. Since it is clear that the main difficulties in the construction of the regularizer in this situation are concentrated near singular points of the manifold in question, we construct the corresponding local algebra, which we call *the local cusp algebra.* For constructing such an algebra we use the noncommutative analysis created by V. Maslov [2] and developed further by V. Maslov and his collaborators (see [3]–[8] and the bibliography therein). Namely, using the notion of non-standard charcteristics introduced by V. Maslov [9] we represent the operator under investigation as a function of two ordered operators and then use the noncommutative calculus mentioned above.

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1 Preliminary considerations

1.1 Geometry of the problem and ellipticity

1. Let M be a smooth manifold with a finite number of points $\{m_1, \dots, m_N\}$ of cusp type and let $\hat{H}$ be a differential operator of order μ on this manifold. This means that:

- In the complement of the set $\bigcup_{j=0}^{N} \{m_j\}$ of all cusp points the manifold M has the structure of a C^∞-manifold.

- Associated to each point m_j are a neighborhood $U_j \subset M$ and a smooth manifold Ω_j such that U_j is topologically equivalent to the cone

  $$([0,1] \times \Omega_j) \,/\, (\{0\} \times \Omega_j) \tag{1}$$

  and the smooth structure on the cone coincides with that on the manifold M in the complement of the vertex m_j of this cone. We denote by (r, ω) the coordinates on U_j induced by the representation (1), where $r \in [0,1]$ and ω are local coordinates on the manifold Ω (see Figure 1).

- Associated to each point m_j is a positive integer k_j such that local expressions of operators $\hat{H}$ near m_j are

  $$\hat{H}_j = r^{-\mu(k_j+1)} \sum_{l=0}^{\mu} \hat{A}_l(r) \left(-r^{k_j+1} \frac{d}{dr} \right)^l , \tag{2}$$

  where $\hat{A}_l(r)$ are smooth differential operators on Ω of order $\mu - l$, and the dependence of the operator $\hat{A}_l(r)$ in the variable r is C^∞ up to the point

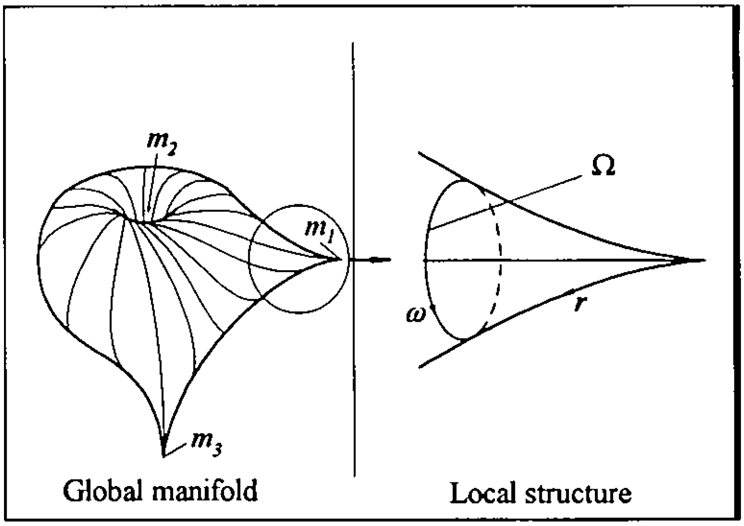

Figure 1. Manifolds with singularities of cusp type.

$r = 0$. The numbers k_j are called the *multiplicities* of the cusp points m_j, $j = 1, \ldots, N$.

In particular, we shall consider elliptic operators of the form (2) in the following sense:

Definition 1 An operator $\hat{H}$ of the above type is called *elliptic* if both:

1. It is elliptic in the usual sense at all points of the complement $M \backslash \{m_1, \ldots .m_N\}$.
2. The family of operators
$$\hat{H}\,(p) = \sum_{l=0}^{\mu} \hat{A}_l\,(0)\,p^l \tag{3}$$
is a strictly elliptic analytic family of operators on the manifold Ω in the sense of Agranovich-Vishik ([10]).

We remark that, under the assumption of ellipticity of the operator $\hat{H}$, the family $\hat{H}(p)$ of operators in (3) is meromorphically invertible in the complex plane $\mathbf{C}$ with the coordinate p.

Theorems showing the finite-dimensionality of the kernel and cokernel of elliptic operators will be established in special weighted Sobolev spaces which will be described below. We remark that the choice of these spaces for proving the finiteness theorems is governed by the asymptotic expansions of solutions to the corresponding homogeneous equation obtained in [1]. Due to the results there, these Sobolev spaces must have an exponential weight at each singular point m_j of the manifold M of an order k_j.

It is known that the proof of finiteness theorem can be accomplished in the framework of an operator algebra including the operators of the considered type and regularizers for such operators. Then the finiteness theorem is an easy consequence of the existence of a regularizer and the corresponding embedding theorems.

1.2 Operator algebra and noncommutative analysis

Due to the locality principle, it is clear that it suffices to construct the above mentioned algebra in a neighborhood of a singular point of the underlying manifold. So, we focus our attention on the construction of the *local cusp algebra* near a cusp point. Let us consider the construction of this algebra in more detail (since we are working in a neighborhood of some fixed cusp point, we omit the corresponding index j).

1. First of all, our future algebra must contain differential operators of the form (2). Omitting the inessential factor $r^{-\mu(k+1)}$, we can write down the expression of the operator in the form

$$\sum_{l=0}^{\mu} \hat{A}_l(r)\left(-ir^{k+1}\frac{d}{dr}\right)^l = \hat{H}\left(r, -ir^{k+1}\frac{d}{dr}\right). \tag{4}$$

One can see that this operator is a function of the two operators

$$B = r \quad \text{and} \quad A = -ir^{k+1}\frac{d}{dr}, \tag{5}$$

where the operator A acts first, and the operator B acts second. The latter remark is necessary since the two operators (5) do not commute:

$$[B, A] = \left[r, -ir^{k+1}\frac{d}{dr}\right] = ir^{k+1}.$$

(The square brackets denote the commutator of the corresponding operators.) So, we shall write down the operator (4) in the form

$$\hat{H} = \hat{H}\left(\overset{2}{r}, \overset{1}{\overline{-ir^{k+1}\frac{d}{dr}}}\right), \tag{6}$$

where the indices over the operators define the order of their action (for complicated operators like $-ir^{k+1}d/dr$ we use the bar to define the range of action of the corresponding index; this bar will be omitted in the case when the notation of an operator consists of a single letter.)

For example, if the function $\hat{H}(r,p)$ is a polynomial in the variable p, say

$$\hat{H}(r,p) = \sum_{j=0}^{\mu} \hat{A}(r)p^j,$$

then one has

$$\hat{H}\left(\overset{2}{r}, \overset{1}{\overline{-ir^{k+1}\frac{d}{dr}}}\right) = \sum_{j=0}^{\mu} \hat{A}(r)\left(-ir^{k+1}\frac{d}{dr}\right)^j,$$

and

$$\hat{H}\left(\overset{1}{r}, \overset{2}{\overline{-ir^{k+1}\frac{d}{dr}}}\right) = \sum_{j=0}^{\mu} \left(-ir^{k+1}\frac{d}{dr}\right)^j \hat{A}(r).$$

It is natural to construct the local algebra as the algebra of functions of operators (5). We remark that the symbol function $\hat{H}(x,\xi)$ involved into relations (4) and (6) is a polynomial in the variable ξ, corresponding to the operator $-ir^{k+1}d/dr$, and, hence, the definition of the operator (6) is clear (see the examples above). However, if we need to write down the regularizers for such operators in the form of functions of operators (5), we must consider more general symbols, which requires the exact definition of function of noncommutative operators (5).

This definition can be given as follows. First, we remark that both B and A are symmetric in the Hilbert space $L_2\left(\mathbf{R}_+, r^{-k-1}\right)$ defined by the norm

$$\|f(r)\|^2 = \int_0^\infty |f(r)|^2 \frac{dr}{r^{k+1}}.$$

Let

$$e^{itB} \text{ and } e^{i\tau A}$$

be the two one-parameter groups[1] corresponding to operators (5). Definition of the

[1]These groups will be well-defined, for example, if the operators A and B are self-adjoint. Unfortunately, this fails for the operator $-ir^{k+1}d/dr$ in the space introduced above. However, it is not essential in our heuristic consideration since it is possible to modify this operator up to a self-adjoint operator which coincides with $-ir^{k+1}d/dr$ near the origin (we recall that we construct a *local* algebra). This will be done in the main part of the paper.

function $F\left(\overset{2}{B},\overset{1}{A}\right)$ can be given in the form

$$F\left(\overset{2}{B},\overset{1}{A}\right)=\int e^{itB}e^{i\tau A}\tilde{F}\left(t,\tau\right)\,dtd\tau, \tag{7}$$

where $\tilde{F}\left(t,\tau\right)$ is the Fourier transform of the function $F\left(x,\xi\right)$ in the variables $\left(x,\xi\right)$. Clearly, the class of functions used in the latter definition requires the exact description. This description will be presented below, and here we denote by Smbl the class of admissible symbols for the definition (7), and by Op the corresponding class of operators.

The main problem in constructing an algebra of operators of the form (7) is to compute the symbol of the composition $\hat{F}\circ\hat{G}$ and the symbols of adjoint operators $\hat{F}^{*}$ and $\hat{G}^{*}$ via symbols $F\left(x,\xi\right)$ and $G\left(x,\xi\right)$ of the operators $\hat{F}$ and $\hat{G}$. Formally it can be expressed as follows. Formula (7) defines a linear mapping

$$\varphi\ :\ \mathrm{Smbl}\rightarrow\mathrm{Op}$$

which is an isomorphism on the image and, hence, the composition law in the algebra Op defines some (noncommutative) composition in the algebra Smbl. Similar, the conjugation in the algebra Op defines a conjugation in the symbol algebra. Our aim is to compute an explicit expressions for these operations.

The method of noncommutative operators supplies us with a procedure to compute the sought-for composition law. Namely, one must first compute the operators of the so-called *left ordered representation*, that is, the two operators l_A and l_B satisfying the following conditions

$$\begin{aligned}\mathrm{smbl}\left(A\hat{F}\right)&=l_A\left[F\left(x,\xi\right)\right],\\ \mathrm{smbl}\left(B\hat{F}\right)&=l_B\left[F\left(x,\xi\right)\right]\end{aligned}$$

for any operator $\hat{F}$ with symbol $\hat{F}\left(y,\xi\right)$. Here smbl $\left(\hat{F}\right)$ denotes the symbol of the operator $\hat{F}$. The general formula for the composition will then be

$$\mathrm{smbl}\left(\hat{F}\hat{G}\right)=F\left(\overset{2}{\overline{l_B}},\overset{1}{\overline{l_A}}\right)G\left(x,\xi\right)$$

([8, p. 98]). The operators l_A and l_B are called the operators of the left ordered representation. Let us compute these operators for A and B considered above.

