MAPPING-CLASS-GROUP ACTION VERSUS GALOIS ACTION ON PROFINITE FUNDAMENTAL GROUPS

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ABSTRACT. Let X be an algebraic curve of genus g, n-punctured, defined over a number field K. Then, the profinite or the pro-l completion of the topological fundamental group of X admits two actions; the action of the profinite completion of the mapping class group of the orientable surface of topological type (g,n) and the action of the absolute Galois group of K. This paper compares these two. In the profinite case, it is shown that the intersection of the images of these two actions is trivial if X is affine and its fundamental group is non-abelian. On the contrary, in the pro-l case, there are many curves such that the image of the Galois action contains the image of the mapping-class-group action. It is proved that the set of points corresponding to such curves is dense in the moduli space of (g,n)-curves.

1. MAPPING-CLASS-GROUP ACTION AND GALOIS ACTION

Let X be a smooth geometrically connected curve of genus g, n-punctured, defined over a number field $K \subset \mathbb{C}$ (simply called a (g,n)-curve over K). We assume that each puncture is a K-rational point in the smooth compactification X^* of X.

Let Π denote the fundamental group of $X(\mathbb{C})$ in the classical topological sense. The *mapping class group* of type (g,n), denoted by $\Gamma_{g,n}$, is defined to be the group of isotopy classes of orientation preserving self-diffeomorphisms of $X^*(\mathbb{C})$ which fix the n points pointwise. This definition induces a natural homomorphism $\rho_\Gamma : \Gamma_{g,n} \to \mathrm{Out}(\Pi) := \mathrm{Aut}(\Pi)/\mathrm{Inn}(\Pi)$, which is known to be injective.

Let $\hat{\Pi}$ denote the profinite completion of Π. Since Π is finitely generated, $\mathrm{Out}(\hat{\Pi})$ is profinite, and hence ρ_Γ induces

$$\rho_{\hat{\Gamma}} : \widehat{\Gamma_{g,n}} \to \mathrm{Out}(\hat{\Pi}).$$

On the other hand, there is an (outer) action of the absolute Galois group G_K of K

$$\rho_{G_K,X} : G_K \to \mathrm{Out}(\hat{\Pi}).$$

This is deduced from the following diagram:

$$\begin{array}{ccccccccc} 1 & \to & \hat{\Pi} & \to & \pi_1^{alg}(X) & \to & G_K & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \mathrm{Inn}(\hat{\Pi}) & \to & \mathrm{Aut}(\hat{\Pi}) & \to & \mathrm{Out}(\hat{\Pi}) & \to & 1, \end{array}$$

where the middle arrow is the conjugation on the normal subgroup $\hat{\Pi}$ (see [7]). Here Aut denotes the group of continuous automorphisms. Note that $\rho_{G_K,X}$ has enough information to distinguish the algebraic structure of X [30][16][17], while ρ_Γ, and hence $\rho_{\hat{\Gamma}}$, depend only on the topological type of X.

Composing with $\mathrm{Out}(\hat{\Pi}) \to \mathrm{Out}(\Pi^l)$, where Π^l is the pro-l completion of Π, we obtain $\rho^l_{G_K,X} : G_K \to \mathrm{Out}(\Pi^l)$ and $\rho^l_{\hat{\Gamma}} : \widehat{\Gamma_{g,n}} \to \mathrm{Out}(\Pi^l)$.

The main results of this paper are the following two theorems, which show that opposite phenomena occur in the profinite case and in the pro-l case.

Theorem 1.1. *The intersection of the image of $\rho_{G_K,X}$ with the image of $\rho_{\hat{\Gamma}}$ is trivial, if X is a (g,n)-curve with $2g-2+n \geq 1$ and $n \geq 1$.*

We conjecture that the condition $n \geq 1$ is superfluous.
On the contrary, in the pro-l case, we prove

Theorem 1.2. *Fix any (g,n) with $2g-2+n \geq 1$, and consider the following property on a curve X: X is a (g,n)-curve over K and the image of $\rho^l_{G_K,X}$ contains the image of $\rho^l_{\hat{\Gamma}}$. Then, the set of points on the moduli space of (g,n)-curves corresponding to such curves, with K not fixed, is dense with respect to the complex topology.*

If the moduli space of (g,n)-curves is a rational variety, then most (g,n)-curves over any fixed number field K satisfy this property, i.e., it holds outside a thin set in the K-rational points of the moduli in the sense of Hilbert's irreducibility theorem.

