Non-symmetric orthogonal geometry of Grothendieck rings of coherent sheaves on projective spaces

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Abstract

In this paper we consider orthogonal geometry of the free $\mathbb{Z}$-module $K_0(\mathbb{P}_n)$ with respect to the non-symmetric unimodular bilinear form

$$\chi(E,F) = \sum(-1)^\nu \dim \operatorname{Ext}^\nu(E,F).$$

We calculate the isometry group of this form and describe invariants of its natural action on $K_0(\mathbb{P}_n)$. Also we consider some general constructions with non-symmetric unimodular forms. In particular, we discuss orthogonal decomposition of such forms and the action of the braid group on a set of semiorthonormal bases. We formulate a list of natural arithmetical conjectures about semiorthogonal bases of the form χ.

§1. Introduction.

1.1. The helix theory and the problem of description of exceptional sheaves on $\mathbb{P}_n$. The helix theory is a cohomology technique to study derived categories of coherent sheaves on some algebraic varieties. It appears first in [GoRu] and [Go1] as the way to construct the *exceptional bundles* on $\mathbb{P}_n$, i.e. locally free sheaves E such that

$$\dim \mathrm{Ext}^0(E,E) = 1,\ \mathrm{Ext}^i(E,E) = 0\ \forall i \geq 1$$

Since then the helix theory was developed in the context of general triangulated categories in [Go2],[Go3],[Bo1],[Bo2].[BoKa]. The main idea of this theory is to consider *exceptional bases* of a triangulated category, i.e. collections of objects $\{E_0, E_1, \ldots, E_n\}$ that generate the category and have the following properties

$$\dim \mathrm{Hom}^0(E_i,E_i) = 1\,, \quad \mathrm{Hom}^\nu(E_i,E_i) = 0\ \forall \nu \neq 1$$

$$\mathrm{Hom}^\mu(E_i,E_j) = 0\ \forall \mu \text{ and } \forall i > j.$$

The simplest example of a such collection is the collection

$$\{\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n)\}$$

of invertible sheaves on $\mathbb{P}_n$. The main problem is to describe all such collections. The most important fact in the study of this problem is that there exists an action of the braid group on the set of exceptional collections of a given length. Transformations of exceptional collections by generators of the braid group are called *mutations*. The mutations make possible to construct an infinite set of exceptional collection starting from a given one (see [Go1], [Go2]).

Thus a description of all exceptional sheaves on $\mathbb{P}_n$ splits into three steps. We have to prove the following three conjectures.

1.1.1. CONJECTURE. *Any exceptional object in the bounded derived category of coherent sheaves on $\mathbb{P}_n$ is quasiisomorphic to a shifted image of an exceptional locally free sheaf.*

1.1.2. CONJECTURE. *Any exceptional collection (in particular, each exceptional object itself) in the bounded derived category of coherent sheaves on $\mathbb{P}_n$ can be included in an exceptional basis of the derived category.*

1.1.3. CONJECTURE. *In bounded derived category of coherent sheaves on $\mathbb{P}_n$ the braid group acts transitively on exceptional collections of any given length.*

All these three conjectures hold on $\mathbb{P}_2$ (see [GoRu], [Go2]), and the third conjecture holds on $\mathbb{P}_3$ for exceptional collections of maximal length (generating the derived category, – see [No]). Discussion of these problems in the general context and the survey of corresponding results see in [Go4].

1.2. Subject of this paper. In this paper we consider an arithmetical analog of problems formulated above. Let us consider the Grothendieck group $K_0(\mathbb{P}_n)$ as a free $\mathbb{Z}$-module of finite rank with non-symmetric unimodular bilinear form

$$\chi(E,F) = \sum(-1)^\nu \dim \mathrm{Ext}^\nu(E,F).$$

1.2.1. DEFINITION. *A collection of vectors* $\{e_1, e_2, \dots, e_k\} \subset K_0(\mathbb{P}_n)$ *is called exceptional or semiorthonormal if the Gram matrix of the form* χ *at this collection is upper triangular with units on the main diagonal.*

Obviously, any exceptional collection of sheaves produces an semiorthonormal collection of vectors in K_0. If we are going to work only in terms of K_0 and χ, then we can not distinguish such collections and their images with respect to the action of isometries of the form χ.

1.2.2. DEFINITION. *$\mathbb{Z}$-linear operator* $\varphi\colon K_0(\mathbb{P}_n) \longrightarrow K_0(\mathbb{P}_n)$ *is called isometric if* $\chi(v,w) = \chi(\varphi v, \varphi w)$ $\forall v, w$. *The group of all isometric operators is denoted by Isom and is called the **isometry group**.*

In §3,§4 we will prove that this group is an unipotent Abelian algebraic group of dimension $[(n+1)/2]$. It has two connected components, and the component of the identity is a direct sum of standard 1-dimensional additive groups. We write explicit formulas for the natural action of isometries on $K_0(\mathbb{P}_n)$ and describe invariants of this action. All this may be considered as the first step in the direction of the following conjecture.

1.2.3. CONJECTURE. *A vector* e *such that* $\chi(e,e) = 1$ *represents (up to the action of isometries) a class of exceptional sheaf if and only if it can be included in some semiorthonormal basis of* K_0.

In §2 we consider some general constructions of non-symmetric orthogonal geometry. In particular, we define an action of the braid group on the set of all semiorthonormal collections of a given length, and introduce the notions of the *canonical operator* and the *canonical algebra* of given bilinear form, which play an important role in the general classification of bilinear non-degenerate forms. We discuss this classification (over an algebraically closed field of characteristic zero) in §3 and give a general approach to the problems like ones considered in [Ru].

This part of paper is the first little step in direction of the following conjecture, which reformulate the main conjecture of helix theory in terms of linear algebra.

1.2.4. CONJECTURE. *Any semiorthonormal basis of* $K_0(\mathbb{P}_n)$ *may be obtained from any other one by changing signs of basic vectors and the action of braid group and isometries.*

§2. Non-symmetric orthogonal geometry.

2.1. Notations. Let M be a free $\mathbb{Z}$-module of a finite rank equipped with an integer bilinear form $M \times M \longrightarrow \mathbb{Z}$, which we denote by

$$\langle *, * \rangle : \; v, w \mapsto \langle v, w \rangle .$$

Submodules of M will be usually denoted by $U, V, W, \ldots$ Restriction of the bilinear form onto submodule $U \subset M$ is denoted by $\langle *, * \rangle_U$.

For a given submodule $U \subset M$ the submodules

$$U^{\perp} = \{ w \in M \,|\, \langle w, u \rangle = 0 \; \forall u \in U \}$$

$$^{\perp}U = \{ w \in M \,|\, \langle u, w \rangle = 0 \; \forall u \in U \}$$

are called *right* and *left* orthogonals of U. So, $\langle U, U^{\perp} \rangle = \langle {}^{\perp}U, U \rangle = 0$.

Fixing some basis $e = \{e_1, e_2, \ldots, e_n\} \subset M$ we denote by χ or $\chi(e)$ the corresponding Gram matrix (the element $\chi_{ij} = \langle e_i, e_j \rangle$ is placed in i-th row and j-th column of this matrix).

We denote by M^* the dual module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$.

2.2. Unimodularity and correlations. For any given bilinear form on M we can consider two linear operators:

left correlation $\lambda : \; M \longrightarrow M^* : \; v \mapsto \langle v, * \rangle$

and

right correlation $\varrho : \; M \longrightarrow M^* : \; v \mapsto \langle *, v \rangle$.

The bilinear form on M is uniquely determinated by each of them. In fact, the Gram matrix χ coincides with a matrix of right correlation written with respect to a pair of dual bases of M and M^*, and a matrix of the left correlation is its transpose. Hence, we get the following proposition:

2.2.1. PROPOSITION. *The conditions:*

(A) left correlation is an isomorphism;

(B) right correlation is an isomorphism;

(C) $\det \chi = \pm 1$;

are pairwise equivalent.

□

We call a bilinear form on M to be *unimodular* if it satisfies the above conditions.

In this paper we will always assume that the form on M is unimodular.

Note that if we identify M with M^{**} in the usual way, then the dual operator to each of two correlations coincides with the other correlation:

$$\varrho^* : \; M^{**} = M \longrightarrow M^* \quad \text{is equal to} \quad \lambda : \; M \longrightarrow M^*$$

$$\lambda^*: M^{**} = M \longrightarrow M^* \quad \text{is equal to} \quad \varrho: M \longrightarrow M^*$$

2.3. Canonical operator. Using correlations, we associate with any unimodular bilinear form on M a linear operator

$$\kappa = \varrho^{-1}\lambda: M \longrightarrow M.$$

This operator is called *canonical*. It is uniquely determinated by the following condition:

$$\langle v, w \rangle = \langle w, \kappa v \rangle \ \forall v, w \in M.$$

A matrix of κ in any basis e of M is expressed in terms of the Gram matrix $\chi = \chi(e)$ by the formula

$$\kappa = \chi^{-1}\chi^t$$

Note that the map $\chi \mapsto \chi^{-1}\chi^t$ is equivariant with respect to the standard actions of linear automorphisms of M on bilinear forms and on linear operators.

The canonical operator is isometric:

$$\langle v, w \rangle = \langle w, \kappa v \rangle = \langle \kappa v, \kappa w \rangle \ \forall v, w \in M$$

2.4. Dual operators and canonical algebra. One can associate with any linear operator $\varphi: M \longrightarrow M$ a pair of its dual. They are uniquely determinated by the following conditions:

the *left dual operator* ${}^\vee\varphi$: $\langle {}^\vee\varphi v, w \rangle = \langle v, \varphi w \rangle$

the *right dual operator* $\varphi^\vee$: $\langle v, \varphi^\vee w \rangle = \langle \varphi v, w \rangle$.

Expressions for their matrices are given by

$${}^\vee\varphi = (\chi^{-1})^t \varphi^t \chi^t ; \qquad \varphi^\vee = \chi^{-1}\varphi^t\chi.$$

In general, ${}^\vee\varphi \neq \varphi^\vee$ for a non-symmetric form on M. But direct computation shows that the following proposition holds.

