A note on the fuchsian groups of genus zero

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Introduction

For every genus-zero fuchsian group Γ we construct the canonical central extension $\hat{\Gamma}$ and discuss some algebro-geometric by-products of this construction (see § 3, Main Theorem).

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1. Basic definitions, facts and notations

Let ϕ be the hermitian form $|z_1|^2 - |z_2|^2$ of signature $(1,1)$ in $\mathbf{C}^2$ and $U(1,1)$ be the group $Aut\ \phi$. Define the cone $\hat{B} = \{z \in \mathbf{C}^2 | \phi(z) < 0\}$ and let B be the unit ball - the image of the cone $\hat{B}$ in the projective space $PC^2 = P^1(\mathbf{C})$. The group $PU(1,1)$ is the group $Aut\ B$ of all biholomorphic automorphisms of B.

Remark 1. The 1-dimensional complex hyperbolic space B can be identified with 2-dimensional real hyperbolic space with curvature -4. This means, for example, that the area of an convex $r-gon$ P in B with angles $\alpha_1, \cdots, \alpha_r$ equals $a(P) = \frac{\pi}{4}\left(r - 2 - \sum_{i=1}^{r} \alpha_i\right)$.

Main definition. We say that a discrete subgroup Γ of $Aut\ B$ is a genus-zero fuchsian group if

(i) the quotient space B/Γ is compact (has finite volume)
(ii) the quotient space $B/\Gamma \simeq \mathbf{P}^1(\mathbf{C})$ (its natural point-compactification $\widetilde{B/\Gamma} \simeq \mathbf{P}^1(\mathbf{C})$)

For the sake of simplicity we will assume that the quotient B/Γ is compact, but all our arguments are also good in general case.

In what follows Γ stands for a genus-zero cocompact fuchsian group in one-dimensional complex ball B.

Algebraic structure of the group Γ is known.

Fact 1. Γ admits corepresentation

$$\Gamma = \langle \gamma_1, \cdots, \gamma_r | \gamma_1^{n_1} = \cdots = \gamma_r^{n_r} = 1,\ \gamma_1 \cdots \gamma_r = 1\}$$

with integers $n_i \geq 2,\ i = 1, \cdots, r$.

We will call γ_i the canonical generators of the group Γ and write sometimes $\Gamma = \Gamma(\gamma_1, \cdots, \gamma_r)$.

Fact 2. Cyclic subgroups $\langle \gamma_1 \rangle, \cdots, \langle \gamma_r \rangle$ are representatives of all elliptic conjugacy classes of Γ.

Remark 2. The volume of the quotient space B/Γ equals $\frac{1}{2}\pi(r - 2 - \Sigma\ 1/n_i)$.

The important definition

The element $\gamma \in U(1,1)$ is called the complex reflection if the set $Fix\ \gamma$ of γ-fixed points in $\mathbf{C}^2$ is a complex line.

We say that a discrete subgroup G of $U(1,1)$ is the complex reflection group, or *cr*-group, if it is generated by complex reflections.

2. The group $\hat{\Gamma}$ and its properties

Let Γ be a genus-zero fuchsian group in $\mathbf{B}$ and $\gamma \neq e* -$ an elliptic element of Γ. Then γ has the unique fixed point $[x]$ in B. Let $\tau \in \mathbf{C}$ be the differential of γ at the fixed point $[x] \subset B$.

Lemma 1. *For any elliptic element $\gamma \in \Gamma$ there exists the unique reflection $\hat{\gamma} \in U(1,1)$ such that*

(i) *the image of $\hat{\gamma}$ under the canonical homomorphism $\pi : U(1,1) \to PU(1,1)$ is γ*
(ii) *the eigenvalues of $\hat{\gamma}$ are $(1,\tau)$ and the eigenvector with eigenvalue 1 lies in the cone $\hat{B}$.*

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Write x for the complex line in $\mathbf{C}^2$ corresponding to the γ-fixed point $[x] \subset B \subset P\mathbf{C}^2$. Choose a vector $e \in \mathbf{C}^2$ such that $\phi(e,x) = 0$, $\phi(e,e) = 1$. Then define the complex reflection $\hat{\gamma}$, we are looking for, by the formula: $z \mapsto z - (1-\tau)\phi(z,e)e$.