First, we have

$$B\hat{F} = \overset{2}{B}F\left(\overset{2}{B}, \overset{1}{A}\right)$$

since the operator B in the function $F\left(\overset{2}{B}, \overset{1}{A}\right)$ acts after the operator A. Hence, the operator l_B is simply multiplication by the variable x:

$$l_B = x.$$

Later on, the expression for $A\hat{F}$ is

$$A\hat{F} = \overset{3}{A}F\left(\overset{2}{B}, \overset{1}{A}\right),$$

and, to compute the symbol of the last operator one has to commute the operators $\overset{3}{A}$ and $\overset{2}{B}$. This can be done with the help of the commutation formula (see [8, p. 62]):

$$\overset{3}{A}F\left(\overset{2}{B}, \overset{1}{A}\right) = \overset{1}{A}F\left(\overset{2}{B}, \overset{1}{A}\right) - \overset{3}{\overline{[B, A]}}\,\frac{\delta F}{\delta x}\left(\overset{2}{B}, \overset{4}{B}, \overset{1}{A}\right),$$

where

$$\frac{\delta F}{\delta x}\left(x', x'', \xi\right) = \frac{F\left(x', \xi\right) - F\left(x'', \xi\right)}{x' - x''}$$

is a difference derivative of the function $F(x, \xi)$ in the variable x. In our case we have

$$\begin{aligned}
\overset{3}{A}F\left(\overset{2}{B}, \overset{1}{A}\right) &= \overset{1}{A}F\left(\overset{2}{B}, \overset{1}{A}\right) - \overset{3}{\overline{iB^{k+1}}}\frac{\delta F}{\delta x}\left(\overset{2}{B}, \overset{4}{B}, \overset{1}{A}\right) \\
&= \overset{1}{A}F\left(\overset{2}{B}, \overset{1}{A}\right) - \overset{2}{\overline{iB^{k+1}}}\,\frac{\partial F}{\partial x}\left(\overset{2}{B}, \overset{1}{A}\right)
\end{aligned}$$

since the commutator $[B, A] = iB^{k+1}$ commutes with B, and the difference derivative becomes the usual derivative for equal arguments $x'' = x'$. Hence, the expression for the operator l_A is

$$l_A = \xi - ix^{k+1}\frac{\partial}{\partial x},$$

and we arrive at the following composition formula:

$$\operatorname{smbl}\left(\hat{F}\hat{G}\right) = F\left(\overset{2}{\overline{l_B}}, \overset{1}{\overline{l_A}}\right) G(x, \xi) = F\left(\overset{2}{x}, \overset{1}{\overline{\xi - ix^{k+1}\frac{\partial}{\partial x}}}\right) G(x, \xi).$$

We remark that the formula for an adjoint operator is almost self-evident:

$$\left[F\left(\overset{2}{B},\overset{1}{A}\right)\right]^{*}=\bar{F}\left(\overset{1}{B},\overset{2}{A}\right),$$

where the bar stands for complex conjugation (in the operator-valued case it must be replaced by conjugation in the space of coefficients).

2. Now, to construct a regularizer for the operator

$$\hat{H}\left(\overset{2}{r},\overset{1}{\overline{-ir^{k+1}\frac{d}{dr}}}\right),$$

one should solve the equation

$$\hat{H}\left(\overset{2}{x},\overset{1}{\overline{\xi-ix^{k+1}\frac{\partial}{\partial x}}}\right)\hat{R}\left(x,\xi\right)=1. \tag{8}$$

Clearly, solving the latter equation is too a complicated task to be done. Fortunately, one does not have to solve this equation exactly, since for constructing a regularizer it is sufficient to solve it up to a smoothing operator. This can be done in the asymptotic space scale generated by the operator $A=-ir^{k+1}d/dr$. We must also take into account that, due to the results of the paper [1], solutions to the equation

$$\hat{H}\left(\overset{2}{r},\overset{1}{\overline{-ir^{k+1}\frac{d}{dr}}}\right)u=0$$

have exponential type of order k. So, we introduce the space scale E_σ^s with the norm

$$\|u\|_{s,\sigma}^2=\int_0^\infty\int_\Omega \exp\left\{-\frac{2\sigma}{kr^k}\right\}\left|\left(1+A^2+\Delta_\omega\right)^{s/2}\right|^2 u\left(r,\omega\right)\frac{dr\,d\omega}{r^{k+1}}, \tag{9}$$

where Δ_ω is the positive Laplace operator on the manifold Ω (we have taken into account that the operator $\hat{H}$ acts on functions with values in a function space on the manifold Ω). Multiplication by the function $\exp\left(-\sigma/\left(kr^k\right)\right)$ determines an isomorphism between the space scale E_σ^s and the space scale $E^s=E_0^s$ defined by the norm

$$\|u\|_s^2=\int_0^\infty\int_\Omega\left|\left(1+A^2+\Delta_\omega\right)^{s/2}u\left(r,\omega\right)\right|^2\frac{dr\,d\omega}{r^{k+1}};$$

that is, the mapping

$$\exp\left\{-\frac{\sigma}{kr^k}\right\} \,:\, E^s_\sigma \to E^s \tag{10}$$

is an isomorphism. In addition, the following diagram commutes

$$\begin{tikzcd}
E^s_\sigma \arrow[r, "A"] \arrow[d, "e^{-\frac{\sigma}{kr^k}}"'] & E^{s-1}_\sigma \arrow[d, "e^{-\frac{\sigma}{kr^k}}"] \\
E^s \arrow[r, "A + i\sigma"] & E^{s-1}
\end{tikzcd}$$

Hence, the definition of functions of operators (B, A) as well as the investigation of their properties can be performed for $\sigma = 0$, and then all the information obtained can be taken to the space scale E^s_σ with the help of isomorphism (10) provided only that the symbol class consists of analytic functions admitting the shift by $i\sigma$ in the complex plane.

Now, expanding the operator

$$\hat{H}\left(\overset{2}{x}, \overset{1}{\overline{\xi - ix^{k+1}\frac{\partial}{\partial x}}}\right)$$

involved in equation (8) in powers of the operator $-ix^{k+1}\partial/\partial x$ (this can be easily done since this operator commutes with multiplication by ξ), we arrive at the following asymptotic form of equation (8):

$$\left[\hat{H}(x,\xi) + \frac{\partial \hat{H}}{\partial \xi}(x,\xi)\left(-ix^{k+1}\frac{\partial}{\partial x}\right) + \dots\right]\hat{R}(x,\xi) = 1.$$

This can be solved by the usual methods by expanding $\hat{R}(x,\xi)$ in a series of functions homogeneous in ξ with decreasing order of homogeneity.

As can be seen from the above diagram, we have to work with symbols $F(x,\xi)$ that are analytic in ξ and belong to the corresponding symbol class on almost every horizontal line $\{\operatorname{Im}\xi = \text{const}\}$. We allow these symbols to have polar singularities in ξ such that the number of poles in each strip $a \le \operatorname{Im}\xi \le b$ is finite.

1.3 Statement of the problem

Now we can describe the exact statement of the problem. To do this we globalize the function spaces introduced above in the following way. Let χ_j be smooth finite functions in neighborhoods U_j of the singular points m_j $(j = 1, \ldots, N)$ of the manifold M such that $\chi_j \equiv 1$ near m_j and the supports of χ_j do not intersect one another. Then the system of functions

$$\{\chi, \chi_1, \ldots, \chi_N\}$$

with

$$\chi = 1 - \sum_{j=1}^{N} \chi_j$$

form a partition of unity on M. Let us consider the corresponding decomposition

$$f = \chi f + \sum_{j=1}^{N} \chi_j f$$

for any function f on the manifold M. By $E^s_{\sigma,\gamma}(M)$, we denote the Banach space of functions on M determined by the finiteness of the norm

$$\|f\|^2_{s,\sigma,\gamma} = \|\chi f\|^2_s + \sum_{j=0}^{N} \|\chi_j f\|^2_{s,\sigma,\gamma} ;$$

here $\|\cdot\|_s$ is the usual Sobolev norm on the manifold M and $\|\cdot\|_{s,\sigma,\gamma}$ is given by

$$\|u\|_{s,\sigma,\gamma} = \left\|r^{-\gamma(k_j+1)} u\right\|_{s,\sigma}$$

near each cusp point m_j, where $\|\cdot\|_{s,\sigma}$ is the norm in the above introduced space E^s_σ.

The aim of the present paper is, in particular, to prove the finiteness theorem (Fredholm property) for the operator

$$\hat{H} \; : \; E^s_{\sigma,\gamma}(M) \to E^{s-\mu}_{\sigma,\gamma-\mu}(M) .$$

The construction of the corresponding local cusp algebra and the proof of the finiteness theorem appear in Sections 2 and 3.

2 Local cusp algebra

We construct a local cusp algebra in the neighborhood of each cusp point using the special coordinates $(r,\omega) \in [0,r_0] \times \Omega$ described above. We recall that in these coordinates the operator $\hat{H}$ has the form

$$\hat{H} = \sum_{l=0}^{\mu} \hat{A}_l(r) \left(-r^{k+1}\frac{d}{dr}\right)^l \tag{11}$$

up to the inessential factor $r^{-\mu(k+1)}$ (see formula (2) above). Here the $\hat{A}_l(r)$ are smooth differential operators of order $\mu - l$ on the manifold Ω smoothly depending on the parameter $r \in [0,r_0]$.

The algebra will be constructed in certain Hilbert scales. The zeroth space of such a scale is exactly the above introduced space E_σ^0 (see (9)). The latter is a weighted L_2-space on $[0,r_0] \times \Omega$, determined by the norm[2]

$$\|f\|_\sigma^2 = \int_0^{r_0}\int_\Omega e^{-\frac{2\sigma}{kr^k}} |f(r,\omega)|^2 \frac{dr\, d\omega}{r^{k+1}}$$

with a $\sigma \in \mathbf{R}$.

As it was already mentioned in the previous section, we reduce the problem to the case $\sigma = 0$. This is accomplished by considering the function $e^{-\sigma/kr^k} f(r,\omega)$ instead of $f(r,\omega)$. Since

$$-r^{k+1}\frac{d}{dr}e^{\frac{\sigma}{kr^k}} = e^{\frac{\sigma}{kr^k}}\left(-r^{k+1}\frac{d}{dr}+\sigma\right),$$

one sees that the transformed operator has the same structure

$$\hat{H}' = \sum_{l=0}^{\mu} \hat{A}_l(r)\left(-r^{k+1}\frac{d}{dr}+\sigma\right)^l = \sum_{l=0}^{\mu} \hat{A}'_l(r)\left(-r^{k+1}\frac{d}{dr}\right)^l,$$

and the corresponding family is

$$\hat{H}'(r,p) = \sum_{l=0}^{\mu} \hat{A}_l(r)(p+\sigma)^l = \sum_{l=0}^{\mu} \hat{A}'_l(r)\,p^l. \tag{12}$$

[2]Since all the constructions of the local algebra will be carried out near one of the singular points of the manifold M, we shall omit the corresponding index j. So, we write k instead of k_j, Ω instead of Ω_j, and so on.

We remark that the ellipticity requirement is fulfilled for the transformed operator whenever it is fulfilled for the initial one. In what follows, we assume that this requirement is fulfilled and we omit the primes over all operators in question (such that $\hat{H}$, $\hat{H}(r,p)$, $A_l(r)$ and so on).

Let us impose a requirement on the choice of the number σ. First, we note that, due to the ellipticity condition above, the operator family (3) is meromorphically invertible in the complex plane $\mathbf{C}$ with the coordinate p. We suppose that no pole of the family $\hat{H}(r,p)$ lies on the vertical line

$$L_\sigma = \{p \in \mathbf{C} |\ \mathrm{Re}\, p = \sigma\},$$

and, what is more, the norm

$$\left\| \left(1+p^2+\Delta_\omega\right)^{\mu/2} \left[\hat{H}(r,p)\right]^{-1} \right\|_{H^s(\Omega)\to H^s(\Omega)} \tag{13}$$

is bounded uniformly in (r,p), $p \in L_\sigma$. For the transformed family (12) this means that no pole of the inverse family lies on the imaginary axis in the plane $\mathbf{C}$ with the corresponding estimate. Clearly, this must be fulfilled for sufficiently small r_0.