2.4.1. PROPOSITION. *The conditions*

(A) ${}^\vee\varphi = \varphi^\vee$

(B) ${}^{\vee\vee}\varphi = \varphi$

(C) $\varphi^{\vee\vee} = \varphi$

(D) $\varphi\kappa = \kappa\varphi$

are pairwise equivalent.

□

An operator φ is called *reflexive* if $\varphi^{\vee\vee} = \varphi$. Reflexive operators form a subalgebra in the algebra of all linear endomorphisms of M. This algebra coincides with the centralizer of the canonical operator. We call it a *canonical algebra* of a bilinear form on M and denote by $\mathcal{A}$.

Obviously, the following sets of operators belong to $\mathcal{A}$:
$\mathcal{A}^+ = \{\varphi \mid {}^\vee\varphi = \varphi = \varphi^\vee\}$ — the submodule of *selfdual* operators;
$\mathcal{A}^- = \{\varphi \mid {}^\vee\varphi = -\varphi = \varphi^\vee\}$ — the submodule of *antiselfdual* operators;
$Isom = \{\varphi \mid {}^\vee\varphi = \varphi^{-1} = \varphi^\vee\}$ — the subgroup of *isometric* operators.

Let us now consider the vector space $M_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} M$ and the corresponding canonical algebra $\mathcal{A}_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{A}$. Evidently, $\mathcal{A}_{\mathbb{Q}}$ splits into direct sum of the subspaces of selfdual and antiselfdual operators

$$\mathcal{A}_{\mathbb{Q}} = \mathcal{A}^+_{\mathbb{Q}} \oplus \mathcal{A}^-_{\mathbb{Q}}.$$

Isom is an algebraic group in the sense of [Se] and the arguments of [Se] (ch.1,th.5) give us immediately the following proposition.

2.4.2. PROPOSITION. *Subspace of all antiselfdual operators coincides with the Lie algebra of algebraic group of all isometric operators:*

$$\mathcal{L}ie(Isom_{\mathbb{Q}}) = \mathcal{A}^-_{\mathbb{Q}}$$

□

On the other side, we have

2.4.3. PROPOSITION. *Two bilinear forms $\langle *,*\rangle_1$ and $\langle *,*\rangle_2$ on a given $\mathbb{Z}$-module M have the same canonical operator if and only if there exists an operator ψ, which is selfdual with respect to both forms and satisfies the identity*

$$\langle v,w\rangle_1 = \langle v,\psi w\rangle_2 \ \forall v\ , w \in M.$$

PROOF. Taking the dual to the identity

$$\kappa = \varrho_1^{-1}\lambda_1 = \varrho_2^{-1}\lambda_2$$

we get

$$\kappa^* = \varrho_1\lambda_1^{-1} = \varrho_2\lambda_2^{-1}.$$

Hence, $\psi = \varrho_2\varrho_1^{-1} = \lambda_2\lambda_1^{-1}$ satisfies the identities

$$\langle v,\psi w\rangle_2 = \langle v,w\rangle_1 = \langle \psi v,w\rangle_2.$$

□

2.4.4. COROLLARY. *There exists the 1-1 correspondence between unimodular bilinear forms, which have same canonical operators, and unimodular operators, which are selfdual with respect to any one of these forms.*

□

2.5. Admissible submodules, orthogonal projections and mutations. All constructions of this and two following sections are trivial reformulations of the technique of orthogonal decomposition in triangulated categories (see [Go2] and [BoKa1]).

A submodule $U \subset M$ is called *admissible* if it satisfies any of the following equivalent conditions:

(A) the restricted form $\langle *, * \rangle_U$ is unimodular;

(B) there exists a linear projection $\mathrm{Lp}_U : M \longrightarrow U$ such that

$$\langle v, u \rangle_M = \langle \mathrm{Lp}_U v, u \rangle_U \; \forall u \in U \; \forall v \in V;$$

(C) there exists a linear projection $\mathrm{Rp}_U : M \longrightarrow U$ such that

$$\langle u, v \rangle_M = \langle u, \mathrm{Rp}_U v \rangle_U \; \forall u \in U \; \forall v \in V;$$

(D) $M = U \oplus {}^{\perp}U$;

(E) $M = U^{\perp} \oplus U$;

(equivalences (A)⇔(B)⇔(D) and (A)⇔(C)⇔(E) are standard in linear algebra and we omit proofs).

Operators Lp_U from (B) and Rp_U from (C) are called *left* and *right orthogonal projections onto* U respectively. Note that they are left and right adjoint operators to the inclusion $U \hookrightarrow M$.

Of course, if a submodule $U \subset M$ is admissible, then both its orthogonals ${}^{\perp}U$ and $U^{\perp}$ are admissible too. If a vector $v \in M$ is written in the form

$$v = v_{U^{\perp}} + v_U \;, \qquad \text{where } v_{U^{\perp}} \in U^{\perp} \;,\; v_U \in U,$$

then obviously

$$\mathrm{Rp}_U(v) = v_U \;, \quad \mathrm{Lp}_{U^{\perp}}(v) = v_{U^{\perp}}.$$

Hence, we have the identities

$$v = \mathrm{Lp}_{U^{\perp}}(v) + \mathrm{Rp}_U(v) = \mathrm{Lp}_U(v) + \mathrm{Rp}_{{}^{\perp}U}(v) \;,$$

which give decompositions of any vector $v \in M$ as an element of $U^{\perp} \oplus U$ and as an element of $U \oplus {}^{\perp}U$ respectively. It we rewrite these identities in the form

$$\mathrm{Lp}_{U^{\perp}}(v) = \underbrace{\mathrm{Lp}_U(v) - \mathrm{Rp}_U(v)}_{\text{from } U} + \underbrace{\mathrm{Rp}_{{}^{\perp}U}(v)}_{\text{from } {}^{\perp}U}$$

or in the form

$$\mathrm{Rp}_{^\perp U}(v) = \underbrace{\mathrm{Lp}_{U^\perp}(v)}_{\text{from } U^\perp} + \underbrace{\mathrm{Rp}_U(v) - \mathrm{Lp}_U(v)}_{\text{from } U}$$

then we see that

$$\mathrm{Rp}_{^\perp U} \circ \mathrm{Lp}_{U^\perp} = \mathrm{Rp}_{^\perp U} \qquad \text{and} \qquad \mathrm{Lp}_{U^\perp} \circ \mathrm{Rp}_{^\perp U} = \mathrm{Lp}_{U^\perp} \,.$$

Hence, the restriction of $\mathrm{Lp}_{U^\perp}$ onto the submodule $^\perp U$ and the restriction of $\mathrm{Rp}_{^\perp U}$ onto the submodule $U^\perp$ are linear isomorphisms between these submodules inverse to each other.

We call them *left mutation of* $^\perp U$ and *right mutation of* $U^\perp$ with respect to U and denote by $\mathrm{Lm}_U : \, ^\perp U \longrightarrow U^\perp$ and $\mathrm{Rm}_U : \, U^\perp \longrightarrow {^\perp U}$. They are well defined by the properties:

$$\mathrm{Lm}_U : \mathrm{Rp}_{^\perp U} v \mapsto \mathrm{Lp}_{U^\perp} v \;\; \forall v \in M$$

$$\mathrm{Rm}_U : \mathrm{Lp}_{U^\perp} v \mapsto \mathrm{Rp}_{^\perp U} v \;\; \forall v \in M.$$

Moreover, direct computation shows that they are isometries. In fact, $\forall\, u_1^\perp, u_2^\perp \in U^\perp$, which are decomposed in $U \oplus {^\perp U}$ as $u_\nu^\perp = u_\nu + {^\perp u_\nu}$ $(\nu = 1, 2)$ we have

$$\begin{aligned} \langle u_1^\perp, u_2^\perp \rangle &= \langle u_1 + {^\perp u_1}, u_2^\perp \rangle = \langle {^\perp u_1}, u_2^\perp \rangle = \langle {^\perp u_1}, u_2 + {^\perp u_2} \rangle = \\ &= \langle {^\perp u_1}, {^\perp u_2} \rangle. \end{aligned}$$

So, we get

2.5.1. PROPOSITION. *Left and right mutations with respect to admissible submodule $U \subset M$ are isometric isomorphisms between $^\perp U$ and $U^\perp$ inverse to each other.*

□

2.6. Semiorthogonal direct sums. Let M_1 and M_2 be two modules equipped with unimodular bilinear forms $\langle *, * \rangle_1$ and $\langle *, * \rangle_2$. Suppose that direct sum $M = M_1 \oplus M_2$ is equipped with bilinear form $\langle *, * \rangle_M = \langle *, * \rangle$ such that $M_1 = M_2^\perp$ (i.e. $\langle M_2, M_1 \rangle = 0$) and the restrictions of $\langle *, * \rangle$ onto M_1 and M_2 coincide with $\langle *, * \rangle_1$ and $\langle *, * \rangle_2$ respectively. In this case we call M to be a *semiorthogonal sum* of M_1, M_2.

The form on M is automatically unimodular too and we have

$$\langle u_1 + u_2, v_1 + v_2 \rangle = \langle u_1, v_1 \rangle_1 + \langle u_1, v_2 \rangle_M + \langle u_2, v_2 \rangle_2 \,,$$

where the second summand can be expressed in two ways:

$$\langle u_1, v_2 \rangle_M = \langle \mathrm{Lp}_2 u_1, v_2 \rangle_2 = \langle u_1, \mathrm{Rp}_1 v_2 \rangle_1$$

(here $\mathrm{Lp}_2 = \mathrm{Lp}_{M_2}|_{M_1}$ and $\mathrm{Rp}_1 = \mathrm{Rp}_{M_1}|_{M_2}$ are the orthogonal projections).

Hence, the structure of semiorthoghonal sum is uniquely determinated by any one of two adjoint to each other linear operators

$$\mathrm{Lp}_2 : M_1 \longrightarrow M_2 \qquad \text{or} \qquad \mathrm{Rp}_1 : M_2 \longrightarrow M_1 .$$

2.6.1. PROPOSITION. *Let $M = M_1 \oplus M_2$ be a semiorthogonal sum. The canonical operator $\kappa = \kappa_M$ is represented in terms of this decomposition by the following matrix*

$$\begin{pmatrix} \kappa_1 - \mathrm{Rp}_1 \circ \kappa_2 \circ \mathrm{Lp}_2 & -\mathrm{Rp}_1 \kappa_2 \\ \kappa_2 \mathrm{Lp}_2 & \kappa_2 \end{pmatrix} ,$$

where κ_1 and κ_2 are the canonical operators on M_1 and M_2 respectively.

