EXPONENTS AND SUSPENSION

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This paper is dedicated to my son Thomas

Abstract. Let X be a finite CW-complex. We show that the image of the homotopy groups of X under suspension have an exponent at every prime. As a corollary we recover Long's result that finite H-spaces have exponents at all primes. We show that the stable homotopy groups of X have an exponent at p if and only if X is rationally equivalent to a point. This allows us to construct many examples of spaces with infinite dimensional rational homotopy groups and without an exponent at any prime. These examples further support Moore's conjecture.

1. Introduction

Let G be an abelian group. We say that G has an exponent at a prime p if there exists r such that $p^r(p - \text{torsion in } G) = 0$. Let X be a topological space. We say that X has an exponent at p if $\pi_*(X)$ has an exponent at p. A famous conjecture of Moore links the existence of an exponent for X with its rational homotopy.

Conjecture 1.1. *Let X be a simply connected finite CW-complex. Then for every prime p, X has an exponent at p if and only if $\pi_*(X) \otimes \mathbb{Q}$ is a finite dimensional vector space.*

McGibbon-Wilkerson [11] showed that if X is a simply connected finite CW-complex such that $\pi_*(X) \otimes \mathbb{Q}$ is a finite dimensional vector space then it has an exponent at almost all primes. This is a weak form of one direction of the conjecture. There are many other examples of spaces for which Moore's conjecture holds. In this paper we will only use the following (which predates the conjecture by about 25 years).

Theorem 1.2. *For every integer n and prime p, S^n has an exponent at p.*

Proof: For $n = 1$ the result is trivial. For $n > 1$ see [8] for $p = 2$ and [18] for odd primes. □

It should be pointed out that the best possible exponent of p^r for $\pi_*(S^{2r+1})$ and $p > 2$ were obtained in the papers [15, 4, 12]. The lower bound having been obtained by Gray [6]. At $p = 2$ the best upper bound has been obtained by Selick [17] and the best lower bound by Mahowald [10].

Some other examples where the conjecture holds are: Moore Spaces [3], H-spaces [9], torsion free suspensions [16], two cell complexes [13] and two cones at almost all primes [1]. It also turned out that Moore's amazing conjectural dichotomy has an analogue in rational homotopy theory. In [5] Felix and Halperin showed that the rational homotopy groups of a finite simply connected CW-complex are either

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finite dimensional or grow exponentially. This paper is motivated by a desire to understand better the conjecture of Moore. We prove:

Theorem 1.3. *Let X be a finite connected CW-complex. Then for every prime p the p-torsion in image $\pi_*(X) \to \pi_*(\Omega\Sigma X)$ has an exponent.*

So although the space itself may or may not have an exponent at p its image under suspension always does. This means that exponent questions are very unstable, they all depend on homotopy classes which disappear after one suspension. As a corollary of our theorem we recover Long's result on H-spaces.

Theorem 1.4. *Let X be a finite H-space. Then for every prime p, X has an exponent at p.*

Long's result appears in his thesis but unfortuneatly has never been published. So in addition to giving an alternative proof of this important result we also make it more physically accesable.

In the last section we determine when the stable homotopy groups of a finite complex have an exponent.

Proposition 1.5. *Let X be a simply connected CW-complex. Then for every prime p, $\pi_*^s(X)$ has an exponent at p if and only if $X_{(0)} \simeq *$.*

This result is used to prove that a number of spaces with infinite dimensional rational homotopy do not have exponents at any primes, as expected by Moore's conjecture.

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2. The Image of Suspension

We work in the homotopy category of spaces having the homotopy type of connected CW-complexes. This means that maps homotopic in the underlying category are considered equal and so too homotopy commutative diagrams are considered commutative. For simplicity we will often say that a space is of a certain form when we actually only mean that it is homotopy equivalent to a space of a certain form. For a map $f : X \to Y$ of spaces we let $C(f)$ denote the mapping cone of f.

We will let $h_* : \pi_*(\_) \to H_*(\_)$ denote the Hurewicz homomorphism. We will denote the rationalization functor by $\_{(0)}$.