2. Profinite case

A key to prove Theorem 1.1 is the following (Theorem 2.2 in [15], the case of $n=1$). Let us denote by $\mathbb{P}^1_{01\infty}$ the projective line minus three points $0, 1, \infty$.

Theorem 2.1. *Let L be any subfield of $\mathbb{C}$. Let X be a smooth geometrically connected (g,n)-curve over L with $2g-2+n \geq 1$, with at least one L-rational puncture. Then, the kernel of $\rho_{G_L,X}$ is contained in the kernel of $\rho_{G_L,\mathbb{P}^1_{01\infty}}$.*

Note that the statement is still true for pro-l case.
The other essential ingredient, which is not true for pro-l case, is the following Theorem by Belyĭ[3].

Theorem 2.2. *$\rho_{G_{\mathbb{Q}},\mathbb{P}^1_{01\infty}}$ is injective.*

Note that from Theorems 2.1 and 2.2, it follows $G_K \subset \mathrm{Out}(\hat{\Pi})$.
The following is due to Takayuki Oda.

Lemma 2.1. *Let $\mathcal{M}_{g,n}$ denote the moduli stack of (g,n)-curves. Then, we have the following exact sequence:*

$$1 \to \pi_1^{alg}(\bar{X}) \to \pi_1^{alg}(\mathcal{M}_{g,n+1}) \to \pi_1^{alg}(\mathcal{M}_{g,n}) \to 1.$$

(See [22] for the fundamental group of stacks.)

Proof. It is enough to prove the exactness of its geometric part

$$1 \to \pi_1^{alg}(\bar{X}) \to \pi_1^{alg}(\overline{\mathcal{M}_{g,n+1}}) \to \pi_1^{alg}(\overline{\mathcal{M}_{g,n}}) \to 1.$$

This is the profinite completion of the well-known exact sequence

$$1 \to \Pi \to \Gamma_{g,n+1} \to \Gamma_{g,n} \to 1,$$

and since $\hat{\Pi}$ is center free, the exactness is preserved by the profinite completion (see [1]). $\square$

Proof of Theorem 1.1.

Proof. Let B be a geometrically connected variety over K, and $C \to B$ be a smooth family of (g,n)-curves. This means that there is a family of proper smooth $(g,0)$-curves $C^* \to B$ over K, there are n sections $s_1, s_2, \dots, s_n : B \to C^*$ such that the union of their images is etale over B, and C is the complement of the sections in C^*. Assume that there is an $x \in B(K)$, whose fiber is X. Then, we have a short exact sequence

$$1 \to \pi_1^{alg}(\bar{B}, \bar{x}) \to \pi_1^{alg}(B, \bar{x}) \to G_K \to 1,$$

where $\bar{B} := B \otimes \bar{K}$. This sequence splits by the section $s_x : G_K \to \pi_1^{alg}(B, \bar{x})$. Let

$$\rho : \pi_1^{alg}(B, \bar{x}) \to \mathrm{Out}(\hat{\Pi})$$

be the monodromy of $C \to B$ with base point $\bar{x}$. This is given by the short exact sequence

$$1 \to \pi_1^{alg}(\bar{X}) \to \pi_1^{alg}(C) \to \pi_1^{alg}(B) \to 1. \tag{2.1}$$

Here, the exactness on the left follows from Lemma 2.1, by considering the map $C \to \mathcal{M}_{g,n+1}$ induced by the classifying map for $C \to B$.

We define $\bar{\rho}$ to be the restriction of ρ to its geometric part $\pi_1^{alg}(\bar{B}, \bar{x})$, and $\rho_x := \rho \circ s_x$. Note that $\rho_x = \rho_{G_K, X}$.

Let L be the function field of B, and fix an embedding $L \hookrightarrow \mathbb{C}$. Then, the absolute Galois group G_L of L acts on $\pi_1(C_{\bar{\eta}})$, where η is the generic point of B and $\bar{\eta}$ is its geometric point with respect to $L \subset \mathbb{C}$. It factors through $G_L \twoheadrightarrow \pi_1(B, \bar{\eta}) \cong \pi_1(B, \bar{x}) \to \mathrm{Out}\hat{\Pi}$, by the exactness of (2.1). Let us take an element in the intersection of the image of $\bar{\rho}$ and ρ_x, and let it be $\bar{\rho}(z) = \rho_x(\sigma)$. Let $\tilde{z}$ be an element of G_L mapped to z and $\tilde{\sigma}$ mapped to $s_x(\sigma)$ in $\pi_1^{alg}(B, \bar{x})$. Then $\bar{\rho}(z)\rho_x(\sigma^{-1}) = 1$ says that $\tilde{z}\tilde{\sigma}^{-1}$ is in the kernel of ρ_{G_L, X_L}. By Theorem 2.1, it lies in the kernel of $\rho_{G_L, \mathbb{P}^1_{01\infty L}}$. By the following commutativity and by Belyi's Theorem 2.2, we have $\sigma = 1$.