So, in this section we carry out the definition of the local algebra for $\sigma = 0$. Clearly, this definition can be rewritten for the arbitrary values of σ with the help of the isomorphism (unitary for $s = 0$)

$$\exp\left\{-\frac{\sigma}{kr^k}\right\} \ :\ E^s_\sigma \to E^s_0$$

described in the previous section.

Under the above described transformation, the zeroth space of our future space scale becomes

$$E^0 = L_2\left((0,r_0]\times\Omega,\ r^{-(k+1)}\right) \tag{14}$$

with the norm

$$\|f\|_0^2 = \int_0^{r_0}\int_\Omega |f(r,\omega)|^2 \frac{dr\,d\omega}{r^{k+1}}.$$

We wish to construct the regularizer in the space scale generated by the operator[3] $-r^{k+1}d/dr$. This means that the remainder must lie in the domain of the operator $\left(-r^{k+1}d/dr\right)^N$ for sufficiently large N. We cannot write down the expressions for the norms in all spaces of the scale at the moment, since the operator $-r^{k+1}d/dr$ must be modified for purely technical reasons; this will also affect the function spaces. The full description of the scale in question will be given in the following subsection.

[3]In fact, we also need some decay conditions as $r \to 0$, but we do not include these conditions in the definition of the scale. The desired estimates will be obtained as a by-product in constructing the regularizer.

2.1 Construction of the local algebra

We see that the operator $\hat{H}$ given by (11) is a function of the operators r and $-r^{k+1}d/dr$, and the asymptotics must be constructed in the scale generated by the second of these operators. So, it is natural to search for the regularizer in the form of a function of these two operators as well. The difficulty is that the operator $-r^{k+1}d/dr$ is not a generator in the scale in question.

To explain this phenomenon we recall ([8]) that for the definition of functions of two noncommutative operators one should use the corresponding one-parameter groups. Let us compute the group U_t corresponding to the operator $-r^{k+1}d/dr$. Let us search for this group in the form

$$(U_t f)(r) = f(R(r,t)).$$

Then, since one must have

$$\frac{d}{dt}(U_t f)(r) = U_t\left(-r^{k+1}\frac{df}{dr}\right)(r) = \left(-r^{k+1}\frac{df}{dr}\right)(R(r,t))$$

we obtain the following equation for the function $R(r,t)$:

$$\frac{\partial R(r,t)}{\partial t} = -R^{k+1}(r,t).$$

The solution of this equation is

$$R(r,t) = \frac{r}{\left[1+kr^k t\right]^{1/k}},$$

and one can see that this group is local, that is, not determined on the whole line $\mathbf{R}$. This happens due to the fact that the trajectories of the corresponding vector field can come to infinity by finite time. One can check that this phenomenon takes place due to the growth of the coefficient r^{k+1} of the considered operator $-r^{k+1}d/dr$ at infinity. Since we are interested in the investigation of the behavior of solutions near the origin, this problem is purely technical, and we can change our operator outside the segment $[0,r_0]$ in the arbitrary way. This is exactly what is done in the sequel.

So, we modify the operators in question in the following way. First, we shall consider functions on the entire semiaxis $r \in (0,+\infty)$.

Let $\varphi(r)$ be a smooth function of the variable $r \in \mathbf{R}$ such that the following conditions are fulfilled:

$$\varphi(r) = \begin{cases} r^{k+1}, & |r| \le r_0, \\ (-1)^{k+1}C, & r \le -2r_0, \\ C, & r \ge 2r_0, \end{cases}$$

with a constant C (see Figure 2).

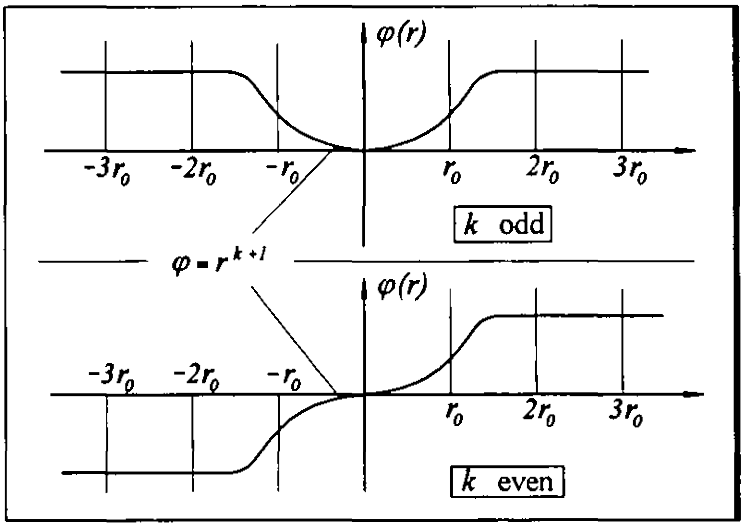

Figure 2. Graph of the function $\varphi(r)$

Now we introduce the operators[4]

$$\begin{aligned} A &= -i\varphi(r)\frac{d}{dr}, && (15) \\ B &= r && (16) \end{aligned}$$

(note that the operator A coincides with the initial operator $-ir^{k+1}d/dr$ near the origin). Clearly, these functions satisfy the following (nonlinear) relations

$$[A,B] = -i\varphi(B), \tag{17}$$

where the square brackets denote, as usual, the commutators of the corresponding operators.

Next, let us extend the function[5] $\hat{H}(r,p)$ to all values of $r \in (-\infty,+\infty)$ so that $\hat{H}(r,p)$ is smooth and $\hat{H}^{-1}(r,p)$ has no poles for real p; in addition we assume that $\hat{H}(r,p)$ is independent of r for sufficiently large $|r|$. Then for $r \in [0,r_0]$ we clearly have

$$\hat{H} = \sum_{l=0}^{\mu} \hat{A}_l(B)(-iA)^l = \hat{H}'\left(\overset{2}{B},\overset{1}{A}\right).$$

[4]We have inserted the factor i in (15) to deal with self-adjoint operators and symbols that are functions of a real variable.

[5]We assume that the change of symbol $H(r,p) \mapsto H(r,-ip)$ corresponding to the above change of operator A is already made.

Now let us give a precise definition of function spaces in which our operators will be considered. Denote by

$$E^0 = L^2\left((0,+\infty)\times\Omega,\varphi^{-1}(r)\right)$$

the space of functions $f(r,\omega)$ with the finite norm

$$\|f\|_0^2 = \int_0^\infty\int_\Omega |f(r,\omega)|^2 \frac{dr\,d\omega}{\varphi(r)}.$$

Note that in a neighborhood of zero $\varphi^{-1}(r) = r^{-(k+1)}$, so for functions supported in such a neighborhood the latter space coincides with the space (14) defined above.

To define all other spaces of our future space scale, we need the following assertion:

Lemma 1 *The operators A and B are self-adjoint in E^0.*

Proof. The operator B generates the one-parameter group given by the multiplication by $\exp\{irt\}$ which is obviously unitary.

Next, for each $r \in (0,+\infty)$ the solution $R(r,t)$ of the Cauchy problem

$$\dot{R} = \varphi(R), \quad R|_{t=0} = r$$

is defined on the entire axis $t \in (-\infty,+\infty)$. Consequently, the operator A generates the one-parameter group of translations

$$\left(e^{itA}f\right)(r,\omega) = f(R(r,t),\omega). \tag{18}$$

One can show that this group is unitary either by straightforward computations or by verifying (again by straightforward computations) that A is symmetric on the dense subset $C_0^\infty \subset E^0$. □

Note that the function $\varphi(B)$ in (17) is now well-defined as a smooth function of a self-adjoint operator B.

Let us now introduce the Hilbert space E^s as follows: E^s is the space of functions $f(r,\omega)$ with finite norm[6]

$$\|f\|_s^2 = \int_0^\infty\int_\Omega \left|\left(1+A^2+\Delta_\omega\right)^{s/2} f(r,\omega)\right|^2 \frac{dr\,d\omega}{\varphi(r)},$$

[6]Since we have reduced our problem to the case $\sigma = 0$, we omit the corresponding index.

where $s \in \mathbf{Z}_+$, and Δ_ω is the positive Beltrami-Laplace operator on Ω. Since the operator A is self-adjoint in E^0, this norm is well-defined for any $s \in \mathbf{R}$; for purely technical reasons we shall use it only for $s \in \mathbf{Z}$.

Later on, it is easy to see that the scalar product

$$(f,g)_0 = \int_0^\infty \int_\Omega f(r,\omega)\,\bar{g}(r,\omega)\,\frac{dr d\omega}{\varphi(r)}$$

establishes the isomorphism

$$(E^s)^* \simeq E^{-s}.$$

We shall consider also the operators A and B in the two-parameter Hilbert space scale $F^{s,l}$. This scale is defined by the finiteness of the norm

$$\|u\| = \int_{-\infty}^{+\infty} \int_\Omega \left| \left(1+r^2\right)^{l/2} \left(1 + A^2 + \Delta_\omega\right)^{s/2} u(r,\omega) \right|^2 \frac{dr\, d\omega}{\varphi(r)}.$$

We remark that this scale coincides with the above scale E^s for $l = 0$: $E^s = F^{s,0}$.

Lemma 2 *The following assertions are valid:*

i) *The operators A and B are bounded in the scale $F^{s,l}$; more precisely, the operators*

$$\begin{aligned} A &: F^{s,l} \to F^{s-1,l}, \\ B &: F^{s,l} \to F^{s,l-1} \end{aligned}$$

are bounded for any s, l.

ii) *The one-parameter groups* $\exp\{itA\}$ *and* $\exp\{itB\}$ *are bounded in each $F^{s,l}$, more precisely,*

$$\|\exp\{itA\}\|_{F^{s,l} \to F^{s,l}} \le C_{sl}(1+|t|)^{|l|}$$

and

$$\|\exp\{itB\}\|_{F^{s,l} \to F^{s,l}} \le C_{sl}(1+|t|)^{|s|} \tag{19}$$

for each s, l with a positive constant C_{sl}.

Proof. i) Since

$$[A,B] = -i\varphi(B)$$

is a bounded operator, and all higher commutators $[[A,B],B]$, ... vanish, we see that the operator A is an operator of first order in the scale $F^{s,l}$ with respect to the first parameter. The assertion on the boundedness of the operator B is quite similar.

ii) Consider first the group $\exp\{itB\}$. The proof of estimate (19) for positive s goes by induction in s. For $s=0$, the statement is true; let us show how the induction step $s=0 \implies s=1$ is carried out (the subsequent calculations are quite evident). Since $B=r$, $A=-i\varphi(r)d/dr$, one has

$$A\exp\{itB\} = -i\varphi(r)\frac{\partial}{\partial r}\exp\{itr\} = t\varphi(B)\exp\{itB\} + \exp\{itB\}A.$$

Hence,

$$\begin{aligned}\|A\exp\{itB\}u\|_{0,l} &\leq \|\exp\{itB\}Au\|_{0,l} + |t|\,\|\varphi(B)\exp\{itB\}u\|_{0,l}\\ &\leq \|Au\|_{0,l} + \text{const}\,|t|\,\|u\|_{0,l}\\ &\leq C(1+|t|)\,\|u\|_{1,l}\end{aligned}$$

for $u\in C_0^\infty$ and, hence, for all $u\in F^{1,l}$, as required. For negative s the desired estimate follows by duality.