PROOF. Let

$$\kappa_M = \begin{pmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{pmatrix} ,$$

where $\kappa_{\mu\nu} : M_\nu \longrightarrow M_\mu$. For any $u = u_1 + u_2$, $v = v_1 + v_2$ consider the identity

$$\langle u, v \rangle = \langle v, \kappa u \rangle .$$

We write its left side as

$$\begin{aligned} \langle u, v \rangle &= \langle u_1, v_1 \rangle_1 + \langle \mathrm{Lp}_2 u_1, v_2 \rangle_2 + \langle u_2, v_2 \rangle_2 = \\ &= \langle v_1, \kappa_1 u_1 \rangle + \langle v_2, \kappa_2 \mathrm{Lp}_2(u_1) \rangle + \langle v_2, \kappa_2 u_2 \rangle \end{aligned}$$

and its right side as

$$\langle v, \kappa u \rangle = \langle v_1, \kappa_{11} u_1 + \kappa_{12} u_2 \rangle + \langle v_1, \mathrm{Rp}_1(\kappa_{21} u_1 + \kappa_{22} u_2) \rangle + \langle v_2, \kappa_{21} u_1 + \kappa_{22} u_2 \rangle .$$

Comparing the components, we obtain

$$\kappa_1 u_1 = \kappa_{11} u_1 + \kappa_{12} u_2 + \mathrm{Rp}_1 \kappa_{21} u_1 + \mathrm{Rp}_1 \kappa_{22} u_2$$

$$\kappa_2 \mathrm{Lp}_2 u_1 + \kappa_2 u_2 = \kappa_{21} u_1 + \kappa_{22} u_2$$

It follows from the second equation that $\kappa_{22} = \kappa_2$ and $\kappa_{21} = \kappa_2 \circ \mathrm{Lp}_2$. Hence the first equation may be rewritten as

$$\begin{cases} \kappa_1 &= \kappa_{11} + \mathrm{Rp}_1 \circ \kappa_2 \circ \mathrm{Lp}_2 \\ 0 &= \kappa_{12} + \mathrm{Rp}_1 \circ \kappa_2 \end{cases} .$$

□

2.6.2. DEFINITION. *A pair (U, V) of submodules $U \subset M$, $V \subset M$ is called semiorthogonal if both are admissible and $\langle v, u \rangle = 0$ $\forall u, v \in M$.*

Let $W = U \oplus V$ be the direct sum of two admissible submodules forming a semiorthogonal pair (U, V). Of course, W is admissible too and $W^\perp = U^\perp \cap V^\perp$, ${}^\perp W = {}^\perp U \cap {}^\perp V$.

2.6.3. PROPOSITION. *In above notations* $\mathrm{Lp}_{W^\perp} = \mathrm{Lp}_{U^\perp} \circ \mathrm{Lp}_{V^\perp}$.

PROOF. For any $w^\perp \in W^\perp = U^\perp \cap V^\perp$ we have $\forall m \in M$:

$$\langle m, w^\perp \rangle = \langle \mathrm{Lp}_{V^\perp} m, w^\perp \rangle = \langle \mathrm{Lp}_{U^\perp} \mathrm{Lp}_{V^\perp} m, w^\perp \rangle.$$

On the other side, $\mathrm{im}\,(\mathrm{Lp}_{U^\perp} \circ \mathrm{Lp}_{V^\perp}) \subset W^\perp$ (this follows from the assumption that $U \subset V^\perp$).

□

By the same way we conclude

2.6.4. PROPOSITION. $\mathrm{Rp}_{\perp W} = \mathrm{Rp}_{\perp V} \circ \mathrm{Rp}_{\perp U}$.

□

These propositions give

2.6.5. COROLLARY. *In above notations*

$$\mathrm{Lm}_W = \mathrm{Lm}_U \circ \mathrm{Lm}_V$$

$$\mathrm{Rm}_W = \mathrm{Rm}_V \circ \mathrm{Rm}_U.$$

□

2.7. Semiorthonormal collections and Braid Group action. The simplest example of an admissible submodule is a 1-dimensional free submodule $\mathbb{Z}e \in M$ generated by a vector e with $\langle e, e \rangle = 1$. In this case we have

$$\begin{aligned} \mathrm{Rp}_e v &= \langle e, v \rangle e\,, & \mathrm{Lp}_e v &= \langle v, e \rangle e\,, \\ \mathrm{Lm}_e v &= \mathrm{Lp}_{e^\perp} v = v - \langle e, v \rangle e\,, & \mathrm{Rm}_e v &= \mathrm{Rp}_{\perp e} v = v - \langle v, e \rangle e\,. \end{aligned}$$

If $W = {}^\perp e$, then it follows from (2.6) that the scalar product on the semiorthogonal sum $M = \mathbb{Z}e \oplus W$ is uniquely determinated by the scalar product $\langle *, * \rangle_W$ on W and the vector $\ell = \mathrm{Lp}_W e \in W$.

By (2.6.1), the canonical operator $\kappa = \kappa_M$ is represented in terms of the decomposition $M = \mathbb{Z}e \oplus W$ by the matrix

$$\kappa_M = \begin{pmatrix} \lambda & \psi \\ \kappa_W(\ell) & \kappa_W \end{pmatrix},$$

where

$$\psi \colon W \longrightarrow \mathbb{Z}e \colon w \mapsto -\langle e, \kappa_2 w \rangle \cdot e = \langle \ell, \kappa_2 w \rangle_W \cdot e$$

and

$$\lambda = 1 - \langle e, \kappa_2 \ell \rangle_M = 1 - \langle \ell, \kappa_2 \ell \rangle_W = 1 - \langle \ell, \ell \rangle_W\,.$$

On the other hand $\langle \kappa_M e, e \rangle = \langle \lambda e, e \rangle + \langle \kappa_W(\ell), e \rangle = \lambda$. We get

2.7.1. PROPOSITION. $\mathrm{tr}\,(\kappa_M) = \mathrm{tr}\,(\kappa_W) + \langle \kappa_M e, e \rangle = \mathrm{tr}\,(\kappa_W) + 1 - \langle \ell, \ell \rangle$ *and* $\langle \kappa_M e, e \rangle = 1 - \langle \ell, \ell \rangle$.

□

Of course, not only one vector but any semiorthonormal collection of $(k+1)$ vectors $\{e_0, e_1, \ldots, e_k\}$ (i.e. such that $\langle e_\mu, e_\nu\rangle = 0\ \forall \mu > \nu$, $\langle e_\nu, e_\nu\rangle = 1\ \forall \nu$) generate an admissible submodule W. For a such submodule W we have

$$\mathrm{Lm}_W = \mathrm{Lm}_{e_0} \circ \mathrm{Lm}_{e_1} \circ \cdots \circ \mathrm{Lm}_{e_k},$$

$$\mathrm{Rm}_W = \mathrm{Rm}_{e_k} \circ \mathrm{Rm}_{e_{k-1}} \circ \cdots \circ \mathrm{Rm}_{e_0}$$

Let us consider a module M generated by a semiorthonormal pair (a, b). Define the *left* and *right mutations* of this pair by the formulas

$$L(a,b) \stackrel{\mathrm{def}}{=} (\mathrm{Lm}_a b, a) = (b - \langle a,b\rangle a, a)$$

$$R(a,b) \stackrel{\mathrm{def}}{=} (b, \mathrm{Rm}_b a) = (b, a - \langle a,b\rangle b)$$

Note that these mutations may be considered as results of two Gram-Schmidt orthogonalisations applied to the non-semiorthogonal pair (b, a) in two possible ways.

It follows from (2.5.1) that

$$RL(a,b) = LR(a,b) = (a,b).$$

Now, let us consider a module M generated by a semiorthonormal triple (a, b, c) and its submodule W generated by (a, b). In this case ${}^\perp W$ is generated by c. We can calculate the mutation $\mathrm{Lm}_W c$ in two ways: taking in (2.6.5) U generated by a, V generated by b, or taking U generated by $\mathrm{Lm}_a b$, V generated by a. The results must be the same, and we get some kind of the *triangle equation*:

$$\mathrm{Lm}_a \mathrm{Lm}_b c = \mathrm{Lm}_{\mathrm{Lm}_a b} \mathrm{Lm}_a c.$$

To present last two identities in more conceptual form let us consieder a submodule $W \subset M$ generated by semiorthonormal collection $\{e_0, e_1, \ldots, e_k\}$. Denote by L_ν and R_ν the operations, which change the pair $(e_{\nu-1}, e_\nu)$ by its left and right mutations respectively ($\nu = 1, 2, \ldots, k$). From our identities it follows immediately the following

2.7.2. PROPOSITION. *Operations L_ν and R_ν satisfy the identities:*

$$\begin{array}{rcll} R_\nu L_\nu &=& L_\nu R_\nu = \mathrm{Id} & \\ L_\nu L_{\nu-1} L_\nu &=& L_{\nu-1} L_\nu L_{\nu-1} & \text{for} \quad \nu = 2,3,\ldots,k \\ L_\mu L_\nu &=& L_\nu L_\mu & \text{for} \quad \mu,\nu : |\mu - \nu| > 1 \end{array}$$

□

2.7.3. COROLLARY. *The braid group acts by left mutations of neighboring pairs on the set of semiorthonormal bases of an admissible submodule.*

□

§3. Decomposition of bilinear forms via canonical operator.

3.1. Notations. In this paragraph we consider a vector space V over an algebraically closed field K of characteristic 0 equipped with a non-degenerate bilinear form $\langle *, * \rangle$.

The pair $\{V, \langle *, * \rangle\}$ is called *decomposable*, if $V = U \oplus W$, the restrictions $\langle *, * \rangle_U$ and $\langle *, * \rangle_W$ are non-degenerate (in particular, $U \neq 0$, $W \neq 0$), and U, W are *biorthogonal* to each other, i.e. $\langle U, W \rangle = \langle W, U \rangle = 0$.