$$\begin{array}{ccccc} \tilde{z}\tilde{\sigma}^{-1} & \in & G_L & \to & \mathrm{Out}(\pi_1^{alg}\mathbb{P}^1_{01\infty \bar{L}}) \\ \downarrow & & \downarrow & & || \\ \sigma^{-1} & \in & G_K & \to & \mathrm{Out}(\pi_1^{alg}\mathbb{P}^1_{01\infty \bar{K}}). \end{array}$$

Thus, the intersection of the image of $\bar{\rho}$ and the image of $\rho_{G_K, X}$ is trivial. Theorem 1.1 follows from this. Take a sufficiently large n' such that the moduli stack $\mathcal{M}_{g,n'}$ of (g,n')-curves lies in the category of scheme. Such n' exists by Knudsen [13]. Take B to be this moduli space. Then, $\pi_1^{alg}(\bar{B})$ surjects to the fundamental group of the moduli stack $\pi_1^{alg}(\overline{\mathcal{M}_{g,n}})$. But the image of $\bar{\rho}$, i.e., the image of $\rho_{\hat{\Gamma}}$ intersects trivially with the image of $\rho_{G_K, X}$, as proved above. □

Remark 2.1. *Let* $\mathrm{Out}^*(\Pi)$ *be the subgroup of* $\mathrm{Out}(\Pi)$ *consisting of the elements stabilizing the conjugacy class of the inertia group at each puncture. Then* $\Gamma_{g,n} \subset \mathrm{Out}^*(\Pi)$ *is of index two (c.f.* [32, Theorem 5.7.1]*), and any orientation-reversing diffeomorphism together with* $\Gamma_{g,n}$ *generates* $\mathrm{Out}^*(\Pi)$*. Since* $\rho_{G_K, X}$ *is injective, the above theorem can be group theoretically stated as follows. If* $\sigma \in G_K \subset \mathrm{Out}(\hat{\Pi})$ *can be approximated by elements in* $\mathrm{Out}^*(\Pi)$*, then it is the unit or a conjugate of the complex conjugation. This is because* σ^2 *must lie in the image of* $\rho_{\hat{\Gamma}}$*, hence must be trivial, and then* σ *is trivial or a conjugate to the complex conjugation* [2].

Remark 2.2. *The above proof says that the kernel of ρ coincides with the kernel of $\bar{\rho}$, which is given in a purely group theoretic way. We do not know whether it is injective or not. This problem can be considered as an analogue of the congruence subgroup problem: does each finite index subgroup of the mapping class group contain the kernel of $\Gamma_{g,n} \to \mathrm{Out}(\Pi) \to \mathrm{Out}(\Pi/N)$, for some finite index characteristic subgroup N of Π? M. Asada proved the injectivity for $g = 0, 1$ and general n (an article is in preparation).*

3. Pro-l case

We begin with group theoretical preliminaries.

Definition 3.1. A profinite group G having a finite-index pro-l subgroup P is said to be *almost pro-l*.

A subgroup and a quotient group of almost pro-l groups are again almost pro-l.

Lemma 3.1. *If an almost pro-l group G is finitely generated, then it has a finite index normal subgroup H with the following property:*

For any profinite group Γ and any homomorphism $\Gamma \to G$, if $\Gamma \to G \to G/H$ is surjective then $\Gamma \to G$ is surjective.