■

We will say that the complex reflection $\hat{\gamma}$ is the complex reflection lift of the elliptic element $\gamma \in \Gamma$. As well we will call an element $\hat{g} \in \mathcal{U}(1,1)$ a lift of an element $g \in P\mathcal{U}(1,1)$ if $\pi(\hat{g}) = g$.

Let $\hat{\Gamma}$ be the subgroup of $U(1,1)$ generated by the lifts of all elliptic elements of Γ.

Lemma 2.

(i) $\hat{\Gamma}$ *is a discrete reflection subgroup of* $U(1,1)$
(ii) $\hat{\Gamma}\hat{B} = \hat{B}$
(iii) *the center* $Z\left(\hat{\Gamma}\right)$ *of the group* $\hat{\Gamma}$ *is finite*
(iv) $\hat{\Gamma}/Z\left(\hat{\Gamma}\right) \simeq \Gamma$
(v) $\hat{\Gamma}$ *acts as a discrete subgroup in the cone* $\hat{B}$.

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By Fact 1 the determinants of the elements of $\hat{\Gamma}$ take only a finite number of values. This proves (iii). The rest is straightforward.

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Lemma 3. *Let* Γ *be* $\Gamma(\gamma_1, \cdots, \gamma_r)$. *Then*

(i) *the group $\hat{\Gamma}$ is generated by the elements $\hat{\gamma}_1, \cdots, \hat{\gamma}_r$*
(ii) *every complex reflection in the group $\hat{\Gamma}$ is conjugate to a power of some of the $\hat{\gamma}_i$, $i = 1, \cdots, r$*

(iii) *the center* $Z\left(\hat{\Gamma}\right)$ *is generated by the product* $\hat{\gamma}_1 \cdots \hat{\gamma}_r$.

■

Suppose γ is an elliptic element of Γ. Then by Fact 2 there exists an element $g \in \Gamma$ such that $\gamma = g\, \gamma_i^{l_i} g^{-1}$. Write $g = \gamma_{i_1} \cdots \gamma_{i_k}$ as a product of the canonical generators γ_i, $i = 1, \cdots, r$ and define the element $\hat{g}$ of $\hat{\Gamma}$ by $\hat{g} = \hat{\gamma}_{i_1} \cdots \hat{\gamma}_{i_k}$. Obviously $\hat{\gamma} = \hat{g}\, \hat{\gamma}_i^{l_i} \hat{g}^{-1}$. Note that this proves (ii) and shows that every complex reflection of $\hat{\Gamma}$ is generated by the elements $\hat{\gamma}_1, \cdots, \hat{\gamma}_r$, and (i) follows by definition of the group $\hat{\Gamma}$. To prove (iii) choose an element $\hat{g} \in Z\left(\hat{\Gamma}\right)$. Then $g = 1$ and we may therefore apply the Fact 1 to conclude that the element $\hat{g}$ has a presentation $\hat{g} = \Pi\ w_i$, where the word w_i has the form: $w_i = P\ \hat{\gamma}_1 \cdots \hat{\gamma}_r P^{-1}$ or $w_i = Q\ \hat{\gamma}_i^{n_i} Q^{-1}$. But $\hat{\gamma}_i^{n_i} = 1$ in the group $\hat{\Gamma}$, and the element $\hat{\gamma}_1 \hat{\gamma}_2 \cdot \hat{\gamma}_r$ lies in the center $Z\left(\hat{\Gamma}\right)$.

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<u>**The Main Lemma.**</u> *Let* $\hat{Z}$ *be the generator of the center of* $\hat{\Gamma}$. *Then* $\hat{Z} = e^{\pm 2\sqrt{-1}\cdot\lambda} \cdot E_2$ *and*