We are going to classify in all indecomposable pairs $\{V, \langle *, * \rangle\}$ up to isometric isomorphisms. The following proposition shows that the answer may be given in terms of the canonical operator of the form on V.

3.1.1. PROPOSITION. *Let $\{V, \langle *, * \rangle_V\}$, $\{U, \langle *, * \rangle_U\}$ be two spaces with non-degenerate bilinear forms. They are isometrically isomorphic to each other if and only if there exists an isomorphism $\psi\colon U \longrightarrow V$ such that $\psi \kappa_V = \kappa_U \psi$, where κ_U and κ_V are the canonical operators of the forms on U and V.*

PROOF. We may assume that two different forms $\langle *, * \rangle_1$ and $\langle *, * \rangle_2$ on the same vector space V are given and that these two forms have the same canonical operator κ. It is sufficient to prove that in this case there exists a linear isomorphism $\varphi\colon V \longrightarrow V$ such that

$$\langle v, w \rangle_1 = \langle \varphi v, \varphi w \rangle_2 \ \ \forall v\,, w \in V.$$

In (2.4.3) we have seen that there exists a selfdual operator $\psi\colon V \longrightarrow V$ such that

$$\langle v, w \rangle_1 = \langle v, \psi w \rangle_2 \ \ \forall v\,, w \in V.$$

Over an algebraically closed field K of characteristic 0 we can find a polynomial $F(t) \in K[t]$ such that the operator $\varphi \stackrel{\text{def}}{=} F(\psi)$ satisfy the equation $\varphi^2 = \psi$. Since φ is selfdual too, we obtain $\langle v, w \rangle_1 = \langle \varphi v, \varphi w \rangle_2 \ \ \forall v\,, w \in V$.
□

So, non-degenerate non-symmetric bilinear form over algebraically closed field of characteristic zero is uniquely determinated by Jordan normal form ot its canonical operator. We will describe the correspondence between the root decomposition of the canonical operator and biorthogonal decomposition of original bilinear form. These results are not new and actually they may be extracted from classical books [HoPe] (Book 2, Ch.IX) and [Ma].

For any linear operator $\varphi\colon V \longrightarrow V$ we will usually denote by λ, μ, ... its eigenvalues and by φ_λ, φ_μ, ... — corresponding differerences $\varphi - \lambda E$, $\varphi - \mu E$, ... The root subspaces, which corresponds to these eigenvalues, will be denoted by V_λ, V_μ, ... So,

$$V_\mu = \bigcup_{n \in \mathbb{N}} \ker \varphi_\mu^n.$$

3.2. Decomposition of an isometry. Let $\varphi\colon V \longrightarrow V$ be an isometric operator with eigenvalues λ, μ and let

$$v_m \overset{\varphi_\lambda}{\mapsto} v_{m-1} \overset{\varphi_\lambda}{\mapsto} \cdots \overset{\varphi_\lambda}{\mapsto} v_0 \overset{\varphi_\lambda}{\mapsto} v_{-1} = 0$$

$$w_k \overset{\varphi_\mu}{\mapsto} w_{k-1} \overset{\varphi_\mu}{\mapsto} \cdots \overset{\varphi_\mu}{\mapsto} w_0 \overset{\varphi_\mu}{\mapsto} w_{-1} = 0$$

be any two Jordan chains for operators $\varphi_\lambda = \varphi - \lambda E$ and $\varphi_\mu = \varphi - \mu E$. Then for any $0 \le i \le m$, $0 \le j \le k$ we have

$$\langle v_i, w_j\rangle = \langle \varphi v_i, \varphi w_j\rangle = \langle \lambda v_i + v_{i-1}, \mu w_j + w_{j-1}\rangle$$

and hence

$$(1 - \lambda\mu)\langle v_i, w_j\rangle = \lambda\langle v_i, w_{j-1}\rangle + \mu\langle v_{i-1}, w_j\rangle + \langle v_{i-1}, w_{j-1}\rangle.$$

Using decreasing induction we obtain that for $\lambda\mu \ne 1$

$$\langle v_i, w_j\rangle = \langle w_j, v_i\rangle = 0.$$

We have proved

3.2.1. PROPOSITION. *If $\lambda\mu \ne 1$, then two root subspaces V_λ and V_μ of any isometry $\varphi\colon V \longrightarrow V$ are biorthogonal to each other.*

□

3.2.2. COROLLARY. *Let $\varphi\colon V \longrightarrow V$ be an an isometry of a space V equipped with non-degenerate bilinear form. Then V splits into biorthogonal direct sum of subspaces W_μ, where:*

– for $\mu = \pm 1$ W_μ coincides with the root subspace V_μ of φ and restriction of the original form onto W_μ is nondegenerate;

– for $\mu \ne \pm 1$ W_μ coincides with the direct sum of root subspaces $V_\mu \oplus V_{\mu^{-1}}$ and the original form restricts trivially onto each of these two root subspaces and induces a nondegenerate pairing between them.

□

In order to clarify the action of φ on the subspaces $W_\mu = V_\mu \oplus V_{\mu^{-1}}$ we denote V_μ by V_+ and $V_{\mu^{-1}}$ by V_-. Let us identify V_-^* with V_+ using non-degenerate pairing $\langle V_+, V_-\rangle$. So, we can consider the dual to a linear operator $f\colon V_+ \longrightarrow V_+$ as the operator $f^*\colon V_- \longrightarrow V_-$ defined by the formula

$$\langle v_+, f^* v_-\rangle = \langle f v_+, v_-\rangle \quad \forall v_+ \in V_+ \;\; \forall v_- \in V_-.$$

Finally, consider two nilpotent operators

$$\varphi_+ \overset{\text{def}}{=} (\varphi - \mu E)|_{V_+} : V_+ \longrightarrow V_+ \,;$$

$$\varphi_- \overset{\text{def}}{=} (\varphi - \mu^{-1} E)|_{V_-} : V_- \longrightarrow V_- \,.$$

In the case $\mu = \pm 1$ we put $V_+ = V_- = V_\mu$ and $\varphi_+ = \varphi_- = \varphi_\mu$.

3.2.3. PROPOSITION. *The following formulae hold:*

(A) $\varphi_+^* = -\mu\varphi^{-1}|_{V_-}\varphi_-$;

(B) $\ker(\varphi_-^k) = \operatorname{Ann}(\operatorname{im}(\varphi_+^k))$;

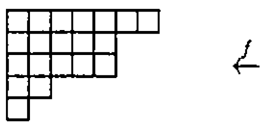
(C) $\operatorname{im}(\varphi_-^k) = \operatorname{Ann}(\ker(\varphi_+^k))$;

PROOF. The first formula is checked by direct computation.

The other two follow from the first one by using the fact that for any pair of dual linear operators $f\colon V \longrightarrow V$ and $f^*\colon V^* \longrightarrow V^*$ we have $\ker(f) = \operatorname{Ann}(\operatorname{im}(f^*))$ and $\operatorname{im}(f) = \operatorname{Ann}(\ker(f^*))$. In our case one have to put $f = \varphi_+^k$, $f^* = (-\mu)^k\varphi^{-k}|_{V_-}\varphi_-^k$ and to note that $\ker(f^*) = \ker(\varphi_-^k)$ and $\operatorname{im}(f^*) = \operatorname{im}(\varphi_-^k)$, because $(-\mu)^k\varphi^{-k}|_{V_-}$ is an isomorphism, which commute with φ_-.
□

3.2.4. COROLLARY. *Nilpotent operators* φ_+ *and* φ_- *have the same Jordan normal form (the same cycle type).*

PROOF. It is convenient to represent an Jordan basis of a given nilpotent operator f on a space W by the Young diagram like the following one:



The cells of this diagram are in 1-1 correspondence with the basic vectors of Jordan basis and f takes each cell to its left neighboring and takes the cells from the first left column to zero.

In terms of such representation the sum $S_k(f)$ of lengths of the first left k columns is equal to the $\dim\ker(f^k)$. From the other hand, the number of cells forming these k columns coincides with the number of cells forming a basis of a direct complement to the subspace $\operatorname{im}(f^k)$ (these cells are at the right side of the Young diagram). So,

$$S_k(f) = \dim\ker(f^k) = \dim W - \dim\operatorname{im}(f^k).$$

It follows from above proposition that in our case

$$\dim\ker(\varphi_+^k) = \dim(V_-) - \dim\operatorname{im}(\varphi_-^k).$$

Hence, $S_k(\varphi_+) = S_k(\varphi_-)\ \forall k$ and our operators have the same Young diagram.
□

3.3. Decomposition of the canonical operator. Suppose now that V is indecomposable and apply the previous results to the canonical operator $\varphi = \kappa$. We see that there are two cases.

In the first case, which we call for a moment *the* μ*-case*,

$$V = V_+ \oplus V_- \quad \kappa|_{V_+} = \mu E + \eta_+ \quad \kappa|_{V_-} = \mu^{-1}E + \eta_-,$$

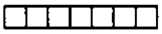
where $\mu \neq \pm 1$ and η_+, η_- are nilpotent operators of the same cycle type. The restrictions $\langle *, * \rangle_{V_+}$, $\langle *, * \rangle_{V_-}$ are zeros and pairing $\langle V_+, V_- \rangle$ is non-degenerate.

In the second case, which we call for a moment *the ε-case*

$$\kappa = \varepsilon E + \eta,$$

where $\varepsilon = \pm 1$ and η is nilpotent.

The exact description of Jordan normal form of $\eta_\pm$ and η in these two cases will be given in two consequent propositions below.

3.3.1. PROPOSITION. *If V is indecomposable and the μ-case takes place, then η_+ and η_- have only one Jordan cycle of the same length, i.e. the Young diagrams of $\eta_\pm$ are of the form*  .

PROOF. Denote by $K_\pm$ the kernel subspaces $\ker(\eta_\pm) \subset V_\pm$. We fix some Jordan basis for η_+ in V_+ and denote by C_+ the direct complement to the image subspace $\operatorname{im}(\eta_+) \subset V_+$ induced by this choice, so $V_+ = C_+ \oplus \operatorname{im}(\eta_+)$. Let

$$e_k^+ \overset{\eta_+}{\mapsto} e_{k-1}^+ \overset{\eta_+}{\mapsto} \cdots \overset{\eta_+}{\mapsto} e_0^+ \overset{\eta_+}{\mapsto} e_{-1}^+ = 0$$

be the Jordan chain of maximal length for η_+ (corresponding to the upper row of the Young diagram of η_+), L_+ be its linear span, and W_+ be the linear span of all others basic vectors, i.e. $V_+ = L_+ \oplus W_+$.