Proof. Take a pro-l group $P \subset G$ in Definition 3.1. We may assume P to be normal in G by replacing it with the intersection of all the conjugates in G if necessary. Being a finite index subgroup of finitely generated group, it is finitely generated. Take the Frattini subgroup of P, i.e., put $H := [P, P]P^l$. (C.f. Proposition in §10.6 in [29].) □

Lemma 3.2. *For any finitely generated pro-l group G, $\mathrm{Aut}(G)$ and $\mathrm{Out}(G)$ is almost pro-l.*

Proof. It is enough to show this for $\mathrm{Aut}(G)$, and then it follows from Corollary 7 in [1]. □

Consider

$$\rho^l : \pi_1^{alg}(B) \to \mathrm{Out}(\hat{\Pi}) \to \mathrm{Out}(\Pi^l)$$

in the same situation as in the proof of Theorem 1.1, i.e., $C \to B$ is a smooth family of (g, n)-curves, where B is a geometrically connected variety over K.

Lemma 3.3. *The image of ρ^l in $\mathrm{Out}(\Pi^l)$ is finitely generated.*

Proof. Take the section $s_x : G_K \to \pi_1^{alg}(B)$. Then $\rho^l(\pi_1^{alg}(B))$ is generated by $\rho^l(\pi_1^{alg}(\bar{B}))$ and $\rho^l s_x(G_K)$. It is sufficient to prove that each of these is finitely generated. The former is finitely generated, since so is $\pi_1^{alg}(\bar{B})$. For the latter we proceed as follows. Take a finite extension L/K, so that $\rho^l(G_L)$ is pro-l. By smooth base change theorem [7, Expose XIII], $\rho^l \circ s_x : G_L \to \mathrm{Out}(\Pi^l)$ ramifies only at the primes $\mathfrak{p}$ of $\mathcal{O}_L$ such that $\mathfrak{p}$ divides l or X has bad reduction at $\mathfrak{p}$. Thus, only finitely many primes may ramify. Let S be this finite set of primes. Then, $\rho^l s_x$ factors through $G_L \to G_{L,S} := \pi_1^{alg}(\mathrm{Spec}\mathcal{O}_L - S)$. By class field theory, l-Frattini quotient of $G_{L,S}$, namely $G_{L,S}/([G_{L,S}, G_{L,S}]G^l_{L,S})$, is finite, and so is the l-Frattini quotient of $\rho^l s_x(G_L)$. Thus, $\rho^l s_x(G_L)$ is finitely generated, and so is $\rho^l s_x(G_K)$. □

Proposition 3.1. *There is a finite etale connected cover*

$$B' \to B$$

with the following property. Let L be a finite extension of K and y be an L-rational point of B. If y is unramified in $B' \to B$ and the fiber above y is connected, then

$$\rho^l(\pi_1^{alg}(B)) = \rho^l s_y(G_L).$$

Proof. Let G be $\rho^l(\pi_1^{alg}(B)) \subset \mathrm{Out}(\Pi^l)$. By Lemma 3.3, one may use Lemma 3.1 for G to obtain $H \subset G$. Take the inverse image of H by the surjection $\rho^l : \pi_1^{alg}(B) \to G$, and let B' be the cover of B corresponding to H. Thus B' is connected. By the connectedness of the fiber above y, we know that

$$G_L \xrightarrow{s_y} \pi_1^{alg}(B_L) \xrightarrow{\rho^l} G \to G/H$$

is surjective. This and Lemma 3.1 showed that $\rho^l s_y(G_L) \subset G = \rho^l(\pi_1^{alg}(B))$ is an isomorphism. □

Proof of Theorem 1.2.

Proof. Let $n' \geq n$ be a sufficiently large integer such that $\mathcal{M}_{g,n'}$ is a scheme. Let B be an open affine subvariety of $\mathcal{M}_{g,n'}$, and let $C \to B$ be the family of (g,n)-curves induced from $B \to \mathcal{M}_{g,n}$. Then, the dimension of B is $N := 3g - 3 + n'$. Apply Proposition 3.1 to obtain $B' \to B$. Take any quasi-finite dominant morphism $B \to \mathbb{P}^N$. By Hilbert's irreducibility theorem, there is a thin set $\subset \mathbb{P}^N(K)$, outside which every fiber in $B' \to B \to \mathbb{P}^N$ is connected and unramified. We denote by $S \subset \mathbb{P}^N(K)$ the complement of the thin set. Take any point p outside this thin set, and take a (unique) point y in B above p. Let L be the field $k(y)$. Then, the curve C_y/L obtained as the fiber of $C \to B$ at y satisfies the unramified-connectedness condition of Proposition 3.1, thus (C_y, L) satisfies the condition of Theorem 1.2, because $\pi_1^{alg}(\bar{B})$ surjects to $\pi_1^{alg}(\overline{\mathcal{M}_{g,n}})$. Now S is dense in $\mathbb{P}^N(\mathbb{C})$ with respect to the complex topology. Then, since $B(\mathbb{C}) \to \mathbb{P}^N(\mathbb{C})$ is locally isomorphic outside a closed subscheme, the set of geometric points lying in the inverse image of S is dense in $B(\mathbb{C})$, which completes the proof of the first statement in the theorem.