Let $S^n \longrightarrow Y \longrightarrow X \xrightarrow{p} S^n + 1$ be a cofibration sequence. Then there is a coaction $\psi : X \to X \vee S^{n+1}$. For any Z this induces a map $\psi^* : [S^{n+1}, Z] \times [X, Z] \to [X, Z]$. We need the following fact about the coaction.

Lemma 2.1. *Let $\phi : \Omega Z \times \Omega Z \to \Omega Z$ denote the multiplication and use the notation of the last paragraph. Then for $f \in [S^{n+1}, \Omega Z]$ and $g \in [X, \Omega Z]$*

$$\psi^*(f, g) = \phi(p^* f \times g) = \phi(g \times p^* f)$$

Therefore we also have that $p^*(f) = 0$ if and only if $\phi^*(f, g) = g$.

Proof: Let $\psi' : \Sigma X \to \Sigma X \vee \Sigma X$ denote the coproduct and $T : \Sigma X \vee \Sigma X \to \Sigma X \vee \Sigma X$ denote the transpose. Then the first statement follows from adjointness and the fact that the following diagram commutes.

$$\begin{tikzcd}
\Sigma X \arrow[r, "\psi'"] \arrow[d, "\psi'"] \arrow[dr, "\Sigma\psi"] & \Sigma X \vee \Sigma X \arrow[d, "id\vee\Sigma p"] \\
\Sigma X \vee \Sigma X \arrow[r, "(id\vee\Sigma p)T"'] & \Sigma X \vee \Sigma S^{n+1}
\end{tikzcd}$$

This diagram commutes for the same reason that a double suspension has a homotopy commutative coproduct. □

Lemma 2.2. *Let X be a finite complex and Z any space. Let $f, g : X \to \Omega Z$ be maps such that $f_{(0)} \simeq g_{(0)}$. Then there exists an integer s such that $sf \simeq sg$.*

Proof: The lemma is true for $X = *$. Now assume the statement is true for any $X = Y$ with n cells, any Z and any maps $f, g : Y \to \Omega Z$. Now let Y' be any space with $n+1$ cells and write $Y' = C(h)$ for a map $h : S^n \to Y$ and Y with n cells. Let $f', g' : C(h) \to \Omega Z$ be any maps and let f and g denote their restrictions to Y. So by the induction hypothesis there is an integer s such that $sf \simeq sg$. So any extensions sf' and sg' differ by an element α of $\pi_* H$. Using Lemma 2.1 we see that since $f'_{(0)} \simeq g'_{(0)}$ there exists an integer l such that $lp^*(\alpha) = 0$. We have that $sf' + \alpha = sg'$. From Lemma 2.1 $l(sf' + \alpha) = lsf' + l\alpha$. So we get that $slf' \simeq slg'$. Thus the lemma is proved by induction. □

Lemma 2.3. *Let X be simply connected. Let $f, g \in \pi_*(\Omega X)$ be such that $h_*(f) = h_*(g)$. Then there exists an integer l such that $lf = lg$.*

Proof: $\Omega X_{(0)}$ is a product of Eilenberg-Mac Lane spaces. Therefore $h_* f = h_* g$ implies that $f_{(0)} \simeq g_{(0)}$. The lemma then follows from Lemma 2.2. □

Lemma 2.4. *Let X be a space such that ΣX is a wedge of spheres and Moore spaces. Let $f : S^r \to X$ be any map. Then there exists an integer l such that $\Sigma C(lh)$ is a wedge of spheres and Moore spaces.*

Proof: Let $X(j)$ be a space such that $\tilde{H}_*(X(j)) = 0$ if $* \neq j$ and $\tilde{H}_j(X(j)) = H_{j-1}(X)$. Then $\Sigma X \simeq \vee_{j\geq 2} X(j)$. Let $i(j) : X(j) \to \Sigma X$ be the inclusions into the wedge. $h_* : \pi_*(X(j)) \to H_*(X(j))$ is surjective. (Actually it is an isomorphism in dimension j.) Let $h' : S^r \to X(r)$ be a map such that $h_*(i(r)h') = h_*(\Sigma f)$. Let l be the integer from Lemma 2.3 such that $li(r)h' = l\Sigma h$. $\Sigma C(lh) = C(\Sigma lh) = C(li(r)h') = \vee_{j\neq r} X_j \vee C(lh')$. $C(lh')$ is a wedge of spheres and Moore spaces. Therefore $\Sigma C(lh)$ is as well and we have proved the lemma. □