It follows from proposition 3.2.3 that the pairing $\langle C_+, K_- \rangle$ is non-degenerate. We fix the basis of K_-, which is dual to the basis of C_+ fixed above, and consider the vector e_0^- of this basis such that $\langle e_k^+, e_0^- \rangle = 1$ and $\langle e^+, e_0^- \rangle = 0$ for all other basic vectors $e^+ \in C_+$. It follows from (3.2.3) and (3.2.4) that automatically $e_0^- \in \operatorname{im}(\eta_-^k)$. Hence, this vector can be included in a Jordan cain

$$e_k^- \overset{\eta_-}{\mapsto} e_{k-1}^- \overset{\eta_-}{\mapsto} \cdots \overset{\eta_-}{\mapsto} e_0^- \overset{\eta_-}{\mapsto} e_{-1}^- = 0.$$

Each vector e_j^- of this chain is determinated by e_{j-1}^- not uniquely but modulo K_-. Since the pairing $\langle C_+, K_- \rangle$ is non-degenerate, we can modify this chain (in the unique way!) in order to have $e_j^- \in C_+^\perp$ $\forall j \geq 1$. Denote by L_- the linear span of the chain chosed in the such way. We have, in particular,

$$\langle C_+ \cap W_+, L_- \rangle = 0.$$

It is easy to check that L_- is biorthogonal to W_+. Actually, any $w^+ \in W_+$ can be written as $w^+ = \eta_+^m c^+$, where $c^+ \in C_+ \cap W_+$. Hence, $\forall j$ we have:

$$\langle w^+, e_j^- \rangle = \langle \eta_+^m c^+, e_j^- \rangle = (-\mu)^m \langle c^+, \eta_-^m \kappa^{-m} e_j^- \rangle = 0,$$

because $\langle c^+, L_- \rangle = 0$ and L_- is invariant under the action of κ and η_-. Orthogonality in the opposite direction follows immediately:

$$\langle L_-, w^+ \rangle = \langle w^+, \kappa L_- \rangle = \langle w^+, L_- \rangle = 0.$$

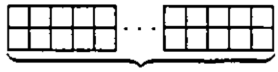
Starting from the others basic vectors of K_- we can construct in the same way as above a direct decomposition $V_- = L_- \oplus W_-$ such that $\langle L_+, W_- \rangle = \langle W_-, L_+ \rangle = 0$. Hence, the subspace $L_+ \oplus L_- \subset V$ is a biorthogonal direct summand in V. Since V is indecomposable, we have $V = L_+ \oplus L_-$.

□

3.3.2. PROPOSITION. *If V is indecomposable and the ε-case takes place, then either η has the Young diagram of the form*

$n+1$

and $\varepsilon = (-1)^n$ or η has the Young diagram of the form



$n+1$

and $\varepsilon = (-1)^{n+1}$.

PROOF. Let $\eta^n \neq 0$ but $\eta^{n+1} = 0$. It follows from proposition 3.2.3 that the bilinear form

$$(v, w) \stackrel{\text{def}}{=} \langle v, \eta^n w \rangle$$

is well defined and non-degenerate on the factor-space $V/\ker(\eta^n)$. The calculation:

$$\begin{aligned}(w, v) &= \langle w, \eta^n v \rangle = \langle \eta^n v, w \rangle = \langle v, (\eta^\vee)^n \kappa w \rangle = (-\varepsilon)^n \langle v, \kappa^{-n} \eta^n \kappa w \rangle = \\ &= (-\varepsilon)^n \langle v, (\varepsilon E + \eta)^{1-n} \eta^n w \rangle = (-\varepsilon)^n \varepsilon^{1-n} \langle v, \eta^n w \rangle = \\ &= (-1)^n \varepsilon (v, w)\end{aligned}$$

shows that this form is symmetric for $\varepsilon = (-1)^n$ and is skew-symmetric for $\varepsilon = (-1)^{n+1}$.

In the first case we can find an orthonormal basis of $V/\ker(\eta^n)$ with respect to this symmetric form. If we construct a Jordan basis for η in V starting with this orthonormal basis of $V/\ker(\eta^n)$, then we will be in a position to apply the arguments from the proof of the previous proposition. Exactly as above we can modify the Jordan chain of maximal length in such a way that its linear span will be detached as biorthogonal direct summand. Hence, in this case κ has only one Jordan cycle and its length is modulo 2 different from the eigenvalue of κ.

In the second case we can decompose $V/\ker(\eta^n)$ with respect to symplectic form $(*, *)$ into direct sum of standard 2-dimensional symplectic planes, which are orthogonal to each other. If we fix a symplectic basis in one of these planes, then, as above, we can construct Jordan chains ended in these two vectors in such a way that its linear span will be detached as biorthogonal direct summand. Hence, in this case κ has only two Jordan cycle of the same length and this length is modulo 2 equal to the eigenvalue of κ.

□

3.4. Forms of type 2. Note that the second case of proposition 3.3.2 can be considered as a particular case of the proposition 3.3.1. Namely, in both cases the structure of the space V is described by the following definition.

3.4.1. DEFINITION. *Let K be an arbitrary field, V be a vector space of even dimension $\dim V = 2k$ over K, and $\mu \in K$, $\mu \neq (-1)^{k+1}$. Non-degenerate indecomposable bilinear form on the space V is called to be of the type 2, if*

$$V = V_+ \oplus V_- \,. \quad \kappa|_{V_\pm} = \mu^{\pm 1} E + \eta_\pm \,,$$

where $\dim V_+ = \dim V_- = k$, $\eta_\pm^k = 0$, $\eta_\pm^{k-1} \neq 0$, the pairing $\langle V_+, V_- \rangle$ is non-degenerate, and both restrictions $\langle *, * \rangle_{V_+}$, $\langle *, * \rangle_{V_+}$ are identically equal to zero.

It is easy to see that non-degenerate indecomposable forms of the type 2 actually exist for any $k \in \mathbb{N}$ and $\mu \in K$. Moreover, from the proofs given above one can extract some standard form for their Gram matrix.

3.4.2. COROLLARY. *Each non-degenerate indecomposable form of the type 2 has at some appropriate basis the Gram matrix*

$$\left(\begin{array}{c|c} 0 & \begin{matrix} & & & & \mu \\ & 0 & & \mu & 1 \\ & & \iddots & \iddots & \\ & \mu & 1 & & 0 \\ \mu & 1 & & & \end{matrix} \\ \hline \begin{matrix} & & & 1 \\ & 0 & 1 & \\ & \iddots & & \\ & 1 & 0 & \\ 1 & & & \end{matrix} & 0 \end{array}\right)$$

PROOF. First consider a form of the type 2. In this case $V = V_+ \oplus V_-$, $\dim V_+ = \dim V_- = k$. Let us fix an arbitrary Jordan basis

$$e_k^+ \overset{\eta_+}{\mapsto} e_{k-1}^+ \overset{\eta_+}{\mapsto} \cdots \overset{\eta_+}{\mapsto} e_0^+ \overset{\eta_+}{\mapsto} e_{-1}^+ = 0$$

for η_+ in V_+. Since the pairing $\langle \ker(\eta_-^j), V_+/\mathrm{im}\,(\eta_+^j) \rangle$ is non-degenerate $\forall j$, we can find a sequence of vectors $v_j \in \ker(\eta_-^j)$ (where $j = 0, 1, \ldots, k$) such that

$$\begin{aligned} \langle v_j, e_{k-j}^+ \rangle &= 1 \\ \langle v_j, e_{k-\nu}^+ \rangle &= 0 \quad \text{for } \nu = 0, 1, \ldots, (j-1). \end{aligned}$$

Note that by (3.2.3) we have also

$$\langle v_j, e_{k-\nu}^+ \rangle = 0 \quad \text{for } \nu = (j+1), (j+2), \ldots, k.$$

These relations determinate the scalar products in the opposite order too:

$$\begin{aligned}\langle e^+_{k-\nu}, v_j\rangle &= \langle v_j, \kappa e^+_{k-\nu}\rangle = \mu\langle v_j, e^+_{k-\nu}\rangle + \langle v_j, e^+_{k-\nu-1}\rangle = \\ &= \begin{cases} \mu & \text{for } \nu = j \\ 1 & \text{for } \nu = j-1 \\ 0 & \text{for all other } \nu \end{cases} .\end{aligned}$$

We see that the Gram matrix of the basis $\{e^+_0, \dots, e^+_k, v^+_0, \dots, v^+_k\}$ has the form what is needed.

On the other hand, simple direct computation shows that the canonical operator of this Gram matrix actually has two Jordan cycles of length k with eigenvalues μ and μ^{-1}.
□

3.5. Forms of type 1. Actually, much more interesting for us are the forms, which satisfy the remainder first condition from proposition 3.3.2.

3.5.1. DEFINITION. *Non-degenerate indecomposable bilinear form on a space V of dimension $\dim V = n+1$ is called to be of the type 1, if*

$$\kappa = (-1)^n E + \eta .$$

where $\eta^{n+1} = 0$, $\eta^n \neq 0$.

We will show in §4 that the form on $K_0(\mathbb{P}_n)$ is of type 1. Let us consider the forms of type 1 in a more details. We put in this section $K = \mathbb{Q}$, because this case will be used in the next paragraph, but actually all results are true for any fild of characteristic zero.