Now, let us turn our mind to the group $\exp\{itA\}$. Similar to the above considered case, it suffices to estimate the norm

$$\|B\exp\{itA\}\|_{s,0}$$

via the 1-norm $\|u\|_{s,1}$ of the function u. We use the direct expression for the group $\exp\{itA\}$ given by (18). So,

$$B\exp\{itA\} = ru\,(R(r,t),\omega)\,.$$

Since $\varphi(r)$ is bounded, we have

$$|R(r,t)| \leq r + C\,|t|$$

for any r and t with a positive constant C. Using this estimate, it is easy to see that

$$\|B\exp\{itA\}\|_{s,0} \leq C\,(1+|t|)\,\|u\|_{s,1}\,.$$

The induction on l completes the proof. □

Now, let us introduce the space of operator-valued symbols. In what follows we denote by (x,ξ) the coordinates corresponding to operators A and B in the symbol space.

Definition 2 By $S^\infty = S^\infty\left(\mathbf{R}^2, H^s(\Omega)\right)$, we denote the space of functions $f(x,\xi)$, $(x,\xi) \in \mathbf{R}^2$ with values in the Sobolev scale $\{H^s(\Omega)\}$ which satisfy the following condition: there exists an integer $m = m(f)$ (depending on f) such that for any n and s the quantity

$$C_{ns}(f) = \sum_{\alpha+\beta\leq n} \sup_{(x,\xi)} \left\| \left(1 + x^2 + \xi^2 + \Delta_\omega\right)^{-m/2} \frac{\partial^{\alpha+\beta} f}{\partial x^\alpha \partial \xi^\beta}(x,\xi) \right\|_{H^s \to H^s} \tag{20}$$

is finite.

Remark 1 The introduced symbol class is quite natural in the considered situation since the operators A and B are generators of groups of tempered growth in the considered scale (see Theorem IV.3 in the book [8]).

Clearly,

$$S^\infty = \bigcup_m S^m \equiv \bigcup_m \left(\bigcap_{n,s} S^m_{ns} \right),$$

where S^m_{ns} is the space of functions with the finite right-hand side in (20). Therefore, we can equip S^∞ with the corresponding convergence closely following the procedure described in Example IV.1 in [8, p. 252]. Note that

$$\hat{H}(x,\xi) \in S^\mu\left(\mathbf{R}^2, H^s(\Omega)\right).$$

Now, it follows from the description of S^∞-generators given in [8, Subsection IV.2.3] that A and B are S^∞-generators (and, hence, functions $\hat{F}\left(\overset{2}{B}, \overset{1}{A}\right)$ are well-defined) in the scale $\left\{F^{s,l}\right\}$ (the fact that our symbols are operator-valued leads to no additional difficulties).

Let us consider now operators $\hat{F}\left(\overset{2}{B}, \overset{1}{A}\right)$ in the space scale E^s. The following assertion takes place:

Theorem 1 *Let $\hat{F}(x,\xi)$ be an operator-valued symbol such that the quantity*

$$\left\| \left(1 + \xi^2 + \Delta_\omega\right)^{-\frac{m-|\alpha|}{2}} \frac{\partial^{\alpha+\beta} \hat{F}(x,\xi)}{\partial x^\alpha \partial \xi^\beta} \right\|_{H^s \to H^s} \tag{21}$$

is bounded uniformly in (x,ξ) for any α, β, s with some fixed value of m. Then the operator

$$\hat{F} = \hat{F}\left(\overset{2}{B}, \overset{1}{A}\right) : E^s \to E^{s-m} \tag{22}$$

is bounded for all values of s.

Proof. Using Theorem IV.5 of [8, p. 279], one can see that the operator

$$\hat{F}\left(\overset{2}{B},\overset{1}{A}\right)\ :\ E_0 \to E_0$$

is bounded provided that $m < -1$. It is also easy to see that for such values of m this operator is bounded in the whole scale E^s. Actually, this fact can be proved by induction on s; we shall illustrate only the first induction step (all subsequent steps are performed in a similar way). To prove the boundedness of the operator

$$\hat{F}\left(\overset{2}{B},\overset{1}{A}\right)\ :\ E_1 \to E_1 \tag{23}$$

it is sufficient to estimate the norm $\left\|A\hat{F}u\right\|_0$ via $\left\|\hat{F}Au\right\|_0$ and $\|u\|_0$. This can be done with the help of the representation operators (see [8, p. 97]; the computation of these operators in the considered case is given in the next subsection)

$$\begin{aligned} l_A &= \xi - i\varphi(x)\frac{\partial}{\partial x}, \\ l_B &= x. \end{aligned}$$

One has[7]

$$A\left[\!\left[\hat{F}\left(\overset{2}{B},\overset{1}{A}\right)\right]\!\right] = \hat{G}\left(\overset{2}{B},\overset{1}{A}\right),$$

where

$$\hat{G}(x,\xi) = l_A\hat{F}(x,\xi) = \xi\hat{F}(x,\xi) - i\varphi(x)\frac{\partial\hat{F}(x,\xi)}{\partial x}.$$

Hence,

$$\left\|A\hat{F}u\right\|_0 = \left\|\hat{F}Au + \hat{H}\left(\overset{2}{B},\overset{1}{A}\right)u\right\| \le \left\|\hat{F}Au\right\|_0 + \left\|\hat{H}\left(\overset{2}{B},\overset{1}{A}\right)\right\|_0$$

with

$$\hat{H}(x,\xi) = -i\varphi(x)\frac{\partial\hat{F}(x,\xi)}{\partial x}.$$

Since the function $h(x,\xi)$ satisfies conditions of Theorem 1, we have

$$\left\|\hat{H}\left(\overset{2}{B},\overset{1}{A}\right)\right\|_0 \le C\|u\|_0,$$

[7] By $[\![\cdot]\!]$, we denote the so-called autonomous brackets which delimit the range of action of the Feunmann indices (see [8, p. 15]).

and, finally,

$$\left\|A\hat{F}u\right\|_0 \leq \left\|\hat{F}Au\right\|_0 + C\left\|u\right\|_0 .$$

This completes the proof of the boundedness of operator (23). Similar, one can prove that the operator

$$\hat{F}\left(\overset{2}{B},\overset{1}{A}\right) \ : \ E^s \ \rightarrow \ E^{s-m}$$

is bounded for any s.

We remark that the estimate (22) stated in the theorem is not yet proved, since there is a loss of $1+\varepsilon$ in the order of the operator. However, order-exact estimate (22) can be proved now quite similar to the proof of Theorem IV.6 in [8, p. 282]; there is no need to reproduce this purely technical proof. □

Remark 2 Clearly, all the above stated results are valid also in the space scale E^s_σ defined by the norm

$$\|f\|^2_{\sigma,s} = \int\limits_{-\infty}^{+\infty}\int\limits_{\Omega} \exp\left\{\sigma\psi(r)\right\}\left|\left(1+A^2+\Delta_\omega\right)^{s/2} f\left(r,\omega\right)\right|^2 \frac{dr\,d\omega}{\varphi(r)},$$

where the function ψ is defined in such a way that $\psi'(r) = \varphi^{-1}(r)$. As it was explained in the beginning of this section, the presence of the weight $\exp\left\{\sigma\psi(r)\right\}$ leads only to the shift in the ξ-plane.

Let now Smbl(k) be a space of symbols $\hat{H}\left(r,p\right)$ such that:

1) $\hat{H}\left(r,p\right)$ is meromorphic in p in the whole plane $\mathbf{C}$ with the coordinate p, and infinitely smooth in r up to the origin;

2) $\hat{H}\left(r,p\right)$ satisfies estimate (21) on each line $L_\sigma = \{\operatorname{Re} p = \sigma\}$ which does not contain poles of this function.

As it will be shown in the next subsection, the set Op(k) of operators $\hat{H}\left(\overset{2}{B},\overset{1}{A}\right)$ with $\hat{H}\left(r,p\right) \in$ Smbl$\left(k\right)$ form an algebra. We introduce the following definition:

Definition 3 The algebra Op(k) is called the *k-th order local cusp algebra.*

2.2 Left ordered representation of the local algebra

Now, in order to find the right regularizer for the operator in question, we must solve asymptotically the equation

$$\left[\!\left[\hat{H}\left(\overset{2}{B},\overset{1}{A}\right)\right]\!\right]\circ\left[\!\left[\hat{R}\left(\overset{2}{B},\overset{1}{A}\right)\right]\!\right]=1,$$

where $\hat{R}(x,\xi)$ is the symbol of the regularizer. To this end, we must compute the symbol of the operator on the left in the latter relation. First, let us evaluate the operators of the left ordered representation of the local cusp algebra generators $\overset{1}{A}$, $\overset{2}{B}$. This can be accomplished by a standard procedure (see the examples in [8, Section II.2]). We have

$$B\left[\!\left[R\left(\overset{2}{B},\overset{1}{A}\right)\right]\!\right]=\overset{3}{B}R\left(\overset{2}{B},\overset{1}{A}\right)=\overset{2}{B}R\left(\overset{2}{B},\overset{1}{A}\right),$$

and

$$\begin{aligned}A\left[\!\left[R\left(\overset{2}{B},\overset{1}{A}\right)\right]\!\right] &= \overset{3}{A}R\left(\overset{2}{B},\overset{1}{A}\right)=\overset{3}{A}R\left(\overset{4}{B},\overset{1}{A}\right)\\ &\quad+\left(\overset{2}{B}-\overset{4}{B}\right)\overset{3}{A}\frac{\delta R}{\delta x}\left(\overset{2}{B},\overset{4}{B},\overset{1}{A}\right)\\ &= \overset{1}{A}R\left(\overset{2}{B},\overset{1}{A}\right)+\overset{3}{\overline{[A,B]}}\frac{\delta R}{\delta x}\left(\overset{2}{B},\overset{4}{B},\overset{1}{A}\right)\\ &= \overset{1}{A}R\left(\overset{2}{B},\overset{1}{A}\right)-i\varphi\left(\overset{2}{B}\right)\frac{\partial R}{\partial x}\left(\overset{2}{B},\overset{1}{A}\right).\end{aligned}$$

Hence, the operators of the left ordered representation are

$$\begin{aligned} l_A &= \xi-i\varphi(x)\frac{\partial}{\partial x},\\ l_B &= x. \end{aligned}\tag{24}$$

Lemma 3 *Operators* (24) *are* S^∞*-generators.*

Proof. The assertion about l_B is clear. Let us consider l_A. We have

$$e^{il_At}\hat{F}(y,\xi)=e^{it\xi}\hat{F}(x(y,t),\xi),$$

where $x(y,t)$ is the solution to the following Cauchy problem

$$\dot{x} = \varphi(x), \ x|_{t=0} = y. \tag{25}$$

It is the behavior of the solution to (25) as $t \to \pm\infty$ that is of importance. If $y = 0$, then $x(y,t) = 0$ and everything is all right. To be definite, let us consider the case $x > 0$ (the opposite case can be considered quite in the same way; note that, intuitively, this case is inessential since the spectrum of our problem lies in $\{x > 0\}$). Then for $t \to +\infty$, beginning with some $t_0 = t_0(y)$, we have

$$x(y,t) = x(y,t_0) + \text{const } (t - t_0),$$

so, only the behavior of the solution as $t \to -\infty$ is essential. Clearly, to investigate this behavior it suffices to consider small values of x where $\varphi(x) = x^{k+1}$. Then the solution to (25) has the form

$$x(y,t) = \left\{\frac{y^k}{1 - kty^k}\right\}^{1/k}, \ t \to -\infty.$$

So, we see that the derivatives

$$\frac{\partial^{\alpha+\beta} x(y,t)}{\partial y^\alpha \partial t^\beta}$$

grow at most polynomially in y and t and that the differentiation by y does not increase the growth in y. Now, we see that l_A generates a group of tempered growth in S^∞, and, hence, this operator is an S^∞-generator in S^∞ (see [8, p. 268] for details). □

We arrive at the following standard theorem of the method of ordered representation.