Lemma 2.5. *Let $S^r \xrightarrow{h} X \xrightarrow{i} Y$ be a cofibration sequence. Assume that X is finite. Assume that there exists a finite type space W such that ΣJW is a wedge of spheres, an integer n, a map g, a multiplicative map f and a commutative diagram*

$$\begin{tikzcd}
X \arrow[r, "E^1"] \arrow[d, "g"] & JX \arrow[d, "n"] \\
JW \arrow[r, "f"'] & JX.
\end{tikzcd}$$

Then there exists a finite type space W' such that $\Sigma JW'$ is a wedge of spheres, an integer n', a map g', a multiplicative map f' and a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "E^1"] \arrow[d, "g'"] & JY \arrow[d, "n'"] \\ JW' \arrow[r, "f'"] & JY. \end{tikzcd}$$

Proof: Let l be the integer from Lemma 2.4 corresponding to the map gh. Since f and Ji are multiplicative the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "E^1"] \arrow[d, "g"] & JX \arrow[r, "Ji"] \arrow[d, "n"] & JY \arrow[d, "n"] \\ JW \arrow[r, "f"] \arrow[d, "l"] & JX \arrow[r, "Ji"] \arrow[d, "l"] & JY \arrow[d, "l"] \\ JW \arrow[r, "f"] & JX \arrow[r, "Ji"] & JY \end{tikzcd}$$

Also the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "E^1"] \arrow[d, "i"] & JX \arrow[r, "nl"] \arrow[d, "Ji"] & JX \arrow[d, "Ji"] \\ Y \arrow[r, "E^1"] & JY \arrow[r, "nl"] & JY \end{tikzcd}$$

We have a pushout

$$\begin{tikzcd} X \arrow[r] \arrow[d, "lg"] & Y \arrow[d, "k"] \\ JW \arrow[r] & C(lgh) \end{tikzcd}$$

Let $\phi : C(lgh) \to JY$ be the map induced by $nlE^1 : Y \to JY$ and $J(i)f : JW \to JY$. Clearly

$$\begin{tikzcd} Y \arrow[r, "E^1"] \arrow[d, "k"] & JY \arrow[d, "nl"] \\ C(lgh) \arrow[r, "\phi"] & JY \end{tikzcd}$$

commutes. Lemma 2.4 implies that there exists a homotopy equivalence $\psi : \Sigma C(lgh) \to \Sigma W' \vee U$ such that W' is a finite type wedge of spheres and U is a wedge of Moore spaces. Let ψ^{-1} be some homotopy inverse. Let $i : \Sigma W' \to \Sigma W' \vee U$ be the inclusion and $p : \Sigma W' \vee U \to \Sigma W'$ be the projection. Let $\theta = \psi^{-1} i p \psi$. Then $\theta_{(0)} \simeq id$. Let $\phi' : JC(lgh) \to JY$ be the mulitplicative extension of ϕ. Then $(nlE^1)_{(0)} \simeq (E^1 \Omega\theta k)_{(0)}$. Therefore by Lemma 2.2 there exists s such that

$$snlE^1 \simeq s\phi'\Omega\theta E^1 k \simeq \phi' s\Omega\theta E^1 k(1) \quad (1).$$

The second homotopy follows since ϕ' is multiplicative.

Let $g' = \Omega(p\psi)E^1 k : Y \to JW'$, $n' = snl$ and $f' = \phi'\Omega(\psi^{-1} i)$. Clearly $\Sigma JW'$ is a finite type wedge of spheres. Also f' is multiplicative and (1) implies that the relevent diagram commutes. $\square$

Lemma 2.6. *Let X be a finite complex. Then there exists a space Y such that ΣY is a finite type wedge of spheres, an integer n and a homotopy commutative diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & JX \arrow[d, "n"] \\ JY \arrow[r, "\theta"] & JX. \end{tikzcd}$$