We fix a vector space V of dimension $\dim V = n+1$ over $\mathbb{Q}$ and denote the sign $(-1)^n$ by ε. Since the canonical operator of a form of type 1 has the form $\kappa = \varepsilon E + \eta$ and nilpotent operator η has a Jordan chain of length $n+1$, the centralizer of the canonical operator in $\mathrm{End}_{\mathbb{Q}}(V)$ (i.e. the *canonical algebra* $\mathcal{A}$, see 2.4) coincides with the commutative subring

$$\mathcal{A} = \mathbb{Q}[\eta]/\eta^{n+1} \subset \mathrm{End}_{\mathbb{Q}}(V) .$$

In order to study the involution v (see 2.4) it is more convenient to choose an other generator of the canonical algebra. Namely, let

$$\zeta = \frac{\varepsilon\kappa - E}{\varepsilon\kappa + E} = \frac{1}{2}\varepsilon\eta(E + \frac{1}{2}\varepsilon\eta)^{-1}.$$

We have

$$\kappa = \varepsilon\frac{1+\zeta}{1-\zeta}; \qquad \eta = 2\varepsilon(\zeta + \zeta^2 + \dots + \zeta^n)$$

and κ, η, ζ are uniquely determinated by each other. Obviously, ζ is nilpotent, has the same Jordan normal form as η, $\ker(\eta^i) = \ker(\zeta^i)\,\forall i$, and $\mathbb{Q}[\zeta] = \mathbb{Q}[\eta]$.

By proposition 3.2.3 the involution $\vee$ is uniquely determinated by the condition $\eta^\vee = -\varepsilon\kappa^{-1}\eta$. Hence, this involution coincides with the involution of the ring $\mathbb{Q}[\zeta]$, which takes ζ to $-\zeta$, and acts on $\mathcal{A} = \mathbb{Q}[\zeta]/\zeta^{n+1}$ by the rule

$$f(\zeta)^\vee = f(-\zeta) \quad \forall f \in \mathbb{Q}[\zeta]\,.$$

In particular, the Gram matrix of any Jordan basis $\{e_i\}$ for ζ is uniquely determinated by its right column by the simple rule:

$$\langle e_i, e_j\rangle = \langle \zeta^{n-i}e_n, \zeta^{n-j}e_n\rangle = \begin{cases} (-1)^j\varepsilon\langle e_{i+j-n}, e_n\rangle & \text{for } n \le (i+j) \le 2n \\ 0 & \text{for } (i+j) < n \end{cases}. \tag{f.3.5A}$$

We see that the subspace $\mathcal{A}_+ \subset \mathcal{A}$ of all selfdual operators coincides with the subspace of all operators $f(\zeta)$ represented by even polynomials f. The subspace $\mathcal{A}_- = \mathcal{L}ie(Isom) \subset \mathcal{A}$ of all antiselfdual operators is generated by $k = \left[\frac{n+1}{2}\right]$ odd powers $\zeta, \zeta^3, \dots, \zeta^{2k-1}$.

3.5.2. PROPOSITION. *The isometry group of a form of type 1 is Abelian and has two connected components. The component of the identity is isomorphic to the direct product of* $\left[\frac{n+1}{2}\right]$ *standard 1-dimensional additive unipotent algebraic groups.*

PROOF. It follows from above remarks that the exponential map

$$(t_1, t_2, \dots, t_k) \mapsto e^{t_1\zeta}e^{t_2\zeta^3}\dots e^{t_k\zeta^{2k-1}}$$

gives an isomorphism between affine additive group of rank $\left[\frac{n+1}{2}\right]$ and the connected component of the identity $Isom_0$.

Isom can be defined as an algebraic subvariety in the affine space $\mathcal{A}$ by equation

$$f^\vee \cdot f = f(-\zeta)f(\zeta) = 1\,.$$

If we use the coefficients $(a_0, a_1, a_2, \dots, a_n)$ of polynomials

$$f(\zeta) = a_0 + a_1\zeta + \dots + a_k\zeta^k \in \mathcal{A}$$

as coordinates on $\mathcal{A}$, then this equation is equivalent to the system of $k+1$ quadratic equations

$$\begin{cases} a_0^2 &= 1 \\ 2a_2^2 &= -a_1^2 \\ 2a_4^2 &= -2a_1a_3 - a_2^2 \\ \dots & \\ 2a_{2k} &= -2a_1a_{2k-1} - 2a_2a_{2k-2} - \dots - 2a_{k-1}a_{k+1} - a_k^2 \end{cases}.$$

We see that for any choose of a value $a_0 = \pm 1$ and any fixed values of $\left[\frac{n+1}{2}\right]$ odd coefficients $a_{2\nu+1}$ there exist a unique collection of values of even coefficients such that $f(\zeta)$ is isometric. Hence, the projection of the affine space $\mathcal{A}$ onto the subspace generated by odd coordinates gives an isomorphism of *Isom* with the disjoint union of two such subspaces.

□

Recall that in proof of the proposition (3.3.2) we consider non-degenerate bilinear form $(v,w) = \langle v, \eta^n w\rangle$ over the factor $V/\ker(\eta^n)$. If the original bilinear form $\langle *, *\rangle$ is of type 1, then this form $(*, *)$ is symmetric and factor $V/\ker(\eta^n)$ is 1-dimensional. Hence, the number $\varrho \stackrel{\text{def}}{=} (v,v) = \langle v, \eta^n v\rangle$ modulo multiplication by squers does not depend on $v \in V/\mathrm{im}\,(\eta)$.

3.5.3. PROPOSITION. *For any indecomposable rational form of type 1 on $(n+1)$-dimensional vector space there exists a Jordan basis of ζ over quadratic extension $\mathbb{Q}(\sqrt{\varrho/2^n})$ such that the Gram matrix of the form at this basis is equal to*

$$\begin{pmatrix} & & & & & 1 \\ & 0 & & & -1 & 1 \\ & & & 1 & -1 & \\ & & -1 & 1 & & \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & & 0 & \\ (-1)^n & (-1)^{n-1} & & & & \end{pmatrix} .$$

PROOF. Let us fix an arbitrary Jordan basis

$$e_n \stackrel{\zeta}{\mapsto} e_{n-1} \stackrel{\zeta}{\mapsto} \cdots \stackrel{\zeta}{\mapsto} e_0 \stackrel{\zeta}{\mapsto} e_{-1} = 0$$

for ζ. We are going to find a selfdual operator

$$f = b_0 + b_2\zeta^2 + b_4\zeta^4 + \cdots + b_{2k}\zeta^{2k}$$

such that

$$\langle fe_i, e_n\rangle = \begin{cases} 1 & \text{for} \quad i = 0,1 \\ 0 & \text{for} \quad i \geq 2 \end{cases} .$$

In order to do this note that $\langle fv, fw\rangle = \langle v, gw\rangle$, where

$$g = f^* f = a_0 + a_2\zeta^{*2} + a_4\zeta^{*4} + \cdots + a_{2k}\zeta^{*2k}$$

is selfdual too. Using orthogonality conditions (3.2.3) we obtain:

$$\langle e_{2\nu}, ge_n\rangle = a_0\langle e_{2\nu}, e_n\rangle + a_2\langle e_{2\nu}, e_{n-2}\rangle + a_4\langle e_{2\nu}, e_{n-4}\rangle + \cdots + a_{2\nu}\langle e_{2\nu}, e_{n-2\nu}\rangle .$$

Hence, we can choose the constants $\{a_2\nu\}$ such that

$$\langle e_{2\nu}, e_n\rangle = \begin{cases} 1 & \text{for} \quad \nu = 0 \\ 0 & \text{for} \quad \nu \geq 1 \end{cases} .$$

Moreover, we can take $a_0 = \langle e_o, e_n\rangle^{-1}$. In order to get f from g we have to solve the system

$$\begin{cases} a_0 &= b_0^2 \\ a_2 &= 2b_0b_2 \\ a_4 &= 2b_0b_4 + b_2^2 \\ & \dots \\ a_{2k} &= 2b_0b_{2k} + 2b_2b_{2k-2} + 2b_4b_{2k-4} + \cdots \end{cases} .$$

It is possible over the qudratic extension $\mathbb{Q}(\xi)$, where

$$\xi^2 = \langle e_0, e_n\rangle = \varepsilon\langle e_n, e_0\rangle = \varepsilon\langle e_n, \zeta^n e_n\rangle = \frac{1}{2^n}\langle e_n, \eta^n e_n\rangle = \frac{1}{2^n}\varrho\,.$$

Finally, using formula (f.3.5A), for basic vectors with odd indices we get:

$$\begin{aligned} 2\langle e_{2\nu-1}, e_n\rangle &= \langle e_{2\nu-1}, e_n\rangle - \varepsilon\langle e_n, e_{2\nu-1}\rangle = \langle e_n, \kappa e_{2\nu-1}\rangle - \langle e_n, \varepsilon e_{2\nu-1}\rangle = \\ &= \langle e_n, (\kappa - \varepsilon E)e_{2\nu-1}\rangle = 2\varepsilon(\langle e_n, e_{2\nu-2}\rangle + \langle e_n, e_{2\nu-3}\rangle + \cdots) = \\ &= 2(\langle e_{2\nu-2}, e_n\rangle - \langle e_{2\nu-3}, e_n\rangle + \langle e_{2\nu-4}, e_n\rangle - \cdots)\,, \end{aligned}$$

Hence,

$$\langle e_{2\nu-1}, e_n\rangle = \begin{cases} 1 & \text{for} \quad \nu = 1 \\ 0 & \text{for} \quad \nu \geq 2 \end{cases}$$

and by (f.3.5A) we obtain the Gram matrix what is needed.
□

§4. $K_0(\mathbb{P}_n)$ in more details.

4.1. Notations. In this section we consider in more details the module $\mathcal{K}_n = K_0(\mathbb{P}_n)$ with the natural unimodular bilinear form

$$\langle E, F\rangle = \sum(-1)^\nu \dim \mathrm{Ext}^\nu(E, F),$$

Denote by $\mathcal{P}_n \subset \mathbb{Q}[t]$ the subspace of all polynomials of degree $\leq n$, and let $\mathcal{M}_n \subset \mathcal{P}_n$ be the $\mathbb{Z}$-submodule of all polynomials taking integer values at all integer points. We will call such polynomials *numerical*. Evidently, $\mathcal{P}_n = \mathcal{M}_n \otimes_{\mathbb{Z}} \mathbb{Q}$.

The map

$$h\colon \mathcal{K}_n \longrightarrow \mathcal{M}_n\colon\ E \mapsto h_E(t) = \chi(E \otimes \mathcal{O}(t))\ (\text{for } t \in \mathbb{Z}),$$

which takes a coherent sheaf E to its Hilbert polynomial $h_E(t)$, is an isomorphism of $\mathbb{Z}$-modules. We identify $\mathcal{K}_n$ with $\mathcal{M}_n$ by this isomorphism.