$$\lambda = \frac{\pi}{2}(r - 2 - \Sigma\ 1/n_i) = vol\ B/\Gamma.$$

To make the proof of the Main Lemma more clear we would like to recall some facts about reflections in the hyperbolic plane B (see Remark 1). If w is such a reflection then we can lift w to the extended group $\tilde{U}(1,1) = U(1,1) \cup \sigma$, where σ is the complex conjugation on $C^2 : \sigma z = \overline{z}$. It is clear that the cone $\hat{B}$ is σ-invariant and that σ normalises the group $U(1,1)$. Any lift of w will have the form $\hat{w} = \sigma \circ \hat{g}$ with $\hat{g} \in U(1,1)$. Note that for every lift $\hat{w}$ we have $\hat{w}^2 = E_2$. The proof if easy: $\hat{w}^2 = \overline{\hat{g}}\hat{g} = \lambda E_2$ because $w^2 = \text{id}$ in the group $\widetilde{PU}(1,1)$ ($= PU(1,1) \cup \sigma$). But $\hat{g} \in U(1,1)$ and hence $\lambda = 1$ by the direct matrix calculation. Now, if we have two reflections w_1 and w_2 then their product $\gamma = w_1 w_2$ might have the fixed point in B (precisely the intersection point of the axes of w_1 and w_2 in B). In this case for any choice of the lift $\hat{w}_1$ we can find the unique lift $\hat{w}_2$ such that $\hat{w}_1 \hat{w}_2 = \hat{\gamma}$ where $\hat{\gamma}$ is the canonical complex reflection lift of γ.

Further, let P be a convex $r - gon$ in B, 'counter clockwise' oriented and with angles $\pi\alpha_1, \cdots, \pi\alpha_r$. Let $w_1, \cdots, w_r$ be the reflections in its sides and put $\gamma_1 = w_1 w_2$, $\gamma_2 = w_2 w_3, \cdots, \gamma_r = w_r w_1$. Note that $\gamma_1 \cdots \gamma_r = \text{id}$ in the group $PU(1,1)$. We can assume that $\hat{\gamma}_1 = \hat{w}_1 \hat{w}_2$, $\hat{\gamma}_2 = \hat{w}_2 \hat{w}_3, \cdots, \hat{\gamma}_{r-1} = \hat{w}_{r-1} \hat{w}_r$ but then $\hat{\gamma}_r = \lambda_p \hat{w}_r \hat{w}_1$. We say that the complex number λ_p is the deficiency number of the $r - gon$ P. Suppose that we fix orientation of P as above. Then we have

Lemma 4. *The deficiency number of the* $r - gon$ P *in* B *equals* $e^{2\sqrt{-1}\,\mu}$, *where*

$$\mu = \frac{\pi}{2}(\Sigma\ \alpha_i - (r-2)) = -2 \text{ area } P.$$

■

If $r = 3$ the claim of the lemma can be proved by direct and short calculation. In general case we will proceed by induction as follows. Let l be the diagonal which connects the vertices of P with numbers 1 and 3. The diagonal l dissects the $r - gon$ P into two pieces: triangle P_1 and $(r-2)gon$ P_2.

Let w be the reflection in l. Then we have

(1) $\lambda_p = \hat{w}_1 \hat{w}_2 \cdot \hat{w}_2 \hat{w}_3 \cdots \cdot \hat{w}_r \hat{w}_1 \lambda_p =$

(2) $= (\lambda p_1 \hat{w}\hat{w}_1\hat{w}_1\hat{w}_2\hat{w}_2\hat{w}) \cdot (\hat{w}\hat{w}_3 \cdot \hat{w}_3\hat{w}_4 \cdot \cdots \cdot \hat{w}_r\hat{w} \cdot \lambda p_2) =$

(1)$=\lambda_{p_1} \cdot \lambda_{p_2}$

(1) because $\hat{w}_i^2 = E_2$

(2) because if $\mu_1\hat{w}\hat{w}_1$ and $\mu_2\hat{w}_r\hat{w}$ are complex reflections then their product $\mu_1\mu_2\hat{w}_r\hat{w}\hat{w}\hat{w}_1 = \mu_1\mu_2\hat{w}_r\hat{w}_1$ is also the complex reflection (in fact, both of them have the common fixed line in the cone $\hat{B}$), and the same remark concerns the pair $\hat{w}_2\hat{w}$, $\hat{w}\hat{w}_3$.

Thus we have shown $\lambda_p = \lambda_{p_1} \cdot \lambda_{p_2}$ and the induction arguments plus the area additivity complete the proof.

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Now let us come back for the proof of the Main lemma.

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Let $T(\Gamma)$ be the Teichmüller space of the marked group $\Gamma = \Gamma(\gamma_1, \cdots, \gamma_r)$ and $[\Gamma']$ be any point of $T(\Gamma)$. Than it is easy to prove that $\lambda(\Gamma) = \lambda(\Gamma')$. Therefore we can make a special choice of Γ and prove the lemma for this particular nice case. Let C be a Coxeter group in B generated by reflections in the sides of the convex $r-gon$ P with the angles $\frac{\pi}{n_1}, \cdots, \frac{\pi}{n_r}$. Further let Γ be the subgroup of C generated by the elements $\gamma_1 = w_1w_2$, $\gamma_2 = w_2w_3, \cdots, \gamma_r = w_rw_1$ (here we use the conventions and notations of lemma 4). The rest part of the proof of the Main lemma is straightforward by the result of lemma 4.