Proof: The Lemma follows from Lemma 2.5 and induction on the cells of X. □

Theorem 2.7. *Let X be a connected finite CW-complex. Then for every prime p and every $r \geq 1$, the image of the map induced by the iterated suspension* image $\pi_*(X) \to \pi_*(\Omega^n \Sigma^n X)$ *has an exponent at p.*

Proof: Fix a prime p. From Lemma 2.6 we have a finite type wedge of spheres, W maps f and g and an integer n such that the following diagram commutes

$$\begin{tikzcd} X \arrow[r] \arrow[d, "g"] & JX \arrow[r] \arrow[d, "n"] & \Omega^n \Sigma^n X \arrow[d, "n"] \\ \Omega W \arrow[r, "f"] & JX \arrow[r] & \Omega^n \Sigma^n X \end{tikzcd}$$

[7] implies that ΩW is homotopy equivalent to a finite type product of loops on spheres. Since X is finite dimensional this implies there is a finite set I and a factorization of g

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & \prod_{i \in I} \Omega S^{n(i)} \arrow[d] \\ & \Omega W. \end{tikzcd}$$

By Theorem 1.2, $\prod_{i \in I} \Omega S^{n(i)}$ has a homotopy exponent for all primes and $\pi_*(\prod_{i \in I} \Omega S^{n(i)}) \otimes \mathbb{Q}$ is finite dimensional. Therefore the image of gf has an exponent. In other words there exists an integer s such that p^s(p-torsion in image $\pi_*(gf)) = 0$. Therefore np^s(p-torsion in image $\pi_*(E^1)) = 0$. Since our fixed p was arbitrary this proves the theorem in case $r = 1$. The same arguement works for $r > 1$. □

We then recover the result of Long.

Corollary 2.8. *Any finite connected H-space has an exponent.*

Proof: X is an H-space if and only if $E^1 : X \to JX$ has a homotopy retraction. So if X is an H-space then $\pi_*(E^1)$ is injective. The corollary then follows directly from the theorem. □

We can make the corollary slightly stronger.

Corollary 2.9. *Any finite space with abelian π_1 whose p-localization is an H-space has an exponent at p.*

Proof: Same as 2.8. □

Can this statement be made even slightly stronger?

Question 2.10. Let H be a connected H-space with finitely generated mod p cohomology. Does H have an exponent at p?

In contract to Theorem 2.7 the map $\pi_*(\Omega X) \to \pi_*(\Omega\Sigma\Omega X)$ is injective. In fact it seems that the image of $\pi_*(\Omega X)$ under stabalization is large and tends to not have an exponent when X doesn't. This leads us to the following question.

Question 2.11. Let X be a finite simply connected CW-complex. Assume that $\pi_*(X) \otimes \mathbb{Q}$ is infinite dimensional. Then is it true that for every p the image of the map induced by stabalization $\pi_*(\Omega X) \to \pi_*(Q\Omega X)$ does not have an exponent?

3. STABLE EXPONENTS

Recall that $\pi_*^s(X)$ denotes the stable homotopy groups of X. Also $X^{\wedge i}$ denotes the i fold smash of X with itself. The statments in the first two paragraphs of the last section hold for this section as well.

Proposition 3.1. *Let X be a simply connected CW-complex. Then $\pi_*^s(X)$ has an exponent if and only if $X_{(0)} \simeq *$.*

Proof: First assume that $X_{(0)} \simeq *$. Then Barrett [2] showed that X has a finite suspension order (in his terminology X has finite characteristic). It follows that $\pi_*^s(X)$ has that same exponent (and of course perhaps a smaller one). Now assume that $X_{(0)} \not\simeq *$. Then there exists a decomposition of $\Sigma^\infty X$:

$$S^r \xrightarrow{f} X(1) \longrightarrow X(2) = C(f)$$

$$A \xrightarrow{g} X(2) \longrightarrow C(g) = \Sigma^\infty X$$

such that there exists an n such that $nf = 0$, $X(1)$ is a finite spectrum and $A_{(0)} \simeq *$. Since $\pi_*^s(S^r)$ has no exponent it follows that ker f_* has no exponent. So since $\pi_*(X(1) \otimes \mathbb{Q})$ is finite dimensional we see using the homotopy long exact sequence that $\pi_*(X(2))$ has no exponent. As observed above there exists an l such that $l\pi_*(A) = 0$. So again we use the homotopy long exact sequence to see that $\pi_*(\Sigma^\infty X) = \pi_*^s(X)$ has no exponent. $\square$