It is easy to check that this identification takes the $\mathbb{Z}$-basis of $\mathcal{K}_n$ consisting of the restrictions of the structure sheaf onto subspaces:

$$\{\mathcal{O}_{\mathbb{P}_n}, \mathcal{O}_{\mathbb{P}_{n-1}}, \ldots, \mathcal{O}_{\mathbb{P}_1}, \mathcal{O}_{\mathbb{P}_0}\}$$

to standard binomial $\mathbb{Z}$-basis

$$\{\gamma_n(t), \gamma_{n-1}(t), \ldots, \gamma_0(t)\}$$

of $\mathcal{M}_n$ consisting of

$$\begin{aligned} \gamma_k(t) &= h_{\mathcal{O}_{\mathbb{P}_k}}(t) = \binom{t+k}{k} = \frac{1}{k!}(t+1)(t+2)\cdots(t+k) \\ & \qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad k = 1, 2, \ldots, n \\ \gamma_0(t) &= h_{\mathcal{O}_{\mathbb{P}_0}}(t) \equiv 1 \end{aligned}$$

The restriction operator $E \mapsto E|_{\mathbb{P}_{n-1}}$ onto a hyperplane $\mathbb{P}_{n-1} \subset \mathbb{P}_n$ is represented in terms of $\mathcal{M}_n$ by *left difference operator*

$$\nabla = 1 - e^{-D}\colon \mathcal{M}_n \longrightarrow \mathcal{M}_n\colon\ f(t) \mapsto \nabla f(t) \stackrel{\text{def}}{=} f(t) - f(t-1),$$

where $D = d/dt$. Note that the polynomials γ_ν form a Jordan chain for this operator, i.e. $\nabla^m \gamma_\nu = \gamma_{\nu-m}$.

4.2. Canonical algebra. We denote by $\mathcal{A}_n$ the canonical algebra of all reflexive operators with respect to the form on $\mathcal{M}_n$ comming from $\mathcal{K}_n$ under our identification. Recall that it coincides with centralizer of κ in $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{M}_n, \mathcal{M}_n)$. To describe $\mathcal{A}_n$ we describe first its vectorisation $\mathcal{A}_n \otimes \mathbb{Q}$, i.e. the centralizer of κ in $\mathrm{Hom}_{\mathbb{Q}}(\mathcal{P}_n, \mathcal{P}_n)$.

The canonical operator of the bilinear form on $\mathcal{K}_n$ coincides with the Serre-Verdier dualizing operator, which takes a class of coherent sheaf E to a class $(-1)^n E(-n-1)$. Under the isomorphism h this operator is identified with the operator

$$\kappa = (-1)^n e^{-(n+1)D} : \mathcal{M}_n \longrightarrow \mathcal{M}_n : \ f(t) \mapsto (-1)^n f(t-n-1)$$

Hence, the canonical operator has the form $\kappa = (-1)^n \mathrm{Id} + \eta$, where

$$\eta = (-1)^n (e^{-(n+1)D} - 1) \tag{f.4.2A}$$

is nilpotent operator such that $\eta^n \neq 0$, but $\eta^{n+1} = 0$. So, in terms of the previous paragraph, we get

4.2.1. PROPOSITION. *$\mathcal{M}_n \otimes \mathbb{Q}$ is the space of type 1.*

□

In particular, the centralizer of κ in $\mathrm{Hom}_{\mathbb{Q}}(\mathcal{P}_n, \mathcal{P}_n)$ is equal to $\mathbb{Q}[\eta]/\eta^{n+1}$. Since $\mathbb{Q}[\eta]/\eta^{n+1} = \mathbb{Q}[D]/D^{n+1}$ by (f.4.2A), we get

4.2.2. COROLLARY. *$\mathcal{A}_n \otimes \mathbb{Q} = \mathbb{Q}[D]/D^{n+1}$, where $D = d/dt$*

□

4.2.3. COROLLARY. *$\mathcal{A}_n = \mathbb{Z}[\nabla]/\nabla^{n+1}$, where $\nabla = 1 - e^{-D}$.*

PROOF. Of course, $\mathcal{A}_n \otimes \mathbb{Q} = \mathbb{Q}[D]/D^{n+1} = \mathbb{Q}[\nabla]/\nabla^{n+1}$. So, we have to prove that an operator

$$A = a_0 + a_1 \nabla + \cdots + a_n \nabla^n : \ \mathcal{P}_n \longrightarrow \mathcal{P}_n$$

takes $\mathcal{M}_n$ into $\mathcal{M}_n$ if and only if all $a_\nu \in \mathbb{Z}$. To do this we apply A to γ_ν and evaluate at the point $t = 0$. We get $a_\nu = A\gamma_\nu(0)$. Hence,

$$A\gamma_\nu \in \mathcal{M}_n \Leftrightarrow a_\nu \in \mathbb{Z}.$$

□

4.3. Tensoring and dualizing. There are two more algebraic structures on the module $\mathcal{K}_n$ — the structure of the ring with respect to the tensor product of locally free sheaves and the involution $*$ taking a locally free sheaf E to its dual $E^* = \mathcal{H}om(E, \mathcal{O})$. We carry these operations over the module $\mathcal{M}_n$ by isomorphism h and denote by $\otimes$ and $*$ as well.

Since tensoring by the restriction of the structure sheaf onto hyperplane is represented in terms of $\mathcal{M}_n$ by the operator ∇, we get immediately that

$$\gamma_{n-\nu} \otimes \gamma_{n-\mu} = \gamma_{n-(\nu+\mu)}.$$

Note, that γ_n is the unit element with respect to tensor product.

Let us define the linear map $\mathrm{ch}^{-1}: \mathcal{A}_n \longrightarrow \mathcal{M}_n$ by the rule

$$\mathrm{ch}^{-1}: A \mapsto A\gamma_n$$

Evidently, this map is an isomorphism of $\mathbb{Z}$-modules. Moreover, the following proposition holds:

4.3.1. PROPOSITION. $\mathrm{ch}^{-1}(AB) = \mathrm{ch}^{-1}(A) \otimes \mathrm{ch}^{-1}(B)$, *i.e. the map* ch^{-1} *is an isomorphism of $\mathbb{Z}$-algebras, where the multiplication on $\mathcal{M}_n$ is given by the tensor product and the multiplication on $\mathcal{A}_n$ is given by the composition of operators (or multiplication of formal power series modulo ∇^{n+1}).*

PROOF. It is sufficient to check the formula for basic operators $A = \nabla^k$, $B = \nabla^m$, but in this case it is obvious.

□

The inverse isomorphism $\mathrm{ch}: \mathcal{M}_n \longrightarrow \mathcal{A}_n$ will be called the *Chern character*. We identify $\mathcal{M}_n$ (and $\mathcal{K}_n$) with canonical algebra $\mathcal{A}_n$ by this isomorphism and carry the involution $*$ over $\mathcal{A}_n$ as well. We are going to compare this involution with the involution $\vee$, which takes a reflexive operator to its dual with respect to the bilinear form in the sense of (2.4).

4.3.2. PROPOSITION. *For any $A = A(D) \in \mathcal{A}_n \otimes \mathbb{Q}$ we have*

$$A^\vee(D) = A^*(D) = A(-D).$$

PROOF. The involution induced by the rule $D \mapsto -D$ takes the canonical operator κ to its dual κ^{-1}, and hence, this involution coincides with the involution $\vee$ of the canonical algebra.

Further, for *translation operator* $T = e^D: f(t) \mapsto f(t+1)$ we have $T^\vee = T^{-1}$. This means that $\vee$ acts on $\mathcal{M}_n$ by taking

$$T^k\gamma_n(t) = \gamma_n(t+k) = h_{\mathcal{O}_{\mathbb{P}_n}(k)}(t)$$

to

$$T^{-k}\gamma_n(t) = \gamma_n(t-k) = h_{\mathcal{O}_{\mathbb{P}_n}(-k)}(t).$$

Hence, $\vee$ coincides with $*$.

□

Since for any pair of locally free sheaves we have

$$\langle E, F\rangle = \chi(E^* \otimes F) = h_{E^*\otimes F}(0),$$

we get for the scalar product on $\mathcal{M}_n$ the formula

$$\langle f, g\rangle = f^* \otimes g\,(0).$$

Hence, we obtain

4.3.3. COROLLARY. *In terms of operator D, the scalar product on $\mathcal{A}_n$ is given by*

$$\langle A(D), B(D)\rangle = A(-D)B(D)\gamma_n(0).$$

□

4.4. Standard basises and their Gram matrices. We have seen in the previous paragraph that there exists a basis $\{\Xi_0, \Xi_1, \ldots, \Xi_n\}$ over some quadratic extension $\mathcal{A}_n \otimes \mathbb{Q}(\xi)$ with Gram matrix

$$\begin{pmatrix} & & & & & 1 \\ & 0 & & & -1 & 1 \\ & & & 1 & -1 & \\ & & -1 & 1 & & \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & & 0 & \\ (-1)^n & (-1)^{n-1} & & & & \end{pmatrix}.$$

Recall that the number ξ was defined by quadratic equation

$$\xi^2 = \frac{1}{2^n}\langle 1, \eta^n\rangle = \left(\frac{n+1}{2}\right)^n D^n\gamma_n(0) = \left(\frac{n+1}{2}\right)^n.$$

Hence, for even n the basis in question is retional and for odd n it exists over $\mathbb{Q}(\sqrt{(n+1)/2})$.

Recall also that this basis coincides with some Jordan basis for the operator

$$\zeta = \frac{(-1)^n\kappa - E}{(-1)^n\kappa + E} = -\tanh\left(\frac{n+1}{2}D\right)$$

and has a form $\{\varphi\zeta^n, \varphi\zeta^{n-1}, \ldots, \varphi\zeta, \varphi\}$, where $\varphi = \varphi(D)$ is an appropriate self-dual operator. Unfortunately, I do not know any general explicit formula for these basic operators. Indeed, it is not difficult to calculate them for concrete n. For example, on $K_0(\mathbb{P}_2) \otimes \mathbb{Q}$ we can take

$$\left\{\frac{3}{2}D^2, -D, \frac{2}{3} - \frac{1}{3}D^2\right\}$$

We will suppose that some such basis is fixed in $\mathcal{A}_n \otimes \mathbb{Q}$ for each n and will denote it by $\{\Xi_0, \Xi_1, \ldots, \Xi_n\}$, where $\zeta\Xi_\nu = \Xi_{\nu-1}$. Coordinates $\{z_0, z_1, \ldots, z_n\}$ with respect to this basis may be considered as some characteristic classes of sheaves and it would be very interesting to investigate their geometrical sense.