■

3. The quotient space $\hat{B}/\hat{\Gamma}$ and its properties

Let $\hat{L}$ be the restriction of the line bundle $O_{\mathbf{P}^1}(-1)$ to the complex ball B (recall that B comes as $\hat{B}/\mathbf{C}^*$, $\hat{B} \subset \mathbf{C}^2$). Then $\hat{L}$ is the $\hat{\Gamma}$-line bundle and we have ([2])

$$\left\{\text{the quotient space } \hat{B}/\hat{\Gamma}\right\} \simeq \left\{\text{the quotient space } \hat{L} - \{\text{zero section}\}/\hat{\Gamma}\right\}$$

($\simeq$ means "the same thing as ..."). Let $m = |Z\left(\hat{\Gamma}\right)|$ be the order of the center. We see that

$$\left\{\hat{L} - \{\text{zero section}\}/\hat{\Gamma}\right\} \simeq \left\{\hat{L}^{\otimes m} - \{\text{zero section}\}/\Gamma\right\}.$$

Notice that the action of Γ on line bundle $\hat{L}^{\otimes m}$ has the following nice property: every elliptic element $\gamma \in \Gamma$ acts trivially on the fiber over γ-fixed point $[x]$ in B. The proof is immediate: the complex reflection lift $\hat{\gamma}$ fixes by definition the complex line x over $[x]$. The shotup of this arguments is the lemma 5.

Lemma 5.

$$\left\{\text{the quotient space } \hat{B}/\hat{\Gamma}\right\} \simeq \{L - \{\text{zero section}\}\}$$

where L *is a line bundle on* $\mathbf{P}^1 \simeq B/\Gamma$.

■

The proof is clear.

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The next proposition will show the line bundle L.

Proposition. $L = O_{\mathbf{P}^1}(N)$, *where*

$$N = \frac{|Z(\hat{\Gamma})|}{2} \cdot \left((r-2) - \sum_{i=1}^{r} 1/n_i\right). \qquad (*)$$

■

The problem of calculating the self-intersection number of zero section in our case is classical. Its solution actually goes back to H. Poincaré [4]. In sequel it was rediscovered many times by different authors (see for example [1], [3], [5], [6] ...). The appendix to the M. Yoshida's paper [6] is a good palce to find the modern proof of the formula (*) along the H. Poincaré lines.

■

Let $\widetilde{L}$ be the surface which comes after blowing down the zero section of the line bundle L (note that by Proposition the self-intersection number of this section is negative). Now we come to the object of our prime interest. We call a zero-genus fuchsian group Γ an excellent group if $\widetilde{L}$ is a smooth surface. In this case $\widetilde{L} \simeq \mathbf{C}^2$ by lemma 5.

Main Theorem. *A zero-genus fuchsian group* $\Gamma = \Gamma(\gamma_1, \cdots, \gamma_r)$ *is excellent* $\Longleftrightarrow$ *the numbers* $n_1, \cdots, n_r$ *are the solutions of diophantine equation*

$$\frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_r} + \frac{2}{m} = (r-2). \qquad (**)$$

■

It follows from the Main lemma and the Proposition.

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Our last statement is in the style of I. Dolgachev's paper [2]. Define the Γ-automorphy-factor $a(\gamma, z)$ by the formula $a(\gamma, z) = \frac{1}{(cz+d)^m}$, where $m = |Z(\hat{\Gamma})|$, $z \in B$, and the γ-lift $\hat{\gamma}$ equals to the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{U}(1,1)$. Let $A(\Gamma, a)$ stands for the graded algebra of Γ-automorphic forms with respect to the automorphy factor $a(\gamma, z)$.

Theorem. *The group* Γ *is excellent iff the algebra* $A(\Gamma, a)$ *is the (weighted) polynomial algebra* $\mathbf{C}[X_1, X_2]$.

Concluding Remark. Needless to say that the arguments of the Main Lemma apply as well to the case of a finite group in $\mathbf{P}^1(\mathbf{C})$ and recover the results of § 9.6 and 11.7 in the H.S. Coxeter's book "Regular Complex Polytopes".

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