Theorem 3.2. *Let X be any space. Assume there that $\Sigma^\infty X_{(p)} \simeq A \vee B$. Also assume that $A_{(0)} \not\simeq *$ and that A is the p-localization of a finite spectrum. Then for every integer $n \geq 2$, $S^n \vee X$ has no exponent at p.*

Proof: Consider the fibration constructed by Ganea

$$\Sigma\Omega S^n \wedge \Omega X \to S^n \vee X \to S^n \times X.$$

Since this fibration splits after looping it is enough to observe that the p-torsion in the homotopy groups of the fibre have no exponent. However since $\Sigma\Omega S^n \simeq \bigvee_{i>0} S^{ni+1}$ we see that the fibre has arbitrarly large suspensions of A as retracts. So the theorem follows from Proposition 3.1. $\square$

Corollary 3.3. *Let ΣX be any finite space such that $(\Sigma X)_{(0)} \not\simeq *$. Then for every $n \geq 2$ and prime p, $S^n \vee \Sigma X$ does not have an exponent at p.*

Proof: We can take A to be the suspension spectrum of ΣX. $\square$

In fact this result can be extended to a signifigantly larger class of spaces using a result of Porter [14]. Following him we let $T_i(A_1, \ldots, A_n)$ denote the subset of $A_1 \times \cdots \times A_n$ with at least i coordinates equal to the base point. So for example T_0 is the product and T_{n-1} is the wedge.

Theorem 3.4. *Assume that for $j \leq l$, $\Sigma X_j \not\simeq_{(0)} *$, $i > n-l$ and that $r \geq 2$. Then for every p, $T_i(S^r, \Sigma X_1, \ldots, \Sigma X_n)$ does not have an exponent at p.*

Proof: Let $F \to T_i(A_1, \ldots, A_n) \to T_0(A_1, \ldots, A_n)$ be a fibration sequence. Porter shows that this fibration splits after looping and that $F \simeq \vee_S \Sigma^{n-i}\Omega A_{i_1} \wedge \cdots \wedge \Omega A_{i_j}$, where S is some set and $i_r \in \{1, \ldots, n\}$. In particular for some $r \leq l$ some $\Sigma^{n-i}\Omega S^r \wedge \Omega\Sigma X_1 \wedge \cdots \wedge \Omega\Sigma X_n$ will be retract of F in our case. Now the theorem follows from James' decomposition $\Sigma\Omega\Sigma X \simeq \Sigma \vee_{i>0} X^{\wedge i}$ (see [19] for a proof) and Proposition 3.1. $\square$

We can see that the condition $i > n-l$ in the theorem is necessary by letting $i = n-l$, $\Sigma X_j = S^r$ if $j \leq l$ and $\Sigma X_j = P^r(p)$ if $j > l$.

Definition 3.5. Let X be a finite CW complex. If for each prime p there are only finitely many distinct (up to suspension) indecomposable pieces of the p-localization of $\bigvee_{i \geq 1} X^{\wedge i}$ then we say that X is smash finite.

Note that the notion of indecomposability is well defined because of [20].

Proposition 3.6. *Let $X = \Sigma^2 Y$ be a finite complex. Suppose $\tilde{H}_*(X, \mathbb{Q})$ is at least two dimensional and that $X_{(p)}$ is smash finite. Then $\pi_*(X) \otimes \mathbb{Z}_{(p)}$ has no exponent.*

Proof: Assume that X is p-local. [21] Theorem 1.6 says that $JX \simeq \prod_j J(L_{k_j}(X)) \times ?$. Each of the $L_{k_j}(X)$ is a retract of $X^{\wedge n}$ for some n. Since X is a finite suspension [20] implies that $L_{k_j}(X)$ is a wedge of the spaces from the list of Definition 3. Also the connectivity of the $L_{k_j}(X)$ goes to infinity with j. So Proposition 3.1 is enough to complete the proof of the theorem. $\square$

Question 3.7. Is every p-localization of any finite suspension smash finite?

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