Theorem 2 *Suppose that $\hat{H}(x,\xi)$ and $\hat{R}(x,\xi)$ are two functions from $S^\infty(H^s(\Omega))$. Then the symbol $\hat{K}(x,\xi)$ of the composition $\left[\!\left[\hat{H}\left(\overset{2}{\hat{B}}, \overset{1}{\hat{A}}\right)\right]\!\right] \circ \left[\!\left[\hat{R}\left(\overset{2}{\hat{B}}, \overset{1}{\hat{A}}\right)\right]\!\right]$ is equal to*

$$\hat{K}(x,\xi) = \hat{H}\left(\overset{2}{l_B}, \overset{1}{l_A}\right)\hat{R}(x,\xi).$$

Proof. This is a special case of Theorem II.1 in [8, p. 98]. □

Remark 3 From the computational viewpoint, we are interested in the computation of the symbol of the composition only for small values of x. If $\hat{H}(x,\xi)$ is a polynomial in ξ, then the computation can be done directly after replacing the operator A by the operator $-ir^{k+1}d/dr$.

Actually, let us consider the composition

$$
\left[\!\!\left[\hat H\left(\overset{2}{r}, \overset{1}{\overline{-ir^{k+1}\frac{d}{dr}}}\right)\right]\!\!\right] \circ \left[\!\!\left[\hat R\left(\overset{2}{r}, \overset{1}{\overline{-ir^{k+1}\frac{d}{dr}}}\right)\right]\!\!\right] u(r)
$$

$$
= \sum_{l=0}^{\mu} \hat A_l(r)\left(-ir^{k+1}\frac{d}{dr}\right)^l \left[\!\!\left[\hat R\left(\overset{2}{r}, \overset{1}{\overline{-ir^{k+1}\frac{d}{dr}}}\right)\right]\!\!\right] u(r). \tag{26}
$$

It is clear that, due to the Leibnitz differentiation formula, each operator $-ir^{k+1}d/dr$ outside the autonomous brackets on the right in (26) acts at the coefficients of the operator $\hat R$ as well as at the function $u(r)$. Namely,

$$
\left(-ir^{k+1}\frac{d}{dr}\right)\left[\!\!\left[\hat R\left(\overset{2}{r}, \overset{1}{\overline{-ir^{k+1}\frac{d}{dr}}}\right)\right]\!\!\right] u(r) =
$$

$$
\left\{\left(-ir^{k+1}\frac{d\hat R}{dr}\right)\left(\overset{2}{r}, \overset{1}{\overline{-ir^{k+1}\frac{d}{dr}}}\right) + \overset{1}{\overline{\left(-ir^{k+1}\frac{d}{dr}\right)}}\,\hat R\left(\overset{2}{r}, \overset{1}{\overline{-ir^{k+1}\frac{d}{dr}}}\right)\right\} u(r) =
$$

$$
\left\{\left(\xi - ix^{k+1}\frac{d}{dx}\right)\hat R(x,\xi)\Big|_{x=\overset{2}{r},\,\xi=\overset{1}{\overline{-ir^{k+1}d/dr}}}\right\} u(r).
$$

Using this formula for each factor $-ir^{k+1}d/dr$ on the right in (26), we obtain

$$
\left[\!\!\left[\hat H\left(\overset{2}{r}, \overset{1}{\overline{-ir^{k+1}\frac{d}{dr}}}\right)\right]\!\!\right] \circ \left[\!\!\left[\hat R\left(\overset{2}{r}, \overset{1}{\overline{-ir^{k+1}\frac{d}{dr}}}\right)\right]\!\!\right] u(r) =
$$

$$
\left\{\sum_{l=0}^{\mu} \hat A_l(x)\left(\xi - ix^{k+1}\frac{d}{dx}\right)^l \hat R(x,\xi)\Big|_{x=\overset{2}{r},\,\xi=\overset{1}{\overline{-ir^{k+1}d/dr}}}\right\} u(r) =
$$

$$
\left\{\hat H\left(\overset{2}{l_B}, \overset{1}{l_A}\right)\hat R(x,\xi)\Big|_{x=\overset{2}{r},\,\xi=\overset{1}{\overline{-ir^{k+1}d/dr}}}\right\} u(r),
$$

as required.

The assertion of Theorem 2 shows that the set of operators $\hat F\left(\overset{2}{B}, \overset{1}{A}\right)$ with symbols from $\mathrm{Smbl}(k)$ forms an algebra. This is exactly the local cusp algebra we wanted to construct. This algebra is a $*$-algebra in the sense of the following affirmation:

Theorem 3 *Let $\hat F(x,\xi)$ be a symbol subject to the conditions of Theorem* 1. *Then* *the adjoint to the operator*

$$\hat{F}\left(\overset{2}{B},\overset{1}{A}\right) : E^s \to E^{s-m}$$

is given by the formula

$$\left[\hat{F}\left(\overset{2}{B},\overset{1}{A}\right)\right]^* = \hat{F}^*\left(\overset{1}{B},\overset{2}{A}\right) : E^{-s+m} \to E^{-s},$$

where $\hat{F}^*(x,\xi)$ *is an adjoint symbol (we recall that* $\hat{F}(x,\xi)$ *is an operator-valued function). Later on, the asymptotic formula*

$$\operatorname{smbl}\left\{\left[\hat{F}\left(\overset{1}{B},\overset{2}{A}\right)\right]^*\right\} = \sum_{j=0}^{\infty}\frac{1}{j!}\left(-i\varphi(x)\frac{\partial}{\partial x}\right)^j\left(\frac{\partial}{\partial \xi}\right)^j \hat{F}^*(x,\xi) \tag{27}$$

is valid.

Proof. It suffices to prove only formula (27) since all other affirmations of the theorem are evident. Due to the *permutation index formula* (see [8, p. 61]), we have

$$\begin{aligned}
\hat{F}^*\left(\overset{1}{B},\overset{2}{A}\right) &= \hat{F}^*\left(\overset{2}{B},\overset{1}{A}\right) + \overset{3}{\overline{[A,B]}}\,\frac{\delta^2\hat{F}^*}{\delta x\delta\xi}\left(\overset{4}{B},\overset{2}{B},\overset{5}{A},\overset{1}{A}\right)\\
&= \hat{F}^*\left(\overset{2}{B},\overset{1}{A}\right) - i\,\overset{3}{\overline{\varphi\left(\overset{2}{B}\right)}}\,\frac{\delta^2\hat{F}^*}{\delta x\delta\xi}\left(\overset{4}{B},\overset{2}{B},\overset{5}{A},\overset{1}{A}\right)\\
&= \hat{F}^*\left(\overset{2}{B},\overset{1}{A}\right) - i\varphi\left(\overset{2}{B}\right)\frac{\partial}{\partial x}\frac{\delta\hat{F}^*}{\delta\xi}\left(\overset{2}{B},\overset{3}{A},\overset{1}{A}\right).
\end{aligned}$$

Similar, commuting the operators $\overset{2}{B}$ and $\overset{3}{A}$ in the latter formula we arrive at the result

$$\hat{F}^*\left(\overset{1}{B},\overset{2}{A}\right) = \hat{F}^*\left(\overset{2}{B},\overset{1}{A}\right) - i\varphi\left(\overset{2}{B}\right)\frac{\partial^2\hat{F}^*}{\partial x\partial\xi}\left(\overset{2}{B},\overset{1}{A}\right) + \dots,$$

where by dots we have denoted the difference derivatives of higher order. Continuation of this process leads to the proof of the desired formula. □

2.3 Construction of right regularizer

As follows from Theorem 2, to obtain a right inverse for $\hat{H}\left(\overset{2}{B},\overset{1}{A}\right)$ one has to solve the equation

$$\hat{H}\left(\overset{2}{x},\overset{1}{\overline{\xi - i\varphi\left(x\right)\frac{\partial}{\partial x}}}\right)\hat{R}\left(x,\xi\right) = 1.$$

To obtain a right regularizer, we must solve this equation asymptotically. Intuitively, since the asymptotics for operators must be "in powers of A", the asymptotics for symbols must be "in powers of ξ". Of course, this plausible reasoning must be justified, for which we must estimate the remainder.

We use the following properties of the symbol $\hat{H}\left(x,\xi\right)$: this function is a polynomial of order μ in ξ, and, moreover, the quantity

$$\left\|\left(1+\xi^2+\Delta_\omega\right)^{(-\mu+\beta)/2}\frac{\partial^{\alpha+\beta}\hat{H}}{\partial x^\alpha\partial\xi^\beta}\right\|_{H^s(\Omega)\to H^s(\Omega)}$$

is bounded uniformly in (x,ξ) for any s, α, and β.

Now, the construction of the regularizer goes as follows. Note that, since

$$\left[\xi, -i\varphi\left(x\right)\frac{\partial}{\partial x}\right] = 0,$$

we can easily write out the Taylor expansion of the operator

$$\hat{H}\left(\overset{2}{x},\overset{1}{\overline{\xi - i\varphi\left(x\right)\frac{\partial}{\partial x}}}\right)$$

around the point (x,ξ):

$$\hat{H}\left(\overset{2}{x},\overset{1}{\overline{\xi - i\varphi\left(x\right)\frac{\partial}{\partial x}}}\right) = \sum_{j=0}^{\mu}\frac{1}{j!}\frac{\partial^j\hat{H}}{\partial\xi^j}\left(x,\xi\right)\left(-i\varphi\left(x\right)\frac{\partial}{\partial x}\right)^j.$$

Let us search for the symbol of the regularizer in the form

$$\hat{R}^{(K)}\left(x,\xi\right) = \sum_{j=0}^{K}\hat{R}_j\left(x,\xi\right),$$

where the orders of the symbols $\hat{R}_j(x,\xi)$ decrease as j increases. This means that the norm

$$\left\| \left(1+\xi^2+\Delta_\omega\right)^{(-\mu+j+\beta)/2} \frac{\partial^{\alpha+\beta}\hat{R}_j}{\partial x^\alpha \partial \xi^\beta} \right\|_{H^s(\Omega)\to H^s(\Omega)} \tag{28}$$

is bounded uniformly in (x,ξ) for any j, s, α, and β. Then we obtain the system

$$\begin{aligned}
\hat{H}(x,\xi)\,\hat{R}_0(x,\xi) &= 1, \\
\hat{H}(x,\xi)\,\hat{R}_j(x,\xi) &= -\sum_{l=1}^{j} \frac{1}{l!}\frac{\partial^l \hat{H}}{\partial \xi^l}\left(-i\varphi(x)\frac{\partial}{\partial x}\right)^l \hat{R}_{j-l}(x,\xi),\; j\ge 1,
\end{aligned}$$

which can be easily solved inductively:

$$\begin{aligned}
\hat{R}_0(x,\xi) &= \left[\hat{H}(x,\xi)\right]^{-1}, \\
\hat{R}_j(x,\xi) &= -\left[\hat{H}(x,\xi)\right]^{-1} \sum_{l=1}^{j} \frac{1}{l!}\frac{\partial^l \hat{H}}{\partial \xi^l}\left(-i\varphi(x)\frac{\partial}{\partial x}\right)^l \hat{R}_{j-l}(x,\xi),\; j\ge 1.
\end{aligned}$$

Since the norm

$$\left\| \left(1+\xi^2+\Delta_\omega\right)^{\mu/2} \left[\hat{H}(x,\xi)\right]^{-1} \right\|_{H^s(\Omega)\to H^s(\Omega)} < \infty$$

is bounded uniformly in (x,ξ) (see formula (13) above), the boundedness of the norm (28) for the operator $\hat{R}_j(x,\xi)$ easily follows by induction. So, we obtain

$$\hat{H}\left(\overset{2}{x}, \overset{1}{\overline{\xi - i\varphi(x)\frac{\partial}{\partial x}}}\right) \hat{R}^{(K)}(x,\xi) = 1 + \hat{Q}_{(K)}(x,\xi),$$

where the symbol $\hat{Q}_{(K)}(x,\xi)$ has the form

$$\hat{Q}_{(K)}(x,\xi) = -\sum_{l=1}^{K} \frac{1}{l!}\frac{\partial^l \hat{H}}{\partial \xi^l}\left(-i\varphi(x)\frac{\partial}{\partial x}\right)^l \hat{R}_{K-l}(x,\xi) \tag{29}$$

and the norm

$$\left\| \left(1+\xi^2+\Delta_\omega\right)^{(-\mu+K+\beta)/2} \frac{\partial^{\alpha+\beta}\hat{Q}_{(K)}(x,\xi)}{\partial x^\alpha \partial \xi^\beta} \right\|_{H^s(\Omega)\to H^s(\Omega)} < \infty$$

is bounded uniformly in (x, ξ) for any N, α, β, s.