If $\mathcal{M}_{g,n}$ is rational, we take a rational open subvariety $B \subset \mathcal{M}_{g,n}$ which is an open subvariety of $\mathbb{P}^N$. In this case, the above argument shows that every $y \in B(K)$ outside a thin set gives (C_y, K) with the desired property. □

4. Problems and Motivations

4.1. **Problems.** In Theorem 1.2, there are curves which do not satisfy the conclusion. Suppose that the automorphism group $\mathrm{Aut}_K(X)$ is nontrivial. Then, the image $\rho^l_{G_K,X}(G_K)$ must commute with the image of $\mathrm{Aut}_K(X)$. The centralizer of nontrivial element in $\Gamma_{g,n}$ is much smaller than $\Gamma_{g,n}$, (except for $(g,n) = (2,0)$ with the hyperelliptic involution and $(1,1)$ with the elliptic involultion, c.f. [10]). Sometimes it is even trivial (see [4]). Thus the conclusion of the theorem can never hold for such curves. What can be said on the locus of the curves satisfying this property? For example, we can consider the following problem.

Problem 4.1. *Is the locus in the moduli space consisting of the curves X such that $\rho^l_{G_K,X}(G_K)$ contains an open subgroup of $\rho^l_{\hat{\Gamma}}(\widehat{\Gamma_{g,n}})$ independent of the choice of prime l? Moreover, for any such curve X, does $\rho^l_{G_K,X}(G_K)$ contain $\rho^l_{\hat{\Gamma}}(\widehat{\Gamma_{g,n}})$ for all primes l but finite?*

This locus is not Zariski open. For example, assume $(g,n)=(1,1)$. If an elliptic curve E has a potential complex multiplication (CM), then the Galois action on the abelianization of $\pi_1^{alg}(\bar{E})$ does not contain the action of $SL_2(\mathbb{Z}_l)$ (which is the image of the action of $\widehat{\Gamma_{1,1}}$ on the abelianization). Since CM points are dense, the locus is not open. If $(g,n)=(1,0)$, then by Serre's Theorem [28], the Galois action coincides with the action of an open subgroup of $GL_2(\mathbb{Z}_l)$ if and only if the elliptic curve is not CM. Thus, the locus is independent of l.

Problem 4.2. *Is it possible that the centralizer of* $\mathrm{Aut}_K(X)$ *in the image of* $\rho_{\hat{\Gamma}}$ *(or* $\rho^l_{\hat{\Gamma}}$*) becomes trivial? If this occurs, it would be interesting to consider the centralizer as an approximation of the image of* G_K.

4.2. **Motivations.** Motivations of this study came from two works by Takayuki Oda.

Monodromy of the inertia for bad reduction. Oda proved that the action of the inertia can be described by Dehn twists (see (2.7) in [25], c.f. [23]).

Theorem 4.1. *Let* l *be a prime,* A *be a mixed characteristic strictly henselian discrete valuation ring with residue characteristic prime to* l *with maximal ideal* $\mathfrak{p}$. *Let* $\mathcal{X}\to A$ *be a family of stable curves, with generic fiber smooth and having stable bad reduction at* $\mathfrak{p}$. *Let* L *be the quotient field of* A. *Then, the monodromy*

$$\pi_1^{alg}(\mathrm{Spec}L)\to \mathrm{Out}(\pi_1^l(\mathcal{X}_{\bar\eta}))$$

factors through $\pi_1^l(\mathrm{Spec}L)\cong\mathbb{Z}_l(1)$ *and the image is generated by a (nontrivial) product of Dehn twists along the vanishing cycles modulo* $\mathfrak{p}$.

This gives an explicit element in the intersection of the mapping class group action and the Galois action. Stimulated from this result, we considered which mapping class can be realized by the Galois action. Our Theorem 1.2 says that in many cases any mapping class can be realized.