An other rational basis of $\mathcal{A}_n \otimes \mathbb{Q}$, which is useful for calculations, consists of *Adams operators*

$$\Psi_k(D) \stackrel{\text{def}}{=} \frac{D^k}{k!} \quad (\text{where } k = 0, 1, \ldots n).$$

Note, that for this basis we have $\Psi_k^* = (-1)^k\Psi_k$ too.

Let $A, B \in \mathcal{A}_n \otimes \mathbb{Q}$ be decomposed by Ψ_ν as

$$A = \sum a_\nu \Psi_\nu\,, \quad B = \sum b_\nu \Psi_\nu.$$

Then we can written

$$A^* B = \sum \alpha_\nu(A, B)\Psi_\nu,$$

where each bilinear form $\alpha_k(A, B)$ depends on only first $(k+1)$ coefficients

$$(a_0, a_1, a_2, \ldots, a_k) \quad \text{and} \quad (b_0, b_1, b_2, \ldots, b_k)$$

of A, B. The precise expression for α_k is

$$\alpha_k(A, B) = \sum_{\nu=0}^{k}(-1)^\nu \binom{k}{\nu} a_\nu b_{k-\nu}.$$

Note that $\alpha_k(A, B)$ is symmetric for even k and skew-symmetric for odd k.

4.4.1. PROPOSITION. *The original bilinear form on* $\mathcal{A}_n \otimes \mathbb{Q}$ *is decomposed by the forms* $\alpha_k(A, B)$ *in the following way:*

$$\langle A, B\rangle = \frac{1}{n!}\sum_{k=0}^{n} \sigma_{n-k}(1, 2, \ldots, n)\alpha_k(A, B),$$

where $\sigma_{n-k}(1, 2, , \ldots, n) = \sum_{1\le\nu_1<\nu_2<\cdots<\nu_k\le n} \nu_1\nu_2\cdots\nu_k$ *is the value of* $(n-k)$*-th elementary symmetrical polynomial at the integer point* $(1, 2, \ldots, n)$.

PROOF. It is easy to check that

$$\frac{D^k}{k!}(t+1)(t+2)\cdots(t+n) = \sigma_{n-k}((t+1), (t+2), , \ldots, (t+n)).$$

Hence,

$$\begin{aligned} \langle A, B\rangle &= A^* B\gamma_n(0) = \sum \alpha_k(A, B)\frac{D^k}{k!}\frac{1}{n!}(t+1)(t+2)\cdots(t+n)|_{t=0} \\ &= \frac{1}{n!}\sum \sigma_{n-k}(1, 2, , \ldots, n)\alpha_k(A, B). \end{aligned}$$

□

For example, Gram matrices of Adams basises for $\mathbb{P}_2$, $\mathbb{P}_3$, $\mathbb{P}_4$ are the following:

$$\frac{1}{2}\begin{pmatrix} 2 & 3 & 1 \\ -3 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \frac{1}{6}\begin{pmatrix} 6 & 11 & 6 & 1 \\ -11 & -12 & -3 & 0 \\ 6 & 3 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\frac{1}{24}\begin{pmatrix} 24 & 50 & 35 & 10 & 1 \\ -50 & -70 & -30 & -4 & 0 \\ 35 & 30 & 6 & 0 & 0 \\ -10 & -4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

4.5. Isometries and their invariants. We have seen in the previous paragraph that the isometry group *Isom* has two connected components and the component of the identity coincides with the image of the exponential map applyed to the subspace $\mathcal{A}_-\otimes\mathbb{Q}$ of all antiselfdual operators. In terms of operator D, this subspace consists of all operators $A(D)$ represented by odd power series.

4.5.1. PROPOSITION. *Two nondegenerate operators $A, B \in \mathcal{A}_n \otimes \mathbb{Q}$ belong to the same orbit of the natural action of Isom on $\mathcal{A}_n \otimes \mathbb{Q}$ by multiplications if and only if $A^*A = B^*B$ in $\mathcal{A}_n \otimes \mathbb{Q}$.*

PROOF. If $A = B\Phi$ for some $\Phi \in Isom$, then

$$A^*A = B^*B\Phi^*\Phi = B^*B,$$

because $\Phi^*\Phi = 1$.

At the same time, if $A^*A = B^*B$ and A and B are invertible, then $\Phi = AB^{-1}$ satisfies the condition

$$\Phi^* = A^*B^{*-1} = BA^{-1} = \Phi^{-1},$$

and hence, it is isometric.

□

4.6. The rank. Let us define the *rank functional*

$$\mathrm{rk} : \mathcal{A}_n \longrightarrow \mathbb{Z}$$

by the rule:

$$\mathrm{rk}\,(A) \stackrel{\mathrm{def}}{=} \langle A, \nabla^n\rangle = \varepsilon\langle \nabla^n, A\rangle \ \ \forall A = A(\nabla) \in \mathcal{A}_n .$$

It follows from the orthogonality conditions (3.2.3) that

$$\mathrm{rk}\,(x_0 + x_1\nabla + \cdots + x_n\nabla^n) = x_0 .$$

Hence, $\forall\, A, B$ we have $\mathrm{rk}\,(A)\mathrm{rk}\,(B) = \langle A, \nabla^n B\rangle = \varepsilon\langle \nabla^n A, B\rangle$.

Geometrically, if the operator A corresponds to the class of a locally free sheaf, then $\nabla^n A$ corresponds to its restriction onto a point. Hence, the rank defined above coincides in this case with the usual rank of locally free sheaf.

Since $\nabla = 1 - e^{-D}$ and $\nabla^n = \varepsilon D^n$ we can calculate rank in terms of D by the formula

$$(\mathrm{rk}\,A)^2 = \varepsilon\langle A, D^n A\rangle .$$

So, $\operatorname{rk}(a_0\Psi_0 + a_1\Psi_1 + \cdots + a_n\Psi_n) = a_0$.

In terms of an other useful operator $\zeta = -\tanh((n+1)D/2)$ we have

$$D = -\frac{1}{n+1}\log\left(\frac{1+\zeta}{1-\zeta}\right) \quad \text{and} \quad D^n = \left(\frac{-2}{n+1}\right)^n \zeta^n .$$

Hence,

$$(\operatorname{rk} A)^2 = \left(\frac{2}{n+1}\right)^n \varepsilon\langle A, \zeta^n A\rangle .$$

So, $\left(\operatorname{rk}(z_0\Xi_0 + z_1\Xi_1 + \cdots + z_n\Xi_n)\right)^2 = \left(\frac{2}{n+1}\right)^n z_n^2$.

4.7. $K_0(\mathbb{P}_2)$ and Markov chain. Let $\mathcal{M}$ be a free $\mathbb{Z}$-module of rank 3 equipped with an unimodular integer bilinear form. Jordan normal form of corresponding the canonical operator on $\mathcal{M} \otimes \mathbb{C}$ may be only one of the following:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\lambda \neq \pm 1$. These three cases are distinguished by a value of the trace $\operatorname{tr}(\kappa)$, which is equal to 3, -1 and $1 + \lambda + \lambda^{-1}$, where $\lambda \neq \pm 1$, respectively.

Suppose now that the form on $\mathcal{M}$ admits some semiorthonormal basis with Gram matrix

$$\chi = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} . \tag{f.4.7A}$$

Easy computation gives $\operatorname{tr}(\kappa) = \operatorname{tr}(\chi^{-1}\chi^t) = 3 - a^2 - b^2 - c^2 + abc$. We get

4.7.1. PROPOSITION. *An integer bilinear form (f.4.7A) is of type 1 (in the sense of previous paragraph) if and only if the numbers $\{a, b, c\}$ satisfy the tripled Markov equation:*

$$a^2 + b^2 + c^2 = abc .$$

□

It is well known (see [Ca]) that all solutions of tripled Markov equations are obtained from the initial solution $\{3, 3, 3\}$ by use of the following two procedures:

(A) changing signs of any two numbers;

(B) changing a value of one of numbers via Vieta theorem:

$$a \mapsto bc - a\,, \ \text{or}\ b \mapsto ac - b\,, \ \text{or}\ c \mapsto ab - c\,.$$

Note now that we can change signs of any two elements of Gram matrix by changing a sign of one of basic vectors. Further, the following three mutations of a semiorthonormal basis $\{e_0, e_1, e_2\}$ (see 2.7):

$$\begin{aligned} L_1: \{e_0, e_1, e_2\} &\longmapsto \{e_1 - \langle e_0, e_1\rangle e_0\,,\ e_0\,,\ e_2\} \\ L_2: \{e_0, e_1, e_2\} &\longmapsto \{e_0\,,\ e_2 - \langle e_1, e_2\rangle e_1\,,\ e_1\} \\ R_2: \{e_0, e_1, e_2\} &\longmapsto \{e_0\,,\ e_2\,,\ e_1 - \langle e_1, e_2\rangle e_2\} \end{aligned}$$

change Gram matrix by the rules

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & -a & c-ab \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & b-ac & a \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & b & a-bc \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence, we have proved

4.7.2. PROPOSITION. *Any semiorthonormal basis of an integer bilinear form of type 1 can be transformed using the braid group action and changing signs of basic vectors to the semiorthonormal basis with Gram matrix*

$$\begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}.$$

□

The last matrix coincides with the Gram matrix of the basis $\{\mathcal{O}, \mathcal{T}(-1), \mathcal{O}(1)\}$ of $K_0(\mathbb{P}_2)$, where $\mathcal{T}(-1)$ is the twisted tangent sheaf. We get

4.7.3. COROLLARY. *There exists a unique up to integer isometries integer bilinear form of type 1, which admits a semiorthonormal basis. This form coincides with the form on $K_0(\mathbb{P}_2)$.*

□

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