From this, it follows that the operator $\hat{Q}_{(K)}\left(\overset{2}{B}, \overset{1}{A}\right)$ is bounded as an operator in spaces

$$\hat{Q}_{(K)}\left(\overset{2}{B}, \overset{1}{A}\right) \;:\; E^{s} \;\to\; E^{s+\mu+K}$$

for any s (the proof can be carried out similar to that of Theorem IV.6 in [8, p. 282]). However, we can even obtain a better result. It can be also proved by induction on j, that each term in the sum on the right in (29) contains exactly K factors $\varphi(x)\frac{\partial}{\partial x}$, arranged throughout the product. Let us consider the behavior of this function as $x \to 0$, where $\varphi(x) = x^{k+1}$. Let us commute all $\varphi(x)$ in the product

$$\varphi(x)\frac{\partial}{\partial x}\cdots\varphi(x)\frac{\partial}{\partial x}\cdots\varphi(x)\frac{\partial}{\partial x}\cdots\varphi(x)\frac{\partial}{\partial x}\cdots \tag{30}$$

to the left and estimate the least power of x thus obtained. The commutation of each $\partial/\partial x$ "kills" x in the first power. Since there are K operators of the form $\partial/\partial x$ involved into the expression in question, the "worst" term (in which all $\partial/\partial x$ were killed by commutation with x) will contain the power[8]

$$x^{K(k+1)} : x^{K-1} = x^{Kk+1}.$$

So,

$$\hat{Q}_{(K)}\left(\overset{2}{B}, \overset{1}{A}\right) = \overset{3}{\overline{B^{Kk+1}}}\,\tilde{Q}_{(K)}\left(\overset{2}{B}, \overset{1}{A}\right),$$

where the operator

$$\tilde{Q}_{(K)}\left(\overset{2}{B}, \overset{1}{A}\right) \;:\; E^{s} \;\to\; E^{s+\mu+K}$$

is bounded for any s.

2.4 Construction of left regularizer

The left regularizer for the equation in question can be constructed with the help of the similar technique. Namely, we search for the left regularizer in the form

$$\hat{L} = \hat{L}\left(\overset{1}{B}, \overset{2}{A}\right)$$

[8] We have taken into account that the first factor $\varphi(x)$ and the last derivative $\partial/\partial x$ in (30) do not take part in the process of "killing" the powers of x.

putting the same operators in the inverse order. Hence, we need to solve the following operator equation:

$$\left[\!\left[\hat{L}\left(\overset{1}{B},\overset{2}{A}\right)\right]\!\right]\circ\left[\!\left[\hat{H}\left(\overset{2}{B},\overset{1}{A}\right)\right]\!\right]=1.$$

Let us find the *right* ordered representation of the tuple $\overset{1}{B},\overset{2}{A}$. We have

$$\left[\!\left[\hat{L}\left(\overset{1}{B},\overset{2}{A}\right)\right]\!\right]B=\hat{L}\left(\overset{1}{B},\overset{2}{A}\right)\overset{1}{B},$$

so that

$$r_B=x.$$

Later on,

$$\begin{aligned}\left[\!\left[\hat{L}\left(\overset{1}{B},\overset{2}{A}\right)\right]\!\right]A &= \hat{L}\left(\overset{1}{B},\overset{2}{A}\right)\overset{0}{A}\\ &= \hat{L}\left(\overset{-1}{B},\overset{2}{A}\right)\overset{0}{A}+\left(\overset{1}{B}-\overset{-1}{B}\right)\overset{0}{A}\frac{\delta\hat{L}}{\delta x}\left(\overset{1}{B},\overset{-1}{B},\overset{2}{A}\right)\\ &= \hat{L}\left(\overset{1}{B},\overset{2}{A}\right)\overset{2}{A}+i\varphi\left(\overset{1}{B}\right)\frac{\partial\hat{L}}{\partial x}\left(\overset{1}{B},\overset{2}{A}\right),\end{aligned}$$

whence

$$r_A=\xi+i\varphi(x)\frac{\partial}{\partial x}.$$

Now, the equation for the symbol $\hat{L}(x,\xi)$ of the left regularizer takes the form

$$\hat{H}\left(\overset{1}{x},\overset{2}{\overline{\xi+i\varphi(x)\frac{\partial}{\partial x}}}\right)\hat{L}(x,\xi)=1.$$

Again we search for the solution of the latter equation in the form of the series

$$\hat{L}(x,\xi)=\hat{L}_0(x,\xi)+\hat{L}_1(x,\xi)+\ldots,$$

thus obtaining

$$\begin{aligned}\hat{H}(x,\xi)\hat{L}_0(x,\xi) &= 1,\\ \hat{H}(x,\xi)\hat{L}_j(x,\xi) &= -\sum_{l=1}^{j}\frac{1}{l!}\left(i\varphi(x)\frac{\partial}{\partial x}\right)^l\left[\frac{\partial^l\hat{H}}{\partial\xi^l}\hat{L}_{j-l}\right](x,\xi),\ j\geq 1\end{aligned}$$

— the most significant difference in this recurrent system compared with that for the left regularizer is that the operator $i\varphi(x)\partial/\partial x$ acts on $\partial^l\hat{H}/\partial\xi^l$ as well.

The rest of the construction makes no difference with that for the right regularizer and we omit it altogether.

3 Construction of global regularizer

In spite of the fact that the construction of a global regularizer is quite standard after we have constructed local regularizers in neighborhoods of all singular points of the manifold M, we shall briefly present this construction for the paper to be self-contained.

First, with the help of a cut-off function χ in a neighborhood of each singular point m_j of the manifold M, we construct a local regularizer in a neighborhood of each cusp point of the manifold M. Namely, the following statement is valid:

Theorem 4 *For any value of K, the operator* [9]

$$\sum_{j=0}^{K} \hat{R}_j \left(\overset{2}{r}, \overset{1}{\overline{-ir^{k+1}\frac{d}{dr}}} \right)$$

is a regularizer for the operator $\hat{H}$ up to the order K in the neighborhood of the corresponding singular point. This means that

$$\hat{H} \sum_{j=0}^{K} \hat{R}_j \left(\overset{2}{r}, \overset{1}{\overline{-ir^{k+1}\frac{d}{dr}}} \right) = \hat{Q}_1 + \hat{Q}_2,$$

where the operator $\hat{Q}_1$ is an operator of order $-K$ in the space scale $E^{s}_{\sigma,\gamma}(M)$:

$$\hat{Q}_1 \; : \; E^{s}_{\sigma,\gamma}(M) \to E^{s+K}_{\sigma,\gamma}(M),$$

and the operator $\hat{Q}_2$ is a pseudodifferential operator of zeroth order on the manifold M which equals 1 on the space of functions concentrated in some neighborhood of the considered singular point, and equals zero on the space of functions with supports outside some larger neighborhood.

[9] More precisely, one should consider the operator

$$\chi(r) \sum_{j=0}^{K} \hat{R}_j \left(\overset{2}{r}, \overset{1}{\overline{-ir^{k+1}\frac{d}{dr}}} \right) \chi(r),$$

where $\chi(r)$ is a cut-off function equal 1 in some neighborhood of the considered singular point and vanishing outside some larger neighborhood. To be short, we omit the corresponding cut-off functions.

Denote by $\hat{R}^j$, $j = 1, \ldots, N$ the above constructed regularizers in neighborhoods of the points m_j, $j = 1, \ldots, N$. Then one has the composition formula

$$\hat{H} \sum_{j=0}^{N} \hat{R}^j = \hat{Q}'_1 + \hat{Q}'_2,$$

where $\hat{Q}'_1$ is an operator of order $-K$ in the space scale $E^s_{\sigma,\gamma}(M)$, and $\hat{Q}'_2$ is a pseudodifferential operator of zero order whose symbol equals 1 in some neighborhood of the set $\{m_1, \ldots m_N\}$ of singular points of the manifold M, and vanishes identically outside some larger neighborhood of this set.

Let us search for the global regularizer for the operator $\hat{H}$ in the form

$$\hat{R} = \sum_{j=0}^{N} \hat{R}^j + \hat{T}, \tag{31}$$

where $\hat{T}$ is a pseudodifferential operator on the manifold M identically vanishing in a neighborhood of the set of its singular points. For the operator $\hat{T}$ we have

$$\hat{H}\hat{R} = \hat{H}\left(\sum_{j=0}^{N} \hat{R}^j + \hat{T}\right) = \hat{Q}'_1 + \hat{Q}'_2 + \hat{H}\hat{T} = 1.$$

We can neglect the operator $\hat{Q}'_1$ since this operator is of arbitrary large negative order, and write down the following equation for the operator $\hat{T}$:

$$\hat{H}\hat{T} = 1 - \hat{Q}'_2.$$

So, the operator $\hat{T}$ can be chosen in the form

$$\hat{T} = \chi \hat{H}^{-1}\left(1 - \hat{Q}'_2\right),$$

where χ is a function on M equal to 1 in a neighborhood of the support of the operator $\left(1 - \hat{Q}'_2\right)$, and vanishing identically near the set of singular points of M and $\hat{H}^{-1}$ is a regularizer for the operator $\hat{H}$ on the smooth part of the manifold M. Clearly, the latter operator is well-defined and, for such a determination of the operator $\hat{T}$, the operator $\hat{R}$ given by (31) is a (right) global regularizer for the operator $\hat{H}$ on the manifold M.

The left regularizer for the operator $\hat{H}$ can be constructed in a quite similar manner.

The existence of the regularizers for the operator $\hat{H}$ together with the boundedness of the corresponding operators leads us to the following statement:

Theorem 5 (finiteness theorem). *Let $\hat{H}$ be an elliptic operator on the manifold M with singularities of the cusp type in the sense of Definition 1. Then the operator*

$$\hat{H} \; : \; E^{s}_{\sigma,\gamma}(M) \to E^{s-m}_{\sigma,\gamma-m}(M)$$

possesses the Fredholm property.