Oda's conjecture. It is natural to consider how much larger $\rho^l_{G_K,X}(G_K)$ is than $\rho^l_{\hat\Gamma}(\widehat{\Gamma_{g,n}})$, in the case where the conclusion of Theorem 1.2 holds. Takayuki Oda conjectured the following (c.f. [26], and several seminars including those at RIMS around 1992), which motivated this study. For the definitions and properties of the filtrations, refer to [24].

Assume that B is the moduli stack of (g,n)-curves and C is the universal (g,n)-curves over B in §3 (c.f. [13][22]). Let Π^l be the pro-l fundamental group of a (g,n)-curve. Then, Π^l has a natural central filtration called *weight filtration*. For a group Γ, a group homomorphism $\rho:\Gamma\to\mathrm{Out}\Pi^l$ induces *induced filtration* on Γ, $\Gamma\supset\Gamma[0]\supset\Gamma[1]\supset\Gamma[2]\supset\cdots$. We denote

$$gr\Gamma:=\bigoplus_{m\geq 1} gr_m\Gamma:=\bigoplus_{m\geq1}\Gamma[m]/\Gamma[m+1],$$

which has a natural structure of a Lie algebra. By $\rho^l:\pi_1^{alg}(\mathcal{M}_{g,n})\to\mathrm{Out}\Pi^l$, we obtain induced filtrations $\pi_1^{alg}(\mathcal{M}_{g,n})[m]$, $\pi_1^{alg}(\overline{\mathcal{M}_{g,n}})[m]$. We define $\mathrm{G}_{\mathbb{Q}}[g,n,l;m]$ to be the image of $\pi_1^{alg}(\mathcal{M}_{g,n})[m]$ by $\pi_1^{alg}(\mathcal{M}_{g,n})\to\mathrm{G}_{\mathbb{Q}}$. Its graded quotients also yield a Lie algebra, which we denote $gr\mathrm{G}_{\mathbb{Q}}[g,n,l]$.

Conjecture 4.1. *Assume* $2g-2+n\geq1$. *Then, the filtration* $\mathrm{G}_{\mathbb{Q}}[g,n,l;m]$ *does not depend on the choice of* g,n.

¿From this, Oda also conjectured the existence of obstructions to the surjectivity of Johnson homomorphisms (introduced by Johnson[11][12], and named in [18]) other than the Morita's obstructions[19]. Conjecture 4.1 is nearly solved [15][20][9]: (1) $\cap_m \mathrm{G}_{\mathbb{Q}}[g,n,l;m]$ is independent of g,n, (2) $\mathrm{G}_{\mathbb{Q}}[g,n,l;m]$ and $\mathrm{G}_{\mathbb{Q}}[g',n',l;m]$ are commensurable; i.e., their intersection is open in each group. The existence of the obstruction to the surjectivity of Johnson morphism was also proved in [20].

Thus, the graded Lie algebra $gr\mathrm{G}_{\mathbb{Q}}[g,n,l] \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ is independent of (g,n). Note that $gr\mathrm{G}_{\mathbb{Q}}[0,3,l] \otimes \mathbb{Q}_l$ is the Galois realization of the "motivic" graded Lie algebra (c.f. §5 in [8], and [5]), which is conjecturally isomorphic to a free Lie algebra generated by a countably many generators, one for each $gr_{4k+2}\mathrm{G}_{\mathbb{Q}}[0,3,l]$, $k = 1,2,3,\ldots$ (see calculative results in [14][31]). Oda's conjecture gives a possibility to study this Lie algebra by using general (g,n).

There are researches on a Hodge-realization of this Lie algebra, e.g. [6]. It seems an interesting problem to consider a Hodge analogue of Oda's conjecture.

Note that $\rho_{G_K,X}$ gives an induced filtration $G_K[m]$ on G_K. If $G_K[0]/G_K[1]$ is $GSp(g,\mathbb{Z}_l)$ and $G_K[m]/G_K[m+1]$ is nontrivial for some $m \geq 1$, then Galois centralizer in $\mathrm{Out}(\Pi^l)$ is finite (see [21, Theorem 3.3] and its proof). Theorem 1.2 shows that for infinitely many curves the graded quotient of the induced filtration of G_K is bounded below, because so is the filtration of $\Gamma_{g,n}$ [27]. Hence this theorem implies the finiteness of the Galois centralizer for infinitely many curves, but now this finiteness is a consequence of more general pro-l Grothendieck conjecture for curves [16][17].

The profinite version of Oda's conjecture seems to have no sense (holds trivially without filtration, and no way to give a suitable filtration) in the presence of Theorem 1.2.

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