4 Asymptotic expansions of solutions

In this section, we shall prove the resurgent character of solutions to equation

$$\hat{H}u = f \tag{32}$$

provided that the right-hand part f of this equation is infinitely exponentially flat near a singular point of the manifold M. This means that the function f admits the estimate

$$|f(r,\omega)| \leq C_a \exp\left\{-\frac{a}{r^{k_j}}\right\}$$

with a positive constant C_a for any real number a.

Remark 4 The requirement of infinite flatness of the right-hand part can be replaced by the requirement of resurgent character of this function (cf. [11]). We shall not construct the corresponding theory here.

Clearly, investigating the asymptotics of solutions on manifolds with cusps near some singular point of this manifold, one should use the k-Laplace transform where k is the order of the corresponding cusp point. So, the representation of solution to equation (32) will be

$$u = \mathcal{L}_k\left[U(p,\omega)\right] = \frac{1}{2\pi i}\int\limits_{L_\sigma} e^{-\frac{p}{kr^k}} U(p,\omega)\; dp.$$

Now one should investigate the analytic properties of the function

$$U(p,\omega) = \mathcal{B}_k\left[u(r,\omega)\right].$$

As it will be shown below (and as one can guess from the results of the paper [1]) this function will not be a meromorphic one. The only thing one has to verify is that this function will be endlessly continuable[10], that is, that the corresponding solution

[10]Roughly speaking, the function is called to be endlessly continuable, if it has not more than a discrete set of singularities on its Riemannian surface. The exact definitions the reader can find in the book [12].

is a resurgent function in the variable r. If so, the asymptotics of the solution has the form derived in the paper [1] by the authors.

The aim of this section is to prove the following affirmation:

Theorem 6 *In a neighborhood of each singular point of the manifold M any solution to equation* (32) *possesses an endlessly continuable k-Borel transform. In other words, all solutions to equation* (32) *are resurgent functions in the variable r. All the singularities of this Borel transform are contained in the set of singularities of the operator $\hat{H}^{-1}(0,p)$.*

The rest part of this section is devoted to the proof of this theorem.

4.1 Preliminary transformation

Let $u(r)$ be a solution of equation (32) and let m be a singular point of the manifold M.

Consider a cut-off function $\chi(r)$ equal to 1 in a sufficiently small neighborhood of the point m and having its support in some larger neighborhood of this point (we suppose that this last neighborhood does not contain other singular points of M and that the variables (r,ω) can be used in this neighborhood). Denote

$$v(r) = \chi(r)\,u(r). \tag{33}$$

Our aim is to prove that the function $v(r)$ has an endlessly continuable k-Borel transform.

We recall [12] that the k-Borel transform of a function $v(r)$ is defined by the formula

$$V(p) = \mathcal{B}_k[v(r)] = \int_0^{r_0} \exp\left(\frac{p}{kr^k}\right) v(r)\,\frac{dr}{r^{k+1}} \tag{34}$$

(the result of the integration is independent of the choice of the value r_0 up to an entire summand). To examine the properties of the function $V(p)$ we must use equation (32). First, we rewrite this equation with respect to the function $v(r)$ given by (33). Clearly, we have

$$\hat{H}v = \tilde{f},$$

where $\tilde{f}(r)$ possesses all properties of the function $f(r)$ (in particular, it is infinitely flat near the point m), but is concentrated in a neighborhood of the point m. So, we arrive at the equation for the function $v(r)$ of the form[11]

$$\hat{H}\left(\overset{1}{r}, \overset{2}{\overline{-r^{k+1}\frac{d}{dr}}}\right) v(r) = \sum_{l=0}^{\mu}\left(-r^{k+1}\frac{d}{dr}\right)^{l} \hat{A}(r)\, v(r) = \tilde{f}(r)$$

(we have omitted the inessential factor $r^{-\mu(k+1)}$).

Expanding the coefficients of the operator $\hat{H}$ in powers of r up to the first order, we obtain the equation

$$\left[\hat{H}_0\left(-r^{k+1}\frac{d}{dr}\right) + \overset{1}{r}\,\hat{H}\left(\overset{1}{r}, \overset{2}{\overline{-r^{k+1}\frac{d}{dr}}}\right)\right] v(r) = \tilde{f}(r). \tag{35}$$

The k-Borel transform of this equation will be a starting point of our proof. However, before the application of the k-Borel transform to (35) we shall investigate the properties of function (34) which follow from the results of the previous sections.

4.2 A priori properties of the Borel transform of solution

The following affirmation is valid[12]:

Lemma 4 *Let $v(r)$ be an element of the space $E_\sigma(\mathbf{R}_+)$. Then the function $V(p) = \mathcal{B}_k[v(r)]$ is holomorphic in the half-plane $\{\operatorname{Re} p < -\sigma\}$. Moreover, in this half-plane the estimate*

$$|V(p)| \le C_\varepsilon \exp\left\{\frac{\operatorname{Re} p}{k r_0^k}\right\}$$

is valid in each closed half-plane $\{\operatorname{Re} p \le -\sigma - \varepsilon\}$ with a positive constant C_ε. Here r_0 is a number such that $v(r) \equiv 0$ for $r \ge r_0$.

---

[11] For purely technical reasons we have written down the equation for the function $v(r)$ using the inverse order of operators. Clearly, the coefficients $A_j(r)$ differ from those involved into the initial equation, but all their properties are just the same. Moreover, the operator $\hat{H}(0, -ir^{k+1}d/dr)$ is the same as for the initial operator.

[12] We formulate this result for a scalar-valued function $v(r)$. Nevertheless, the same result is valid for functions with values in a Banach space as well.

Proof. Let $\{\operatorname{Re} p \leq -\sigma - \varepsilon\}$. Then one has

$$\begin{aligned} |V(p)| &\leq \int_0^{r_0} \left| \exp\left\{\frac{p}{kr^k}\right\} v(r) \right| \frac{dr}{r^{k+1}} \\ &= \int_0^{r_0} \exp\left\{\frac{\operatorname{Re} p + \sigma}{kr^k}\right\} \left| \exp\left\{-\frac{\sigma}{kr^k}\right\} v(r) \right| \frac{dr}{r^{k+1}}. \end{aligned}$$

Using the Cauchy-Schwarz inequality, we obtain the estimate

$$\begin{aligned} |V(p)| &\leq \left\{ \int_0^{r_0} \exp\left\{\frac{2(\operatorname{Re} p + \sigma)}{kr^k}\right\} \frac{dr}{r^{k+1}} \right\}^{1/2} \|v\|_\sigma \\ &= \left\{ \int_0^{r_0} \exp\left\{-\frac{2\varepsilon}{kr^k}\right\} \exp\left\{\frac{2(\operatorname{Re} p + \sigma + \varepsilon)}{kr^k}\right\} \frac{dr}{r^{k+1}} \right\}^{1/2} \|v\|_\sigma \\ &\leq C_\varepsilon \exp\left\{\frac{\operatorname{Re} p + \sigma + \varepsilon}{kr_0^k}\right\} \|v\|_\sigma . \end{aligned}$$

The latter estimate proves the lemma. □

4.3 Multiplication by a function in the Borel representation

To apply the k-Borel transform to equation (35), one needs to investigate the action of the operator of multiplication by a function $A(r)$ in the Borel representation. For simplicity, we shall carry out our considerations for scalar-valued functions though all the results are obtained in the same way for functions with values in a Banach space as well.

First, we consider the action of multiplication by powers of r. The following affirmation is valid.

Lemma 5 *The formula*

$$\mathcal{B}_k\left[r^j v(r)\right] = \int_{-\infty}^{p} \frac{(p-q)^{\frac{j}{k}-1}}{k^{j/k}\Gamma(j/k)} V(q)\, dq \tag{36}$$

is valid for each positive integer j and any function $V(p)$ subject to the conditions of Lemma 4. Here the integration is fulfilled over a contour along which the real part of p tends to $-\infty$.

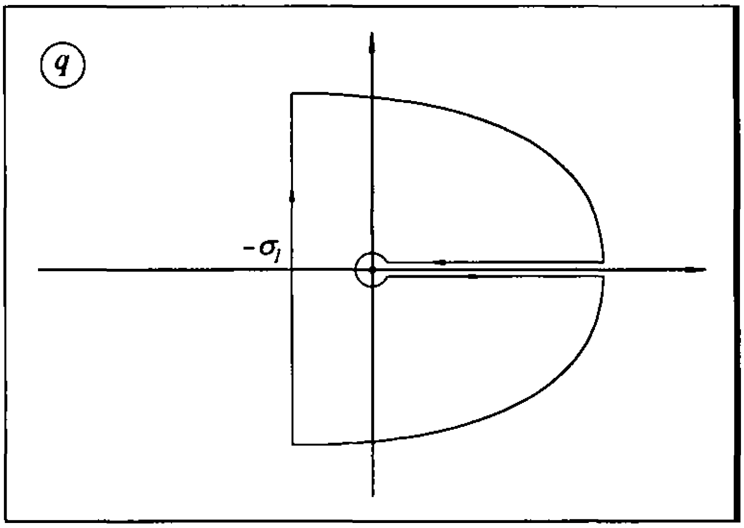

Figure 3. Deformation of the integration contour.

Proof. Let us consider first the case $j < k$. First of all, we have

$$\begin{aligned}\int\limits_{-\sigma_1-i\infty}^{-\sigma_1+i\infty} \exp\left\{-\frac{q}{kr^k}\right\} q^{\frac{j}{k}-1}\,dq &= \left(1-\exp\left\{\frac{2\pi ij}{k}\right\}\right)\int\limits_0^{\infty} \exp\left\{-\frac{q}{kr^k}\right\} \\ \times q^{\frac{j}{k}-1}\,dq &= \left(1-\exp\left\{\frac{2\pi ij}{k}\right\}\right)\Gamma\left(\frac{j}{k}\right)k^{\frac{j}{k}}r^j\end{aligned}$$

for each $\sigma_1 > 0$. Here we have used the Jordan lemma for the contour drawn on Figure 3. Substituting the latter expression into the integral

$$\mathcal{B}_k\left[r^j v(r)\right] = \int\limits_0^{r_0} \exp\left\{\frac{p}{kr^k}\right\} r^j v(r)\,\frac{dr}{r^{k+1}}$$

and changing the integration order, we obtain

$$\begin{aligned}\mathcal{B}_k\left[r^j v(r)\right] &= \frac{1}{\left(1-\exp\{2\pi ij/k\}\right)\Gamma(j/k)\,k^{j/k}} \\ &\times \int\limits_{-\sigma_1-i\infty}^{-\sigma_1+i\infty}\left\{\int\limits_0^{r_0}\exp\left\{\frac{p-q}{kr^k}\right\}v(r)\,\frac{dr}{r^{k+1}}\right\} q^{\frac{j}{k}-1}\,dq\end{aligned}$$

$$= \frac{1}{(1-\exp\{2\pi i j/k\})\,\Gamma(j/k)\,k^{j/k}} \int\limits_{-\sigma_1-i\infty}^{-\sigma_1+i\infty} V(p-q)\,q^{\frac{j}{k}-1}\,dq.$$

Performing once more the above described deformation of the integration contour and taking into account that the function $V(p)$ is holomorphic for sufficiently large negative values of $\operatorname{Re} p$, we arrive at relation (36).

Later on, let $j = k$. In this case we have

$$\begin{aligned} \frac{\partial}{\partial p}\mathcal{B}_k\left[r^k v(r)\right] &= \frac{\partial}{\partial p}\int\limits_0^{r_0} \exp\left\{\frac{p}{kr^k}\right\} r^k v(r)\,\frac{dr}{r^{k+1}} \\ &= \frac{1}{k}\int\limits_0^{r_0} \exp\left\{\frac{p}{kr^k}\right\} v(r)\,\frac{dr}{r^{k+1}} = \frac{V(p)}{k}. \end{aligned}$$

Taking into account the decay properties of the function $V(p)$ as $\operatorname{Re} p \to -\infty$, we have

$$\mathcal{B}_k\left[r^k v(r)\right] = \frac{1}{k}\int\limits_{-\infty}^{p} V(q)\,dq,$$

which clearly coincides with (36) for $j = k$.

Finally, the proof of relation (36) for $j > k$ can be carried out by induction on j. Suppose that the needed formula is already proved for $j = j_1$ and $j = j_2$. Then one has

$$\begin{aligned} &\mathcal{B}_k\left[r^j v(r)\right] = \mathcal{B}_k\left[r^{j_1+j_2} v(r)\right] \\ &= \int\limits_{-\infty}^{p} \frac{(p-q)^{(j_1/k)-1}}{k^{\frac{j_1}{k}}\Gamma(j_1/k)} \left\{ \int\limits_{-\infty}^{q} \frac{(q-q')^{(j_2/k)-1}}{k^{j_2/k}\Gamma(j_2/k)} V(q')\,dq' \right\} dq. \end{aligned} \tag{37}$$

The latter integral can be rewritten as a multiple integral over the domain drawn on Figure 4. The convergence of this integral follows from the estimate of Lemma 4. So, one can change the integration order in the integral (37), thus obtaining

$$\mathcal{B}_k\left[r^j v(r)\right] = \int\limits_{-\infty}^{p} \left\{ \int\limits_{q'}^{p} \frac{(p-q)^{(j_1/k)-1}(q-q')^{(j_2/k)-1}}{k^{(j_1+j_2)/k}\Gamma(j_1/k)\,\Gamma(j_2/k)}\,dq \right\} V(q')\,dq'.$$

Since

$$\int\limits_{q'}^{p} (p-q)^{(j_1/k)-1}(q-q')^{(j_2/k)-1}\,dq = \frac{\Gamma(j_1/k)\,\Gamma(j_2/k)}{\Gamma((j_1+j_2)/k)}\,(p-q')^{((j_1+j_2)/k)-1},$$

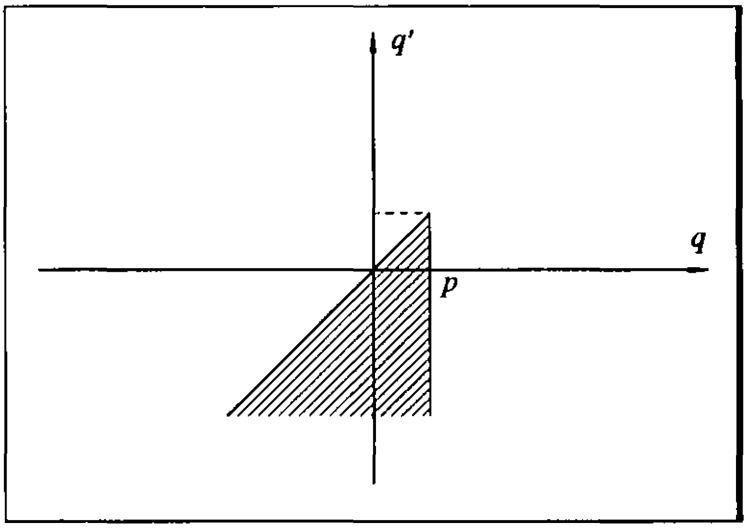

Figure 4. Integration domain.

we have

$$\mathcal{B}_k\left[r^j v(r)\right] = \int\limits_{-\infty}^{p} \frac{(p-q')^{((j_1+j_2)/k)-1}}{k^{(j_1+j_2)/k}\Gamma\left((j_1+j_2)/k\right)} V(q')\, dq'.$$

So, the affirmation of the lemma is valid for $j = j_1 + j_2$. □

Now we can derive the formula for the Borel representation of the multiplication by an arbitrary function $A(r)$ holomorphic in a neighborhood of the origin. Denote by R the radius of convergence of the Taylor series

$$A(r) = \sum_{l=0}^{\infty} A_j r^j.$$

Then, formally, the multiplication by $A(r)$ is transformed by the k-Borel transform into the convolution with the function

$$A_0\delta(p) + A'(p),$$

where

$$A'(p) = \sum_{l=1}^{\infty} A_j \frac{p^{(j/k)-1}}{k^{j/k}\Gamma(j/k)}. \tag{38}$$

The following assertion takes place:

Proposition 1 *Let $A(r)$ be a holomorphic function near the origin, and let R be its radius of convergence. Then the formula*

$$\mathcal{B}_k\left[A(r)\,v(r)\right] = \left(A_0\delta(p) + A'(p)\right) * \mathcal{B}_k\left[v(r)\right]$$

is valid for a constant A_0 and a function $A'(p)$. Moreover, the function $A'(p)$ given by (38) *is an entire function subject to the estimate*

$$|A'(p)| \leq C \exp\left\{(1+\varepsilon)\frac{|p|}{kR^k}\right\}. \tag{39}$$

Proof. Clearly, it is sufficient only to prove the latter estimate. We have

$$|A'(p)| \leq \sum_{j=1}^{\infty} |A_j| \frac{|p|^{(j/k)-1}}{k^{j/k}\Gamma(j/k)} \leq \sum_{j=1}^{\infty} \frac{C}{R^j} \frac{|p|^{(j/k)-1}}{k^{j/k}\Gamma(j/k)}.$$

Splitting the latter sum in the following way

$$\sum_{j=1}^{\infty} \frac{C}{R^j} \frac{|p|^{(j/k)-1}}{\Gamma(j/k)} = \sum_{l=0}^{k-1} \left\{ \left|\frac{p}{Rk}\right|^{\frac{l}{k}} \sum_{j=0}^{\infty} \frac{1}{R^{kj}k^j} \frac{|p|^j}{\Gamma(((l/k)+j+1)} \right\}$$

and evaluating each sum on the right in the latter relation, we arrive at the estimate

$$|A'(p)| \leq C\,|p|^{\frac{k-1}{k}} \exp\left\{\frac{|p|}{kR^k}\right\}.$$

Estimate (39)readily follows from the latter estimate. □

4.4 Proof of the theorem on endless continuability

Let us apply the k-Borel transform to equation (35). Due to the result of Proposition 1, we obtain the following equation for the k-Borel transform $V(p)$ of the function $v(r)$:

$$\left[\hat{H}_0(p) + \overline{\frac{1}{(d/dp)^{-1/k}}}\,\hat{H}_1\left(\overline{\frac{1}{(d/dp)^{-1/k}}}, \overset{2}{p}\right)\right] V(p) = F(p), \tag{40}$$

where

$$\overline{\frac{1}{(d/dp)^{-1/k}}}\,\hat{H}_1\left(\overline{\frac{1}{(d/dp)^{-1/k}}}, \overset{2}{p}\right) V(p) = \int_{-\infty}^{p} \mathcal{H}(p-q,p)\,V(q)\,dq.$$

The function $\mathcal{H}(q,p)$ is defined via the function $r\hat{H}_1(r,p)$ in the same way as the function $A'(p)$ is defined via $A(p)$, the variable p being a parameter. Namely,

$$\mathcal{H}(q,p) = \sum_{j=1}^{\infty} \hat{H}_j(p) \frac{q^{(j/k)-1}}{k^{j/k}\Gamma(j/k)}$$

if

$$r\hat{H}_1(r,p) = \sum_{j=1}^{\infty} \hat{H}_j(p) r^j.$$

Dividing equation (40) by $\hat{H}_0(p)$, we arrive at the equation

$$V(p) + \int_{-\infty}^{p} \hat{H}_0^{-1}(p)\mathcal{H}(p-q,p)V(q)\,dq = \hat{H}_0^{-1}(p)F(p)$$

(the integral on the right converges for sufficiently small r_0 due to the estimates of the type (39) for the function $\mathcal{H}(p-q,p)$). This equation will be used for the proof of the fact that the function $V(p)$ is an endlessly continuable function in p. First of all, we remark that the function $V(p)$ is already known to be a holomorphic function in p in the half-plane $\operatorname{Re} p < -\sigma$. We shall consider $V(p)$ in this region as a known function.

Consider an arbitrary small neighborhood U of the set of poles of the function $\hat{H}_0^{-1}(p)$ (see Figure 5). In the complement $\mathbf{C} \setminus U$ of this neighborhood we rewrite the equation in the form

$$V(p) + \int_{p_0}^{p} \hat{H}_0^{-1}(p)\mathcal{H}(p-q,p)V(q)\,dq =$$

$$= \hat{H}_0^{-1}(p)F(p) - \int_{-\infty}^{p_0} \hat{H}_0^{-1}(p)\mathcal{H}(p-q,p)V(q)\,dq. \qquad (41)$$

Since in the considered region the kernel $\hat{H}_0^{-1}(p)\mathcal{H}(p-q,p)$ of this equation is bounded everywhere except $p=q$ and has a weak singularity at this point, the usual estimates for successive approximation method show that the Volterra equation (41) is solvable in $\mathbf{C} \setminus U$. Since U can be chosen arbitrary small, this fact proves the endless continuability of the function $V(p)$. This completes the proof of the theorem.

Remark 5 One can also prove that the constructed solution is of exponential type with order k in the whole plane $\mathbf{C}$. We leave the corresponding estimates to the reader.

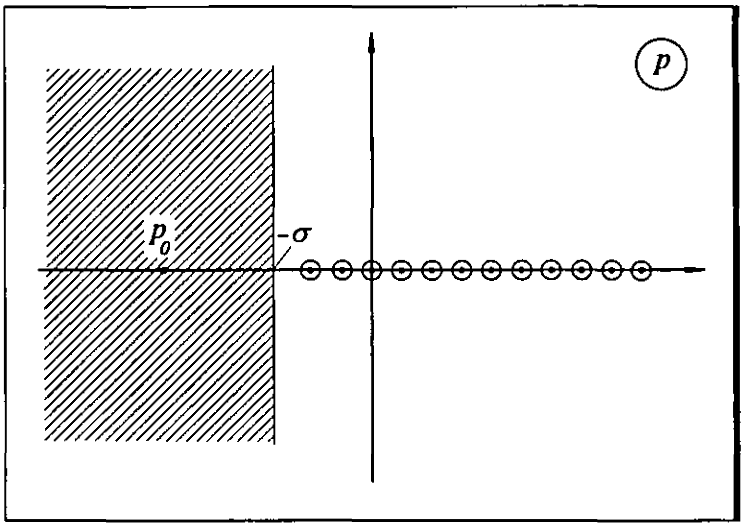

Figure 5. Analytic continuation of $V(p)$.

Remark 6 As we have already written, the function $V(p)$ has no singularities in the half-plane $\mathrm{Re}\, p < \sigma$ even at those points where the operator family $\hat{H}_0^{-1}(p)$ has singularities. However, this remark concerns only one sheet of the Riemannian surface of $V(p)$, from which we have started the process of analytic continuation. On all other sheets of the Riemannian surface of $V(p)$ the function $V(p)$ can have singularities at poles of $\hat{H}_0^{-1}(p)$ even for $\mathrm{Re}\, p < \sigma$.

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